WHAT IS...

a Borel Reduction?

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ABSTRACT. Borel reductions provide a method of proving that certain problems are impossible using countably infinitary techniques based on countable information and provide a hierarchy of difficulty for classification problems. This is illustrated with examples, including a recent result that a classification problem in dynamical systems proposed by von Neumann in 1932 is impossible to solve with inherently countable tools.

Mathematics is uniquely capable of producing *impossibility results*. The most famous examples include the impossibility of

- proving the parallel postulate
- squaring the circle
- solving a general quintic polynomial
- solving the word problem for finitely presented groups.

What do these results have in common? They have rules that determine what methods are considered legal for a solution. For example, the quintic is *unsolvable by radicals*. Explicitly there is no algebraic formula for solving the general quintic that uses expressions of the form $a^{1/n}$ ($a \in \mathbb{Q}$). Quintics are trivially solvable if you allow expressions that stand for solutions to arbitrary equations. Similarly it is impossible to square the circle *using ruler and compass*; it is impossible to prove the parallel postulate *using the other Euclidean axioms*, and so forth.

The notion of *unsolvability* has various alternate meanings, including the related notion of *independence*. In the context of the *word problem*, being solvable would mean the existence of a *recursive algorithm* for deciding whether two words in the generators represent the same element of the group. Heuristically, this would mean that there is a protocol using inherently finite information that converges in finite time with a yes-no answer to the question.

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In contrast, here we describe a method for proving an emerging form of impossibility result that says

Doing **X** is impossible using
inherently countable techniques.

Note that being unsolvable using *inherently countable* techniques is a much stronger result than unsolvability using inherently finite techniques. Moreover the objects we describe here give a "hierarchy of difficulty" for many types of problems in mathematics.

What precisely does the phrase *inherently countable technique* mean? The context is Polish Spaces—those spaces whose topology can be induced by a complete separable metric. The collection of Borel sets is the smallest σ-algebra that contains the open sets. The Borel sets can be viewed as the broadest class of sets for which membership can be modeled as passing a countable—possibly transfinite—protocol of yes/no questions asked of an arbitrary countable collection of basic open sets. Thus the statement that "A is *not* Borel" says that there is **no** inherently countable method of determining membership in A. The natural setting for considering the Borel/non-Borel distinction is that of *analytic sets*, where a subset A of a Polish space X is *analytic* if it is the continuous image of a Borel subset B of a Polish space Y. Similarly, C is coanalytic if $Y \setminus C$ is analytic.

An example of an impossibility result of this sort is due independently to Kaufman and Solovay, who in 1983-84 showed that the collection of closed sets of uniqueness for trigonometric series is not a Borel set. (A set $E \subseteq [0, 1]$

is a *set of uniqueness* if whenever $\sum c_n e^{2\pi inx} = 0$ on $[0,1] \setminus E$ the series $\sum c_n e^{2\pi inx}$ is identically 0.) Hence the classical problem of deciding whether the complement of a given closed set determines the values of a trigonometric series is simply not possible using anything resembling even a countable transfinite computation. Following these results there have been a plethora of similar results in many areas, including one by Beleznay and the author in 1995 showing that the classically studied collection of so-called *distal* dynamical system is not a Borel set.

B. Weiss and the author[1] hope to publish soon a proof that the program initiated by von Neumann in 1932 ([3]) to classify the statistical behavior of Lebesgue measure-preserving diffeomorphisms of the 2-torus is impossible to carry out using inherently countable techniques. It is currently unknown if isomorphism for diffeomorphisms is strictly above graph isomorphism.

Borel reduction is the main tool for proving certain procedures are impossible. It originated in the late 1980s in the work of Friedman and Stanley (1989) and independently Harrington, Kechris, and Louveau. The idea starts with the cliche that:

> To solve A you reduce it to a problem B which you already know how to solve.

Turning this on its head:

> To show that solving B is impossible, you start with a *known* impossible problem A and reduce it to B.

Formally:

Definition. Let A and B be subsets of Polish spaces X and Y. Then A is *Borel reducible* to B if and only if there is a Borel function $f : X \to Y$ such that for $x \in X$:

$$x \in A \text{ if and only if } f(x) \in B.$$

Thus if A is not Borel, B cannot be either, since the inverse image of a Borel set by a Borel function is Borel. The function f is a *Borel reduction.*

Define $A \leq_{\mathcal{B}} B$ if A is Borel reducible to B. Then $\leq_{\mathcal{B}}$ is transitive since one can compose Borel reductions. Defining the equivalence relation $A \sim_{\mathcal{B}} B$ if $A \leq_{\mathcal{B}} B$ and $B \leq_{\mathcal{B}} A$ we see that $\leq_{\mathcal{B}}$ induces a partial ordering of the $\sim_{\mathcal{B}}$ equivalence classes.

The heuristic above interprets $A \leq_{\mathcal{B}} B$ as saying that B is at least as complicated as A (with respect to countably feasible computations) and $A \sim_{\mathcal{B}} B$ as saying that they have the same complexity. Among analytic sets, there is a $\leq_{\mathcal{B}}$-maximal equivalence class, called the *complete* analytic sets.

For Borel reductions to be useful we must have an example of a non-Borel set A to start with. There are many choices. One canonical example can be found by taking X to be the space of connected acyclic countable graphs (allowing infinite valence) and $A \subseteq X$ to be the set of graphs with a nontrivial end (an end is an infinite path through the graph). Equivalently we can take X to be the space of rooted connected countable trees and A to be the collection of *ill-founded trees*—those trees that have an infinite branch. (Figure 1 represents a tree with an infinite branch.) In each example, the set A is complete analytic and not Borel. Thus if there is a Borel reduction of A to any set B then B is not Borel (and by transitivity B is also complete).

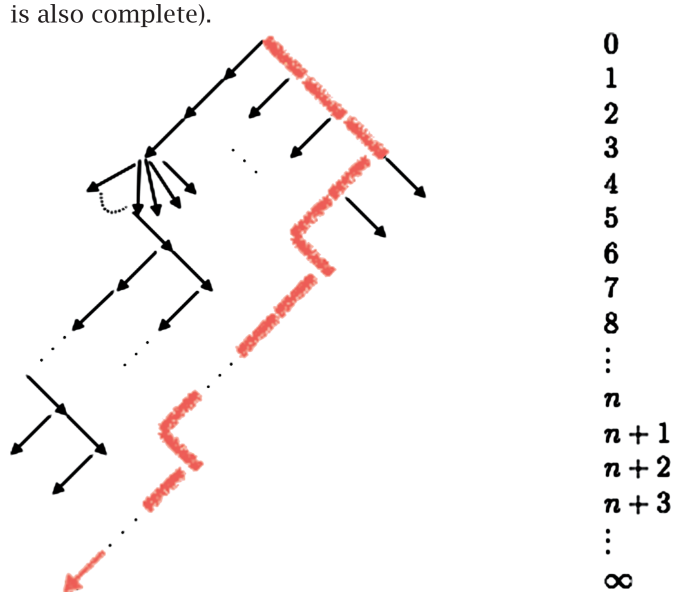

Figure 1. The set of trees with an infinite path, like the tree pictured here, is an example of a complete analytic subset of the space of trees that is not Borel—i.e., cannot be determined by a countable process based on countable information.

An extension of the ordering $\leq_{\mathcal{B}}$ from subsets to relations is its two-dimensional version, which we write $\leq^2_{\mathcal{B}}$. For $E \subseteq X \times X$ and $F \subseteq Y \times Y$, we let $E \leq^2_{\mathcal{B}} F$ if and only if there is a Borel $f : X \to Y$ such that for $(x_1, x_2) \in X$:

$$x_1 E x_2 \text{ if and only if } f(x_1) F f(x_2).$$

The function f is again called a *Borel reduction.*

Classification problems are the most common objects of study here, because they are naturally given by equivalence relations, such as those coming from attaching invariants to collections of objects being studied. Saying that one classification problem E is Borel reducible to another classification problem F is a precise way of saying that determining whether $y_1 F y_2$ is at least as hard as determining whether $x_1 E x_2$. This subject has been studied extensively over the last thirty years by many mathematicians (see [2]).

Analytic equivalence relations fall into five basic intersecting categories (see Figure 2): countable equivalence relations, S^∞-actions, Polish group actions, Borel, and non-Borel. The first three are qualitative:

$$\begin{gathered}\{\text{countable equivalence relations}\}\\ \cap \\ \{S^\infty\text{-actions}\} \\ \cap \\ \{\text{Polish group actions}\}\end{gathered}$$

To these we add the Borel/non-Borel distinction. The countable equivalence relations are all Borel, hence this

[1] *"Measure Preserving Diffeomorphisms of the Torus are Unclassifiable,"* https://arxiv.org/abs/1705.04414

distinction only applies to S^∞-actions, Polish group actions, and those equivalence relations that are neither.

We now define these classes, give examples of each type, and describe which are more complex than others. Many more examples are completely understood; we only scratch the surface of the subject.

Countable equivalence relations

A Borel equivalence relation with countable classes is called a *countable* equivalence relation. It is a theorem of Feldman and Moore (1975) that every such equivalence relation is the orbit relation of a countable group of Borel isomorphisms.

Group actions

The most ubiquitous examples of equivalence relations come from group actions. If G is a Polish group acting on a Polish space X in a Borel manner, then we get the *orbit equivalence relation,* namely $x \sim y$ if and only if there is a $g \in G, gx = y$. Especially important classes of Polish group actions include those of S^∞, the group of permutations of the natural numbers, the group of unitary operators on a separable Hilbert space, the group MPT of measure-preserving transformations of $[0,1]$, and groups of homeomorphisms of compact separable metric spaces.

S^∞-actions

We let S^∞ be the group of permutations of the natural numbers. We illustrate the importance of S^∞-actions with an example. We can identify a countable group $G = \langle g_n : n \in \mathbb{N}\rangle$ with its multiplication table $\{(l,m,n) : g_l \cdot g_m = g_n\}$. Defining $\chi_G : \mathbb{N}\times\mathbb{N}\times\mathbb{N} \to \{0,1\}$ by setting $\chi_G(l,m,n) = 1$ if and only if $g_l \cdot g_m = g_n$, we get an element of $\{0,1\}^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ which, endowed with the product topology, is a compact space homeomorphic to the Cantor set. Let S^∞ act on

$$CG = \{\chi_G : G \text{ is a countable group}\}$$

by setting $(\phi\chi)(l,m,n) = \chi(\phi^{-1}l, \phi^{-1}m, \phi^{-1}n)$. Let $G = \langle g_n\rangle_n$ and $H = \langle h_n\rangle_n$ be isomorphic. Then there is a $\phi \in S^\infty$ such that this isomorphism takes g_n to $h_{\phi(n)}$. Thus $\phi\chi_G = \chi_H$. For two countable groups G and H we've shown:
G is isomorphic to H if and only if χ_G and χ_H are in the same S^∞ orbit.

We conclude that the isomorphism relation for countable groups is naturally encoded as the orbit equivalence relation of an S^∞-action.

Clearly there is nothing special here about groups: *for any class of countable algebraic structures the isomorphism relation is coded by an S^∞-action.* Being Borel reducible to an S^∞-action is thus equivalent to being able to assign countable algebraic structures as invariants. Showing that a given classification problem is *not* reducible to an S^∞-action is an impossibility result interpreted as saying there are no complete algebraic invariants for the equivalence relation. In the mid 1990s, Hjorth gave a general method for doing this—the method of *turbulence*.

Polish group actions

More generally, many classification problems are given as orbit equivalences of Polish group actions. Commonly the group action is some form of *conjugacy*. We now place some benchmarks into the setting being described (Figure 2).

At the bottom of $\leq^2_{\mathcal{B}}$

Since the Cantor set can be injected into every perfect Polish space, the identity equivalence relation on $\{0,1\}^{\mathbb{N}}$ (the diagonal relation), denoted $\mathbf{Id}_{2^{\mathbb{N}}}$, is at the bottom of the $\leq^2_{\mathcal{B}}$ ordering. A given relation E being reducible to $\mathbf{Id}_{2^{\mathbb{N}}}$ is equivalent to being able to attach complete numerical invariants to the equivalence classes of E in a Borel way.

Another important benchmark is E_0: the equivalence relation of eventual agreement of sequence of 0's and 1's. A fundamental result is due to Harrington, Kechris, and Louveau, who proved for a Borel equivalence relation F that either E_0 is reducible to F or F has complete numerical invariants (i.e., is reducible to $Id_{2^{\mathbb{N}}}$).

Maximal relations in a class

Several of the classes have a *maximal* equivalence relation—in the sense that every equivalence relation in that class is reducible to it. We describe these as follows:

For countable equivalence relations

Let F_2 be the free group on 2 generators. Then we can identify the power set of F_2 with the product space $\{0,1\}^{F_2}$ and let F_2 act by left translation on the exponent. (This is the *Bernoulli Shift* for F_2.) The resulting equivalence relation is denoted E_∞. It has countable classes, and every countable Borel equivalence relation is reducible to E_∞.

Another natural example of a maximal Borel equivalence relation among those with countable classes was identified by Hjorth and Kechris (2000): the relation of conformal equivalence among (noncompact) Riemann surfaces.

A third example is isomorphism for finitely generated groups.

For S^∞-actions

A graph whose vertices are natural numbers can be identified with an element of $\{0,1\}^{\mathbb{N}\times\mathbb{N}}$ by setting $\mathrm{X}_G(n,m) = 1$ if and only if n and m are connected by an edge. By letting S^∞ act on the exponent, we code the equivalence relation of *isomorphism of countable graphs.* Every S^∞-action is reducible to isomorphism of countable graphs.

For Polish group actions

Becker and Kechris (1996) proved that for every Polish group there is a $\leq^2_{\mathcal{B}}$-maximal orbit equivalence relation. It then follows from a result of Uspenskiy, showing there

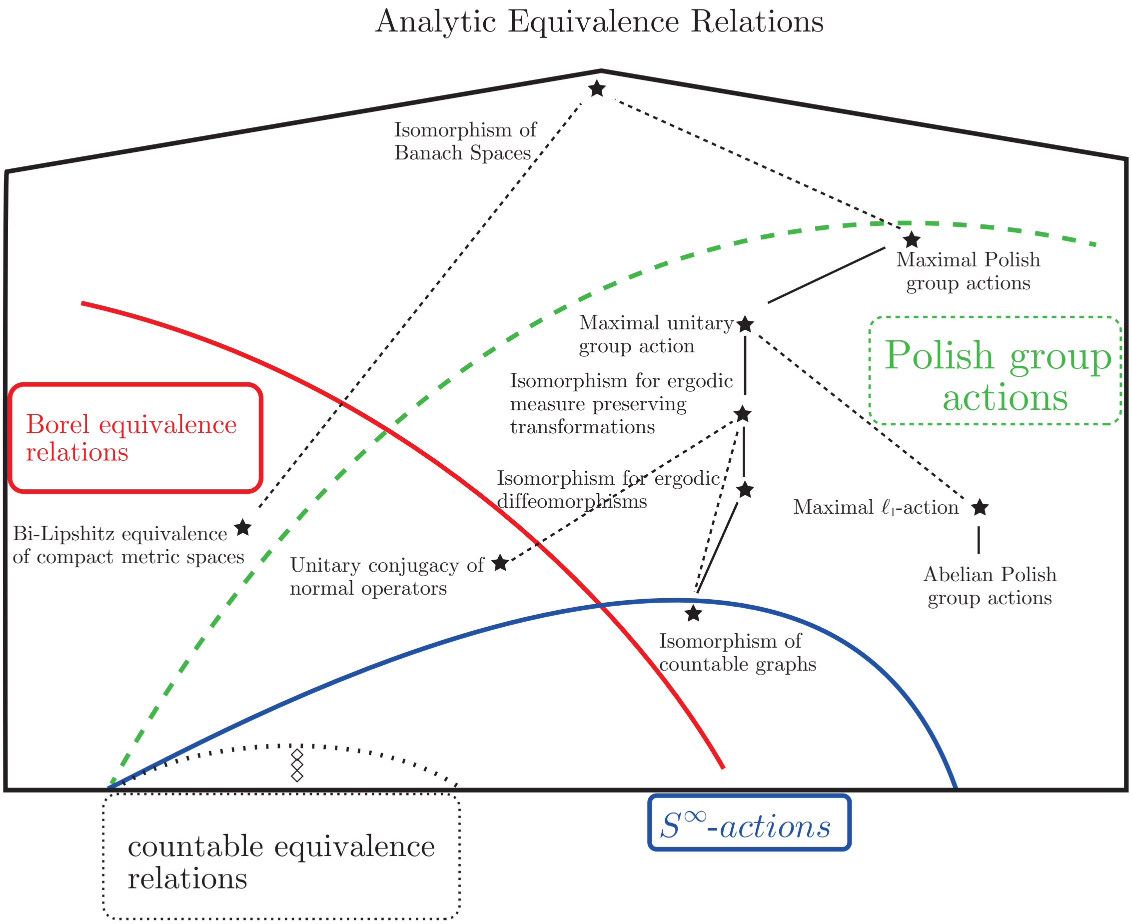

Figure 2. Five basic types of equivalence relations: analytic (the whole box), Borel (in red), those induced by Polish group actions (in green), those induced by S^∞-actions (in blue) and those that have countable classes (at the bottom). In the diagram a line indicates the lower equivalence relation is Borel reducible to the upper equivalence relation and a dotted line indicates the upper equivalence relation is *not* reducible to the lower relation. Other relationships remain open.

is a universal Polish group, that there is a maximal equivalence relation among all Polish group actions.

Borel equivalence relations

Friedman and Stanley (1998) showed that there is no maximal Borel equivalence relation.

For Borel Polish group actions

Hjorth, Kechris, and Louveau (1998) showed there were no maximal Borel Polish group orbit equivalence relations.

Analytic equivalence relations

Harrington proved the existence of a maximal analytic equivalence relation, but it wasn't until the remarkable work of Ferenczi, Louveau, and Rosendal (2009) that a natural example was given. It is *isomorphism for Banach Spaces.*

Placing mathematical examples in the ordering

Many well-known classification results have been placed into the Borel Reducibility ordering. We now give only a tiny sample of the known examples, ending with a recent solution of von Neumann's classification problem for measure-preserving diffeomorphisms.

At the bottom are the countable equivalence relations—those that have countable classes. These are always induced by Borel actions of countable groups. Among many possibilities we take as typical examples questions from the classification of finite-rank torsion-free abelian groups. Thomas showed that they form a collection of problems of strictly increasing complexity as the rank increases. Define the following equivalence relations.

$\cong_{fg}$ the isomorphism relation on finitely generated groups

$\cong_n$ the isomorphism of torsion-free abelian groups of rank n

$\cong_n^p$ the isomorphism relation on p-local abelian groups of rank n.

The relationships between these equivalence relations are given in Figure 3:

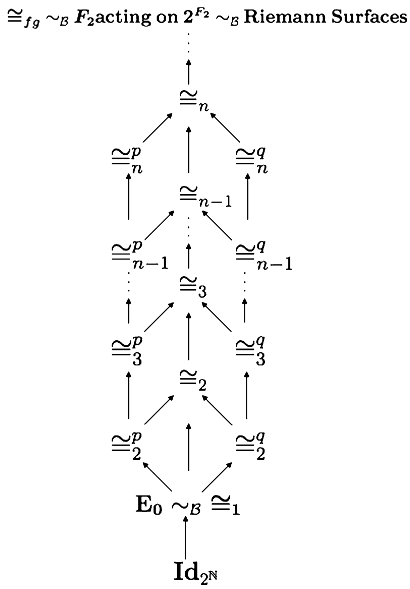

Figure 3. The Borel reducibility ($\leq_{\mathcal{B}}^2$ ordering) of some countable equivalence relations. The central spine consists of isomorphism for torsion free abelian groups of rank n. For primes p the relation $\cong_n^p$ is strictly reducible to $\cong_n$ and if $p \neq q$ are prime the relations $\cong_n^p$ and $\cong_n^q$ are $\leq_{\mathcal{B}}^2$ incomparable.

We now consider the following examples of equivalence relations with uncountable classes (Figure 2):

Unitary conjugacy for normal operators

Here the classical spectral theorem shows the equivalence relation is Borel and it is trivially reducible to the maximal unitary group action. This relation is strictly below isomorphism for measure-preserving transformations.

Bi-Lipschitz equivalence of metric spaces

Rosendal (2005) showed that the relation on pairs of metric spaces given by having a Lipschitz homeomorphism with a Lipschitz inverse is a Borel equivalence relation that is not reducible to a Polish group action.

ℓ_1-actions

Every action of an abelian Polish group can be reduced to an action of the abelian group $\ell_1(\mathbb{N})$ with pointwise addition, hence to the maximal ℓ_1-action. This in turn can be reduced to the maximal unitary group action by results of Gao and Pestov.

Isomorphism for MPTs

This is the equivalence relation of isomorphism (the conjugacy action of MPT) of ergodic measure-preserving transformations of $[0,1]$. Classifying this equivalence relation was proposed by Halmos in 1956. In 2008 Rudolph, Weiss, and the author [1] showed that this equivalence relation is not Borel. The author observed that the graph isomorphism problem can be reduced to isomorphism of ergodic measure-preserving transformations. Furthermore, with Weiss (2003), the author showed the equivalence relation is *turbulent*, hence strictly above every equivalence relation induced by an S^∞-action.

Open problems

We now note some open problems. We give two questions related to geometry and end with a problem internal to the subject.

Classification up to homeomorphism:

Von Neumann was concerned with classifying the statistical behavior of diffeomorphisms. Hence the relevant equivalence relation was *isomorphism by measure-preserving transformations*. In 1967, Smale suggested classifying diffeomorphisms of surfaces up to *conjugation by homeomorphisms*. This spawned a large and successful literature that solved the problem for structurally stable diffeomorphisms, but not in general.

Let M be a compact surface. Where does the equivalence relation *conjugacy by homeomorphism* of pairs of diffeomorphisms of M sit in Figure 2? In particular *is it Borel*?

Classifying smooth $\mathbb{R}^4$ structures:

Taubes proved in 1987 that there are a continuum of smooth structures on $\mathbb{R}^4$ up to equivalence by diffeomorphisms. What is the complexity of this equivalence relation on smooth structures?

What happens at the top?

Many problems, such as isomorphism of ergodic diffeomorphisms of the 2-torus, are reducible to the maximal Polish group action, but it is not known if the reductions are strict. While it seems unlikely to practitioners, it could be that the problems shown are all $\leq_{\mathcal{B}}^2$-equivalent.

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ABOUT THE AUTHOR

Matthew Foreman sailed his C&C 44 across the Atlantic, and around Europe and the Mediterranean. He also circumnavigated Newfoundland. In a different sailboat, he rounded Cape Horn.