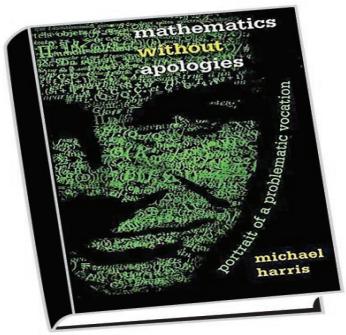
BOOK REVIEW



Mathematics without Apologies

A Review by John McCleary

Communicated by Thomas Garrity

Mathematics without Apologies: Portrait of a Problematic Vocation
By Michael Harris
Princeton University Press, 2015
464 pages

"Is any wonder that, in popular culture's more serious precincts, the mathematician has become the romantic figure of choice?"

—*From* Mathematics without Apologies

In *A Mathematician's Apology*, G. H. Hardy asks: "Why is it really worth while to make a serious study of mathematics? What is the justification of a mathematician's life?" Michael Harris takes up these questions 75 years later in a world where Hardy would have felt alien. Nowadays mathematics, even pure mathematics, has made significant changes to the lives of everyone. Information is available instantly and securely almost everywhere. Economic forces influence individual lives more directly. Hardy would certainly recognize that mathematics is not well understood by most. Harris's unapologetic answers to Hardy's questions are placed in our "age when everything has to prove immediate profitability."

Harris bases his account on his own career, not as autobiography, but as raw material for his journalistic impulse. He also draws on his insatiable thirst for ideas—in philosophy, sociology, film, music, religion, and more. Frankly, to review this book in detail is beyond my pay grade. The torrent of associations Harris makes produces a "deliberately elusive style." But a book on mathematics that can leave many readers with many different impressions is welcome. Let me share mine.

How do we understand mathematical activity? Especially pure mathematics? I recall a spirited discussion of "What is mathematics?" in an institute's coffee room. The positions were *a philosophy*, for how could the most trivial computation require knowledge of the homotopy groups of spheres; *a chess game*, for we need not know what we are moving, only how to move it correctly, and cleverly; and *a garden*, for we plant ideas and coax them to grow and propagate. Harris's answer is in his subtitle: "Portrait of a Problematic Vocation." Problems are the stuff that determines what being a mathematician is about.

In the practice of mathematics, some activity (philosophical insight, chess move, or weeding) takes place—at a blackboard, in conversation, at a desk—after which mathematicians produce a proof, a notion that has evolved considerably since its inception in ancient Greece. A written proof, however, cannot be identified with what it means to do mathematics. Atiyah has written "I may think that I understand, but the proof is the check that I have understood, that's all. It is the last stage in the operation—an ultimate check—but it isn't the primary thing at all." Harris goes on to say that "answers are less important than how they change the way we look at the questions."

So what questions are the right ones? How does a mathematician find problems? The first chapter of the book opens with a quote from Hilbert's Paris 1900 ICM address—"to lift the veil behind which the future lies." Did the Hilbert problems, as he suggested, lead to "an advancement of science"? Here we have a chicken-and-egg problem. In the framework that Harris outlines in the book, Hilbert was a charismatic leader, whose followers found value in his suggestions. By working on these prob-

John McCleary is professor on the Elizabeth Stillman Williams chair of mathematics at Vassar College. His email address is mccleary@vassar.edu.

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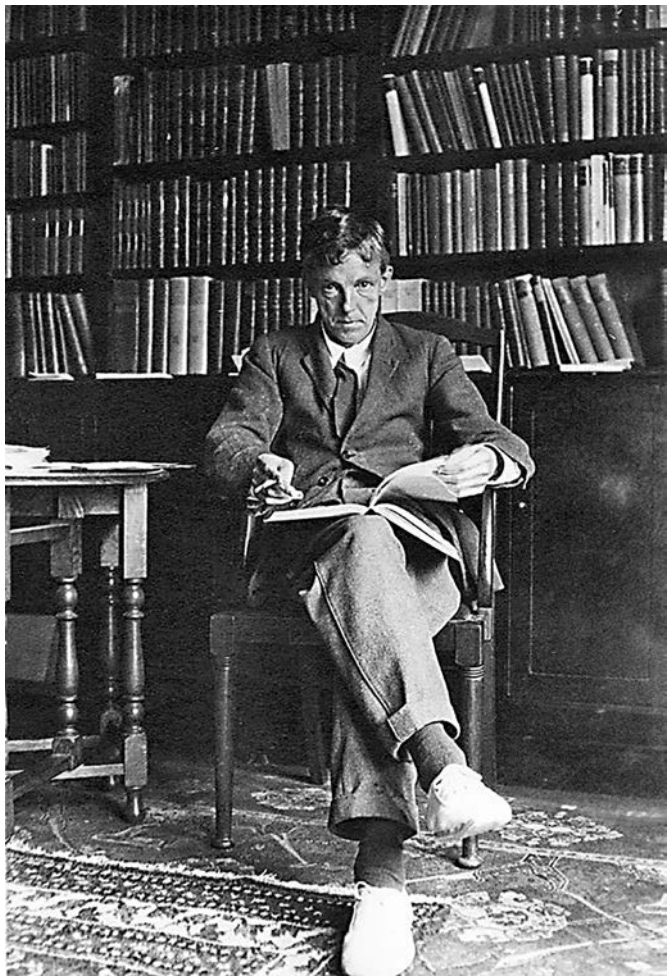
lems individuals could increase their own stature, their *charisma*, in the mathematical community. Today there are many research programs that generate problems for groups of specialists. Harris is part of a number theory tribe, for which he has contributed significantly to the Langlands program. Generalizing from his own example, he describes mathematical socialization through the lens of Max Weber's sociological analysis of authority. Legal or traditional authority does not motivate mathematicians to prove theorems; charisma does. Harris introduces *routinized charisma*. For Weber this is how the charisma of a leader turns into a new bureaucracy. For mathematics, those with charisma shape mathematical activity—by the problems leaders work on or dispense and by the public value placed on such work. Success in mathematics begins when ability is identified, usually by a teacher or parent, training begins, and the milestones of an academic career are passed—PhD, post-doc, tenure, prizes, better positions—each step with enhanced charisma.

In the wider world charisma is rewarded, usually with wealth. Mathematicians and scientists generally appear to be outside this kind of valuation. For example, Gauss wrote that there is "a pure, disinterested joy in study." Harris focuses on a trilogy of motivations signaled by the words *good, truth, and beauty*. The good, called the "Golden Goose" by Stephen Shapin, is the promise that research in pure mathematics may one day bring unanticipated applications that benefit humankind. Hardy dismisses the "Golden Goose" by suggesting that it is the least interesting parts of mathematics that find their way into applications. The higher parts of mathematics are unsullied by applications. Hardy writes, just five years before Hiroshima, "No one has yet discovered any warlike purpose to be served by the theory of numbers or relativity, and it seems very unlikely that anyone will do so for many years."

Truth, the certitude of mathematical results, has been a philosophical touchstone since Descartes. Harris has a lot to say about the rise of interest in machine verification of mathematics and the danger of proofs that are so complex they defy human verification. There is great satisfaction in discovering a truth, old or new. But providing the bedrock of a philosophical program does not motivate most mathematicians.

That leaves beauty, the deepest source for Hardy, and the most challenging motivation for mathematical activity. Challenging because it calls into question how external support of mathematics is justified. Hardy tells us that "a mathematician, like a painter or poet, is a maker of patterns." Throwing mathematicians in with the painters and poets leads to justification of mathematics as an art. Harris situates his discussion of beauty in the notion of play. Play needs to take place in a "relaxed field," a notion introduced by ethologist Gordon Burghardt to describe animal behavior when not preoccupied with food, mating, territory, or predators—"the opposite of stress." Cantor's *Paradies* and Hausdorff's *Spielraum* are the relaxed fields of thought where mathematics arises. And when it does, it is a source of pleasure. Harris makes a good case for the mathematician's sense of beauty to be located in the rush of pleasure that accompanies mathematical activity. Today neuroscientists have identified the places activated in our brains by pleasure. In fact, in a small study employing fMRI imaging, the field A1 of the mOFC (medial orbito-frontal cortex) lights up when presented with beautiful mathematics. The mOFC is the same part of the brain that responds to other experiences of beauty.



In his *A Mathematician's Apology*, Hardy asks: "Why is it really worth while to make a serious study of mathematics?"

If mathematicians are motivated by play, how can the vocation be justified to those who support it? The golden eggs of pure mathematics are plentiful. Chapter 4 presents an introduction to mathematical finance as an example of the modeling of reality, and as a Faustian bargain. Students are attracted by high-paying positions as quants, and academic mathematicians train such students to think carefully, analytically, and critically about mathematics that will "make money for the firm." The pressure on the mathematical community to feed the needs of industry appears in several places in the book. Harris, from his charismatic position, participated in workshops about the future of mathematics hosted by the *decideurs*, who acknowledge the difficulty of measuring pure mathemati-

cal research as a resource—but measure they will, posing a threat to the relaxed field enjoyed by mathematicians.

In Hardy's time, mathematicians were immersed in the beginnings of World War II. The ethical challenges were clear, and of course many contributed to the war effort. For Hardy, however, "real mathematics," that is, pure mathematics, "has no effects on war." He was justifying not only his place in mathematics, but his response to the war. By contrast, 35 years later, in the midst of the Cold War, Stanislaw Ulam published his *Adventures of a Mathematician*, another account of what is it like to be a mathematician, in this case, an applied mathematician. His apology is brief:

"…I failed to realize fully the immense importance of nuclear armament and the influence it would have on the course of world events.… I would describe myself as having taken a middle course between completely naive idealism and extreme jingoism. I followed my instincts (or perhaps lack of instincts) and was mainly interested in the scientific aspects of the work."

Ulam's book is entertaining with its great cast of prominent mathematicians and physicists who led the A-bomb and H-bomb efforts—a relaxed field of a different sort.

Michael Harris takes up Hardy's question about the purpose of mathematics 75 years later in a world where Hardy would have felt alien.

For Harris, it is the "profoundly contingent" nature of mathematics that worries him, as he told an interviewer at CIRM. Without taking foundational questions seriously, mathematicians will be faced with machines that will produce a substitute for mathematics. The humanity of mathematics is indispensable, and it needs to be better understood and encouraged.

The other chapters of the book take up the humanity of mathematics with gusto. There is a bonus chapter in which Harris offers an insightful critical reading of Thomas Pynchon's novels organized around the conic sections. Chapter 6 focuses on the film *Rites of Love and Math* by Edward Frenkel and Riene Graves, in which a mathematical formula of love leads to the death of its discoverer, but not before he has tattooed it on his lover. Harris takes the film as an occasion to consider the perception of the (mad/martyr) mathematician by the film-going public, giving *Rites* a countercultural quality.

Chapter 7 goes to the place left empty by Atiyah—if a proof is the last stage of the operation, what is the first step? Much is made of the use of "scare quotes," a form of punctuation that indicates where an analogy is straining to communicate an unproved relation or an undefined object. This is the "what if?" that is often part of a mathematical conversation. The discussion covers a dizzying selection of topics, including a description of the Langlands program, higher category theory, and *avatars*—the reflections of one theory in another. Chapter 8 plays with the meaning of the word "trick" inside and outside mathematics. The playful nature of this discussion supports Harris's image of the relaxed field and beauty as pleasure. A quote of Steven Pinker is a good summary. Pinker describes music as "cheesecake," "unlike anything in the natural world because it is a brew of megadoses of agreeable stimuli which we concocted for the express purpose of pressing our pleasure buttons." Harris counts pure mathematics a cheesecake.

Woven through the book are the Greek letter chapters entitled "How to explain number theory at a dinner party." The Number Theorist N.T. explains a large swath of number theory at a dinner party to a Performing Artist P.A., who challenges Number Theorist in dialogue to higher levels of clarity. Performing Artist also provides a foil for Number Theorist's analogies of mathematical objects. From primes and the irrationality of their square roots through algebraic numbers, Galois theory, transcendental numbers, congruences, and elliptic curves, the theme of numbers as answers to questions about numbers is explored. The monologues and dialogues depict how Number Theorist thinks, a sort of sampler of the rest of the book.

Chapter 9 is my favorite. The excitement of being on the trail of an idea, an idea that came in a dream, is engagingly written. As in Villani's *Birth of a Theorem*, the reader is not expected to keep up with the mathematics, but the thrill is evident. I can add another dream result to the discussion—Ulam's obituaries in various newspapers report that his insight that shock waves would play a role in the design of the H-bomb came in a dream. The unconscious as a source of ideas is not new, but another piece of the puzzle of the sources of mathematics.

Chapter 10, "No Apologies," contains a recapitulation of the main points of the book. The narrative ends with an afterword in which Felix Hausdorff (aka Paul Mongré) appears. A poet, dramatist, philosopher, and mathematician, Hausdorff is a model for the kind of free thinker that Harris admires and whose example mathematicians can follow.

There are more than 70 pages of endnotes. I dutifully consulted them as I read, and I found many references that I will return to.

The goal of Harris's *Mathematics without Apologies* is not "to justify mathematics; no one needs another *Mathematician's Apology*." Nor is it to explain mathematics to "ordinary people" (as the British philosopher Crispin Wright would put it), but to tell "what it is like to be a mathematician, freely choosing a tradition to which to adapt, not to serve the Powerful Beings of market rationality …" The image I leave with is his final one, that each veil we turn aside has behind it another veil: mathematics is about problems and the problems that are born from their answers. The book is about "how hard it is to write a book about mathematics," and failing to tell us what mathematics is, Harris has "failed better," to paraphrase Samuel Beckett, and given us a rich, playful, and personal portrait that will raise significant questions for every reader, and give each of us even more problems to try to solve.

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John McCleary

ABOUT THE AUTHOR

John McCleary teaches and writes mathematics. He also sings, and works at singing better.