
of the American Mathematical Society
December 2018

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$\Delta \hat{u}=0$ in $M$
$\partial_{h} \hat{u}=\sigma \hat{u}$ an $\partial M$

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## The Mathematics of Richard Schoen



December 2018

## FEATURED



Ad Honorem Yves Meyer


Felix Browder (1927-2016)

We close the year with tributes to Rick Schoen, who won the 2017 Wolf Prize, and Yves Meyer, who won the 2017 Abel Prize. We remember Felix Browder, former president of our Society, recognized by President Clinton with the National Medal of Science. We finish our term as editors with much gratitude (see page 1414), especially for you our readers, and best wishes to the incoming Editor-in-Chief Erica Flapan (see page 1412) and all.
-Frank Morgan, Editor-in-Chief

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About the Cover: The cover showcases images from the articles featuring Rick Schoen, Yves Meyer, and Felix Browder. The checkerboard illustrates "WHAT IS....a Multilayer Network?" The blackboard background comes from a talk by Schoen on "Minimal Surfaces and Eigenvalues" at a July conference at Stanford, captured by Christina Sormani.

# THE MATHEMATICS OF VOTING AND ELECTIONS: A HANDS-ON APPROACH SECOND EDITION 

Jonathan K. Hodge, Grand Valley State University, Allendale, MI, and Richard E. Klima, Appalachian State University, Boone, NC The Mathematics of Voting and Elections: A HandsOn Approach, Second Edition, is an inquiry-based approach to the mathematics of politics and social choice. The aim of the book is to give readers who might not normally choose to engage with mathematics recreationally the chance to discover some interesting mathematical ideas from within a familiar context, and to see the applicability of mathematics to real-world situations. Through this process, readers should improve their critical thinking and problem solving skills, as well as broaden their views of what mathematics really is and how it can be used in unexpected ways. In addition to making small improvements in all the chapters, this second edition contains several new chapters. Of particular interest might be Chapter 12 which covers a host of topics related to gerrymandering.

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## The Mathematics of Richard Schoen



Communicated by Christina Sormani

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## Hubert L. Bray and William P. Minicozzi II

## Preface

For more than forty years Richard Schoen has been a leading figure in geometric analysis, connecting ideas between analysis, geometry, topology, and physics in fascinating and unexpected ways. In 2017 Richard Schoen was awarded the Wolf Prize for these fundamental contributions and for his "understanding of the interconnectedness of partial differential equations and differential geometry." In this article we survey some of his many fundamental ideas.


Figure 1. Schoen was the tenth of thirteen children, shown here in fourth grade at the Sharpsburg Elementary School, at graduation from Fort Recovery High School in 1968, and at graduation summa cum laude from the University of Dayton in 1972.


Figure 2. Rick Schoen, pictured here soon after winning the Guggenheim Fellowship in 1996, also won the Wolf Prize in 2017.

Rick Schoen was born in 1950 in Celina, Ohio. He was the tenth in a family of thirteen children growing

[^0]up on a farm (Figure 1). He enjoyed farm work and has described driving a tractor to plow the fields as "great for thinking." His mother encouraged the children in their schooling, and his father was always inventing things. His older brothers, Hal and Jim, were both math majors and inspired him to study mathematics.

In 1972 Schoen (Figure 1) graduated summa cum laude from the University of Dayton and received an NSF Graduate Fellowship. In March 1977 Rick received his PhD from Stanford University under the direction of Leon Simon and Shing-Tung Yau and soon after received a Sloan Postdoctoral Fellowship. His early work was on minimal surfaces and harmonic maps. By the time Schoen received his PhD , he had already proven major results, including his 1975 curvature estimates paper with Simon and Yau.

In the late 1970s Schoen and Yau developed new tools to study the topological implications of positive scalar curvature. This work grew out of their study of stable minimal surfaces, eventually leading to their proof of the positive mass theorem in 1979. Altogether, their work was impressive for the way it connected neighboring fields, first using analysis to understand geometry and then using geometry to understand physics.


Figure 3. Schoen with his collaborators on the theory of stable minimal surfaces in manifolds of positive scalar curvature: Fischer-Colbrie and Fields Medalist Yau (2015). Yau and Schoen applied stable minimal surfaces to prove the positive mass theorem in 1979.

In the early 1980s, Schoen published a number of fundamental papers on minimal surfaces and harmonic maps. His work on minimal surfaces includes an influential Bernstein theorem for stable minimal surfaces with Doris Fischer-Colbrie. Schoen met his future wife, FischerColbrie, in Berkeley, where she received her PhD in 1978. They have two children, Alan and Lucy, both of whom graduated from Stanford.

Other works from the early 1980s include an extremely useful curvature estimate for stable surfaces, a uniqueness theorem for the catenoid, and a partial regularity
theory for stable hypersurfaces in high dimensions with Simon. In 1982, Schoen and Karen Uhlenbeck proved the partial regularity of energy-minimizing harmonic maps. In 1983 Schoen was awarded the very prestigious MacArthur Prize Fellowship.

Schoen is also very well known for his celebrated solution to the remaining cases of the Yamabe problem in 1984, this time using a theorem from physics, namely the positive mass theorem, to solve a famous problem in geometry. The resulting fundamental theorem in geometry, that every smooth Riemannian metric on a closed manifold admits a conformal metric of constant scalar curvature, had been open since the 1960s. This work was cited in 1989 when Schoen received the Bocher Prize of the American Mathematical Society. His work on scalar curvature at this time set the direction for the field for the next twenty-five years.

Schoen was elected to the American Academy of Arts and Sciences in 1988 and the National Academy of Sciences in 1991. He has been a Fellow of the American Association for the Advancement of Science since 1995 and won a Guggenheim Fellowship in 1996. Rick was elected Vice President of the AMS in 2015. He was awarded the Wolf Prize in Mathematics for 2017, shared with Charles Fefferman. In 2017 he was also awarded the Heinz Hopf Prize, the Lobachevsky Medal and Prize, and the Rolf Schock Prize, to mention only a few of his awards.


Figure 4. Schoen with Simon at Stanford in 2018. Schoen is Simon's most successful doctoral student, with 137 mathematical descendants.

Starting around 1990 Schoen began two major programs. The first was to develop a theory of harmonic maps with singular targets, starting with a joint paper with Mikhail Gromov, where they used harmonic maps to establish $p$-adic superrigidity for lattices in groups of rank one. In a series of papers, Schoen and Nick Korevaar laid the foundations for a general theory of mappings to NPC spaces, established the basic existence and regularity results, and applied their theory to settle problems in a number of areas of mathematics. The second big program
was a variational theory of Lagrangian submanifolds, including the existence and regularity theory, done in a series of papers with Jon Wolfson.

Over the last decade, Schoen has continued to make major contributions to geometric analysis and general relativity. Among other results in general relativity, Schoen has made fundamental contributions to the constraint equations (with Corvino and others), which dictate the range of possible initial conditions for a spacetime, proved theorems on the topology of higher-dimensional black holes (with Galloway), and proved the positive mass theorem in dimensions greater than seven (with Yau). In geometric analysis he has several important results with Simon Brendle on Ricci flow, including the proof of the differentiable sphere theorem, as well as a compactness theorem for the Yamabe equation with Marcus Khuri and Fernando Codá Marques.


Figure 5. Schoen at the Banff International Research Station in 2016 with his students Pengzi Miao, Christos Mantoulidis, Daniel Pollack, Alessandro Carlotto, Sumio Yamada, Dan Lee, and Hubert Bray.

Schoen has written two books and roughly eighty papers and has solved an impressively wide variety of major problems and conjectures. He has supervised around forty students and counting and has hosted many postdocs. Even with his great success, Schoen is still one of the hardest working people in mathematics, giving us all the distinct impression that he must love it. His impact on mathematics, both in terms of his ideas and the example he sets, continues to be tremendous.

This article contains surveys of a sampling of Schoen's mathematical works, interspersed with personal recollections. Many of Schoen's accomplishments are not surveyed here-there were just too many to attempt that feat. Nevertheless, we hope that the reader will get a taste of the mathematical genius that is Richard Schoen.

In the next sections, Michael Eichmair and Lan-Hsuan Huang describe Schoen's work on scalar curvature, followed by a personal note by Shing-Tung Yau about their collaboration on the positive mass theorem. William Minicozzi describes the work of Schoen-Uhlenbeck, followed by a personal note by Karen Uhlenbeck about their collaboration. Rob Kusner describes Schoen's work on classical minimal surfaces in Euclidean space, and Fernando Codá Marques describes his work on the Yamabe problem.


Figure 6. At the June 2009 Mathematics Research Communities Conference in Snowbird, Utah. First row: Catherine Williams, Vincent Bonini, Leobardo Rosales, Jeff Jauregui, Po-Ning Chen. Second row: George Lam, Jacob Bernstein, Chen-Yun Lin, Andrew Bulawa, Graham Cox, Joel Kramer, Iva Stavrov, Rick Schoen, Christine Breiner, Lan-Hsuan Huang, Michael Eichmair, Lu Wang.

Chikako Mese describes his work on harmonic maps into NPC spaces, and Ailana Fraser describes their joint work on Steklov eigenvalues. Any of these sections may be read as its own distinct contribution.

## Michael Eichmair and Lan-Hsuan Huang

## Scalar Curvature, Minimal Surfaces, and the Positive Mass Theorem

Here we describe some of Richard Schoen's early work with Shing-Tung Yau on the use of minimal surfaces in the study of three-dimensional geometry. We also describe the impact of their contributions on our own careers.

To set the stage, we briefly recall several notions of curvature. Let $(M, g)$ be a Riemannian manifold of dimension $n$.

Let $\pi$ be a 2-plane in the tangent space of $M$ at $p$. The sectional curvature of $\pi$ may be computed from the spread of geodesics $\gamma_{i}$ starting at $p$ with orthonormal initial velocities $e_{i}$ such that $\pi=e_{1} \wedge e_{2}$ :

$$
K_{p}(\pi)=\lim _{t \rightarrow 0} 6\left(\frac{t-\operatorname{dist}\left(\gamma_{i}(t), \gamma_{j}(t)\right) / \sqrt{2}}{t^{3}}\right)
$$

The sectional curvature is positive when these geodesics bend together and negative when they drift apart when

[^1]compared with Euclidean space. Starting with an orthonormal basis $e_{1}, \ldots, e_{n}$ of the tangent space at $p$, we can compute the (symmetric) Ricci curvature tensor as
$$
\operatorname{Ric}_{p}\left(e_{i}, e_{i}\right)=\sum_{j \neq i} K_{p}\left(e_{i} \wedge e_{j}\right)
$$
and the scalar curvature as
$$
R_{p}=\sum_{i=1}^{n} \operatorname{Ric}_{p}\left(e_{i}, e_{i}\right)=\sum_{i \neq j} K_{p}\left(e_{i} \wedge e_{j}\right)
$$

The scalar curvature can also be computed from the deficit between the volume $\omega_{n} r^{n}$ of a Euclidean ball of radius $r>0$ and the volume of a geodesic ball $B_{p}(r)=\{x \in M: \operatorname{dist}(x, p)<r\}$ in $(M, g):$

$$
R_{p}=\lim _{r \rightarrow 0} 6(n+2)\left(\frac{\omega_{n} r^{n}-\operatorname{vol}\left(B_{p}(r)\right)}{\omega_{n} r^{n+2}}\right)
$$

Thus $R_{p}>0$ means that a small geodesic ball in $(M, g)$ has less volume than a Euclidean ball of the same radius.


Figure 7. The spatial Schwarzschild geometry, with vanishing scalar curvature, provides the prototype for the concept of mass in general relativity. The sphere at the neck (in red) is a stable minimal surface called the horizon.

Let $\Sigma \subset M$ be a hypersurface. The variation $\nabla_{X} v$ of the unit normal $v: \Sigma \rightarrow T M$ along a field $X$ that is tangent to $\Sigma$ measures how $\Sigma$ bends in ambient space. These variations are recorded synthetically in a symmetric tensor on $\Sigma$,

$$
A_{\Sigma}(X, Y)=g\left(\nabla_{X} v, Y\right)
$$

called the second fundamental form. The principal curvatures $\lambda_{1}, \ldots, \lambda_{n-1}$ of $\Sigma$ at a given point are the eigenvalues of this tensor with respect to the inner product induced on $\Sigma$ by $g$. Their sum

$$
H_{\Sigma}=\lambda_{1}+\cdots+\lambda_{n-1}
$$

is called the mean curvature $H_{\Sigma}$ of $\Sigma$.
We examine mean curvature more carefully. Let us assume, for definiteness, that $\Sigma$ is closed. Every smooth
function, $f \in C^{\infty}(\Sigma)$, with sufficiently small $C^{1}$-norm gives rise to a surface

$$
\Sigma_{f}=\left\{\exp _{\sigma} f(\sigma) v(\sigma): \sigma \in \Sigma\right\}
$$

near $\Sigma$. In fact, every surface close to $\Sigma$ has this form. Taylor expansion gives

$$
\begin{gathered}
\operatorname{area}\left(\Sigma_{f}\right)=\operatorname{area}(\Sigma)+\int_{\Sigma} H_{\Sigma} f+\frac{1}{2} \int_{\Sigma} H_{\Sigma}^{2} f^{2} \\
+\frac{1}{2} \int_{\Sigma}\left|\nabla_{\Sigma} f\right|^{2}-\left(\left|A_{\Sigma}\right|^{2}+\operatorname{Ric}(v, v)\right) f^{2} \\
+O\left(\|f\|_{C^{1}(\Sigma)}^{3}\right)
\end{gathered}
$$

Assume now that $\Sigma$ has least area among all nearby surfaces. In particular, the first variation of area is zero. By the preceding formula, this is equivalent to

$$
H_{\Sigma}=0 .
$$

Moreover, the second variation of area is nonnegative. In view of our expansion, this amounts to the stability inequality, i.e.,

$$
\int_{\Sigma}\left(\left|A_{\Sigma}\right|^{2}+\operatorname{Ric}(v, v)\right) f^{2} \leq \int_{\Sigma}\left|\nabla_{\Sigma} f\right|^{2}
$$

for all $f \in C^{\infty}(\Sigma)$.
In general, we call a surface minimal if its mean curvature vanishes. Stable minimal surfaces are those that satisfy both the first and the second derivative tests for least area.


Figure 8. Schoen and Yau's famous manipulation of the second variation formula for a stable minimal surface, $\Sigma$, to involve the ambient scalar curvature, $R_{M}$, the intrinsic Gauss curvature, $R_{\Sigma}$, and the norm of the second fundamental form squared, $|A|^{2}$.

Using parallel surfaces for area comparison, where $f \equiv$ constant, is tempting and hard to resist. It gives

$$
\int_{\Sigma}\left|A_{\Sigma}\right|^{2}+\operatorname{Ric}(v, v) \leq 0
$$

J. Simons observed from this that there can be no stable minimal surfaces if the Ricci curvature is positive. Schoen and Yau [4] manipulated the integrand further, using the Gauss equation

$$
R=2 K_{\Sigma}+\left|A_{\Sigma}\right|^{2}-H_{\Sigma}^{2}+2 \operatorname{Ric}(v, v)
$$

and the Gauss-Bonnet formula

$$
\int_{\Sigma} K_{\Sigma}=2 \pi \chi(\Sigma)
$$

to conclude that, for a stable minimal surface,

$$
\frac{1}{2} \int_{\Sigma} R+\left|A_{\Sigma}\right|^{2} \leq \int_{\Sigma} K_{\Sigma}=2 \pi \chi(\Sigma)
$$

Inspecting this identity, Schoen and Yau observed that the following two conditions-one metric and one topological-cannot hold simultaneously:

- The scalar curvature $R$ is positive along $\Sigma$.
- The genus of $\Sigma$ is positive; i.e., $\chi(\Sigma) \leq 0$.

Consider $(M, g)$ where $M=\mathbb{S}^{1} \times \mathbb{S}^{1} \times \mathbb{S}^{1}$ is the 3 torus. From standard results in geometric measure theory, among all surfaces in $M$ that are homologous to $\Sigma_{0}=$ $\mathbb{S}^{1} \times \mathbb{S}^{1} \times$ point $\}$, there is one that has least area in $(M, g)$; see Figure 9. By Stokes' theorem, this surface has a component $\Sigma$ for which

$$
\int_{\Sigma}\left(d \theta^{1}\right) \wedge\left(d \theta^{2}\right) \neq 0
$$

where $\theta^{1}, \theta^{2}$ are the angles on the first two factors. Assume that $\Sigma$ has genus zero. The restrictions of the forms $d \theta^{1}$ and $d \theta^{2}$ and consequently of $d \theta^{1} \wedge d \theta^{2}$ are exact on $\Sigma$. In particular, the latter should integrate to zero on $\Sigma$. This contradiction shows that $\Sigma$ has positive genus. In view of the dichotomy discussed above, these ideas of Schoen and Yau give
Theorem 1 (Schoen and Yau [3]). The 3-torus does not admit a metric of positive scalar curvature.


Figure 9. There is a stable minimal surface $\Sigma$ (in red) lying in a torus $M$ (in black), which is homologous to $\Sigma_{0}$ (in blue).

As shown by Kazdan-Warner in 1975, a Riemannian metric with nonnegative scalar curvature is either Ricci flat or there is a metric with positive scalar curvature nearby. Theorem 1 is thus really a rigidity result for Riemannian metrics of nonnegative scalar curvature. That is, every 3 -dimensional torus with nonnegative scalar curvature is flat.

Schoen and Yau extended these ideas in their proof of the positive mass theorem:

Theorem 2 (Schoen and Yau [4]). Let ( $M, g$ ) be a complete Riemannian 3-manifold that is asymptotically flat,

$$
g_{i j}=\delta_{i j}+O\left(|x|^{-q}\right)
$$

for some $q>1 / 2$, with nonnegative integrable scalar curvature. Then the $A D M$-mass of $(M, g)$ is nonnegative, and it is zero if and only if $(M, g)$ is flat Euclidean space.

The ADM-mass (after R. Arnowitt, S. Deser, and C. Misner),

$$
m_{A D M}=\lim _{r \rightarrow \infty} \frac{1}{16 \pi r} \int_{\{|x|=r\}} \sum_{i, j=1}^{3}\left(\partial_{i} g_{i j}-\partial_{j} g_{i i}\right) x^{j},
$$

is a geometric invariant of $(M, g)$ that measures the deviation from Euclidean at infinity. The example of Schwarzschild where $M=\left\{x \in \mathbb{R}^{3}:|x| \geq m / 2\right\}$ and

$$
g_{i j}=u^{4} \delta_{i j} \quad \text { with } u(x)=1+\frac{m}{2|x|}
$$

for some $m>0$ is particularly important. The boundary of $M$ is a stable minimal surface and is called the horizon. Note that the ADM-mass is equal to $m$.

Schoen and Yau first consider the special case of harmonic asymptotics where, outside a bounded set,

$$
g_{i j}=u^{4} \delta_{i j} \quad \text { with } u(x)=1+\frac{m}{2|x|}+O\left(|x|^{-2}\right)
$$

and where the scalar curvature is positive everywhere. (The reduction of the proof of the positive mass theorem to the special case of such harmonic asymptotics is proven in a 1981 paper of Schoen and Yau using a density argument.) If $m=m_{A D M}$ is negative, the slab

$$
\left\{-\Lambda<x^{3}<\Lambda\right\}
$$

is a mean-convex region for $\Lambda>0$ sufficiently large. See Figure 10.


Figure 10. In their proof of the positive mass theorem, Schoen and Yau find a least area surface, $\Sigma_{r}$, with boundary, $\partial \Sigma_{r}$, lying in a cylinder of radius $r$, before taking $r \rightarrow \infty$ to create a complete stable minimal surface.

Schoen and Yau go ahead and construct least area surfaces in the slab with respective boundary

$$
\partial \Sigma_{r}=\left\{\left(x^{1}, x^{2}, 0\right):\left|\left(x^{1}, x^{2}\right)\right|=r\right\} .
$$

As $r \rightarrow \infty$, these surfaces limit to a complete stable minimal surface $\Sigma$ that is asymptotic to a horizontal plane. Since $\Sigma$ is unbounded, one cannot simply choose the constant test function 1 in the second variation inequality as
before. Schoen and Yau apply an approximation argument involving a logarithmic cut-off trick to argue that, still,

$$
\int_{\Sigma}\left|A_{\Sigma}\right|^{2}+\operatorname{Ric}(v, v) \leq 0
$$

Using the Gauss equation as before, this gives

$$
0<\frac{1}{2} \int_{\Sigma} R \leq \int_{\Sigma} K_{\Sigma} .
$$

The Cohn-Vossen inequality bounds the right-hand side by $2 \pi \chi(\Sigma)$. Now, using that $\Sigma$ is asymptotic to a plane, they conclude that $\chi(\Sigma) \leq 0$, a contradiction.

This shows that $m_{A D M} \geq 0$. To characterize the case of equality, Schoen and Yau use a perturbation argument.

These ideas of Schoen and Yau also extend to higher dimension.

Theorem 3 (Schoen and Yau [3]). Let $M$ be a closed manifold of dimension $n \leq 7$ so there exists a map $M \rightarrow \mathbb{T}^{n}=$ $\mathbb{S}^{1} \times \cdots \times \mathbb{S}^{1}$ of nonzero degree. Then $M$ does not admit a metric of positive scalar curvature.

The proof is by induction on the dimension. We may assume that $n \geq 4$. Using area-minimization, the stability inequality, and the Gauss equation as above, Schoen and Yau show that there is a hypersurface $\Sigma \subset M$ that admits a map of nonzero degree into the torus, $\mathbb{T}^{n-1}$, such that

$$
\frac{1}{2} \int_{\Sigma}\left(R+\left|A_{\Sigma}\right|^{2}\right) f^{2} \leq \int_{\Sigma}\left|\nabla_{\Sigma} f\right|^{2}+\frac{1}{2} R_{\Sigma} f^{2}
$$

for all $f \in C^{\infty}(\Sigma)$. Using the assumption that $R>0$, they conclude that the Yamabe operator

$$
L_{\Sigma}=-\Delta_{\Sigma}+\frac{n-3}{4(n-2)} R_{\Sigma}
$$

has positive first eigenvalue on $\Sigma$. If $u \in C^{\infty}(\Sigma)$ is a positive first eigenfunction, then the scalar curvature of the conformal metric $u^{\frac{4}{n-3}} g_{\Sigma}$ on $\Sigma$ is positive. This contradicts the induction hypothesis.

The restriction to low dimensions is on account of singularities that area-minimizing hypersurfaces may form. In recent work, Schoen and Yau present a powerful approach that extends their ideas to all dimensions.

Schoen and Yau have introduced a watershed of ideas relating scalar curvature, minimal surfaces, topology, and physics. Their insights have been hugely influential, and certainly a short survey cannot do them justice. We conclude by sketching briefly a few examples of how their ideas have influenced our own work.

The positive mass theorem is a remarkable rigidity result for Euclidean space: There really is no way at all to deform the Euclidean metric on a compact set while keeping the scalar curvature nonnegative. The following result due to Otis Chodosh and Eichmair establishes a related rigidity result that was conjectured by Schoen:

Theorem 4 ([1]). The only asymptotically flat 3-manifold that has nonnegative scalar curvature and which contains an unbounded area-minimizing surface is flat Euclidean space.


Figure 11. Huang, Fischer-Colbrie, and Schoen at Stanford (2007). Fischer-Colbrie and Schoen's paper "On the structure of complete stable minimal surfaces in 3-manifolds of positive scalar curvature" is one of Schoen's five most highly cited papers.

This result has already been used in the solution of several other conjectures related to the geometry of 3manifolds in joint work of Chodosh and Eichmair with Alessandro Carlotto, with Vlad Moraru, and with Yuguang Shi and Haobin Yu.

The reduction in the proof of the positive mass theorem to harmonic asymptotics described above illustrates how subtle results about the geometry of asymptotically flat manifolds can be obtained by looking at special cases.

This observation has found many useful applications. Huang studied further properties of the special classes that play a key role in studying the geometric center of mass, building on the foundational work for special data by Gerhard Huisken and Yau. This work was completed in her 2009 Stanford dissertation under Schoen's supervision. Schoen, Mu-Tao Wang, and Huang applied these ideas to study the angular momentum and disproved a conjectural mass and angular momentum inequality.

From the point of view of general relativity, asymptotically flat manifolds are just the beginning of the story. General initial data for the Einstein equations consists of triplets $(M, g, h)$, where $(M, g)$ is a Riemannian manifold and $h$ is a symmetric $(0,2)$-tensor that represents the spacetime second fundamental form of the slice $(M, g)$ through the evolving spacetime. ${ }^{1}$

In joint work with Dan A. Lee and Schoen [2], the authors proved a spacetime version of the positive mass theorem for initial data sets in dimensions $\leq 7$. The strategy is modeled on that for the time-symmetric case given by Schoen and Yau [5]. In our case, marginally outer trapped surfaces (MOTS) take the place of minimal surfaces. Since MOTS do not arise as minimizers of a geometric variational problem, we depend on existence,

[^2]regularity, and compactness results for two-sided surfaces $\Sigma \subset M$ that solve a geometric boundary value problem of the form
$$
H_{\Sigma}(x)=F\left(x, v_{\Sigma}(x)\right)
$$
for all $x \in \Sigma$ with $\partial \Sigma=\Gamma$. Here, $F: M \times T M \rightarrow \mathbb{R}$ smooth and $\Gamma \subset M$ closed of codimension two are given. A theory that satisfyingly extends the classical theory for area-minimizing surfaces to this broader, nonvariational setting is developed in Eichmair's thesis under Schoen at Stanford in 2008. The accompanying rigidity result for Minkowski spacetime was obtained by Lee and Huang.


Figure 12. Lan-Hsuan Huang, Ailana Fraser, Michael Eichmair, Rick Schoen, Shing-Tung Yau, Robert Bartnik, Mu-Tao Wang, and Xiaodong Wang in Melbourne (2010).

On a personal note, we have very fond memories of our time working as doctoral students with Rick at Stanford-2003-2008 (Eichmair) and 2004-2009 (Huang). He would be in the office seven days a week, typically from quite early in the morning until around 4 pm . (There was an ongoing search for tennis partners in the department for the late afternoons.) This example has rubbed off on his PhD students too. We worked hard in a nurturing environment that could not have been more down-toearth about mathematics. We aspired to be like Rick. Those years laid the foundations for our own research careers, and we are very grateful.

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## Shing-Tung Yau

## Memories of Working with Rick Schoen

I am pleased to write on behalf of Rick Schoen on the occasion of his Wolf Prize, which is clearly a welldeserved award in recognition of the tremendous insight and strength he has long demonstrated in the field of geometric analysis.

I have known Rick for forty-four years, and he is certainly one of my very best friends. I am proud that Leon Simon and I were his thesis advisors at Stanford. Together, Schoen, Leon Simon, Karen Uhlenbeck, S. Y. Cheng, Richard Hamilton, Clifford Taubes, and I made great strides in the 1970s in developing the modern theory of geometric analysis. This was a truly exciting period for the subject of differential geometry, laying the foundation for major developments in topology, algebraic geometry, and mathematical physics over the last four decades.


Figure 13. The collaborators Yau and Schoen. Yau writes that "more than half of my important papers were either written jointly with Rick or influenced by him."

In the following I shall share some reminiscences involving Rick that date back to the 1970s.

Both Simon and I arrived at Stanford in the fall of 1973; fortunately we had offices across the hallway

[^3]from each other. Rick Schoen was just starting out as a graduate student. A trenchant observation by Rick regarding the second variation formula for area functional for deformation of minimal hypersurfaces in Euclidean space led to our first collaboration, during which we managed to find a curvature estimate for stable minimal hypersurfaces up to dimension five. This result provided a direct proof of the famous Bernstein conjecture in these dimensions; it also turned out to be important for later work on the existence of mini-max hypersurfaces in a low-dimensional compact manifold.

This paper marked the beginning of our very successful collaboration over the last forty years. More than half of my important papers were either written jointly with Rick or influenced by him. We have a similar way of approaching problems in mathematics, and we are both excited by the power of nonlinear analysis in geometry, topology, and mathematical physics.

I was excited by the rigidity theorem of Mostow when I was a graduate student. I thought the theory of harmonic maps could offer a good approach for replacing the quasiconformal map argument of Mostow. (This was finally achieved in 1991 by Jost and me and also by Mok, Young, and Siu based on the similar strategy of Calabi.) I suggested to Rick that we explore the theory of harmonic maps as a complement to the theory of minimal surfaces. In the period of 1974 to 1975, we studied the existence of harmonic maps and their geometric applications.

We exploited the uniqueness theory of harmonic maps when the image manifolds have nonpositive curvature in order to prove the rigidity of group actions on manifolds, reproving and generalizing some of the previous theorems in my thesis, as well as theorems pertaining to finite group actions on general manifolds. One extremely important development concerned the proof of the existence of harmonic maps from a Riemann surface into any manifold so long as there is a nontrivial map from the Riemann surface into the manifold inducing a nontrivial map on the fundamental group of the manifolds. This was remarkable because we were able to make sense of the action on the fundamental group for maps that are not necessarily continuous but have finite energy only.


Figure 14. Yau and Schoen with Isadore Singer in San Diego, where they worked late into the night on their lecture series.

We were led to believe that there should be some form of existence theorem for harmonic maps from the two-sphere. But before we started to work on that, we saw an announcement by Sacks and Uhlenbeck, who had proved such an existence theorem using ingenious ideas on treating the bubbling phenomena, and they can also treat the case with fundamental group. A couple of years later I, along with Siu, made use of the sphere theorem of Sacks-Uhlenbeck to prove the existence of rational curves in an algebraic manifold with positive bisectional curvature. The idea rests on a study of the second variational formula of minimal surfaces, similar to the case I studied with Rick, as mentioned above.

Rick and I also made use of these ideas to prove that the three-dimensional torus does not support a metric with positive scalar curvature. That had been a major open problem at the time. Our proof, in turn, led to Rick's and my subsequent proof of the positive mass conjecture in general relativity.

This theorem started the extensive applications of ideas from geometry and geometric analysis to general relativity. But the flow of ideas between geometers and general relativists can be very rewarding for all parties. I remembered that Hawking asked me to look into the four-dimensional analogue of the positive mass conjecture, which he and Gibbons called the positive action conjecture.

While pondering the problem posed by Hawking, Rick and I hit upon an amazing fact: the second variational formula of a minimal hypersurface can give rise to a conformal deformation of the minimal hypersurface, thus providing a way to achieve metrics of positive scalar curvature on the hypersurfaces. This important observation allowed us to do dimensional reduction as a way to understand higher-dimensional manifolds with positive scalar curvature. We were able to utilize such ideas to solve certain nonlinear equations, which are called the Jang equations in general relativity. These equations allow us, for example, to link the concept of black hole to mass in general spacetime. Rick and I were able to demonstrate rigorously that if the matter density is large in some fixed region, a black hole will form.

Since there is a regularity problem for minimal hypersurfaces for dimensions greater than seven, we have to restrict our theory to lower dimensions. Fortunately, about ten years ago we were able to remove the dimension restriction for the positive mass conjecture and also as pertains to the structure of manifolds with positive scalar curvature.

During our attempt to solve the positive mass conjecture, Rick and I naturally became interested in understanding the structure of manifolds with positive scalar curvature, which can, in turn, help us describe the structure of the topology of our universe. The first thing that Rick and I did was to determine what class of topology in a three-dimensional manifold can support metrics with positive scalar curvature. We found in early 1978 that it is possible to preserve metrics with positive scalar curvature among such manifolds. And we thought that it should be


Figure 15. Richard Schoen, Robert Bartnik, Leon Simon, and Shing-Tung Yau.
possible to generalize the procedure to perform surgery on submanifolds of codimension 3 whose normal bundle is trivial.

I gave a plenary talk at the 1978 Helsinki Congress, where I discussed some of the results that Rick and I had obtained. On my way to Helsinki, I mentioned the surgery result to Blaine Lawson, who later worked with Gromov, using a more straightforward argument to reproduce a similar result. These results became the key to understanding the structure of manifolds with positive scalar curvature.

For me, working with Rick has been wonderful in so many ways. I remember that we spent the summer of 1979 together in Stanford while finishing the writing of our paper on the full positive mass conjecture. We worked during the day, often stopping to eat at a Chinese restaurant called Moon Palace. In the evening we stayed and swam in the house owned by a friend of Doris's, Rick's girlfriend and future wife. The weather was excellent, and we really had a good time.

Rick and I also gave a lecture series in Princeton, Berkeley, and San Diego, the contents of which were published in two books. We often went over the substance of these lectures a day before we delivered them. Some of the materials we presented were new, and sometimes our preparations lasted all the way to midnight and even beyond. Unfortunately not all of the materials were recorded or preserved. We can recall some of the statements that we could prove but not the full proof.

I've so far talked mainly about some of the work that Rick and I have done together. But he has, of course, made many spectacular achievements on his own, including the old conjecture of Yamabe, where Rick applied the positive mass conjecture in a brilliant manner. The subject of conformal deformation became an important area of research after this work by Rick. And he is clearly the leader of the whole field.

His proof with Brendle on the quarter-pinching of positively curved manifolds brought a dramatic resolution to a conjecture that had lasted for seventy years. All these accomplishments reveal the great insight and strength of Rick as a mathematician of tremendous originality. Rick


Figure 16.Yau and Schoen look out into the future.
has had many other important successes throughout his career that I could discuss here. But the work that I have discussed already demonstrates why Rick Schoen is a resoundingly worthy recipient of the Wolf Prize.

## William P. Minicozzi II

## Schoen-Uhlenbeck Theory for Harmonic Maps

In the early 1980s Schoen and Uhlenbeck proved a number of major results on harmonic maps in two papers that shaped the development of the field. The first paper [SU1] proved the crucial $\epsilon$-regularity, proved a sharp bound on the Hausdorff dimension of the singular set, and gave new insight into when singularities can occur. The second paper [SU2] proved complete boundary regularity and solved the Dirichlet problem for harmonic maps. These papers are extraordinarily clear and beautifully written, and the techniques remain important today.

A map $u: M^{m} \rightarrow N^{n}$ between Riemannian manifolds is harmonic if it is a critical point for the energy functional:

$$
E(u)=\int_{M}|\nabla u|^{2} .
$$

This is one of the most natural geometric variational problems. Taking $N^{n}=\mathbb{R}$ yields the special case of harmonic functions. Alternatively, taking $M^{m}=\mathbb{R}$ yields the special case of geodesics parametrized by a multiple of arclength. Taking $M^{m}=\mathbb{R}^{2}$ yields the special case of nice parametrizations of minimal surfaces. See Figure 17. Harmonic maps have played an important role in geometry, topology, and mathematical physics.

The energy $E(u)$ can be defined even when $u$ is not smooth; it suffices to have $|\nabla u|^{2}$ defined almost everywhere and integrable. Thus, harmonic maps can be defined weakly, and a central question becomes the regularity of a weak solution. A point $x$ is said to be regular if there is a neighborhood $B_{r}(x)$ where $u$ is smooth. The singular set $S$ is the complement of the regular set; it is automatically closed by definition. Both geodesics and harmonic functions are necessarily smooth, but general

M


Figure 17. The standard map from $M=\mathbb{S}^{2}$ into the horizon at the neck of Schwarzschild space $N^{3}$ is a harmonic map as well as minimal.
harmonic maps-even ones that minimize energy-can have singularities.

In 1948 Morrey [M] showed that energy-minimizing harmonic maps from a surface $M$ must be Hölder continuous and therefore smooth as long as $M$ and $N$ are smooth. For the next thirty-five years, there were a number of results proving regularity for special classes of target manifolds $N$. For example, in the 1964 paper [ES, Eells-Sampson] proved the existence and regularity (smoothness) of harmonic maps when the target $N$ has nonpositive curvature. Hamilton solved the corresponding Dirichlet problem for manifolds with boundary.

There have been a number of great subsequent results on harmonic maps. Hardt [H] provides a nice survey.


Figure 18. Minicozzi and Schoen in 2018.
The Size of the Singular Set
The simplest example of a singular harmonic map is the map

$$
u: M^{3}=B(0,1) \subset \mathbb{R}^{3} \rightarrow \mathbb{S}^{2}
$$

defined by

$$
u(x)=\frac{x}{|x|} .
$$

This map minimizes energy for its boundary values (e.g., among all maps that are the identity map when restricted to $\partial M^{3}=\mathbb{S}^{2}$ ). However it is clearly not continuous at the origin. Since it is smooth everywhere else, the singular set $S$ is the single point $\{0\}$. Notice that

$$
\operatorname{dim}(S)=0=3-3=\operatorname{dim}(M)-3,
$$

so we say $S$ has codimension 3 .
Schoen and Uhlenbeck showed that this is what happens in general:
Theorem 5 (Schoen-Uhlenbeck). If $u: M^{m} \rightarrow N^{n}$ is energy minimizing and $E(u)$ is finite and if $u(M)$ is contained in a compact subset of $N$, then

$$
\operatorname{dim}(S) \leq m-3
$$

If $m=3$, then $S$ is discrete.
An immediate corollary of Theorem 5 is that $u$ is entirely smooth when $M$ is a 2-dimensional surface. This special case was proven in 1948 by Morrey. Around the same time as Schoen and Uhlenbeck, Giaquinta and Giusti proved the special case of Theorem 5 when the image of $u$ is contained within a coordinate chart.

One of the central tools developed by Schoen and Uhlenbeck, and an absolutely fundamental result on its own, was the following $\epsilon$-regularity theorem:

Theorem 6. There exists $\epsilon>0$ so that if $u: M^{m} \rightarrow N^{n}$ is energy minimizing on the ball $B(p, r) \subset M$ for some $r \leq 1$ and

$$
r^{2-m} \int_{B(p, r)}|\nabla u|^{2} \leq \epsilon,
$$

then $u$ is Hölder continuous on $B(p, r / 2)$.
Schoen and Uhlenbeck actually proved something stronger than Theorem 6: there is a uniform Hölder bound that depends on $M, N, r$, and $\epsilon$ (and, in particular, not on $u$ ). Once the map $u$ is continuous then higher regularity theory shows that $u$ is smooth [S].

The quantity they estimate in Theorem 6 is the scaleinvariant energy:

$$
r^{2-m} \int_{B(p, r)}|\nabla u|^{2}=f_{B(p, r)}|\nabla u|^{2} / r^{2}
$$

It is scale invariant in the sense that it does not change if we change $u$ by dilating the domain.

It follows immediately from the $\epsilon$-regularity theorem that each singular point $p$ has the property that

$$
r^{2-m} \int_{B(p, r)}|\nabla u|^{2}>\epsilon>0
$$

for $r$ arbitrarily small. When $m=2$, it follows immediately that there are only finitely many singular points. Namely, any ball, no matter how small, about a singular point contains energy at least $\epsilon>0$, so the number of singular points is at most the energy of $u$ divided by $\epsilon$.

In higher dimensions, Schoen and Uhlenback apply a covering argument to prove that

$$
\operatorname{dim}(S) \leq m-2
$$

Improving the dimension bound for $S$ to $m-3$ requires a more detailed blowup analysis, using monotonicity and scaling and a version of the Almgren-Federer dimension reduction argument of geometric measure theory. Energyminimizing maps have the following crucial monotonicity property: the scale-invariant energy is nondecreasing as a function of $r$ :

$$
r_{1}^{2-m} \int_{B\left(p, r_{1}\right)}|\nabla u|^{2} \leq r_{2}^{2-m} \int_{B\left(p, r_{2}\right)}|\nabla u|^{2} \text { for } r_{1}<r_{2}
$$

This monotonicity fails for a general weakly harmonic map, though it does hold for an important class of harmonic maps called stationary harmonic maps.

Using the monotonicity, Schoen and Uhlenbeck show that at each $p$ there exists a tangent map,

$$
u_{p}: B(0,1) \subset T M_{P}=\mathbb{R}^{m} \rightarrow N
$$

defined by blowing up about $p$ :

$$
u_{p}(x)=\lim _{\sigma_{i} \rightarrow 0} u\left(\exp _{p}\left(\sigma_{i} x\right)\right)
$$

At a regular point $p$, the tangent map $u_{p}$ is trivial:

$$
u_{p}(x)=\lim _{\sigma_{i} \rightarrow 0} u(p)+g\left(\nabla u, \sigma_{i} x\right)=u(p)
$$

In the simplest example where $u(x)=x /|x|$, we see that at $p=0$ the tangent map is

$$
u_{p}(x)=\lim _{\sigma_{i} \rightarrow 0} u\left(\sigma_{i} x\right)=\lim _{\sigma_{i} \rightarrow 0} \frac{\sigma_{i} x}{\left|\sigma_{i} x\right|}=u(x)
$$

In general the tangent map is also homogeneous,

$$
u_{p}(x)=w_{p}(x /|x|),
$$

and $w_{p}: \mathbb{S}^{m-1} \rightarrow N^{n}$ is harmonic.
This gives a criterion for regularity:
If there are no nontrivial harmonic maps from $\mathbb{S}^{k}$ to $N$ for every $k \leq m-1$, then every energyminimizing map from $B(0,1) \subset \mathbb{R}^{m}$ to $N$ must be smooth.

As a consequence there is a large class of target manifolds $N$ where the harmonic maps are smooth, including, for instance, $N$ with sectional curvature $K \leq 0$, as proven in Eells-Sampson.

## Boundary Regularity and the Dirichlet Problem

The Dirichlet problem is the search for a harmonic map $u: M \rightarrow N$ with prescribed values of $u: \partial M \rightarrow N$ on the boundary of $M$. In the setting where $M$ is an interval this is equivalent to the problem of finding a length-minimizing geodesic between two points. When $M$ is a surface and the energy is, thus, conformally invariant, the Dirichlet problem is closely related to the Plateau problem for minimal surfaces. See Figure 19.

The 1982 results of Schoen and Uhlenbeck [SU1] give interior regularity for the solution $u$ but say nothing about regularity along the boundary. In 1983 Schoen and Uhlenbeck [SU2] proved complete boundary regularity for energy-minimizing maps:


Figure 19. A soap film is a solution to the Dirichlet problem of finding a conformal harmonic map with prescribed boundary.

Theorem 7 (Schoen-Uhlenbeck). If $M$ is compact with $C^{2, \alpha}$ boundary, $u$ is energy minimizing (and $E(u)<\infty$ ), the image of $u$ is in a compact subset of $N$, and the restriction of $u$ to $\partial M$ is $C^{2, \alpha}$, then $u$ is $C^{2, \alpha}$ in a neighborhood of $\partial M$.

One consequence is that $S$ is a compact subset of the interior of $M$. The boundary regularity comes from the nonexistence of nontrivial smooth harmonic maps from the hemisphere which map the boundary to a point. Boundary regularity for surfaces was proven by Morrey in his work on the Plateau problem for minimal surfaces [M].

Using these results, Schoen and Uhlenbeck solved the Dirichlet problem.

Theorem 8 (Schoen-Uhlenbeck). Let $M$ be compact with $C^{2, \alpha}$ boundary as in Theorem 7 and suppose that $\partial N=\varnothing$. If $v: M \rightarrow N$ has

$$
\int|v|^{2}+|\nabla v|^{2}<\infty
$$

and the restriction of $v$ to $\partial M$ is $C^{2, \alpha}$, then there is an energy-minimizing harmonic map $u: M \rightarrow N$ that equals $v$ on $\partial M$. The singular set $S$ for $u$ is a compact subset of the interior of $M$ and

$$
\operatorname{dim}(S) \leq \operatorname{dim}(M)-3
$$



Figure 20. Schoen's students André Neves and Bill Minicozzi discussing mathematics as his long-time colleague, Brian White, listens at Stanford in 2018.

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Figure 21. Rick Schoen with his doctoral student Yng-Ing Lee and her doctoral students in 2015.

## Karen Uhlenbeck

## Working with Rick Schoen

The academic year 1979-80 at the Institute for Advanced Study in Princeton was a special year in differential geometry organized by S.-T. Yau. This year was indeed special for most of the participants, who met over a long period of time with their colleagues. They learned both classic and new open problems and a new set of techniques to apply to them. This period was a flowering of many methods of using nonlinear partial differential equations to approach problems in topology, differential geometry, and algebraic geometry. S.-T. Yau was a leader and driving force at this time. We all got to know each other, talk mathematics, and incidentally play volleyball and establish an eating co-op. Rick and I were both in residence for the year.


Figure 22. Schoen at IAS in 1979, where he first began working with Uhlenbeck on regularity using monotonicity.

I had since my student days been interested in minimization problems for maps between manifolds, and Rick brought to our collaboration a knowledge of minimal surface theory. Together we had a good working knowledge of nonlinear elliptic PDE. We did not start out with a deliberate goal of understanding harmonic maps, although we had seminars on them during the year. There were two background references we were both well aware

[^4]of. A step in the construction of minimal surfaces in manifolds was the solution of the Dirichlet problem for harmonic maps of domains of the plane into arbitrary manifolds. Morrey's regularity theorem for minimizing the energy integral showed that the minimum of energy, which is by construction almost continuous, must lie in a Hölder space. We had both already used harmonic maps from surfaces to find minimal surfaces in manifolds. An already classic paper by Eells and Sampson from the 1960s discussed maps from an arbitrary dimensional space into manifolds of negative curvature, but their heat flow technique shed little light on harmonic maps into arbitrary manifolds from spaces of dimension greater than two. The standard regularity theorems only applied once one had a continuous solution, and direct minimization yields maps with only one derivative in $L^{2}$.


Figure 23. Uhlenbeck with her husband, the mathematician Bob Williams, in 2017.

At the very end of the year, Rick and I had an "aha moment" and our collaboration took off. We realized two things. First, the monotonicity formula for harmonic maps, which comes from variations by diffeomorphisms of the base manifold, showed that the minima were almost continuous. Minima lie in a Morrey or Besov space. Second, on balls of small energy, we could construct a comparison function to improve the estimate by smoothing out the boundary values a bit using mollifiers. So we got a modification of the classic theorem of Morrey to push us over into stronger estimates implying Hölder continuity. By covering and counting, we could show that the energy minimizer was smooth on the complement of a set of Hausdorff codimension 2.

In our papers, the monotonicity formula was the basis for all our arguments. We called it the scaling inequality, which is just as good a term, but not one that became accepted. I wonder today that neither Rick nor I nor the referee realized the accepted name for our basic estimate.

We also had a disagreement as to how to make the estimates. I wanted to construct a comparison minimizer, while Rick used the Euler-Lagrange equations and harmonic functions into the ambient manifold. He won, as in the later steps we did not assume minimization.

This was a great starting point. Later on, work of Tian et al. on Yang-Mills and work of Chang and Yang on


Figure 24. Uhlenbeck at the Institute for Advanced Study in 1995.
biharmonic maps between manifolds showed a similar result, but it is technically tricky to carry the result further in those cases.

During the next year, via long distance, we were able to refine this result. By letting the rescaling go to zero, one can easily construct a tangent map from $S^{n-1}$ to the image manifold, which has similar properties. In dimension 3, the tangent maps from $S^{2}$ to $N$ are smooth, and approximation of the map by a tangent map shows the singular set in $M$ consists of isolated points. Rick knew of a dimension-reducing argument of Federer in geometric measure theory. I was very anxious about this result but trusted Rick's confidence. The final result was that the singular set was of finite codimension-3 Hausdorff measure. We wrote two additional papers on Dirichlet boundary values and minimizing maps between spheres. I had become interested in the reduction of harmonic maps to ordinary differential equations using extra symmetry, and so I was pleased to be able to include these.

It was a pleasure to work with Rick. He brought valuable background information and consistent enthusiasm to our collaboration. The following image has stayed with me. Rick was until very recently a baseball player who played regularly. I have always thought that he brought his consistent practice of mathematics, insistence on carrying out the game to the end, and a real team spirit from baseball into mathematics.


Figure 25. Uhlenbeck writes that Schoen brought "a real team spirit from baseball into mathematics" always "carrying out the game to the end."

## Rob Kusner

## Classical Minimal Surfaces

Following his work with Yau on the existence of incompressible minimal surfaces and the topology of 3-manifolds of positive scalar curvature and their breakthrough proof of the positive mass theorem in general relativity, Rick Schoen turned his attention to questions about classical minimal surfaces in $\mathbb{R}^{3}$.

The simplest complete minimal surface in $\mathbb{R}^{3}$ is the plane, and early in the twentieth century S. Bernstein characterized it as the only entire minimal graph. The generalization of this to higher dimensions became known as the Bernstein problem, which drove much work in the field, culminating in the 1968 Annals of Mathematics paper by Jim Simons and further explored in Rick's 1975 debut paper in Acta Mathematica with Leon Simon and Yau. Minimal graphs are stable-in fact, area-minimizinghypersurfaces. Both papers make use of the stability inequality from the second variation formula to bound the second fundamental form, leading to the solution of the Bernstein problem and to the regularity of such hypersurfaces (off a singular set of high codimension).

Even in $\mathbb{R}^{3}$ it's natural to ask whether the plane is the only complete stable surface. Schoen proved this with Doris Fischer-Colbrie in 1980 as a corollary to their results on complete stable surfaces in 3-manifolds of nonnegative scalar curvature [2]. A short proof of this generalized Bernstein theorem was also given by Manfredo do Carmo and C. K. Peng around the same time.

Another classical minimal surface problem concerns the uniqueness of the catenoid (see Figure 26). It can be motivated by a simple experiment. Imagine dipping a pair of rings into a soap solution. If the rings are

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Figure 26. A catenoid is the only minimal surface of revolution (up to scaling and besides the plane).
close enough, they bound a connected least-area surface, a soap film, that is an embedded annulus (if the rings were linked, this annulus would be embedded with a full twist). In 1956 Max Shiffman proved a remarkable fact about this minimal annulus in case the boundary rings are perfectly round circles that lie in parallel planes: all the intermediate parallel planes must also meet the minimal annulus in round circles! This means the resulting minimal annulus must either be part of a catenoid (in the special case where the boundary circles are coaxial) or be part of a Riemann staircase (Figure 27).


Figure 27. A Riemann staircase is a minimal surface foliated by round circles and straight lines (these come in a 1-parameter family, up to scaling).

More generally, if the boundary is a pair of convex curves in parallel planes, then Shiffman proved any minimal annulus bounded by them is still foliated by convex curves of intersection with the intermediate parallel planes.

Could there be another nonannular compact, connected minimal surface spanning a pair of convex curves in parallel planes?

Surprisingly, this remains an open problem, yet-in the spirit of the slogan "As all physicists know, and some mathematicians believe..."-the answer is conjectured (by Bill Meeks) to be "No."

The first answers in this direction were provided by Schoen in [4] in 1983. Applying the strong maximum principle of Eberhard Hopf and A. D. Alexandrov's "moving planes" reflection method, Schoen proved that any such minimal surface inherits the reflection symmetries of its boundary in the following sense: if the fundamental pieces of the boundary are "graphical" over the reflection planes, then so are the surface pieces. In particular, if there is a pair of orthogonal mirror planes carrying each such boundary curve into itself, then the minimal surface is necessarily annular (genus 0). See Figure 28.


Figure 28. The connected minimal surface whose boundary is a pair of squares lying in parallel planes must be an annulus (genus 0 ).

Even for nonconvex boundary curves like the one in Figure 29 one can still get this graphical property for the mirror symmetric pieces of the surface as long as the boundary itself is composed of graphical pieces over the corresponding reflection plane, as in Figures 30-31, but in that case the surface could indeed have higher genus.


Figure 29. Connected minimal surfaces with these nonconvex boundaries are depicted in Figures 30-31.


Figure 30. This connected minimal surface is an annulus (genus 0 ), which is a twice-punctured sphere.


Figure 31. This connected minimal surface is a twice-punctured torus (genus 1).

Schoen then considered the case of complete minimal surfaces without boundary; topological and geometric conditions on ends of the surface stand in for the boundary conditions. He introduced the notion of regular at infinity, meaning that each end of the minimal surface can be expressed as a graph having the asymptotic expansion (with respect to suitable coordinates $x, y, z$ for $\mathbb{R}^{3}$, with $x^{2}+y^{2}=r^{2}$ )

$$
z(x, y)=a \log r+\frac{b x+c y}{r^{2}}+O\left(\frac{1}{r^{2}}\right)
$$

Physically, the logarithmic growth a corresponds to the strength of the overall soap film surface-tension force pulling at that end; if it is nonzero, that end is asymptotic to a catenoid scaled proportionally to $a$ and is called a catenoidal end.

Using force balancing to align the ends and applying the Alexandrov reflection method, Schoen proved the following uniqueness theorem:

Theorem 9 (Schoen's Uniqueness Theorem [4]). The only complete, connected minimal surface in $\mathbb{R}^{3}$ with 2 ends, each regular at infinity, is the catenoid.

A decade later, Pascal Collin removed Schoen's "regularity at infinity" hypothesis. Meanwhile Korevaar, Solomon, and the author [3] proved the soap bubble analogue of this theorem: a complete, finite topology surface with 2 ends, properly embedded in $\mathbb{R}^{3}$ with constant mean curvature, must be a Delaunay unduloid (as in Figure 32).


Figure 32. Korevaar, Kusner, and Solomon proved the soap bubble analogue of Schoen's soap film uniqueness theorem: a complete, finite topology surface with 2 ends, properly embedded in $\mathbb{R}^{3}$ with constant mean curvature, must be a Delaunay unduloid [3].

To get a sense of the amazing progress-the discovery of new examples and development of new methodssince 1983, note that Rick [4] remarked, "While there are a number of examples of finite total curvature surfaces known, some being regular at infinity, it is not known whether an embedded example exists besides the catenoid and the plane." Within a year, Celso Costa had mooted this, constructing the first example of such a minimal surface in more than two centuries!

The situation for infinite topology minimal surfaces in $\mathbb{R}^{3}$ is much more complicated. There are 1 -ended examples like the singly periodic Scherk minimal surfaces (Figure 33) and the triply periodic Schwarz minimal surfaces of infinite genus. The latter can be constructed by rotating the "rectangular catenoid" in Figure 28 by a half-turn about each boundary edge. Both these examples come in families: changing the angle between the "wings" of the Scherk surfaces gives a 1-parameter family, while changing the edge ratio and separation of the rectangles gives a 2-parameter family of Schwarz surfaces.

For more than two ends, there are subtle topological obstructions discovered by Collin, Meeks, Rosenberg, and the author [1]: a complete, properly embedded minimal surface in $\mathbb{R}^{3}$ can have only countably many ends (like the


Figure 33. Infinite topology minimal surfaces include the singly periodic Scherk surfaces. These come in a 1-parameter family (up to scaling) determined by the angle between "wings" of the surface. We see a doubled plane or (suitably rescaled) catenoid in the limit as the angle tends to zero.

Riemann staircase in Figure 27, which has two limit ends and countably many middle ends).

Although Schoen's uniqueness theorem characterizes the catenoid topologically, it's also interesting to consider a coarser, measure-theoretic characterization, in terms of the density at infinity:

$$
\lim _{R \rightarrow \infty} \frac{\operatorname{Area}(\Sigma \cap B(0, R))}{\pi R^{2}}
$$

where $\Sigma$ is the minimal surface and $B(0, R)$ is a ball of radius $R$ in the ambient space.

Each end of a minimal surface that is regular at infinity has density 1 at infinity, and so the catenoid has density 2 at infinity. By monotonicity of area density, it can be seen that any minimal surface (besides a pair of planes) in $\mathbb{R}^{3}$ with density 2 at infinity must be embedded.

Are there any other minimal surfaces with density 2 at infinity besides the catenoid? There are! Indeed, the 1-parameter family of singly periodic Scherk surfaces (see Figure 33) also do, and the catenoid appears (upon suitable rescaling) in the limit as the "wing" angle tends to zero. It's conjectured that these are the only examples:

Conjecture 10. The catenoid and the family of singly periodic Scherk surfaces are the only complete, connected, embedded minimal surfaces in $\mathbb{R}^{3}$ with density 2 at infinity.

It's not clear who should be credited with this conjecture or even whether it is true. It first came up in a lunchtime conversation between Rick and me-the youngest of Rick's students from UC Berkeley-after we had all moved to UC San Diego for a few years in the mid-1980s. I recall Rick being characteristically agnostic but aware of the key issues, as if he'd thought about it for some time, and ready to share his insights with whomever he thought would have a good chance of solving the problem. Most of our interactions were like that, whether an


Figure 34. Kusner blowing an unduloidal soap bubble (at the end of the straw) in 2003.
hour-long phone call or a detailed handwritten letter (so 1980s, long before email), dropping by his office or out for a long lunch (preferably outdoors-so California, so not Massachusetts), on the volleyball or basketball court. Rick always treated me like a peer, letting me wander and find my own way, or occasionally like a teammate, providing gentle teasing which urges you to improve and simultaneously says he thinks you're tough enough to take it. No wonder Rick has had so many students. I'm grateful to be among them.

## Websites with Graphics:

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Matthias Weber:www.indiana.edu/~minima1/archive/ GeometrieWerkstatt: service.ifam.uni-hannover.de /~geometriewerkstatt/ga11ery/

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## Fernando Codá Marques

## The Yamabe Problem

In 1984 Richard Schoen [3] solved the remaining cases of the famous Yamabe problem with the introduction of a remarkable idea. The Yamabe problem is the question of whether any given compact Riemannian manifold $\left(M^{n}, g\right)$ admits a conformal metric $\tilde{g}$ with constant scalar curvature. In two dimensions the answer is affirmative due to the uniformization theorem. In high dimensions the problem was proposed in 1960 by Yamabe [4], who mistakenly thought he had found a solution. Trudinger (1968) pointed out the error and fixed it when the scalar curvature is nonpositive. The case when $n \geq 6$ and the manifold is not locally conformally flat was proven in important work of Aubin (1976). Schoen finished the cases when $n=3,4$, or 5 or when $g$ is locally conformally flat in any dimension by discovering a surprising connection with the positive mass theorem of general relativity. We refer the reader to Lee and Parker [2] for a complete account of this history.

For $n \geq 3$, the Yamabe Problem is equivalent to finding a positive solution $u \in C^{\infty}(M)$ of the nonlinear elliptic partial differential equation

$$
\begin{equation*}
\Delta_{g} u-c(n) R_{g} u+c(n) \lambda u^{\frac{n+2}{n-2}}=0 \tag{1}
\end{equation*}
$$

where $\Delta_{g}$ is the Laplace operator of $g, R_{g}$ is the scalar curvature of $g$,

$$
c(n)=\frac{(n-2)}{4(n-1)}
$$

and $\lambda \in \mathbb{R}$. Any positive solution $u$ gives rise to $a$ conformal metric

$$
\tilde{g}=u^{\frac{4}{n-2}} g
$$

with constant scalar curvature $R_{\tilde{g}}=\lambda$.
This is in fact a variational problem. One can associate to every metric $\tilde{g}=u^{\frac{4}{n-2}} g$ in the conformal class [ $g$ ] of $g$ the Einstein-Hilbert action:

$$
\mathcal{R}(\tilde{g})=c(n) \frac{\int_{M} R_{\tilde{g}} d v_{\tilde{g}}}{\left(\int_{M} d v_{\tilde{g}}\right)^{\frac{n-2}{n}}} .
$$

Applying (1) and integrating by parts, we see that the Einstein-Hilbert action can also be written as

$$
\mathcal{R}(\tilde{g})=\frac{\int_{M}\left(\left|\nabla_{g} u\right|^{2}+c(n) R_{g} u^{2}\right) d v_{g}}{\left(\int_{M} u^{\frac{2 n}{n-2}} d v_{g}\right)^{\frac{n-2}{n}}} .
$$

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Figure 35. Schoen solved the Yamabe problem by discovering a surprising connection with the positive mass theorem.

This functional is clearly bounded from below. A metric $\tilde{\mathcal{g}}$ is a critical point of $\mathcal{R}$ restricted to the conformal class [ $g$ ] if and only if $\tilde{g}$ has constant scalar curvature.

The Yamabe problem is hard because the equation (1) has a critical exponent for the Sobolev embeddings. The Sobolev space $W^{1,2}(M)$ is continuously embedded in $L^{\frac{2 n}{n-2}}(M)$ and is compactly embedded in $L^{q}(M)$ for every

$$
1 \leq q<\frac{2 n}{n-2} \quad \text { but not for } q=\frac{2 n}{n-2}
$$

For this reason the direct method of the calculus of variations, which tries to extract a limit out of a minimizing sequence of the functional, is doomed to fail.

The difficulty can also be explained on the geometric side. If $\bar{g}$ denotes the round metric on $S^{n}$, then $\varphi^{*}(\bar{g})$ has constant scalar curvature equal to $n(n-1)$ for any diffeomorphism $\varphi$ of the sphere. If $\varphi$ is a conformal map, then

$$
\varphi^{*}(\bar{g})=u_{\varphi}^{\frac{4}{n-2}} \bar{g}
$$

for some positive solution $u_{\varphi}$ of the Yamabe equation (1) with $g=\bar{g}$ and $\lambda=n(n-1)$. It is also possible to prove that $u_{\varphi}$ is a minimizer of $\mathcal{R}$. The group

$$
\operatorname{Conf}\left(S^{n}, \bar{g}\right)
$$

of conformal diffeomorphisms of the round sphere is noncompact, and in fact for any given $p \in S^{n}$ one can


Figure 36. Schoen with his academic grandson, Fernando Codá Marques.
choose a sequence

$$
\varphi_{i} \in \operatorname{Conf}\left(S^{n}, \bar{g}\right)
$$

such that the corresponding functions

$$
u_{\varphi_{i}} \rightarrow 0
$$

in compact subsets of $S^{n} \backslash\{p\}$, while

$$
\begin{equation*}
\sup _{B_{\varepsilon}(p)} u_{\varphi_{i}} \rightarrow \infty \tag{2}
\end{equation*}
$$

as $i \rightarrow \infty$ for any $\varepsilon>0$. When condition (2) holds for some sequence of solutions $u_{i}$ we say $u_{i}$ blows up at $p$.

The usual approach to the Yamabe problem consists of first finding a positive solution $u_{p}$ to the subcritical equation

$$
\Delta_{g} u-c(n) R_{g} u+c(n) \lambda u^{p}=0
$$

with

$$
p<\frac{n+2}{n-2}
$$

which can be done by minimizing the corresponding functional and then studying the convergence of $u_{p}$ as

$$
p \rightarrow \frac{n+2}{n-2}
$$

The Yamabe quotient of $(M, g)$, defined to be the conformal invariant

$$
Q(M, g)=\inf \{\mathcal{R}(\tilde{g}): \tilde{g} \in[g]\}
$$

plays a key role. Note that

$$
Q(M, g)=\inf _{u \in C^{\infty}(M)} \frac{\int_{M}\left(\left|\nabla_{g} u\right|^{2}+c(n) R_{g} u^{2}\right) d v_{g}}{\left(\int_{M} u^{\frac{2 n}{n-2}} d v_{g}\right)^{\frac{n-2}{n}}}
$$

One always has

$$
Q(M, g) \leq Q\left(S^{n}, \bar{g}\right)
$$

as can be seen by choosing appropriate local test functions, and when the strict inequality

$$
Q(M, g)<Q\left(S^{n}, \bar{g}\right)
$$

holds, the subcritical solutions $u_{p}$ converge to a solution $u$ of (1). Hence in order to solve the Yamabe problem it suffices to prove that

$$
Q(M, g)<Q\left(S^{n}, \bar{g}\right)
$$

holds for any $(M, g)$ that is not conformally equivalent to the round sphere. If $n \geq 6$ and the manifold is not locally conformally flat, Aubin proved the strict inequality by constructing local test functions supported near a point where the Weyl tensor is nonzero.

The cases when $n=3,4,5$ or when $g$ is locally conformally flat in any dimension are more difficult because they require the construction of a global test function. The key insight of Schoen was to realize that the test function would have to be Green's function $G_{p}$ of the conformal Laplacian

$$
L_{g}=\Delta_{g}-c(n) R_{g}
$$

smoothed out near its singularity $p$. One can suppose scalar curvature is positive, $R_{g}>0$, in which case Green's function is also positive. Since

$$
L_{g} G_{p}=0 \quad \text { outside } p
$$

the conformal metric

$$
\hat{g}=G_{p}^{\frac{4}{n-2}} g
$$

has zero scalar curvature. The fact that, near $p$,

$$
G_{p}(x)=d(x, p)^{2-n}+\text { lower order terms }
$$

also implies that the metric $\hat{g}$ is asymptotically flat. See Figure 37.


Figure 37. The main idea in Schoen's solution of the Yamabe problem is to invoke an asymptotically flat manifold obtained by blowing up the original manifold at a point. On the left is $(M, g)$ with the point $p$, and on the right is the blow-up, $(M-\{p\}, \hat{g})$, with an asymptotically flat end.

The mass $m$ of $\hat{g}$, as in the positive mass theorem, magically comes into play, and the strict inequality

$$
Q(M, g)<Q\left(S^{n}, \bar{g}\right)
$$

becomes a consequence of the positivity of $m$. The assertion that $m>0$ unless $\hat{g}$ is flat is the positive mass theorem.

In 1988, Schoen raised the question of whether the full set of solutions to equation (1) is compact in the $C^{k}$ topology (for any $k$ ) when the manifold is not conformally


Figure 38. Fernando Codá Marques with his doctoral advisor, José Escobar, in 2003. Escobar completed his doctorate with Schoen in 1986 and is known in part for his work on the Yamabe problem with boundary.
equivalent to the standard sphere. This is basically a problem of establishing a priori estimates for the solutions. Schoen was motivated by a potential use of the Pohozaev identity and the positive mass theorem as obstructions to the blow-up phenomenon. The a priori estimates predicted that the Weyl tensor and its derivatives to order [ $(n-6) / 2]$ should vanish at a blow-up point.

The a priori estimates were obtained much later for $n \leq 24$ in a paper [1] I wrote with Khuri and Schoen. They were known to hold for $n \leq 7$ (Druet, Y.Y Li, L. Zhang, Marques) and turned out to be false for $n \geq 25$ (Brendle, Brendle-Marques). In 2009, Khuri, Schoen, and I [1] discovered that the Pohozaev identity leads to a certain quadratic form (there is one for every dimension $n$ ) that is positive definite if $n \leq 24$ (it is not in higher dimensions), in which case we succeeded in proving the Weyl vanishing and the a priori estimates.

## On a Personal Note

I had the privilege of learning directly from Rick during the academic year 2005-2006, when I visited Stanford University. It was a transformative experience in my career. I am indebted to him for having helped shape my vision of mathematics.

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Figure 39. In addition to his important joint work with Schoen and Khuri on the Yamabe problem, Codá Marques is also famous for solving the Willmore conjecture with André Neves.

## Chikako Mese

## Harmonic Maps in Singular Geometry and Rigidity

Rick Schoen opened up a new frontier in geometric analysis by introducing harmonic maps in the singular setting through the series of papers [GS], [KS1], [KS2] written in collaboration with M. Gromov and N. Korevaar. More specifically, Rick initiated the study of harmonic maps into nonpositively curved metric spaces (NPC spaces) and introduced new techniques that were used to settle outstanding questions in rigidity problems. Here we will describe this work and further applications.

One of the main applications of harmonic maps is in geometric rigidity. Let $M$ be a compact or a finite volume Riemannian manifold with universal cover $\tilde{M}$, and let $\rho: \pi_{1}(M) \rightarrow \operatorname{Isom}(X)$ be a homomorphism from the fundamental group of $M$ into the group of isometries of a contractible metric space $X$. We say this setup is rigid if there exists a totally geodesic map $u: \tilde{M} \rightarrow X$ that is $\rho$-equivariant:

$$
u(\gamma(x))=\rho(\gamma)(u(x)), \quad \forall \gamma \in \Gamma, \forall x \in \tilde{M}
$$

Some of the most influential rigidity results concern special cases of this setup. In the original Mostow rigidity theorem, $\tilde{M}$ and $X$ are both $n$-dimensional hyperbolic manifolds with $n \geq 3$, and $\rho$ is a discrete, faithful, and co-compact isometric action. Mostow also generalized this result by considering cases when $\tilde{M}=G / K$ and $X$ are other symmetric spaces. The celebrated Margulis superrigidity further extended Mostow's work by allowing

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for a more general homomorphism $\rho$ and established the arithmeticity of lattices $\Gamma$ in groups $G$ when

$$
\operatorname{rank}(G / K) \geq 2
$$

The rank is the dimension of the maximal Euclidean space that can be isometrically embedded in $\tilde{M}=G / K$. Margulis also proved the higher rank non-Archimedean superrigidity by considering lattices in $p$-adic groups which, unlike their Archimedean counterpart, do not act on smooth Riemannian manifolds but instead act on Bruhat-Tits' Euclidean buildings. Euclidean buildings form an important class of NPC spaces, which we define in the next paragraph.


Figure 40. A geodesic space $X$ is an NPC space if every geodesic triangle in $X$ is thinner than a comparison geodesic triangle in the Euclidean plane $\mathbb{E}^{2}$.

A complete metric space ( $X, d$ ) is a geodesic space if every pair of points $p, q \in X$ is joined by a geodesic whose length is the distance between the points. A geodesic space $(X, d)$ is an NPC space if it also has nonpositive curvature in the sense of triangle comparison: a geodesic triangle in $X$ with vertices $p, q, r$ is thinner than a comparison geodesic triangle in Euclidean 2-plane $\mathbb{E}^{2}$ (as in Figure 40). More precisely, if $\bar{p}, \bar{q}, \bar{r} \in \mathbb{E}^{2}$ are such that

$$
d(p, q)=|\bar{p}-\bar{q}|, d(q, r)=|\bar{q}-\bar{r}|, d(r, p)=|\bar{r}-\bar{p}|,
$$

then

$$
d\left(p_{t}, r\right) \leq|((1-t) \bar{p}+t \bar{q})-\bar{r}|,
$$

where $t \mapsto p_{t}$ for $t \in[0,1]$ is the constant speed parameterization of the geodesic from $p$ to $q$. The study of NPC spaces (more generally, spaces with curvature bounded from above by $\kappa$ known as CAT( $\kappa$ ) spaces) was initiated by the foundational work of A. D. Alexandrov and further brought to prominence by Gromov.

Hadamard manifolds (complete and simply connected Riemannian manifolds with sectional curvatures bounded from above by 0 ) are examples of NPC spaces. A simple example of an NPC space that is not a manifold is a tripod $T$ that is the union of three copies of the interval $[0, \infty)$ identified at zero, as in Figure 41. The distance between two points $a$ and $b$ lying in different copies of $[0, \infty)$ is defined to be $a+b$. Note that any pair of points in $T$ is contained in a totally geodesic isometric embedding of $\mathbb{R}$ into $T$.

An $n$-dimensional Euclidean building $X$ is a more complicated example of an NPC space, but one can think of such buildings as higher-dimensional versions of the


Figure 41. The tripod, $T$, is an example of an NPC space that is not a manifold.
tripod in the sense that these metric spaces have the property that any two points $p, q \in X$ are contained in an image of a totally geodesic and isometric embedding of $\mathbb{E}^{n}$ in $X$. However, not all NPC spaces have this property. Other important examples come from different areas of mathematics, such as geometric group theory and Teichmüller theory. For example, hyperbolic buildings arise in the study of hyperbolic groups of Gromov, and the Weil-Petersson completion of a Teichmüller space of a surface of genus $\geq 2$ is an NPC space.

Korevaar and Schoen [KS1], [KS2] consider maps u: $M \rightarrow X$ from an $m$-dimensional Riemannian manifold, $m$, to an NPC space, $X$. For simplicity, we will let $M$ be a compact Lipschitz domain in Euclidean space. To define energy, consider the square of the difference quotient at a point $x \in M$ and integrate it over unit vectors $V$ to define an $\varepsilon$-approximate energy density function

$$
e_{\varepsilon}(x)=\int_{S^{m-1}}\left(\frac{d(u(x), u(x+\varepsilon V))}{\varepsilon}\right)^{2} d \sigma(V) .
$$

When the measures $e_{\varepsilon}(x) d x$ have uniformly bounded total mass then they converge weakly as $\varepsilon \rightarrow 0$ to a measure of the form $e(x) d x$ where $|\nabla u|^{2}:=e(x)$ is an integrable function. We define the energy functional by setting

$$
E(u)=\int_{M}|\nabla u|^{2} d x
$$

In this way, one can define the Sobolev space $W^{1,2}(M, X)$ (and in a similar way, $W^{1, p}(M, X)$ for $p>1$ and $B V(M, X)$ for $p=1$ ) and the notion of energy-minimizing maps.

Gromov-Schoen and Korevaar-Schoen proved the existence of energy-minimizing maps for the Dirichlet
problem and certain equivariant problems. Of particular importance to rigidity is that energy-minimizing maps are essentially unique (e.g., a $\rho$-equivariant energyminimizing map is unique if $X$ is negatively curved but may only be unique up to parallel translation along a flat subspace generally).

The uniqueness follows from quadrilateral comparison inequality (a special case of Reshetnyak's theorem):

$$
\begin{aligned}
d^{2}\left(m_{p, s}, m_{q, r}\right) \leq & \frac{1}{2} d^{2}(p, q)+\frac{1}{2} d^{2}(r, s) \\
& -\frac{1}{4}(d(p, s)-d(q, r))^{2}
\end{aligned}
$$

for the ordered sequence $\{p, q, r, s\} \subset X$ where $m_{x, y}$ is the midpoint between $x$ and $y$. See Figure 42.


Figure 42. A quadrilateral comparison implies uniqueness for energy-minimizing maps.

This in turn implies the convexity of the energy functional:

$$
\begin{aligned}
\int\left|\nabla\left(\frac{u+v}{2}\right)\right|^{2} d x \leq & \frac{1}{2} \int|\nabla u|^{2} d x+\frac{1}{2} \int|\nabla v|^{2} d x \\
& -\frac{1}{4} \int|\nabla d(u, v)|^{2} d x
\end{aligned}
$$

where $u, v \in W^{1,2}(M, X)$ and

$$
\left(\frac{u+v}{2}\right)(x)=m_{u(x), v(y)} .
$$

Note that if $u$ and $v$ are energy minimizing, then the map $\frac{u+v}{2}$ is also energy minimizing and $d(u, v)$ is a constant.

The uniqueness of energy-minimizing maps suggests that if one wants to prove rigidity, then the first step should be to find a $\rho$-equivariant energy-minimizing map and then try to prove that it is totally geodesic. (This idea is mentioned in Yau's contribution to this article as the motivating source for his work with Schoen on harmonic maps.) Indeed, building upon the work of Y.-T. Siu and K. Corlette, Gromov and Schoen used harmonic maps in the singular setting to study the $p$-adic case of the superrigidity problem and established nonArchimedean superrigidity and consequent arithmeticity for lattices of certain rank-1 groups. A crucial issue in this application is to understand the regularity and singularity of harmonic maps. Natural questions that arise are: Under what conditions are harmonic maps regular? If they are not regular, what can we say about the structure of the singular set?

Korevaar and Schoen proved the following regularity theorem:

Theorem 11. An energy-minimizing map $u:(\Omega, g) \rightarrow$ ( $X, d$ ) from a Lipschitz Riemannian domain into an NPC space is locally Lipschitz continuous, where the Lipschitz constant of $u$ at $x \in \Omega$ is dependent on the geometry of $(\Omega, g)$, the distance of $x$ to $\partial \Omega$, and the total energy of $u$.

On the one hand, Lipschitz regularity is the optimal result when the target is assumed to be only an NPC space. On the other hand, the crucial idea in Gromov-Schoen [GS] is that energy-minimizing maps behave better than in Theorem 11 when the target has a certain manifold structure.

For a map $u: \Omega \rightarrow X$ into an NPC space, define the regular set $\mathcal{R}(u)$ of $u$ as the set of points in $\Omega$ that possess a neighborhood mapping into an image of a totally geodesic and isometric embedding of a smooth Riemannian manifold and define the singular set $S(u)$ as its complement. For example, the leaf space of the vertical foliation of the quadratic differential $z d z^{2}$ on $\mathbb{C}$ endowed with the distance function defined by the vertical measure is isometric to the tripod $T$ (see Figure 43). The natural projection map

$$
u: \mathbb{D} \rightarrow \mathrm{T}
$$

from the unit disk centered at 0 is an energy-minimizing map. It takes any neighborhood away from the origin into at most two copies of $[0, \infty)$. Thus, $u$ is locally a harmonic function away from the origin. The image of any neighborhood of the origin is not a manifold. Consequently, $\mathcal{R}(u)=\mathbb{D} \backslash\{0\}$ and $S(u)=\{0\}$.


Figure 43. The energy-minimizing map $u$ takes each leaf (drawn in blue) to its unique point in the tripod $T$.

The regularity theorem of Gromov-Schoen asserts similar behavior for harmonic maps into Euclidean buildings.
Theorem 12. The singular set of an energy-minimizing map и : $\Omega \rightarrow X$ from a Lipschitz Riemannian domain into a Euclidean building is of Hausdorff codimension at least 2.

This is a delicate result and depends heavily on the special structure of the target space. We now give an example from Gromov-Schoen [GS] of a harmonic map with singular set of codimension < 2. Unlike 2-dimensional Euclidean buildings, the 2 -dimensional target space in this example has no isometric totally geodesic embedding of $\mathbb{R}^{2}$.

Example 13. As in Figure 44 , let $\Omega=\mathbb{D}$ be the standard disk and let $X=C o n e(\Gamma)$ be the metric cone over a curve $\Gamma$ of length $2 \pi \alpha$ with vertex denoted by $P_{0}$. If $\alpha>1$, there exist closed nondegenerate and disjoint intervals $I_{1}, I_{2} \subset$ $\Gamma$ such that the length of each component of $\Gamma \backslash\left(I_{1} \cup I_{2}\right)$ is at least $\pi$. If $C\left(I_{i}\right)$ denotes the convex hull of $I_{i} \cup P_{0}$ in Cone $(\Gamma)$, then $C\left(I_{i}\right)$ is a sector and $K=C\left(I_{1}\right) \cup C\left(I_{2}\right)$ is the convex hull of $I_{1} \cup I_{2}$ in Cone (Г). Let

$$
u: \overline{\mathbb{D}} \rightarrow \text { Cone }(\Gamma)
$$

be the energy-minimizing map such that

$$
\left.u\right|_{\partial \mathbb{D}}: \partial \mathbb{D} \rightarrow \partial K=\partial C\left(I_{1}\right) \cup \partial C\left(I_{2}\right)
$$

is a constant-speed parameterization of the join of the closed curves $\partial C\left(I_{1}\right)$ and $\partial C\left(I_{2}\right)$. Observe that $u(\overline{\mathbb{D}}) \subset K$ since $K$ is geodesically convex and that $\overline{\mathbb{D}} \backslash u^{-1}\left(P_{0}\right)$ is not connected since $K \backslash\left\{P_{0}\right\}$ is not connected. Since $u^{-1}\left(P_{0}\right)=$ $S(u)$, we conclude that

$$
\operatorname{dim}_{\mathcal{H}}(S(u))=\operatorname{dim}_{\mathcal{H}}\left(u^{-1}\left(P_{0}\right)\right) \geq 1 .
$$



Figure 44. The singular set $S(u)$ (depicted in red on the left) of the energy-minimizing map $u$ from a disk, $\mathbb{D}^{2}$, into a cone, Cone ( $\Gamma$ ), over a curve, $\Gamma$, of length greater than $2 \pi$, with prescribed boundary (in dark blue), has Hausdorff dimension at least 1.

The key ingredient in the proof of Theorem 12 is the monotonicity property of energy-minimizing maps. This relies on the monotonicity property from the SchoenUhlenbeck theory combined with the convexity of the distance function in an NPC space $X$. Using monotonicity, Gromov-Schoen [GS] construct homogeneous maps approximating energy-minimizing maps similar to the tangent maps in the Schoen-Uhlenbeck theory. Furthermore, they introduce the notion of a homogeneous degree-1 map $l: \Omega \rightarrow X_{0}$ being effectively contained in an essentially regular totally geodesic subspace $X_{0}$. The metric space $X_{0}$ is called essentially regular if any harmonic map into $X_{0}$ is well approximated near a point by a homogeneous degree-1 map. The map $l$ being effectively contained in $X_{0}$ says that most points in the image of $l$ are sufficiently far away from $X \backslash X_{0}$. For example, the union of two copies of $[0, \infty)$ in $T$ (which we will call $L$ ) is essentially regular (since it can be isometrically identified to $\mathbb{R}$ ), and the identity map $l: \mathbb{R} \rightarrow L \approx \mathbb{R}$ is effectively contained in $L \subset T$.

Theorem 12 is derived from the following:

Theorem 14. Let $X_{0}$ be an essentially regular totally geodesic subspace of $X$ and let $l: \Omega \rightarrow X_{0}$ be a homogeneous degree 1 map effectively contained in $X_{0}$ with $l\left(x_{0}\right)=P_{0} \in$ $X_{0}$. If an energy-minimizing map $и: \Omega \rightarrow X$ is sufficiently close tol in $B_{r_{0}}\left(x_{0}\right)$, then $u\left(B_{\sigma_{0}}\left(x_{0}\right)\right) \subset X_{0}$ for some $\sigma_{0}>0$.

The proof of Theorem 14 [GS] is an analytical tour de force. It is reminiscent of the proof of the SchoenUhlenbeck $\epsilon$-regularity theorem, but it additionally overcomes the difficulty that there is no PDE to work with in the singular setting. The idea is to inductively compare successive blow-up scalings of $u$ with the energy-minimizing maps into $X_{0}$. The delicate argument combines the convexity property of the distance function in an NPC space with the properties of $X_{0}$ and $l$.


Figure 45. Schoen (far right) with some of his PhD students at Park City in 2013 at the 23rd Annual PCMI Summer Session. First row: Chikako Mese, Dan Lee, Xiaodong Wang. Second row: Michael Eichmair, Justin Corvino, Ailana Fraser, Hugh Bray, Alessandro Carlotto, Rick Schoen. Third row: Peter Topping (actually a student of Mario Micallef, Schoen's first student).

Rick's innovations opened up a completely new avenue in the applications of harmonic map theory. In particular, in a joint work of the author with Georgios Daskalopoulos, Rick's ideas were pushed forward to approach rigidity problems that could not be previously solved by other methods: superrigidity of hyperbolic buildings and holomorphic rigidity of Teichmüller space.

Daskalopoulos and Mese [DM1] extended the GromovSchoen regularity theory to include important NPC spaces other than Euclidean buildings. This result implies, as originally conjectured by Gromov, the superrigidity of hyperbolic buildings. Another application concerns the Weil-Petersson completion $\overline{\mathcal{T}}$ of Teichmüller space $\mathcal{T}$. The analysis of harmonic maps into $\overline{\mathcal{T}}$ is difficult because the Weil-Petersson metric is degenerate near the boundary.

Inspired by Rick's proof of Theorem 14, in a series of papers that culminated in [DM2], we introduced new techniques and proved the long-standing holomorphic rigidity conjecture of Teichmüller space. Loosely speaking, we proved that the action of the mapping class group
uniquely determines the Teichmüller space as a complex manifold. The main ingredient in the proof is the surprising discovery that the Weil-Petersson completion $\overline{\mathcal{T}}$ of Teichmüller space contains essentially regular subspaces in a sense similar to Euclidean buildings despite its nonlocal compactness and degenerating geometry near the boundary.

## On a Personal Note

Rick was a terrific thesis advisor for me at Stanford, and I have many fond memories of my time as a graduate student. I am grateful for his generosity, vast mathematical knowledge, and many words of wisdom that he has shared with me. Rick's powerful ideas have guided me throughout my career as a mathematician.

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Figure 46. Rick Schoen with he and wife Doris Fischer-Colbrie's two children Alan and Lucy.

## Ailana Fraser

## Steklov Eigenvalues and Free Boundary Minimal Surfaces

One of the fundamental problems in spectral geometry is to determine sharp eigenvalue bounds. This is a subject with a long tradition and is a classical subject both in geometry and PDE. Work for compact surfaces extends over a period of more than forty years, with contributions by many authors. Determining sharp eigenvalue bounds is related to finding extremal metrics for the eigenvalue problem. The existence of extremal metrics has been a very difficult subject, and Richard Schoen has contributed in a fundamental way.


Figure 47. R. Schoen and A. Fraser, Workshop on General Relativity and Geometric Analysis, Australia, 2010. Can you find the kangaroo?

For surfaces $M$ with boundary, the eigenvalue problem that leads to a geometrically interesting variational problem is the Steklov eigenvalue problem:

$$
\left\{\begin{array}{lll}
\Delta_{g} u & =0 & \text { on } M \\
\frac{\partial u}{\partial \eta} & =\sigma u & \text { on } \partial M
\end{array}\right.
$$

where $g$ is a Riemannian metric on $M, \eta$ is the outward unit normal vector to $\partial M, \sigma \in \mathbb{R}$, and $u \in C^{\infty}(M)$.

Steklov eigenvalues are eigenvalues of the Dirichlet-toNeumann map, $\Lambda$, which takes a given smooth function, $f: \partial M \rightarrow \mathbb{R}$, finds its harmonic extension, $u: M \rightarrow \mathbb{R}$, satisfying

$$
\begin{aligned}
& \Delta u=0 \quad \text { on } M \\
& u=f \quad \text { on } \partial M
\end{aligned}
$$

and gives as an output the normal derivative of that harmonic extension,

$$
\Lambda(f)=\frac{\partial u}{\partial \eta}
$$

[^5]

Figure 48. Examples of free boundary minimal surfaces in the ball include the equatorial plane and the critical catenoid.

The Dirichlet-to-Neumann map is a self-adjoint elliptic pseudo-differential operator of order 1 and therefore has discrete spectrum

$$
\sigma_{0}=0<\sigma_{1} \leq \sigma_{2} \leq \cdots \leq \sigma_{k} \leq \cdots \rightarrow \infty
$$

The Steklov problem has a long history and many people have contributed; please see Girouard and Polterovich [GP] for a recent survey.

Some basic questions are:

1) Assuming we fix the boundary length to be 1 , what is the metric on $M$ that maximizes the first eigenvalue?
2) Does such a metric exist?
3) If so, what can we say about its geometry?

If we fix a surface $M$ of genus $\gamma$ with $k$ boundary components, define

$$
\sigma^{*}(\gamma, k)=\sup _{g} \sigma_{1}(g) L_{g}(\partial M)
$$

where the supremum is over all smooth metrics on $M$. By a 1954 result of Weinstock, $\sigma^{*}(0,1)=2 \pi$, and the supremum is achieved by the Euclidean disk. In general, there is a coarse upper bound due to Fraser-Schoen [FS1] and Kokarev:

$$
\sigma^{*}(\gamma, k) \leq \min \{2 \pi(\gamma+k), 8 \pi[(\gamma+3) / 2]\}
$$

It turns out that there is an intimate connection between maximizing metrics and minimal surfaces in the Euclidean unit ball $B^{n}$ that are proper in the ball and that meet the boundary of the ball orthogonally. Such surfaces are referred to as free boundary minimal surfaces since they arise variationally as critical points of the area among surfaces in the ball whose boundaries lie on $\partial B^{n}$ but are free to vary on $\partial B^{n}$. Classical examples (see Figure 48) include the equatorial plane disk and the critical catenoid, the unique portion of a suitably scaled catenoid that defines a free boundary surface in $B^{3}$.

Free boundary minimal surfaces $\Sigma$ in $B^{n}$ are characterized by the condition that the coordinate functions are Steklov eigenfunctions with eigenvalue 1:

$$
\begin{aligned}
\Delta x_{i}=0 & \text { on } \Sigma \\
\nabla_{\eta} x_{i}=x_{i} & \text { on } \partial \Sigma .
\end{aligned}
$$

Moreover, if we assume that we have a smooth metric $g$ that realizes the maximum, then there are independent first eigenfunctions $u_{1}, \ldots, u_{n}$ such that the map

$$
u=\left(u_{1}, \ldots, u_{n}\right)
$$

defines a proper conformal map from $M$ into $B^{n}, n \geq 3$. The image $\Sigma=u(M)$ is a free boundary minimal surface in $B^{n}$, and the maximizing metric can be realized by the induced metric [FS2].

The question of existence of a maximizing metric is extremely difficult: a major achievement of our work [FS2] is the proof that for any compact surface $M$ of genus zero with boundary, a smooth maximizing metric $g$ exists. More generally, existence of a maximizing metric on any surface $M$ with boundary is proved, provided the conformal structure is controlled for any metric near the maximum. The proof involves a canonical regularization procedure that produces a special maximizing sequence for which a carefully chosen set of eigenfunctions converges strongly in $H^{1}$ to a limit. It is then shown that the limit defines a continuous map that is stationary for the free boundary problem. Higher regularity follows from minimal surface theory.

For surfaces of genus zero with arbitrarily many boundary components, we prove boundedness of the conformal structure for nearly maximizing metrics and hence existence of a maximizing metric:
Theorem 15. For any $k \geq 1$ there exists a smooth metric $g$ on the surface of genus 0 with $k$ boundary components with the property $\sigma_{1}(g) L_{g}(\partial M)=\sigma^{*}(0, k)$.


Figure 49. Schoen presenting his work with Fraser at Stanford in 2018.

In the case of the annulus and the Möbius band, we explicitly characterize the maximizing metric. This follows from a characterization of the critical catenoid and the "critical Möbius band," an explicit free boundary minimal embedding of the Möbius band into $B^{4}$ by first eigenfunctions, as the only free boundary minimal immersions of the annulus and Möbius band into $B^{n}$ by first eigenfunctions.

As a result we have sharp eigenvalue bounds:
Theorem 16. For any metric annulus $M$,

$$
\sigma_{1} L \leq\left(\sigma_{1} L\right)_{c c}
$$

with equality if and only if $M$ is equivalent to the critical catenoid. In particular,

$$
\sigma^{*}(0,2)=\left(\sigma_{1} L\right)_{c c} \approx 4 \pi / 1.2
$$



Figure 50. Free boundary minimal surface $\Sigma_{k}$ of genus 0 with $k$ boundary components in $B^{3}$ as $k \rightarrow \infty$. For large $k, \Sigma_{k}$ is approximately a pair of nearby parallel plane disks joined by $k$ half-catenoidal boundary bridges.

Theorem 17. For any metric Möbius band $M$,

$$
\sigma_{1} L \leq\left(\sigma_{1} L\right)_{c m b}=2 \pi \sqrt{3},
$$

with equality if and only if $M$ is equivalent to the critical Möbius band.

For surfaces of genus 0 with $k \geq 3$ boundary components, while we don't have an explicit characterization of the maximizing metrics, we [FS2] show that maximizing metrics arise from free boundary surfaces in $B^{3}$ which are embedded and star-shaped with respect to the origin. We analyze the limit as the number of boundary components tends to infinity, as in Figure 50, and obtain an asymptotically sharp eigenvalue bound:

Theorem 18 ([FS2, Fraser and Schoen]). The sequence $\sigma^{*}(0, k)$ is strictly increasing in $k$ and

$$
\lim _{k \rightarrow \infty} \sigma^{*}(0, k)=4 \pi
$$

For each $k$ a maximizing metric is achieved by a free boundary minimal surface $\Sigma_{k}$ in $B^{3}$ of area less than $2 \pi$. The limit of these minimal surfaces as $k$ tends to infinity is a double disk. See Figure 50.

The only previously known free boundary minimal surfaces in $B^{3}$ were the equatorial disk and the critical catenoid. As a consequence of Theorem 18, we have the following existence theorem for free boundary minimal surfaces in the ball.
Corollary 19. For every $k \geq 1$ there is an embedded free boundary minimal surface in $B^{3}$ of genus 0 with $k$ boundary components. Moreover these surfaces are embedded by first eigenfunctions.

Recently Folha-Pacard-Zolotareva, Kapouleas-Li, and Kapouleas-Wiygul have used gluing techniques and Ketover has used an equivariant min-max construction to prove existence of further new examples of free boundary minimal surfaces in $B^{3}$.

## On a Personal Note

I was a PhD student of Richard Schoen's at Stanford University from 1993 to 1998. One could not hope for a better advisor, and I will feel forever fortunate and grateful to have had the opportunity to work with him. Not only is the mathematics exciting, but another aspect that is so special about working with him is how motivating and inspiring he is in discussions and lectures. It has been a privilege to collaborate with him many years later.

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## Ad Honorem Yves Meyer



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## Ronald Coifman, Guest Editor

## Introduction

Yves Meyer was awarded the 2017 Abel Prize. His work has impacted mathematics in a broad and profound way. Perhaps even more importantly, he has led a broad, multifaceted, worldwide network of research collaborations of mathematics with music, chemistry, physics, and signal processing. He has made seminal contributions in a number of fields, from number theory to applied mathematics.

Meyer started his career on the interface between Fourier analysis and number theory. Early in his career he introduced the theory of model sets [1], which have become an important tool in the mathematical study of aperiodic order two years before the discovery of Penrose pavings by Roger Penrose and ten years before the discovery of quasi-crystals by Dan Shechtman.

Around 1975 he initiated the field of nonlinear Fourier analysis as a tool for organizing and analyzing nonlinear functional transformations of mathematics. He developed all the fundamental tools and concepts necessary to understand the nonlinear dependence of solutions of boundary value problems on the shape of the boundary. In particular, together with Coifman and McIntosh [4], he solved in 1982 the last outstanding problem of classical harmonic analysis by proving the continuity of the Cauchy integral operator on Lipschitz curves. It had been the key obstacle to the solvability of boundary value problems for Lipshitz domains (e.g. domains with corners). His methodologies prepared the way for Bony's para-differential calculus, Wu's proof of the existence of water waves in three dimensions, and the proof of Kato's conjecture, essentially changing the landscape of analysis.

Around 1984 Meyer [6]-[8] discovered the relation between the analytic tools used in harmonic analysis and various signal processing algorithms used in seismic exploration. In his Abel Prize interview [DS], Meyers said, "Morlet, Grossmann, and Daubechies were in a sense ahead of me in their work on wavelets. So I was the 'Quatrieme Mousquetaire.' They were Les Trois Mousquetaires." He recognized their work as a rediscovery of Calderón's formulas in harmonic analysis, thereby bridging fifty years of multiscale harmonic analysis with "wavelets." This discovery led later to the construction of the Meyer wavelet basis, an orthonormal basis of functions localized in space and frequency. His work inspired Daubechies to discover the compactly supported orthonormal wavelet bases, which profoundly affected the field of engineering, leading in subsequent work to nonlinear adapted Fourier analysis and signal processing [3]-[5].

The technological impact has been remarkable. For example, the current JPEG 2000 standard for image compression has evolved from the wavelet tools invented by Meyer. The field of wavelet analysis has thousands

[^6]of papers in areas of application ranging from signal processing to medical diagnostics. Modern engineering depends on his methods. Numerical analysis uses his tools for efficient numerical computation of linear and nonlinear maps.

More recently Meyer has introduced new tools for analysis of the Navier-Stokes equations for fluid flow, discovering remarkable profound mechanisms relating oscillation to stability and blowups.

Some of Meyer's close collaborators have kindly provided some descriptions of his work, with a goal of covering a broad panorama of analysis. Stéphane Mallat, who formalized with Meyer the orthogonal multiresolution framework, describes Meyer's celebrated work on wavelets. His student Stéphane Jaffard (1989), who wrote the account [J] of his Abel Prize for the Société Mathématique de France Gazette des Mathématiciens, focuses on time-frequency analysis. Alexander Olevskii describes Meyer's Sets, which modeled quasi-crystals before they were discovered. My own contribution describes Meyer's work on nonlinear Fourier analysis. His student Albert Cohen (1990) describes Meyer's impact on sparse analysis.

Meyer's work is characterized by an extraordinary depth, solving longstanding problems, and starting new fields of mathematics and applications. These seminal, broad contributions have had a profound impact on different areas of science and establish him as a major figure in mathematics.

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## Ronald Coifman

## Yves Meyer's Work on Nonlinear Fourier Analysis: Beyond Calderón-Zygmund

It is a privilege to be able to relate ideas, explorations, and visions that Yves, his collaborators, and his students developed over the last forty years, for some of which I was an active participant and observer.

We have had a lot of fun and excitement in this adventure, exploring and discovering beauty and structure.

I will focus my narrative on the simplest illustrations and examples of Yves Meyer's foundational contributions to nonlinear harmonic analysis, and try to illuminate and motivate some key programmatic issues that continue and build beyond the Calderón-Zygmund vision and program. It was Zygmund's view that harmonic analysis provides the infrastructure linking all areas of analysis, from complex analysis to partial differential equations to probability and geometry.

In particular he pushed forward the idea that the remarkable tools of complex analysis, such as contour integration, conformal mappings, and factorization, used to provide miraculous proofs in real analysis should be deciphered and converted to real variable tools. Together with Calderón, he bucked the trend for abstraction, prevalent at the time, and formed a school pushing forward this interplay between real and complex analysis. A principal bridge was provided by real variable methods, multiscale analysis, Littlewood Paley theory, and related Calderón representation formulas, later rediscovered by Morlet and others. They will be discussed here in relation to wavelets.

## Bilinear Convolutions, the Calderón Commutator, Complex Analysis, and Paraproducts

In order to understand some of the basic ideas and methods introduced by Meyer and to illustrate the scope of the program, we start with the basic example of "paracalculus" introduced by Calderón as a bilinear operator needed to extend smooth pseudo-differential calculus to rough environments. He managed by an analytical tour de force using complex function theory to prove that the Calderón commutator defined below is a bounded operator on $L^{2}$.

Given a Lipschitz function $A$ on the real line (so $\left.A^{\prime}:=a \in L^{\infty}(\mathbb{R})\right)$ one formally defines the linear operator $C_{1}(f)$ by the formula

$$
\begin{align*}
C_{1}(f) & =\int_{\mathbb{R}} \frac{A(x)-A(y)}{(x-y)^{2}} f(y) d y  \tag{1}\\
& =(A|d / d x|-|d / d x| A) f=[A,|d / d x|](f),
\end{align*}
$$

where the meaning of the absolute value of the derivative operator is given as

$$
\begin{equation*}
|d / d x| f(x)=1 / 2 \pi \int_{\mathbb{R}} \exp (i x \xi)|\xi| \hat{f}(\xi) d \xi \tag{2}
\end{equation*}
$$

and $\hat{f}$ is the Fourier transform of $f$.
This is the first commutator of Calderón. The simplest particular case is obtained when $A(x)=x$ and $C_{1}(f)$ becomes the classical Hilbert transform. He introduced an auxiliary related operator, linking complex function theory and Fourier analysis as we now describe.

Let $a$ and $f$ be two periodic functions on the circle of power series type, with $a$ bounded and $f$ in $L^{2}$. Let $h$ satisfy $h^{\prime}=a f^{\prime}$.

Calderón's theorem is equivalent to the statement that $h$ is in $L^{2}$. Written in terms of Fourier coefficients of $a, f$, and $h$ this equation becomes

$$
\begin{equation*}
\hat{h}_{k}=\sum_{0<j \leq k}(j / k) \hat{a}_{k-j} \hat{f}_{j} . \tag{3}
\end{equation*}
$$

Observe that $h(\theta)=\int_{0<t<\theta} a(t) f^{\prime}(t) d t$ and that the derivative is in the sense of distributions as $f$ is only in $L^{2}$. Meyer came out with the following-remarkably simple-proof. Write

$$
j / k=s=1 / \pi \int_{\mathbb{R}} s^{i \gamma} \frac{d \gamma}{1+\gamma^{2}} .
$$

Subsituting in (3) we get

$$
\hat{h}_{k}=1 / \pi \int_{\mathbb{R}} \sum_{0<j \leq k}(j / k)^{i \gamma} \hat{a}_{k-j} \hat{f}_{j} \frac{d \gamma}{1+\gamma^{2}}
$$

leading to the representation of $h$ as

$$
h=\pi(a, f)=1 / \pi \int_{\mathbb{R}} M_{-\gamma}\left(a M_{\gamma} f\right) \frac{d \gamma}{1+\gamma^{2}}
$$

where $M_{y} f(\theta)=\sum_{0<k} k^{i \gamma} \hat{f}_{k} e^{i k \theta}$ is a contraction on $L^{2}$, proving that in the simpler case when $a$ is a bounded function and $f$ is in $L^{2}$ that $h$ is square integrable. Together we obtained the full strength of Calderón's theorem using real variable methods on $M_{-y}$ as a singular integral operator on $H^{1}$.

An important property hidden in the paraproduct is the weak continuity of this bilinear expression. Observe that the product of functions is not a bilinear operation that is weakly continuous in the arguments: consider $\sin (n x)$, which converges weakly to 0 while $\sin (n x) \cdot \sin (n x)$ converges weakly to $1 / 2$. On the other hand, since each Fourier coefficient of the product of two functions of analytic type is a combination of a finite number of coefficients, it is clear that the paraproduct is weakly continuous in the arguments. The analysis done by Meyer and collaborators introduced real variable versions of the commutators or bilinear operators described above. These are fundamental building blocks for higher-dimensional nonlinear analysis, starting with bilinear pairings.

The simplest real variable version of the Calderón paraproduct is given in terms of wavelets or Haar functions as

$$
h=\pi(a, f)=\sum_{I} m_{I}(a)<f, h_{I}>h_{I},
$$

where $h_{I}$ are the Haar basis functions supported on the dyadic interval $I$ and $m_{I}(a)$ is the mean value of a, on that interval [2].

## Applications to Analytic Dependence

One class of problems in nonlinear Fourier analysis concerns the nonlinear analytic dependence of linear operators on functional arguments. As we will see, such problems are deeply connected to all aspects of harmonic analysis [CM, 4]. We shall focus on the seminal example of the Riemann mapping functional on rectifiable curves in the complex plane.

The fundamental mathematical question is whether this dependence is analytic on some families of curves. A particular example blending all of this is the following: We wish to understand the flow lines of water above a riverbed as in Figure 1, and their dependence on the modification of the shape of the bed. Since the flow lines are the images of horizontal lines under the Riemann mapping from the upper half plane to the region above the curve, we need to understand the dependence of the Riemann mapping on the curve. The link between the geometric description of the curve and the analytic problem is provided by the Cauchy transform. This operator is at the center of complex analysis, and provides a vehicle to understand the infinite-dimensional Banach manifold of rectifiable curves satisfying the chord-arc condition.


Figure 1. Flowlines above a riverbed. How are they going to be affected by the red bump?

So consider a Jordan curve in terms of its arc length parameterization, i.e.,

$$
z(s)=\int^{s} e^{i \alpha(t)} d t,
$$

and its corresponding Cauchy integral operator as:

$$
C(\alpha, f)(s)=p \cdot v \cdot(1 / 2 \pi i) \int_{\mathbb{R}} \frac{f(t) e^{i \alpha(t)} d t}{(z(s)-z(t))}
$$

The natural space for which this operator-valued function of $\alpha$ is analytic is the space of functions of bounded mean oscillation (BMO). In fact the first derivative of $C(t \alpha, f)$ at $t=0$ is a bounded operator on $L^{2}$ if and only if $\alpha$ is in BMO. The condition of $\alpha(s)$ having a small norm in BMO is equivalent to the geometric chord-arc condition $|s-t|<(1+c)|z(s)-z(t)|$ for small $c$.

We are now going to sketch the relationship of the question of analyticity to complex analysis, more surprisingly to operator functional calculus, and to the BMO manifold of chord-arc curves. This whole theory provides remarkable linkages between functional analysis, operator theory, singular integrals, and geometry.

Consider a monotone change of variable of the form $h(x)=\int^{x}(1+a(t)) d t$. We conjugate the operator $d / d x$ with the change of variable defined by $h(x)$ to obtain the operator $1 /(1+a) d / d x=U_{h} d / d x U_{h}^{-1}$, where $U_{h} f(x)=f\left(h^{-1}(x)\right)$. We now consider the operator $\operatorname{sgn}(d / d x)$ defined as

$$
\operatorname{sgn}(d / d x) f=1 / 2 \pi \int_{\mathbb{R}} \exp (i x \xi) \operatorname{sgn} \xi \hat{f}(\xi) d \xi
$$

We let $h(x)=x+A(x)$ to conclude that

$$
\operatorname{sgn}\left((1 /(1+a) d / d x)=\int_{\mathbb{R}} \frac{f(t)(1+a(t)) d t}{x-t+(A(x)-A(t)}\right.
$$

We see that the Cauchy integral for the curve above the graph of $A(x)$ is given by the same expression with $a$ replaced by $i a$.

More generally if we replace $a$ by za, the analyticity in $a$ is equivalent to a bound in $c^{k}$ for the $k$-th Taylor coefficient in $z$, known as the $k$-th commutator. We also can show easily that the analytic dependence of the operator $\operatorname{sgn}((1 /(1+a) d / d x)$ on the coefficient $1 /(1+a)$ is equivalent to the boundedness of the Cauchy integral operator [1].

Returning to the Riemann mapping, we observe that the Cauchy transform is an oblique projection from $L^{2}$ onto the space of boundary restrictions (to the curve) of holomorphic functions on one side of curve. The corresponding orthogonal projection, called the Szego projection, can be easily expressed as a series involving the Cauchy transform, and also using the Riemann mapping itself, thereby leading to an analytic expression and a proof that the Riemann mapping depends analytically on the curve, provided that the Cauchy transform does. The distance between operators corresponding to two curves can be shown to be equivalent to the bounded mean oscillation norm of the difference between their arguments, thereby defining the most general geometry on the infinite-dimensional manifold of curves for which all of these objects are real analytic.

Several remarkable features appear here. The Cauchy transform on chord-arc curves is an operator-valued functional of the argument of the curve; it can be expanded in a convergent power series of operators on BMO. The norm on BMO is the operator norm defined by the first linear term. It defines the largest Banach space for which the nonlinear operator-valued transform is analytic.

Specifically for the Riemann map functional from the curve $z(s)=\int^{s} e^{i \alpha(t)} d t$ given in terms of the argument $\alpha(s)$ to the argument $\beta(s)$ of the change of variable $h(s)=\int^{s} e^{\beta(t)} d t$ defined by the Riemann map is an analytic functional, in the BMO topology of both $\alpha(s)$ and $\beta(s)$.


Figure 2. Coifman and Meyer having fun while working on the $k$ th commutators.

This theme of discovery of the natural functional space appears again in Meyer's work for the Navier-Stokes equations and other nonlinear PDE. The multilinear operators arising in the power series can be analyzed and decomposed directly, using Fourier or other transforms. This approach provides insight and could enable efficient numerical implementations. The Cauchy integral generalizes directly to higher dimensions, for example as a double layer potential operator, or more generally, as the restriction to a submanifold of a Calderón-Zygmund operator of the appropriate homogeneity. The long-standing problem of short-time existence of water waves in 2 or 3 dimensions was solved by Sijue Wu using these higherdimensional extensions. This example is related to a range of questions on the dependence and properties of functions of operators, or more generally on the dependence of spectral theory or resolvents on the coefficients.

The example of a variable-coefficient Laplace operator $\Delta$ in divergence form is related to the Kato conjecture, which states that the domain of $\Delta^{1 / 2}$ is the space of functions having a gradient in $L^{2}$. This conjecture in one variable is equivalent to the boundedness of the Cauchy integral, as proved by Coifman, McIntosh, and Meyer [1], and then extended to the general case close to the identity and finally resolved by Pascal Auscher, Steve Hofmann, Michael Lacey, Alan McIntosh, and Philippe Tchamitchian in 2002.

## A Few Representative Works by Meyer

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## ABOUT THE AUTHOR

Ronald (Raphy) Coifman has been a close friend and collaborator of Yves Meyer for the last forty years, jointly encouraging a global network of harmonic analysts.

## Stéphane Mallat

## Orthogonal Wavelets: Bridging Beyond Mathematics

Yves Meyer is a pure mathematician. This seemed obvious to us in the amphitheatre of 400 students at École Polytechnique in Paris, where Yves was enthusiastically describing the beauty of harmonic analysis as "a Mozart symphony." It was 1984, and Meyer already had the aura of an exceptional mathematician, with spectacular results in number theory and harmonic analysis. We thought that he was living in the limbo of abstractions, but we were wrong. I later realized that his intellectual home is transient, a nomad he sometimes calls himself, which is a root of his profound originality.

In 1986 it was under the umbrella of the Bourbaki seminar that Yves published his discovery of wavelet orthonormal bases, with $\mathrm{C}^{\infty}$ functions having a fast decay. This was not a 100-year-old conjecture, perhaps not looking spectacular for such a mathematical artist, but nonetheless opening a door to a world of science and mathematics. It led to more than a hundred thousand papers in many fields of science and mathematics, books, patents, industrial applications, and start-ups. He did not just open a door, but also penetrated the world of applications and led the way for a whole generation of mathematicians. Another spectacular outcome was to bridge pure and applied mathematics in France, the nation of Bourbaki.

But let me return to 1986. It was known that one could build an orthonormal basis of $\mathbf{L}^{2}(\mathbb{R})$ with functions that are dilations by $2^{j}$ and translations by $2^{j} n$ of a single real-valued function $\psi(t)$ for $(j, n) \in \mathbb{Z}^{2}$ :

$$
\left\{\psi_{j, n}(t)=2^{-j / 2} \psi\left(2^{-j}\left(t-2^{j} n\right)\right)\right\}_{(j, n) \in \mathbb{Z}^{2}}
$$

As a result, any function $f \in \mathbf{L}^{2}(\mathbb{R})$ can be expanded as

$$
f=\sum_{(j, n) \in \mathbb{Z}^{2}}\left\langle f, \psi_{j, n}\right\rangle \psi_{j, n},
$$

[^7]where wavelet coefficients exhibit the local variations of $f$ at scale $2^{j}$ in the neighborhood of $t=2^{j} n$ :
$$
\left\langle f, \psi_{j, n}\right\rangle=\int f(t) \psi_{j, n}(t) d t
$$

In 1909, Alfréd Haar constructed the first wavelet basis, generated by a piece-wise constant wavelet

$$
\psi=2^{-1 / 2}\left(1_{[0,1 / 2]}-1_{[1 / 2,1]}\right) .
$$

It is supported on $[0,1]$ and discontinuous.
It was also known that an orthonormal wavelet basis could be generated by the Shannon wavelet $\psi$ whose Fourier transform $\hat{\psi}(\omega)$ is compactly supported

$$
\hat{\psi}=1_{[-2 \pi,-\pi]}+1_{[\pi, 2 \pi]} .
$$

The Shannon wavelet $\psi(t)$ is $\mathbf{C}^{\infty}$ but has a slow decay because its Fourier transform is discontinuous. The fact that it defines a wavelet orthonormal basis is closely related to the Shannon sampling theorem.

Yves knew about wavelets through his encounter with the physicist Alex Grossmann and the geophysicist Jean Morlet, who had introduced redundant families of regular wavelets with fast decay that spanned the space $\mathbf{L}^{2}(\mathbb{R})$ without being orthogonal. But is it possible to find a regular wavelet $\psi$ having a fast decay, which generates an orthonormal basis of $\mathbf{L}^{2}(\mathbb{R})$ ? In 1986, this seemed highly unlikely otherwise it would certainly have been discovered before. The Balian-Low theorem proved that orthogonal bases composed of localized complex exponentials could not be constructed with regular functions having fast decay. To verify a similar property for wavelets, Yves tried to build an orthogonal basis with a regular and welllocalized wavelet, hoping to find the reason why it did not exist. To his surprise, he discovered wavelets $\psi$ that are $\mathbf{C}^{\infty}$ with a fast decay and that generate orthonormal bases. Meyer wavelets resemble Shannon wavelets, with a Fourier transform that has a compact support but that is $\mathbf{C}^{\infty}$. As a result $\psi(t)$ is also $\mathbf{C}^{\infty}$ with a fast decay: the best of both worlds, illustrated in Figure 1. Yves also extended this construction to define wavelet bases of functions of several variables, which led to important applications in image processing.

These wavelet constructions could have been anecdotal if Yves had not realized that these bases were unconditional bases of most classical functional spaces, from $\mathbf{L}^{\mathbf{p}}(\mathbb{R})$ for $p<1<\infty$ to Sobolev or Hölder spaces. Hence one can read the regularity of a function $f$ from the amplitude of its wavelet coefficients $\left|\left\langle f, \psi_{j, n}\right\rangle\right|$ and their decay as the scale $2^{j}$ goes to 0 . At fine scales, wavelet coefficients become very small in domains where $f$ is regular. If $f$ has few singularities, then at fine scales $2^{j}$ it has few large wavelet coefficients $\left\langle f, \psi_{j, n}\right\rangle$. One can thus approximate a piecewise regular function $f$ by keeping relatively few large coefficients $\left\langle f, \psi_{j, n}\right\rangle$ in its wavelet decomposition.

These properties opened the possibility to transport pure harmonic analysis to applications. It provided a mathematical framework to understand a wide range of scientific and industrial applications. But Yves did not


Figure 1. Graph of a Meyer wavelet. It is a $\mathrm{C}^{\infty}$ function with a fast decay, whose translations and dilations define an orthonormal basis of $L^{2}(\mathbb{R})$.
stay in the cocoon of his native mathematics community, and crossed the door with enthusiasm. Similarly to Calderón and Zygmund in harmonic analysis, he became the concert master of a wide research program, providing a vision and connections between mathematics and multiple fields of science, and providing space for many researchers to contribute. This generosity drew a whole generation of young mathematicians along this path, without frontiers between pure and applied mathematics, science, and engineering applications. His student Pierre-Gilles Lemarié together with Guy Battle began by showing that other wavelet bases could be constructed with polynomial splines, having exponential decay.

But this is not
> a story where mathematics and other sciences grew together a story where the world was illuminated by mathematics, but where mathematics and other sciences grew together. Engineers had not been waiting arms crossed. Applications were too important. For telecommunications, you need to multiplex signals-to aggregate multiple signals into a single one for transmission and then be able to separate them. In 1977, two engineers, Esteban and Galand, had introduced a procedure to split $N$ coefficients of a sequence $x[n]$ for $0 \leq n<N$ into two sequences of $N / 2$ coefficients each, computed with a convolution of $x$ with two "filters" $h$ and $g$ while keeping one sample out of two:

$$
x_{1}[n]=x \star h[2 n] \text { and } x_{2}[n]=x \star g[2 n] .
$$

Finding conditions on $h$ and $g$ to recover $x$ from $x_{1}$ and $x_{2}$ became a major research issue in signal processing. Necessary and sufficient "conjugate mirror filter" conditions were established by Smith and Barnwell on the Fourier transforms $\hat{h}(\omega)$ and $\hat{g}(\omega)$ of the two filters:

$$
|\hat{h}(\omega)|^{2}+|\hat{h}(\omega+\pi)|^{2}=2 \text { and } \hat{g}(\omega)=e^{-i \omega} \hat{h}^{*}(\omega+\pi) .
$$

Although seemingly unrelated to wavelets, these filters then appeared to be at the core of wavelet orthonormal bases.

In computer vision, similar questions appeared in the 1980s, but for other reasons. To recognize objects in images, researchers proposed to detect edges, which appear at different scales. Reducing the resolution of images is important to eliminate fine details when they are not needed. In 1983, Peter Burt developed a pyramidal algorithm that iteratively averages the image with a filter $h$, while detecting multiscale edges with another filter $g$ derived from $h$.

In 1986, I had left École Polytechnique to do a PhD in image processing at the University of Pennsylvania. I was learning image pyramids and conjugate mirror filters when a friend gave me Meyer's paper on orthogonal wavelets. The connections became apparent. Multiresolution image pyramids could be formalized as projections in embedded subspaces of $L^{2}(\mathbb{R})$, providing progressively finer approximations of functions as the scale decreases. Moreover, connection across scales appeared to be governed by the conjugate mirror filters $h$ and $g$ studied in signal processing, which also insure orthogonality properties. As a result, wavelets generating an orthonormal basis $\mathbf{L}^{2}(\mathbb{R})$ have a Fourier transform obtained by cascading the Fourier transform of these filters:

$$
\hat{\psi}(\omega)=2^{-1 / 2} \hat{g}\left(2^{-1} \omega\right) \prod_{p=2}^{\infty} \hat{h}\left(2^{-p} \omega\right) .
$$

This also implies that the wavelet coefficients $\left\langle f, \psi_{j, n}\right\rangle$ of a discretized function $f$ can be computed with a fast algorithm, by cascading convolutions and subsampling with $h$ and $g$, as in filter bank algorithms used by signal processing engineers. For appropriate filters, its requires fewer operations than the Fast Fourier Transform.

I sent my manuscript to Yves, who enthusiastically brought me to Chicago where he was working in the office of Zygmund. In three days, he solved all remaining mathematical problems. Yves had previously realized that there was an embedding-space structure underlying wavelets. We gave a sufficient condition to guarantee convergence in $\mathbf{L}^{2}(\mathbb{R})$ of the infinite filter product and obtain a wavelet that generates an orthonormal basis. This meant that new wavelet orthonormal bases could be constructed by defining filters that satisfy the conjugate mirror conditions discovered in signal processing. A necessary and sufficient condition was then found by Albert Cohen, a student of Yves, to relate wavelets to these filters.

At this point, many signal processing engineers looked at wavelets as a useless mathematical abstraction. What should they care about functions in $\mathbf{L}^{2}(\mathbb{R})$ when all
computations are performed over finite sequences, with filters they had discovered before?

A first answer was given by the remarkable work of Ingrid Daubechies. Ingrid showed that more conditions had to be imposed on conjugate mirror filters to obtain regular wavelets of compact support. The Daubechies filters reduced computations, and the regularity of the resulting wavelets appeared to be important to avoid introducing visible artifacts when compressing images. In collaboration with Albert Cohen and Christian Feauveau, she designed the wavelet filters that have been adopted in the image compression standard JPEG-2000. This compression algorithm decomposes images in a wavelet basis and represents the non-zero coefficients with an efficient coding scheme. It can code images with 20 times fewer bits, with barely visible artifacts, because few wavelet coefficients of large amplitude need to be coded. The position of these large coefficients are located in near edges, as shown in Figure 2.

A second reason why mathematics became useful came from relations with Hölder regularity exponents, proved by Stéphane Jaffard, another student of Yves. He found necessary conditions and sufficient conditions to characterize the regularity of a function $f$ at any point $t$, from the decay of wavelet coefficients in a neighborhood of this point. It gave a mathematical framework to understand relations between wavelet coefficients and signal properties and to analyze very irregular functions such as multifractals.

A surprising connection appeared with statistics. David Donoho and Iain Johnstone realized that the unconditional basis properties proved by Meyer were exactly what was needed to suppress additive noise from signals having sparse wavelet coefficients. They showed that a simple thresholding, which set to zero the smallest wavelet coefficients of a noisy signal, is a nearly optimal non-linear estimator over large classes of signals. It opened a new field in non-linear statistics, which is still alive today.

Applications in numerical analysis came from the work of Gregory Beylkin, Raphy Coifman, and Vladimir Rocklin, who proved that many singular operators are represented by sparse matrices in a wavelet orthonormal basis. Matrix multiplications can then be computed with far fewer operations, for applications to calculations of solutions of partial differential equations, integral equations, and variational problems.

Wavelet orthonormal bases have found a multitude of applications, in chemistry, physics, many branches of information processing, and mathematics. They provided new tools in statistics and approximation theory, to specify multiscale properties of random processes or to develop fast numerical analysis algorithms. This is how the free nomadism of a curious and incredibly talented mathematician has had such high impact and created unexpected openings between many fields within and outside mathematics.


Figure 2. Top: original image $f$. It is an array of pixels that encode the image intensity. Bottom: each small image displays as dark points the wavelet coefficients that have a large amplitude, at a particular scale and orientation. Large coefficients are located where the image intensity has a sharp transition, near contours. Most coefficients are nearly zero (white), which is why they are compressed with much fewer bits than the original image pixels.

## Image Credits

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## Stéphane Jaffard

## Time-Frequency Analysis, Chirps, and Local Regularity

By the middle of the twentieth century the limitations of classical Fourier analysis for signal processing were patent. For example, musical recordings, which are a succession of notes of limited duration, clearly call for a localized Fourier analysis, as do chirps, a generic denomination which covers signals $f$ that locally look like a pure frequency, evolving slowly and smoothly with time. These defining conditions for chirps can be formalized as

$$
\begin{equation*}
f(t)=\operatorname{Re}\left(a(t) e^{i \varphi(t)}\right), \tag{4}
\end{equation*}
$$

where the modulation factor $a$ and the instantaneous frequency $\varphi$ satisfy

$$
\left|\frac{a^{\prime}(t)}{a(t)}\right| \ll \varphi^{\prime}(t) \text { and } \quad\left|\varphi^{\prime \prime}(t)\right| \ll\left(\varphi^{\prime}(t)\right)^{2} .
$$

The most important examples of chirps are supplied by gravitational waves (Figure 1): The very first one detected, in September 2015, was of the form $\left|t-t_{0}\right|^{-1 / 4} \cos (\omega \mid t-$ $\left.t_{0}\right|^{5 / 8}+\eta$ ).

Analysis of such data should involve time-frequency analysis. The signal is first localized by multiplying it with "windows." Gaussians are a natural choice because of their optimal localization in space and frequency. Then a Fourier analysis of the localized signal is performed. This idea was introduced in the 1940s by Nobel laureate D. Gabor and leads to the short-time Fourier transform of a function $f$, defined as

$$
G_{f}(x, \xi)=\int_{\mathbb{R}} f(t) \varphi(t-x) e^{-2 i \pi t \xi} d t,
$$

where $\varphi$ is the window. A continuous transform is computationally inefficient, and this raises the question of the existence of appropriate orthogonal decompositions, and of the corresponding fast transforms. Orthonormal bases cannot follow directly by sampling $G_{f}$ because of the Balian-Low theorem (1981), which states that if a

[^8]


Figure 1. The gravitational wave as recorded by LIGO on September 14, 2015, after denoising; and the corresponding chirp as predicted by general relativity.
window $\varphi$ is both smooth and well localized then, for any choice of $a$ and $b$, a system of the form

$$
\varphi(t-a k) e^{i b n t} \quad k, n \in \mathbb{Z}
$$

is either incomplete or over-complete. Another Nobel laureate, K. Wilson, suggested a way to overcome this obstruction, by allowing for a double localization around opposite frequencies. Such bases, where the complex exponentials are simply replaced by sines and cosines, were constructed by Daubechies, Jaffard, and Journé and supply orthonormal bases of $L^{2}(\mathbb{R})$ of the form:
$\varphi_{n, k}(t)=\left\{\begin{array}{cl}\varphi_{0, k}(t)=\varphi(t-k) \quad k \in \mathbb{Z}, \\ \sqrt{2} \varphi\left(t-\frac{k}{2}\right) \cos (2 \pi n t) & \text { if } k+n \in 2 \mathbb{Z}, \\ \sqrt{2} \varphi\left(t-\frac{k}{2}\right) \sin (2 \pi n t) & \text { if } k+n \in 2 \mathbb{Z}+1 .\end{array}\right.$
S. Klimenko, who was the designer of Coherent Waveburst, the algorithm used in the signal processing part of the gravitational wave detection, chose this basis because it meets the following requirements:

- The window $\varphi$ can be Meyer's [LM] scaling function (Figure 2). This choice is motivated by the fact that a window with compact support in the Fourier domain allows elimination of noise components


Figure 2. Meyer's scaling function and an element of the corresponding Wilson basis. Meyer's scaling function stands at the junction between two major extensions of Fourier analysis at the end of the twentieth century, time-scale and time frequency analysis. It is the building stone that allows generation of orthonormal bases in both settings.
that lie away from the main Fourier area of interest.

- Fast decomposition algorithms.
- Fast translation algorithms: one needs to compare the two signals recorded in the two LIGO detectors, which do not arrive exactly at the same time, and therefore need to be shifted.
- Gravitational waves are sparse in Wilson bases.

This procedure is now referred to as the Wilson-Daubechies-Meyer transform.

Variants of Wilson bases were independently constructed by H. Malvar. They also are time-frequency orthonormal bases of $L^{2}(\mathbb{R})$ of the form

$$
\varphi_{n, k}(t)=\varphi(t-k) \cos \left[\pi\left(n+\frac{1}{2}\right)(t-k)\right] \quad k, n \in \mathbb{Z} .
$$

The corresponding decomposition, called MDCT (Modified Discrete Cosine Transform), is currently used in audio compression formats, e.g. MP3 or MPEG2 AAC. Malvar bases of adaptive lengths were introduced by R. Coifman and Y. Meyer [CM]: They are of the form

$$
\varphi_{n, k}(t)=\varphi_{k}(t) \cos \left[\frac{\pi}{l_{k}}\left(n+\frac{1}{2}\right)\left(t-a_{k}\right)\right],
$$

where the windows $\varphi_{n, k}$ start at $a_{k}$ and have arbitrary lengths $l_{k}$. These bases found a remarkable application in speech segmentation: E. Wesfreid and V. Wickerhauser devised an entropy minimization criterion that allows the lengths of the windows to adapt to the changes in the signal and thus performs automatic segmentation.

A different point of view can be developed for modeling chirps, where the instantaneous frequency $\varphi(t)$ in (4) diverges at a point $x_{0}$, leading to functions that have a singularity at $t_{0}$. This leads to pointwise singularities that are typically of the form

$$
c_{h, \beta}(t)=\left|t-t_{0}\right|^{h} \sin \left(\frac{1}{\left|t-t_{0}\right|^{\beta}}\right) .
$$

Based on the heuristic supplied by such toy-examples, Meyer [JM1] developed a general framework for such behaviors, where the sine function is replaced by a fairly arbitrary oscillating function, and showed that they are characterized by precise estimates on the wavelet coefficients. A remarkable application is supplied by the analysis of Riemann's function

$$
\mathcal{R}(t)=\sum_{n=1}^{\infty} \frac{\sin \left(\pi n^{2} t\right)}{n^{2}},
$$

which was proposed by Riemann as a candidate for a continuous nowhere differentiable function. It took a century to disprove Riemann's intuition and show that $\mathcal{R}$ is differentiable at rational points of the form ( $2 p+$ 1) / $(2 q+1)$ (J. Gerver, 1970). Meyer considerably improved our comprehension of the behavior of Riemann's function in the neighbourhood of these points by exhibiting a complete chirp expansion about $t_{0}=1$ :
$\mathcal{R}(1+t) \sim-\frac{t}{2}+\sum_{k \geq 1}|t|^{k+1 / 2} g_{k}\left(\frac{1}{t}\right) \quad$ where $\quad g_{k} \sim \mathcal{R}^{(-k)}$
(see Figure $3 ; g_{k}$ is essentially a primitive of order $k$ of the Riemann function itself!).

These explorations were the first stones that paved the way towards a classification of the pointwise singularities of everywhere irregular functions, now referred to as multifractal functions. Meyer [M1], [M2] introduced a new regularity exponent, the weak scaling exponent, which has the remarkable property of being covariant with respect to fractional primitives or derivatives. He revisited Wilson bases, constructed new bridges with wavelet decompositions, and built a whole variety of bases that can be tailored to particular chirp behaviors. With Jaffard [JM2] he uncovered an unexpected relationship between sparsity in a wavelet basis and pointwise regularity: They showed that generically, in the sense of Baire category, functions that have sparse wavelet expansions (the decreasing rearrangement of the sequence of their wavelet coefficients has fast decay) are multifractal, and they determined their multifractal spectrum (i.e. the Hausdorff dimensions of their pointwise Hölder singularities); Jaffard, Meyer, and Ryan [JMR] provide a user-friendly review on all these topics.

Meyer has been at the center of the effervescent and prolific multidisciplinary network that made the success of wavelets. He is a living proof that the limits between


Figure 3. The Riemann function, and a zoom around the chirp at $t_{0}=1$. Meyer exhibited a complete chirp expansion near the rare differentiable points.
pure and applied science do not exist, and he repeatedly has shown that deep mathematical concepts can be the key to spectacular applications. He is famous for being a passionate lecturer. Extremely generous, he always pushed his many students and collaborators to the front; each of them can testify to the importance he laid to the transmission of science and the values of intellectual integrity, humanism, and tolerance. In a time driven by material values and short range-profit, Meyer stands as an example for younger generations of scientists.

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Figure 1 dx.doi.org/10.7935/K5MW2F23.
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ABOUT THE AUTHOR
Stéphane Jaffard did his PhD under the supervision of Yves Meyer at the very beginning of wavelet theory. His main subject of research is harmonic and multifractal analysis and their applications in signal and image processing.
Stéphane Jaffard

## Alexander Olevskii

## Meyer's Sets and Related Problems

Many years ago Yves Meyer introduced remarkable concepts and constructions, which allowed him to solve some important problems in harmonic analysis. After many years these concepts and ideas remain important and are applicable to new problems.

Below I am going to focus on some aspects of this new development, in which Meyer is one of the main actors.

## Poisson Summation Formula

We start with the classical Poisson formula:

$$
\sum_{\lambda \in \Lambda} \hat{f}(\lambda)=\frac{1}{|\Lambda|} \sum_{\lambda^{*} \in \Lambda^{*}} f\left(\lambda^{*}\right)
$$

Here $\Lambda$ is a lattice in $\mathbb{R}^{n},|\Lambda|$ is the volume of its fundamental parallelepiped, $\Lambda^{*}$ is the dual lattice, $f$ is any function in the Schwartz class $S(\mathbb{R})$, and $\hat{f}$ its Fourier transform.

Equivalently, if $\mu$ is the sum of unit masses on a lattice (a Dirac comb), then its Fourier transform $\hat{\mu}$ (in sense of distributions) is again a Dirac comb. The spectrum of $\mu$ (:= the support of $\hat{\mu}$ ) in this case is the dual lattice.

The Poisson formula has many applications. In particular, it plays a seminal role in the $X$-ray diffraction.

The following problem is important: do there exist some other discrete measures $\mu$ such that the distributional Fourier transform $\mu$ is also a discrete measure? Here one looks for measures $\mu$ that are not finite sums of Dirac combs, translated and modulated. This problem was studied in the 1950s by J.-P. Kahane - S. Mandelbrot and by J.-P. Guinand.

[^9]

Figure 1. The Voronoi cells of a model set provide a nonperiodic tiling, such as this Penrose tiling in Keskuskatu Square, Helsinki.

## Meyer's Model Sets

Yves Meyer [1] discovered a family of non-periodic discrete measures with point spectrum.

The construction is based on a concept of model sets, obtained from lattices by the cut-and-project procedure.

Here is the definition in the simplest case. In the plane $(x, y)$, take a lattice $\Gamma$ in the general position. Fix a set $Q$ (a "window") that is a finite union of intervals on the $y$-axis. Consider the set of lattice points that lie in $\mathbb{R} \times Q$, and project them onto the $x$-axis. This way one gets a Meyer model set $M$ in $\mathbb{R}$. A similar construction can be done in $\mathbb{R}^{n}$, starting with a lattice $\Gamma$ in $\mathbb{R}^{n} \times \mathbb{R}^{m}$.

Here are some properties of these model sets $M$ :

- Every $M$ is a nonperiodic uniformly discrete set. Uniform discreteness means that the pairwise distances between the different elements are separated from zero.
- The "Voronoi cells" corresponding to $M$ provide a tiling of $\mathbb{R}^{n}$ by non-overlapping translates of a finite family of polyhedra. A well-known example of such a tiling is due to Penrose (Figure 1). De Bruijn proved that the Penrose tiling can be obtained by the cut-and-project procedure from a 5-D lattice.
- Every model set $M$ supports a measure $\mu$ with point spectrum.

Here is a proof of the last property in the simplest case: let $M$ be a 1-D model obtained from a lattice $\Gamma$ and window $I$. Take a function $f \in S(\mathbb{R})$ supported on $I$. Set

$$
\mu:=\sum_{x, y \in \Gamma} f(y) \delta_{x}
$$

Then $\mu$ is supported on $M$. Using the Poisson formula in $\mathbb{R}^{2}$ one can verify that

$$
\begin{equation*}
\hat{\mu}=\sum_{(u, v) \in \Gamma^{*}} \hat{f}(v) \delta_{u} . \tag{5}
\end{equation*}
$$

So, $\hat{\mu}$ is a discrete measure with a dense support $S$.

Model Sets and Reconstruction of Signals
Here I discuss a new application of the model sets as sampling sets.

For a bounded set $S \subset \mathbb{R}$, denote by $P W_{S}$ (the PaleyWiener space with spectrum $S$ ) the space of functions (signals) $f \in L^{2}(\mathbb{R})$ whose Fourier transform is supported on $S$.

Given a uniformly discrete set $\Lambda$, one wishes to recover every $f \in P W_{S}$ from its values (samples) on $\Lambda$.

The following inequality would allow one to solve this problem in a stable way:

$$
\begin{equation*}
\|f\|^{2} \leq C(S, \Lambda) \sum_{\lambda \in \Lambda}|f(\lambda)|^{2}, \quad \forall f \in P W_{S} \tag{6}
\end{equation*}
$$

Given $S$, for which sets $\Lambda$ does this condition hold? When $S$ is an interval, A. Beurling proved that the validity of (2) is essentially equivalent to the condition that $\Lambda$ must be "dense" in a certain sense. H. Landau extended the necessity of the density condition to the case of disconnected spectra. However, in general no reasonable sufficient condition can be expressed in terms of some density of $\Lambda$. The geometric structure of $S$ plays a crucial role. Hence, a special construction of a "sampling set" $\Lambda$ is needed in the general case.

Is it possible to find $\Lambda$ so that (2) holds for every spectrum $S$ of given measure, independently of its structure and localization? This problem was put forward by A. Olevskii and A. Ulanovskii, who proved in 2006 that for compact spectra such "universal sampling sets" do exist. Such sets were constructed as certain perturbations of lattices. A new remarkable proof of this result was found in 2008 by B. Matey and Y. Meyer, who showed that appropriate model sets satisfy the same property. For more about the subject see Olevskii and Ulanovskii [2].

## Quasicrystals

It is surprising that Meyer's concept of model sets turned out to be a recognized mathematical model for a physical phenomenon discovered much later. In the early 1980s Dan Shechtman and his colleagues found aperiodic atomic structures whose diffraction patterns consist of isolated spots. For this discovery he received the 2011 Nobel Prize. Such a phenomenon has been perceived as a contradiction to classical crystallography. The structures got the name "quasicrystals."

A parallel with Meyer's model sets is clear: each M is an aperiodic uniformly discrete set and it supports a measure $\mu$ with point spectrum. Since $\hat{f}$ in (1) is a Schwartz function, one can prove that the "visible" part of the spectrum, which is the set of spectral atoms whose size exceeds a fixed small number, is uniformly discrete (consists of spots!).

Similarly, discrete aperiodic measures with pure point spectrum are often called "Fourier quasicrystals." Meyer sets provide important examples of such sets.

On the other hand, if both the support and the spectrum of a measure $\mu$ in $\mathbb{R}^{n}$ are uniformly discrete, then it is a finite sum of translates and modulates of a Dirac comb, as proved by N. Lev and A. Olevskii [4]. Their result shows


Dan Shechtman, Nobel Laureate for the discovery of quasicrystals, for which Meyer's model sets serve as mathematical models.
that in some natural situations a quasicrystal can not be observed.

For $n=1$ the result is true in full generality, in higher dimensions for positive measures. In particular, the support of such measures is periodic. This answered a problem raised by J. Lagarias [3].

## Nonclassical Poisson Formulas

The characterization of measures as the sums of Dirac combs remains true even if the support of $\mu$ is just locally finite (whilst the spectrum is still uniformly discrete). On the other hand, there are measures such that both the support and the spectrum are locally finite but not periodic [4]. Further examples of this kind were found by M. Kolountzakis and Y. Meyer.

Each such measure generates a Poisson-type summation formula. A spectacular one, due to Meyer [5], has the form

$$
\hat{\mu}=c \mu
$$

where $\mu$ is a purely atomic measure on $\mathbb{R}$ supported at the points $\pm|k+a|$, where $a \in \mathbb{R}^{3} \backslash \mathbb{Z}^{3}, k$ runs over $\mathbb{Z}^{3}$, and the masses of atoms are effectively defined.

Recently Meyer has moved forward substantially this line of research. He found a series of non-classic Poisson formulas with special arithmetics of the nodes.

Note that all examples known so far, in one way or another, are based on the classic Poisson formula. It would be interesting to know whether there are examples of a different nature.

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Photo of Alexander Olevskii used with his permission.


## ABOUT THE AUTHOR

Alexander Olevskii received his ScD degree in 1966 at Moscow State University. His research has been concentrated on problems of classical harmonic analysis and related areas.

## Alexander <br> Olevskii

## Albert Cohen

## The Emergence of Sparse Analysis through the Works of Yves Meyer

Since the turn of the century, the concept of sparsity has become prominent in many areas of applied mathematics, at the crossroad with other disciplines such as signal and image processing. Loosely speaking, sparse approximation refers to the possibility of representing certain complex objects by very few numbers up to a controlled loss of accuracy. This task is usually made possible once a proper mathematical representation of these objects has been provided, for example through a change of basis.

Such economical representations play an obvious role in data compression. They also found powerful applications in statistical estimation and inverse problems. The spectacular development of compressed sensing led to a deeper mathematical understanding of how sparsity

[^10]could be exploited in many such applications. While Yves Meyer's work was not primarily focused on such areas, his contributions played a key role in the emergence and spreading of this concept and in the development of its mathematical foundations.

The present paper includes parts of a more detailed survey on the research works of Meyer published in the Abel volume [2], which can be consulted for further references.

## A Golden Decade

The process of analyzing and representing an arbitrary function $f$ by means of more elementary functions has been at the heart of fundamental and applied advances in science and technology for several centuries. In more recent decades, implementation of this process on computers by fast algorithms has become of ubiquitous use in scientific computing. In the foundational example of the univariate Fourier expansions these elementary building blocks are the 1-periodic complex exponential functions defined by

$$
\begin{equation*}
e_{n}(t)=e^{i 2 \pi n t}, \quad n \in \mathbb{Z} \tag{1}
\end{equation*}
$$

and they form an orthonormal basis of $L^{2}(] 0,1[)$.
The functions $e_{n}$ are perfectly localized in frequency but have no localization in time, since their modulus is equal to 1 independently of $t$. This property constitutes a major defect when trying to efficiently detect the local frequency content of functions by means of Fourier analysis. It also makes Fourier representations numerically ineffective for functions that are not smooth everywhere. For example, the Fourier coefficients $c_{n}(f)$ of a 1-periodic piecewise smooth function $f$ with a jump discontinuity at a single point $t_{0} \in[0,1]$ decay like $|n|^{-1}$, which affects the convergence of the Fourier series on the whole of $\mathbb{R}$.

Wavelet bases are orthonormal bases of $L^{2}(\mathbb{R})$ with the general form

$$
\begin{equation*}
\psi_{j, k}(x)=2^{j / 2} \psi\left(2^{j} x-k\right), \quad j \in \mathbb{Z}, k \in \mathbb{Z} \tag{2}
\end{equation*}
$$

where $\psi$ is a function such that $\int_{\mathbb{R}} \psi=0$. Due to their increased resolution as the scale level $j$ tends to $+\infty$, they are better adapted than Fourier bases for capturing local phenomena. The most basic example of the Haar system, which corresponds to $\psi=X_{[0,1 / 2[ }-X_{[1 / 2,1[ }$, has been known since 1911. In this example, the function $\psi$ suffers from a lack of smoothness, also reflected by the slow decay at infinity of its Fourier transform.

The construction of modern wavelet theory took place during the decade of 1980-1990. It benefited greatly from ideas coming from various (and sometimes completely disjoint) sources: theoretical harmonic analysis, approximation theory, computer vision and image analysis, computer aided geometric design, digital signal processing. One of the fundamental contributions of Yves Meyer was to recognize and organize these separate developments into a unified and elegant theory.

After some attempts to disprove their existence, Meyer turned the table in 1985 and gave a beautiful construction
of orthonormal wavelet bases that belong to the Schwartz class
(3)

$$
S(\mathbb{R}):=\left\{f \in C^{\infty}(\mathbb{R}): \sup _{x \in \mathbb{R}}|x|^{k}\left|f^{(l)}(x)\right|<\infty, k, l \geq 0\right\}
$$

and are therefore well localized both in time and frequency. Another orthonormal wavelet basis with smoothness and localization properties had been obtained earlier in the work of Jan-Olov Strömberg. By its elegant simplicity, Meyer's construction was celebrated as a milestone.

A major turning point occurred in 1986 when Stéphane Mallat [5] introduced the natural framework that was the key to the general construction of wavelets, as well as to fast decomposition and reconstruction algorithms. A multiresolution approximation is a dense nested sequence of approximation spaces

$$
\begin{equation*}
\{0\} \cdots \subset V_{j-1} \subset V_{j} \subset V_{j+1} \subset \cdots L^{2}(\mathbb{R}) \tag{4}
\end{equation*}
$$

generated by a so-called scaling function $\varphi$ in the sense that $\left(2^{j / 2} \varphi\left(2^{j}-k\right)\right)_{k \in \mathbb{Z}}$ is a Riesz basis of $V_{j}$. A countable family $\left(e_{k}\right)_{k \in \mathcal{F}}$ in a Hilbert space $V$ is called a Riesz basis if it is complete and there exists constants $0<c \leq C<\infty$ such that

$$
\begin{equation*}
c \sum_{k \in \mathcal{F}}\left|x_{k}\right|^{2} \leq\left\|\sum_{k \in \mathcal{F}} c_{k} e_{k}\right\|_{V}^{2} \leq C \sum_{k \in \mathcal{F}}\left|x_{k}\right|^{2} \tag{5}
\end{equation*}
$$

holds for any finitely supported coefficient sequence $\left(x_{k}\right)_{k \in \mathcal{F}}$, and therefore by density for any sequence in $\ell^{2}(\mathcal{F})$.

In this framework, the generating wavelet $\psi$ is then constructed so that the functions $\left(2^{j / 2} \psi\left(2^{j} \dot{-} k\right)\right)_{k \in \mathbb{Z}}$ constitute a Riesz basis for a complement $W_{j}$ of $V_{j}$ into $V_{j+1}$. This approach allowed in particular the construction of compactly supported orthonormal wavelets by Ingrid Daubechies [4].

The multiresolution analysis framework was immediately extended by Stéphane Mallat and Yves Meyer to multivariate functions, by tensorizing the spaces $V_{j}$ in the different variables. This leads to multivariate wavelet bases of the form

$$
\begin{equation*}
\psi_{\varepsilon, j, k}^{\varepsilon}=2^{d j / 2} \psi_{\varepsilon}\left(2^{j} \cdot-k\right), \quad j \in \mathbb{Z}, k \in \mathbb{Z}^{d} \tag{6}
\end{equation*}
$$

for $\varepsilon=\left(\varepsilon_{1}, \ldots, \varepsilon_{d}\right) \in\{0,1\}^{d} \backslash\{(0, \ldots, 0)\}$, where
(7)
$\psi_{\varepsilon}\left(x_{1}, \ldots, x_{d}\right):=\psi_{\varepsilon_{1}}\left(x_{1}\right) \cdots \psi_{\varepsilon_{d}}\left(x_{x}\right), \quad \psi_{0}:=\varphi, \psi_{1}:=\psi$.
Adaptation of these bases to more general bounded domains of $\mathbb{R}^{d}$ as well as to various types of manifolds came in the following years, again based on the multiresolution concept.

All these developments are well documented in the classical textbooks [4],[8]. A major stimulus was the vision of powerful applications in areas as diverse as signal and image processing, statistics, and fast numerical simulation. This perspective was confirmed in the following decades. Meyer played a key role in identifying the mathematical properties that are of critical use in such applications, in particular the ability of wavelets to characterize a large variety of function spaces. As we next discuss, these properties naturally led to the concept of sparse approximation. He was also one of the first to point
out some intrinsic limitations of wavelets and promote alternative strategies.


Yves Meyer receives Abel Prize from King Harald of Norway.

## Function Spaces and Unconditional Bases

When expanding a function $f$ into a given basis $\left(e_{n}\right)_{n \geq 0}$, a desirable feature is that the resulting decomposition $f=\sum_{n \geq 0} x_{n} e_{n}$ is numerically stable: operations such as perturbations, thresholding or truncation of the coefficients $x_{n}$ should effect the norm of $f$ in a well-controlled manner. Such prescriptions can be encapsulated in the following classical property.

A sequence $\left(e_{n}\right)_{n \geq 0}$ in a separable Banach space $X$ is an unconditional basis if the following properties hold:
(i) It is a Schauder basis: every $f \in X$ admits a unique expansion $\sum_{n \geq 0} x_{n} e_{n}$ that converges towards $f$ in $X$.
(ii) There exists a finite constant $C \geq 1$ such that for any finite set $F \subset \mathbb{N}$,

$$
\begin{equation*}
\left|x_{n}\right| \leq\left|y_{n}\right|, \quad n \in F \Longrightarrow\left\|\sum_{n \in F} x_{n} e_{n}\right\|_{X} \leq C\left\|\sum_{n \in F} y_{n} e_{n}\right\|_{X} . \tag{8}
\end{equation*}
$$

The property (8) means that membership of $f$ in $X$ only depends on the moduli of its coordinates $\left|x_{n}\right|$. In other words, multiplier operators of the form

$$
\begin{equation*}
T: \sum_{n \geq 0} x_{n} e_{n} \rightarrow \sum_{n \geq 0} c_{n} x_{n} e_{n} \tag{9}
\end{equation*}
$$

should act boundedly in $X$ if $\left(c_{n}\right)_{n \geq 0}$ is a bounded sequence. Orthonormal and Riesz bases are obvious examples of unconditional bases in Hilbert spaces.

While the trigonometric system (1) is a Schauder basis in $L^{p}$ (] 0,1 [) when $1<p<\infty$, it does not constitute an unconditional basis when $p \neq 2$, and it is thus not possible to characterize the space $L^{p}$ through a property of the moduli of the Fourier coefficients. The same situation is met for classical smoothness spaces, such as the Sobolev spaces $W_{\text {per }}^{m, p}$ ( $] 0,1[$ ) that consist of 1-periodic functions
having distributional derivatives up to order $m$ in $L_{\mathrm{loc}}^{p}$ : apart from the Hilbertian case $p=2$, for which one has

$$
\begin{equation*}
f \in W_{\text {per }}^{m, 2}(] 0,1[) \Longleftrightarrow \sum_{n \in \mathbb{Z}}\left(1+|n|^{2 m}\right)\left|c_{n}(f)\right|^{2}<\infty, \tag{10}
\end{equation*}
$$

no such characterization is available when $p \neq 2$.
Meyer showed that, in contrast to the trigonometric system, wavelet bases are unconditional bases for most classical function spaces that are known to possess one. The case of $L^{p}$ spaces for $1<p<\infty$ is treated by the following observation: if the general wavelet $\psi$ has $C^{1}$ smoothness, the multiplier operator (9) by a bounded sequence belongs to a classical class of integral operators introduced by Calderón and Zygmund, which are proved to act boundedly in $L^{p}\left(\mathbb{R}^{d}\right)$. Conversely, Meyer showed that Calderón-Zygmund operators are "almost diagonalized" by wavelet bases in the sense that the resulting matrices have fast off-diagonal decay. This property plays a key role in the numerical treatment of partial differential and integral equations by wavelet methods.

The characterization of more general function spaces by the size properties of wavelet coefficients is particularly simple for an important class of smoothness spaces introduced by Oleg Besov. There exist several equivalent definitions of Besov spaces. The original one uses the $m$-th order $L^{p}$-modulus of smoothness

$$
\begin{equation*}
\omega_{m}(f, t)_{p}:=\sup _{|h| \leq t}\left\|\Delta_{h}^{m} f\right\|_{L^{p}} \tag{11}
\end{equation*}
$$

where $\Delta_{h}^{m}$ is the $m$-th power of the finite difference operator $\Delta_{h}: f \mapsto f(\cdot+h)-f$. For $s>0$, any integer $m>s$, and $0<p, q<\infty$, a function $f \in L^{p}\left(\mathbb{R}^{d}\right)$ belongs to the space $B_{q}^{s, p}\left(\mathbb{R}^{d}\right)$ if and only if the function $g: t \rightarrow$ $t^{-s} \omega_{m}(f, t)_{p}$ belongs to $L^{q}\left(\left[0, \infty\left[, \frac{d t}{t}\right)\right.\right.$. One may use
$\|f\|_{B_{q}^{s, p}}:=\|f\|_{L^{p}}+|f|_{B_{q}^{s, p}}, \quad$ with $\quad|f|_{B_{q}^{s, p}}:=\|g\|_{L^{q}\left(\left[0, \infty\left[, \frac{d t}{T}\right)\right.\right.}$,
as a norm for such spaces, also sometimes denoted by $B_{q}^{S}\left(L^{p}\left(\mathbb{R}^{d}\right)\right)$. Roughly speaking, functions in $B_{q}^{s, p}\left(\mathbb{R}^{d}\right)$ have up to $s$ (integer or not) derivatives $L^{p}$. The third index $q$ may be viewed as a fine tuning parameter, which appears naturally when viewing Besov spaces as real interpolation spaces between Sobolev space: for example, with $0<s<m$,

$$
\begin{equation*}
B_{q}^{s}\left(L^{p}\right)=\left[L^{p}, W^{m, p}\right]_{\theta, q}, \quad s=\theta m \tag{13}
\end{equation*}
$$

Particular instances are the Hölder spaces $B_{\infty}^{s, \infty}=C^{s}$ and Sobolev spaces $B_{p}^{s, p}=W^{s, p}$, when $s$ is not an integer or when $p=2$ for all values of $s$.

Let $\left(\psi_{\lambda}\right)$ denote a multivariate wavelet basis of the type (6), where for simplicity $\lambda$ denotes the three indices $(e, j, k)$ in (6). Denoting by $|\lambda|:=j=j(\lambda)$ the scale level of $\lambda=(e, j, k)$, we consider the expansion

$$
\begin{equation*}
f=\sum_{|\lambda| \geq 0} d_{\lambda} \psi_{\lambda} \tag{14}
\end{equation*}
$$

where the coarsest scale level $|\lambda|=0$ also includes the translated scaling functions that span $V_{0}$.

The characterization of $B_{q}^{s, p}\left(\mathbb{R}^{d}\right)$ established by Meyer for such expansions requires some minimal prescriptions: one assumes that for an integer $r>s$ the univariate
generating wavelet $\psi$ and scaling functions $\varphi$ that defines (6) have derivatives up to order $r$ that decay sufficiently fast at infinity, for instance faster than any polynomial rate, and that $\int_{-\infty}^{+\infty} t^{k} \psi(t) d t=0$ for all $k=0,1, \ldots, r-1$. Then, one has the norm equivalence
(15)

$$
\|f\|_{B_{q}^{s p}}^{s,} \sim\|\varepsilon\|_{\ell q},
$$

where the sequence $\varepsilon=\left(\varepsilon_{j}\right)_{j \geq 0}$ is defined by

$$
\begin{equation*}
\varepsilon_{j}:=2^{s j} 2^{\left(\frac{d}{2}-\frac{d}{D}\right) j}\left\|\left(d_{\lambda}\right)_{|\lambda|=j}\right\|_{\ell_{p}} . \tag{16}
\end{equation*}
$$

A closely related characterization of Besov spaces uses the Littlewood-Paley decomposition

$$
\begin{equation*}
f=S_{0} f+\sum_{j \geq 0} \Delta_{j} f, \quad \Delta_{j} f:=S_{j+1} f-S_{j} f, \tag{17}
\end{equation*}
$$

where $\widehat{S_{j} f}(\omega):=\Theta\left(2^{-j} \omega\right) \hat{f}(\omega)$ with $\Theta$ a smooth compactly support function with value 1 for $|\omega| \leq 1$. It has the same form as above, with now $\varepsilon_{j}:=\left\|\Delta_{j} f\right\|_{L^{p}}$. In the wavelet characterization the dyadic blocks are further discretized into the local components $d_{\lambda} \psi_{\lambda}$. Similar results have been obtained for Besov spaces defined on general bounded Lipschitz domains $\Omega \subset \mathbb{R}^{d}$ with wavelet bases adapted to such domains.

The norm equivalence (15) shows that membership of $f$ in Besov spaces is characterized by simple weighted summability properties of its wavelet coefficients. In the particular case $q=p$, this equivalence takes the very simple form

$$
\begin{equation*}
\|f\|_{B_{p}^{s, p}} \sim\left\|\left(2^{\left(s+\frac{d}{2}-\frac{d}{p}\right)|\lambda|} d_{\lambda}\right)\right\|_{\ell p} . \tag{18}
\end{equation*}
$$

As an immediate consequence, classical results such as the critical Sobolev embedding $B_{p}^{s, p} \subset L^{2}$ for $s=\frac{d}{p}-\frac{d}{2}$ take the trivial form of the embedding $\ell^{p} \subset \ell^{2}$ for $p<2$. While this embedding is not compact, an interesting approximation property holds: when retaining only the $n$ largest coefficients in the wavelet decomposition of $f$, the resulting approximation $f_{n}$ satisfies

$$
\begin{equation*}
\left\|f-f_{n}\right\|_{L^{2}} \leq C n^{-r}\|f\|_{B_{p}^{s p}}, \quad r:=\frac{s}{d} . \tag{19}
\end{equation*}
$$

This follows immediately from the fact that, for $p<2$, the decreasing rearrangement of $\left(d_{k}\right)_{k \geq 1}$ of a sequence $\left(d_{\lambda}\right) \in \ell^{p}$ satisfies the tail bound

$$
\begin{equation*}
\left(\sum_{k \geq n} d_{k}^{2}\right)^{1 / 2} \leq n^{\frac{1}{2}-\frac{1}{p}}\left\|\left(d_{\lambda}\right)\right\|_{\rho \rho} . \tag{20}
\end{equation*}
$$

This last estimate shows that $\ell^{p}$ summability governs the compressibility of a sequence, in the sense of how fast it can be approximated by $n$-sparse sequences. The theory of best $n$-term wavelet approximation, generalizing the above remarks, has been developed by Ronald DeVore and his collaborators, in close relation with other nonlinear approximation procedures such as free knot splines or rational approximation.

A particularly useful feature of nonlinear wavelet approximation is that piecewise smooth signals, such as images, can be efficiently captured since the large coefficients are only those of the few wavelets whose supports contain the singularities. This is an instance of sparse approximation which aims to accurately capture functions by a small number of well chosen coefficients in a basis


Yves Meyer at his Abel Prize Ceremony.
or dictionary expansion. Sparse approximation in unconditional bases was identified by David Donoho as a key ingredient for powerful applications in data compression and statistical estimation, in particular through thresholding algorithms that he developed jointly with Iain Johnstone, Gérard Kerkyacharian, and Dominique Picard. Pushed into the forefront by the work of Meyer, Donoho, and DeVore, sparse approximation became within a few years a prominent concept in signal processing and scientific computing.

## Taking Off from the Wavelet World

The estimate (20) shows that the rate $n^{-r}$ of best $n$-term approximation of a function, using an orthonormal or Riesz basis $\left(\psi_{\lambda}\right)$, is implied by the $\ell^{p}$ summability of is coefficient sequence ( $d_{\lambda}$ ) with $\frac{1}{p}=r+\frac{1}{2}$. A more refined analysis shows that this rate is exactly equivalent to the slightly weaker property that $\left(d_{\lambda}\right)$ belongs to $w \ell^{p}$, which means that its decreasing rearrangement has the decay property

$$
\begin{equation*}
d_{k} \leq C k^{-1 / p} . \tag{21}
\end{equation*}
$$

The spaces $\ell^{p}$ and $w \ell^{p}$ are thus natural ways of quantifying sparsity of a function when decomposed in an arbitrary orthonormal or Riesz basis ( $\psi_{\lambda}$ ) of a Hilbert space.

In the case of wavelet bases, these summability properties are equivalent to Besov smoothness. From an applicative point of view, a more natural question is:
given a class of functions $\mathcal{K}$ in a Hilbert space, which basis should be used in order to obtain the sparsest possible representations of the element of this class? In view of the previous observations, this basis should be picked so that the coefficient sequence of any element of $\mathcal{K}$ belongs to $w \ell^{p}$, for the smallest possible value of $p$.

One class of particular interest for modeling real images is the space $\mathrm{BV}(Q)$ of bounded variation functions on the unit cube $Q:=[0,1]^{2}$ that consists of functions $f \in L^{1}(Q)$ such that $\nabla f$ is a finite measure. In particular, if $\Omega \subset Q$ is a set of finite perimeter, the characteristic function $\mathrm{X}_{\Omega}$ belongs to $\mathrm{BV}(Q)$. More generally, piecewise smooth images with edge discontinuities across curves of finite length have bounded variation. While the space $B V(Q)$ admits no unconditional basis, we showed together with DeVore, Pencho Petrushev, and Hong Xu that it can be "almost" characterized by its decomposition in a bivariate wavelet basis $\left(\psi_{\lambda}\right)$ in the following sense: if $f=\sum d_{\lambda} \psi_{\lambda}$, one has

$$
\begin{equation*}
\left(d_{\lambda}\right) \in \ell^{1} \Rightarrow f \in \operatorname{BV}(Q) \Longrightarrow\left(d_{\lambda}\right) \in w \ell^{1} . \tag{22}
\end{equation*}
$$

In view of the previous remarks this shows that for general images of bounded variation, the rate of best $n$-term approximation in wavelet bases is $n^{-1 / 2}$. It can be shown that this rate is also the best that can be achieved by any basis. In particular, no polynomial rate can be achieved when using the Fourier basis. In this sense wavelets appear as the optimal tool for piecewise smooth images with edges of finite length.

The situation becomes quite different if one considers images with edges enjoying some geometric smoothness in addition to finite length. The simplest model consists of piecewise constant images with straight edges. For such images, Meyer [9] noticed that the decreasingly rearranged Fourier coefficients decay at rate
(23)

$$
c_{k} \leq C k^{-1} \log (k)
$$

therefore comparable to wavelet coefficients up to the logarithmic factor. When going to a higher dimensional cube $Q=[0,1]^{d}$, this rate persists for Fourier coefficients while wavelet representations become less effective.

A more elaborate model consists of the functions which are piecewise $C^{m}$ with edge discontinuities having $C^{n}$ geometric smoothness. For such classes $\mathcal{K}(n, m)$, both wavelet and Fourier decompositions can be outperformed by more sophisticated representations into functions that combine local support with directional selectivity. One representative example are the curvelets, introduced by Emmanuel Candès and Donoho, which have the form
(24) $\psi_{\lambda}=2^{3 j / 2} \psi\left(D^{j} R_{j}^{l} \cdot-k\right), \quad k \in \mathbb{Z}, \quad l=0, \ldots, 2^{j}-1$,
where $D$ is the anisotropic dilation matrix $\binom{40}{02}$ and $R_{j}$ the rotation of angle $2^{-j-1} \pi$. The anisotropic scaling and angular selectivity allow to better capture the geometry of edges, leading to improved sparsity: for example, it is known that
(25) $\quad f \in \mathcal{K}(2,2) \Longrightarrow\left(d_{\lambda}\right) \in \ell^{p}, \quad p>2 / 3$,
where $d_{\lambda}$ are the coefficients of $f$ in the curvelet expansion. The value $2 / 3$ is optimal for this class. Other
representation methods have since then been proposed and studied for better capturing geometry: contourlets, shearlets, bandlets, anisotropic finite elements.

Returning to univariate signals, one object of longterm interest to Meyer is signals whose "instantaneous frequency" evolves with time in some controlled manner. Such signals are called chirps and take the general form

$$
\begin{equation*}
f(t)=\operatorname{Re}\left(a(t) e^{i \varphi(t)}\right), \tag{26}
\end{equation*}
$$

where $\left|\frac{a^{\prime}(t)}{a(t)}\right| \ll\left|\varphi^{\prime}(t)\right|$ and $\left|\varphi^{\prime \prime}(t)\right| \ll\left|\varphi^{\prime}(t)\right|^{2}$.
Typical examples of chirp are ultrasounds emitted by bats and recordings of voice signals, but the most famous one is the gravitational wave signal first detected in 2015, which has for a large part the behaviour

$$
\begin{equation*}
f(t) \sim\left|t-t_{0}\right|^{-1 / 4} \cos \left(\left|t-t_{0}\right|^{5 / 8}+\varphi_{0}\right) . \tag{27}
\end{equation*}
$$

Wavelets are not the right tool for sparse representation of chirps. Time-frequency analysis such as the shorttime Fourier transform provides more natural tools, once proper orthonormal bases have been provided.

The first example of such a basis was originally suggested by Kenneth Wilson and formalized by Ingrid Daubechies, Stéphane Jaffard, and Jean-Lin Journé: an orthonormal basis of $L^{2}(\mathbb{R})$ is constructed by taking for all $n \in \mathbb{Z}$ the functions $\varphi_{0, n}(t)=\varphi(t-n)$ and (28)

$$
\varphi_{l, n}(t)= \begin{cases}\sqrt{2} \varphi\left(t-\frac{n}{2}\right) \cos (2 \pi l t) & l \geq 0, l+n \in 2 \mathbb{Z} \\ \sqrt{2} \varphi\left(t-\frac{n}{2}\right) \sin (2 \pi l t) & l>0, l+n \in 2 \mathbb{Z}+1\end{cases}
$$

The generating function $\varphi$ should satisfy certain symmetry properties. One possible choice is the scaling function associated with the orthonormal wavelet basis of Meyer, which is defined by $\hat{\varphi}=\sqrt{\kappa}$, where $\kappa$ is the symmetric and smooth cut-off function. A variant of this system, where the family is made redundant by additional dilations, was proposed in the papers of Sergei Klimenko and his collaborators for the sparse representation of gravitational waves and used for their detection.

In recent years, sparse approximation has also been intensively exploited for the treatment of high-dimensional approximation. Problems that involve functions of a very large number of variables are challenged by the so-called "curse of dimensionality": the complexity of standard discretization methods blows up exponentially as the number of variables grows. Such problems arise naturally in learning theory, partial differential equations, and numerical models depending on parametric or stochastic variables. Wavelet representations are not well suited for extracting sparsity in such high-dimensional applications. This motivated the development of better adapted tools, such as sparse grids, sparse polynomials, and sparse tensor formats.

## Compressed Sensing and Quasi-Crystals

The most usual approach for obtaining a sparse approximation of a discrete signal represented by a vector $x \in \mathbb{R}^{N}$ is to choose an appropriate basis, compute the coefficients of $x$ in this basis, and then retain only the $n$ largest of these, with $n \ll N$.

This approximation process is adaptive since the indices of the retained coefficients vary from one signal to another. The view expressed by Candès, Justin Romberg, and Terence Tao [1] and Donoho is that since only a few of these coefficients are needed in the end, it should be possible to compute only a few non-adaptive linear measurements in the first place and still retain the information needed in order to build a compressed representation. These ideas have led since the turn of the century to the very active area compressed sensing.

If $m$ is the number of linear measurements, the observed data has the form

$$
\begin{equation*}
y=\Phi x, \tag{29}
\end{equation*}
$$

where $\Phi$ is an $m \times N$ matrix. Any $n$-sparse vector $x$ is uniquely characterized by its measurement if and only if no $2 n$-sparse vector lies in the kernel of $\Phi$. In other words, any submatrix of $\Phi_{T}$ obtained by retaining a set $T \subset\{1, \ldots, N\}$ of columns with $\#(T)=2 n$ should be injective. It is easily seen that a generic $m \times N$ matrix satisfies this property provided that $m \geq 2 n$, and therefore $m=2 n$ linear measurements are in principle sufficient to be able to reconstruct $n$-sparse vectors. However, the reconstruction from $2 n$ measurements will generally be computationally untractable when $N$ is large and unstable due to the fact that $\Phi_{T}^{*} \Phi_{T}$, even if invertible, can be very ill-conditioned.

Stability and computational feasibility can be recovered at the expense of a stronger condition introduced by Candès, Romberg, and Tao: the matrix $\Phi$ satisfies the restricted isometry property (RIP) of order $k$ with parameter $0<\delta<1$ if and only if
(30) $\quad(1-\delta)\|z\|_{2}^{2} \leq\left\|\Phi_{T} z\right\|_{2}^{2} \leq(1+\delta)\|z\|_{2}^{2}, \quad z \in \mathbb{R}^{k}$,
for all set $T \subset\{1, \ldots, N\}$ such that $\#(T)=k$. Under such a property with $k=2 n$ and $\delta<1 / 3$, it was shown that an $n$-sparse vector can be stably reconstructed from its linear measurements by a convex optimization algorithm, which consists in searching for the solution of (29) with minimal $\ell^{1}$-norm.

Measurement matrices $\Phi$ of size $m \times N$ that satisfy RIP to order $k$ are known to exist in the regime $m \sim k \log (N / k)$. Therefore, with $k=2 n$ the measurement budget $m$ is linear in $n$ up to logarithmic factors. However the constructions of such matrices rely upon probabilistic arguments: they are realizations of random matrices for which it is proved that RIP of order $2 n$ holds with high probability under this type of regime. Two notable examples are the matrix with entries consisting of independent centered Gaussian variables of variance $1 / m$ and the matrix obtained by picking at random $m$ rows from the $N \times N$ discrete Fourier transform matrix. The currently available deterministic constructions of
matrices satisfying RIP of order $k$ require the non-optimal regime $m \sim k^{2}$ up to logarithmic factors.

In recent years, Meyer studied the problem of sampling continuous bandlimited signals with unknown Fourier support, which may be viewed as an analog counterpart to the above compressed sensing problem. For any set $E \subset \mathbb{R}^{d}$, we denote by $\mathcal{F}_{E}$ the Paley-Wiener space of functions $f \in L^{2}\left(\mathbb{R}^{d}\right)$ such that their Fourier transform

$$
\begin{equation*}
\hat{f}(\omega)=\int_{\mathbb{R}^{d}} f(x) \exp (-i 2 \pi \omega \cdot x) d x \tag{31}
\end{equation*}
$$

is supported on $E$. Sampling theory for such functions has been motivated since the 1960s by the development of discrete telecommunications. It is well known, since the foundational work of Claude Shannon and Harry Nyquist, that regular grids, which are full-rank lattices

$$
\begin{equation*}
L=B \mathbb{Z}^{d}, \tag{32}
\end{equation*}
$$

where $B$ is a $d \times d$ invertible matrix, are particularly suitable for the sampling of certain band-limited functions. This may be seen as a direct consequence of the Poisson summation formula, which says that for any sufficiently nice function $f$,

$$
\begin{equation*}
|L| \sum_{\lambda \in L} f(\lambda) e^{i 2 \pi(\lambda, \omega}=\sum_{\lambda^{*} \in L^{*}} \hat{f}\left(\omega+\lambda^{*}\right) . \tag{33}
\end{equation*}
$$

Here $|L|:=|\operatorname{det}(B)|$ is the measure of the fundamental volume of $L$, and $L^{*}=\left(B^{t}\right)^{-1} \mathbb{Z}^{d}$ its dual lattice. This formula shows that, if $E \subset \mathbb{R}^{d}$ is a compact set with translates $\left(E+\lambda^{*}\right)_{\lambda^{*} \in L^{*}}$ having intersections of null measure, functions with Fourier transform supported in $E$ are then stably determined by their sampling on $L$. Such sets $E$ should in particular have measure smaller than the density of $L$, that is,

$$
\begin{equation*}
|E| \leq \operatorname{dens}(L):=\left|L^{*}\right|=|L|^{-1} . \tag{34}
\end{equation*}
$$

One elementary example, for which equality holds in the above, is the fundamental volume of the lattice $L^{*}$, that is,

$$
\begin{equation*}
E_{L^{*}}=\left(B^{*}\right)^{-1}\left([0,1]^{d}\right), \tag{35}
\end{equation*}
$$

or any of its translates.
A theory of stable sampling on more general discrete sets was developed in the 1960s by Henry Landau and Arne Beurling. The possibility of reconstructing any $f \in \mathcal{F}_{E}$ from its samples over a discrete set $\Lambda \subset \mathbb{R}^{d}$ is described by the property of stable sampling: there exists a constant $C$ such that

$$
\begin{equation*}
\|f\|_{L^{2}}^{2} \leq C \sum_{\lambda \in \Lambda}|f(\lambda)|^{2}, \quad f \in \mathcal{F}_{E} . \tag{36}
\end{equation*}
$$

Landau proved that a necessary condition for such a property to hold is that

$$
\begin{equation*}
\operatorname{dens}(\Lambda) \geq|E|, \tag{37}
\end{equation*}
$$

where

$$
\begin{equation*}
\operatorname{dens}(\Lambda)=\liminf _{R \rightarrow \infty} \inf _{x \in \mathbb{R}^{d}} \frac{\#(\Lambda \cap B(x, R))}{|B(x, R)|} \tag{38}
\end{equation*}
$$

is the lower density of $\Lambda$, which is the usual density dens( $\Lambda$ ) when the standard limit exists.


Yves Meyer giving an exuberant interview post Abel Prize Ceremony.

A continuous signal is $s$-sparse in the Fourier domain if it belongs to $\mathcal{F}_{E}$ for some set of Lebesgue measure $|E| \leq s$. Stable reconstruction of any $s$-sparse signal from its sampling on a discrete set $\Lambda$ requires that this set has the property of stable sampling for all sets of measure $|E| \leq r:=2 s$. Such sets $\Lambda$ are called universal sampling sets. Obviously, they should have density larger than $r$, but this condition cannot be sufficient. The case of a regular lattice $L$ is instructive: on the one hand, the set $E_{L^{*}}$ has measure $\left|E_{L^{*}}\right|=|L|^{-1}=\operatorname{dens}(L)$ and satisfies the stable sampling property in view of (33). On the other hand, other sets $E$ with the same or even smaller measure could have their translates by $\Lambda^{*}$ overlapping with nonzero measure, which is a principle obstruction to these properties. This phenomenon is well known in electrical engineering as aliasing. This shows that universal sampling sets cannot be regular lattices.

Alexander Olevskii and Alexander Ulanovskii gave the first construction of a set $\Lambda$ of uniform density that has the stable sampling property for any set $E$ such that

$$
\begin{equation*}
|E|<\operatorname{dens}(\Lambda) \tag{39}
\end{equation*}
$$

Meyer had the intuition that the mathematical models of quasicrystals that emerged from his early work on harmonic analysis and number theory [7] could provide a natural alternative solution to this problem.

One such model is obtained by the following cut and project scheme that was implicit in earlier work on algebraic number theory: the set of interest is obtained by projecting a "slice" cut from a higher-dimensional lattice in general position. More precisely, if $L$ is a full rank lattice of $\mathbb{R}^{d+m}$ for some $d, m>0$, we denote by $p_{1}(x) \in \mathbb{R}^{d}$ and $p_{2}(x) \in \mathbb{R}^{m}$ the components of $x \in \mathbb{R}^{d+m}$ such that $x=\left(p_{1}(x), p_{2}(x)\right)$ and assume that $p_{1}$ is a bijection between $L$ and $p_{1}(L)$ with dense image. A similar property is assumed for $p_{2}$. Let $K \subset \mathbb{R}^{m}$ be a Riemann integrable compact set of positive measure. The associated model
set $\Lambda=\Lambda(L, K) \in \mathbb{R}^{d}$ is defined by

$$
\begin{equation*}
\Lambda:=\left\{p_{1}(x): x \in L, p_{2}(x) \in K\right\} \tag{40}
\end{equation*}
$$

The density of a model set $\Lambda=\Lambda(L, K)$ is uniform and given by

$$
\begin{equation*}
\operatorname{dens}(\Lambda)=\frac{|K|}{|L|} \tag{41}
\end{equation*}
$$

Basarab Matei and Meyer [6] showed that the stable sampling property holds for any $E$ under the condition (39) for model sets $\Lambda:=\Lambda(L, K) \subset \mathbb{R}^{d}$ such that $K$ is a univariate interval. Such sets are called simple quasicrystals. They are universal sampling sets and may therefore be used for the reconstruction of $s$-sparse signals with $2 s<\operatorname{dens}(\Lambda)$. A remarkable fact is that, in contrast to compressed sensing matrices, their construction does not rely on any probabilistic argument.

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Albert Cohen's current research interests include nonlinear and high-dimensional approximation theory, statistics, signal-imagedata processing, numerical analysis, and inverse problems.

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## Felix Browder (1927-2016)



Felix Browder received the National Medal of Science and served as president of the American Mathematical Society.

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## Haïm Brezis, Guest Editor

Felix Browder, who passed away exactly two years ago, was awarded the National Medal of Science by President Bill Clinton for "his pioneering work in nonlinear functional analysis and its applications to partial differential equations, and for leadership in the scientific community,... and in promoting science and math education for all." Indeed, there were two Felix Browders! There was Felix, the solver of nonlinear problems, who made major contributions to fixed point theory and was the leading architect of the theory of monotone operators and its generalizations. Some of his ideas have been successfully used in tackling equations arising in real-world models. See below the descriptions of his research by former student Roger D. Nussbaum and me. And there was Felix the visionary scientist who was able to foresee original directions and generously support emerging young talents. He was a great chairperson of the math department at the University of Chicago, and a very influential vice-president for research at Rutgers. Felix played an energizing role at the AMS throughout his career and served as its president for the years 1999-2000. In this capacity, and also as an extremely active member of the National Academy of Sciences, he lobbied Congress for additional funding for mathematical research and education. On another front, Felix developed close ties with a number of French mathematicians. His impact on the "post-Bourbaki" flourishing school of partial differential equations (PDEs) in France cannot be overestimated. All who knew Felix emphasize the friendly intellectual atmosphere radiating from him. No matter what topic, be it mathematics, philosophy, or history, Felix would communicate his joyful appetite and sheer pleasure of knowledge. In what follows, some of Felix's friends, colleagues, and former students share their memories and discuss the major contributions of this extraordinary mathematician.

## See also:

Nomination for Felix Browder for AMS President Elect by Jerry Bona, September 1997 Notices, https://www.ams .org/notices/199708/from.pdf.

Reflections on the future of mathematics (retiring presidential address), June-July 2002 Notices, https://www.ams.org/journa1s/notices/200206 /fea-browder.pdf.
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Felix Browder, mathematician shadowed by his father's life as a Communist, dies at eighty-nine, The Washington Post, December 15, 2016, https://tinyur1.com /y9s7uhvh.

## Photo Credit

Opening photo of Felix Browder courtesy of Rutgers University.

## The Early Life of Felix Browder

Felix Earl Browder was born July 31, 1927 in Moscow, Russia, and died December 10, 2016 in Princeton, New Jersey. His father, Earl Browder, an American political activist, visited Russia in the 1920s as a representative of the Communist Trade Unions in the United States. There, he met and married Raissa Berkmann, born in a Jewish family, who had a law degree from the University of St. Petersburg. In the early 1930s the family settled in Yonkers, New York, where Felix attended high school. Felix was a child prodigy; he is said to have read at least a book a day from the time he was five years old. His only sport in high school was the debating team. He was a shark. His younger brother Bill vividly remembers the triumphal atmosphere, when their parents returned home with the conquering hero. At high school graduation, at the age of sixteen, he collected almost every academic prize and was awarded a New York State Regents scholarship. In the same year he entered MIT and, after only two years of study, graduated in 1946.

For his graduate studies, Felix attended Princeton University, where he worked on his PhD under the supervision of Solomon Lefschetz, a leading topologist who also made fundamental contributions to the theory of nonlinear ordinary differential equations. Felix submitted his doctoral thesis, The Topological Fixed Point Theory and Its Applications in Functional Analysis, and was awarded his PhD at the age of twenty.

## The Difficult Years

The first positions held by Felix were instructorships at MIT (1948-1951) and at Boston University (1951-52), followed by a temporary position at the Institute for Advanced Study (IAS) in Princeton. During the McCarthy era it was perilous to be called "Browder": since the early 1930s his father Earl had been the leader of the American Communist Party and even its candidate for the presidential election of 1936. The renowned physicist Robert Oppenheimer was at the time the director of the Institute. Originally he pushed through the appointment of Felix, but later, when he was himself under investigation, he declined to sign a deferment request from military draft for Felix. Ironically, Earl had been expelled from the party six years earlier! In 1953, his case was brought to the Committee on Un-American Activities of the US House of Representatives. Norman Levinson, an MIT professor, made a very forceful deposition in support of Felix: "...He is the best student we had ever had in mathematics in the ninety years of existence of this institution...." To no avail-Felix was drafted to the army in 1953 and spent most of his two years of service pumping gas. It is fun to
note that while at Fort Bragg, he published several papers with no institutional listing but only the laconic address, Fayetteville, North Carolina.

## The Chicago Years

Felix was professor at the University of Chicago from 1963 to 1986. For eleven of those years he served as chair of the mathematics department.

His efforts as chair attest to his vision and dedication to excellence. He initiated many appointments of the highest quality-including those of Spencer Bloch, Luis Caffarelli, Charles Fefferman, Carlos Kenig, and Karen Uhlenbeckwhich turned the Chicago math department into one of the most prestigious centers in the US. He was enormously supportive of young talented people and eager to promote them quickly to full professorship-maybe because of the difficulties he encountered at the beginning of his career? During these years he had an immense esteem for Alberto Calderón and much admiration for harmonic analysis; clearly he foresaw the impact this field would have on the future development of partial differential equations (PDEs). He had also established close contacts with leading Chicago physicists, such as the Nobel laureate S. Chandrasekhar and L. Kadanoff. Throughout his tenure in Chicago, there was a nonstop flow of distinguished international visitors. These were also the golden years of his mathematical research.

## The Rutgers Years

Felix was appointed vice-president for research at Rutgers University in 1986 and occupied this position until 1991. Subsequently, he remained at Rutgers until his death. Felix was especially proud to have attracted to Rutgers-in his capacity as VP-the legendary Soviet mathematician Israel M. Gelfand. Felix encouraged and supported Gelfand's Correspondence Program for Mathematical Education. He was influential in other stellar appointments in Applied Mathematics, such as Bernard Coleman, Ingrid Daubechies, Martin Kruskal, and Norman Zabusky. Felix was also instrumental in getting a very strong group of string theorists, the so-called "String Quartet," to the Rutgers physics department. He prompted the creation of the Center for Nonlinear Analysis under the leadership of Haïm Brezis with the active participation of Abbas Bahri, Sagun Chanillo, and Yanyan Li. At the same time he initiated the successful NSF Science and Technology Center, which involved Rutgers, Bell Labs, Bellcore, and Princeton University, and which led to the formation of DIMACS, the Rutgers Center for Discrete Mathematics and Theoretical Computer Science. Felix clearly enjoyed and excelled in his responsibilities as VP for research; unfortunately his activities stopped due to the arrival of a new president who had different priorities for Rutgers.

## Browder Family

Felix's wife, Eva nee Tislowitz, died in 2015. She had arrived in the US alone at age nine as a refugee from Austria just before the Holocaust. Eva met Felix at MIT, where she was a student and he was a young instructor.


Felix (center) with (from left to right) son Thomas, Ivar Ekeland, Haïm Brezis, wife Eva, and son Bill, at a reception after Felix received an honorary degree at the Sorbonne in 1990. Felix's impact on the post-Bourbaki flourishing school of PDEs in France cannot be overestimated.

While working as an administrator at Rutgers, Eva was also involved in the National Outreach Program in Mathematics inspired by I. M. Gelfand. Felix is survived by two younger brothers: Andrew Browder, who is professor emeritus at Brown University, and William Browder, a wellknown topologist, who is professor emeritus at Princeton University. He is also survived by two sons: Tom Browder, a physics professor at the University of Hawaii, and Bill Browder, a bold international financier. Felix had five grandchildren. The oldest one, Joshua Browder, is the inventor of "DoNotPay," a program that allows motorists to appeal their parking tickets automatically.

## The Mathematics of Felix Browder

Felix worked in three different directions:
i) Linear PDEs and functional analysis.
ii) Fixed point and degree theories.
iii) Monotone operators and beyond.
i) Linear PDEs and functional analysis. This topic preoccupied Felix from 1952 until 1962. He was concerned with questions of existence and regularity estimates (in terms of Sobolev and Schauder norms) for linear elliptic PDEs. These results overlap with the simultaneous ones by Agmon-DouglisNirenberg. He also made decisive contributions to the spectral analysis of nonselfadjoint elliptic differential operators. More importantly, Felix cemented the bridge connecting PDEs and functional analysis (at a time when functional analysis was becoming increasingly abstract and detached from PDEs). In his own words:
..It is our purpose to present a general treatment in unified terms of these various PDEs and to exhibit explicitly the common methodological basis of the discussion of apparently diverse sorts of problems. In rough terms, this basis consists of the


Figure 1. The 1909 Brouwer (and later the Schauder) fixed point theorems prove that a continuous function $f: K \rightarrow K$ on a compact convex set, $K$, in a Euclidean (or Banach space) has a fixed point. Browder's fixed point theorem does not require $K$ to be compact. In his theorem, $K$ is closed, bounded, and convex in a uniformly convex Banach space and $f: K \rightarrow K$ is a nonexpansive map:
$\|f(x)-f(y)\| \leq\|x-y\|$ for all $x, y \in K$.
combination of general principles from functional analysis with concrete analytical a priori estimates...(Math. Ann., 1959)
This approach paved the way to a similar strategy Felix used in nonlinear problems.

One should also give credit to Felix for publicizing in the West important works of the Russian PDE community. In particular Felix translated the book by S. L. Sobolev, Applications of Functional Analysis in Mathematical Physics (Amer. Math. Soc., 1963), which received the following compliment in Math. Reviews: "the translation is exceedingly smooth." (At that time there was a massive effort to translate Russian math papers into English and this was often done by people with a limited math background.)


Figure 2. In Browder's fixed point theorem, the closed, bounded, convex set $K$ is assumed to lie in a uniformly convex Banach space: for all $\varepsilon \in(0,2]$ there exists a $\delta>0$ such that for all unit vectors:
$\|v-w\| \geq \varepsilon \Rightarrow\|(v+w) / 2\| \leq 1-\delta$.
ii) Fixed point and degree theories after the classic fixed point theorems of Brouwer and Schauder (Figure 1). Felix worked on this subject throughout his
entire research life, from his PhD in 1948 until 1984. He made beautiful contributions in two directions. Firstly, he wrote a number of influential papers on nonexpansive mappings (maps with Lipschitz constant one). For example, he proved in 1965 that any nonexpansive self-mapping of a bounded, closed, convex subset of a uniformly convex Banach space has a fixed point (Figure 2). This result has interesting applications, e.g. in the study of periodic solutions of differential equations. Incidentally, it is still an open problem, more than fifty years later, whether a similar conclusion holds in all reflexive Banach spaces. Secondly, Felix achieved substantial progress in our understanding of topological fixed points and degree theory; see for example his 1983 expository paper in the Bulletin of the American Mathematical Society. A typical result is described by R. Nussbaum in his text "Felix Browder and a Useful Result in Fixed Point Theory" (page 1402 herein).
iii) Monotone operators and beyond. Felix has been the leading architect of the theory of monotone operators, its generalizations, and its applications to nonlinear PDEs. This theory is one of the most elegant and powerful tools in the study of nonlinear problems, and a cornerstone of nonlinear functional analysis. Felix has written many papers and two monographs ([B4] and [B6]) on this subject, spanning the period 1963-1997. Some of his early works were already very influential in the mid-1960s, e.g. in France, where J. Leray and J.-L. Lions published a paper under the title "Quelques Résultats de Višik sur les Problèmes Elliptiques Non Linéaires par les Méthodes de Minty-Browder" [LL]. Felix perceived immediately that this abstract theory could be successfully used in tackling nonlinear PDEs arising e.g. in differential geometry, physics, mechanics, biology, engineering, ecology, climate, finance, etc. Felix was not himself an "applied mathematician," but his ideas have had a lasting impact on real-world problems. On this subject see the contribution by H. Brezis "Felix Browder and Monotone Operators" (page 1403 herein).

Felix Browder and the AMS: A Life of Commitment Felix Browder played an active role in the Society throughout his career. He served as president of the AMS for the years 1999-2000. In this capacity he lobbied Congress for additional funding for mathematical research and education. Throughout his life, Felix never missed an opportunity to be a spokesperson for the scientific community in the presence of politicians. In particular he was fond of his continued exchanges with Rush Holt, a US Representative for New Jersey. The list of AMS committees on which he served is too long to present here. His effective work as editor of the Bulletin and his service on the Science Policy Committee have left a permanent mark. He organized innumerable special sessions at regional and national meetings, as well as major international


Felix received the National Medal of Science in 1999. President Clinton had had a hard time placing the medal around the neck of the previous recipient, so Felix wisely removed his glasses beforehand, and Clinton remarked, "You see...these scientists they learn by experience," generating a big laugh.
conferences sponsored by the AMS. The proceedings that he edited are a very useful source of information. Let us mention in particular:
-Nonlinear Functional Analysis, Chicago, 1968; proceedings published by the AMS in 1970.
-Mathematical Developments Arising from Hilbert Problems, De Kalb, Illinois 1974; proceedings published by the AMS in 1976.
-The Mathematical Heritage of H. Poincaré, Bloomington, Indiana, 1980; proceedings published by the AMS in 1983.

- Nonlinear Functional Analysis and its Applications, Berkeley, 1983; proceedings published by the AMS in 1986.
- Mathematical Challenges of the 21st Century, UCLA, 2000; proceedings published in the AMS Bulletin.


## Honors Received

In 1999, Felix was awarded the National Medal of Science by President Bill Clinton, for "his pioneering work in nonlinear functional analysis and its applications to partial differential equations, and for leadership in the scientific community." The citation added: "Through his accomplishments Browder laid the groundwork for the mathematics needed to study the array of complexities and intricacies we find in our biological and physical world. Throughout his career Browder has demonstrated unwavering commitment in broadening the interactions among the scientific disciplines and in promoting science and math education for all."

Other honors include his election to the National Academy of Science in 1973-he served as a member of the Council 1992-1995 and the Governing Board of the

National Research Council 1994-1995. He was elected a fellow of the American Academy of Arts and Sciences in 1959 and awarded an honorary degree by the University of Paris-Sorbonne in 1990. An international conference honoring Felix was held at Rutgers in October 2001.

## Browder, Knowledge, and the General Public

Felix was always eager to share with a wide general public his thoughts on the importance of mathematics, its history, its place in a changing world, and the major challenges it faces. During his tenure in Chicago Felix established a rule that colloquium speakers should include a historical perspective. To get a quick idea of his preoccupations it suffices to glance at the titles of some papers he wrote: "The Relevance of Mathematics" (1976), "Does Pure Mathematics Have a Relation to the Sciences?" (1976), "Mathematics and Society-a Historical View" (1977), "Mathematics and the Sciences" (1985), "Of Time, Intelligence and Institutions" (1992), "Reflections on the Future of Mathematics" (2002).

Particularly fascinating was Felix's unique breadth of interests: of course, mathematics, but also physics and science, history, religion, philosophy, political science, economics, literature, and you name it. He enjoyed discussing his views in long private conversations. He also taught philosophy classes at Rutgers for many years. His personal library had over 35,000 books.

## Image Credits

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Sorbonne photo courtesy of Haïm Brezis.
National Medal of Science photo courtesy of the National Science Technology Medals Foundation.

## Roger D. Nussbaum

## Felix Browder and a Useful Result in Fixed Point Theory

Fixed point theory was a life-long mathematical interest of Felix, beginning with his 1948 Princeton PhD dissertation. Here I would like to discuss a small, but very useful segment of Felix's work in this area-a sequence of three papers [1], [2], [3] rather unimaginatively titled "On a Generalization of the Schauder Fixed Point Theorem," "Another Generalization of the Schauder Fixed Point Theorem," and "A Further Generalization of the Schauder Fixed Point Theorem." All of these papers fit into the category of asymptotic fixed point theory, that is, results in which fixed point theorems for a map $f$ are obtained with the aid of assumptions on the iterates $f^{n}$ of $f$.

Specifically, I shall discuss without proofs results in [3] concerning the existence of "nonejective fixed points." As Felix mentions, his theorem is motivated by a question about [2] in a letter from G. Stephen Jones, who had used results from the first paper [1]. Jones's

[^11]

Figure 1. An ejective fixed point, $x_{0}$, of a map $f: C \rightarrow C$ has a neighborhood $U \subset C$, such that for all other $x \in U$, there exists $j=j(x)$ such that the $j^{\text {th }}$ iteration, $f^{(j)}(x)$, is not in $U$.
original motivation came from the so-called Wright-Jones nonlinear differential-delay equation
(1)

$$
x^{\prime}(t)=-\alpha x(t-1)(1+x(t))
$$

Jones had used fixed point theory to prove that if $\alpha>\pi / 2$, equation (1) has a nonconstant "slowly oscillating periodic solution" $x(\cdot)$.
E. M. Wright's name is associated with equation (1) because of his seminal paper [4], but it is interesting to note that Wright apparently had no idea that (1) has nonconstant slowly oscillating periodic solutions for $\alpha>\pi / 2$.

Other applications of theorems from [1] were given by Halanay and Yoshizawa.

If $C$ is a subset of a Banach space $X$ and $f: C \rightarrow C$ is a continuous map with a fixed point $x_{0}$ (so $f\left(x_{0}\right)=x_{0}$ ), $x_{0}$ is called an "ejective fixed point" in [3] (see Figure 1) if there exists an open neighborhood $U$ of $x_{0}$ in $C$ (open in the relative topology on $C$ ) such that for every $x \in U \backslash\left\{x_{0}\right\}$ there exists a positive integer $j=j(x)$ such that $f^{j}(x) \notin U$.

If $C \subset X$ is convex, we shall call $C$ infinite-dimensional if $C$ is not contained in some finite-dimensional affine linear subspace of $X$.
Theorem 1 ([3]). Let $C$ be a compact, convex infinitedimensional subset of a Banach space $X$ and $f: C \rightarrow C$ a continuous map. Then $f$ has a fixed point which is not ejective.

If $C \subset X$ is compact and convex and contained in a finite-dimensional affine linear subspace of $X$, we let $Y$ denote the smallest affine linear subspace that contains $C$ and let $\partial(C)$ denote the boundary of $C$ in $Y$ and int ( $C$ ) denote the interior of $C$ in $Y$.

The following theorem is not stated by Browder [3] but follows easily by the same ideas.

Theorem 2. Let C be a compact, convex finite-dimensional subset of a Banach space $X$ and assume $C$ is not a single
point. Let $f: C \rightarrow C$ be a continuous map. Then either (i)f has a fixed point in int (C) or (ii)f has a fixed point in $\partial(C)$ which is not ejective.

If $D \subset \mathbb{C}=\mathbb{R}^{2}$ and $D=\{z \in \mathbb{C}| | z \mid \leq 1\}$, define $f: D \rightarrow D$ by $f(r \exp (i \theta))=\sqrt{r} \exp (i \theta+i \pi / 2), 0 \leq r \leq 1$. One can check that 0 is the only fixed point of $f$ in $D$ and 0 is an ejective fixed point of $f$; so, in the finite-dimensional case, $f$ may fail to have a nonejective fixed point.

There have been numerous generalizations of the above results, and there have been applications, for example, to the study of nonlinear differential-delay equations.

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## Image Credit

Figure 1 by Pen Chang.

## Haïm Brezis

## Felix Browder and Monotone Operators

The concept of monotone operator in Hilbert spaces was introduced by G. Minty [M] whose original motivation came from graph theory, electrical networks, and linear programming. Felix, who had a strong background in linear PDEs, foresaw immediately the enormous potential applicability of this abstract theory to nonlinear PDEs. This is already conspicuous in the first papers [B1], [B2] he wrote on this subject.

It is easy to explain some of the original ideas and far-reaching ramifications of this rich theory. Let us start with an elementary observation. Assume $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous nondecreasing function. Then the function $x \mapsto x+\varphi(x)$ is one-to-one and onto $\mathbb{R}$. This fact was extended by G. Minty to real Hilbert spaces $H$ as follows. A mapping $A: H \rightarrow H$ is called monotone provided
(2) $\left(A\left(u_{1}\right)-A\left(u_{2}\right), u_{1}-u_{2}\right) \geq 0$ for all $u_{1}, u_{2} \in H$.

Assuming $A$ is continuous and monotone, G. Minty [M] proved that $I d+A$ is one-to-one and onto $H$. Here is a typical extension due to F. Browder:

Theorem 1 ([B5]). Assume $A: H \rightarrow H$ is continuous, monotone, and satisfies

$$
\begin{equation*}
\lim _{|u| \rightarrow \infty}|A(u)|=\infty \tag{3}
\end{equation*}
$$

Then $A$ is onto.
This theorem paved the way to a large collection of generalizations and applications to nonlinear PDEs. Firstly, Felix observed that the monotonicity condition (2) is natural in the framework of elliptic and parabolic boundary value problems. Secondly, he noticed that


Figure 1. Browder made great advances in the study and applications of monotone maps, $A$, such that $\left(A x_{1}-A x_{2}, x_{1}-x_{2}\right) \geq 0$. Positive semidefinite linear operators on Hilbert spaces are easily seen to be monotone, but monotone maps need not be linear. More interesting examples are gradients of convex functions.
condition (3) amounts to the fact that for every given $f \in H$ the equation $A(u)=f$ admits a priori estimates-a recurrent key phrase in PDEs. Thirdly, he popularized and made extensive use of a remarkable device (originally due to Minty $[\mathrm{M}]$ ) that is both very elementary and extremely effective in a wide range of situations involving a passage to the limit (e.g. from finite to infinite-dimensional spaces). Here it is, used to derive the above theorem from Minty's result. Given $f \in H$ and $\varepsilon>0$ there exists a unique $u_{\varepsilon}$ solution of

$$
\begin{equation*}
\varepsilon u_{\varepsilon}+A\left(u_{\varepsilon}\right)=f \tag{4}
\end{equation*}
$$

The monotonicity of $A$ yields $\left(A\left(u_{\varepsilon}\right)-A(0), u_{\varepsilon}\right) \geq 0$, so that $\varepsilon\left|u_{\varepsilon}\right| \leq|f-A(0)|$. Applying (4) and (3) we see that $\left|u_{\varepsilon}\right|$ remains bounded as $\varepsilon \rightarrow 0$, and thus a subsequence $u_{\varepsilon_{n}}$ converges weakly to some $u$. Unfortunately nonlinear operators are badly behaved under weak convergence (a well-known source of difficulties in nonlinear problems); thus this fact alone does not guarantee that $A\left(u_{\varepsilon_{n}}\right)$ converges, even weakly, to $A(u)$. Here enters monotonicity in its full glory. It allows one to produce solutions by a subtle mechanism that requires only bounds in some weak norms. By monotonicity, for all $v \in H$,

$$
\left(A(v)-A\left(u_{\varepsilon}\right), v-u_{\varepsilon}\right) \geq 0
$$

so that by (4)

$$
\left(A(v)-f+\varepsilon u_{\varepsilon}, v-u_{\varepsilon}\right) \geq 0
$$

Passing to the limit as $\varepsilon_{n} \rightarrow 0$ yields

$$
\begin{equation*}
(A(v)-f, v-u) \geq 0 \tag{5}
\end{equation*}
$$

Choosing in (5) $v=u+t w$, with $t>0$ and $w \in H$, gives $(A(u+t w)-f, w) \geq 0$. Letting $t \rightarrow 0$ we obtain $A(u)=f$, the desired conclusion.

The original setting of the above theorem has been generalized in many ways, thereby increasing tremendously its applicability to PDEs. Here are some typical directions:
i) The Hilbert space $H$ is replaced by a Banach space $V$. Maps $A: V \rightarrow V^{*}$ are monotone provided they satisfy the condition

$$
\left\langle A\left(u_{1}\right)-A\left(u_{2}\right), u_{1}-u_{2}\right\rangle \geq 0 \text { for all } u_{1}, u_{2} \in V
$$

ii) Nonlinear compact perturbations of monotone operators are admissible (see Browder [B2]). This has been


Work on compact perturbations of monotone operators by Browder led to further extensions by Leray, Lions, and Brezis, shown here with Browder many years later, in 2003.
pushed even further by J. Leray and J. L. Lions [LL] and H . Brezis [Bre], thus extending the classical LeraySchauder theory, which relies heavily on compactness. Another standard tool in nonlinear problems is the "variational" approach, which consists of minimizing convex-type functionals. Since gradients of convex functionals are monotone operators, the current state of the art provides a unified "roof," which requires no compactness and no variational structure. The impact of Browder's ideas cannot be overestimated. In 1969 Lions published an influential book Quelques Méthodes de Résolution des Problèmes aux Limites Non linéaires [L], which is a collection of techniques used in solving nonlinear PDEs. Out of a total of 550 pages, about 150 pages are dedicated to monotonicity methods, and the bibliography includes twenty papers by F. Browder.
iii) The concept of accretive maps is another natural extension of monotone maps. Recall that in a Banach space $X$ the semiscalar product is defined by

$$
[x, y]=\lim _{t \downarrow 0} \frac{1}{2 t}\left(\|x+t y\|^{2}-\|x\|^{2}\right)
$$

A map $A: X \rightarrow X$ is called accretive provided it satisfies the condition

$$
\left[u_{1}-u_{2}, A\left(u_{1}\right)-A\left(u_{2}\right)\right] \geq 0 \text { for all } u_{1}, u_{2} \in X
$$

Evolution equations associated with accretive maps play an important role because they generate semigroups of nonlinear contractions. Their systematic study was initiated by F. Browder in his pioneering paper [B3] (see also [B6]) and pursued by many people including T. Kato, J. L. Lions, M. Crandall, A. Pazy, Y. Komura, T. Liggett, V. Barbu, Ph. Bénilan, L. C. Evans, and myself. It has countless applications to problems coming from physics and mechanics. Recent uses include image processing, motion by mean curvature, etc.
iv) Maps $A$ need not be defined on the whole space $X$, but just on a domain $D(A)$. This is especially relevant in applications to PDEs. In concrete examples $D(A)$ consists of smooth functions, while $X$ may include rough functions.
v) One of the major achievements of the twentieth century in linear PDEs has been the systematic study of the concept of weak solutions in the sense of distributions. This notion requires an adjoint operator acting on a class of smooth testing functions. Such an approach has no analogue in nonlinear problems and the concept of weak solution becomes a very delicate issue even in simple models such as $u_{t}+u u_{x}=0$ (Burgers equation) or $|\nabla u|=$ $f(x)$ (Eikonal equation). Typically, such problems do not have classical $\left(C^{1}\right)$ solutions and they admit too many weak solutions. It is necessary to select among these "fake" solutions the physical solution. Here again, monotone operators-and especially the weak formulation (5)-can be extremely useful. Property (5) suitably adapted, may provide a mechanism to detect the physically interesting weak solutions. For example, (5) adapted to the semiscalar product in $X=L^{1}$, picks up the "entropy solution" (in the sense of Lax-Oleinik theory of shock waves) for the Burgers equation.

To conclude, let's mention briefly an example of a problem that has raised considerable interest in recent years and confirms the lasting impact of Browder's ideas. Consider the fully nonlinear equation

$$
\begin{equation*}
F\left(x, u, D u, D^{2} u\right)=f(x) \text { in } \Omega \subset \mathbb{R}^{N}, \tag{6}
\end{equation*}
$$

where $F$ satisfies the degenerate ellipticity condition (7)
$\left\{\begin{array}{r}F(x, u, p, X) \geq F(x, v, p, Y) \\ \text { whenever } u \geq v \text { and } X \leq Y \\ \quad \text { in the sense of symmetric matrices. }\end{array}\right.$
It is easy to see (using the maximum principle) that the operator $A: D(A) \subset X \rightarrow X$, where $X=L^{\infty}(\Omega), D(A)=$ $\left\{u \in C^{2}(\bar{\Omega}) ; u=0\right.$ on $\left.\partial \Omega\right\}$, and $A u=F\left(x, u, D u, D^{2} u\right)$, is accretive. The solvability of the equation

$$
\begin{equation*}
A(u)=f, \text { for a given (smooth) } f \tag{8}
\end{equation*}
$$

is a challenging task, especially in view of the fact that weak solutions in the sense of distribution have no meaning whatsoever. Returning to the above discussion (in particular (5)) it is natural to introduce a totally new concept of weak solution: a function $u \in L^{\infty}(\Omega)$ is a weak solution of (6) provided it satisfies
(9) $\quad[v-u, A(v)-f] \geq 0$ for all $v \in D(A)$,
where [,] denotes the semiscalar product in $L^{\infty}(\Omega)$. Condition (9) corresponds basically to the notion of viscosity solution in the sense of M . Crandall and P. L. Lions (see e.g. the expository paper [CIL]). Of course, this is just the beginning of the story. Studying the existence, uniqueness, and regularity of solutions of (6) is a formidable task-still partially under investigationrooted in the seminal plans of Felix Browder.

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Photo of Haïm Brezis with Felix Browder courtesy of Rutgers University.

## Amy Cohen

## Remembering Felix Browder

My thesis advisor, Murray Protter, introduced me to Felix Browder one day before I left Berkeley for Cornell in the summer of 1971. I moved to Rutgers in 1972, never expecting that I'd meet Browder again. However, Browder came to Rutgers as vice president for research in 1986.

Browder was instrumental in bringing I. M. Gelfand to Rutgers in the late 1980s. While Gelfand is primarily known as a mathematician, Browder knew of Gelfand's work in applications of computer science. Browder introduced him to Rutgers faculty, who became co-workers in fields ranging from automated medical diagnostics to machine reading of handwriting.

At Rutgers, Browder encouraged and supported Gelfand's Mathematical Correspondence Program. This program continued the Gelfand School by Correspondence in the former Soviet Union, which connected grad students in Moscow with rural students whose mathematical education had progressed beyond what their school teachers could guide.

After leaving his vice presidency, Browder moved into an office in the math department. He worked on mathematics with colleagues. As his health failed, we saw less of him.

[^12]I'll end with some personal anecdotes. As a member of the senior administration Browder learned that I had agreed to serve as acting dean of the Rutgers unit for returning adult bachelors degree candidates. When he next met me on campus he demanded, "What's a nice girl like you doing in a job like yours?" I must have answered correctly, since he later made me an AMS nominee to the Committee on the Undergraduate Program in Mathematics. Later I shamelessly asked him (and other senior faculty) to contribute financially to a math department project to enhance the success of first year students who placed into Calculus I but had "nonacademic indicators of risk." To my delight, Browder and others did contribute so the project could start before we received grant support.

## Jean Mawhin

Besides his fundamental contributions to partial differential equations, Felix Browder is recognized as one of the founding fathers of nonlinear functional analysis, with numerous and fundamental contributions to its monotone, variational, and topological aspects. When he was still in Chicago, Felix invited me to talk at his seminar. After the lecture, Jerry Bona told me with a big smile: "Felix took notes during your lecture. You should expect to see some generalizations soon." It was indeed the case, and I was both happy and proud to have inspired him in a very small part of his outstanding work.

## Roger Temam

I have always known Felix Browder, ever since I started to do research. Felix was a regular visitor to Paris and I always attended his seminars. After I graduated he invited me several times to visit him at the University of Chicago. The mathematics department was very diverse and inspiring, and it was a great pleasure to spend time there. Felix had a great influence on the theory of partial differential equations and functional analysis.

I would like to add a personal witness to the many witnesses that colleagues and friends will tell. When I was assistant professor working on my own thesis, a colleague, who was assistant professor in theoretical mechanics was stuck trying to write his thesis on shell theory. After some discussions where he told me the problem he was interested in, I gave him one day a paper of Felix. The paper contained exactly the theorem that he needed, and within a few months the colleague completed his thesis, and he moved on to become associate and then full professor. Although he probably never met Felix, he considered Felix as a sort of savior!

[^13]
## Richard Beals

## Advice from Felix

Everyone who knew Felix knew him not only as a brilliant intellect, but as a fount of stories, particularly stories about mathematicians. I will counter with a story about Felix. The spring of 1964 was a wonderful time to be on the job market. I had just completed a thesis under Felix's direction, and had three tempting instructorship offers, call them A, B, and C. Felix was at the Institute that year, so I went there to get his advice. Generous with both advice and time, he devoted three quarters of an hour to making a convincing case for choosing A rather than B or C. Felix's mind was not only brilliant, but supple. He went on for another forty-five minutes making the case for B, rather than A or C. As you have probably surmised, there followed yet another forty-five minutes on the case for C rather than A or B. I had much to think about on the train back to New Haven. A year later Felix made the whole A-B-C question moot by getting me to Chicago on a one-year visiting appointment that turned into tenure there. I have much to thank him for.

## Louis Nirenberg

The death of Felix is a great loss for mathematics and for all who knew him. I first met Felix around the time that he received his PhD. We quickly became friends, and our friendship lasted all these years. Our mathematical interests had much in common. In addition to his research I admired many things about Felix. He read and seemed to remember everything. He was a superb chairman at the math department at the University of Chicago, and at Rutgers he was influential in bringing Gelfand and Brezis there. Talking with him was always a great pleasure, and always very informative.

## Henri Berestycki

After sitting on my PhD committee in Paris, Felix brought me to the University of Chicago as a young Dickson instructor. This turned out to be a major experience for me that oriented my way of doing mathematics henceforth. When I arrived at the University of Chicago from France, he showed me to my apartment, explained how things worked around there with typical Felix warmth. When I asked him about safety as I had been alarmed by some reports (and after all this was Al Capone's city), he pointed to a park in the distance and said, "This park is so dangerous that even murderers do not venture there."

[^14]

Louis Nirenberg, pictured here with Felix and John Nash at Rutgers in 2007, said that Felix "seemed to remember everything."

When I arrived in Chicago, Felix asked me to teach an advanced graduate course on bifurcation theory. He grasped early on the importance of the subject as he understood the connections with other fields, not only in mechanics or physics but even in social sciences. And thanks to Felix on this occasion, I was able to meet economists.

A personal recollection of him has been a lesson for me on how to deal with apprentice mathematicians throughout my career. I once went along when Felix took a famous visiting speaker out to dinner at a fancy restaurant. This mathematician started to bully me a bit, poking at me simply I guess because I was young. I will always remember how Felix stopped this at once, nicely but quite firmly. Another quality of Felix was his immense trust. Once he had formed an opinion about you, he would go on to trust you in a generous manner.

Felix maintained an extraordinary lively and friendly intellectual atmosphere, creating a kind of Latin quarter by himself in his house in Hyde Park where he would often have people over. His library enchanted me. He constantly added volumes to it. I think that he viewed all his many trips as foraging expeditions for his unique collection. Particularly fascinating was his breadth of interest: his books covered nearly all fields of knowledge. This corresponded to an inclination I always felt myself and it brought us close. I felt elated by his example of a great mathematician who had such a large scope of interest and knowledge. Once, as I was teasing him that it was impossible to have read all these books, I picked one out of his shelves and opened it at a random page. He knew exactly what was on it! Mathematics, science, history, politics, philosophy-it seemed that there was no bound to his intellectual appetite and curiosity.

In his unique erudition, there was one thing that struck me particularly. It was the happiness of it all. It made him extraordinarily joyous to discuss an intellectual theme, and no matter what the topic, he would consider it with the same devoted attention and communicatively joyful appetite.

I vividly remember Felix grinning with joy on the many occasions of intellectual discussions. Be it when he could see some beautiful mathematical idea or because you brought in an interesting element or fact or argument in the conversation or because he could make some connection, you could always notice his smile. It was clear that knowledge and understanding gave him great joy.

## Barbara Mastrian

I recall meeting Felix for the first time in 1991, in my office in the Rutgers mathematics department, which I shared with his secretary Sue. I was a new employee, and had heard about Felix but had not met him. I recall the feeling of Felix walking into the office and giving Sue a letter to type. I felt like I was meeting someone important and distinguished. Felix had a powerful presence, and was demanding of excellent work. He brought that letter back to Sue several times before she got it to his specifications.

A few years later I met Eva. She had come in the mathematics department to get some paperwork notarized, showed me pictures of her then infant grandson Josh, who was in Russia at the time. Periodically Eva would come in and show pictures and chat. Later after Eva and Felix were less mobile I would visit them at their home, update Felix on department functions and activities, and bring well wishes from faculty, along with piles of emails for Felix to go through.

My lasting impression of Felix was when I told him about Professor Gelfand's memorial, and he insisted on attending. He wanted to give a talk at the memorial, and even though it was a difficult task to get him there, as he was in a wheelchair, he made the event. He gave an impressive long talk about Gelfand.

## Joel Lebowitz

One of the activities in which Felix played a central role was getting Gelfand to Rutgers. I remember Gelfand coming to see me at the Academy Hotel in Moscow where I was staying as a visitor to the Soviet Academy of Science. I transmitted to him the invitation from Felix to join Rutgers, and I wrote a letter to the American consul in Moscow for Gelfand to take to the consulate when applying for a visa.

## Sagun Chanillo

Felix never shied away from bold decisions and action. This was a man with impeccable taste who took risks. A year after I arrived at Rutgers, Calderón was invited to a colloquium. At the dinner afterwards, Felix with a

[^15]grand flourish pulled out an envelope from his jacket pocket, turned to Calderón, and said, "Alberto, this is for you." Calderón opened the envelope and his eyebrows rose several inches. At that point Felix added, "Alberto you deserve it. I hope you can come." Everyone at the table was stunned and taken by surprise, as nobody had been consulted, there was no committee, and no regard to "what if we fail and what will be the fallout?" Once Felix had a thought and a goal, it would seem he went after it in a bold and decisive way.

Felix was well known as a polymath and raconteur. Years ago, Felix and I found ourselves together at a Thanksgiving supper at the house of Abbas Bahri. The topic moved to something I deeply care about, ancient Indian astronomy. Felix had of course read the monumental treatise by Otto Neugebauer, The Exact Sciences in Antiquity, and quite remarkably also a translation of the Surya Siddhanta, which is Sanskrit for 'the treatise of the sun.' So he proceeded to tell me what Indian astronomy owed to Babylonian astronomy and vice versa. He also knew of the work of Frits Staal at Berkeley who was a pioneer in such studies. I have not met many since who could speak with such authority on so esoteric a subject.

## Fred Roberts

## Some Reflections on Felix Browder and the Formation of DIMACS

Not many people realize the critical role that Felix Browder played in the formation of DIMACS, the Center for Discrete Mathematics and Theoretical Computer Science, at Rutgers.

Felix came to Rutgers as vice president for research in 1986, and one of his early initiatives was to spearhead the Rutgers proposal for a National Science Foundation Science and Technology Center (STC). The STC program had three major components: cutting edge science in some field, education closely tied to research, and technology transfer. It required university, industry, and government partners.

When the STC program was announced, Rutgers, Princeton, and what were then AT\&T Bell Labs and Bellcore (Bell Communications Research) had all developed strong programs in discrete mathematics and theoretical computer science, but they were only interacting sporadically and informally. Felix had the insight to see that a New Jersey Center for Discrete Mathematics and Theoretical Computer Science already existed in some sense. Felix managed to get Danny Gorenstein of Rutgers to agree to serve as director (a critical development), and he was joined by Ron Graham of Bell Labs and later Bob Tarjan of Princeton. Felix also played an important role in getting the New Jersey Commission on Science and Technology to join the team as the government partner.

The Rutgers-led proposal was a winner in the first-ever STC competition, one of 11 winners in all fields of science.

[^16]

Browder played a critical role in the proposal that led to the formation of DIMACS, the host of the pictured $E+M=C^{2}$ workshop held in January 2017.

This led to the formation of DIMACS, the Center for Discrete Mathematics and Theoretical Computer Science, in early 1989. It is fair to say that we would not have gotten started on this without Felix. He played a pivotal role.

When Rutgers received the NSF award for DIMACS, it was at the time the largest award the university had ever gotten (some SUS22 million) and a major center kickoff conference was held, with New Jersey Governor Tom Kean giving a keynote address.

Felix was there, but stayed in the background. He didn't need the public thanks that many of us who have benefited from DIMACS owed him.

Felix continued to play an important role as DIMACS grew and developed. He also found it useful to leverage the existence of DIMACS to recruit Israel Gelfand to Rutgers, arranging for DIMACS to provide both space and resources for Gelfand's many visitors. The scientific and educational collaboration between DIMACS and Gelfand continued until his death and indeed continues today through the work of Tanya Gelfand to get her late husband's educational books published. This too is part of the legacy of Felix Browder.

Felix remained interested in DIMACS long after he stepped down as vice president for research. He often asked how things were going. One of the last times I saw him was when he became aware of the Simons Foundation announcement for an Institute for the Theory of Computation and invited me to strategize about how DIMACS could apply. I visited him at home, and we had a wide-ranging discussion. It was still another reminder of how Felix stayed engaged with the center he played such a great role in founding.

## Jerry Bona

## Browder the Professor

During my years as a young mathematician at the University of Chicago, I sat for three full courses from Felix. His lectures were fluid and his blackboard technique superb. He filled the room with his personality and kept the small audiences utterly focused. I still retain the notes from these courses. I came away from his lectures wondering how on earth he came up with the material, as most of it was his.

It was during the third course, which was on topological degree theory, when I was part of the Browder's social circle, that I had the temerity to ask Eva one evening if Felix prepared his lectures at home. Her response was immediate and forceful: no, he did not prepare lectures at home and she had no idea when he did. I found out a few weeks later. I was sitting in Felix's office discussing some mathematical point. Felix noticed that it was almost time for class. He immediately left off our conversation and concentrated for about a minute. He then rose and we walked together to class. All the while, he was clearly preparing the lecture.

I had the answer to my question: he prepared on the way to class, despite the complexity of what he was presenting! It was less than 15 meters from Felix's office to his favorite classroom. So once in a while, he would get stuck. You were then in for a treat. He would move to the side a little and start addressing the board and you got to see exactly how this great mind worked. It was a revelation to see the true inner workings of the beautiful, smoothly running mathematical theory that he was developing.

Probably this style did not go down so well in undergraduate, or even first-year graduate classes, but for someone a little more advanced and intensely interested, it was transformative.

## Наїm Brezis

In 1964 I asked G. Choquet in Paris to give me a PhD thesis topic. He told me to learn fixed point theory. He was perhaps hoping that I would find connections between fixed points and extreme points-his main research interest at the time. A few months later, I went back to G. Choquet, told him that I had read many papers on fixed points, and asked him politely, "Monsieur le Professeur, what should I do next?" On his desk was a huge envelope he had just received from Felix Browder. Choquet knew Felix personally and held him in high esteem, but did not have much interest in these topics. So, instead of throwing away the envelope, he handed it to me and said, "This might have connections with fixed points."

These brief minutes turned out to be a defining moment in my career. I started reading the twenty or so reprints, all published in 1963-1964. Most of the papers had

[^17]two parts. The first one was written in the language of abstract functional analysis, and fixed points were mentioned here and there; I was comfortable with their content. The second part was concerned with applications to PDEs, and I was totally lost. Under Bourbaki's influence, PDEs were simply not taught in Paris.

One day, in between two reading sessions at the library of the Institut Henri Poincaré, I noticed a flyer advertising a month-long summer school on PDEs in Montreal. 'F. Browder' was on the list of speakers; the other names, such as S. Agmon and G. Stampacchia-leading experts in PDEs-were totally unknown to me. The school was about to start the following week and the registration deadline had passed long ago. I mentioned it to my father, who immediately bought me a plane ticket. Luckily, I was admitted-even offered a room in the dorms-and started listening to Felix's classes. I introduced myself to Felix, who was pleased to hear that the package he had sent to Choquet had not been lost. We spoke for only a few minutes, but they were precious, particularly because Felix gave me ten more freshly published papers. At that time Felix was at the peak of his creativity, writing over twenty papers a year.

I greatly benefited from attending these classes. Felix taught me to enjoy PDEs through the eyes of a functional analyst. Originally most of functional analysis grew out of PDEs, but eventually it became increasingly abstract and detached from PDEs. In the mid-1960s, PDEs were hardly legitimate in Paris. J. Leray-who made celebrated contributions to PDEs in the 1930s-had shifted to other fields. J.-L. Lions was teaching some PDEs, disguised under the title "Numerical Analysis." His classes were held in "exile," in a building miles away from the "Holy of Holies," the legendary Institut Henri Poincaré.

Returning to Paris from Montreal, I realized that Felix's work had a rejuvenating impact on some French mathematicians, in particular, Leray and Lions. In fact, they had just published a PDE paper relying heavily on techniques developed by Felix. I studied it with great interest, and I understood it thanks to the background I had acquired from Felix's papers.

I also came across a new paper by Lions and G. Stampacchia concerning variational inequalities. The concept of weak solution that they had adopted was via integration by parts (not surprisingly, since Lions had been a student of L. Schwartz, the inventor of distributions); they could prove existence, but not uniqueness. Influenced by the ideas of Browder, I proposed a slightly different concept of weak solution, for which I could establish both existence and uniqueness (see page 1403 "Felix Browder and Monotone Operators"). Lions and Stampacchia were impressed and offered to collaborate with me on various projects using the same technique. Ironically, my first papers, at age twenty-three, were dealing with PDEs!

In 1967 Felix came to Paris for a conference organized by Lions. This time we had lengthy conversations! Felix invited me to give a talk at an AMS meeting to be held in Chicago the following year. My lecture was scheduled for the first day, and I came prepared to give a presentation on


Felix and Yves Meyer in Paris in 1998 at the celebration for J.-L. Lions' seventieth birthday. Felix had developed close ties with a number of French mathematicians.
a blackboard. When I arrived at the Hilton Grand Ballroom, I searched frantically for the blackboards. Instead, there were strange gadgets that had not yet reached France: overhead projectors. My limited English added to the catastrophe. Nevertheless, Felix kindly invited me to spend the following week at the University of Chicago, where I gave a more relaxed talk on a blackboard. He was extremely receptive and made numerous comments. Subsequently, I spent a month every year with him in Chicago. Strictly speaking, I was not Felix's student, but I consider myself as an "adopted child" of Felix; very naturally he became my mentor.

When in Chicago, I enjoyed working with Felix for a few hours and then walking to one of his favorite used bookstores in Hyde Park: Powell's, O'Gara (with its dusty high shelves, and a cat), or 57th St Books (in a basement). Felix loved to buy books; he also spent hours at the copy machine in the library, copying papers, and then carefully cutting the black margins to make them look neater.

Felix had developed close ties with a number of French mathematicians. I remember the constant flow of visitors from France invited by Felix in the 1970s and early 1980s. Of course, senior professors (such as J. -L. Lions and R. Thom) were on his list. But also for many junior French mathematicians (such as I. Ekeland, R. Temam, H. Berestycki, and the late A. Bahri), the gate to the USA was Chicago. Felix was a superb host-even fully mobilized when a young instructor required special help in critical medical matters. In addition, Felix was the driving force in organizing two conferences celebrating two legendary French mathematicians: Henri Poincaré and Elie Cartan. On the other side of the Atlantic, Felix was awarded a well-deserved honorary degree at the Sorbonne in 1990. He often visited Paris, by himself or with his family. His favorite hotel was the Parisiana-Panthéon, run by two old ladies; they kept telling him fascinating stories about the famous Russian mathematician, Luzin, who had stayed there at the beginning of the 20th century. Felix
read French fluently and was buying large numbers of books in the Latin Quarter. Needless to add that shipping them back home at the end of each visit was a major challenge!

When Felix moved to Rutgers in 1986, I followed him as a long-term distinguished visiting professor. Our mathematical collaboration became greatly reduced due to his time-consuming responsibilities as vice-president for research. Our last joint paper, "Partial Differential Equations in the 20th Century," written in 1997, had been commissioned by an encyclopedia on the history of science. We spent more time than expected working on this project; it was an unusual activity for both of us-especially for me. I watched with admiration as Felix proposed offhand a detailed table of contents and a list of topics to be discussed. He had a deep understanding of the evolution of concepts, and the main novelties each period produced. We also carefully read papers by Hilbert and Poincaré dating back to the beginning of the 20th century and discovered with great surprise that some preconceptions had to be reexamined. Felix was sharpminded and had his personal views on almost every topic.


Felix and Eva at their home in New Jersey, circa 2002.

In our discussions, I found Felix always very respectful of my opinions, even if he did not share them. The same was true for religious beliefs. He regularly insisted that we go for lunch to a kosher restaurant to accommodate my dietary restrictions. If we worked on a Friday, he would send me back home early enough so that I could properly keep the Sabbath.

As years passed, our personal relationship grew deeper and sweeter. Occasionally, Felix would buy a book for me if he thought I might enjoy it. Eva and Felix had an open house for my whole family. Felix had a genuine interest for the work of my wife, the Israeli writer and poet, Michal Govrin. And Eva graciously agreed to be interviewed about her life by our daughter Rachel for a school project.

## Photo Credits

Photo of Louis Nirenberg and John Nash with Felix Browder courtesy of Barbara Mastrian.
DIMACS photo by Tamra Carpenter.
Photo of Felix Browder with Yves Meyer courtesy of Jean Mawhin. Felix and Eva Browder photo courtesy of Haïm Brezis.

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# Erica Flapan to Top Notices Masthead 

Sophia D. Merow



Incoming Notices editor Erica Flapan studies the rigid and flexible geometry of molecules.

Erica Flapan, the Lingurn H. Burkhead Professor in the Department of Mathematics at Pomona College, will begin a three-year term as editor of Notices starting with the January 2019 issue.

She's psyched.
Flapan gets a thrill when she proves a theorem or initiates a cohort of Pomona students into the mysteries of topology. But when she learned she'd be the next editor of Notices-and began thinking about what that would entail-Flapan felt a different kind of excitement.
"I think doing something new at my age is good," she says. "It's stressful but it's also very exciting to be learning

[^18]something completely new and also communicating with people on a different level."

Flapan completed her undergraduate education at Hamilton College and earned her PhD under Daniel R. McMillan at the University of Wisconsin-Madison for a dissertation titled "Non-Periodic Knots and Homology Spheres." She was a post-doc at Rice and UC-Santa Barbara before joining the Pomona College faculty in 1986.

Flapan works in spatial graph theory, the area of topology that studies knots embedded in $S^{3}$ and other 3 -manifolds. She is particularly interested in symmetries of non-rigid molecules and the role that knotting plays in DNA recombination and protein folding. She relishes working in an area that is "simultaneously very pure and also applied."

Flapan's CV lists 56 research papers and five books, three of which were published by the AMS (see Figure 1). She won the Mathematical Association of America's Deborah and Franklin Tepper Haimo Award for Distinguished College or University Teaching of Mathematics in 2011 and was named an inaugural Fellow of the AMS in 2012. She spent eleven summers as an instructor at Carleton College's Summer Math Program, and from 2015 to 2017 traveled the country giving talks as MAA Pólya Lecturer. Her service to the AMS includes stints on the Committee on Education, the Committee on Professional Ethics, and the editorial committee of the Student Mathematical Library, and as a Member at Large of the Council.

Flapan sees membership in the AMS as an identity matter, not a question of the tangible benefits reaped by forking over annual dues. "I see the AMS as the organization that represents mathematicians who are actively doing mathematics. I think that every mathematician who is interested in research should want to be a member of the AMS and be part of shaping its future."
"The Notices is the best way that the AMS has to communicate with its members and potential members," says Flapan. "The Notices has the potential to reach out to a broader audience of mathematicians so that they feel that the AMS speaks to their interests, and hence they will become motivated to join."

Besides experience wrangling colleagues into producing quality mathematical content (see, again, Figure 1), Flapan boasts other characteristics bound to prove useful

## COMMUNICATION



Figure 1. As leader of the Undergraduate Faculty Program at the Park City Mathematics Institute in 201 1, Flapan was charged with guiding a group of faculty to collaboratively produce materials that could be used at a broad range of institutions. The resulting volume has 18 coauthors!
to her as the editor of Notices: She has a lot of energy, a penchant for organization, and a lot of ideas.

Flapan describes Notices as "a news magazine for mathematicians" and wants the bulk of its articles to cover trends and happenings in current mathematics. And 'mathematics' to Flapan means pure and applied.
"I feel that the line [between the two] is much more fuzzy than people think," she says.

Flapan's Notices will include a handful of quarterly columns-one by Danny Calegari and Igor Pak, another showcasing important mathematical discoveries that have taken place in national laboratories and agenciesand an Early Career Section. Spearheaded by Angela Gibney, the Early Career Section will give graduate students, postdocs, and tenure track faculty access to the accrued wisdom of their mathematical forebears.
"It's completely different from anything that's been in the Notices before," says Flapan.

Flapan wants the experience of writing for Notices to be interesting and satisfying, and hopes authors will enjoy crafting content for the publication's wide audience. Flapan is encouraging individual authors to write in their own styles and thinks readers will benefit from the diversity that will no doubt result.
"I want readers to go in with the mindset that there's going to be a variety of articles," she says, "and hopefully there will be something you find interesting and will be at a level or in a style that you really appreciate."

## Image Credits

Photo of Erica Flapan courtesy of Stephan Garcia.
Figure 1 courtesy of the AMS.

FEATURED TITLES FROM THE
EUROPEAN MATHEMATICAL SOCIETY


Boundary Behavior of Solutions to Elliptic Equations in General Domains
Vladimir G. Maz'ya, Linköping University, Sweden, and University of Liverpool, UK
This book is a detailed exposition of the author and his collaborators' work on boundedness, continuity, and differentiability properties of solutions to elliptic equations in general domains, that is, in domains that are not a priori restricted by assumptions such as "piecewise smoothness" or being a "Lipschitz graph".

EMS Tracts in Mathematics, Volume 30; 2018; 441 pages; Hardcover; ISBN: 978-3-03719-190-3; List US\$98; AMS members US\$78.40; Order code EMSTM/30


A Brief Introduction to Spectral Graph Theory

Bogdan Nica, McGill University, Montreal, Canada
This text is an introduction to spectral graph theory and can be used as an invitation to algebraic graph theory.
EMS Textbooks in
Mathematics, Volume 21; 2018 168 pages; Hardcover; ISBN: 978-3-03719-188-0; List US\$48; AMS members US\$38.40; Order code EMSTEXT/21

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## Goodbye

## from FrankMorgan, Editor-in-Chief, 2016-2018



The first issue of the Notices published under Frank Morgan.

It has been a privilege to serve as your Editor and work with authors, editors, and reviewers who contribute so generously to the cause of mathematics. One editor reported how a Fields Medalist "sent his first version within two weeks and accepted all my edits without the slightest complaint... wonderful to work with." Another contributor whose nice submission didn't fit in the Black History Month issue responded to the news only with thanks. One editor drove 90 minutes to personally ask someone to write a column we needed as soon as possible. Reviewers kindly recognize our tight schedule and generally respond within a few days.

More illustrations and the new full color version of Notices require major effort and expense. Managing Editor Rachel Rossi, former Deputy Editor Allyn Jackson, Contributing Writer Elaine Kehoe, and the staff in Providence have worked wonders and made major improvements. Special thanks go to my assistant Sophia Merow. Of course our best friends are you our readers, and we thank you for reading, sharing, and doing mathematics.

We're grateful for the larger and wider involvement of our community. Many invited lecturers at national, sectional, and some international meetings now generously provide inviting lecture samplers ahead of time. From 2015 to 2018, the total number of articles has risen from 82 to 114 ; the number of female authors has risen from 35 to 106 (from $16 \%$ to $38 \%$ ); and the number of minority authors has risen from 17 to 63 (from 8\% to 23\%).

We wish all future authors, editors (especially the new Editor Erica Flapan), and readers much happiness and success in carrying forward the noble enterprisemathematics.



Notices


Notices


## LETTER TO THE EDITOR

## Young Cathleen Morawetz and the Fields Medal

I would like to add a small, non-mathematical historical note tangentially linking the late Cathleen Morawetz, featured in the August 2018 Notices, to the Fields Medal. ${ }^{1}$

Cathleen was the daughter of mathematician John Lighton Synge, who was the executor of John Charles Fields's estate. On his deathbed, Fields entrusted Synge with the unfinished task of bringing into existence the mathematics medal that he had been planning for some time. Although Fields had played around with various designs for the medal, he had not settled upon one at the time of his death in 1932.

Whether at his own behest or that of Fields, Synge met with sculptor R. Tait McKenzie in 1932 at McKenzie’s home in Almonte, Ontario, in the Ottawa Valley. Cathleen Morawetz later recalled to me the drive from Toronto to Almonte as a girl of nine with her father-a long and difficult drive in those years, in those cars, over those roads. She remembers meeting Tait McKenzie and running around his wooded property while Synge described Fields's wishes and gave the commission to McKenzie.

Neither Fields, nor Synge, nor McKenzie had any idea then that the lithe young girl would herself become a distinguished mathematician.

-Elaine McKinnon Riehm Burlington, Ontario, Canada

(Received August 21, 2018)

[^19]
## THE GRADUATE STUDENT SECTION

## Liverituvai

## Wei Ho Interview

## Conducted by Alexander Diaz-Lopez



Wei Ho is assistant professor at the University of Michigan. Recipient of several NSF grants, as well as a Sloan Research Fellowship, Wei works on algebraic geometry and its application to arithmetic problems.

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Diaz-Lopez: When did you know you wanted to be a mathematician?

Ho: At the beginning of college, I was very undecided about what major to choose. I was so certain I would not major in math that in my first semester I took a distribution requirement from which math majors were exempt! By the end of my first year, however, I realized that all my favorite topics in my classes were the most mathematical: quantum mechanics, symmetry groups in inorganic chemistry, game theory. After working in an organic chemistry lab for the summer, I also found out that I was so clumsy that I might blow myself up if I continued in a lab science. So, during my sophomore year, I took (and enjoyed) some more math classes and switched by the end of the year. I spent the rest of my undergraduate years feeling like I was playing catch-up to the "real" math majors who had taken the hardest freshman math sequence, but I eventually realized that starting a year-or even many years-later does not matter and there is no "right" path.

> Starting a year-or even many years-later does not matter, and there is no "right" path.

At every stage of my education and career, I have doubted whether to continue in academia, mostly due to my own insecurities (I'm told this is the infamous "imposter syndrome"). I seriously considered working as a quant at a hedge fund after college, dropping out of grad school to work for a tech company, and finding a consulting or tech job after grad school and again after my postdoc years. I am quite happy with my choice now, but I think there are many viable careers for someone with a background in math (and I would encourage graduate students to investigate all types of jobs as early as possible).

## I think there are many viable careers for someone with a background in math．

Diaz－Lopez：Who else en－ couraged or inspired you？

Ho：My parents encouraged my non－stop questions about math and everything else as a child，and they prioritized my brother＇s and my educa－ tion over all else．They taught us to think analytically and precisely and question every－ thing；while these are useful traits for a mathematician， I＇ve learned these pedantic tendencies sometimes need to be suppressed in social situ－ ations！
I feel fortunate that many older mathematicians have been very kind and encouraging to me for no apparent reason，and there are so many people who had a profound influence on my mathematical path．I＇ll name just a few here．When I was deciding whether to switch to math， Daniel Allcock＇s number theory class was fun and ac－ cessible－and made me believe I could actually do math． During my year abroad after college，I loved Burt Totaro＇s class on geometric invariant theory，which strongly influ－ enced my research directions in graduate school．My PhD advisor Manjul Bhargava and postdoc mentors Joe Harris and Johan de Jong have all been incredibly supportive throughout my career，and talking to any of them about a problem I＇m mulling over is always fun and stimulating．

Someone who probably has no idea of her impact on me is Linda Chen，who was the first Asian－American woman mathematician I remember meeting．While Asians are not considered an underrepresented minority in math，there seem to be very few Asian－American mathematicians，and of course even fewer such women．The stereotypes gave me yet another reason for self－doubt：maybe the emphasis on obedience and duty（＂聽話 and be 乘＂）in my Chinese cultural upbringing made me less able to think outside the box in my research，while my privileged and comfortable American childhood made me too lazy？These qualms have not gone away completely，but they have made me further appreciate how much representation matters，es－ pecially for people from groups that are underrepresented in whatever ways．

Diaz－Lopez：How would you describe your work to a graduate student？

Ho：I tend to think a lot about classical algebraic ge－ ometry constructions that can be adapted to arithmetic settings．Perhaps we understand some construction very well over the complex numbers，but more interesting phenomena appear when you change the base field to the rational numbers．

Most of the motivation for my work comes from simple questions in number theory and algebraic geometry．For example，given a multivariate polynomial with rational coefficients，how many rational solutions does it have？ Maybe you can find the answer to be zero，or a finite number，or even＂infinity＂；but can you describe the set
of solutions（＂rational points＂）more precisely in any way？ Sometimes it＇s easier to think about lots of polynomials all at once，in a big family，i．e．，study a＂moduli space＂of the polynomials．Then you can try to study the structure of the relevant moduli space to answer your original question．

This strategy has proved immensely useful in a field now called arithmetic statistics，which studies the distri－ bution of arithmetic invariants in families．For instance， we can associate a nonnegative integer，called the rank，to an＂elliptic curve＂defined over the rational numbers（the rank of the finitely generated abelian group made from the rational points of the curve）．If we order all such el－ liptic curves（there are infinitely many）in a sensible way， can we predict what proportion of the curves have rank 0 ，or rank 21？Do any have rank 100？Mathematicians don＇t know the answer to these questions yet，but study－ ing moduli spaces related to elliptic curves has given us some insight into this distribution．We also have various heuristic predictions and even some data to support them， thanks to much improved computational software over the last decades．

Diaz－Lopez：What theorem are you most proud of and what was the most important idea that led to this break－ through？

Ho：This question is very hard to answer，since I tend to find all my theorems trivial once I know how to prove them！

I do remember being very excited about my recent work with Levent Alpoge，since the turning point was the trivial observation that $2^{2}=4$ ．The paper is about understanding the number of integer solutions $(x, y)$ to equations of the form $y^{2}=x^{3}+A x+B$ ，as $A$ and $B$ are varying over integers （with $4 A^{3}+27 B^{2}$ nonzero）．Levent previously proved that when you order these so－called elliptic curves by the size of their coefficients， then the average number of integral points is bounded．To understand the distribution even better，you might study the higher moments of the number of integral points．We show that the second moment is also bounded，by combining a classical explicit construction of Mordell with some analytic techniques and a theorem of Bhargava－Shankar on the aver－ age size of Selmer groups of elliptic curves．Originally，we had hoped to prove this re－ sult with a rather complicated method，so we were both very happy when we realized there was a much simpler way that actually worked．

Another paper that has a fun backstory is a very short one with Johan de Jong．Johan and I had been meeting semi－regularly to chat about interesting math，and the paper came out of just a one－hour discussion．We real－

## THE GRADUATE STUDENT SECTION

ized that some constructions I knew from other projects could be adapted to solve a problem he liked. It was so simple that we tried very hard (but failed) to make the paper just one page long. But this was the first time where I concretely saw how random knowledge in one very specialized corner of math could be unexpectedly useful in a different area; it is helpful to pick up ideas from as many different fields as possible, even if they don't seem immediately relevant.

Diaz-Lopez: What advice do you have for graduate students?

Ho: Grad students are often stressed out about choosing a perfect advisor with a perfect problem. Neither exists. Choose an advisor based on compatibility, both mathematical and otherwise. And the problem you start with is almost definitely not going to be the problem you end with; it will evolve as you work on it.

Talk to people! Learn from your fellow grad students, as well as postdocs and faculty. Organize your own learning seminars if they aren't already happening.

Go to seminars. Go to summer schools.

## Go to conferences. Read widely.

 Go to conferences. Read widely. Read survey papers, e.g., in the Bulletin of the AMS. You won't understand nearly anything at first, but it's useful to be exposed to lots of ideas-and to lots of people who have different ways of doing math.If you aren't set on academia, explore your other options early. Do a summer internship if you can. Learn to code.

Diaz-Lopez: All mathematicians feel discouraged occasionally. How do you deal with discouragement?

Ho: I would say it is more than occasional! Research can be quite frustrating because much of your time is spent going down seemingly incorrect paths, but I try to remember that those supposed dead ends often are useful later.

I tend to have many projects at different stages. While it is sometimes hard to switch between topics (and most days I don't actively think hard about more than one project), I find it useful to switch to another when I am stuck on one. Having collaborators-who often are friends alsohelps a lot with the discouragement. Finally, teaching and service responsibilities can make me feel productive even if I didn't get anywhere on my research on a given day!

Another strategy is to just do something completely non-mathematical. I go to ballet classes or play video games when my brain is not working well. It took me many years to not feel guilty whenever I was not working. As a research mathematician, I could in theory be working $24 / 7$. But of course, I wouldn't be productive all of that time, and it was important to learn to make guilt-free time for friends, family, and outside interests. For example, I read novels and rarely work on airplanes.

Diaz-Lopez: Any final comment or advice?
Ho: I think grad students should learn early that their own mathematical adventure will be unique, so it is com-
pletely okay to not follow in other people's footsteps. You don't have to do things the way everyone else seems to, if something else works for you.

Most importantly, just have fun!

## Photo Credit

Wei Ho photo by Bhargav Bhatt.

## ABOUT THE INTERVIEWER

Alexander Diaz-Lopez, having earned his PhD at the University of Notre Dame, is now assistant professor at Villanova University. Diaz-Lopez was the first graduate student member of the Notices Editorial Board.


Alexander Diaz-Lopez

# a Multilayer Network? 

Mason A. Porter<br>Communicated by Cesar E. Silva

We are surrounded by networks. People communicate with other people using online social networks like Facebook and Twitter (and occasionally even in person in offline social networks). They also travel or perform daily routines using transportation networks. Animals interact with each other in numerous ways in their social networks. Plants and fungi transport nutrients through networks.


Figure 1. A graph consists of nodes (which I show as disks) that are connected to each other by edges (which I show as arcs).

The simplest type of network, which I show in Figure 1, is a graph $G=(V, E)$ [3], where the nodes (or "vertices") are elements of the set $V$ of $N$ entities in a network and $E \subseteq V \times V$ is a set of edges (or "links" or "ties") that encode pairwise interactions between the entities. A graph can be either undirected or directed. One can encode the information in a graph $G$ as an $N \times N$ adjacency matrix A, whose entry $A_{i j}$ is equal to 1 if there is an edge from node $i$ to node $j$ and is otherwise equal to 0 . In an undirected network, $A_{i j}=1$ if and only if $A_{j i}=1$. One can learn a lot about a graph $G$, and about many dynamical processes on it, by studying the properties (e.g., the eigenvalues) of its associated adjacency matrix A. One can also assign

[^20]weights to edges to represent connections with different strengths (e.g., stronger friendships, larger transportation capacity, and so on) by defining $w: E \longrightarrow X$. The most common choice is $X=\mathbb{R}_{+}$, so that all edge weights are positive real numbers.

The study of networks in the form of graphs has a long, rich history in a variety of fields, including mathematics, statistics, computer science, sociology, physics, ecology, economics, and many others [3]. However, most real networks are much more complicated than ordinary graphs. For example, the nodes, edges, and edge weights can change in time; there can be multiple types of nodes or multiple types of relationships; and nodes can represent entities at different levels of granularity (e.g., a mathematics department, an applied-mathematics group, or an individual). One common type is a multirelational network, such as the air-transportation and social networks in Figure 2, in which there are multiple types of edges between nodes. We depict the social network in Figure 2b as an edge-colored multigraph, where different colors (i.e., annotations or "labels") represent different types of relationships: friendship, arguments, horseplay, and so on.

The formalism of multilayer networks [2], a generalization of graphs, was developed recently to help study multitudinous types of networks and to unify them into one framework. Many of these, such as multirelational networks in sociology and interconnected networks of different subsystems in engineering, have been studied for decades, but the development of the multilayer-network formalism to analyze such systems is very recent.

Even with its relatively short history, the study of multilayer networks has become very prominent. In briefly introducing this idea, I mostly follow the terminology and conventions from the review article [2].

A multilayer network $M=\left(V_{M}, E_{M}, V, \mathrm{~L}\right)$, as illustrated in Figure 3, has an underlying set $V$ of $N$ physical nodes (representing entities), often labeled $1,2,3, \ldots, N$, that manifest on layers in $L$ that are constructed from


Figure 2. Two examples of multilayer networks: (a) part of an air-transportation network, in which each layer has flights from a different airline; and (b) a social network of individuals, with layers representing different types of relationships between them, in a bank-wiring room.


Figure 3. (a) An example of a multilayer network $M=\left(V_{M}, E_{M}, V, \mathrm{~L}\right)$ with four physical nodes and two aspects, which have corresponding elementary-layer sets $L_{1}=\{A, B\}$ and $L_{2}=\{X, Y\}$. The four layers of $M$ are $(A, X)$, $(A, Y),(B, X)$, and $(B, Y)$. Each layer includes a subset of the physical nodes in $V$. The set of state nodes is $V_{M}=\{(1, A, X),(2, A, X),(3, A, X),(2, A, Y),(3, A, Y),(1, B, X),(3, B, X),(4, B, X),(1, B, Y)\}$. One can connect nodes to each other in a pairwise manner both within layers and across layers. I show intralayer edges as solid lines and interlayer edges as dotted lines and arcs. (b) The graph $G_{M}=\left(V_{M}, E_{M}\right)$ associated the multilayer network $M$. I again show intralayer edges as solid lines and interlayer edges as dotted lines. The adjacency matrix of this graph, which has accompanying labels (in both nodes and edges) from the layer information, is the multilayer network's supra-adjacency matrix. (See Figure 4 for an example.) Intralayer edges, which correspond to an ordinary type of edge in a graph, are associated with nonzero entries in the diagonal blocks of a supra-adjacency matrix, whereas interlayer edges are associated with nonzero entries in off-diagonal blocks.
elementary-layer sets $L_{1}, \ldots, L_{d}$, where $d$ is the number of "aspects" (i.e., types of layering). One layer in $L$ is a combination, through the Cartesian product $L_{1} \times \cdots \times L_{d}$, of an elementary layer from each aspect. In Figure 3, the sets of elementary layers are $L_{1}=\{A, B\}$ and $L_{2}=\{X, Y\}$. The set of node-layer tuples (sometimes called "state nodes")
in $M$ is $V_{M} \subseteq V \times L_{1} \times \cdots \times L_{d}$, and the set of multilayer edges is $E_{M} \subseteq V_{M} \times V_{M}$. The edge $((i, \alpha),(j, \beta)) \in E_{M}$ indicates that there is an edge from node $i$ on layer $\alpha$ to node $j$ on layer $\beta$ (and vice versa, if $M$ is undirected). Each aspect of $M$ represents a type of layering: a type of social tie, a point in time, and so on. For example,

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a multirelational network that does not change in time, such as the bank-wiring network in Figure 2b, has one aspect; a multirelational network that has layers covering multiple time points has two aspects; and so on. To consider weighted edges, one proceeds as in ordinary graphs by using a function $w: E_{M} \longrightarrow X$.

For example, suppose that Figure 3 represents a multilayer network of collaborations and citations among scientists. In this network, Buffy (physical node 1), Willow (2), Angel (3), and Wesley (4) are writing papers on the mathematical theory of vampire slaying and using this information in vampire-slaying expeditions. The elementary layers $A$ and $B$ encode different types of interactions: paper coauthorships $(A)$ and joint expeditions (B). Suppose that the elementary layers $X$ (2017) and $Y$ (2018) represent years. Thus, the intralayer edge between state nodes $(1, A, X)$ and $(2, A, X)$ signifies that Buffy and Willow went on a joint vampire-slaying expedition in 2017. Let's suppose that interlayer edges, which are often harder to interpret than intralayer ones, represent the use of information from a paper or an exposition. Such an an interaction is directed, although I don't indicate any directions on the edges in Figure 3. To give an example, the edge from $(3, A, Y)$ to $(4, B, X)$ represents the fact that, in a 2018 paper, Angel used information from one of Wesley's 2017 expeditions.

Each unweighted multilayer network with $d$ aspects and the same number of nodes in each layer has an associated adjacency tensor $\mathcal{A}$ of order $2(d+1)$. Analogous to the case of ordinary graphs, each directed edge in $E_{M}$ is associated with a 1 entry of $\mathcal{A}$ (undirected edges are each associated with two such entries) and the other entries (the "missing" edges) are 0. If a multilayer network does not have the same number of nodes in each layer, one can add empty nodes so that it does, but the edges attached to such nodes are "forbidden" edges. When studying multilayer networks, missing edges and forbidden edges need to be treated differently (e.g., when normalizing quantities such as clustering coefficients or measures of node-layer or edge importance). One can flatten $\mathcal{A}$ into a "supra-adjacency matrix" $\mathbf{A}_{M}$, which is the adjacency matrix of the graph $G_{M}$ associated with $M$ (as in Figure 3b). Intralayer edges are associated with entries on the diagonal blocks of a supra-adjacency matrix, and interlayer edges are associated with matrix entries on the off-diagonal blocks. Figure 4, which illustrates a type of multilayer network known as a cognitive social structure, gives an example supra-adjacency matrix and associated multilayer network. In practice, most numerical computations with multilayer networks employ supraadjacency matrices.

Multilayer networks allow one to investigate a diverse set of complicated network architectures and to integrate different types of data into one mathematical object. Two key types of multilayer networks arise from (i) labeling edges or (ii) labeling nodes. When one labels edges, one thinks of edges in different layers as representing different types of relationships. This is the case for a "multiplex network," a type of multilayer network in which the
only permitted types of interlayer edges are those that connect manifestations of the same physical node in different layers. A special case of a multiplex network is an edge-colored multigraph, like the one in Figure 2b. Interlayer edges in a multiplex network occur on diagonal elements of off-diagonal blocks in a supra-adjacency matrix (as in Figure 4b). By contrast, when one labels nodes, one can think of different layers as representing different subsystems (in "interconnected networks" or "networks of networks," such as in coupled infrastructure networks), and one can have interlayer edges with nonzero supra-adjacency matrix elements in both the diagonal and off-diagonal entries of the off-diagonal blocks.

Multilayer networks have rich structural properties, and dynamical processes on them behave in fascinating ways-including experiencing novel phase transitions, where system behavior changes qualitatively. For further discussion of dynamical processes on multilayer networks, see [2] and a recent survey article [1] on spreading processes on multilayer networks. An important idea is that interlayer edges are fundamentally different from intralayer edges, and it is often less straightforward to assign weights from data to interlayer edges than to intralayer ones. A conceptually easy situation is a multimodal transportation network, in which one might calculate an interlayer edge weight based on how long it takes to change modes of transportation (e.g., from the subway to a bus). For communication on a social network, one might construe an interlayer edge as representing a transition probability between different modes of communication. For other applications, interlayer edges can run into significant conceptual difficulties, and researchers struggle with how to make sense of them. For example, in protein interaction networks, a layer can represent a type of interaction; there are dependencies across layers, and interlayer edges can encode them, but how does one determine meaningful values for the weights of those edges?

Examining consequences of the relative weights of intralayer and interlayer edges has also led to interesting theoretical results. A valuable example started with a paper by Filippo Radicchi and Alex Arenas (Nature Physics, 2013); it has been built on subsequently by them and others [2]. Considering a multiplex network, Radicchi and Arenas constructed the combinatorial supra-Laplacian matrix $\tilde{\mathbf{L}}_{M}=\mathbf{D}_{M}-\mathbf{A}_{M}$, where $\mathbf{D}_{M}$ is the diagonal supramatrix that has node-layer strengths along the diagonal. Each diagonal entry of $\tilde{\mathbf{L}}_{M}$ consists of the sum of the corresponding row in $\mathbf{A}_{M}$, and each nondiagonal element of $\tilde{\mathbf{L}}_{M}$ consists of the corresponding element of $\mathbf{A}_{M}$ multiplied by -1 . Using the case in which counterpart nodes in each pair of layers are connected with a homogeneous interlayer edge weight as an illustrative example, Radicchi and Arenas showed that $\tilde{\mathbf{L}}_{M}$ 's smallest nontrivial eigenvalue $\Lambda_{2}$, which is related to many structural and dynamical features of the corresponding multilayer network $M$, has two distinct regimes when examined as a function of the relative weights of the interlayer and intralayer edges. They also showed that there is a discontinuous phase

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(a)

(b)


Figure 4. (a) Representation of a cognitive social structure as a multilayer network (without drawing any interlayer edges). In a cognitive social structure, each "perceiver" can have a different view of which edges exist in a network. Such data can arise, for example, from survey questions in which one asks people to fill in edges whenever they think a given type of relationship exists between two people. The depicted network has two aspects: perceiver and type of social connection. (b) Illustration of (a subset of) the supra-adjacency matrix that represents this cognitive social structure. This example has a nested block-diagonal structure: the interior block-diagonal structures (in blue and green) correspond to intralayer connections, and the two large blocks on the block diagonal have edges between different perceivers when the edge-type (i.e., friendship or advice) aspect is fixed to one of the two alternatives. One can represent this multilayer network as a multiplex network, because the only nonzero off-block-diagonal entries occur in the diagonal entries of the depicted blocks. (In the figure, I show nodes and layers only for persons 1-4.)
transition between those regimes. In one regime, $\Lambda_{2}$ is independent of the intralayer network structure and is thus determined by the weight of the interlayer edges. In the other regime, $\Lambda_{2}$ is bounded above by a constant multiplied by the smallest nontrivial eigenvalue of the unweighted superposition of the layers. This thread of work has important implications for interpretation of results in many investigations of multilayer networks, including for the behavior of dynamical processes on multilayer networks, evaluating node importances in such networks, and partitioning such networks into dense "communities" of nodes.

Before closing, it is also worth highlighting that there are two categories of dynamical processes on multilayer networks: (i) a single process that is defined on a multilayer network and (ii) interacting dynamical processes that are defined separately on different layers of such a network [1]. An important example of the first category is a random walk, where the qualitative behavior depends on the relative speeds and probabilities of intralayer versus interlayer steps. To examine the time scales of diffusion of such a dynamical process, one calculates the smallest nontrivial eigenvalues of $\tilde{\mathbf{L}}_{M}$, related supra-matrices, and matrices that are associated with individual network layers. Another example of this category of process is
the spread of memes on social media. An example of the second category of dynamical process is interactions between multiple strains of a disease.

Excellent available software to both visualize and analyze multilayer networks includes MuxViz by Manlio De Domenico (http://muxviz.net; in R) and Pymnet by Mikko Kivelä (http://www.mkive7a.com/pymnet/; in Python).

The study of multilayer networks-including their structure, dynamical processes on them, and numerous applications-is among the most vibrant areas of network science. They offer a promising avenue both for further mathematical study and for numerous applications.

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## ABOUT THE AUTHORS

Mason Porter studies networks， complex systems，nonlinear sys－ tems，and their applications．He enjoys playing games of all sorts and is very fond of fantasy，base－ ball（Go Dodgers！），making witty re－ marks，and inserting music lyrics and other allusions into his scien－


Mason Porter


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## BOOK REVIEW

# Humanizing Mathematics and its Philosophy 

A Review by Joseph Auslander



Communicated by Cesar E. Silva

## Humanizing Mathematics and its Philosophy: Essays Celebrating the 90th Birthday of Reuben Hersh Bharath Sriraman, editor <br> Birkhäuser, 2017 <br> Hardcover, 363 pages <br> ISBN: 978-3-319-61230-0

Although he was not trained as a philosopher, Reuben Hersh has emerged as a major figure in the philosophy of mathematics. His Humanist view is probably the most cogent expression contrasting the dominant philosophies of Platonism and Formalism.

Hersh has had a remarkable career. He was born in the Bronx, to immigrant working class parents. He graduated from Harvard at age nineteen with a major in English and worked for some time as a journalist, followed by a period of working as a machinist. He then attended the Courant Institute from which he obtained a PhD in 1962, under the supervision of Peter Lax. Hersh settled into a conventional academic career as a professor at The University of New Mexico, pursuing research mostly in partial differential equations. But beginning in the 1970s he wrote a number of expository articles, in Scientific American, Mathematical Intelligencer, and Advances in Mathematics. Subsequently,

[^21]he wrote and coauthored with Philip Davis and Vera JohnSteiner several books and developed what he terms a Humanist philosophy of mathematics. He has also been a frequent book reviewer for Notices. ${ }^{1}$


Reuben Hersh with co-author Vera John-Steiner signing copies of their book Loving and Hating Mathematics: Challenging the Myths of Mathematical Life at the Princeton University Press Exhibit, JMM 2011.

This Humanist point of view is presented in several of his books, most notably in 1997's What Is Mathematics, Really? A (too) brief summary: Mathematics is part

[^22]of human culture "like literature, religion, and banking," mathematical knowledge isn't infallible, and there are different versions of proof and rigor. Moreover, a philosophy of mathematics should be guided by what mathematicians actually do.

The Humanist view is contrasted with Platonism and Formalism. Platonism is the position that mathematical objects and truths exist independently of individuals, and the obligation of mathematicians is to discover these truths. Formalism regards mathematics as the study of formal deductive systems, and mathematical truth is just provability in the system.

Bharath Sriraman has collected twenty-five essays from mathematicians and philosophers, an educator, and a linguist for a Festschrift volume to celebrate Hersh's ninetieth birthday. As one might expect, this is an extremely varied collection of articles. Several contain heartfelt appreciation of Hersh. Some attempt to develop and elucidate his Humanist philosophy. Others just mention him in passing or not at all. A few contain some nontrivial mathematics.

In the preface, Sriraman says that Hersh asked the contributors to speculate on the future of mathematics, mathematical education, and the philosophy of mathematics and in particular to address the prediction of Paul Cohen that "at some unspecified future time, mathematicians would be replaced by computers." ${ }^{2}$

The book starts out with a delightful interview with Hersh and a collection of photos from various periods of his life. This is followed by three short articles by Hersh, reprinted from other collections.
"Pluralism as modeling and confusion" begins by pointing out that "while in mathematics complete consensus is the norm," in the philosophy of mathematics it's the opposite. The practice of the philosophy of mathematics consists in "choosing a position and fighting for it." Hersh on the other hand advocates Pluralism, "peaceful coexistence" among different philosophies, "a radical new idea and...a great idea. It is a philosophy of the philosophy of mathematics."

He recalls his own well-known statement that "the typical mathematician is a Platonist on weekdays and a Formalist on Sundays."

Hersh justifies his position by comparing it with what occurs in mathematics itself: $L^{2}$ is a Hilbert space whereas $L^{p}$ for $p \neq 2$ is not, Euclidean geometry accepts the parallel postulate in contrast to non-Euclidean geometry, and different surfaces have different curvatures. (But this comparison is disingenuous. In the mathematical subjects just mentioned we are considering different mathematical objects, while the different philosophical positions are looking at the same object.)

Hersh's article "'Now' has an infinitesimal positive duration" tries to make sense of this word: "a time interval shorter than any...positive interval, yet longer than any infinitesimal." It includes a worthwhile historical survey, starting with Aristotle, continuing with Leibniz and the

[^23]19th century analysts, and culminating in Abraham Robinson's non-standard analysis.

A lot is packed into the Monthly review of David Tall's book How Humans Learn to Think Mathematically. Hersh's and Tall's similar views are interwoven. There is much discussion of "the math we teach in school," all the way from elementary school to graduate school. This in turn requires us to understand "mathematical reality," and also allows Hersh to recall another of his well-known theses


James A. Donaldson and Peter Lax, Hersh's PhD advisor, with Hersh and John-Steiner at the Princeton University Press Exhibit, JMM 2011.
that "mathematical entities are equivalence classes of mental models."

For me, one of the highlights of this volume is William Byer's article "Can you say what mathematics is?" It cogently elucidates Hersh's view of mathematics as a human creation but goes beyond it. I am tempted to fill the next several paragraphs with quotations from this article. He begins by posing Bill Thurston's question "What is mathematics?" but then switches to the "easier" question "What is number?" Not just real number, which is a definition in analysis, but "number" in general-a concept which may well change as we explore different ways of approaching it. On the other hand, it is not completely arbitrary. This leads to the question of whether mathematics is "objective." Well, it is and it isn't. Certainly "mathematicians no matter what their race, creed, gender, or culture...agree... about the sum of the angles of a plane triangle in Euclidean geometry." But it is nonobjective because "(human) mathematicians bring it into existence." Here, as elsewhere in the article, Byers emphasizes the notion of ambiguity (as he did brilliantly in his books How Mathematicians Think and The Blind Spot.)

Pursuing this question further in the next section "What is objectivity?" Byers distinguishes between "strong" and "weak" objectivity. Strong objectivity means it does not depend on mind. Weak objectivity means that it is "free from prejudice and arbitrary opinion but not independent of intelligence."

The next several sections are concerned with conceptual systems. While this is (necessarily) difficult to pin down precisely, it refers to "a mathematical structure like the real numbers or topological spaces looked at from
the inside." It is similar to the notion of a paradigm in the philosophy of science. (Somewhat surprisingly this is the only article in the volume that refers to the physicistphilosopher Thomas Kuhn.)

An example of two different but related conceptual systems are the counting numbers and the rational numbers. "Ask a child how many numbers there are between 2 and 3 and his or her answer will tell you which [Conceptual System] they are currently living in." This very good section would have been even better if it included more advanced mathematics.

A philosophy of mathematics is, in fact, a conceptual system. It follows that "asking whether a philosophy of math is right or wrong...is not a good question." The "incompatibilities between different philosophies" are "inevitable and...valuable." Clearly this is related to the view of "Pluralism" in the philosophy of mathematics considered elsewhere in the volume.

The penultimate section deals with artificial intelligence and computer-generated mathematics. Byers forthrightly rejects the myth that "the human being is a machine and mind is algorithmic," as well as the prediction of Paul Cohen that mathematicians will eventually be replaced by computers. The final section tries to answer Hersh's questions as to whether mathematicians can contribute to the philosophy of math (yes) and whether philosophers can say anything to practicing mathematicians (maybe).

Just to mention a couple of points of disagreement. Byers asserts that "what you find in a research article or a textbook is not mathematics in the same way that what you find in a musical score is not music." "The reader has to add context, meaning, and understanding... Something has to click in your own mind to make this potential mathematics into real mathematics." Of course a reader has to add context but, in fact, a research article is mathematics. The analogy with music doesn't hold, in that case we are awaiting a performance not something to click.

In the section on objectivity Byers says "real mathematics is impermanent" and "those who view mathematics as...unchangeable are doomed to disappointment." Sure. But then he asserts that "no one cares" about metrization theorems in topology. Experience shows that sometimes unfashionable fields reassert themselves and become important again.

I have not discussed everything in this beautifully written article. Anyone interested in the philosophy of mathematics will profit from reading and engaging with it.

One of Hersh's contributions, introduced in "What is mathematics, really?" and discussed by several of the authors, is the distinction between the "front" and "back" of mathematics. This was inspired by the sociologist Erving Goffman who distinguished between the front and back of a restaurant (the dining area is the front and the kitchen is the back). The front of mathematics is "mathematics in finished form, lectures, textbooks, and journals" whereas the back is "mathematics among working mathematicians...in offices or at cafe tables. Mainstream philosophy doesn't know that mathematics has a back."

The article by Delariviere and Van Kerkhove, "Artificial mathematician," deals with the prediction by Paul Cohen
about mathematicians being replaced by computers. It takes the form of a number of amusing dialogues, mostly between an ideal and an artificial mathematician (AM). Of course AM is a computer; the question is whether it can do mathematics. The first dialogue concludes with the statement by the ideal: "While humans are motivated by the meaning of mathematics, you are motivated by rule-following procedures without understanding what you're doing." The question of understanding ("easy to make, but hard to elucidate") plays an important role in this article. Parallel to this dialogue is one between an ideal and an artificial restaurant owner, taking off from Hersh's analogy of the front and back of mathematics with the front and back of a restaurant. Here a "chef's insight" is analogous to that of a mathematician (in contrast to a computer). There's even a funny comparison of the book by Pólya How to Solve It with an imagined "famous book" by "Bolya" How to Cook it.

There are also dialogues involving a Functionalist epistemologist and a subcognitive scientist. Some of these deal with the discovery process ("the ability to recognize a good thing when you stumble upon it").

There is much discussion of proof-more than navigating a formal system-and the need for "a procedure for deriving interesting theorems" (via interesting routes) in analogy with the "dirty aspects of the kitchen." From this one might expect the authors to dismiss the idea of computers doing mathematics. On the contrary, they consider the counterintuitive idea of informal computing and conclude by asserting "the possibility that computers could play a...meaningful role in mathematical practicenot just as a method of inquiry but as fellow inquirers, as artificial mathematicians."

In spite of its whimsical tone, this is a serious and insightful article, although not always easy reading.

Ian Stewart's article "Xenomath!" is exasperating. Stewart is one of our finest mathematical writers and his discussion of Hersh's Humanist point of view is outstanding. But he goes off the deep end in his extended consideration of alien mathematics. This would be fine for a science fiction story, but here I find it rather silly.

Here's something of Stewart's I do like: Hersh's "suggestion that mathematics is dependent on human conventions...appears to smack of relativism. [But] Reuben's position implies nothing of the kind. [Mathematics] is by no means arbitrary. Nothing new is incorporated into it unless it passes stringent reality checks...supported by proofs. ...However, Platonism is seductive, because that's what it feels like the vivid impression that the answer is already out there and we're just "discovering what it is.'"

Stewart points out that this is true if "out there" means "the correct consequences of whichever axiom system... we happen to prefer."

A major contribution is Stewart's pointing out "just how firmly [mathematics] rests on human perceptions and conventions." This includes the architecture of the human body: "our visual senses present the world to us as a twodimensional projection. [Our] coordinate system reflects our body plan...we stand upright...our arms extend sideways." We like dualities "perhaps because we're a bisexual
species." Mathematics "is a tangled tale of concepts being imported from the outside world, reworked by a human mind, and exported back."

Elena Marchisotto's ambitious article "A case study in Reuben Hersh's philosophy: Bézout's theorem" actually does a substantial amount of mathematics. Her aim is "to examine a piece of mathematics through a Humanist lens." This is accomplished by considering the statement and proofs of Bézout's theorem, "the precise number of points of intersection of two plane curves," and generalizations thereof, "a conversation through the centuries" (Euler, Bézout, Monge, Poncelet, culminating in Weil). A lot depends on nailing down the correct definitions.

All of this "gives an appreciation of mathematics as a process, during which progress is both impeded and stimulated by...collective consciousness" and "illustrates the social nature of mathematics and the avenues that emerge because of it." It is an illustration of Hersh's concept of the front and back of mathematics.

Marchisotto concludes by discussing the role of the computer in mathematics. Of course there is recognition of its importance. But "there are limits...It is not curious. It cannot follow hunches...It cannot replicate the social interaction that in Reuben's view is essential to the growth of mathematics."

The article also includes an appreciation and summary of Hersh's career. I think this article is spot on.

Carlo Cellucci's extensive and impressive article "Varieties of maverick philosophy of mathematics" pays tribute to Hersh ("a champion of the maverick philosophy of mathematics") and also expresses some differences with him. While the differences are certainly not trivial, I don't perceive them as very substantial (with possibly one exception, noted below). For example, Hersh characterizes mathematics as the subject where 'proof or disproof' brings unanimous agreement by all qualified experts" whereas Cellucci would just say "by the majority of qualified experts." Hersh asserts that the philosophy of mathematics is "the working philosophy of the professional mathematician...the researcher, teacher, or user of mathematics." Cellucci points out that this "working philosophy" varies "from period to period...from school to school...from mathematician to mathematician." Cellucci agrees with Hersh that mathematics has a front and a back but disagrees slightly about what the back actually is. Hersh says it's "mathematics as it appears...in informal settings...in an office behind closed doors" whereas for Cellucci it’s "the creative work...the discovery work."

Hersh and Cellucci agree that mathematics is not "about truth and certainty," basing this conclusion on Gödel's second incompleteness theorem. Rather it's about plausibility "compatible with existing knowledge,...the best we can achieve." Where they apparently differ concerns Hersh's preference for deductive proof while Cellucci supports analytic proof. The former consists of "deductive derivations from primitive premises "going down to the proposition to be proved." Analytic proofs are "nondeductive derivations from plausible hypotheses. Their aim is to discover plausible hypotheses capable of giving a solution to the problem,...both a method of discovery and a method of justification."

Cellucci asserts (perhaps unfairly) that if deductive proof is mathematicians's proof then "it is impossible to prove propositions that cannot be deduced from established mathematics," that "mathematicians can be replaced by computers completely," and that "all mathematical knowledge can ultimately be deduced from some elementary mathematical propositions such as $1+1=2$."

An example of analytic proof is provided by Ken Ribet's contribution to the solution of Fermat's last theorem. What Ribet showed is that the Taniyama-Shimura conjecture implies Fermat's last theorem. So it depended upon a "hypothesis" that had not yet been proved. Then Wiles and Taylor proved the Taniyama-Shimura conjecture, the solution depending on the axioms of set theory. Another example (not mentioned by Cellucci) is given by the proof of theorems on the assumption of the Riemann hypothesis.

There is much more to Cellucci's fine article: an extensive discussion of proofs using diagrams; a summary of the history of the philosophy of mathematics, going back to Plato and Aristotle; and an interesting brief discussion of "normal" and "revolutionary" mathematics, the latter requiring "hypotheses which cannot be deduced from established mathematics [and] open up new areas of mathematics."
"What is Mathematics and What Should It Be?" by Doron Zeilberger is infuriating and wrongheaded. He claims that Greek mathematics was a "major setback." This is followed by a lengthy section, "A Brief History of Mathematics as a sequence of (Unsuccessfully!) Trying to Answer Stupid Questions," including proving the parallel postulate, solving a quintic equation by radicals, developing a rigorous foundation for calculus, and devising an algorithm for the solutions of Diophantine equations. He says that today's mathematics is not a science, but a religion, because it depends on rigorous proofs. Mathematicians "do not care
about truth; they only care about playing their (artificial!) game." He says that mathematics can become a science by "taking full advantage of computers" and "should abandon the dichotomy between conjecture and theorem."

BUT, to give the devil his due, Zeilberger does raise some really interesting issues about infinity in his discussion of calculus and of Gödel's theorem. "We live in a finite and discrete world and the infinite and continuous are mere optical illusions." Statements about infinite sets are "a posteriori meaningless."

The linguist William Labov has a lovely and outstanding short article "The philosophy of Reuben Hersh: a nontechnical assessment." Labov and Hersh were both students at Harvard in the 1940s, where their connection was mostly political, and they maintained intermittent contact over the years. Labov summarizes some of his own research, while lamenting his insufficient mathematical training. In spite of their different scholarly areas, Labov notes some similarities in his and Hersh's research based on the work of the sociologist Emile Durkheim. He concludes by recalling a chance encounter with a nurse in Santa Fe, who remembered with gratitude Hersh's support of a nurses' strike.

There are two other articles which can be characterized as political. Michael Harris's "Do mathematicians have responsibilities?" has woven through it an eloquent appreciation of Hersh as a mathematician and a human being (although they apparently have never met). Harris discusses the misuse of mathematics, not only its military applications, but also "embodied artificial intelligence," treating "human beings as a means rather than an end." He refers to "the dominant ethos of Silicon Valley, where the sum total of human experience is treated as data to be mined for content." This is definitely related to Paul Cohen's vision of mathematicians being replaced by computers. Harris does note that some mathematicians have pushed back against this "instrumentalist" view, citing the debate in the Notices on the role of the NSA in the wake of the Snowden revelations. ${ }^{3}$ He concludes by quoting the economist Thomas Piketty about "the obsession with mathematics...acquiring the appearance of objectivity without having to answer the far more complex questions posed by the world we live in."

Chandler Davis's "Friends and Former Comrades" mentions several distinguished mathematicians, including Hersh and himself, who passed through the Communist movement (and I should mention that that was my first connection to Hersh almost seventy years ago). It includes a tribute to Lee Lorch for his uncompromising fight against racism in academia and our profession (which cost him several jobs), while at the same time distancing himself from Lee's continued devotion to the Soviet Union. While he is no longer a Communist, Chandler calls for "reaching out...when capitalism, self-immolating, threatens to take everything else with it."

Chandler has another fine article in the volume, "Can something just happen to be true?" It begins with his "feel-

[^24]ing that 5279 is prime is...accidental" and goes on to say that his "attitude towards mathematical assertions is at odds with [his] attitude toward the world of experience." He considers "the appearance of regularity" which sometimes guides one towards a correct proof, and at other times has led to "conjectures which turned out to fail." And an interesting discussion of the "dragon curve" and its relation to computer experiments versus rigorous proof.

An article by another Davis (Martin), "Gödel's legacy," is not explicitly related to Hersh's work, but it is a welcome contribution. It contains succinct discussions of algorithmic computability and determinacy, as well as the relation of the Zermelo-Fraenkel axioms to the continuum hypothesis.

Both of these eminent Davises are also in their nineties.
Three of the articles concern education, and they could not be more different from one another. Nel Noddings' "A gift to teachers" echoes Hersh's call for mathematics students to be broadly educated, "expanding the teaching of mathematics into vital interdisciplinary studies." Bonnie Gold's "School mathematics and 'real' mathematics" presents a number of interesting ideas for introducing non-rote problems into the K-12 curriculum. She feels that the algorithms (such as for long division) that dominate elementary and high school classes are not real mathematics (some of us might disagree). And Alexandre Borovik's ambitious and quirky "Mathematics for makers and mathematics for users" begins by posing the "difficult" question, "What is mathematics education, really?" The article is wide ranging, its "aim to start a discussion and pose more questions than to give answers." A couple of provocative statements: "Banks and insurance companies need a numerate workforce-but even more so they need innumerate customers"; "a learner of mathematics is a dog trainer." The latter is related to consideration of the subconscious and neurophysiology, which in turn concerns the relation between mathematics and language.

Jody Azzouni's "Does reason evolve? (Does the reasoning in mathematics evolve?)" is rich and thoughtful. It is the longest article in the volume, and not easily summarized. He begins by referring to Hersh's stressing "the great distance...between the reality of professional mathematical practice...and the reasoning in formal languages... that philosophers have largely characterized mathematical proof in terms of." The latter is called the "derivationist account." This certainly has advantages. "The existence of formal derivations explains...why mathematicians are so agreeable to one another"-comparatively speaking. "Apart from mistakes, there is no space for disagreement;...mathematical results...have eternal shelf lives."

But in the next section Azzouni recognizes problems with this position. "Not only have the standards of mathematical proofs mutated over the ages...but there have been heated disputes...over appropriate proof methods." "Reasoning itself...is a social phenomenon which historically changes with the passage of time."

In a sense combining these two approaches, Azzouni speaks of "informal rigorous mathematical proof, 'rigorous' in the sense that mathematicians are convinced that an effectively recognizable derivation...(in the ideological
neighborhood of the proof they've inspected) has been shown to exist." He also considers the possibility that "in the future robots will be checking our proofs," but then casts some doubt about whether this will actually occur.

There is an extensive discussion of baseball and other games. There are several sections on diagrammatic proofs, including infinite diagrams. An example is the construction of the Koch snowflake, which is a potentially infinite process. Another example is the "mutilated chessboard," the proof that if the two squares at each end of a diagonal are removed it is impossible to cover the remaining board with 31 dominoes. (Here Azzouni misses an opportunity to explain why introducing some additional structure, namely the coloring of the board, enables one to solve this problem.)
"Wittgenstein, mathematics, and the temporality of technique" by Paul Livingston is concerned with Wittgenstein's question of whether there is an occurrence of 777 in the decimal expansion of $\pi$. (It is now known that there is such, but one can ask the question about any sequence.) Are such questions even meaningful before they are answered? (Does "God" know the answer?) There is no mention of the unsolved problem of whether $\pi$ is a normal number, which of course would imply much more.

There are several articles, mostly by philosophers, that I just don't understand. It seems to me that these articles were not written for mathematicians. This is not meant as a criticism of the articles per se, but rather as an explanation of my inability to deal with them in depth. I hope some readers of this review will contribute a letter or article to the Notices elucidating them, especially Michele Friend's article "Mathematical theories as models," which develops Hersh's idea of Pluralism.


Hersh, in 2016, next to the Santa Fe Jewish Cemetery's remembrance wall.

The volume concludes with a special three-page contribution by Hersh "On the nature of mathematical entities," a neat summary of his position. The assertion that "accepted theorems are absolutely certain...is naive [mathematics] is a human artifact and...can never claim final perfection... There are three sides of mathematical entities-social, mental, and neural."

The final two sentences in the article (and therefore in the book): "These multiplicities are not logical contradictions. They are the different ways we know things-any kinds of things-including mathematical things, which are manifested as cultural items, as personal experience, and/or as currents in our flesh and blood."

Here is my own take on this. Maybe mathematics is not "absolutely certain," but it's surely more certain than social science, and even more certain than biology and physics. While there are occasional disagreements about the correctness of a purported result, such are regarded as anomalies, and if the claim is important, we scramble to resolve the disagreement.

A few final thoughts. As I mentioned at the beginning of this review, there is great variation among these articles, and there doesn't seem to have been a unifying theme. Among those who deal with Hersh's humanist philosophy there is the expected general agreement. It might have been good to have a dissenting view.

Also, it would have been appropriate to include a discussion of Hersh's mathematical research.

Finally, regarding the Paul Cohen prediction that mathematicians will be replaced by computers: Almost all of the authors who deal with it reject it. But one must ask, what does it mean? Look, nobody doubts that computers are becoming increasingly important in both pure and applied mathematics. Still, how are we to interpret such an assertion? Do we expect computers to survey the literature, make conjectures, and then prove them or find counterexamples? Or will a real live mathematician feed the question into the computer and expect it to churn out the answer? Or maybe (and I have heard this asserted) the entire "theorem, proof, counterexample" procedure will be abandoned.

Something the next collection of essays can consider.

## Photo Credits

Photo of Hersh at The Brookdale, Santa Fe by Neal Singer. Author photo courtesy of Barbara Meeker. All other photos courtesy of Reuben Hersh.

## ABOUT THE REVIEWER

Joseph Auslander's research is in topological dynamics, and he also has an interest in philosophy of mathematics.


Joseph Auslander

## AMS Mathematical Moments \& Mathematics of Planet Earth



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# Report on 2016-2017 Academic Recruitment, Hiring, and Attrition 

Amanda L. Golbeck, Thomas H. Barr, and Colleen A. Rose

Each year in academic mathematical sciences departments around the United States, new full-time faculty are recruited, and a subset of those positions are filled. The hiring infuses a new cohort of mathematical scientists actively engaged in research and teaching. At the same time, others retire, take jobs elsewhere, or die, and this process removes a segment of the population of mathematical scientists. This report provides a snapshot of that process to aid in understanding the current status of such variables as: hiring rates, gender distribution, position type, and prior experience. Along with current data the report provides historical context to aid the reader in discerning trends and patterns. For further details, including all tables generated to prepare this report, please see www.ams.org/annualsurvey.

A total of 955 mathematical sciences departments participated in this survey. This report is based on the completed questionnaires received from the 500 departments that reported recruiting to fill doctoral tenure-track and non-tenuretrack positions during the academic year 2016-2017 for employment beginning in the fall of 2017. An additional 7 departments ( 2 Stats, 2 Masters, and 3 Bachelors) reported conducting recruitment and hiring during this time but did not return a completed questionnaire. Those departments are not included in the analysis.

## Overview of Recruitment

The 2016-17 data shows that 1,999 positions were under recruitment (in 2015-16 this figure was 1,994). Most groups reported increases in recruitment, though Math Private Large ( $\downarrow 25 \%$ ), Applied Math ( $\downarrow 15 \%$ ), and Masters and Bachelors ( $\downarrow 6 \%$ ), reported decreases.

Figure R.1: Positions Under Recruitment in Mathematical Sciences by Tenure Eligibility


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Figure R.2: Positions Under Recruitment in Mathematics Departments by Highest Mathematical Sciences Degree Offered


During the 2016-17 academic year, the estimated number of full-time positions under recruitment in mathematical sciences departments was 1,999 . This figure breaks down as follows: 841 tenure-track mathematics positions, 927 non-tenure-track mathematics positions, 135 tenure-track statistics or biostatistics positions, and 96 non-tenuretrack statistics or biostatistics positions. See Figure R. 1 for comparisons. In the period from 2012 to 2017, the overall percentage of positions under recruitment that were tenure-track ranged from $48 \%$ to $53 \%$, with the highest percentages in 2012-2013 and 2013-2014 of this range of time.

- Overall features in the 2016-2017 cycle:
- The estimated number of positions under recruitment was 1,999; this figure represents a slight increase from last year's estimate of 1,994 positions.
- Women account for 31\% of those hired; down from 32\% in 2015-16.
- Since 2010 recruitment has increased 65\% in all Mathematical Sciences, increased 63\% in Math, and increased 75\% in Stats.
- Tenure-track positions under recruitment:
- Open tenure-track positions increased 3\% overall from 2015-16.
- $49 \%$ (976) of all positions under recruitment were tenure-track. Of these 976 positions, $85 \%$ (831) were open to new PhDs, and $21 \%$ (201) were at the rank of associate/full professor.
- Non-tenure-track positions under recruitment:
- Non-tenure-track positions decreased $2 \%$ overall, to 1,023 from 1,042 the previous year.
- $51 \%(1,023)$ of all positions under recruitment were non-tenure-track.

In Math the number of positions under recruitment $(1,768)$ in 2016-2017 is comparable with the previous year $(1,774)$. See Figure R.2. Over the period since 2006-07 recruitment in Doctoral departments has increased by 13\%, in Masters departments decreased by $14 \%$, and in Bachelors departments decreased by $10 \%$. In the same ten-year period, the net number of mathematics positions under recruitment has decreased by $1 \%$.

In Stats, the number of positions under recruitment in 2016-2017 was 231 a $5 \%$ increase over 2015-16. Since 2012-13 positions under recruitment has fluctuated between 220 and 253.

## Positions Filled

A total of 1,697 full-time positions in Mathematical Sciences were filled during the 2016-17 academic cycle, 1,526 from Mathematics Departments and 171 from Statistics or Biostatistics. Figure F. 1 gives a breakdown. The total for Math is up $59 \%$ from the 2009-10 cycle. For Stats, the number of filled positions is up 30\% from 2009-10. One interesting feature implicit in these data is that the success rate for filling mathematical sciences tenure-track positions over the period $2012-17$ is about $81 \%$, whereas the success rate for non-tenure-track is about $95 \%$.

Figure F. 2 gives a breakdown on hiring by gender and department grouping. Percentages generally are obtained by comparison with Figure R1. Here are further highlights and comparisons from the data:

Figure F.1: Positions Filled in Mathematical Sciences


- Overall features of hires in mathematical sciences:
- Women hold $31 \%$ (530) of positions filled.
- Of all hires, $45 \%$ (763) were tenure-track; women constitute $32 \%$ (241) of these.
- Of all hires, $55 \%$ (934) were non-tenure track; women constitute $31 \%$ (289) of these.
- Math and Stats breakdown:
- In Math overall, 1,526 of 1,768 positions ( $86 \%$ ) were filled; $31 \%$ of Math positions were filled by women.
- In Stats, 171 of 231 positions ( $74 \%$ ) were filled; $34 \%$ of Stats positions were filled by women.
- Tenure-track hires in mathematical sciences:
- Of the tenure-track positions under recruitment, $78 \%$ (763) were filled.
- Of tenure-track positions filled, $70 \%$ (531) were filled by doctoral faculty (i.e., not new PhDs). Of these positions filled by doctoral faculty, $27 \%$ went to women. In comparison with last year, all groups reported increases in tenure-track hires of doctoral faculty except Applied Math ( $\downarrow 24 \%$ ), Statistics ( $\downarrow 4 \%$ ), Masters $(\downarrow 9 \%)$, and Bachelors ( $\downarrow 13 \%$ ).
- Of the $30 \%$ of tenure-track hires who were new PhDs, $43 \%$ were women.
- Of tenure-track hires, $15 \%$ (114) had a non-tenure-track position last year; of these individuals, $23 \%$ were women.
- Of tenure-track hires, $35 \%$ (267) held a postdoc last year, and $25 \%$ of these postdocs were women.
- Non-tenure-track hires
- Of the 1,023 non-tenure-track positions under recruitment, $91 \%$ were filled. In comparison to last year, all groups reported decreased hiring of non-tenure-track faculty except Math Public Small ( $\uparrow 26 \%$ ), Math Private Large ( $\uparrow 9 \%$ ), and Masters ( $\uparrow 22 \%$ ).
- Of non-tenure-track hires, $39 \%$ (360) were filled by doctoral faculty (excluding new PhDs); $29 \%$ of these doctoral faculty hires were women.
- Of non-tenure-track hires, $49 \%$ (461) were filled by new PhDs ; $25 \%$ of these new PhD hires were women.

Figure F.2: Gender of Tenure-track and Non-tenure-track Hires by Department Grouping


## ANNUAL SURVEY

- Non-tenure-track hires (continuted)
- Of non-tenure-track hires, $12 \%$ (113) were filled by non-doctoral faculty; $61 \%$ of these non-doctoral hires were women. Fourty-two percent of these non-doctoral, non-tenure-track hires were in Masters departments.
- Of non-tenure-track hires, $24 \%$ (224) are temporary (one-year); $32 \%$ of these temporary hires are women. About half of all temporary hires were in Bachelors departments.
- Of non-tenure-track hires, $41 \%$ (382) were in postdoctoral positions; $23 \%$ of these postdocs were women.
- Women hires (see Figure F.2):
- Of all hires, $31 \%$ (530) were women; of these women, Bachelors departments hired $38 \%$, and Doctoral Math departments hired 38\%.
- In the Doctoral Math Group, women hires increased by $6 \%$ to 200.
- All groups reported increases in the number of women hires over last year except Math Private Small ( $\downarrow 40 \%$ ), Applied Math ( $\downarrow 30 \%$ ), Statistics ( $\downarrow 34 \%$ ), and Bachelors ( $\downarrow 14 \%$ ) .
- The number of women hired into tenure-track positions increased slightly to 241 from 238; the number hired into non-tenure-track positions increased by $12 \%$ to 289.
- Women accounted for 32\% of all tenure-track and 31\% of all non-tenure track hires; in 2015-2016 these percentages were, respectively, $31 \%$ and $26 \%$.


## Faculty Attrition

Figure A. 1 shows the variation in attrition from deaths and retirements among full-time faculty for the academic years 2012-13 through 2016-17. On average over the period shown, the percentage of faculty in doctoral departments retiring or dying each year is about $1.9 \%$, and in Masters and Bachelors departments that percentage is about $2.7 \%$.

 divided by the number of full-time faculty at the start of the academic year.

During the same period, in the respective groups, the percentages of tenured faculty who retired averaged $3.3 \%$ for Doctoral Math departments, $4.7 \%$ for Bachelors and Masters, and $5.8 \%$ for Stats. As reported in previous years, departments continue to report the majority of those retiring as members of the tenured faculty. For instance, for 2015-2017 approximately, $82 \%, 83 \%$, and $82 \%$ of these faculty have retired.

Here are a few other highlights from the attrition data from the 2016-2017 cycle:

- Overall retirements by tenured faculty increased by 9\% to 469
- Deaths and retirements increased by 6\% to 599
- Overall deaths and retirements break down by departmental grouping as follows:
- $44 \%$ (263) were from Bachelor
- $29 \%$ (175) were from Doctoral Math
- $16 \%$ (97) were from Masters
- $11 \%$ (64) were from Stats


## Department Grouping Response Rates

In this report, Mathematical Sciences departments are those in four-year institutions in the US that refer to themselves with a name that incorporates (with a few exceptions) "Mathematics" or "Statistics" in some form. For instance, the term includes, but is not limited to, departments of "Mathematics," "Mathematical Sciences," "Mathematics and Statistics," "Mathematics and Computer Science," "Applied Mathematics," "Statistics," and "Biostatistics." Also, Mathematics (Math) refers to departments that (with exceptions) have "mathematics" in the name; Stats refers to departments that incorporate (again, with exceptions) "statistics" or "biostatistics" in the name but do not use "mathematics." The streamlining of language here militates against the possible objection to foreshortening the full subject names.

Starting with reports on the 2012 AMS-ASA-IMS-MAASIAM Annual Survey of the Mathematical Sciences, the Joint Data Committee implemented a new method for grouping doctorate-granting Mathematics departments. These departments are first grouped into those at public institutions and those at private institutions. These groups are further subdivided based on the size of their doctoral program as reflected in the average annual number of PhDs awarded between 2000 and 2010, based on their reports to the Annual Survey during that period.

For further details on the change in the doctoral department groupings, see the article in the October 2012 issue of Notices of the AMS at www.ams.org/journa7s/ notices/201209/rtx120901262p.pdf.

Math Public Large consists of departments with the highest annual rate of production of PhDs , ranging between 7.0 and 24.2 per year. Math Public Medium consists of departments with an annual rate of production of PhDs, ranging between 3.9 and 6.9 per year.
Math Public Small consists of departments with an annual rate of production of PhDs of 3.8 or less per year.
Math Private Large consists of departments with an annual rate of production of PhDs, ranging between 3.9 and 19.8 per year.
Math Private Small consists of departments with an annual rate of production of PhDs of 3.8 or less per year.
Applied Math consists of doctoral-degree-granting applied mathematics departments.
Statistics consists of doctoral-degree-granting statistics departments.
Biostatistics consists of doctoral-degree-granting biostatistics departments.
Masters contains US departments granting a master's degree as the highest graduate degree.
Bachelors contains US departments granting a baccalaureate degree only.
Doctoral Math contains all US math public, math private, and applied math mathematics departments granting a PhD as the highest graduate degree.
Mathematics (Math) contains all Math Public, Math Private, Applied Math, Masters, and Bachelors Groups above.
Stats consists of all doctoral-degree-granting statistics and biostatistics departments.
Listings of the actual departments that compose these groups are available on the AMS website at www. ams.org/ annual-survey/groups.

Response Rates by Survey Groups
Faculty Recruitment \& Hiring Response Rates*

| Group | Received (\%) |
| :--- | :---: |
| Math Public Large | 24 of 26 with 23 recruiting (92\%) |
| Math Public Medium | 36 of 40 with 35 recruiting (90\%) |
| Math Public Small | 61 of 68 with 43 recruiting (90\%) |
| Math Private Large | 21 of 24 with 19 recruiting (88\%) |
| Math Private Small | 23 of 28 with 17 recruiting (82\%) |
| Applied Math | 18 of 23 with 14 recruiting (78\%) |
| Statistics | 45 of 59 with 32 recruiting (76\%) |
| Biostatistics | 34 of 45 with 24 recruiting (76\%) |
| Masters | 119 of 174 with 65 recruiting (68\%) |
| Bachelors | 574 of 1020 with 228 recruiting (56\%) |
| Total | 955 of 1507 with 500 recruiting (63\%) |

* Doctoral programs that do not formally "house" faculty and their salaries are excluded from this survey.


## Other Information

The interested reader may view additional details on the results of this survey and prior year trends by visiting the AMS website at www. ams.org/annual-survey.

## Acknowledgments

The Annual Survey attempts to provide an accurate appraisal and analysis of various aspects of the academic mathematical sciences scene for the use and benefit of the community and for filling the information needs of the professional organizations. Every year, college and university departments in the United States are invited to respond. The Annual Survey relies heavily on the conscientious efforts of the dedicated staff members of these departments for the quality of its information. On behalf of the Joint Data Committee and the Annual Survey Staff, we thank the many secretarial and administrative staff members in the mathematical sciences departments for their cooperation and assistance in responding to the survey questionnaires. Comments or suggestions regarding this Survey Report may be emailed to the committee at ams-survey@ams.org.

## KELLY MCKINNIE

Associate Professor, Department of Mathematical Sciences, University of Montana, Missoula. AMS member since 2000.
"I visited my congressional representative on Capitol Hill as a member of the AMS Committee on Science Policy in March 2015. It was a very valuable experience in which I met other mathematicians interested in science policy, learned in greater detail how the mathematical sciences are funded in the US, and
 met with the congressional representative from my district (the whole state of Montana) and from other districts as well. One of the most valuable things that came from the visit was holding a meeting with my department when I returned, giving them information on the state of funding from Congress for the mathematical sciences."

## 2018 ICM AWARDS

ICM prize citations with biographical sketches by Elaine Kehoe

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# Kashiwara Awarded Chern Medal 

MASAKI Kashiwara of Kyoto University was awarded the 2018 Chern Medal at the 2018 International Congress of Mathematicians in Rio de Janeiro, Brazil, "for his outstanding and foundational contributions to algebraic analysis and representation theory sustained over a period of almost fifty years."

## The work of Masaki Kashiwara



Masaki Kashiwara's mark on current mathematics is exceptional. It extends from microlocal analysis, representation theory, and combinatorics to homological algebra, algebraic geometry, symplectic geometry, and integrable systems. Most notable are his decisive contributions to theory of $D$-modules and his creation of crystal bases, which have shaped modern representation theory.
Over a span of almost fifty years he has established the theory and applications of algebraic analysis. Introduced by Sato around 1960, algebraic analysis is a framework in which systems of linear differential equations are formulated as modules over a ring $D$ of differential operators and are analyzed with algebraic means such as rings, modules, sheaves, and categories. Sato's idea of $D$-modules was greatly developed by Kashiwara in his 1971 thesis and has become a fundamental tool in many branches of mathematics. With Sato and Kawai, he developed microlocal analysis, the study of partial differential equations, locally combining space and Fourier variables, and working on the cotangent bundle of the base manifold. In the 1980s with Schapira he further introduced and developed microlocal sheaf theory.

One of his early major results was his 1980 construction of the Riemann-Hilbert correspondence, a far-reaching generalization of Hilbert's twenty-first problem about the existence of a linear differential equation on the projective line with prescribed monodromy. Greatly generalizing work of Deligne, Kashiwara (and later, independently, Mebkhout) established an equivalence between the derived category of regular holonomic $D$-modules on a complex algebraic variety ("systems of linear differential equations") and the derived category of constructible sheaves on the same variety ("solutions"), thereby creating a fundamental bridge between algebra and topology.

The Riemann-Hilbert correspondence found a remarkable application to a problem in representation theory: the Kazhdan-Lusztig conjecture, proved independently by Brylinski-Kashiwara and Beilinson-Bernstein. This beautiful synthesis of algebra, analysis, and geometry may be viewed as a precursor to geometric representation theory in its modern form. With Tanisaki, Kashiwara generalized the conjecture yet further to infinite dimensional KacMoody Lie algebras, a critical ingredient for the completion of Lusztig's program in positive characteristic.

Kashiwara's 1990 discovery of crystal bases is another landmark in representation theory. Quantum groups are deformations of the enveloping algebras of Kac-Moody Lie algebras originating from certain lattice models in statistical mechanics. Here the deformation parameter $q$ is temperature, with $q=0$ corresponding to absolute zero. Crystal bases are bases of representations of quantum groups at $q=0$ which encode essential information about the representations. Kashiwara proved that for all $q$, crystal bases lift uniquely to global bases, which turned out to coincide with the canonical bases discovered earlier by Lusztig for the $q=0$ and generic $q$ cases. Being a powerful combinatorial tool, crystal bases have had great impact with many applications, including solving the classical problem of decomposing tensor products of irreducible representations.

Kashiwara's work, with many different coauthors, continues to be groundbreaking. He has been at the forefront of recent developments on crystal bases, including the categorification of representations of quantized enveloping algebras. Other highlights include the solution of two well-known open problems about $D$-modules: the "codimension 3 conjecture" (2014) and the extension of the Riemann-Hilbert correspondence to the irregular case (2016). His work on sheaf quantization of Hamiltonian isotopies (2012) opened the way to applications of microlocal sheaf theory to symplectic geometry.

Kashiwara has many other results too numerous to mention. For example, he obtained major results on representations of real Lie groups, and on infinite-dimensional Lie algebras, in particular in connection with integrable systems and the $K P$ hierarchy.

Kashiwara's influence extends far beyond his published work. Many of his informal talks have initiated important subjects, and his ideas have been a source of inspiration for many people. Several of his books have become essential references, his book on sheaves with Schapira being regarded as the bible of the subject. He has served as di-
rector of the Research Institute for Mathematical Sciences (RIMS, Kyoto) and was vice president of the International Mathematical Union from 2003 to 2006.

Kashiwara's work stands out in depth, breadth, technical brilliance, and extraordinary originality. It is impossible to imagine either algebraic analysis or representation theory without his contributions.

## Biographical Sketch

Masaki Kashiwara was born in Yuki, Ibaraki, Japan in 1947 and received his PhD from Kyoto University in 1974 under the direction of Mikio Sato. He was associate professor at Nagoya University (1974-1977). He became associate professor at RIMS, Kyoto University, in 1978 and full professor in 1984. Since 2010 he has been emeritus professor at Kyoto and project professor at RIMS, Kyoto. His awards and honors include the Iyanaga Prize (1981), the Asahi Prize (1988), the Japan Academy Prize (1988), the Fujihara Award (2008), and the Kyoto Prize in Basic Sciences for Outstanding Contributions to a Broad Spectrum of Modern Mathematics (2018). He is a member of the French Academy of Sciences and of the Japan Academy.

## Photo credit

Photo of Masaki Kashiwara courtesy of Inamori Foundation.


## Donoho Awarded Gauss Prize

David L. Donoho of Stanford University was awarded the Gauss Prize at the 2018 International Congress of Mathematicians in Rio de Janeiro, Brazil, for "his fundamental contributions to the mathematical, statistical, and computational analysis of important problems in signal processing."

## Citation



David Donoho
Large amounts of data are today generated at an increasingly accelerated pace. Processing it by sampling, compression and denoising has become an essential undertaking.

David Donoho has throughout his remarkable research career helped us make sense of data, which often is in the form of signals and images. His research transcends boundaries between mathematics, statistics, and data science. The contributions range from deep mathematical and statistical theories to efficient computational algorithms and their applications.

Already in his early research Donoho was reaching outside of the mainstream of classical applied mathematics and statistics. He understood the importance of sparse representation and optimization in signal processing. He also recognized the power of wavelet type representations for a variety of tasks in signal and image analysis. One example is his work with Johnstone, where they exploit sparsity in wavelet representations together with soft thresholding for enhanced signal estimation and denoising.

The development, analysis and application of curvelets by Donoho and collaborators introduced another powerful tool in sparse image representation. While wavelets can be seen as a generalization of delta functions and Fourier expansions by using a basis that represents both location and frequency, curvelets go further by adding localization in orientation. Much of what wavelets do for one-dimensional signals, curvelets can do for multidimensional data. Efficient representation and processing of images with edges are natural and successful applications.

Compressed sensing is a technique for efficiently sampling and reconstructing a signal by exploiting sparsity in an incoherent representation and thus beating the classical limit on the required sampling rate imposed by the Nyquist-Shannon sampling theorem. This technique has,
for example, been applied to shorten magnetic resonance imaging scanning sessions. Donoho and collaborators have contributed to develop and refine this powerful theory. He showed that one could solve some types of underdetermined linear systems via $L_{1}$-minimization provided that the solution is sufficiently sparse. He derived the existence of sharp transitions for the recovery of sparse signals from special kinds of random measurements.

David Donoho stands out in his ability to bring together pure and applied aspects of mathematics and statistics. He has had a fundamental influence by his original research, and also by his writing and mentoring of students and postdocs.

## Biographical Sketch

David Donoho received his PhD from Princeton University in 1983 under the direction of Peter J. Huber. From 1984 to 1990 he served on the faculty at the University of California Berkeley. He has been at Stanford since 1990. He is a Fellow of the AMS. His honors include a MacArthur Fellowship (1991), the John von Neumann Prize of the Society for Industrial and Applied Mathematics (SIAM; 2001), the SIAM-AMS Norbert Wiener Prize in Applied Mathematics (2010), and the Shaw Prize (2013). He is a Fellow of SIAM and of the American Academy of Arts and Sciences, a foreign associate of the French Académie des Sciences, and a member of the US National Academy of Sciences.

## Photo Credit

Photo of David L. Donoho by Jamie Lozano.

## Nesin Awarded Leelavati Prize

Ali Nesin was awarded the 2018 Leelavati Prize at the 2018 International Congress of Mathematicians in Rio de Janeiro, Brazil, "for his outstanding contributions towards increasing public awareness of mathematics in Turkey, in particular for his tireless work in creating the 'Mathematical Village' as an exceptional, peaceful place for education, research and the exploration of mathematics for anyone." The Leelavati Prize, sponsored by Infosys, recognizes outstanding contributions for increasing public awareness of mathematics as an intellectual discipline and the crucial role it plays in diverse human endeavors.

## The work of Ali Nesin



Ali Nesin

Ali Nesin was born in Istanbul in 1957, and he began his early academic career as a distinguished research mathematician. He graduated in France at Université Paris VII, earned his PhD at Yale University in 1985, and took several temporary positions at prestigious universities in the United States.

On the death of his father, the renowned writer Aziz Nesin, in 1995, he made a conscious decision and returned to his native Turkey. In place of pursuing his academic career in the United States, he decided that it was desirable and worthwhile to contribute to the development of mathematics in Turkey. He pursued his father's work as the director of the Nesin Foundation, a nonprofit institution devoted to providing educational opportunities for children who did not have them, and he created the Nesin Mathematics Village, which is now the main activity of the foundation.

Since his return to Turkey, he has devoted himself to developing the appreciation of mathematics as an element of modern Turkish culture. His activities towards the awareness of mathematics in Turkish society are numerous:

- He was the founder and editor-in-chief of Matematik Dünyast (The World of Mathematics), a magazine for the popularization of mathematics in Turkish, mainly addressed to high school students.
- He is a member of the editorial committee of NTV-Bilim, a Turkish popular science journal.
- He authored nine popular mathematics books (e.g., Mathematics and Games, Mathematics and Nature, Mathematics and Infinity, Mathematics and Truth)
and many popular mathematical articles in several periodicals.
- Several lecture notes and monographs by Ali Nesin received awards and became recommended material by the Turkish Academy of Sciences.
His books, popular journals, editorial work, articles, public lectures, and Internet activities had an enormous influence on enabling a new generation of Turkish mathematicians to flourish.

However, what makes the work of Nesin unique and goes beyond all envisaged activities for the Leelavati Prize is his creation, organization, and development of the Mathematics Village, despite difficult economic and political conditions. The Nesin Mathematics Village is a physical site south of Izmir, near the ancient Greek town of Ephesus, entirely devoted to the mathematical enrichment of gifted students at all levels, from elementary school through graduate level, as well as constituting a major venue for international conferences. It is perhaps most appropriately characterized by Gizem Karaali in The Mathematical Intelligencer as constituting "a possibly unique experiment in building a mathematical community in one of the most unexpected places on earth."

Ali Nesin's work in mathematics education and popularization of mathematics has and continues to have a truly transformative effect on Turkish culture with respect to the profile and appreciation of mathematics.

## Biographical Sketch

Ali Nesin was educated all over the world: high school in Switzerland, the Université Paris VII Jussieu, and Yale University. After receiving his PhD in mathematics from Yale, he taught at the University of California Berkeley (19851986), University of Notre Dame (1987-1988), and the University of California Irvine (1988-1996). He also spent time as a visitor at the Mathematical Sciences Research Institute (MSRI) at Berkeley (1989), as a visiting professor at Bilkent University (1993-1994), and as a visitor at the University of Lyon I (2001-2002). He has been professor at Istanbul Bilgi University since 1996. From 1996 to 2013 he served as chair of the mathematics department, and he reassumed that role in 2018. Since 1995 he has been director of the Nesin Foundation. In 2005 he founded the Nesin Publishing House to publish the books of his father, the writer Aziz Nesin. He has been director of the Corporation of the Turkish Mathematical Society (2007-2009) and was a member of the administrative committee of

## COMMUNICATION

the Turkish Mathematical Society (2007-2009). Among his nonmathematical works are a four-volume publication of correspondence of Aziz and Ali Nesin, a biography of Aziz Nesin, and a number of children's books translated from French. He is also a painter who has held a number of exhibitions.

## Photo Credit

Photo of Ali Nesin courtesy of Ali Nesin.


# Daskalakis Awarded Nevanlinna Prize 

Constantinos Daskalakis of the Massachusetts Institute of Technology was awarded the Nevanlinna Prize at the 2018 International Congress of Mathematicians in Rio de Janeiro, Brazil, "for transforming our understanding of the computational complexity of fundamental problems in markets, auctions, equilibria, and other economic structures. His work provides both efficient algorithms and limits on what can be performed efficiently in these domains."

## Citation



Constantinos Daskalakis

Constantinos Daskalakis has developed a powerful body of results that resolve the computational complexity of some of the central problems in economic theory. These results have also been key to an important emerging theme in computer science, in which algorithms are designed for agents who behave according to their own self-interest.

Daskalakis's early work (with Goldberg and Papadimitriou) showed that finding Nash equilibria is complete for the complexity class PPAD; this implies that it is computationally equivalent in a precise sense to finding Brouwer fixed points, thus establishing a natural converse to Nash's famous construction of equilibria using fixed-point theorems. This computational difficulty in finding the Nash equilibrium in general raises important questions about the role of Nash equilibria as predicted outcomes when players interact in a fashion modeled by a game.

Daskalakis has also made important contributions to the theory of mechanism design, creating algorithms for use by selfish agents. The prior state of the art in this area (including Myerson's Nobel-Prize-winning work on optimal auctions) had been restricted largely to problems where participants' objectives are one-dimensional, described by a single parameter for each participant. Daskalakis (with Cai and Weinberg) provided a general method for transforming multidimensional mechanism design problems to pure problems in algorithm design, in the process developing mathematical tools that he used (with Deckelbaum and Tzamos) to make significant progress on characterizing optimal multidimensional mechanisms, an area that has been open for several decades. The underlying technique relies on a new and powerful geometric interpretation
of a collection of large implicitly defined optimization problems at the heart of the transformation.

In addition to his work on issues at the interface of computational and economic theory, Daskalakis has produced work contributing to an impressive diversity of further areas in the theory of computation. This includes his resolution (with Mossel and Roch) of a fundamental conjecture in the reconstruction of evolutionary trees and his recent approaches to challenges in the theory of machine learning, including (with Tzamos and Zampetakis) new global guarantees for the well-known expectation-maximization heuristic from statistics for mixtures of two Gaussians with known covariances.

## Biographical Sketch

Constantinos Daskalakis was born in Athens, Greece, in 1981. He received his PhD in electrical engineering and computer science from the University of California Berkeley in 2008. His awards and honors include the 2007 Microsoft Graduate Research Fellowship, the 2008 ACM Doctoral Dissertation Award, the 2008 Kalai Game Theory and Computer Science Prize from the Game Theory Society, the 2010 Sloan Fellowship in Computer Science, the 2011 SIAM Outstanding Paper Prize, the 2011 Ruth and Joel Spira Award for Distinguished Teaching, the 2012 Microsoft Research Faculty Fellowship, the 2015 Research and Development Award of the Vatican Giuseppe Sciacca Foundation, the 2017 Google Faculty Research Award, and the 2018 Simons Investigator Award.

## Photo Credit

Photo of Constantinos Daskalakis is in the public domain.

# Fostering Inclusive Communities: Reasons why YOU should organizea Mathematics Research Community 

Katharine A. Ott

We are all accustomed to seeing some variation of the following statement on a conference announcement: "Graduate students, postdocs, and persons from underrepresented groups are especially encouraged to apply." Most of us have fallen into at least one of these categories, whether for a short period of time or in perpetuity. We've been especially encouraged to apply, but have we always felt especially encouraged to belong? I do not mean to say that the sentiment is ill-intentioned, but I think we can agree that this one sentence is not enough to make a meaningful change in the inclusivity of mathematics. I write today to put the spotlight on a program that is helping early career mathematicians establish a sense of belonging through fostering research communities: the Mathematics Research Communities (MRC) program of the AMS.

In existence since 2008, the MRC program serves those who have recently finished their doctorates, as well as graduate students who are close to finishing. It has over 1,300 alumni spread over 40 research cohorts. I am one of those alumni, and a past co-organizer of a program, so I can speak to particular ways the MRC has helped me to feel a sense of belonging. As a participant in 2009, I met a cohort of harmonic analysts who would be my "conference squad" from that day forward. What a relief to see not just familiar faces at a conference, but also to be welcomed by a group of colleagues who you know and trust. As a co-organizer in 2014 I similarly established lasting connections with mathematicians. These colleagues are not so early-career anymore, which has opened even more avenues for us to network: I have been invited to speak in their special sessions, and they have become resources for talking about post-early career issues such as family

[^25]and maternity leave, tenure dossiers, and re-entering the job market.

It is the signature of the MRC program that lasting communities are forged during the week-long summer conferences and sustained by collaborations and conference participation throughout the following year. As Rebecca Everett, participant in the 2018 program Agent-Based Modeling in Biological and Social Systems writes, "I think the camp-like atmosphere of the MRC contributed to the community feeling. There is something about being in the woods sitting around a fire that helps you bond with those around you. Especially when there are s'mores!" While the impact of s'mores should not be discounted, there is another reason why MRC's are so successful at creating communities of early-career researchers: the organizers.

Organizers are instrumental to the MRC program and, in particular, to creating the inclusive communities discussed above. They bring to the table the mathematical content of the program. The very act of sharing one's research problems with early career researchers is an incredibly welcoming and inclusive action. Organizers also provide invaluable professional development (generally through panel discussions). Some of the topics covered in recent programs are how to choose a journal, how to referee a paper, and how to apply for external funding. These aspects of academic life are all too often left unmentioned in graduate programs, which can lead to new researchers feeling unprepared, and in turn, unwelcome.

In short, organizing an MRC is very impactful work, and it is work that you can (and should) do! If you are unfamiliar with the MRC program, more details including the application process are at www.ams.org/programs/ research-communities/mrd. If the arguments presented above have not yet convinced you to organize an MRC program, read on to hear some common excuses and a rebuttal to those excuses.

## Some reasons NOT to organize an MRC, and a counterargument

"I'm not senior enough." Each MRC program (there are typically three or four per year) is organized by a group of 3-5 mathematicians. A successful formula is to have a variety of experience within the organizing team. Junior mathematicians can have key contributions on several fronts, including professional development. Professional development is a main component of the MRC model. Formally, the professional development often takes the form of one or two panel discussions in the evenings during the program. Mathematicians at the level of Assistant Professor are invaluable to these conversations because they are so close to the process of finishing the dissertation, applying for postdocs and jobs, and navigating the early career years. Combining the on-the-ground experience of junior researchers with that of senior organizers, who likely have experience on hiring committees, refereeing journals, and evaluating grant proposals, is a fantastic combination. From a research standpoint, the process of planning the mathematical content of an MRC can elucidate research questions and give you a wider perspective on your research field, all while recruiting new talent to work on problems in the field.
"I don't work with graduate students, or I'm not at a PhD granting institution." The MRC would like to see representation from a wide variety of positions in academia and even outside of academia. Mathematicians working at research institutions, regional universities, and small colleges are encouraged to apply. Indeed, an important role of the organizing team is to prepare junior researchers to enter the profession. We are all well aware of how few tenure track positions are available at research institutions, as well as the fact that many participants will have alternative career paths in mind.

For organizers who do not regularly interact with postdocs or graduate students, the MRC is a unique way to connect with these groups. For anyone at an undergraduate institution like myself, the MRC could be thought of as a REGP (Research Experience with Graduate Students and Postdocs). Other benefits that are attractive to organizers are support to travel to the Joint Mathematics Meetings the year following the MRC, as well access to additional travel money for small gatherings of MRC alumni to further research projects ("micro-conferences").
"My research area isn't suited for this format." Researchers in any area of pure and applied mathematics are welcome to apply to the MRC program. Worried that your field requires too much background knowledge? Lillian Pierce, member of the organizing team of the 2018 program Harmonic Analysis: New Developments on Oscillatory Integrals writes, "We were initially skeptical of how the MRC format would work on a highly technical topic-could participants really grapple with the problems without having a long series of lectures beforehand? But in fact the MRC was very successful, and has stuck with me as an image of a good way to get young researchers going in challenging territory."
"I don't have the time or energy." The AMS provides a great deal of support for organizing an MRC. They will take care of advertising, the online application process, reimbursements, and all of the day-to-day operations (think coffee breaks, technology set-up, photocopying, etc.) That means that organizers are able to focus on the mathematics of the program. Now, it is true that this aspect of organization is intense. "It is a lot of work," explains David Penneys, a participant in 2014 and organizer of the 2018 program Quantum Symmetries: Subfactors and Fusion Categories, "You have to really think about the problems that you're going to do before you get there. You have to have a few really well-thought out projects in your back pocket." But, isn't this the kind of work that as mathematicians we like to do?
"There are plenty of conferences in my field already. How is this any different?" There are three obvious ways in which an MRC program differs from a typical research conference: talks are generally limited to a few (or none), all participants are early career mathematicians, and there is a significant professional development component. If your aim is to disseminate an overview of the latest progress in your field to other experts, a conventional conference may be the way to go. If you are interested in training junior researchers, and if you can imagine the benefits in the long term of fostering a healthy research community in your field of expertise, then an MRC program is the perfect venue.

If this article has piqued your interest, there is plenty of time to gather a group of organizers and prepare to submit a proposal for summer 2021. The proposal deadline for organizers will be in August 2019. The MRC Advisory Board and AMS staff will offer suggestions in advance of the proposal deadline for those who send inquiries. Proposals for the 2020 Mathematics Research Communities are currently under review, and the 2019 programs are accepting applications for participants until February 15, 2019.

## Photo Credit

Author headshot courtesy of Katharine A. Ott.


Katharine A. Ott

## Invitation for 2019 MRC Applications:

The AMS invites applicants for the 2019 MRCs. The eligible group includes those who are within two years of completing a PhD to those who've earned a PhD within the past five years. Find more information about each conference and how to apply on the MRC 2019 webpage: www. ams.org/programs/research-communities/ mrc . The application deadline is February 15, 2019.


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The October Contest Winner Is...
Mason A. Porter, who receives our book award.


Professor Coffee was shocked by the realization that sitting in the new chairs changed the homology of the students.

Driving a tractor is "great for thinking."—Rick Schoen, who grew up on a farm as one of thirteen children.

Nobody Wants Mathematician Parents


There are online an atlas, library, and blog of Irish mathematics and mathematicians:
see www.mathsireland. ie.
Submitted by Colm Mulcahy.

Cartoon by Zach Weinersmith of smbc-comics.com, who has a new book out at sooni shbook. com.

## Mathematics People

## *NSF Division of Mathematical Sciences (DMS) New Rotating Program Directors

Yuliya Gorb, University of Houston, applied and computational mathematics, with particular emphasis on multiscale analysis and simulations.

Pamela Gorkin, Bucknell University, functional analysis, operator theory, linear algebra, and complex analysis.

Pawel Hitczenko, Drexel University, probability theory and applications, particularly in combinatorics, discrete mathematics, and the analysis of algorithms.

Michelle Manes, University of Hawai`i at Mānoa, arithmetic dynamics and arithmetic geometry.

Krishnan (Ravi) Shankar, University of Oklahoma, Norman, Riemannian geometry and topological data analysis.

Janet Striuli, Fairfield University, commutative algebra, homological algebra, and representation theory.

Branislav (Brani) Vidakovic, Georgia Institute of Technology, Bayesian statistics, statistical modeling in wavelet domains, biostatistics, statistical analysis of signals/images, geoscientific and biomedical statistical applications.

Huixia (Judy) Wang, George Washington University, quantile regression, semiparametric regression, extreme value theory, high dimensional inference, and spatial data analysis.
-From letter dated August, 23, 2018, by Juan C. Meza, Director, NSF DMS.


[^26]
## Kra Awarded Noether Lectureship



Bryna Kra

Bryna Kra of Northwestern University has been selected to deliver the 2019 Noether Lecture by the Association for Women in Mathematics (AWM) and the AMS. She was honored for her "profound impact on mathematics, both through her work in the fields of dynamical systems and ergodic theory and through her service to the profession."

The prize citation reads in part: "Kra is best known for her fundamental contributions to ergodic theory. Her 2005 joint paper with Bernard Host titled 'Nonconventional Ergodic Averages and Nilmanifolds' (Annals of Mathematics) settled a long-standing open problem on the existence of the limit of certain multiple ergodic averages, uncovering the role of nilpotent groups and their homogeneous spaces in analyzing configurations in sets of integers. The work inspired many further developments, including structure theorems in ergodic theory, in topological dynamics, and in combinatorics, convergence results for numerous multiple ergodic averages, and the uncovering of recurrence phenomena that imply the existence of patterns in sufficiently large sets of integers. In further work joint with Vitaly Bergelson and Host, they introduce the notion of a nilsequence and use it to provide further structural results in dynamics. It has been adapted to the combinatorial setting, playing an important role in studying patterns in smaller subsets of the integers, for example the set of primes. Continuing her work at the intersection of dynamics and combinatorics, Kra's more recent research lies in topological and symbolic dynamics, studying systems of low complexity. In joint work with Van Cyr, she has the strongest work to date on Nivat's Conjecture, relating a global property of periodicity of a two-dimensional configuration to a locally checkable property on the complexity."

Kra received her PhD from Stanford University under the direction of Yitzhak Katznelson. She is a Fellow of the AMS and of the American Academy of Arts and Sciences. She has been the recipient of an AMS Centennial Fellowship and the Levi L. Conant Prize. She has given many invited lectures, including the AMS Arnold Ross Lecture. She is on the AMS Board of Trustees and has been a member of the Council and Executive Committee of the AMS and the Executive Committee of the AWM, among other service positions. She is an editor of the Bulletin of the AMS, as well as Ergodic Theory and Dynamical Systems, Discrete and Continuous Dynamical Systems, and Discrete Analysis.
-From an AWM announcement

## Prizes of the Mathematical Society of Japan

The Mathematical Society of Japan awarded a number of prizes for the fall of 2018.

Hirofumi Osada of Kyushu Uni-


Hirofumi Osada versity has been awarded the Autumn Prize for his outstanding contributions to studies on stochastic dynamics of infinite particle systems with long range interaction and its rigidity. The Spring Prize and the Autumn Prize are the most prestigious prizes awarded by the MSJ to its members. The Autumn Prize is awarded without age restriction to people who have made exceptional contributions in their fields of research.

The Analysis Prizes were awarded to the following: Shuichi Kawashima of Waseda University for work on the stability analysis of systems of nonlinear partial differential equations with dissipative structure; to NORIO KonNo of Yokohama National University for work on the mathematics of quantum walks and its applications; to AKIHIKO MiYachi of Tokyo Woman’s Christian University for study of Hardy spaces and boundedness for Fourier multiplier operators and pseudodifferential operators.

The Geometry Prizes were awarded to Shouhei Honda of Tohoku University for work on geometric analysis on convergence of Riemannian manifolds and to Yuji Odaka of Kyoto University for study on $K$-stability and moduli theory.

The Takebe Katahiro Prizes were awarded to the following: YOHEI FUJISHIMA of Shizuoka University for research on blow-up sets of solutions for the semilinear heat equation; to JOHANNES JAERISCH of Shimane University for research on ergodic theory and its intensive applications to various fields; to MASAYA MAEDA of Chiba University for work on the asymptotic stability of solitary waves for nonlinear Schrödinger equations; and to Kiwamu Watanabe of Saitama University for studies on the Campana-Peternell conjecture on Fano manifolds with nef tangent bundle.

The Takebe Katahiro Prizes for Encouragement of Young Researchers were awarded to the following: HIRAKU Atobe of the University of Tokyo for automorphic representations and related local and global theta correspondences; to Takayuki Koike of Osaka City University for work on function theory on a neighborhood of a complex submanifold and its application to geometry; to SHUTA NAKAJIMA of Kyoto University for research on first passage percolation; to Yusuke NaKamura of the University of Tokyo for studies of minimal log discrepancy and the minimal model theory over a finite field; to Genki Omori of Saitama University for work on the group structure of the mapping class group of a surface and its subgroups; and to JIn TAKAHASHI of the Tokyo Institute of Technology for work on moving singularities for parabolic equations.
-From an MSJ announcement

## Lucia Awarded Rubio de Francia Prize



Angelo Lucia

Angelo Lucia of the California Institute of Technology has been awarded the 2017 José Luis Rubio Prize for important results in the mathematical aspects of quantum mechanics systems. In particular, his work focuses on studying mathematical models of systems composed of a large number of particles, in which their behavior responds to the rules of quantum mechanics. Lucia tells the Notices: "I am a big fan of trains, especially long distance train trips. I have sometimes shocked people by arriving at conferences after a 24 -hour train ride." The prize is awarded by Royal Spanish Mathematical Society (RSMS) and is intended to recognize young Spanish researchers or researchers who have done their work in Spain.
-From an RSMC announcement

## Mohar Receives <br> RSC Synge Award



Bojan Mohar

Bojan Mohar of Simon Fraser University has been awarded the John L. Synge Award of the Royal Society of Canada for his work in graph theory. The prize citation reads: "Bojan Mohar is a Slovenian-Canadian mathematician who holds a Canada Research Chair position at Simon Fraser University. He is one of the world leaders in graph theory and is well known for his solutions of open problems and conjectures. Interplay of combinatorics, geometry, topology and algebra is visible in most of his work. His deep and transformative results in topological and structural graph theory made a lasting impact not only in topological graph theory but also in theoretical computing and other fields." Mohar received his PhD from the University of Ljubljana in 1986. Since 2005, he has been a Canada Research Chair in Graph Theory at Simon Fraser University in Burnaby, British Columbia. He also holds his office at the Institute of Mathematics, Physics and Mechanics in his native Slovenia, where he returns regularly. The award is given at irregular intervals for outstanding research in any branch of the mathematical sciences.

Previous winners of the Synge Award are:

- James G. Arthur (1987)
- Israel M. Sigal (1993)
- Joel Feldman (1996)
- George A. Elliott (1999)
- Stephen Cook (2006)
- Henri Darmon (2008)
- Bálint Virág (2014)
-From an RSC announcement


## Prizes of the Canadian Mathematical Society

The Canadian Mathematical Society (CMS) has announced a number of awards for 2018.


Thomas Hutchcroft

Thomas Hutcheroft of Cambridge University has been awarded the CMS Doctoral Prize. The citation reads in part: "Together with Asaf Nachmias, Hutchcroft has made remarkable progress in the study of uniform spanning trees on unimodular and planar graphs, answering several open questions raised in a celebrated paper by Benjamini, Lyons, Peres and Schramm. In a solo paper, Dr. Hutchcroft proved that critical percolation almost surely has only finite clusters on all transitive graphs of exponential growth. One of the central open problems in percolation is to prove this property for any transitive graph of at least quadratic growth, and Dr. Hutchcroft's work is an important step in this direction. In his research, Hutchcroft often uses tools from different branches of mathematics, including complex analysis, differential geometry and topology. For example, his paper with Omer Angel, Asaf Nachmias, and Gourab Ray combined hyperbolic triangulations, circle packings, random walks and mass transport in an ingenious way." Hutchcroft received his PhD in 2017 from the University of British Columbia. The Doctoral Prize recognizes outstanding performance by a doctoral student from a Canadian university.


Maksym
Radziwill

MAKSYM RADZIWILL of McGill University received the Coxeter-James Prize for his work in analytic number theory "focusing on the distribution of prime numbers, multiplicative functions and related objects," including work with Matomäki that has had wide-ranging influence, for instance, on the resolution of the Erdős discrepancy problem by Terence Tao, and the first progress on Chowla's conjecture. The prize recognizes young mathematicians who have made outstanding contributions to mathematical research.


Keith Taylor

Keith Taylor of Dalhousie University was honored with the 2018 Graham Wright Award for Distingished Service. According to the prize citation, Taylor "has truly exemplified what this award represents, not just because of his excellent record of research and mentorship, but also through his academic work as associate dean, dean and associate vice president at two universities and through years of fundamental service to the CMS, including a term as president (2012-2014)." His outreach work focuses on "developing pathways to mathematical literacy for underrepresented groups in Saskatchewan and Nova Scotia, and consistently supporting and championing disadvantaged communities." He has been the recipient of the Master Teacher Award from the University of Saskatchewan (2001), where he also championed the Math Readiness Project aimed at bridging the gap between high school and college/university mathematics, especially for students in remote areas. In 1996, Taylor was awarded the President's Educational Site Award for the MRC web course at the University of Saskatchewan. The following year, he received the Student Union Teaching Excellence Award. Taylor received his PhD from the University of Alberta in 1976 under the direction of Anthony T.-M. Lau. Taylor tells the Notices: "I grew up on a farm and the life I have had in mathematics would have been totally unexpected by my teenage self. I remain astonished that one can earn a good living doing something that is this much fun. When taking a break from mathematics, I might be found on a golf course, in a bridge club, at the symphony, or playing with grandchildren."


Anthony To-Ming Lau of the University of Alberta received the David Borwein Distinguished Career Award for his "exceptional, continued, and broad contributions to mathematics, from research central, to the development of abstract harmonic analysis in Canada and internationally, to teaching of such a high calibre that it has been recognized with a 3M Teaching Award, and award-winning service." Lau received his PhD degree from the University of British Columbia in 1969 and has been affiliated with the University of Alberta since that year. His first PhD student was Keith Taylor, recipient of the 2018 Graham Wright Award. Lau served as department chair of Mathematical and Statistical Sciences at the University of Alberta, as well as CMS President (2008-2010). He has served on editorial boards of more than ten journals, including the Canadian Journal of Mathematics and the Canadian Mathematical Bulletin. Lau tells the Notices that one of his childhood teachers helped him to come to the United States to study: "That was probably the best fortune I ever had," he says. "After I finished Grade 11, she made arrangements for me to go to the U.S., to San Jose

City College." He worked as a houseboy and at odd jobs to pay for his education. Lau's wife, Alice, is a librarian, and his son and daughter are both engineering students.


Patrick Ingram

Patrick Ingram of York University and Anastasia Stavrova of St. Petersburg State University have received G. de B. Robinson Awards for outstanding papers. Ingram was honored for his paper "Rigidity and Height Bounds for Certain Post-Critically Finite Endomorphisms of PN," (Canadian Journal of Mathematics 68 (2016), no. 3, 625-654)-according to the prize citation, "the first published work describing the arithmetic of post-critically finite self-maps for higher dimensional spaces." Stavrova was honored for her paper "Non-stable $K_{1}$-functors for Multiloop Groups," (Canadian Journal of Mathematics 68 (2016), no. 1, 150-178). According to the prize citation, her paper is a "fundamental contribution to group theory and Lie theory, which provides a deep understanding of the automorphism groups of multiloop Lie algebras in higher nullity." The Robinson Awards are given for outstanding contributions to the Canadian Journal of Mathematics or the Canadian Mathematical Bulletin.

The Centre for Education in Mathematics and Computing (CEMC) at the University of Waterloo is the recipient of the 2018 Adrien Pouliot Award. The citation describes it as "one of Canada's largest outreach organizations in mathematics and computer science. The focus of the center is to increase interest, enjoyment, confidence, and ability in mathematics and computer science among learners and educators in Canada and internationally." The Pouliot Award is given for "significant and sustained contributions to mathematics education in Canada."
-From CMS announcements

## 2018 Davidson Fellows

Several high school students whose projects involved the mathematical sciences have been named 2018 Davidson Fellows.


Franklyn Wang


David Wu
Scholarships worth US $\$ 25,000$ were awarded to Franklyn WANG of Falls Church, Virginia, for his project "Monodromy Groups of Indecomposable Rational Functions" and to David Wu of Potomac, Maryland, for his project "Nonuniform Distributions of Patterns of Sequences of Primes in Prime Moduli." Wang enjoys watching the New England Patriots and listening to Taylor Swift's music. Wu reports that he recently took an ice skating class: "Considering I was slightly traumatized by breaking my wrist the first time I went ice skating, it went surprisingly smoothly. I can definitely see myself following my little sister's footsteps-she's actually good at ice skating."

A scholarship worth US $\$ 10,000$ was awarded to Grant Sheen of Irvine, California, for his project "An Alternating Minimization Method to Train Neural Network Models for Brainwave Classification." Sheen says: "Taking care of my grandmother during her final stages of Alzheimer's disease inspired me to search for a solution to the communication issues that she suffered from. After researching electroencephalography (EEG), I spent the following years analyzing wireless EEG brainwave data of Alzheimer's subjects, with the goal to make thought recognition a reality. I am an active volunteer at my local senior center. I am also a nationally-ranked foil fencer and have achieved a ' $B$ ' rating."

The following students received honorable mentions: Brian Huang, Fresh Meadows, New York, for "On Sufficient Conditions for Trapped Surfaces in Spherically Symmetric Spacetimes" and Michael Ma, Plano, Texas, for "New Results on Pattern-Replacement Equivalences: Generalizing a Classical Theorem and Revising a Modern Conjecture."

The Davidson Fellows program, a project of the Davidson Institute for Talent Development, awards scholarships to students eighteen years of age or younger who have created significant projects that have the potential to benefit society in the fields of science, technology, mathematics, literature, music, and philosophy.

[^27]
## Royal Society of Canada Elections

The Royal Society of Canada (RSC) has elected two mathematical scientists to its 2018 class of new fellows in the Division of Mathematical and Physical Sciences. They are Uri Ascher of the University of British Columbia and Richard Lockhart of Simon Fraser University.
-From an RSC announcement

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Photos of Franklyn Wang and David Wu courtesy of Davidson Institute for Talent Development.

## Inside the AMS

## AMS Department Chairs Workshop

The annual workshop for department chairs will be held a day before the start of the 2019 Joint Mathematics Meetings in Baltimore, Maryland, on Tuesday, January 15, 2019, from 8:00 am to 6:30 pm at the Marriott Inner Harbor hotel.

Workshop leaders will be Malcolm Adams, former head, Department of Mathematics, University of Georgia; Gloria Marí-Beffa, Associate Dean for the Natural, Physical and Mathematical Sciences, University of Wisconsin-Madison; Douglas Mupasiri, head, Department of Mathematics, University of Northern Iowa; and Jennifer Zhao, chair, Department of Mathematics and Statistics, University of Michigan-Dearborn.

What makes a chair different from any other engaged faculty member in the department? This workshop examines the chair's role in leading a department. The day will be structured to include and encourage networking and sharing of ideas among participants and will include four sessions:

1. Reassessing the relationship between pure and applied mathematics.
2. Math in the data movement.
3. Professional development and evaluating faculty.
4. The next step: moving to higher administration.

The workshop registration fee of US\$200 is in addition to and separate from the Joint Meetings registration. Those interested in attending should register at http://bit. 7y/2MHngPu by December 19, 2018. For further information, please contact the AMS at 202-5881100 or alb@ams.org.

- Anita L. Benjamin

AMS Office of Government Relations

## From the AMS Public Awareness Office

## Celebrating Women Mathematicians: Researchers and Role Models Poster



Notices March 2018 guest editors Margaret A. Readdy and Christine Taylor (https://www.ams. org/women-18) spotlight women mathematicians past and present as a way to commemorate Women's History Month in this newly available poster from the AMS.

This poster features some of the women profiled by Readdy and Taylor in the March issue with their words of advice to young mathematicians, and it is free upon request via the AMS Posters web page at https://www.ams.org/posters. Copies will also be available in the AMS exhibit at the 2019 Joint Mathematics Meetings in Baltimore.

## Making an Impact:

## Transitioning from Math PhD to Journalist

During my last year of graduate school, I was struggling with how I could do "good" in the world-what would my impact be? It was 2016. I had just become pregnant with my daughter when my son turned two, and I felt a furious drive to keep them safe and make the world a better place for them. It took me two long meandering years to find my answer via the AMS-sponsored AAAS Mass Media Fellowship.

I considered political activism, tutoring, management consulting, data science, and project management. I met with the university's career counselor, who was specifically assigned to grad students and postdocs looking for a career outside of academia, and with my alma mater's career services. I read What Color is Your Parachute? and completed the accompanying workbook. I did informational interviews. I wrote blog posts. I felt like I was nowhere.

And then I was just sad and struggling to finish my thesis before the baby was born. I didn't apply for math jobs that fall since I didn't love research or teaching enough to go the uncertain postdoc, tenure-track route of uprooting my family.

I stayed home with the full-time gig? As my toddler would now say, that year wasn't my favorite.


Yen Duong 2018 AMS-AAAS Mass Media Fellow Many people are excellent stay-at-home parents, but also many people play team sports, and it is okay that I do not fit into either of those groups. I decided to stop feeling bad about things I'm not good at and instead continue looking for things I am good at.

Throughout grad school I had attended plenty of alternate career panels at excellent mathematical and professional development conferences, such as the Topology Student Workshop, Underrepresented Students in Topology and Algebra Research Symposium, the regional Women in Mathematics Symposia, and the Women and Mathematics program at the Institute for Advanced Study. During my year as a stay-at-home parent I applied for a data science fellowship, curriculum/tutoring jobs, e-learning stuff, and a research coordinator position. While I wasn't chipper at the time, I am now ecstatic about my failure to land any of those. That meant I could do the Mass Media Fellowship and find the right path for me to make an impact with my particular skills and experiences.

Before I even applied for the fellowship, I emailed the program director to ask about how I, as a mother of a toddler and a baby, could do the ten-week program, in which fellows are assigned to 23 sites scattered across the country from NPR in Washington, DC, to KQED in Sacramento and the St. Louis Post-Dispatch. As it turned out, the director had also had a child during her PhD and very much wanted to support working parents. I ended up applying on the condition that I only wanted to work at Raleigh's The News \& Observer. I took the three-hour train or drive home every weeken on how the bids, prepare dinners, and give my spouse a chance to sleep in (or sleep at all, depending on how the baby was handling her teething that week).

If the fellowship didn't offer so many sites or didn't explicitly include a note on the application about parents of young children, I wouldn't have done it. I wasn't going to leave my baby for a seventh of her life. I was still breastfeeding when the fellowship began, and my site had an easily-accessible lactation room near a sink and paper towels. Though my last location had a lactation room, it required me to check out a key from a library for a two-hour period at a time. That meant schlepping to the library, back up to the lactation room, down to the library, and back up to my office three times a day. I spent most of that summer working at home instead of in the office for convenience, but that probably meant I got less math done. For more on supporting working mothers in our community, please read the July 2018 issue of the online Journal of Humanistic Mathematics.

I appreciated the support of AAAS and happily drank the Kool-Aid at orientation in DC on how the expert witnesses for the Scopes Monkey trial. I was thirsty for answers on how to do good in the world, and I'm happy I stumbled into the cult of science communication. Here I could combine my matical
skill set of reading and teaching myself from dense, jargon-heavy text with my communication passions, which had until now been relegated to my grad school blog, Baking and Math.

On my first day in Raleigh, I went through a corporate orientation with a gaggle of 20-year-old journalism majors, then was shown to my desk. My editor told me to settle in for a few minutes until he could wrap up a few stories. Suddenly the entire newsroom was yelling as the police had just released dash cam video and cell phone video of police officers assaulting an unarmed black man with their flashlights and a police dog. Several reporters started transcribing, others were on the phone with community leaders and the police, and the editors met to figure out a plan to responsibly roll out the news with analysis and commentary. That was the day that journalism grabbed me, and it hasn't let go.

I contacted museums, research institutions, and universities to get started, and by the end of the day, thanks to-, I had a lead on an upcoming art exhibit done in collaboration with a bunch of scientists. That first news story was straightforward and I only had to learn a little bit about gravitational waves to include some science. The next story involved a deep dive into CDC data and a statistics-heavy paper about heroin.

That was the first of many abstruse, dense research papers I read over this summer. I actually found them easier to read than the papers from my thesis research. My teaching background also prepared me for adapting arguments and creating interesting analogies and ways to explain different ideas to different audiences. It was so fun to spend several hours learn to name a few. My mathematical experiences gave me confipaleontology, genetics, and climatology, just to namen't intimidated by protomammals, polygenic scores, dence in my ability to teach myself and words have definitions and I can grasp definitions; the real story, or phenology because I know that all worke terms interact and why we care about them. I would puzzle
 through studies and background research until I was ready for and delighted by my knowledge. Of the 25 stories I wrote this summer, ten whe favorite stories of the summer was a dive into peanut front page of Durham's The Herald-Sur form, for which I interviewed biopharmaceutical companies, allergies and upcoming treatments form on using polio to treat brain cancer was also a medical researchers, parents, and a six hital stories too.

From the grandparents who wanted the photo we ran of a child marveling at the gravitational lensing simulation in that first art exhibit to the caretaking spouse who asked for a copy of the brain cancer research paper, people wanted science news and they wanted to know what's happening in science research that can affect their lives. The more people like science, the more they'll support science. This is a way for me to make the impact I've been searching for-I help persuade people to believe in "real news" and facts and evidence-based reasoning.

Some time ago I talked with Evelyn Lamb, a past AMS-sponsored Mass Media Fellow, about the guilt of leaving academia and not setting an example for women mathematicians. Evelyn, now a well-known math writer, pointed out that she might be doing more good by spreading knowledge and awareness of mathematics through her stories. I'm so grateful to the AMS and AAAS for giving me the opportunity to do the same. Math will always be part of me, and I will always spread my love of it. Thanks to the AMS, I can now do that in a way that better matches my strengths and vision of what I want my life to look like. 'alternative careers' at various meetings. I've accepted a part-time job with the and time with freelance news organization North Carolina Health News Communicators of North Carolina group for freelance science journalism, connecting with the Scinking and My Favorite Theorem podcasts, and had two speakleads. I've appeared on the excellent PhDre one at a state university. None of this would have happened ing gigs this fall, one at a high school ane Fellowship. I cannot imagine what my life would have looked without the support of the AMS and the Feling off point for the rest of my career doing what I love. like without this fellowship-it is the -Yen Duong

## For Your Information

## Departments Coordinate Job Offer Deadlines

For the past nineteen years, the American Mathematical Society has led the effort to gain broad endorsement for the following proposal:

That mathematics departments and institutes agree not to require a response prior to a certain date (usually around February 1 of a given year) to an offer of a postdoctoral position that begins in the fall of that year.
This proposal is linked to an agreement made by the National Science Foundation (NSF) that the recipients of the NSF Mathematical Sciences Postdoctoral Fellowships would be notified of their awards, at the latest, by the end of January.

This agreement ensures that our young colleagues entering the postdoctoral job market have as much information as possible about their options before making a decision. It also allows departmental hiring committees adequate time to review application files and make informed decisions. From our perspective, this agreement has worked well and has made the process more orderly. There have been very few negative comments. Last year, more than 180 mathematical sciences departments and institutes endorsed the agreement.

Therefore we propose that mathematics departments again collectively enter into the same agreement for the upcoming cycle of recruiting, with the deadline set for Monday, January 28, 2019. The NSF's Division of Mathematical Sciences has already agreed that it will complete its review of applications by January 18, 2019, at the latest, and that all applicants will be notified electronically at that time.

The American Mathematical Society facilitated the process by sending an email message to all doctoral-granting mathematics and applied mathematics departments and mathematics institutes. The list of departments and institutes endorsing this agreement was widely announced on the AMS website beginning November 1, 2018, and is updated weekly until mid-January.

We ask that you view this year's formal agreement at www.ams.org/employment/postdoc-offers.htm1 along with this year's list of adhering departments.
lmportant: To streamline this year's process for all involved, we ask that you notify T. Christine Stevens at postdoc-deadline@ams.org if and only if:
(1) your department is not listed and you would like to be listed as part of the agreement;
or
(2) your department is listed and you would like to withdraw from the agreement and be removed from the list.

Please feel free to email us with questions and concerns. Thank you for consideration of the proposal.

-Catherine A. Roberts<br>AMS Executive Director

T. Christine Stevens

AMS Associate Executive Director

# Mathematics Opportunities 

## Listings for upcoming math opportunities to appear in Notices may be submitted to notices@ams.org

## AAAS-AMS Mass Media Fellowships

With placements at national print media such as National Geographic, Wired, Scientific American, The Washington Post, and The Los Angeles Times, national radio and TV such as NPR, CNN, and NOVA, and local newspapers and radio and TV stations, the AAAS Mass Media Fellowship is the premier experience for mathematicians who want to try out science communication.

During the ten-week fellowship, scientists, mathematicians, and engineers are placed in newsrooms around the country to report on today's headlines in the fast-paced world of science journalism.

Fellows observe and participate in the process by which events and ideas become news, improve their communication skills by describing complex technical subjects in a manner understandable to the public, and increase their understanding of editorial decision making and how information is effectively disseminated. They go through a short boot camp of journalism training at the beginning of summer, keep in touch with each other throughout the summer, and meet again at the end to wrap up the Fellowship.

After the Fellowship, alumni participate in science communication in a variety of career fields including science journalism, advocacy, education, and community outreach. In the forty-four-year program history, the more than 700 fellows have become renowned leaders in the scientific community, award-winning TV and documentary producers, authors, producers, and leaders in cutting-edge science communication.

The AMS sponsors one Fellowship placement each year. This year's Fellow, Yen Duong, finished her PhD in geometric group theory at the University of Illinois at Chicago in fall 2017. Over her summer at the Raleigh News and Observer, Yen published more than twenty articles explaining science to the public. From a computer science paper (which included sigma algebras) to psychology papers which used statistical distributions and models to animal science stories, her topics varied in terms of mathematical content but held one thing in common: a respect for and foundation in evidence and logical thinking.

Read about the rest of the 2018 Fellowship class at www. aaas.org/page/2018-mass-media-fe11ows. See more information about the AMS Mass Media fellows at Www. ams.org/programs/ams-fe11owships/media-fe11ow/ massmediafellow.

Applications may be filed at either URL for the AMSAAAS 2019 Mass Media Fellowship during the application period, October 16, 2018-January 15, 2019.
-Rebekah Corlew, PhD
Project Director, Public Engagement, AAAS

## National Defense Science and Engineering Graduate Fellowships

To help increase the number of U.S. citizens trained in disciplines of military importance in science and engineering, the Department of Defense awards National Defense Science and Engineering Graduate Fellowships to individuals who have demonstrated ability and special aptitude for advanced training in science and engineering. The fellowships are awarded for a period of up to three years for study and research leading to doctoral degrees in any of fifteen scientific disciplines. Application forms are available online at www.ndsegfe11owships.org/ application and are due December 7, 2018.
-From an NDSEG announcement

## CRM Intensive <br> Research Programs

The Centre de Recerca Matemàtica (CRM) will hold three Intensive Research Programs in the spring and summer of 2019. An intensive research program in geometry, algebra, and topology (LIGAT) consisting of an Advanced Course in Geometry, Topology, and Algebra (May 27-31, 2019) and a Workshop in Geometry: Multiple Perspectives on Geometric Inequalities (June 3-7, 2019). A program on

## NEWS

Interactions between Representation Theory, Algebraic Topology, and Commutative Algebra consists of an advanced course (June 3-7, 2019) and a conference (June 10-14, 2019). A program on the Anthropological Theory of the Didactic (June 3-July 26, 2019) consists of four advanced courses covering dialogues between theories, teacher education, the curriculum problem, and university education. Seewww.crm.cat/en/Host/SciEvents/IRP/ Pages/CurrentIRP.aspx.
-From a CRM announcement

## STaR Fellowship Program

The Service, Teaching, and Research (STaR) Program of the Association of Mathematics Teacher Educators (AMTE) supports the development of early-career mathematics educators, including their induction into the professional community of university-based teacher educators and researchers in mathematics education. Senior and mid-career mathematics education faculty organize and facilitate STaR events, serving as mentors to Fellows. Applications are due December 1, 2018; see www. amte. net/ star/app7y.
-From an AMTE announcement

## MSRI Summer Research for Women in Mathematics

The Mathematical Sciences Research Institute (MSRI) provides space and funds to groups of women mathematicians to work on research projects at MSRI. Research projects can arise from work initiated at a women's conference or can be freestanding activities. Visits must take place between June 10, 2019, and August 2, 2019. The deadline for applications is December 1, 2018. For more details, seewww.msri.org/web/msri/scientific/ summer-research-for-women-in-mathematics.
-From an MSRI announcement

## Mathuobs,Org



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American Mathematical Society
201 Charles Street
Providence, RI 02904-2294 USA 800.321.4267, ext. 4105 | Email: mathjobs@ams.org

# Classified Advertisements 

## Positions available, items for sale, services available, and more

## MAINE

## Colby College

## Tenure Track Assistant Professor of Mathematics

Tenure-track assistant professor position beginning September 1, 2019. The department of mathematics and statistics seeks an exceptional teacher with a dynamic research program in combinatorics, graph theory, or a closely related area of discrete mathematics. The teaching responsibility is an average of 4.5 courses per year. Candidates must have a PhD in mathematics and must have at least one year of full-time teaching and/or research experience distinct from the PhD program. Applications must include evidence of potential for a strong continuing research program and exceptional teaching and mentoring of undergraduates with diverse backgrounds and demographics.

For a full description of the position, please go to www.colby edu/math.

## MASSACHUSETTS

## Northeastern University College of Science, Department of Mathematics and the College of Computer and Information Science Joint Tenured/Tenure-Track Position, Open Level

The College of Computer and Information Science and the Department of Mathematics, in the College of Science, at Northeastern University invite applications for an open tenure-track/ tenured faculty position at all levels, in Mathematics and Data Science, beginning in Fall 2019.
Northeastern University seeks to build excellence in the area of Mathematics of Data in the Life Sciences. The rationale is to expand the University's existing strength in related areas of the College of Science, and the College of Computer and Information Science, and address the rapidly growing need for the development of new mathematical techniques to process the immense trove of scientific data which have become available in the Life Sciences.

Appointments will be based on exceptional research contributions at the interface between Mathematics and Computer Science

Suggested uses for classified advertising are positions available, books or lecture notes for sale, books being sought, exchange or rental of houses, and typing services. The publisher reserves the right to reject any advertising not in keeping with the publication's standards. Acceptance shall not be construed as approval of the accuracy or the legality of any advertising.
The 2018 rate is $\$ 3.50$ per word with a minimum two-line headline. No discounts for multiple ads or the same ad in consecutive issues. For an additional $\$ 10$ charge, announcements can be placed anonymously. Correspondence will be forwarded.
Advertisements in the "Positions Available" classified section will be set with a minimum one-line headline, consisting of the institution name above body copy, unless additional headline copy is specified by the advertiser. Headlines will be centered in boldface at no extra charge. Ads will appear in the language in which they are submitted.
There are no member discounts for classified ads. Dictation over the telephone will not be accepted for classified ads.
Upcoming deadlines for classified advertising are as follows: November 2018-August 29, 2018; December 2018-September 21, 2018; January 2019— October 17, 2018; February 2019-November 15, 2018; March 2019-December 17, 2018; April 2019-January 17, 2019; May 2019-February 18, 2019. US laws prohibit discrimination in employment on the basis of color, age, sex, race, religion, or national origin. "Positions Available" advertisements from institutions outside the US cannot be published unless they are accompanied by a statement that the institution does not discriminate on these grounds whether or not it is subject to US laws. Details and specific wording may be found on page 1373 (vol. 44).
Situations wanted advertisements from involuntarily unemployed mathematicians are accepted under certain conditions for free publication. Call toll-free 800-321-4AMS (321-4267) in the US and Canada or 401-455-4084 worldwide for further information.
Submission: Promotions Department, AMS, P.O. Box 6248, Providence, Rhode Island 02904; or via fax: 401-331-3842; or send email to classads@ams .org. AMS location for express delivery packages is 201 Charles Street, Providence, Rhode Island 02904. Advertisers will be billed upon publication.
combined with a strong commitment and demonstrated success in teaching.

Outstanding candidates from all areas in Computer and Data Science, Probability and Statistics, Topological Data Analysis, and Discrete and Computational Mathematics will be considered.

Qualifications: A PhD in Computer Science, Mathematics or a closely related field to one of the above-listed areas of expertise by the start date is required. Successful candidates are expected to have or to develop an independently funded research program of international caliber and demonstrated evidence of excellent teaching in undergraduate and graduate courses. Qualified candidates should be committed to fostering diverse and inclusive environments as well as to promoting experiential learning, which are central to a Northeastern University education.

Additional Information: Northeastern University is home to 35,000 full- and part-time degree students and to the nation's premier cooperative education program. The past decade has witnessed a dramatic increase in Northeastern's international reputation for research and innovative educational programs. A heightened focus on interdisciplinary research and scholarship is driving a faculty hiring initiative at Northeastern, advancing its position amongst the nation's top research universities. The College of Computer and Information Science and the College of Science have been major participants in this initiative and will continue their efforts this year, with additional interdisciplinary searches ongoing in related areas. For more information about the College of Computer and Information Science, please visit www.ccis.northeastern.edu, and for the College of Science, please visitwww. northeastern.edu/cos/. The College of Science is strongly committed to fostering excellence through diversity and enthusiastically welcomes nominations and applications from members of groups that have been, and continue to be, underrepresented in academia.

Additional information and instructions for submitting application materials may be found at the following web site: https:// neu. peopleadmin.com/postings/search.

Screening of applications begins immediately. For full consideration, application materials should be received by December 1, 2018. However, applications will be accepted until the search is completed.

Northeastern University is an Equal Opportunity, Affirmative Action Educational Institution and Employer, Title IX University. All qualified applicants will receive consideration for employment without regard to race, color, religion, sex, national origin, disability status, protected veteran status, or any other characteristic protected by the law. Northeastern University is an E-Verify Employer.

## The Williams College Department of Mathematics and Statistics

The Williams College Department of Mathematics and Statistics invites applications for two two-year visiting positions in mathematics, to begin fall 2019. Candidates should have earned a PhD in mathematics, applied mathematics, or a related field by summer, 2019. We will consider candidates with any area of mathematical expertise.

Visiting Assistant Professors are asked to teach four courses per year on our 12-week semester schedule, advise several undergraduate student colloquia (our capstone experience for seniors), and make small contributions to service activities in the department. This set of professional duties provides a window into the experience of being a mathematician in a liberal arts setting.
Our department offers a vibrant undergraduate program with majors in mathematics (including an applied mathematics emphasis) and in statistics. For more information, see https:// math.williams.edu. The multidisciplinary environment is a rich and collegial setting for student education and faculty research. Williams College provides: the opportunity to apply for student research assistant support; a standard, annual allocation
of funds to support travel and research; and a shared computer cluster for parallel computation. Visiting Assistant Professors are also eligible to participate in the college's comprehensive First Three professional development program (https://faculty -networks.williams.edu/networking-opportunities).
Approximately one hour from the Albany, NY airport, Williams College is located in Williamstown, a thriving destination proximate to: three major art museums; theater, music, and dance festivals; community supported agriculture farms; a highly-rated public school system; and many other resources.
The Williams undergraduate student body has 40\% US minority enrollment and nearly $10 \%$ international enrollment. Reflecting the institution's values, our department is diverse and inclusive, with $50 \%$ of our faculty being women, people of color, and/or members of the LGBTQ+ community. We encourage applications from members of underrepresented groups with respect to gender, race and ethnicity, religion, sexual orientation, disability status, socioeconomic background, and other axes of diversity.

Applications should be submitted viawww.mathjobs.org. Your application should include the following components.

1) Please provide a cover letter. This letter might describe your interest in Williams and in the liberal arts, and provide a brief summary of your professional experience and future goals. We ask you to address how your teaching, scholarship, mentorship and/or community service might support Williams's commitment to diversity and inclusion.
2) Please provide a current curriculum vitae.
3) Please provide a teaching statement. Ideally, this statement should be 2-3 pages long, and it might address your teaching philosophy, teaching experience, and any other reflections or relevant information you would like to share.
4) Please provide a brief research statement. Ideally, it should help our faculty, who come from a wide range of mathematical disciplines, understand the nature of your work and think about how to support you during your post-PhD years.
5) Please have at least three recommenders submit letters of recommendation. If possible, at least one of these letters should comment on your experience as a teaching assistant or on any other instructional capacities in which you have served.
We also ask applicants to fill out this brief EEOC demographic survey: https://goo.g1/forms/xqT52JBGKXSonPUn1. While completing this form is voluntary, we hope you will fill it out. Responses will be accessible only by administrators and EEO officers.
If you have questions about this position, contact search committee chair Chad Topaz (cmt6@wi 11iams. edu). Review of applications will begin on or after November 1 and will continue until the positions are filled. All offers of employment are contingent upon completion of a background check. Further information is available at https://faculty.wi11iams.edu/prospective -faculty/background-check-policy.
Williams College is a coeducational liberal arts institution located in the Berkshire Hills of western Massachusetts. The college has built its reputation on outstanding teaching and scholarship and on the academic excellence of its approximately 2,000 students. Please visit the Williams College website (www. williams.edu). Beyond meeting fully its legal obligations for non-discrimination, Williams College is committed to building a diverse and inclusive community where members from all backgrounds can live, learn, and thrive.

## TEXAS

## Baylor University Department of Mathematics

Baylor University is a private Christian university and a nationally ranked research institution, consistently listed with highest hon-
ors among The Chronicle of Higher Education's "Great Colleges to Work For." The University is recruiting new faculty with a deep commitment to excellence in teaching, research and scholarship. Baylor seeks faculty who share in our aspiration to become a tier one research institution while strengthening our distinctive Christian mission as described in our strategic vision, Pro Futuris https://www.bay7or.edu/profuturis/and academic strategic plan, Illuminate (www. baylor. edu/il1uminate). As the world's largest Baptist University, Baylor offers over 40 doctoral programs and has more than 17,000 students from all 50 states and more than 85 countries.
Baylor seeks to fill the following tenured position in the Department of Mathematics, College of Arts \& Sciences:

## Professor and Department Chair

preferably starting with the academic year 2019-2020. The Department seeks applicants, with an appropriate doctoral degree, with a strong record of scholarship and mentoring of students. In particular, we are seeking exceptionally qualified candidates to provide leadership and vision in the department and contribute to the university's focus on STEM and the university's commitment to becoming a Research 1 university. The Baylor Department of Mathematics currently has twenty-three tenured and tenure-track faculty members and seven lecturers. With over twenty-five graduate students, three postdoctoral fellows, and 185 undergraduate majors, the department has a vibrant atmosphere with current strengths in the areas of differential equations, harmonic analysis, mathematical physics, numerical analysis, operator theory, orthogonal polynomials, representation theory, ring theory, and topological dynamics.

Application materials should be submitted online through www.mathjobs.org/jobs and should include a statement indicating qualifications and aspirations for the position; the AMS cover sheet for academic employment; a curriculum vitae and publication list; official doctoral transcripts; a letter indicating support of the Baylor Mission Statement (https://www.baylor .edu/profuturis/index.php?id=88943); and four letters of recommendation. For full consideration, applications are encouraged to be submitted by January 1, 2019, although applications will be accepted until the position is filled.

To learn more about this position, the department, the College of Arts \& Sciences, and Baylor University, please visit these links: https://jobs.bay7or.edu, www.bay7or.edu/math/, or www.baylor.edu/artsandsciences/.

Baylor University is a private not-for-profit university affiliated with the Baptist General Convention of Texas. As an Affirmative Action/Equal Opportunity employer, Baylor is committed to compliance with all applicable anti-discrimination laws, including those regarding age, race, color, sex, national origin, marital status, pregnancy status, military service, genetic information, and disability. As a religious educational institution, Baylor is lawfully permitted to consider an applicant's religion as a selection criterion. Baylor encourages women, minorities, veterans, and individuals with disabilities to apply.

## UTAH

## University of Utah Department of Mathematics

The Department of Mathematics at the University of Utah invites applications for the following positions:

- Full-time tenure-track or tenured appointments at the level of Assistant, Associate, or Full Professor in all areas of mathematics.
- Full-time tenure-track or tenured appointments at the level of Assistant, Associate, or Full Professor in all areas of statistics. These positions are part of a University-wide cluster hiring effort in statistics, with particular emphasis in mathematics,
computer science, and bioengineering. Successful candidates will have strong interdisciplinary interests.
- Three-year Burgess, Kollár, Tucker, and Wylie Assistant Professor Lecturer positions.
Please see our website at www.math.utah.edu/positions for information regarding available positions and application requirements. Applications must be completed through www. mathjobs.org/jobs/Utah. Review of complete applications for tenure-track positions will begin on October 5, 2018, and will continue until the positions are filled. Completed applications for postdoctoral positions received before January 1, 2019, will receive full consideration.

The University of Utah is an Equal Opportunity/Affirmative Action employer and educator. Minorities, women, veterans, and those with disabilities are strongly encouraged to apply. Veterans' preference is extended to qualified veterans. Reasonable disability accommodations will be provided with adequate notice. For additional information about the University's commitment to equal opportunity and access see:www. utah. edu/nondiscrimination/.

## CHINA

## Southern University of Science and Technology (SUSTech) Faculty Positions of Mathematics The Department of Mathematics

The Department of Mathematics at Southern University of Science and Technology (SUSTech) is founded in 2015 with a dual mission of creating a first-class research and education organization for mathematics and providing service courses in support of other academic departments at SUSTech. We currently have 36 full-time faculty members, including 6 Chair Professors \& 7 Full Professors, 3 Associate Professors, 12 Assistant Professors, and 8 teaching faculty members. Research interests of the faculty members cover a broad array of Mathematics including Pure Mathematics, Computational and Applied Mathematics, Probability and Statistics, and Financial Mathematics.

## Call for Applications

We invite applications for full-time faculty positions at all ranks and in all areas of Mathematics, including Financial Mathematics and Statistics. SUSTech has a tenure system. Qualified candidates may apply for appointments with tenure.

Candidates should have demonstrated excellence in research and a strong commitment to teaching. A doctoral degree is required at the time of appointment. A candidate for a senior position must have an established record of research and teaching, and a track-record in securing external funding.

To apply, please visit www. mathjobs.org and look up our job ad for instructions. For an informal discussion about applying to one of our positions, please contact Ms. Xianghui Yu, the Secretary of Department of Mathematics, by phone $+86-755-88018703$ or email: yuxh@sustc.edu.cn.
SUSTech offers competitive salaries, fringe benefits including medical insurance, retirement and housing subsidy, which are among the best in China. Salary and rank will be commensurate with qualifications and experiences of an appointee.

## About the University

Established in 2012, SUSTech is a public institution funded by Shenzhen, a city with a designated special economic zone status in Southern China bordering Hong Kong. As one of China's key gateways to the world, Shenzhen is the country's fastest-growing city in the past three decades. From a small fishing village 30 years ago to a modern city with a population of over 10 million, the city has become the high-tech and manufacturing hub of southern China. It is home to the world's third-busiest container port and

## FURTHERING CONNECTION \& COLLABORATION

IN THE MATHEMATICAL SCIENCES
the fourth-busiest airport on the Chinese mainland. Being a picturesque coastal city, Shenzhen is also a popular tourist destination.

SUSTech is a pioneer in higher education reform in China. Its mission is to become a globally recognized institution that excels in research and promotes innovation, creativity and entrepreneurship. Ninety percent of SUSTech faculty members have overseas work experiences, and sixty percent studied or worked in top 100 universities in the world. The languages of instruction are English and Chinese. Sitting on five hundred acres of subtropical woodland with hills, rivers and a natural lake in Nanshan District of Shenzhen, the SUSTech campus is a beautiful place for learning and research.

The prosperity of Shenzhen is built on innovations and entrepreneurship of its citizens. The city has some of China's most successful high-tech companies such as Huawei and Tencent. SUSTech strongly supports innovations and entrepreneurship, and provides funding for promising initiatives. The university encourages candidates with intention and experience on entrepreneurship to apply.

## Tianjin University, China Tenured/Tenure-Track/Postdoctoral Positions at the Center for Applied Mathematics

Dozens of positions at all levels are available at the recently founded Center for Applied Mathematics, Tianjin University, China. We welcome applicants with backgrounds in pure mathematics, applied mathematics, statistics, computer science, bioinformatics, and other related fields. We also welcome applicants who are interested in practical projects with industries. Despite its name attached with an accent of applied mathematics, we also aim to create a strong presence of pure mathematics. Chinese citizenship is not required.

Light or no teaching load, adequate facilities, spacious office environment and strong research support. We are prepared to make quick and competitive offers to self-motivated hard workers, and to potential stars, rising stars, as well as shining stars.
The Center for Applied Mathematics, also known as the Tianjin Center for Applied Mathematics (TCAM), located by a lake in the central campus in a building protected as historical architecture, is jointly sponsored by the Tianjin municipal government and the university. The initiative to establish this center was taken by Professor S. S. Chern. Professor Molin Ge is the Honorary Director, Professor Zhiming Ma is the Director of the Advisory Board. Professor William Y. C. Chen serves as the Director.

TCAM plans to fill in fifty or more permanent faculty positions in the next few years. In addition, there are a number of temporary and visiting positions. We look forward to receiving your application or inquiry at any time. There are no deadlines.

Please send your resume to mathjobs@tju.edu.cn.
For more information, please visit cam.tju.edu.cn or contact Ms. Erica Liu at mathjobs@tju.edu.cn, telephone: 86-22-2740-6039.
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Shalaine L. Buck, 2013 Waldemar J. Trjitzinsky Memorial Award Recipient, University of New Mexico

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AMERICAN MATHEMATICAL SOCIETY

# New Publications Offered by the AMS 

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## Algebra and Algebraic Geometry

## MEMOIRS

On Fusion Systems of Component Type

Michael Aschbacher

(GMs:

## On Fusion Systems of Component Type

Michael Aschbacher, California Institute of Technology, Pasadena, California

Contents: Introduction; Preliminaries; Some lemmas on fusion systems; Tight embedding; More on tight embedding; Split extensions; Component combinatorics; The proof of Theorem 2; Terminal components; Standard subsystems; Bibliography.
Memoirs of the American Mathematical Society, Volume 257, Number 1236
December 2018, 182 pages, Softcover, ISBN: 978-1-4704-3520-2, 2010 Mathematics Subject Classification: 20D05; 55R35, Individual member US\$48.60, List US\$81, Institutional member US\$64.80, Order code MEMO/257/1236


## Interpolation for Normal Bundles of General Curves

Atanas Atanasov, Harvard University, Cambridge, Massachusetts, Eric Larson, Stanford University, California, and David Yang, Massachusetts Institute of Technology, Cambridge, Massachusetts

Contents: Introduction; Elementary modifications in arbitrary dimension; Elementary modifications for curves; Interpolation and short exact sequences; Elementary modifications of normal bundles; Examples of the bundles $N_{C \rightarrow \Lambda}$; Interpolation and specialization; Reducible curves and their normal bundles; A
stronger inductive hypothesis; Inductive arguments; Base cases; Summary of remainder of proof of Theorem 1.2; The three exceptional cases; Appendix A. remainder of proof of Theorem 1.2; Appendix B. Code for Section 4; Bibliography.
Memoirs of the American Mathematical Society, Volume 257, Number 1234

December 2018, 105 pages, Softcover, ISBN: 978-1-4704-34892, 2010 Mathematics Subject Classification: 14H99, Individual member US\$48.60, List US\$81, Institutional member US\$64.80, Order code MEMO/257/1234

## Analysis



## Covering Dimension

 of C"-Algebras and 2-Coloured ClassificationJoan Bosa, University of Glasgow, Scotland, United Kingdom, Nathanial P. Brown, The Pennsylvania State University, University Park, Pennsylvania, Yasuhiko Sato, Kyoto University, Japan, Aaron Tikuisis, University of Aberdeen, Scotland, United Kingdom, Stuart White, University of Glasgow, Scotland, United Kingdom, and University of Münster, Germany, and Wilhelm Winter, University of Münster, Germany

Contents: Introduction; Preliminaries; A $2 \times 2$ matrix trick; Ultrapowers of trivial $\mathrm{W}^{*}$-bundles; Property (SI) and its consequences; Unitary equivalence of totally full positive elements; 2-coloured equivalence; Nuclear dimension and decomposition rank; Quasidiagonal traces; Kirchberg algebras; Addendum; Bibliography.

Memoirs of the American Mathematical Society, Volume 257, Number 1233

December 2018, 97 pages, Softcover, ISBN: 978-1-4704-3470-0, 2010 Mathematics Subject Classification: 46L05, 46L35, Individual member US\$48.60, List US\$81, Institutional member US\$64.80, Order code MEMO/257/1233

## MEMOIRS

Dilations. Linear Matrix Inequalities, the Matrix Cube Problem and Beta Distributions

む. William Helton Igor Klep
Scott MeCullough
Markus Schweighofer

> Dilations, Linear Matrix Inequalities, the Matrix Cube Problem and Beta Distributions

J. William Helton, University of California, San Diego, Igor Klep, The University of Auckland, New Zealand, Scott McCullough, University of Florida, Gainesville, Florida, and Markus Schweighofer, Universität Konstanz, Germany

Contents: Introduction; Dilations and free spectrahedral Inclusions; Lifting and averaging; A simplified form for $\vartheta ; \vartheta$ is the optimal bound; The optimality condition $\alpha=\beta$ in terms of beta functions; Rank versus size for the matrix cube; Free spectrahedral inclusion generalities; Reformulation of the optimization problem; Simmons' theorem for half integers; Bounds on the median and the equipoint of the Beta distribution; Proof of Theorem 2.1; Estimating $\boldsymbol{\vartheta}(d)$ for Odd $d$.; Dilations and inclusions of Balls; Probabilistic theorems and interpretations continued; Bibliography; Index.
Memoirs of the American Mathematical Society, Volume 257, Number 1232
December 2018, 104 pages, Softcover, ISBN: 978-1-4704-3455-7, 2010 Mathematics Subject Classification: 47A20, 46L07, 13J30; 60E05, 33B15, 90C22, Individual member US\$48.60, List US\$81, Institutional member US\$64.80, Order code MEMO/257/1232


## Harmonic Analysis

Smooth and Non-smooth
Palle E.T. Jorgensen, University of Iowa, Iowa City, IA

There is a recent and increasing interest in harmonic analysis of non-smooth geometries. Real-world examples where these types of geometry appear include large computer networks, relationships in datasets, and fractal structures such as those found in crystalline substances, light scattering, and other natural phenomena where dynamical systems are present.

Notions of harmonic analysis focus on transforms and expansions and involve dual variables. In this book on smooth and non-smooth harmonic analysis, the notion of dual variables will be adapted to fractals. In addition to harmonic analysis via Fourier duality, the
author also covers multiresolution wavelet approaches as well as a third tool, namely, $L^{2}$ spaces derived from appropriate Gaussian processes. The book is based on a series of ten lectures delivered in June 2018 at a CBMS conference held at Iowa State University.
Contents: Introduction. Smooth vs the non-smooth categories; Spectral pair analysis for IFSs; Harmonic analyses on fractals, with an emphasis on iterated function systems (IFS) measures; Four kinds of harmonic analysis; Harmonic analysis via representations of the Cuntz relations; Positive definite functions and kernel analysis; Representations of Lie groups. Non-commutative harmonic analysis; Bibliography; Index.
CBMS Regional Conference Series in Mathematics, Number 128
November 2018, 266 pages, Softcover, ISBN: 978-1-4704-48806, LC 2018030996, 2010 Mathematics Subject Classification: 28A80, 81Q35, 11K70, 60J70, 42C40, 60G22, 37A45, 42B37, AMS members US\$41.60, List US\$52, Order code CBMS/128

## MEMOIRS

Measure and Capacity of Wandering Domains in Gevrey Near-Integrable Exact Symplectic Systems


Measure and Capacity of Wandering Domains in Gevrey Near-Integrable Exact Symplectic Systems

Laurent Lazzarini, Université Paris VI, France, Jean-Pierre Marco, Université Paris VI, France, and David Sauzin, Observatoire de Paris, France

Contents: Introduction; Presentation of the results; Stability theory for Gevrey near-integrable maps; A quantitative KAM result-proof of Part (i) of Theorem D; Coupling devices, multi-dimensional periodic domains, wandering domains; Appendices; Appendix A. Algebraic operations in $\mathcal{O}_{k} ;$ Appendix B. Estimates on Gevrey maps; Appendix C. Generating functions for exact symplectic $C^{\infty}$ maps; Appendix D. Proof of Lemma 2.5; Bibliography.
Memoirs of the American Mathematical Society, Volume 257, Number 1235

December 2018, 106 pages, Softcover, ISBN: 978-1-4704-3492-2, 2010 Mathematics Subject Classification: 53D22, 70H08; 26E10, Individual member US\$48.60, List US\$81, Institutional member US\$64.80, Order code MEMO/257/1235


# Lectures on the Fourier Transform and Its Applications 

Brad G. Osgood, Stanford University, CA

This book is derived from lecture notes for a course on Fourier analysis for engineering and science students at the advanced undergraduate or beginning graduate level. Beyond teaching specific topics and techniques-all of which are important in many areas of engineering and
science-the author's goal is to help engineering and science students cultivate more advanced mathematical know-how and increase confidence in learning and using mathematics, as well as appreciate the coherence of the subject. He promises the readers a little magic on every page.

The section headings are all recognizable to mathematicians, but the arrangement and emphasis are directed toward students from other disciplines. The material also serves as a foundation for advanced courses in signal processing and imaging. There are over 200 problems, many of which are oriented to applications, and a number use standard software. An unusual feature for courses meant for engineers is a more detailed and accessible treatment of distributions and the generalized Fourier transform. There is also more coverage of higher-dimensional phenomena than is found in most books at this level.

A thoroughly enjoyable yet careful mathematical perspective of the underlying concepts and many applications of modern signal analysis.

## -Les Atlas, University of Washington

Osgood leads his readers from the basics to the more sophisticated parts of applicable Fourier analysis with a lively style, a light touch on the technicalities, and an eye toward communications engineering. This book should be a great resource for students of mathematics, physics, and engineering alike.
-Gerald B. Folland, University of Washington
Fourier analysis with a swing in its step.

## -Tom Körner, University of Cambridge

Contents: Fourier series; Fourier transform; Convolution; Distributions and their Fourier transforms; $\delta$ hard at work; Sampling and interpolation; Discrete Fourier transform; Linear time-invariant systems; $n$-Dimensional Fourier transform; A list of mathematical topics that are fair game; Complex numbers and complex exponentials; Geometric sums; Index.
Pure and Applied Undergraduate Texts, Volume 33
March 2019, approximately 702 pages, Hardcover, ISBN: 978-1-4704-4191-3, LC 2017061807, 2010 Mathematics Subject Classification: 42A38, 42B10, 65T50, 94A20, 46F10, AMS members US\$92, List US\$115, Order code AMSTEXT/33


> Multilinear Singular Integral Forms of Christ-Journé Type

Andreas Seeger, University of Wisconsin, Madison, Charles K. Smart, University of Chicago, Illinois, and Brian Street, University of Wisconsin, Madison

Contents: Introduction; Statements of the main results; Kernels; Adjoints; Outline of the proof of boundedness; Some auxiliary operators; Basic $L^{2}$ estimates; Some results from Calderón-Zygmund theory; Almost orthogonality; Boundedness of multilinear singular forms; Proof of the main theorem: Part I; Proof of the main theorem: Part II; Proof of the main theorem: Part III; Proof of the main theorem: Part IV; Proof of the main theorem: Part V; Interpolation; Bibliography.

Memoirs of the American Mathematical Society, Volume 257, Number 1231

December 2018, 142 pages, Softcover, ISBN: 978-1-4704-34373, 2010 Mathematics Subject Classification: 42B20, Individual member US\$48.60, List US\$81, Institutional member US\$64.80, Order code MEMO/257/1231

## Applications



# The Mathematics of Data 

Michael W. Mahoney, University of California, Berkeley, John C. Duchi, Stanford University, CA, and Anna C. Gilbert, University of Michigan, Ann Arbor, Editors

Data science is a highly interdisciplinary field, incorporating ideas from applied mathematics, statistics, probability, and computer science, as well as many other areas. This book gives an introduction to the mathematical methods that form the foundations of machine learning and data science, presented by leading experts in computer science, statistics, and applied mathematics. Although the chapters can be read independently, they are designed to be read together as they lay out algorithmic, statistical, and numerical approaches in diverse but complementary ways.
This book can be used both as a text for advanced undergraduate and beginning graduate courses, and as a survey for researchers interested in understanding how applied mathematics broadly defined is being used in data science. It will appeal to anyone interested in the interdisciplinary foundations of machine learning and data science.
Titles in this series are co-published with the Institute for Advanced Study/Park City Mathematics Institute.
This volume is a co-publication of the AMS, IAS/Park City Mathematics Institute, and Society for Industrial and Applied Mathematics

Contents: P. Drineas and M. W. Mahoney, Lectures on randomized numerical linear algebra; S. J. Wright, Optimization algorithms for data analysis; J. C. Duchi, Introductory lectures on stochastic optimization; P.-G. Martinsson, Randomized methods for matrix computations; R. Vershynin, Four lectures on probabilistic methods for data science; R. Ghrist, Homological algebra and data.
IAS/Park City Mathematics Series, Volume 25
November 2018, 325 pages, Hardcover, ISBN: 978-1-4704-3575-2, LC 2018024239, 2010 Mathematics Subject Classification: 15-02, 52-02, 60-02, 62-02, 65-02, 68-02, 90-02, AMS members US\$83.20, List US\$104, Order code PCMS/25

# Discrete Mathematics and Combinatorics 



# Combinatorial Reciprocity Theorems 

An Invitation to Enumerative Geometric Combinatorics

Matthias Beck, San Francisco State University, CA, and Raman Sanyal, Goethe-Universität Frankfurt, Germany

Combinatorial reciprocity is a very interesting phenomenon, which can be described as follows: A polynomial, whose values at positive integers count combinatorial objects of some sort, may give the number of combinatorial objects of a different sort when evaluated at negative integers (and suitably normalized). Such combinatorial reciprocity theorems occur in connections with graphs, partially ordered sets, polyhedra, and more. Using the combinatorial reciprocity theorems as a leitmotif, this book unfolds central ideas and techniques in enumerative and geometric combinatorics.
Written in a friendly writing style, this is an accessible graduate textbook with almost 300 exercises, numerous illustrations, and pointers to the research literature. Topics include concise introductions to partially ordered sets, polyhedral geometry, and rational generating functions, followed by highly original chapters on subdivisions, geometric realizations of partially ordered sets, and hyperplane arrangements.
Contents: Four polynomials; Partially ordered sets; Polyhedral geometry; Rational generating functions; Subdivisions; Partially ordered sets, geometrically; Hyperplane arrangements; Bibliography; Notation index; Index.
Graduate Studies in Mathematics, Volume 195
December 2018, approximately 314 pages, Hardcover, ISBN: 978-1-4704-2200-4, 2010 Mathematics Subject Classification: 05Axx, 05C31, 05E45, 11P21, 52B05, 52B11, 52B20, 52B45, 52C07, 68R05, AMS members US\$58.40, List US\$73, Order code GSM/195


> An Introduction to Ramsey Theory

Fast Functions, Infinity, and Metamathematics

Matthew Katz, Pennsylvania State University, University Park, PA, and Jan Reimann, Pennsylvania State University, University Park, PA

This book takes the reader on a journey through Ramsey theory, from graph theory and combinatorics to set theory to logic and metamathematics. Written in an informal style with few requisites, it develops two basic principles of Ramsey theory:
many combinatorial properties persist under partitions, but to witness this persistence, one has to start with very large objects. The interplay between those two principles not only produces beautiful theorems but also touches the very foundations of mathematics. In the course of this book, the reader will learn about both aspects. Among the topics explored are Ramsey's theorem for graphs and hypergraphs, van der Waerden's theorem on arithmetic progressions, infinite ordinals and cardinals, fast growing functions, logic and provability, Gödel incompleteness, and the Paris-Harrington theorem.
Quoting from the book, "There seems to be a murky abyss lurking at the bottom of mathematics. While in many ways we cannot hope to reach solid ground, mathematicians have built impressive ladders that let us explore the depths of this abyss and marvel at the limits and at the power of mathematical reasoning at the same time. Ramsey theory is one of those ladders."
This item will also be of interest to those working in logic and foundations.
Contents: Graph Ramsey theory; Infinite Ramsey theory; Growth of Ramsey functions; Metamathematics; Bibliography; Notation; Index.
Student Mathematical Library, Volume 87
November 2018, 207 pages, Softcover, ISBN: 978-1-4704-4290-3, LC 2018024651, 2010 Mathematics Subject Classification: 05D10, 03-01, 03E10, 03B10, 03B25, 03D20, 03H15, All Individuals US\$41.60, List US\$52, Institutional member US\$41.60, Order code STML/87

## General Interest



# Limitless Minds 

Interviews with Mathematicians

Anthony Bonato, Ryerson University, Toronto, ON, Canada

## Every mathematician is a person with a

 story. Limitless Minds tells those stories in an engaging way by featuring interviews with twelve leading mathematicians. They were invited to answer some key questions such as: Who and what were the influences that pointed them towards mathematics? Why do mathematicians devote their lives to discovering new mathematics? How do they see mathematics evolving in the future?The book, written in an accessible style and enriched by dozens of images, offers a rare insight into the minds of mathematicians, provided in their own words. It will enlighten and inspire readers about the lives, passions, and discoveries of mathematicians.
Contents: Interview with Alejandro Adem; Interview with Federico Ardila; Interview with Jennifer Chayes; Interview with Maria Chudnovsky; Interview with Fan Chung Graham; Interview with Ingrid Daubechies; Interview with Nassif Ghoussoub; Interview with Lisa Jeffrey; Interview with Izabella Laba; Interview with Barry Mazur; Interview with Richard Nowakowski; Interview with Ken Ono; Index.

December 2018, 155 pages, Softcover, ISBN: 978-1-4704-47915, 2010 Mathematics Subject Classification: 01A70, 01A65, AMS members US\$23.20, List US\$29, Order code MBK/118


> The Mathematics of Voting and Elections: A Hands-On Approach Second Edition

Jonathan K. Hodge, Grand Valley State University, Allendale, MI, and Richard E. Klima, Appalachian State University, Boone, NC

The Mathematics of Voting and Elections: A Hands-On Approach, Second Edition, is an inquiry-based approach to the mathematics of politics and social choice. The aim of the book is to give readers who might not normally choose to engage with mathematics recreationally the chance to discover some interesting mathematical ideas from within a familiar context, and to see the applicability of mathematics to real-world situations. Through this process, readers should improve their critical thinking and problem solving skills, as well as broaden their views of what mathematics really is and how it can be used in unexpected ways. The book was written specifically for non-mathematical audiences and requires virtually no mathematical prerequisites beyond basic arithmetic. At the same time, the questions included are designed to challenge both mathematical and non-mathematical audiences alike. More than giving the right answers, this book asks the right questions.
The book is fun to read, with examples that are not just thought-provoking, but also entertaining. It is written in a style that is casual without being condescending. But the discovery-based approach of the book also forces readers to play an active role in their learning, which should lead to a sense of ownership of the main ideas in the book. And while the book provides answers to some of the important questions in the field of mathematical voting theory, it also leads readers to discover new questions and ways to approach them.

In addition to making small improvements in all the chapters, this second edition contains several new chapters. Of particular interest might be Chapter 12 which covers a host of topics related to gerrymandering.
Contents: What's so good about majority rule?; Le Pen, Nader, and other inconveniences; Back into the ring; Trouble in democracy; Explaining the impossible; Gaming the system; One person, one vote?; Calculating corruption; The ultimate college experience; Trouble in direct democracy; Proportional (mis)representation; Choosing your voters; Bibliography; Index.
Mathematical World, Volume 30
November 2018, approximately 247 pages, Softcover, ISBN: 978-1-4704-4287-3, 2010 Mathematics Subject Classification: 91B12, AMS members US\$41.60, List US\$52, Order code MAWRLD/30


Figuring Fibers

Carolyn Yackel, Mercer<br>University, Macon, GA, and<br>sarah-marie belcastro, MathILy, Mathematical Staircase, Inc., Holyoke, MA, and Smith College, Northampton, MA, Editors

Pick up this book and dive into one of eight chapters relating mathematics to fiber arts! Amazing exposition transports any interested person on a mathematical exploration that is rigorous enough to capture the hearts of mathematicians. The zenith of creativity is achieved as readers are led to knit, crochet, quilt, or sew a project specifically designed to illuminate the mathematics through its physical realization. The beautiful finished pieces provide a visual understanding of the mathematics that can be shared with those who view them. If you love mathematics or fiber arts, this book is for you!
Contents: C. Yackel, Introduction; D. J. Wildstrom, More granny, less square; K. Calderhead, Gosper-like fractals and intermeshed crochet; C. Yackel, Templeton square truchet tiles; M. D. Shepherd, Variations on snake trail quilting patterns; B. N. Givens, The Chinese remainder theorem and knitting stich patterns; s.-m. belcastro, Knitting torus knots and links; S. L. Gould, Triply periodic polyhedra in Euclidean three-dimensional space; B. E. Nimershiem, Piecing together link complements; Index.

December 2018, 232 pages, Hardcover, ISBN: 978-1-4704-2931-7, LC 2018033586, 2010 Mathematics Subject Classification: 00A05, 00A06, 00A08, 00A66, 05A10, 05C99, 11A99, 52B10, 57M25, 28A80, AMS members US\$32, List US\$40, Order code MBK/117

## Geometry and Topology



# Topological Recursion and its Influence in Analysis, Geometry, and Topology 

Chiu-Chu Melissa Liu, Columbia University, New York, and Motohico Mulase, University of California, Davis, Editors

This volume contains the proceedings of the 2016 AMS von Neumann Symposium on Topological Recursion and its Influence in Analysis, Geometry, and Topology, which was held from July $4-8,2016$, at the Hilton Charlotte University Place, Charlotte, North Carolina.
The papers contained in the volume present a snapshot of rapid and rich developments in the emerging research field known as topological recursion. It has its origin around 2004 in random matrix theory and also in Mirzakhani's work on the volume of moduli spaces of hyperbolic surfaces.
Topological recursion has played a fundamental role in connecting seemingly unrelated areas of mathematics such as matrix models,
enumeration of Hurwitz numbers and Grothendieck's dessins d'enfants, Gromov-Witten invariants, the A-polynomials and colored polynomial invariants of knots, WKB analysis, and quantization of Hitchin moduli spaces. In addition to establishing these topics, the volume includes survey papers on the most recent key accomplishments: discovery of the unexpected relation to semi-simple cohomological field theories and a solution to the remodeling conjecture. It also provides a glimpse into the future research direction; for example, connections with the Airy structures, modular functors, Hurwitz-Frobenius manifolds, and ELSV-type formulas.

Contents: J. E. Andersen, G. Borot, and N. Orantin, Modular functors, cohomological field theories, and topological recursion;
A. Brini, On the Gopakumar-Ooguri-Vafa correspondence for Clifford-Klein 3-manifolds; L. Chen, Bouchard-Klemm-MarinoPasquetti conjecture for $\mathbb{C}^{3}$; A. Chiodo and J. Nagel, The hybrid Landau-Ginzburg models of Calabi-Yau complete intersections; P. Ciosmak, L. Hadasz, M. Manabe, and P. Sułkowski, Singular vector structure of quantum curves; N. Do and M. Karev, Towards the topological recursion for double Hurwitz numbers; O. Dumitrescu and M. Mulase, Quantization of spectral curves for meromorphic Higgs bundles through topological recursion; P. Dunin-Barkowski, Topological recursion and Givental's formalism: Spectral curves for Gromov-Witten theories; P. Dunin-Barkowski, P. Norbury, N. Orantin, A. Popolitov, and S. Shadrin, Primary invariants of Hurwitz Frobenius manifolds; J. N. Esteves, Hopf algebras and topological recursion; B. Fang and Z. Zong, Graph sums in the remodeling conjecture; T. Kimura, Double quantization of Seiberg-Witten geometry and W-algebras; M. Kontsevich and Y. Soibelman, Airy structures and symplectic geometry of topological recursion; D. Korotkin, Periods of meromorphic quadratic differentials and Goldman bracket; D. Lewanski, On ELSV-type formulae, Hurwitz numbers and topological recursion; X. Liu, M. Mulase, and A. Sorkin, Quantum curves for simple Hurwitz numbers of an arbitrary base curve.

Proceedings of Symposia in Pure Mathematics, Volume 100
December 2018, approximately 561 pages, Hardcover, ISBN: 978-1-4704-3541-7, 2010 Mathematics Subject Classification: 14H10, $14 \mathrm{H} 15,14 \mathrm{H} 60,14 \mathrm{~J} 33,14 \mathrm{~N} 10,14 \mathrm{~N} 35,53 \mathrm{D} 45,81 \mathrm{~T} 45$, AMS members US\$106.40, List US\$133, Order code PSPUM/100

## Math Education



## Mathematics: Rhyme

 and Reason
## Mel Currie

Mathematics: Rhyme and Reason is an exploration of the aesthetic value of mathematics and the culture of the mathematics community.
This book introduces budding mathematicians of all ages to mathematical ways of thinking through a series of chapters that
mix episodes from the author's life with explanations of intriguing mathematical concepts and the stories of the mathematicians who discovered them. The chapters can be read independently, and most require only a background in basic high school algebra or geometry to appreciate the topics covered.

Part personal memoir, part appreciation of the poetry and humanity inherent in mathematics, this entertaining collection of stories, theorems, and reflections will be of interest to anyone curious about mathematics and the human beings who practice it.
In the interest of fostering a greater awareness and appreciation of mathematics and its connections to other disciplines and everyday life, MSRI and the AMS are publishing books in the Mathematical Circles Library series as a service to young people, their parents and teachers, and the mathematics profession.
Titles in this series are co-published with the Mathematical Sciences Research Institute (MSRI).
Contents: The riddle; Primes; Some geometry; Mysterious pattern; Some things add up. Some don't.; A tangential remark; Plus or minus; Making the optimal choice; Impossibilities; Magnitudes of infinity; The inevitable (Sperner's Lemma-The Brouwer fixed-point theorem); Consider the sequence (Fibonacci and Golden Ratio); What are the chances?; The Euler line; The dissertation; The next prime number is? (Gandhi's formula); Bulgarian solitaire; Which is bigger? ( $a^{b}$ versus $b^{a}$ ); Fascinating; From the sublime to the ridiculous; A few more words; Photos and pictures; Notation, etc.; Mysterious; Impossibilities; Magnitudes; Fascinating.

## MSRI Mathematical Circles Library, Volume 22

November 2018, 178 pages, Softcover, ISBN: 978-1-4704-4796-0, LC 2018028165, 2010 Mathematics Subject Classification: 00A09, 97A80, AMS members US\$32.80, List US\$41, Order code MCL/22


# Math Circle by the Bay Topics for Grades 1-5 

Laura Givental, United Math Circles Foundation, Berkeley and Stanford, CA, Maria Nemirovskaya, University of Oregon, Eugene, and Ilya Zakharevich, United Math Circles Foundation, Berkeley and Stanford, CA

This book is based on selected topics that the authors taught in math circles for elementary school students at the University of California, Berkeley; Stanford University; Dominican University (Marin County, CA); and the University of Oregon (Eugene). It is intended for people who are already running a math circle or who are thinking about organizing one. It can be used by parents to help their motivated, math-loving kids or by elementary school teachers. We also hope that bright fourth or fifth-graders will be able to read this book on their own.

The main features of this book are the logical sequence of the problems, the description of class reactions, and the hints given to kids when they get stuck. This book tries to keep the balance between two goals: inspire readers to invent their own original approaches while being detailed enough to work as a fallback in case the teacher needs to prepare a lesson on short notice. It introduces kids to combinatorics, Fibonacci numbers, Pascal's
triangle, and the notion of area, among other things. The authors chose topics with deep mathematical context. These topics are just as engaging and entertaining to children as typical "recreational math" problems, but they can be developed deeper and to more advanced levels.

In the interest of fostering a greater awareness and appreciation of mathematics and its connections to other disciplines and everyday life, MSRI and the AMS are publishing books in the Mathematical Circles Library series as a service to young people, their parents and teachers, and the mathematics profession.

This item will also be of interest to those working in general interest.

Titles in this series are co-published with the Mathematical Sciences Research Institute (MSRI).
Contents: Numbers as geometric shapes; Combinatorics; Fibonacci numbers; Pascal's triangle; Area; Selected warmup and challenging problems; Handouts; Bibliography; Index.
MSRI Mathematical Circles Library, Volume 21
December 2018, approximately 179 pages, Softcover, ISBN: 978-1-4704-4785-4, LC 2018025745, 2010 Mathematics Subject Classification: 00A05, 00A07, 00A08, 00A09, 97-01, 97A20, 97A80, 97D50, 97E40, 97E50, AMS members US\$20, List US\$25, Order code MCL/21

## Probability and Statistics



# Algebraic Statistics 

Seth Sullivant, North Carolina State University, Raleigh, NC

Algebraic statistics uses tools from algebraic geometry, commutative algebra, combinatorics, and their computational sides to address problems in statistics and its applications. The starting point for this connection is the observation that many statistical models are semialgebraic sets. The algebra/statistics connection is now over twenty years old, and this book presents the first broad introductory treatment of the subject. Along with background material in probability, algebra, and statistics, this book covers a range of topics in algebraic statistics including algebraic exponential families, likelihood inference, Fisher's exact test, bounds on entries of contingency tables, design of experiments, identifiability of hidden variable models, phylogenetic models, and model selection. With numerous examples, references, and over 150 exercises, this book is suitable for both classroom use and independent study.
This item will also be of interest to those working in applications.
Contents: Introduction; Probability primer; Algebra primer; Conditional independence; Statistics primer; Exponential families; Likelihood inference; The cone of sufficient statistics; Fisher's exact test; Bounds on cell entries; Exponential random graph models; design of experiments; graphical models; Hidden variables; Phylogenetic models; Identifiability; Model selection and Bayesian integrals; MAP estimation and parametric inference; Finite metric spaces; Bibliography; Index.

Graduate Studies in Mathematics, Volume 194
December 2018, 490 pages, Hardcover, ISBN: 978-1-4704-3517-2, LC 2018025744, 2010 Mathematics Subject Classification: 62-01, 14-01, 13P10, 13P15, 14M12, 14M25, 14P10, 14T05, 52B20, 60J10, 62F03, 62H17, 90C10, 92D15, AMS members US\$66.40, List US\$83, Order code GSM/194

## New AMS-Distributed Publications

## Algebra and Algebraic Geometry



> Lectures on Representations of Locally Compact Groups

Ion Colojoară, University of Bucharest, Romania, and Aurelian Gheondea, Bilkent University, Ankara, Turkey, and IMAR, Bucharest, Romania

This is a modern presentation of the theory of representations of locally compact groups. In a small number of pages, the reader can get some of the most important theorems on this subject. Many examples are provided.
Highlights of the volume include:
(1) A generous introduction explaining the origins of group theory and their representations, the motivation for the main problems in this theory, and the deep connections with modern physics.
(2) A solid presentation of the theory of topological groups and of Lie groups.
(3) Two proofs of the existence of Haar measures.
(4) The detailed study of continuous representations on general locally convex spaces, with an emphasis on unitary representations of compact groups on Hilbert spaces.
(5) A careful presentation of induced representations on locally convex spaces and G. W. Mackey's Theorem of Imprimitivity.
About half of the results included in this volume appear for the first time in a book, while the theory of $p$-induced representations on locally convex spaces is new. To facilitate reading, several appendices present the concepts and basic results from general topology, differential manifolds, abstract measures and integration, topological vector spaces, Banach spaces, Banach algebras, $C^{*}$-algebras, and operator theory on Hilbert spaces.
A publication of the Theta Foundation. Distributed worldwide, except in Romania, by the AMS.

## International Book Series of Mathematical Texts

September 2018, 186 pages, Hardcover, ISBN: 978-606-8443-10-2, 2010 Mathematics Subject Classification: 22-01; 22D12, 22D30, AMS members US\$47.20, List US\$59, Order code THETA/24


Families of Berkovich Spaces

Antoine Ducros, Sorbonne Université, Paris, France

This book investigates the variation of the properties of the fibers of a map between analytic spaces in the sense of Berkovich. First, the author studies flatness in this setting; the naive definition of this notion is not reasonable and he explains why. He then describes the loci of fiberwise validity of some usual properties (e.g., Cohen-Macaulay, Gorenstein, geometrically regular). He shows that these are (locally) Zariski-constructible subsets of the source space. For that purpose, he develops systematic methods for "spreading out" in Berkovich geometry, as is done in scheme theory, some properties from a generic fiber to a neighborhood of it.
A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the U.S., Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30\% discount from list.
Astérisque, Number 400
June 2018, 262 pages, Softcover, ISBN: 978-2-85629-885-5, 2010 Mathematics Subject Classification: 14G22, 14A99, AMS members US\$60, List US\$75, Order code AST/400


## The 50th Anniversary of Gröbner Bases

Takayuki Hibi, Osaka University, Japan, Editor

The discovery of the algorithm by Bruno Buchberger in July 1965, the so-called Buchberger algorithm used to compute Gröbner bases of ideals of the polynomial ring, led to the birth of the exciting research area in modern mathematics called computer algebra.
The 8th Mathematical Society of Japan Seasonal Institute (MSJ SI 2015), entitled The 50th Anniversary of Gröbner Bases, was held in July 2015. This volume contains the proceedings of MSJ SI 2015 and consists of 14 papers related to computer algebra, algebraic statistics, D-modules, convex polytopes, and toric ideals. These papers enable readers to explore current trends in Gröbner bases. Young researchers will find a treasury of fascinating research problems which are pending. The foreword was contributed by Bruno Buchberger and includes a secret story on the discovery of the Buchberger algorithm.
This item will also be of interest to those working in discrete mathematics and combinatorics.

Published for the Mathematical Society of Japan by Kinokuniya, Tokyo, and distributed worldwide, except in Japan, by the AMS.

Advanced Studies in Pure Mathematics, Volume 77
August 2018, 436 pages, Hardcover, ISBN: 978-4-86497-052-5, 2010 Mathematics Subject Classification: 13P10; 05E40, 14F10, 14M25, 33F10, 46F10, 52B20, 62H05, 62E15, AMS members US\$54.40, List US\$68, Order code ASPM/77

## Differential Equations



# Boundary Behavior of Solutions to Elliptic Equations in General Domains 

Vladimir G. Maz'ya, Linköping University, Sweden, and University of Liverpool, UK

This book is a detailed exposition of the author and his collaborators' work on boundedness, continuity, and differentiability properties of solutions to elliptic equations in general domains, that is, in domains that are not a priori restricted by assumptions such as "piecewise smoothness" or being a "Lipschitz graph".
The description of the boundary behavior of such solutions is one of the most difficult problems in the theory of partial differential equations. After the famous Wiener test, the main contributions to this area were made by the author. In particular, necessary and sufficient conditions for the validity of imbedding theorems are given, which provide criteria for the unique solvability of boundary value problems of second and higher order elliptic equations. Another striking result is a test for the regularity of a boundary point for polyharmonic equations.
This book will be interesting and useful for a wide audience. It is intended for specialists and graduate students working in the theory of partial differential equations.
This item will also be of interest to those working in analysis.
A publication of the European Mathematical Society (EMS). Distributed within the Americas by the American Mathematical Society.

EMS Tracts in Mathematics, Volume 30
October 2018, 441 pages, Hardcover, ISBN: 978-3-03719-190-3, 2010 Mathematics Subject Classification: 35J40, 31B15, 31B25, AMS members US\$78.40, List US\$98, Order code EMSTM/30

## General Interest



## Operator Theory:

Themes and Variations
Conference Proceedings, Timişoara, June 27-July 2, 2016

Hari Bercovici, Indiana University, Bloomington, Dumitru Gaspar, West University of Timişoara, Romania, Dan Timotin, Romanian Academy, Bucharest, Romania, and FlorianHoria Vasilescu, University of Lille, France, Editors

This volume contains the proceedings of the 26th International Conference on Operator Theory, held from June 27-July 2, 2016, in Timişoara, Romania. It consists of a careful selection of papers. One of the higlights is an extended presentation of the heliciodal method in harmonic analysis.
Other subjects covered include function theory on the unit disc; free holomorphic functions; applications of Toeplitz operators; traces on ideals of operators; geodesics of projections on Hilbert space; preserver problems; Sturm Liouville operators; and Bratteli diagrams.

This item will also be of interest to those working in analysis.
A publication of the Theta Foundation. Distributed worldwide, except in Romania, by the AMS.

## International Book Series of Mathematical Texts

November 2018, 204 pages, Hardcover, ISBN: 978-606-8443-096, 2010 Mathematics Subject Classification: 00B25, 30-06, 42-06, 46-06, 47-06, AMS members US\$44.80, List US\$56, Order code THETA/23


## MEETINGS \& CONFERENCES OF THE AMS <br> DECEMBER TABLE OF CONTENTS

The Meetings and Conferences section of the Notices gives information on all AMS meetings and conferences approved by press time for this issue. Please refer to the page numbers cited on this page for more detailed information on each event. Invited Speakers and Special Sessions are listed as soon as they are approved by the cognizant program committee; the codes listed are needed for electronic abstract submission. For some meetings the list may be incomplete. Information in this issue may be dated.

The most up-to-date meeting and conference information can be found online at: www.ams.org/meetings.
Important Information About AMS Meetings: Potential organizers, speakers, and hosts should refer to page 88 in the January 2018 issue of the Notices for general information regarding participation in AMS meetings and conferences.
Abstracts: Speakers should submit abstracts on the easy-to-use interactive Web form. No knowledge of LATEX is
necessary to submit an electronic form, although those who use LATEX may submit abstracts with such coding, and all math displays and similarily coded material (such as accent marks in text) must be typeset in LATEX. Visit WWW.ams.org /cgi-bin/abstracts/abstract.pl. Questions about abstracts may be sent to absinfo@ams.org. Close attention should be paid to specified deadlines in this issue. Unfortunately, late abstracts cannot be accommodated.

## MEETINGS IN THIS ISSUE

|  | 2019 |  |
| :---: | :---: | :---: |
| January 16-19 JMM Baltimore, Maryland |  |  |
| March 15-17 | Auburn, Alabama |  |
| March 22-24 | Honolulu, Hawaii |  |
| April 13-14 | Hartford, Connecticut |  |
| June 10-13 | Quy Nhon City, Vietnam |  |
| September 14-15 | Madison, Wisconsin |  |
| October 12-13 | Binghamton, New York |  |
| 2020 |  |  |
| January 15-18 | Denver, Colorado | p. 1484 |
| March 13-15 | Charlottesville, Virginia | p. 1484 |
| March 21-22 | Medford, Massachusetts | p. 1484 |
| May 2-3 | Fresno, California | p. 1484 |
| September 12-13 | El Paso, Texas | p. 1484 |


| January 6-9 | Washington, DC | p. 1485 |
| :---: | :---: | :---: |
| July 5-9 | Grenoble, France | p. 1485 |
| July 19-23 | Buenos Aires, Argentina | p. 1485 |
| October 9-10 | Omaha, Nebraska | p. 1485 |
|  | 2022 |  |
| January 5-8 | Seattle, Washington | p. 1485 |
|  | 2023 |  |
| January 4-7 | Boston, Massachusetts | p. 1486 |

See www. ams.org/meetings for the most up-to-date information on the meetings and conferences that we offer.

## ASSOCIATE SECRETARIES OF THE AMS

Central Section: Georgia Benkart, University of WisconsinMadison, Department of Mathematics, 480 Lincoln Drive, Madison, WI 53706-1388; email: benkart@math.wisc.edu; telephone: 608-263-4283.

Eastern Section: Steven H. Weintraub, Department of Mathematics, Lehigh University, Bethlehem, PA 18015-3174; email: steve.weintraub@lehigh.edu; telephone: 610-758-3717.

Southeastern Section: Brian D. Boe, Department of Mathematics, University of Georgia, 220 D W Brooks Drive, Athens, GA 30602-7403, email: brian@math.uga.edu; telephone: 706-5422547.

Western Section: Michel L. Lapidus, Department of Mathematics, University of California, Surge Bldg., Riverside, CA 925210135; email: lapidus@math.ucr.edu; telephone: 951-827-5910.

Visit the AMS Membership Booth to learn more about the benefits of membership: In addition to receiving a discount on books purchased through the online bookstore and at meetings, members are also entitled to receive free shipping on their purchases, free and discounted subscriptions to journals, and access to colleagues via the Member Directory. Join or renew your membership at JMM and receive a complimentary gift!

Availability:
THURSDAY, JANUARY 17 ${ }^{\text {H. }}$ 9:30AM-4:25PM FRIDAY, JANUARY $1^{1{ }^{\text {TH }}}$, 9:30AM-4:25pm

AMS Members, have your professional portrait taken at the AMS Membership Booth! Back by popular demand! The AMS Membership Department has arranged for a photographer to take your profes-
 sional portrait and have it emailed to you in just a few minutes. Consider uploading this photo to your MathSciNet ${ }^{\circledR}$ Author Profile page, using it on your university website, submitting it as the professional photograph for your book publication, or using it as your profile picture in email and on social platforms.

# Meetings \& Conferences of the AMS 

IMPORTANT INFORMATION REGARDING MEETINGS PROGRAMS: AMS Sectional Meeting programs do not appear in the print version of the Notices. However, comprehensive and continually updated meeting and program information with links to the abstract for each talk can be found on the AMS website. See www. ams.org/meetings/.
Final programs for Sectional Meetings will be archived on the AMS website accessible from the stated URL

# Baltimore, Maryland 

## Baltimore Convention Center, Hilton Baltimore, and Baltimore Marriott Inner Harbor Hotel

January 16-19, 2019
Wednesday - Saturday

## Meeting \#1 145

Joint Mathematics Meetings, including the 125th Annual Meeting of the AMS, 102nd Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM)and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).
Associate secretary: Steven H. Weintraub
Announcement issue of Notices: October 2018
Program first available on AMS website: November 1, 2018 Issue of Abstracts: Volume 40, Issue 1

## Deadlines

For organizers: Expired
For abstracts: Expired
The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs /national.htm1.

## Joint Invited Addresses

Sarah Koch, University of Michigan, What is the shape of a rational map? (AMS-MAA Invited Address).

Bryna Kra, Northwestern University, Dynamics of systems with low complexity (AWM-AMS Noether Lecture).

Cathy O'Neil, ORCAA, Big data, inequality, and democracy (MAA-AMS-SIAM Gerald and Judith Porter Public Lecture).

Daniel A Spielman, Yale University, Miracles of Algebraic Graph Theory (AMS-MAA Invited Address).

## AMS Invited Addresses

Jesús A. De Loera, University of California, Davis, Algebraic, Geometric, and Topological Methods in Optimization.

Benedict H. Gross, University of California San Diego, Complex multiplication: past, present, future (AMS Colloquium Lectures: Lecture I).

Benedict H. Gross, University of California San Diego, Complex multiplication: past, present, future (AMS Colloquium Lectures: Lecture II).

Benedict H. Gross, University of California San Diego, Complex multiplication: past, present, future (AMS Colloquium Lectures: Lecture III).

Peter Ozsvath, Princeton University, From knots to symplectic geometry and algebra.

Lior Pachter, California Institute of Technology, Title to be announced.

Karen Hunger Parshall, University of Virginia, The Roaring Twenties in American Mathematics.

Alan S Perelson, Los Alamos National Laboratory, Immunology for mathematicians (AMS Josiah Willard Gibbs Lecture).

Lillian B. Pierce, Duke University, On torsion subgroups in class groups of number fields.

## AMS Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at jointmathematicsmeetings.org/meetings/abstracts /abstract.p7?type=jmm.

Some sessions are cosponsored with other organizations. These are noted within the parenthesis at the end of each listing, where applicable.

25 years of Conferences for African-American Researchers in the Mathematical Sciences (CAARMS times 25), William A. Massey, Princeton University.

A Showcase of Number Theory at Undergraduate Institutions, Adriana Salerno, Bates College, and Lola Thompson, Oberlin College.

Advances and Applications in Integral and Differential Equations, Jeffrey T. Neugebauer, Eastern Kentucky University, and Min Wang, Kennesaw State University.

Advances by Early Career Women in Discrete Mathematics, Jessalyn Bolkema, State University of New York at Oswego, and Jessica De Silva, California State University, Stanislaus.

Advances in Operator Theory, Operator Algebras, and Operator Semigroups, Joseph Ball, Virginia Tech, Marat Markin, California State University, Fresno, Igor Nikolaev, St. John's University, and Ilya Spitkovsky, New York University, Abu Dhabi.

Advances in Quantum Walks, Quantum Simulations, and Related Quantum Theory, Radhakrishnan Balu, US Army Research Lab, Chaobin Liu, Bowie State University, and Takuya Machida, Nihon University, Japan.

Agent-based Modeling in Biological and Social Systems (a Mathematics Research Communities Session), Maryann Hohn, University of California Santa Barbara, Angelika Manhart, Imperial College, London, Christopher Miles, Courant Institute, New York University, and Cole Zmurchok, University of British Columbia.

Algebraic Structures Motivated by Knot Theory, Mikhail Khovanov, Columbia University, and Jozef H. Przytycki and Alexander Shumakovitch, George Washington University.

Algebraic and Geometric Methods in Discrete Optimization, Amitabh Basu, Johns Hopkins University, and Jesus De Loera, University of California, Davis.

Algebraic, Discrete, Topological and Stochastic Approaches to Modeling in Mathematical Biology, Olcay Akman, Illinois State University, Timothy D. Comar, Benedictine University, Daniel Hrozencik, Chicago State University, and Raina Robeva, Sweet Briar College.

Algorithmic Dimensions and Fractal Geometry, Jack H. Lutz, Iowa State University, and Elvira Mayordomo, University of Zaragoza, Spain (AMS-ASL).

Analysis and Geometry of Nonlinear Evolution Equations, Marius Beceanu, University at Albany, State University of New York, and Dan-Andrei Geba, University of Rochester.

Analysis of Fractional, Stochastic, and Hybrid Dynamic Systems with Applications, John R. Graef, University of Tennessee at Chattanooga, G. S. Ladde, University of South Florida, and A. S. Vatsala, University of Louisiana at Lafayette.

Analytic Number Theory, Thomas A. Hulse, Boston College, Angel V. Kumchev and Nathan McNew, Towson University, and John Miller, The Johns Hopkins University.

Arithmetic Statistics, Michael Chou and Robert Lemke Oliver, Tufts University, and Ari Shnidman, Center for Communications Research-Princeton.

Bifurcations of Difference Equations and Discrete Dynamical Systems with Applications, Arzu Bilgin, Recep Tayyip Erdogan University, Turkey, and Toufik Khyat, Trinity College.

Commutative Ring Theory: Research for Undergraduate and Early Graduate Students, Nicholas Baeth, Franklin and Marshall College, and Branden Stone, Hamilton College.

Continued Fractions, Geremías Polanco Encarnación, Hampshire College, James McLaughlin, West Chester University, Barry Smith, Lebanon Valley College, and Nancy J. Wyshinski, Trinity College.

Counting Methods in Number Theory, Lillian Pierce, Duke University, Arindam Roy, Rice University, and Jiuya Wang, University of Wisconsin.

Definability and Decidability Problems in Number Theory, Kirsten Eisenträger, Pennsylvania State University, Deidre Haskell, McMaster University, Ontario, Canada, Jennifer Park, University of Michigan, and Alexandra Shlapentokh, East Carolina University (AMS-ASL).

Differential Equations on Fractals, Patricia Alonso-Ruiz, University of Connecticut, Joe Chen, Colgate University, Luke Rogers, University of Connecticut, Robert Strichartz, Cornell University, and Alexander Teplyaev, University of Connecticut.

Enumerative Combinatorics, Miklos Bona, University of Florida, and Cheyne Homberger, University of Maryland, Baltimore County.

Financial Mathematics, Maxim Bichuch, Johns Hopkins University, Anja Richter, Baruch College, City University of New York, and Stephan Sturm, Worcester Polytechnic Institute.

Geometric and Topological Combinatorics, Anastasia Chavez and Jamie Haddock, University of California, Davis, and Annie Raymond, University of Massachusetts, Amherst.

Geometric and Topological Generalization of Groups, Amrita Acharyya, University of Toledo, and Bikash C. Das, University of North Georgia.

Geometry Labs United: Research, Visualization, and Outreach, Marianne Korten, Kansas State University, and Sean Lawton and Anton Lukyanenko, George Mason University.

Geometry and Dynamics of Continued Fractions, Anton Lukyanenko, George Mason University, and Joseph Vandehey, Ohio State University.

Geometry of Representation Spaces, Sean Lawton, George Mason University, Chris Manon, University of Kentucky, and Daniel Ramras, Indiana University-Purdue University Indianapolis.

Group Representation Theory and Character Theory, Mohammad Reza Darafsheh, University of Tehran, Iran, and Manouchehr Misaghian, Prairie View A\&M University.

Harmonic Analysis, Partial Differential Equations, and Applications, Russell Brown, University of Kentucky, and Irina Mitrea, Temple University.

Harmonic Analysis: Recent Developments on Oscillatory Integrals (a Mathematics Research Communities Session), Xiumin Du, University of Maryland, Taryn C. Flock, University of Massachusetts Amherst, and Yakun Xi, University of Rochester.

History of Mathematics, Sloan Despeaux, Western Carolina University, Jemma Lorenat, Pitzer College, Daniel E. Otero, Xavier University, and Adrian Rice, RandolphMacon College (AMS-MAA-ICHM).

Hopf Algebras and Tensor Categories, Siu-Hung Ng, Louisiana State University, Julia Plavnik, Texas A\&M University, and Henry Tucker, University of California, San Diego.

How to Guard an Art Gallery and Other Discrete Mathematical Adventures (In Memory of T. S. Michael, 1960 to 2016), Joseph Bonin, The George Washington University, Carolyn Chun, US Naval Academy, and Nancy Neudauer, Pacific University.

If You Build It They Will Come: Presentations by Scholars in the National Alliance for Doctoral Studies in the Mathematical Sciences, David Goldberg, Purdue University, and Phil Kutzko, University of Iowa.

Latinx in Math, Alexander Diaz-Lopez, Villanova University, Laura Escobar, University of Illinois, and Juanita Pinzón-Caicedo, North Carolina State University.

Lattice Path Combinatorics and Applications, Christian Krattenthaler, University of Vienna, Austria, and Alan Krinik and Randall J. Swift, California State Polytechnic University.

Localization and Delocalization for Disordered Quantum Systems, Peter D. Hislop, University of Kentucky, Christoph A. Marx, Oberlin College, and Jeffery Schenker, Michigan State University.

Low Complexity Models in Data Analysis and Machine Learning, Emily J. King, University of Bremen, Germany, Nate Strawn, Georgetown University, and Soledad Villar, New York University.

Mappings on Metric and Banach Spaces with Applications to Fixed Point Theory, Torrey M. Gallagher, Bucknell University, and Christopher J. Lennard, University of Pittsburgh.

Mathematical Analysis in Fluid Dynamics, Yanqiu Guo, Florida International University, Jinkai Li, South China Normal University, Jing Tian, Towson University, and Yuncheng You, University of South Florida.

Mathematical Investigations of Spatial Ecology and Epidemiology, Leah Shaw and Junping Shi, College of William and Mary, and Zhisheng Shuai, University of Central Florida.

Mathematical Models in Ecology, Epidemiology, and Medicine, Richard Schugart, Western Kentucky University, and Najat Ziyadi, Morgan State University.

Mathematicians at Sea (in the Sky, or on Land): Defense Applications of Mathematics, Tegan Emerson, Timothy Doster, and George Stantchev, Naval Research Laboratory.

Mathematics in the Realm of Cyber Research, Daniel Bennett, Army Cyber Institute, Paul Goethals, United States Military Academy, and Natalie Scala, Towson University.

Mathematics of Coding Theory and Applications, Hiram Lopez-Valdez and Felice Manganiello, Clemson University, and Gretchen L. Matthews, Virginia Tech.

Multiscale Problems in the Calculus of Variations, Elisa Davoli, University of Vienna, Austria, and Rita Ferreira, King Abdullah University of Science and Technology, Saudi Arabia.

Natural Resources Modeling, Julie Blackwood, Williams College, and Shandelle M. Henson, Andrews University.

Network Science, David Burstein, Swarthmore College, Franklin Kenter, United States Naval Academy, and Feng 'Bill' Shi, University of North Carolina.

New Directions in the Theory of Complex Multiplication, Henri Darmon, McGill University, Samit Dasgupta, University of California, Santa Cruz, and Benedict Gross, Harvard University.

Nonlinear Evolution Equations and Their Applications, Mingchao Cai, Morgan State University, Gisele Mophou Loudjom, University of French West Indies, Guadeloupe, France, and Gaston N'Guerekata, Alexander Pankov, Xuming Xie, and Guoping Zhang, Morgan State University.

Not KNerds: A Community for Knot Theory, Moshe Cohen, Vassar College, Elizabeth Denne, Washington and Lee University, and Adam Lowrance, Vassar College.

Number Theoretic Methods in Hyperbolic Geometry (a Mathematics Research Communities Session), Samantha Fairchild, University of Washington, Junxian Li, Universität Göttingen, and Richard Vradenburgh, University of Virginia.

Number Theory, Arithmetic Geometry, and Computation, Brendan Hassett, Brown University, Drew Sutherland, Massachusetts Institute of Technology, and John Voight, Dartmouth College.

Numerical Methods for PDEs and Applications, Wenrui Hao, Qingguo Hong, and Jinchao Xu, Pennsylvania State University.

Optimal Methods in Applicable Analysis: Variational Inequalities, Low Rank Matrix Approximations, Systems Engineering, Cyber Security, Aritra Dutta, King Abdullah University of Science and Technology, Saudi Arabia, Ram Mohapatra, University of Central Florida, Gayatri Pany, Singapore University of Technology and Design, Singapore, and Nabin Kumar Sahu, Dhirubhai Ambani Institute of Information and Communication Technology, India.

Orthogonal Polynomials, Quantum Probability, Harmonic and Stochastic Analysis, Nobuhiro Asai, Aichi University of Education, Kariya, Japan, Rodica Costin, The Ohio State University, Aurel I. Stan, The Ohio State University at Marion, and Hiroaki Yoshida, Ochanomizu University, Tokyo, Japan.

Partition Theory and Related Topics, Dennis Eichhorn, University of California, Irvine, Tim Huber, University of Texas, Rio Grande Valley, and Amita Malik, Rutgers University.

Problems in Partial Differential Equations, Alex Himonas, University of Notre Dame, and Curtis Holliman, The Catholic University of America.

Quantum Symmetries: Subfactors and Fusion Categories (a Mathematics Research Communities Session), Cain Edie-Michell and Lauren Ruth, Vanderbilt University, and Yilong Wang, Louisiana State University.

Quaternions, Terrence Blackman, Medgar Evers College, City University of New York, and Johannes Familton and Chris McCarthy, Borough of Manhattan Community College, City University of New York.

Recent Advancements in Mathematical Modeling of Cancer, Kamila Larripa, Humboldt State University, and Hwayeon Ryu, University of Hartford.

Recent Advances and Trends in Computable Structure Theory (in honor of J. Remmel), Jennifer Chubb, University of San Francisco, and Tim McNicholl, Iowa State University.

Recent Advances in Biological Modeling and Related Dynamical Analysis, Joshi Raj Hem, Xavier University, and Yanyu Xiao, University of Cincinnati.

Recent Advances in Homological and Commutative Algebra, Neil Epstein, George Mason University, Claudiu Raicu, Notre Dame University, and Alexandra Seceleanu, University of Nebraska.

Recent Advances in Inverse Problems and Imaging, Kui Ren, University of Texas at Austin, and Yang Yang, Michigan State University.

Recent Advances in Regularity Lemmas, Gabriel Conant, University of Notre Dame, Rehana Patel, and Julia Wolf, University of Bristol, UK.

Recent Progress in Multivariable Operator Theory, Dmitry Kaliuzhnyi-Verbovetsky and Hugo Woerdeman, Drexel University.

Research in Mathematics by Early Career Graduate Students, Marat Markin, Morgan Rodgers, Khang Tran, and Oscar Vega, California State University, Fresno.

Research in Mathematics by Undergraduates and Students in Post-Baccalaureate Programs, Darren A. Narayan, Rochester Institute of Technology, Khang Tran, California State University, Fresno, Mark David Ward, Purdue University, and John Wierman, The Johns Hopkins University (AMS-MAA-SIAM).

Riordan Arrays, Alexander Burstein and Dennis Davenport, Howard University, Asamoah Nkwanta, Morgan State University, Lou Shapiro, Howard University, and Leon Woodson, Morgan State University.

Statistical, Variational, and Learning Techniques in Image Analysis and their Applications to Biomedical, Hyperspectral, and Other Imaging, Justin Marks, Gonzaga University, Laramie Paxton, Washington State University, and Viktoria Taroudaki, Eastern Washington University.

Stochastic Analysis and Applications in Finance, Actuarial Science and Related Fields, Julius N. Esunge, University of Mary Washington, See Keong Lee, University of the Sciences, Malaysia, and Aurel I. Stan, The Ohio State University at Marion.

Stochastic Differential Equations and Applications, Carey Caginalp, University of Pittsburgh.

Symbolic Dynamics, Van Cyr, Bucknell University, and Bryna Kra, Northwestern University.

The Mathematics of Gravity and Light (a Mathematics Research Communities Session), Sougata Dhar, University of Maine, Chad R. Mangum, Niagara University, and Nadine Stritzelberger, University of Waterloo.

The Mathematics of Historically Black Colleges and Universities (HBCUs) in the Mid-Atlantic, Edray Goins, Purdue University, Janis Oldham, North Carolina A\&T, Talithia Washington, Howard University, and Scott Williams, University at Buffalo, State University of New York.

Topological Data Analysis: Theory and Applications, Justin Curry, University at Albany, State University of New York, Mikael Vejdemo-Johansson, College of Staten

Island, City University of New York, and Sara Kalisnik Verovsek, Wesleyan University.

Topology, Structure and Symmetry in Graph Theory, Lowell Abrams, George Washington University, and Mark Ellingham, Vanderbilt University.

Using Modeling to Motivate the Study of Differential Equations, Robert Kennedy, Centennial High School, Ellicott City MD, Audrey Malagon, Virginia Wesleyan University, Brian Winkel, SIMIODE, Cornwall NY, and Dina Yagodich, Frederick Community College.

Women in Topology, Jocelyn Bell, Hobart and William Smith Colleges, Rosemary Guzman, University of Chicago, Candice Price, University of San Diego, and Arunima Ray, Max Planck Institute for Mathematics, Germany.

## Auburn, Alabama

## Auburn University

## March 15-17, 2019

Friday - Sunday

## Meeting \#1 146

Southeastern Section
Associate secretary: Brian D. Boe
Announcement issue of Notices: January 2019
Program first available on AMS website: January 31, 2019 Issue of Abstracts: Volume 40, Issue 2

## Deadlines

For organizers: Expired
For abstracts: January 29, 2019
The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs /sectional.htm1.

## Invited Addresses

sharandeep singh, Punjabi University Patiala, Importance of operational Research in our daily life.

Carina Curto, Pennsylvania State University, To be announced.

Ming Liao, Auburn University, To be announced.

## Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at www.ams.org/cgi-bin /abstracts/abstract.p1.

Algebraic and Discrete Methods in Mathematical Biology (Code: SS 21A), Carina Curto, The Pennsylvania State University, Katherine Morrison, University of Northern Colorado, and Nora Youngs, Colby College.

Applications of Algebraic Geometry (Code: SS 25A), Greg Blekherman, Georgia Institute of Technology, Michael Burr, Clemson University, and Tianran Chen, Auburn University at Montgomery.

Clustering Methods and Applications (Code: SS 23A), Benjamin McLaughlin, Naval Surface Warfare Center Panama City Division (NSWCPCD), and Sung Ha Kang, Georgia Institute of Technology.

Combinatorial Matrix Theory (Code: SS 2A), Zhongshan Li, Georgia State University, and Xavier Martínez-Rivera, Auburn University.

Commutative and Combinatorial Algebra (Code: SS 3A), Selvi Kara Beyarslan, University of South Alabama, and Alessandra Costantini, Purdue University.

Developments in Commutative Algebra (Code: SS 1A), Eloísa Grifo, University of Michigan, and Patricia Klein, University of Kentucky.

Differential Equations in Mathematical Biology (Code: SS 7A), Guihong Fan, Columbus State University, Zhongwei Shen, University of Alberta, and Xiaoxia Xie, Idaho State University.

Discrete and Convex Geometry (Code: SS 17A), Andras Bezdek, Auburn University, Ferenc Fodor, University of Szeged, and Wlodzimierz Kuperberg, Auburn University.

Evolution Equations and Applications (Code: SS 9A), Dmitry Glotov, Wenxian Shen, and Paul G. Schmidt, Auburn University.

Experimental Mathematics in Number Theory, Analysis, and Combinatorics (Code: SS 6A), Amita Malik, Rutgers University, and Armin Straub, University of South Alabama.

Geometric Flows and Minimal Surfaces (Code: SS 20A), Theodora Bourni, University of Tennessee, and Giuseppe Tinaglia, King's College London and University of Tennessee.

Geometric Methods in Representation Theory (Code: SS 15A), Jiuzu Hong and Shrawan Kumar, University of North Carolina, Chapel Hill, and Yiqiang Li, University at Buffalo, the State University of New York.

Geometric and Combinatorial Aspects of Representation Theory (Code: SS 19A), Mark Colarusso, University of South, and Jonas Hartwig, Iowa State University.

Geometry and Topology of Low Dimensional Manifolds, and Their Invariants (Code: SS 13A), John Etnyre, Georgia Institute of Technology, Bulent Tosun, University of Alabama, and Shea Vela-Vick, Louisiana State University.

Graph Theory in Honor of Robert E. Jamison's 70th Birthday (Code: SS 4A), Robert A Beeler, East Tennessee State University, Gretchen Matthews, Virginia Tech, and Beth Novick, Clemson University.

Hopf Algebras and Their Applications (Code: SS 10A), Robert Underwood, Auburn University at Montgomery, and Alan Koch, Agnes Scott College.

Mapping Class Groups (Code: SS 27A), Joan Birman, Columbia University, and Kevin Kordek and Dan Margalit, Georgia Institute of Technology.

Mathematical Analysis and Control Theory of Coupled Partial Differential Equation Models (Code: SS 11A), George Avalos and Pelin Gu"ven Geredeli, University of Nebraska-Lincoln, and László Kindrat, University of New Hampshire.

Nonlinear Reaction-Diffusion Equations and Their Applications (Code: SS 18A), Jerome Goddard,II, Auburn

University at Montgomery, Nsoki Mavinga, Swarthmore College, Quinn Morris, Appalachian State University, and R. Shivaji, University of North Carolina at Greensboro.

Probability and Stochastic Processes (Code: SS 5A), Ming Liao, Erkan Nane, and Jerzy Szulga, Auburn University.

Random Discrete Structures (Code: SS 24A), Lutz P Warnke, Georgia Institute of Technology, and Xavier Pérez-Giménez, University of Nebraska-Lincoln.

Recent Advances in Coarse Geometry (Code: SS 12A), Jerzy Dydak, University of Tennessee.

Recent Advances in Numerical Methods for PDEs and PDE-constrained Optimization (Code: SS 26A), Yanzhao Cao, Thi-Thao-Phuong Hoang, and Junshan Lin, Auburn University.

Recent Developments in Graph Theory (Code: SS 16A), Xiaofeng Gu, Jeong-Hyun Kang, David Leach, and Rui Xu, University of West Georgia.

Representations of Lie Algebras, Algebraic Groups, and Quantum Groups (Code: SS 8A), Joerg Feldvoss, University of South Alabama, Lauren Grimley, Spring Hill College, and Cornelius Pillen, University of South Alabama.

The Modeling and Analysis of Spatially Extended Structures (Code: SS 22A), Shibin Dai, University of Alabama, Keith Promislow, Michigan State University, and Qiliang Wu, Ohio University.

Topological Data Analysis, Statistics and Applications (Code: SS 14A), Yu-Min Chung, University of North Carolina at Greensboro, and Vasileios Maroulas, University of Tennessee.

## Honolulu, Hawaii

## University of Hawaii at Manoa

March 22-24, 2019
Friday - Sunday

## Meeting \#1 147

Central Section
Associate secretaries: Georgia Benkart and Michel L. Lapidus
Announcement issue of Notices: January 2019
Program first available on AMS website: February 7, 2019
Issue of Abstracts: Volume 40, Issue 2

## Deadlines

For organizers: Expired
For abstracts: January 29, 2019
The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs /sectional.htm1.

## Invited Addresses

Barry Mazur, Harvard University, On the arithmetic of curves (Einstein Public Lecture in Mathematics).

Aaron Naber, Northwestern University, Analysis of geometric nonlinear partial differential equations.

Deanna Needell, University of California, Los Angeles, Simple approaches to complicated data analysis.

Katherine Stange, University of Colorado, Boulder, Title to be announced.

Andrew Suk, University of California, San Diego, Title to be announced.

## Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at www.ams.org/cgi-bin /abstracts/abstract.p1.

Advances in Iwasawa Theory (Code: SS 12A), Frauke Bleher, University of Iowa, Ted Chinburg, University of Pennsylvania, and Robert Harron, University of Hawaii at Manoa.

Advances in Mathematical Fluid Mechanics (Code: SS 32A), Kazuo Yamazaki, University of Rochester, and Adam Larios, University of Nebraska - Lincoln.

Algebraic Groups, Galois Cohomology, and Local-Global Principles (Code: SS 3A), Raman Parimala, Emory University, Andrei Rapinchuk, University of Virginia, and Igor Rapinchuk, Michigan State University.

Algebraic Number Theory and Diophantine Equations (Code: SS 20A), Claude Levesque, University of Laval.

Algebraic Points (Code: SS 36A), Barry Mazur and Hector Pasten, Harvard University.

Algebraic and Combinatorial Structures in Knot Theory (Code: SS 9A), Sam Nelson, Claremont McKenna College, Natsumi Oyamaguchi, Shumei University, and Kanako Oshiro, Sophia University.

Algebraic and Geometric Combinatorics (Code: SS 17A), Andrew Berget, Western Washington University, and Steven Klee, Seattle University.

Analysis of Nonlinear Geometric Equations (Code: SS 23A), Aaron Naber, Northwestern University, and Richard Bamler, University of California Berkeley.

Analytic and Probabilistic Methods in Convex Geometry (Code: SS 27A), Alexander Koldobsky, University of Missouri, Alexander Litvak, University of Alberta, Dmitry Ryabogin, Kent State University, Vladyslav Yaskin, University of Alberta, and Artem Zvavitch, Kent State University.

Applications of Ultrafilters and Nonstandard Methods (Code: SS 33A), Isaac Goldbring, University of California, Irvine, and Steven Leth, University of Northern Colorado.

Arithmetic Dynamics (Code: SS 29A), Andrew Bridy, Texas A\&M University, Michelle Manes, University of Hawai'i at Manoa, and Bianca Thompson, Harvey Mudd College.

Arithmetic Geometry and Its Connections (Code: SS 51A), Laura Capuano, University of Oxford, and Amos Turchet, University of Washington.

Arithmetic and Transcendence of Special Functions and Special Values (Code: SS 56A), Matthew A. Papanikolas, Texas A\&M University, and Federico Pellarin, Université Jean Monnet, St. Étienne.

Coarse Geometry, Index Theory, and Operator Algebras: Around the Mathematics of John Roe (Code: SS 53A), Erik Guentner, University of Hawai'i at Manoa, Nigel Higson, Penn State University, and Rufus Willett, University of Hawai'i at Manoa.

Coding Theory and Information Theory (Code: SS 39A), Manabu Hagiwara, Chiba University, and James B. Nation, University of Hawaii.

Combinatorial and Experimental Methods in Mathematical Phylogeny (Code: SS 47A), Sean Cleary, City College of New York and the CUNY Graduate Center, and Katherine St. John, Hunter College and the American Museum of Natural History.

Commutative Algebra and its Environs (Code: SS 4A), Olgur Celikbas and Ela Celikbas, West Virginia University, and Ryo Takahashi, Nagoya University.

Computability, Complexity, and Learning (Code: SS 45A), Achilles A. Beros and Bj $\backslash$ orn Kjos-Hanssen, University of Hawai'i at Manoa.

Computational and Data-Enabled Sciences (Code: SS 54A), Roummel Marcia, Boaz Ilan, and Suzanne Sindi, University of California, Merced.

Constructive Aspects of Complex Analysis (Code: SS 7A), Ilia Binder and Michael Yampolsky, University of Toronto, and Malik Younsi, University of Hawaii at Manoa.

Differential Geometry (Code: SS 10A), Vincent B. Bonini, Cal Poly San Luis Obispo, Jie Qing, University of California, Santa Cruz, and Bogdan D. Suceava, California State University, Fullerton.

Dynamical Systems and Algebraic Combinatorics (Code: SS 57A), Maxim Arnold, University of Texas at Dallas, Jessica Striker, North Dakota State University, and Nathan Williams, University of Texas at Dallas.

Emerging Connections with Number Theory (Code: SS 43A), Katherine Stange, University of Colorado, Boulder, and Renate Scheidler, University of Calgary.

Equivariant Homotopy Theory and Trace Methods (Code: SS 58A), Andrew Blumberg, University of Texas, Teena Gerhardt, Michigan State University, Michael Hill, UCLA, and Michael Mandell, Indiana University.

Factorization and Arithmetic Properties of Integral Domains and Monoids (Code: SS 31A), Scott Chapman, Sam Houston State University, Jim Coykendall, Clemson University, and Christopher O'Neill, University California, Davis.

Generalizations of Symmetric Spaces (Code: SS 22A), Aloysius Helminck, University of Hawaii at Manoa, Vicky Klima, Appalachian State University, Jennifer Schaefer, Dickinson College, and Carmen Wright, Jackson State University.

Geometric Approaches to Mechanics and Control (Code: SS 55A), Monique Chyba, University of Hawaii at Manoa, Tomoki Ohsawa, The University of Texas at Dallas, and Vakhtang Putkaradze, University of Alberta.

Geometry, Analysis, Dynamics and Mathematical Physics on Fractal Spaces (Code: SS 8A), Joe P. Chen, Colgate University, Lũ (Tim) Hùng, Hawai'i Pacific University, Machiel van Frankenhuijsen, Utah Valley University, and Robert G. Niemeyer, University of the Incarnate Word.

Homotopy Theory (Code: SS 48A), Kyle Ormsby and Angélica Osorno, Reed College.

Interactions between Geometric Measure Theory, PDE, and Harmonic Analysis (Code: SS 41A), Mark Allen, Brigham Young University, Spencer Becker-Kahn, University of Washington, Max Engelstein, Massachusetts Institute of Technology, and Mariana Smit Vega Garcia, University of Washington.

Interactions between Noncommutative Algebra and Noncommutative Algebraic Geometry (Code: SS 24A), Garrett Johnson, North Carolina Central University, Bach Nguyen and Xingting Wang, Temple University, and Daniel Yee, Bradley University.

Lie Theory in the Representations of Groups and Related Structures - dedicated to the memory of Kay Magaard (Code: SS 14A), Christopher Drupieski, DePaul University, and Julia Pevtsova, University of Washington.

Mapping Class Groups (Code: SS 35A), Asaf Hadari, University of Hawaii.

Mathematical Analysis of Nonlinear Phenomena (Code: SS 16A), Mimi Dai, University of Illinois at Chicago.

Mathematical Methods and Models in Medicine (Code: SS 19A), Monique Chyba, University of Hawaii, and Jakob Kotas, University of Hawaii and University of Portland.

New Trends in Geometric Measure Theory (Code: SS 37A), Antonio De Rosa, Courant Institute of Mathematical Sciences, New York University, and Luca Spolaor, Massachusetts Institute of Technology.

New Trends on Variational Calculus and Non-Linear Partial Differential Equations (Code: SS 44A), Craig Cowan, University of Manitoba, Michinori Ishiwata, Osaka University, Abbas Moameni, Carleton University, and Futoshi Takahashi, Osaka City University.

Nonlinear Wave Equations and Applications (Code: SS 42A), Boaz Ilan, University of California, Merced, and Barbara Prinari, University of Colorado, Colorado Springs.

Numerical Methods for Partial Differential Equations (Code: SS 50A), Evan Gawlik, Michael Holst, and Martin Licht, University of California, San Diego.

Real and Complex Singularities (Code: SS 34A), Leslie Charles Wilson, University of Hawaii, Manoa, Goo Ishikawa, Hokkaido University, and David Trotman, AixMarseille University.

Recent Advances and Applications of Modular Forms (Code: SS 1A), Amanda Folsom, Amherst College, Pavel Guerzhoy, University of Hawaii at Manoa, Masanobu Kaneko, Kyushu University, and Ken Ono, Emory University.

Recent Advances in Lie and Related Algebras and their Representations (Code: SS 28A), Brian D. Boe, University of Georgia, and Jonathan Kujawa, University of Oklahoma.

Recent Advances in Numerical Methods for PDEs (Code: 2249A), Hengguang Li, Wayne State University, and Sara Pollock, University of Florida.

Recent Advances in Numerical Methods for PDEs (Code: SS 49A), Hengguang Li, Wayne State University, and Sara Pollock, University of Florida.

Recent Developments in Automorphic Forms (Code: SS 2A), Solomon Friedberg, Boston College, and Jayce Getz, Duke University.

Recent Trends in Algebraic Graph Theory (Code: SS 26A), Sebastian Cioaba, University of Delaware, and Shaun Fallat, University of Regina.

SYZ Mirror Symmetry and Enumerative Geometry (Code: SS 11A), Siu Cheong Lau, Boston University, Naichung Leung, The Chinese University of Hong Kong, and Hsian-Hua Tseng, Ohio State University.

Several Complex Variables (Code: SS 5A), Peter Ebenfelt, University of California, San Diego, John Erik Fornaess, University of Michigan and Norwegian University of Science and Technology, Ming Xiao, University of California, San Diego, and Yuan Yuan, Syracuse University.

Spaces of Holomorphic Functions and Their Operators (Code: SS 21A), Mirjana Jovovic and Wayne Smith, University of Hawaii.

Sparsity, Randomness, and Optimization (Code: SS 15A), Deanna Needell and Jamie Haddock, University of California, Los Angeles.

Spectral Geometry: The Length and Laplace Spectra of Riemannian Manifolds (Code: SS 25A), Benjamin Linowitz, Oberlin College, and Jeffrey S. Meyer, California State University at San Bernardino.

Stability and Singularity in Fluid Dynamics (Code: SS 40A), Tristan Buckmaster, Princeton University, Steve Shkoller, University of California, Davis, and Vlad Vicol, Princeton University.

Structural Graph Theory (Code: SS 30A), Zixia Song, University of Central Florida, Martin Rolek, College of William and Mary, and Yue Zhao, University of Central Florida.

The Mathematics of Cryptography (Code: SS 18A), Shahed Sharif, California State University, San Marcos, and Alice Silverberg, University of California, Irvine.

Three-dimensional Floer Theory, Contact Geometry, and Foliations (Code: SS 6A), Joan Licata, Australian National University, and Robert Lipshitz, University of Oregon.

Topics at the Interface of Analysis and Geometry (Code: SS 38A), Alex Austin and Sylvester Eriksson-Bique, University of California, Los Angeles.

Valuations on Algebraic Function Fields and Their Subrings (Code: SS 46A), Ron Brown, University of Hawaii, Steven Dale Cutkosky, University of Missouri, and FranzViktor Kuhlmann, University of Szczecin.

What is Happening in Mathematical Epidemiology? Current Theory, New Methods, and Open Questions (Code: SS 52A), Olivia Prosper, University of Kentucky.

# Hartford, Connecticut 

## University of Connecticut Hartford (Hartford Regional Campus)

April 13-14, 2019
Saturday - Sunday

## Meeting \#1 148

Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of Notices: February 2019
Program first available on AMS website: February 21, 2019
Issue of Abstracts: Volume 40, Issue 2

## Deadlines

For organizers: Expired
For abstracts: February 5, 2019
The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs /sectional.htm7.

## Invited Addresses

Olivier Bernardi, Brandeis University, Percolation on triangulations and a bijective path to Liouville quantum gravity.

Brian Hall, Notre Dame University, Eigenvalues of random matrices in the general linear group in the large- $N$ limit.

Christina Sormani, Lehman College and CUNY Graduate Center, Compactness Theorems for Sequences of Riemannian Manifolds.

## Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at www.ams.org/cgi-bin /abstracts/abstract.p1.

Algebraic Number Theory (Code: SS 22A), Harris Daniels, Amherst College, and Alvaro Lozano-Robledo and Erik Wallace, University of Connecticut.

Analysis, Geometry, and PDEs in Non-smooth Metric Spaces (Code: SS 1A), Vyron Vellis, University of Connecticut, Xiaodan Zhou, Worcester Polytechnic Institute, and Scott Zimmerman, University of Connecticut.

Banach Space Theory and Metric Embeddings (Code: SS 11A), Mikhail Ostrovskii, St. John's University, and Beata Randrianantoanina, Miami University.

Chip-firing and Divisor Theory (Code: SS 19A), Caroline Klivans, Brown University, and David Perkinson, Reed College.

Cluster Algebras and Related Topics (Code: SS 12A), Emily Gunawan and Ralf Schiffler, University of Connecticut.

Combinatorial Commutative Algebra and Polyhedral Geometry (Code: SS 13A), Elie Alhajjar, US Military Academy, and McCabe Olsen, Ohio State University.

Computability Theory (Code: SS 2A), Damir Dzhafarov and Reed Solomon, University of Connecticut, and Linda Brown Westrick, Pennsylvania State University.

Convergence of Riemannian Manifolds (Code: SS 17A), Lan-Hsuan Huang and Maree Jaramillo, University of Connecticut, and Christina Sormani, City University of New York Graduate Center and Lehman College.

Discrete Dynamical Systems and Applications (Code: SS 20A), Elliott J. Betrand, Sacred Heart University, and David McArdle, University of Connecticut.

Invariants of Knots, Links, and Low-dimensional Manifolds (Code: SS 15A), Patricia Cahn, Smith College, and Moshe Cohen and Adam Lowrance, Vassar College.

Knot Theory, the Colored Jones Polynomial, and Khovanov Homology (Code: SS 18A), Adam Giambrone, Elmira College, and Katherine Hall, University of Connecticut.

Mathematical Cryptology (Code: SS 8A), Lubjana Beshaj, United States Military Academy, and Jaime Gutierrez, University of Cantabria, Santander, Spain.

Mathematical Finance (Code: SS 14A), Oleksii Mostovyi, University of Connecticut, Gu Wang, Worcester Polytechnic Institute, and Bin Zhou, University of Connecticut.

Modeling and Qualitative Study of PDEs from Materials Science and Geometry. (Code: SS 6A), Yung-Sze Choi, Changfeng Gui, and Xiaodong Yan, University of Connecticut.

Recent Advances in Structured Matrices and Their Applications (Code: SS 16A), Maxim Derevyagin, University of Connecticut, Olga Holz, University of California, Berkeley, and Vadim Olshevsky, University of Connecticut.

Recent Development of Geometric Analysis and Nonlinear PDEs (Code: SS 3A), Ovidiu Munteanu, Lihan Wang, and Ling Xiao, University of Connecticut.

Representation Theory of Quantum Algebras and Related Topics (Code: SS 10A), Drew Jaramillo, University of Connecticut, Garrett Johnson, North Carolina Central University, and Margaret Rahmoeller, Roanoke College.

Special Session on Regularity Theory of PDEs and Calculus of Variations on Domains with Rough Boundaries (Code: SS 5A), Murat Akman, University of Connecticut, and Zihui Zhao, University of Washington.

Special Values of L-functions and Arithmetic Invariants in Families (Code: SS 21A), Ellen Eischen, University of Oregon, Yifeng Liu, Yale University, Liang Xiao, University of Connecticut, and Wei Zhang, Massachusetts Institute of Technology.

Stochastic Analysis and Related Fields (Code: SS 7A), Fabrice Baudoin, University of Connecticut, and Cheng Ouyang, University of Illinois at Chicago.

Stochastic Processes, Random Walks, and Heat Kernels (Code: SS 4A), Patricia Alonso Ruiz, University of Connecticut, and Phanuel Mariano, Purdue University.

Sub-Riemannian and CR Geometric Analysis (Code: SS 9A), Fabrice Baudoin, University of Connecticut, and Luca Capogna, Worcester Polytechnic Institute.

## Quy Nhon City, <br> Vietnam

## Quy Nhon University

June 10-13, 2019
Monday - Thursday
Meeting \#1 149
Associate secretary: Brian D. Boe
Announcement issue of Notices: April 2019
Program first available on AMS website: To be announced Issue of Abstracts: To be announced

## Deadlines

For organizers: November 30, 2018
For abstracts: To be announced
The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs /internmtgs.htm7.

## Invited Addresses

Henry Cohn, Microsoft Research, To be announced.
Robert Guralnick, University of Southern California, To be announced.

Le Tuan Hoa, Hanoi Institute of Mathematics, To be announced.

Nguyen Dong Yen, Hanoi Institute of Mathematics, To be announced.

Zhiwei Yun, Massachusetts Institute of Technology, To be announced.

Nguyen Tien Zung, Toulouse Mathematics Institute, To be announced.

## Madison, Wisconsin

## University of Wisconsin-Madison

## September 14-15,2019

Saturday - Sunday

## Meeting \#1 150

Central Section
Associate secretary: Georgia Benkart
Announcement issue of Notices: June 2019
Program first available on AMS website: July 23, 2019
Issue of Abstracts: Volume 40, Issue 3

## Deadlines

For organizers: February 14, 2019

For abstracts: July 16, 2019
The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs /sectional.htm1.

## Invited Addresses

Nathan Dunfield, University of Illinois, Urbana-Champaign, Title to be announced.

Teena Gerhardt, Michigan State University, Title to be announced.

Lauren Williams, University of California, Berkeley, Title to be announced (Erdős Memorial Lecture).

## Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at www.ams.org/cgi-bin /abstracts/abstract.p1.

Association Schemes and Related Topics - in Celebration of J.D.H. Smith's 70th Birthday (Code: SS 8A), Kenneth W. Johnson, Penn State University Abington, and Sung Y. Song, Iowa State University.

Computability Theory in honor of Steffen Lempp's 60th birthday (Code: SS 6A), Joseph S. Miller, Noah D. Schweber, and Mariya I. Soskova, University of Wiscon-sin-Madison.

Homological and Characteristic $p>0$ Methods in Commutative Algebra (Code: SS 1A), Michael Brown, University of Wisconsin-Madison, and Eric Canton, University of Michigan.

Model Theory (Code: SS 5A), Uri Andrews and Omer Mermelstein, University of Wisconsin-Madison.

Recent Developments in Harmonic Analysis (Code: SS 3A), Theresa Anderson, Purdue University, and Joris Roos, University of Wisconsin-Madison.

Recent Work in the Philosophy of Mathematics (Code: SS 4A), Thomas Drucker, University of Wisconsin-Whitewater, and Dan Sloughter, Furman University.

Several Complex Variables (Code: SS 7A), Hanlong Fang and Xianghong Gong, University of Wisconsin-Madison.

Special Functions and Orthogonal Polynomials (Code: SS 2A), Sarah Post, University of Hawai'i at Manoa, and Paul Terwilliger, University of Wisconsin-Madison.

Uncertainty Quantification Strategies for Physics Applications (Code: SS 9A), Qin Li, University of WisconsinMadison, and Tulin Kaman, University of Arkansas.

## Binghamton, New York

Binghamton University

October 12-13, 2019
Saturday - Sunday
Meeting \#1151
Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of Notices: August 2019
Program first available on AMS website: August 29, 2019
Issue of Abstracts: Volume 40, Issue 3

## Deadlines

For organizers: March 12, 2019
For abstracts: August 20, 2019
The scientific information listed below may be dated. For the latest information, see www. ams.org/amsmtgs /sectional.htm1.

## Invited Addresses

Richard Kenyon, Brown University, Title to be announced.

Tony Pantev, University of Pennsylvania, Title to be announced.

Lai-Sang Young, New York University, Title to be announced.

## Gainesville, Florida

## University of Florida

November 2-3, 2019
Saturday - Sunday

## Meeting \#1 152

Southeastern Section
Associate secretary: Brian D. Boe
Announcement issue of Notices: September 2019
Program first available on AMS website: September 19, 2019
Issue of Abstracts: Volume 40, Issue 4

## Deadlines

For organizers: April 2, 2019
For abstracts: September 10, 2019
The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs /sectional.htm1.

## Invited Addresses

Jonathan Mattingly, Duke University, To be announced.
Isabella Novik, University of Washington, To be announced.

Eduardo Teixeira, University of Central Florida, To be announced.

## Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at www.ams.org/cgi-bin /abstracts/abstract.p1.

Geometric and Topological Combinatorics (Code: SS 1A), Bruno Benedetti, University of Miami, Steve Klee, Seattle University, and Isabella Novik, University of Washington.

## Riverside, California

University of California, Riverside
November 9-10, 2019
Saturday - Sunday

## Meeting \#1 153

Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of Notices: September 2019
Program first available on AMS website: September 12, 2019
Issue of Abstracts: Volume 40, Issue 4

## Deadlines

For organizers: April 9, 2019
For abstracts: September 3, 2019
The scientific information listed below may be dated. For the latest information, see www. ams.org/amsmtgs /sectional.htm1.

## Invited Addresses

Robert Boltje, University of California, Santa Cruz, Title to be announced.

Jonathan Novak, University of California, San Diego, Title to be announced.

Anna Skripka, University of New Mexico, Albuquerque, Title to be announced.

## Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at www.ams.org/cgi-bin /abstracts/abstract.p1.

Inverse Problems (Code: SS 3A), Hanna Makaruk, Los Alamos National Laboratory, and Robert Owczarek, Uni-
versity of New Mexico, Albuquerque and University of New Mexico, Los Alamos.

Random Matrices and Related Structures (Code: SS 2A), Jonathan Novak, University of California, San Diego, and Karl Liechty, De Paul University.

Topics in Operator Theory (Code: SS 1A), Anna Skripka and Maxim Zinchenko, University of New Mexico.

## Denver, Colorado

## Colorado Convention Center

January 15-18, 2020
Wednesday - Saturday

## Meeting \#1 154

Joint Mathematics Meetings, including the 126th Annual Meeting of the AMS, 103rd Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM)
Associate secretary: Michel L. Lapidus
Announcement issue of Notices: October 2019
Program first available on AMS website: November 1, 2019
Program issue of electronic Notices: To be announced Issue of Abstracts: To be announced

## Deadlines

For organizers: April 1, 2019
For abstracts: To be announced

## Charlottesville, Virginia

University of Virginia
March 13-15, 2020
Friday - Sunday
Southeastern Section
Associate secretary: Brian D. Boe
Announcement issue of Notices: To be announced
Program first available on AMS website: To be announced Issue of Abstracts: To be announced

## Deadlines

For organizers: To be announced For abstracts: To be announced

## Medford, Massachusetts

Tufts University

March 21-22, 2020
Saturday - Sunday
Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of Notices: To be announced
Program first available on AMS website: To be announced
Issue of Abstracts: To be announced

## Deadlines

For organizers: To be announced For abstracts: To be announced

## Fresno, California

## California State University, Fresno

May 2-3, 2020
Saturday - Sunday
Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of Notices: To be announced
Program first available on AMS website: To be announced Issue of Abstracts: To be announced

## Deadlines

For organizers: To be announced For abstracts: To be announced

## El Paso, Texas

University of Texas at El Paso
September 12-13, 2020
Saturday - Sunday
Central Section
Associate secretary: Georgia Benkart
Announcement issue of Notices: To be announced Program first available on AMS website: To be announced Issue of Abstracts: To be announced

## Deadlines

For organizers: To be announced For abstracts: To be announced

# Washington, District of Columbia 

Walter E. Washington Convention Center

## January 6-9, 2021

Wednesday - Saturday
Joint Mathematics Meetings, including the 127th Annual Meeting of the AMS, 104th Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).
Associate secretary: Brian D. Boe
Announcement issue of Notices: October 2020
Program first available on AMS website: November 1, 2020 Issue of Abstracts: To be announced

## Deadlines

For organizers: April 1, 2020
For abstracts: To be announced

## Grenoble, France

## Université Grenoble Alpes

July 5-9, 2021
Monday - Friday
Associate secretary: Michel L. Lapidus Announcement issue of Notices: To be announced Program first available on AMS website: To be announced Issue of Abstracts: To be announced

## Deadlines

For organizers: To be announced For abstracts: To be announced

## Buenos Aires, Argentina

The University of Buenos Aires

July 19-23, 2021
Monday - Friday
Associate secretary: Steven H. Weintraub
Announcement issue of Notices: To be announced Program first available on AMS website: To be announced Issue of Abstracts: To be announced

## Deadlines

For organizers: To be announced For abstracts: To be announced

## Omaha, Nebraska

## Creighton University

October 9-10, 2021
Saturday - Sunday
Central Section
Associate secretary: Georgia Benkart
Announcement issue of Notices: To be announced Program first available on AMS website: To be announced Issue of Abstracts: To be announced

## Deadlines

For organizers: To be announced For abstracts: To be announced

## Seattle, Washington

## Washington State Convention Center and the Sheraton Seattle Hotel

January 5-8, 2022
Wednesday - Saturday
Associate secretary: Georgia Benkart
Announcement issue of Notices: October 2021
Program first available on AMS website: To be announced Issue of Abstracts: To be announced

## Deadlines

For organizers: To be announced For abstracts: To be announced

## Boston, Massachusetts

John B. Hynes Veterans Memorial<br>Convention Center, Boston Marriott Hotel, and Boston Sheraton Hotel<br>January 4-7, 2023<br>Wednesday - Saturday<br>Associate secretary: Steven H. Weintraub<br>Announcement issue of Notices: October 2022<br>Program first available on AMS website: To be announced<br>Issue of Abstracts: To be announced

## Deadlines

For organizers: To be announced
For abstracts: To be announced


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Part II: Societal Record, is organized alphabetically under each category heading. This Part provides readers with a searchable listing of all content of record for the Society, including: elections, awards, meetings, news, opportunities, annual AMS reports, surveys, grants, fellowships, etc.

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## 2019 Joint Mathematics Meetings Advance Registration/Housing Form

Name
(please print your name as you would like it to appear on your badge)


Acknowledgment of this registration and any hotel reservations will be sent to the email address(es) given here. Check this box to receive a copy in U.S. Mail:
Affiliation for badge (company/university) Nonmathematician guest badge name: $\quad$ (Note fee of USS 22)

PLEASE NOTE THAT BADGES WILL NOT BE MAILED IN ADVANCE FOR THIS MEETING. YOU MAY OPT TO HAVE YOUR PROGRAM MAILED ON DEC. 12 (SEE BELOW)


## MAA Minicourses (see listing in text)

I would like to attend: I One Minicourse $\square$ Two Minicourses Please enroll me in MAA Minicourse(s) \# $\qquad$ and \#

## Price: US\$ 100 for each minicourse.

(For more than 2 minicourses, call or email the MMSB.)

## Payment

Registration \& Event Total (total from column on left)
Hotel Deposit (only if paying by check) $\qquad$
If you send a hotel deposit check, the
deadline for this form is December 1.
Total Amount To Be Paid
\$
Method of Payment

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$\square$ Credit Card. All major credit cards accepted. For your security, we do not accept credit card numbers by email, fax, or postal mail. If the MMSB receives your registration form by any of these methods, it will contact you at the phone number provided on this form.

Signature:

- Purchase Order \# (please enclose copy)


## Other Information

Mathematical Reviews primary field of interest \#

- I am willing to serve as a judge for the MAA Undergraduate Student Poster Session.
- If you are an undergraduate, are you interested in participating in the Radical Dash, a multi-day scavenger hunt sponsored by the MAA?
$\square$ For planning purposes for the MAA Two-year College Reception, please check if you are a faculty member at a two-year college.
- Please this box if you have a disability requiring special services.

To respect your privacy and to better serve you, please indicate your preferences for the following:

- Please include my name and affiliation on the JMM Participant List.
- Please include my name and postal address on promotional mailing lists.

Registration for the Joint Meetings is not required for the short course but it is required for the minicourses and the Employment Center. To register for the Employment Center, go to www.ams.org/profession/employment-services. For questions, email emp-info@ams.org

## Registration Deadlines

To be eligible for the complimentary hotel room lottery: In time to receive programs in the mail: Hotel reservations with check deposit: Hotel reservations, changes/cancellations through the JMM website:
Advance registration for the Joint Meetings, short course, minicourses, and dinner tickets:
Cancel in time to receive $50 \%$ refund on advance registration, banquets, minicourses, and short course
"no refunds issued after this date.

## Mailing Address/Contact:

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Providence, RI 02940-6887 Fax: 401-455-4004; Email: mmsb@ams.org
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## MD

Please see the hotel information in the announcement or on the web for detailed information on each hotel. To ensure accurate assignments. please rank hotels in order of preference by writing $1,2,3$, etc. in the column on the left and by circling the requested bed configuration. If your requested hotel and room type is no longer available, you will be assigned a room at the next available comparable rate. Please call the MMSB for details on suite configurations, sizes, availability, etc. All reservations, including suite reservations, must be made through the MMSB to receive the JMM rates. Reservations made directly with the hotels before December 14,2018 may be changed to a higher rate. All rates are subject to applicable local and state taxes in effect at the time of check-in; currently $15.5 \%$ state tax. Guarantee requirements: First night deposit by check (add to payment on reverse of form) or a credit card guarantee. Please note that reservations with check deposits must be received by the MMSB by December 1, 2018 People interested in suites should contact the MMSB directly at mmsb@ams. org or by calling 800-321-4267, ext. 4137; ( $401-455-4137$ ).
$\square$ Deposit enclosed (see front of form) Deposit enclosed (see front of form) number provided on the reverse of this form. Date and Time of Arrival

## Name of Other Adult Room Occupant (s)

 - I have disabilities as defined by the ADA that require a sleeping room that is accessible to the physically challenged. My needs are: - I am a member of a hotel frequent-travel club and would like to receive appropriate credit. The hotel chain and card number are:| $\begin{array}{c\|} \hline \text { Order } \\ \text { of choice } \end{array}$ | Hotel | Single | $\begin{array}{\|c\|} \hline \text { Double } \\ 1 \text { bed-2 people } \\ \hline \end{array}$ | $\begin{gathered} \text { Double } \\ 2 \text { beds- } 2 \text { people } \end{gathered}$ | $\begin{gathered} \text { Triple } \\ 3 \text { adults-2 beds } \end{gathered}$ | $\begin{gathered} \text { Quad } \\ 4 \text { adults-2 beds } \end{gathered}$ | Rollaway/Cot Information (add to special requests if reserving online) |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Hilton Baltimore (co-headquarters) | US\$ 179 | US\$ 179 | US\$ 179 | US\$ 199 | US \$219 | No charge for rollaway cots, available in king-bedded rooms only |
|  | Student Rate | US\$ 149 | US\$ 149 | US\$ 149 | US \$169 | US \$189 |  |
|  | Baltimore Marriott Inner Harbor (co-headquarters) | US\$ 169 | US\$ 169 | US\$ 169 | US \$189 | US\$ 209 | No charge for rollaway cots, available in king-bedded rooms only |
|  | Student Rate | US\$ 138 | US\$ 138 | US\$ 138 | US \$158 | US\$ 178 |  |
|  | Sheraton Inner Harbor | US\$ 159 | US\$ 159 | US\$ 159 | US\$ 179 | US\$ 199 | No charge for rollaway cots, available in king-bedded rooms only |
|  | Student Rate | US\$ 145 | US\$ 145 | US\$ 145 | US\$ 165 | US\$ 185 |  |
|  | Renaissance Harborplace Hotel | US\$ 155 | US\$ 155 | US\$ 155 | US\$ 175 | US\$ 195 | Rollaway cots available for US\$ 25 per stay, in kingbedded fooms only |
|  | Student Rate | US\$ 138 | US\$ 138 | US\$ 138 | US\$ 158 | US\$ 178 |  |
|  | Hyatt Regency Baltimore | US\$ 150 | US\$ 150 | US\$ 150 | US\$ 175 | US\$ 200 | Rollaway cots available for US\$ 25 per stay in kingbedded rooms only |
|  | Student Rate | US\$ 140 | US\$ 140 | US\$ 140 | US\$ 165 | US\$ 190 |  |
|  | Royal Sonesta | US\$ 149 | US\$ 149 | US\$ 149 | US\$ 169 | US\$ 189 | Rollaway cots available for US $\$ 30$ per night in kingbedded rooms only. |
|  | Student Rate | US\$ 134 | US\$ 134 | US\$ 134 | US\$ 154 | US\$ 174 |  |
|  | Lord Baltimore Hotel | US\$ 139 | US\$ 139 | US\$ 139 | US\$ 159 | US\$ 179 | Rollaway cots available for US\$20 per night in kingbedded fooms only |
|  | Student Rate | US\$ 129 | US\$ 129 | US\$ 129 | US\$ 149 | US\$ 169 |  |
|  | Hotel Monaco | US\$ 135 | US\$ 135 | US\$ 135 | US\$ 155 | US\$ 175 | Rollaway cots available for US\$ 25 per stay in kingbedded rooms only. |
|  | Student Rate | US\$ 125 | US\$ 125 | US\$ 125 | US\$ 145 | US\$ 165 |  |
|  | Days Inn Baltimore Inner Harbor | US\$ 139 | US\$ 139 | US\$ 139 | N/A | N/A | Rollaway cots available for US\$ 10 per night based on availability |
|  | Student Rate | US\$ 129 | US\$ 129 | US\$ 129 | N/A | N/A |  |
|  | Holiday Inn Inner Harbor | US\$ 139 | US\$ 139 | US\$ 139 | US\$ 159 | US\$ 179 | No charge for rollaway cots, available in king-bedded rooms only |
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Brad G. Osgood, Stanford University, CA
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Pure and Applied Undergraduate Texts, Volume 33; 2019; approximately 702 pages; Hardcover; ISBN: 978 1-4704-4191-3; List US\$115; AMS members US\$92; MAA members US\$103.50; Order code AMSTEXT/33

## Algebraic Statistics

Seth Sullivant, North Carolina State University, Raleigh
This text uses tools from algebraic geometry commutative algebra, combinatorics, and their computational sides to address problems in statistics and its applications.
Graduate Studies in Mathematics, Volume 194; 2018; 490 pages; Hardcover; ISBN: 978-1-4704-3517-2; List US\$83; AMS members US\$66.40; MAA members US\$74.70; Order code GSM/194

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Carolyn Yackel, Mercer University, Macon, GA, Chief Editor, and sarah-marie belcastro, MathILy, Mathematical Staircase, Inc., Holyoke, MA, and Smith College, Northampton, MA Assistant Editor
This book transports the reader on a math-
ematical exploration of fiber arts that is rigorous enough to capture the hearts of mathematicians.

2018; 232 pages; Hardcover; ISBN: 978-1-4704-2931-7; List US\$40; AMS members US\$32; MAA members US\$36; Order code MBK/117

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Anthony Bonato, Ryerson University, Toronto, ON, Canada
Written in an accessible style and enriched by dozens of images, this book offers a rare insight into the minds of mathematicians, provided in their own words.
2018; 155 pages; Softcover; ISBN: 978-1-4704-4791-5; List US\$29; AMS members US\$23.20; MAA members US\$26.10; Order code MBK/118

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Topics for Grades 1-5
Laura Givental, United Math Circles Foundation, Berkeley and Stanford, CA, Maria Nemirovskaya, University of Oregon, Eugene, and Ilya Zakharevich, United Math Circles Foundation, Berkeley and Stanford, CA This book is based on selected topics that the authors taught in math circles for elementary school students, including combinatorics, Fibonacci numbers, Pascal's triangle, and the notion of area, among others. Titles in this series are co-published with the Mathematical

Sciences Research Institute (MSRI).
MSRI Mathematical Circles Library, Volume 21; 2018; 169 pages; Softcover; ISBN: 978-1-4704-4785-4; List US\$25; AMS members US\$20; MAA members US\$22.50; Order code MCL/21

## An Introduction to Ramsey Theory

Fast Functions, Infinity, and Metamathematics
Matthew Katz, Pennsylvania State University, University Park, and Jan Reimann, Pennsylvania State University, University Park
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Student Mathematical Library, Volume 87; 2018; 207 pages; Softcover; ISBN: 978-1-4704-4290-3; List US\$52; All individuals US\$41.60; Order code STML/87

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[^19]:    ${ }^{1}$ The 2018 Fields Medals were reported in the November 2018 Notices,https://www.ams.org/journa1s/notices/201810/ rnoti-p1285.pdf.
    *We invite readers to submit letters to the editor at notices-1etters@ams.org.

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[^22]:    ${ }^{1}$ See for example "How Mathematicians Think" https://www.ams .org/notices/200711/tx071101496p.pdf.

[^23]:    ${ }^{2}$ See also "The Mechanization of Mathematics" by Jeremy Avigad, in the June/July 2018 Notices www.ams.org/journa1s /notices/201806/rnoti-p681.pdf.

[^24]:    3 "Mathematicians Discuss the Snowden Revelations," https:// www.ams.org/notices/201406/rnoti-p623.pdf.

[^25]:    Katharine A. Ott is associate professor of mathematics at Bates College. Her email address is kott@bates. edu.
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    DOI: http://dx.doi.org/10.1090/noti1785

[^26]:    See an interview with Meza in the October 2018 Notices.

[^27]:    -From a Davidson Fellows announcement

