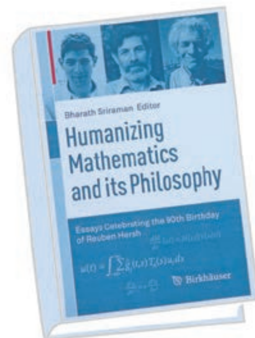
BOOK REVIEW

Humanizing Mathematics and its Philosophy



A Review by Joseph Auslander

Communicated by Cesar E. Silva

Humanizing Mathematics and its Philosophy: Essays Celebrating the 90th Birthday of Reuben Hersh
Bharath Sriraman, editor
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Although he was not trained as a philosopher, Reuben Hersh has emerged as a major figure in the philosophy of mathematics. His Humanist view is probably the most cogent expression contrasting the dominant philosophies of Platonism and Formalism.

Hersh has had a remarkable career. He was born in the Bronx, to immigrant working class parents. He graduated from Harvard at age nineteen with a major in English and worked for some time as a journalist, followed by a period of working as a machinist. He then attended the Courant Institute from which he obtained a PhD in 1962, under the supervision of Peter Lax. Hersh settled into a conventional academic career as a professor at The University of New Mexico, pursuing research mostly in partial differential equations. But beginning in the 1970s he wrote a number of expository articles, in *Scientific American*, *Mathematical Intelligencer*, and *Advances in Mathematics*. Subsequently, he wrote and coauthored with Philip Davis and Vera John-Steiner several books and developed what he terms a Humanist philosophy of mathematics. He has also been a frequent book reviewer for *Notices*.[1]

Reuben Hersh with co-author Vera John-Steiner signing copies of their book *Loving and Hating Mathematics: Challenging the Myths of Mathematical Life* **at the Princeton University Press Exhibit, JMM 2011.**

This Humanist point of view is presented in several of his books, most notably in 1997's *What Is Mathematics, Really?* A (too) brief summary: Mathematics is part

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[1] *See for example "How Mathematicians Think"* https://www.ams.org/notices/200711/tx071101496p.pdf.

of human culture "like literature, religion, and banking," mathematical knowledge isn't infallible, and there are different versions of proof and rigor. Moreover, a philosophy of mathematics should be guided by what mathematicians actually do.

The Humanist view is contrasted with Platonism and Formalism. Platonism is the position that mathematical objects and truths exist independently of individuals, and the obligation of mathematicians is to discover these truths. Formalism regards mathematics as the study of formal deductive systems, and mathematical truth is just provability in the system.

Bharath Sriraman has collected twenty-five essays from mathematicians and philosophers, an educator, and a linguist for a *Festschrift* volume to celebrate Hersh's ninetieth birthday. As one might expect, this is an extremely varied collection of articles. Several contain heartfelt appreciation of Hersh. Some attempt to develop and elucidate his Humanist philosophy. Others just mention him in passing or not at all. A few contain some nontrivial mathematics.

In the preface, Sriraman says that Hersh asked the contributors to speculate on the future of mathematics, mathematical education, and the philosophy of mathematics and in particular to address the prediction of Paul Cohen that "at some unspecified future time, mathematicians would be replaced by computers."[2]

The book starts out with a delightful interview with Hersh and a collection of photos from various periods of his life. This is followed by three short articles by Hersh, reprinted from other collections.

"Pluralism as modeling and confusion" begins by pointing out that "while in mathematics complete consensus is the norm," in the philosophy of mathematics it's the opposite. The practice of the philosophy of mathematics consists in "choosing a position and fighting for it." Hersh on the other hand advocates Pluralism, "peaceful coexistence" among different philosophies, "a radical new idea and…a great idea. It is a philosophy of the philosophy of mathematics."

He recalls his own well-known statement that "the typical mathematician is a Platonist on weekdays and a Formalist on Sundays."

Hersh justifies his position by comparing it with what occurs in mathematics itself: L^2 is a Hilbert space whereas L^p for $p \neq 2$ is not, Euclidean geometry accepts the parallel postulate in contrast to non-Euclidean geometry, and different surfaces have different curvatures. (But this comparison is disingenuous. In the mathematical subjects just mentioned we are considering different mathematical objects, while the different philosophical positions are looking at the same object.)

Hersh's article "'Now' has an infinitesimal positive duration" tries to make sense of this word: "a time interval shorter than any...positive interval, yet longer than any infinitesimal." It includes a worthwhile historical survey, starting with Aristotle, continuing with Leibniz and the 19th century analysts, and culminating in Abraham Robinson's non-standard analysis.

A lot is packed into the *Monthly* review of David Tall's book *How Humans Learn to Think Mathematically*. Hersh's and Tall's similar views are interwoven. There is much discussion of "the math we teach in school," all the way from elementary school to graduate school. This in turn requires us to understand "mathematical reality," and also allows Hersh to recall another of his well-known theses that "mathematical entities are equivalence classes of mental models."

James A. Donaldson and Peter Lax, Hersh's PhD advisor, with Hersh and John-Steiner at the Princeton University Press Exhibit, JMM 2011.

For me, one of the highlights of this volume is William Byer's article "Can you say what mathematics is?" It cogently elucidates Hersh's view of mathematics as a human creation but goes beyond it. I am tempted to fill the next several paragraphs with quotations from this article. He begins by posing Bill Thurston's question "What is mathematics?" but then switches to the "easier" question "What is number?" Not just real number, which is a definition in analysis, but "number" in general—a concept which may well change as we explore different ways of approaching it. On the other hand, it is not completely arbitrary. This leads to the question of whether mathematics is "objective." Well, it is and it isn't. Certainly "mathematicians no matter what their race, creed, gender, or culture…agree…about the sum of the angles of a plane triangle in Euclidean geometry." But it is nonobjective because "(human) mathematicians bring it into existence." Here, as elsewhere in the article, Byers emphasizes the notion of ambiguity (as he did brilliantly in his books *How Mathematicians Think* and *The Blind Spot*.)

Pursuing this question further in the next section "What is objectivity?" Byers distinguishes between "strong" and "weak" objectivity. Strong objectivity means it does not depend on mind. Weak objectivity means that it is "free from prejudice and arbitrary opinion but not independent of intelligence."

The next several sections are concerned with conceptual systems. While this is (necessarily) difficult to pin down precisely, it refers to "a mathematical structure like the real numbers or topological spaces looked at from

[2] *See also "The Mechanization of Mathematics" by Jeremy Avigad, in the June/July 2018* Notices www.ams.org/journals/notices/201806/rnoti-p681.pdf.

the inside." It is similar to the notion of a paradigm in the philosophy of science. (Somewhat surprisingly this is the only article in the volume that refers to the physicist-philosopher Thomas Kuhn.)

An example of two different but related conceptual systems are the counting numbers and the rational numbers. "Ask a child how many numbers there are between 2 and 3 and his or her answer will tell you which [Conceptual System] they are currently living in." This very good section would have been even better if it included more advanced mathematics.

A philosophy of mathematics is, in fact, a conceptual system. It follows that "asking whether a philosophy of math is right or wrong...is not a good question." The "incompatibilities between different philosophies" are "inevitable and...valuable." Clearly this is related to the view of "Pluralism" in the philosophy of mathematics considered elsewhere in the volume.

The penultimate section deals with artificial intelligence and computer-generated mathematics. Byers forthrightly rejects the myth that "the human being is a machine and mind is algorithmic," as well as the prediction of Paul Cohen that mathematicians will eventually be replaced by computers. The final section tries to answer Hersh's questions as to whether mathematicians can contribute to the philosophy of math (yes) and whether philosophers can say anything to practicing mathematicians (maybe).

Just to mention a couple of points of disagreement. Byers asserts that "what you find in a research article or a textbook is not mathematics in the same way that what you find in a musical score is not music." "The reader has to add context, meaning, and understanding… Something has to click in your own mind to make this potential mathematics into real mathematics." Of course a reader has to add context but, in fact, a research article *is* mathematics. The analogy with music doesn't hold, in that case we are awaiting a performance not something to click.

In the section on objectivity Byers says "real mathematics is impermanent" and "those who view mathematics as...unchangeable are doomed to disappointment." Sure. But then he asserts that "no one cares" about metrization theorems in topology. Experience shows that sometimes unfashionable fields reassert themselves and become important again.

I have not discussed everything in this beautifully written article. Anyone interested in the philosophy of mathematics will profit from reading and engaging with it.

One of Hersh's contributions, introduced in "What is mathematics, really?" and discussed by several of the authors, is the distinction between the "front" and "back" of mathematics. This was inspired by the sociologist Erving Goffman who distinguished between the front and back of a restaurant (the dining area is the front and the kitchen is the back). The front of mathematics is "mathematics in finished form, lectures, textbooks, and journals" whereas the back is "mathematics among working mathematicians...in offices or at cafe tables. Mainstream philosophy doesn't know that mathematics has a back."

The article by Delariviere and Van Kerkhove, "Artificial mathematician," deals with the prediction by Paul Cohen about mathematicians being replaced by computers. It takes the form of a number of amusing dialogues, mostly between an *ideal* and an *artificial* mathematician (AM). Of course AM is a computer; the question is whether it can do mathematics. The first dialogue concludes with the statement by the ideal: "While humans are motivated by the meaning of mathematics, you are motivated by rule-following procedures without understanding what you're doing." The question of understanding ("easy to make, but hard to elucidate") plays an important role in this article. Parallel to this dialogue is one between an ideal and an artificial restaurant owner, taking off from Hersh's analogy of the front and back of mathematics with the front and back of a restaurant. Here a "chef's insight" is analogous to that of a mathematician (in contrast to a computer). There's even a funny comparison of the book by Pólya *How to Solve It* with an imagined "famous book" by "Bolya" *How to Cook it*.

There are also dialogues involving a Functionalist epistemologist and a subcognitive scientist. Some of these deal with the discovery process ("the ability to recognize a good thing when you stumble upon it").

There is much discussion of proof—more than navigating a formal system—and the need for "a procedure for deriving interesting theorems" (via interesting routes) in analogy with the "dirty aspects of the kitchen." From this one might expect the authors to dismiss the idea of computers doing mathematics. On the contrary, they consider the counterintuitive idea of informal computing and conclude by asserting "the possibility that computers could play a...meaningful role in mathematical practice—not just as a method of inquiry but as fellow inquirers, as artificial mathematicians."

In spite of its whimsical tone, this is a serious and insightful article, although not always easy reading.

Ian Stewart's article "Xenomath!" is exasperating. Stewart is one of our finest mathematical writers and his discussion of Hersh's Humanist point of view is outstanding. But he goes off the deep end in his extended consideration of alien mathematics. This would be fine for a science fiction story, but here I find it rather silly.

Here's something of Stewart's I do like: Hersh's "suggestion that mathematics is dependent on human conventions...appears to smack of relativism. [But] Reuben's position implies nothing of the kind. [Mathematics] is by no means arbitrary. Nothing new is incorporated into it unless it passes stringent reality checks...supported by proofs. ...However, Platonism is seductive, because that's what it feels like the vivid impression that the answer is already out there and we're just 'discovering what it is.'"

Stewart points out that this is true if "out there" means "the correct consequences of whichever axiom system... we happen to prefer."

A major contribution is Stewart's pointing out "just how firmly [mathematics] rests on human perceptions and conventions." This includes the architecture of the human body: "our visual senses present the world to us as a two-dimensional projection. [Our] coordinate system reflects our body plan...we stand upright…our arms extend sideways." We like dualities "perhaps because we're a bisexual species." Mathematics "is a tangled tale of concepts being imported from the outside world, reworked by a human mind, and exported back."

Elena Marchisotto's ambitious article "A case study in Reuben Hersh's philosophy: Bézout's theorem" actually does a substantial amount of mathematics. Her aim is "to examine a piece of mathematics through a Humanist lens." This is accomplished by considering the statement and proofs of Bézout's theorem, "the precise number of points of intersection of two plane curves," and generalizations thereof, "a conversation through the centuries" (Euler, Bézout, Monge, Poncelet, culminating in Weil). A lot depends on nailing down the correct definitions.

All of this "gives an appreciation of mathematics as a process, during which progress is both impeded and stimulated by…collective consciousness" and "illustrates the social nature of mathematics and the avenues that emerge because of it." It is an illustration of Hersh's concept of the front and back of mathematics.

Marchisotto concludes by discussing the role of the computer in mathematics. Of course there is recognition of its importance. But "there are limits…It is not curious. It cannot follow hunches…It cannot replicate the social interaction that in Reuben's view is essential to the growth of mathematics."

The article also includes an appreciation and summary of Hersh's career. I think this article is spot on.

Carlo Cellucci's extensive and impressive article "Varieties of maverick philosophy of mathematics" pays tribute to Hersh ("a champion of the maverick philosophy of mathematics") and also expresses some differences with him. While the differences are certainly not trivial, I don't perceive them as very substantial (with possibly one exception, noted below). For example, Hersh characterizes mathematics as the subject where 'proof or disproof' brings unanimous agreement by all qualified experts" whereas Cellucci would just say "by the majority of qualified experts." Hersh asserts that the philosophy of mathematics is "the working philosophy of the professional mathematician…the researcher, teacher, or user of mathematics." Cellucci points out that this "working philosophy" varies "from period to period…from school to school…from mathematician to mathematician." Cellucci agrees with Hersh that mathematics has a front and a back but disagrees slightly about what the back actually is. Hersh says it's "mathematics as it appears...in informal settings…in an office behind closed doors" whereas for Cellucci it's "the creative work…the discovery work."

Carlo Cellucci called Hersh (pictured here at The Brookdale, Santa Fe) "a champion of the maverick philosophy of mathematics."

Hersh and Cellucci agree that mathematics is not "about truth and certainty," basing this conclusion on Gödel's second incompleteness theorem. Rather it's about plausibility "compatible with existing knowledge,…the best we can achieve." Where they apparently differ concerns Hersh's preference for deductive proof while Cellucci supports analytic proof. The former consists of "deductive derivations from primitive premises "going down to the proposition to be proved." Analytic proofs are "nondeductive derivations from plausible hypotheses. Their aim is to discover plausible hypotheses capable of giving a solution to the problem,…both a method of discovery and a method of justification."

Cellucci asserts (perhaps unfairly) that if deductive proof is mathematicians's proof then "it is impossible to prove propositions that cannot be deduced from established mathematics," that "mathematicians can be replaced by computers completely," and that "all mathematical knowledge can ultimately be deduced from some elementary mathematical propositions such as 1+1=2."

An example of analytic proof is provided by Ken Ribet's contribution to the solution of Fermat's last theorem. What Ribet showed is that the Taniyama–Shimura conjecture implies Fermat's last theorem. So it depended upon a "hypothesis" that had not yet been proved. Then Wiles and Taylor proved the Taniyama–Shimura conjecture, the solution depending on the axioms of set theory. Another example (not mentioned by Cellucci) is given by the proof of theorems on the assumption of the Riemann hypothesis.

There is much more to Cellucci's fine article: an extensive discussion of proofs using diagrams; a summary of the history of the philosophy of mathematics, going back to Plato and Aristotle; and an interesting brief discussion of "normal" and "revolutionary" mathematics, the latter requiring "hypotheses which cannot be deduced from established mathematics [and] open up new areas of mathematics."

"What is Mathematics and What Should It Be?" by Doron Zeilberger is infuriating and wrongheaded. He claims that Greek mathematics was a "major setback." This is followed by a lengthy section, "A Brief History of Mathematics as a sequence of (Unsuccessfully!) Trying to Answer Stupid Questions," including proving the parallel postulate, solving a quintic equation by radicals, developing a rigorous foundation for calculus, and devising an algorithm for the solutions of Diophantine equations. He says that today's mathematics is not a science, but a religion, because it depends on rigorous proofs. Mathematicians "do not care about truth; they only care about playing their (artificial!) game." He says that mathematics can become a science by "taking full advantage of computers" and "should abandon the dichotomy between conjecture and theorem."

BUT, to give the devil his due, Zeilberger does raise some really interesting issues about infinity in his discussion of calculus and of Gödel's theorem. "We live in a finite and discrete world and the infinite and continuous are mere optical illusions." Statements about infinite sets are "a posteriori meaningless."

The linguist William Labov has a lovely and outstanding short article "The philosophy of Reuben Hersh: a nontechnical assessment." Labov and Hersh were both students at Harvard in the 1940s, where their connection was mostly political, and they maintained intermittent contact over the years. Labov summarizes some of his own research, while lamenting his insufficient mathematical training. In spite of their different scholarly areas, Labov notes some similarities in his and Hersh's research based on the work of the sociologist Emile Durkheim. He concludes by recalling a chance encounter with a nurse in Santa Fe, who remembered with gratitude Hersh's support of a nurses' strike.

There are two other articles which can be characterized as political. Michael Harris's "Do mathematicians have responsibilities?" has woven through it an eloquent appreciation of Hersh as a mathematician and a human being (although they apparently have never met). Harris discusses the misuse of mathematics, not only its military applications, but also "embodied artificial intelligence," treating "human beings as a means rather than an end." He refers to "the dominant ethos of Silicon Valley, where the sum total of human experience is treated as data to be mined for content." This is definitely related to Paul Cohen's vision of mathematicians being replaced by computers. Harris does note that some mathematicians have pushed back against this "instrumentalist" view, citing the debate in the *Notices* on the role of the NSA in the wake of the Snowden revelations.[3] He concludes by quoting the economist Thomas Piketty about "the obsession with mathematics…acquiring the appearance of objectivity without having to answer the far more complex questions posed by the world we live in."

Chandler Davis's "Friends and Former Comrades" mentions several distinguished mathematicians, including Hersh and himself, who passed through the Communist movement (and I should mention that that was my first connection to Hersh almost seventy years ago). It includes a tribute to Lee Lorch for his uncompromising fight against racism in academia and our profession (which cost him several jobs), while at the same time distancing himself from Lee's continued devotion to the Soviet Union. While he is no longer a Communist, Chandler calls for "reaching out...when capitalism, self-immolating, threatens to take everything else with it."

Chandler has another fine article in the volume, "Can something just happen to be true?" It begins with his "feeling that 5279 is prime is…accidental" and goes on to say that his "attitude towards mathematical assertions is at odds with [his] attitude toward the world of experience." He considers "the appearance of regularity" which sometimes guides one towards a correct proof, and at other times has led to "conjectures which turned out to fail." And an interesting discussion of the "dragon curve" and its relation to computer experiments versus rigorous proof.

An article by another Davis (Martin), "Gödel's legacy," is not explicitly related to Hersh's work, but it is a welcome contribution. It contains succinct discussions of algorithmic computability and determinacy, as well as the relation of the Zermelo–Fraenkel axioms to the continuum hypothesis.

Both of these eminent Davises are also in their nineties.

Three of the articles concern education, and they could not be more different from one another. Nel Noddings' "A gift to teachers" echoes Hersh's call for mathematics students to be broadly educated, "expanding the teaching of mathematics into vital interdisciplinary studies." Bonnie Gold's "School mathematics and 'real' mathematics" presents a number of interesting ideas for introducing non-rote problems into the K–12 curriculum. She feels that the algorithms (such as for long division) that dominate elementary and high school classes are not real mathematics (some of us might disagree). And Alexandre Borovik's ambitious and quirky "Mathematics for makers and mathematics for users" begins by posing the "difficult" question, "What is mathematics education, really?" The article is wide ranging, its "aim to start a discussion and pose more questions than to give answers." A couple of provocative statements: "Banks and insurance companies need a numerate workforce—but even more so they need innumerate customers"; "a learner of mathematics is a dog trainer." The latter is related to consideration of the subconscious and neurophysiology, which in turn concerns the relation between mathematics and language.

Jody Azzouni's "Does reason evolve? (Does the reasoning in mathematics evolve?)" is rich and thoughtful. It is the longest article in the volume, and not easily summarized. He begins by referring to Hersh's stressing "the great distance…between the reality of professional mathematical practice…and the reasoning in formal languages… that philosophers have largely characterized mathematical proof in terms of." The latter is called the "derivationist account." This certainly has advantages. "The existence of formal derivations explains…why mathematicians are so agreeable to one another"—comparatively speaking. "Apart from mistakes, there is no space for disagreement;...mathematical results…have eternal shelf lives."

But in the next section Azzouni recognizes problems with this position. "Not only have the standards of mathematical proofs mutated over the ages...but there have been heated disputes…over appropriate proof methods." "Reasoning itself…is a social phenomenon which historically changes with the passage of time."

In a sense combining these two approaches, Azzouni speaks of "informal rigorous mathematical proof, 'rigorous' in the sense that mathematicians are convinced that an effectively recognizable derivation…(in the ideological

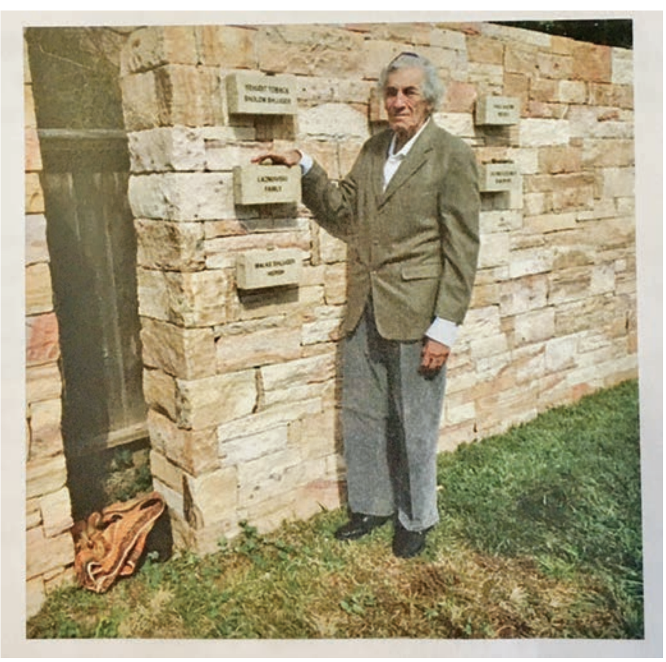
[3] *"Mathematicians Discuss the Snowden Revelations,"* https://www.ams.org/notices/201406/rnoti-p623.pdf.

neighborhood of the proof they've inspected) has been shown to exist." He also considers the possibility that "in the future robots will be checking our proofs," but then casts some doubt about whether this will actually occur.

There is an extensive discussion of baseball and other games. There are several sections on diagrammatic proofs, including infinite diagrams. An example is the construction of the Koch snowflake, which is a potentially infinite process. Another example is the "mutilated chessboard," the proof that if the two squares at each end of a diagonal are removed it is impossible to cover the remaining board with 31 dominoes. (Here Azzouni misses an opportunity to explain why introducing some additional structure, namely the coloring of the board, enables one to solve this problem.)

"Wittgenstein, mathematics, and the temporality of technique" by Paul Livingston is concerned with Wittgenstein's question of whether there is an occurrence of 777 in the decimal expansion of π. (It is now known that there is such, but one can ask the question about any sequence.) Are such questions even meaningful before they are answered? (Does "God" know the answer?) There is no mention of the unsolved problem of whether π is a normal number, which of course would imply much more.

There are several articles, mostly by philosophers, that I just don't understand. It seems to me that these articles were not written for mathematicians. This is not meant as a criticism of the articles per se, but rather as an explanation of my inability to deal with them in depth. I hope some readers of this review will contribute a letter or article to the *Notices* elucidating them, especially Michele Friend's article "Mathematical theories as models," which develops Hersh's idea of Pluralism.



Hersh, in 2016, next to the Santa Fe Jewish Cemetery's remembrance wall.

The volume concludes with a special three-page contribution by Hersh "On the nature of mathematical entities," a neat summary of his position. The assertion that "accepted theorems are absolutely certain…is naive [mathematics] is a human artifact and…can never claim final perfection… There are three sides of mathematical entities—social, mental, and neural."

The final two sentences in the article (and therefore in the book): "These multiplicities are not logical contradictions. They are the different ways we know things—any kinds of things—including mathematical things, which are manifested as cultural items, as personal experience, and/or as currents in our flesh and blood."

Here is my own take on this. Maybe mathematics is not "absolutely certain," but it's surely more certain than social science, and even more certain than biology and physics. While there are occasional disagreements about the correctness of a purported result, such are regarded as anomalies, and if the claim is important, we scramble to resolve the disagreement.

A few final thoughts. As I mentioned at the beginning of this review, there is great variation among these articles, and there doesn't seem to have been a unifying theme. Among those who deal with Hersh's humanist philosophy there is the expected general agreement. It might have been good to have a dissenting view.

Also, it would have been appropriate to include a discussion of Hersh's mathematical research.

Finally, regarding the Paul Cohen prediction that mathematicians will be replaced by computers: Almost all of the authors who deal with it reject it. But one must ask, what does it mean? Look, nobody doubts that computers are becoming increasingly important in both pure and applied mathematics. Still, how are we to interpret such an assertion? Do we expect computers to survey the literature, make conjectures, and then prove them or find counterexamples? Or will a real live mathematician feed the question into the computer and expect it to churn out the answer? Or maybe (and I have heard this asserted) the entire "theorem, proof, counterexample" procedure will be abandoned.

Something the next collection of essays can consider.

Photo Credits

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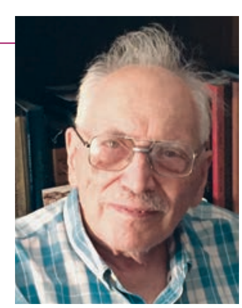
ABOUT THE REVIEWER

Joseph Auslander's research is in topological dynamics, and he also has an interest in philosophy of mathematics.



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