Alan Baker 1939–2018

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Figure 1. Alan Baker

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Alan Baker, Fields Medallist, died on the 4th of February 2018 in Cambridge, England, after suffering a severe stroke a few days earlier.

He achieved a major breakthrough in transcendence theory and applied it to obtain a new and important large class of transcendental numbers (opening the way to the subsequent discovery of several other such classes); developed quantitative versions and applied them to the effective solutions of many classical diophantine equations as well as the resolution of the celebrated Gauss Conjecture on class numbers of imaginary quadratic number fields; and started the study of extensions to elliptic curves (opening the way to later generalizations to abelian varieties and commutative group varieties and in turn their applications to old and new problems in diophantine geometry).

The following [3] is perhaps the most easily stated of his results. Given non-zero k in the ring $\mathbf{Z}$ of rational integers, all solutions x, y in $\mathbf{Z}$ of the so-called "Mordell equation"

$$y^2 = x^3 + k \qquad (1)$$

satisfy

$$\max\{|x|, |y|\} \leq \exp(10^{10}|k|^{10000}). \qquad (2)$$

Despite (1) being around since at least the year 1621, there were no estimates at all for x, y until [2] in 1968. Thus given k one can in principle solve (1) completely; even this fact was not previously known.

Baker's parents Barnet and Bessie (with roots in eastern Europe) lived in Forest Gate in East London, where he was born on August 19, 1939. From a very early age he showed signs of mathematical brilliance and was encouraged by his parents. Already his father was very gifted in this direction. After having attended Stratford Grammar School he went with a scholarship to University College London where he studied mathematics. He finished with a first class degree before he moved to Trinity College (where he would be based for the rest of his life) in Cambridge to study for MA and PhD degrees with Harold Davenport, one of the leading number theorists at the time with many international connections. During this time (between 1962 and 1965) he published eight papers that made his very high potential obvious. He received his PhD in 1964 and one year later was elected Fellow of Trinity in the research category. In 1970 he was awarded the Fields Medal at the International Congress in Nice on the basis of his outstanding work on linear forms in logarithms and its consequences. Since then he received many honours including the prestigious Adams Prize of Cambridge University, the election to the Royal Society (1973) and the Academia Europaea; and he was made an honorary fellow of University College London, a foreign fellow of the Indian Academy of Science, a foreign fellow of the National Academy of Sciences India, an honorary member of the Hungarian Academy of Sciences, and a fellow of the American Mathematical Society.

In 1974 he was elected to a personal chair for Pure Mathematics at the University of Cambridge. Between 1969 and 1988 he supervised a number of outstanding PhD students, and 389 mathematical descendants are listed.

Figure 2. Mate Gyory, Alan Baker, and Rob Tijdeman at a number theory conference in Eger, Hungary, in 1996.

As mentioned, Baker had done substantial work before 1966. For example there was a very interesting article around Roth's Theorem and Mahler's Classification. But it was two papers of 1964 on rational approximation that probably made a more permanent impression on number-theorists. The second of these papers [1] established among other things the striking inequality

$$\left|2^{1/3} - \frac{p}{q}\right| > \frac{10^{-6}}{q^{2.955}} \tag{3}$$

for all rational integers $p, q > 0$. Since Roth it had been known that the exponent 2.955 could be reduced arbitrarily close to 2, and this even for any irrational algebraic number in place of $2^{1/3}$; but that result (the main reason for Roth's own Fields Medal) was not "effective" in the sense that the multiplying constant (10^{-6} in the above) could not then be calculated or even estimated. And to this day no one knows how to do this, even in (3) with exponent 2.3, say.

Baker's result (3) was significant, not because 2.955 is particularly near 2, but because it is strictly less than the degree 3 of $2^{1/3}$, which is the trivial exponent supplied by the much earlier ideas of Liouville. It made it an easy matter to solve completely any diophantine equation

$$x^3 - 2y^3 = m$$

in integers x and y. Before this result, there had been no algorithm at all, just as for (1).

Strangely enough, Baker's proof (using the so-called Padé theory with an extra 3-adic twist) broke down completely for

$$x^3 - 5y^3 = m. \tag{4}$$

He himself found a way round this, and in the process opened up an entirely new area of diophantine approximation, with his wonderful sequence [2] of papers on linear forms in logarithms dating from 1966.

The classical theorem of Hermite-Lindemann is equivalent to the fact that if α is a non-zero algebraic number, and $\log\alpha$ is any non-zero choice of its complex logarithm, then 1 and $\log\alpha$ are linearly independent over the field $\overline{\mathbf{Q}}$ of all algebraic numbers. Similarly the classical theorem of Gelfond–Schneider is equivalent to the fact that if α_1, α_2 are non-zero algebraic numbers, and $\log\alpha_1, \log\alpha_2$ are any choices of logarithms that are linearly independent over the field $\mathbf{Q}$ of rational numbers, then they are linearly independent over $\overline{\mathbf{Q}}$. Nothing was known about even three logarithms until Baker proved (1966-68)

Theorem. *If $\alpha_1, \dots, \alpha_n$ are non-zero algebraic numbers, and $\log\alpha_1, \dots, \log\alpha_n$ are any choices of logarithms which are linearly independent over $\mathbf{Q}$, then*

$$1, \log\alpha_1, \dots, \log\alpha_n \tag{5}$$

are linearly independent over $\overline{\mathbf{Q}}$.

The reader may easily construct simple examples of transcendental numbers not covered by Hermite-Lindemann or Gelfond-Schneider; a less simple example is

$$\int_0^1 \frac{dx}{x^3+1} = \frac{\pi\sqrt{3}}{9} + \frac{\log 2}{3}$$

quoted by Siegel in his famous transcendence monograph.

In a paragon of modesty, clarity, and foresight, Baker wrote in the first of his sequence

Finally, as regards the proof of the theorem, our method depends on the construction of an auxiliary function of several complex variables which would seem to be the natural generalisation of the function of a single variable used in Gelfond's original work. The subsequent treatment employed by Gelfond, *however, is not applicable in the more general context and so it has been necessary to devise a new technique. Nevertheless it will be appreciated that the argument involves many familiar ideas. The method will probably be capable of considerable development for it applies in principle to many other auxiliary functions apart from the one constructed here.*

Let us give an idea of this "new technique," for simplicity taking $n = 3$ and ignoring the extra 1 in (5), so that we have to deduce a contradiction from a relation

$$\beta_1 \log \alpha_1 + \beta_2 \log \alpha_2 = \log \alpha_3 \qquad (6)$$

with β_1, β_2 also algebraic. The auxiliary function $\Phi(z_1, z_2)$ is a polynomial of large degree in

$$e^{z_1}, e^{z_2}, e^{\beta_1 z_1 + \beta_2 z_2} \qquad (7)$$

which indeed generalize in a fairly natural way (although no one had previously written them down) Gelfond's $\Phi(z)$ and $e^z, e^{\beta z}$. Note that from (6) the functions (7) take algebraic values at all points

$$(z_1, z_2) = (s \log \alpha_1, s \log \alpha_2),\ s = 0, 1, 2, \ldots$$

and this is true even of their partial derivatives. That enables Φ to be constructed, with algebraic coefficients not all zero, such that

$$\frac{\partial^{t_1}}{\partial z_1^{t_1}} \frac{\partial^{t_2}}{\partial z_2^{t_2}} \Phi(s \log \alpha_1, s \log \alpha_2) = 0$$

for all non-negative integers s, t_1, t_2 in some large range

$$s \leq S,\ t_1 + t_2 \leq T. \qquad (8)$$

Gelfond had shown by an extrapolation technique on his $\Phi(z)$ that his range $s \leq S, t \leq T$ could be extended to say $s \leq S,\ t \leq 2T$. This step could be then iterated, even indefinitely, to get zeroes of infinite multiplicity and so the required contradiction.

To this day no one knows how to increase T in (8) to $2T$. But Baker, by applying similar extrapolation techniques on all the separate $\Phi_{\tau_1,\tau_2}(z) = \frac{\partial^{\tau_1}}{\partial z_1^{\tau_1}} \frac{\partial^{\tau_2}}{\partial z_2^{\tau_2}} \Phi(z \log \alpha_1, z \log \alpha_2)$, $\tau_1 + \tau_2 \leq T/2$ was able to modify (8) to say

$$s \leq 8S,\ t_1 + t_2 \leq T/2.$$

The number of conditions here is roughly twice that in (8) and so we have gained something. We can iterate but not indefinitely in any profitable way. Already this was a new sort of difficulty, which Baker overcame by getting just as many zeroes as are needed for the contradiction (along the principle that a polynomial of degree D cannot have $D+1$ zeroes).

So Baker's main achievement was to introduce several complex variables (not being afraid of possible Hartogs-style complications), reduce them to a single variable along a line, and supply the missing zero estimates.

Figure 3. Alan Baker with Yu Kunrui on the occasion of Peter Sarnak's 61st birthday conference at the IAS in 2014.

In fact he did more, obtaining positive lower bounds for the absolute value of linear forms

$$\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n \qquad (9)$$

in terms only of certain complexity measures (heights) of the algebraic numbers appearing. This was vital for the applications; and crucial too was that the lower bounds should be sufficiently strong. The earlier bounds sufficed for (1) and (4), but more sophistication was needed for his solution [2,I] of the Gauss Conjecture that the only imaginary quadratic fields with class number $h = 1$ have discriminant Δ at most 163 in absolute value, and even more for his extension [5] to $h = 2$ (there followed a collaboration with Stark who had independently obtained these results) leading finally to $|\Delta| \leq 427$. Similarly the work led to Feldman's improvement on the Liouville exponent for any algebraic number of degree at least three, for example

$$\left| 5^{1/3} - \frac{p}{q} \right| > \frac{10^{-12900}}{q^{2.9999999999998}}$$

(also by which (4) can be solved) due to Baker and Stewart [8]. And with Wüstholz [9] in 1993 Baker took the lower bounds for (9) already extremely close to their modern-day versions.

The foresight in the above quotation was illustrated by Baker himself in making a start [4] on analogues of his Theorem for elliptic functions, where $\log \alpha = \int_1^\alpha \mathrm{d}x/x$ is replaced by an "elliptic logarithm"

$$\int_\infty^\alpha \frac{\mathrm{d}x}{\sqrt{4x^3 - g_2 x - g_3}}.$$

He gave an account of much of this (and more besides) in his book [6], a worthy successor to the classics of Siegel, Gelfond, and Schneider.

Baker single-handedly transformed the subject of transcendence and diophantine approximation, and others have taken things yet further. For example Tijdeman used linear forms in logarithms to show that there are at most finitely many solutions $p > 1, q > 1, r > 1, s > 1$ to Catalan's equation $p^s - q^r = 1$, and then Mihăilescu showed that the only solution is indeed $(p,q,r,s) = (3,2,3,2)$ as Catalan had conjectured. And various authors developed the elliptic and higher analogues culminating in the work of Wüstholz on general commutative group varieties. This further led via so-called "isogeny estimates" to effective versions of Faltings's Finiteness Theorems and the Tate Conjecture for abelian varieties, and even to the solution of geometric problems such as the existence of "small" polarizations. Some of this was in turn described in Baker's book [10] (not to be confused with the attractive [7], much more elementary) with Wüstholz. Since then the material has been found useful also in aspects of the André-Oort Conjecture.

Figure 4. Gerd Faltings and Alan Baker on the occasion of Peter Sarnak's 61st birthday conference at the IAS in 2014.

As mentioned, Baker was firmly based in Cambridge; it seems that college life there suited him especially in the style of Trinity, whose society he enriched. He had a flat in London and enjoyed life there too, for example the theatre. He was enthusiastic about travel, and as his reputation grew he was able to combine this with professional visits to China and many parts of Europe and especially of America. In later life he made regular trips to Switzerland to work with Wüstholz at ETH Zürich. It was there, during a conference in honour of his 60th birthday, that he gave an entertaining and typically candid speech about his life, starting with his recollections of wartime London and ending with his regrets about never marrying.

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