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Memories of a Grandmaster: Michel Raynaud and the Flattening by Blowing Up

Ahmed Abbes

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"A course in algebraic geometry is a course in which one takes out more and more algebra and puts in more and more geometry." This beautiful definition is due to Michel Raynaud. He would often add, with a trace of bitterness and considerable humor: "but these days, students arrive knowing so little algebra there is nothing to take out." His conception of what a course of algebraic geometry should be is perfectly illustrated by his book *Anneaux locaux henséliens* [5], based on a graduate course given at Orsay in 1969, probably still the best place to learn the notion of *étale morphism*. Although accessible to beginning students, this book remains, nearly fifty years after its publication, the definitive reference for the notion of a henselian pair, beyond the case of local rings.

The theme of removing more and more algebra and replacing it by more and more geometry is also present in Raynaud's research. A striking example is his joint article with Laurent Gruson, *Critères de platitude et de projectivité, Techniques de platification d'un module* [8]. When I asked him at some point if he still had any reprints, he answered that there were no more, and that he was very surprised that there had been so much demand. Despite this modesty, so characteristic of Raynaud, this article is one of the masterpieces of algebraic geometry. To this day it remains his most cited article on MathSciNet®. In order to describe its main result, I need first to present the notion of *flatness* to the readers who do not know it. I can do no better than to quote the following excerpt from another of his articles, intended for a wider public, and entitled *Grothendieck et la théorie des schémas* [7], in which he describes his teacher Grothendieck's encounter of the notion of flatness.

> In GAGA [9], Serre gave a wonderful example of a "flat pair," in the course of the passage from algebraic to analytic geometry. With Grothendieck, flatness will play an important role, demonstrated in many ways.
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> First of all, flatness is at the heart of relative geometry. If one thinks of a morphism $f{:}X \to S$ as a family of schemes over fields parameterized by the points of S, flatness of f is the most convenient hypothesis assuring that a fiber of f over a point s of S is "precisely" contained in the scheme theoretic closure of a fiber over a generization of s. Thus flatness appears as a natural link among the fibers in relative geometry. Nevertheless, flatness is not a completely "geometric" notion; it includes, in the end, an algebraic aspect. For example, if S is local and Artinian, the flatness of f translates simply as the statement that the rings of open affine subsets of X are free modules over the ring of S. Furthermore, in the noetherian case, flatness can be tested on Artinian rings, by passing to the limit. This motivated Grothendieck to prove a delicate criterion for flatness, which he suggested to Bourbaki. It appeared in Bourbaki Alg. Com. Chap 3 [1]....
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> A technical advantage of the notion of flatness is that, under reasonable finiteness assumptions (noetherianity, finite presentation), a number

Ahmed Abbes is research director at CNRS, Institut des Hautes Études Scientifiques (IHÉS), Bures-sur-Yvette, France. His email address is abbes@ihes.fr. *He thanks Arthur Ogus for help with the English translation.*

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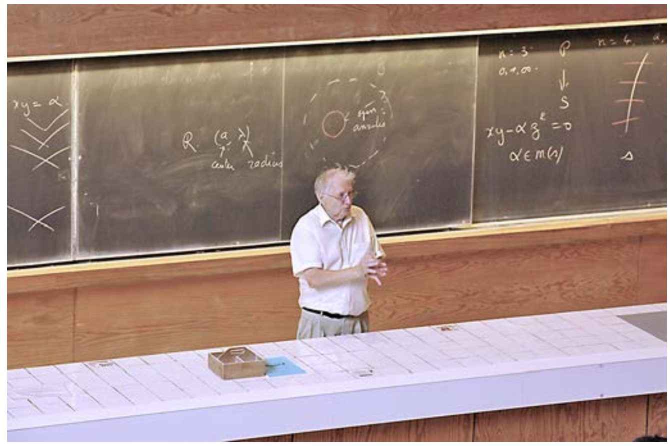
> of standard properties of a morphism $f: X \to S$ can be checked on the fibers, assuming that f is flat. One thus obtains a general method to pass from geometry over a field to relative geometry and conversely, leading to the development of [a] whole chapter—simultaneously useful and easy—of relative commutative algebra….
>
> Flatness, under suitable finiteness conditions, is ideal for the study of local and openness properties. Following the path initiated by Serre in *Algèbre locale et multiplicités*, [10], Grothendieck systematically "modulates" his statements. Thus a good framework is to consider a morphism $X \to S$, locally of finite type, and a sheaf of $\mathcal{O}_X$-modules of finite presentation $\mathcal{M}$, which is S-flat.

As one sees, when Raynaud speaks of a notion in algebraic geometry, it is as if he endows it with a living soul.

One might think that the "good framework" for studying local and openness properties which he describes at the end of the excerpt above is so good that it is exceptional. But in fact in his article with Gruson, he proves that one can always transform a (rather general) relative situation into this good case, by a simple transformation, "blowing up." In order to describe this result in simple terms, I must first do the opposite of Raynaud's approach to teaching algebraic geometry, namely I must return to algebra. Thus let A be a ring, B an A-algebra of finite presentation, and M a B-module of finite presentation. Raynaud and Gruson are interested in the flatness of M as an A-module. They begin by establishing a criterion that reduces the study of flat A-modules to the case of modules that are free and of finite type over smooth A-algebras. The precise statement is a little bit technical, but let us try: the module M is A-flat if and only if, locally for the étale topology on Spec(A) and Spec(B), the A-module M admits a finite composition series the successive quotients of which are the A-modules underlying free finite type modules over A-algebras which are smooth and with geometrically integral fibers. They then show that one can render M flat over A by performing a blow-up on Spec(A) and replacing M by its strict transform along this blow-up. Thanks to their criterion, the key case is when the A-algebra B is smooth with geometrically integral fibers, in which case one can use the Fitting ideals to make M locally free over B. Their results, which have the virtue of extending to the case in which M is merely of finite type over B, provide an alternative to Grothendieck's approach in EGA IV to the study of flatness in a relative case [3].

The main motivation of the work of Raynaud and Gruson came from Tate's rigid geometry. Their techniques extend to formal geometry and provide a systematic procedure for transforming problems in rigid geometry into



Michel Raynaud during a conference in Rennes, France, July 2009.

problems in algebraic geometry [6]. This approach has proved extremely powerful and has had many applications in arithmetic and in algebraic geometry.

At the beginning of this year, I sent my New Year's greetings to Raynaud and attached to my email an article by my student Quentin Guignard, which gives a new proof of his theorem with Gruson on flattening by blowing-up [2]. I didn't get an answer, and in fact I learned several days later that Raynaud had had an emergency operation because of a brain tumor. I did have the opportunity to meet with him a little later. He had come, with his wife and son, to see the office in the new mathematics building in Orsay that had been intended for him to share with Luc Illusie. With his usual lively humor, he began by teasing Luc about a joint past collaboration [4]. He thanked me for my New Year's greetings and the article, and added in a bantering tone of voice: "but what gave him the idea of finding a new proof, mine is after all well understood." He then summarized it in a few sentences in his fascinating style, so alive and inimitable. He was still amazed by the fact that, to flatten a module, one can reduce to the case in which one can make it locally free. This was, alas, our last meeting.

My strongest memory of him, one that will stay with me forever, goes back to July 2000. We were in Azumino, in the Japanese Alps, for a conference on algebraic geometry. I met him a few minutes before the beginning of his exposé; he was, as was his habit, pacing back and forth, somewhat nervously. I said to him, teasing, "What, at your age?" He looked me in the eye and answered, "This fire, if you lose it, you lose everything."

References

[1] N Bourbaki. *Algèbre commutative*, Chapitres 1–9, Hermann (1985).

[2] Q Guignard. A new proof of Raynaud-Gruson's flattening theorem, prepublication (2018), arXiv:1801.01188.

[3] A Grothendieck, J A Dieudonné. Éléments de Géométrie Algébrique, IV Étude locale des schémas et des morphis-

mes de schémas, *Pub. Math. IHÉS* **20** (1964), **24** (1965), **28** (1966), **32** (1967).

[4] L Illusie, M Raynaud. Les suites spectrales associées au complexe de de Rham-Witt, *Publ. Math. de l'IHÉS*, **57** (1983), 73–212. MR0699058

[5] M Raynaud. *Anneaux locaux henséliens*, Lecture Notes in Mathematics **169**, Springer–Verlag (1970). MR0277519

[6] M Raynaud. Géométrie analytique rigide d'après Tate, Kiehl… *Mémoires de la Société Mathématique de France*, **39–40** (1974), 319–327. MR0470254

[7] M Raynaud. Grothendieck et la théorie des schémas, in *Alexandre Grothendieck: a mathematical portrait*, Int. Press (2014), 25–34. MR3236918

[8] M Raynaud, L Gruson. Critères de platitude et de projectivité. Première partie: Platitude en géométrie algébrique, *Invent. Math.* **13** (1971), 1–89. MR0308104

[9] J-P Serre. Géométrie algébrique et géométrie analytique, *Ann. Inst. Fourier, Grenoble* **6** (1955–1956), 1–42. MR0082175

[10] J-P. Serre. *Algébre Locale, Multiplicités: Cours au Collège de France*, 1957–1958, Lecture Notes in Mathematics **11**, Springer–Verlag (1965). MR0201468

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