A Comprehensive Course in Analysis

A review by William T. Ross

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A Comprehensive Course in Analysis (5-volume set)
Barry Simon
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On my top bookshelf is a set of analysis books I often reference to make sure I'm using a theorem correctly or perhaps to review a long-forgotten fact. I have my favorite books for real analysis (Royden, Rudin), functional analysis (Conway, Rudin), operator theory (Akhiezer and Glazman), complex analysis (Alhfors, Rudin, Markushevich), harmonic analysis and function spaces (Duren, Garnett, Koosis), and, of

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course, Kelly's book to shore up my set theory and topology. All of these authors have different writing styles and levels of details for their proofs. When I need to focus further, I refer to my second shelf of books to dig a bit deeper on more specialized knowledge such as Bergman spaces, Dirichlet spaces, Toeplitz operators, Hankel operators, and composition operators.

To help us trim down our burgeoning book collections, there is a comprehensive collection of all things analysis with Barry Simon's *A Comprehensive Course in Analysis*. This course is a five-volume *tour de force* by a preeminent mathematical physicist and educator who cares deeply about how analysis is presented to graduate students and working analysts. This series of books covers the major parts of analysis and can be selectively used as parts of courses as well as a reference—both mathematical and historical. In fact, this undertaking is so enormous, that Simon includes a separate 68-page guide (*The Ultimate Companion....*) for the volumes.

The analysis community is at times so specialized, and maybe too focused on their own work, that at conferences, we often talk past each other rather than to each other. We often lack a set of common ideas and knowledge base. So, in a way, a set of books like Simon's reminds us of what should be the corpus of analysis that we all should know. These books also allow more experienced researchers who are perhaps focused on the minutia of their own work, to expand their analysis foundations so they can, in the future, appreciate a department colloquium talk not directly in their line of work.

Along the way through the pillars and gems of analysis we all should know, Simon gives us plenty of historical vignettes but wisely refrains from getting into the nomenclature wars of who should get proper credit for what. The past is the past. Renaming theorems just to set the record

straight only serves to confuse the reader who probably already knows "X's Theorem" whether or not X was the first to prove it—or prove it at all. For example, Pythagoras probably did not prove the "Pythagorean theorem."

As someone who teaches at a liberal arts college, where we present mathematics as part of a balanced holistic undergraduate education, I certainly value these historical references since they show the development of analysis beyond just the names attached to the theorems. This series of books emphasizes that analysis developed over a long period of time with stumbles and false starts along the way. Simon also points out the historical figures, often with very brief, but well-resourced biographies, to show their influence in the development of analysis. Some, such as Euler and von Neumann, were great. Others, such as Blaschke, were questionable. Bieberbach was just plain rotten.

Simon's presentation sorts it all out and, as stated early on in his first volume, he aims for the best and most insightful proofs that he has gathered up throughout the years. He claims early on that he is not a historian. However, from the careful scholarship and the numerous references to original sources, he sure acts like one and the reader should certainly appreciate his doggedness to chase this all down for us. Indeed, there are 3,701 references in these five volumes, many of them dating back to the beginnings of analysis.

Simon's aim in this project is twofold. First, these books are to be used by instructors of graduate analysis classes such as real analysis, functional analysis, complex analysis, harmonic analysis, and operator theory. With the volume of material, an instructor needs to be selective about what they choose, but Simon gives sage guidance on what needs to be covered and what can be used as bonus material to suit an instructor's particular interests. Second, these books are to be used as references for the researcher who needs to re-heat a forgotten topic (Riemann-Stieltjes integration, self-adjoint extensions of operators) or perhaps fill in a gap in their analysis education (tensor products of Hilbert spaces, Brownian motion, gap series). As Simon is keenly aware of the volume of material he has produced, he helps the reader ferret out what is truly important with his "Big notions and theorems" section at the beginning of every chapter. His "Bonus sections" and "Bonus chapters" indicate that although the reader might not need this material at first, it would be very wise to learn it...eventually.

Since these books are for graduate courses in analysis, Simon assumes a certain amount of mathematical analysis maturity. Simon provides a review of some basics, not so much as to teach or even re-teach this material, but to fill in any gaps and to initiate the reader to both the level of difficulty and to the notation he will use throughout these volumes. Though the material is presented in a thoughtful and friendly way, there is a certain amount of seriousness and erudition to the presentation and the reader, both student and researcher, will definitely need to pay close attention to details and have pencil and paper at the ready.

Simon has a mission to present the best quality versions of theorems (e.g., the broadest version of the Cauchy integral formula)—often at the cost of some extra legwork on the part of the reader who needs to work through some advanced background material.

As part of Simon's dedication to being an analysis educator, there is a very generous helping of problems at the end of each section that serve several purposes. First, they point out the necessity of certain hypotheses of the theorems since often something goes terribly wrong when they are left out. Furthermore, these exercises force the student reader to work through some of the technical details of the proofs. Indeed, there is so much covered in this five-volume set that Simon can't chase down every last detail. Instead, he puts some of these details, or extensions of these details, as exercises with gentle nudges to help the student along. Still further, Simon uses the exercises to give the reader a chance to learn some interesting tidbits that expand what is covered in the chapter. For example, in the complex analysis part of the series (Part II A), there is a wonderful selection of power series problems that display various aspects of the issue of a "natural boundary" via the series of Lambert, Hadamard, and Weierstrass. Simon helps the reader through these examples by giving them bite-size chunks to work through and then assemble as a bona fide theorem at the end. As mentioned earlier, all of this extra material in the exercises is meticulously referenced, often back to its original sources. Of course, there are plenty of exercise (drill) type problems that give the student an opportunity to work through some standard gems.

This course, in full or in parts, was beta tested by a half-dozen or so institutions, and there was plenty of input on these volumes from more than thirty of some of the best names in analysis. The analysis community has spoken. This is the core of analysis. Learn it!

I feel obligated as part of this review to not just talk about Simon's series in general, as I have done thus far. But, if the reader is going to invest in this series, they should know what they are getting. Even though surveying some of the topics covered in these volumes might not be the best way to spend valuable space in a book review, here is some of what you are getting.

Part 1 (*Real Analysis*) begins with some preliminaries (set theory, topology, linear algebra) and continues with topological spaces, measure theory, Lebesgue theory, Hilbert spaces, convexity in Banach spaces, distributions, with "bonus" material on topics that include Haar measure, probability, fixed point theorems, Brownian motion, and Banach space valued functions.

Parts 2 and 2 A (*Basic Complex Analysis, Advanced Complex Analysis*). The first of these two volumes cover the Cauchy theory (Cauchy integral theorem and formula in various stages of generality) and its consequences (argument principle, Rouché's theorem, open mapping theorem, maximum modulus, singularities, harmonic functions) along with

approximation theorems, Riemann mapping theorem, uniformization theorem, and the Paley-Weiner theory. The second covers more specialized, but still important, topics such as the Poincaré metric, applications to number theory, asymptotic methods, univalent functions, and the Nevanlinna theory.

Part 3 (*Harmonic analysis*) is the second largest of these volumes (Part 1 is the largest—but only by a bit). Here, after some calculus, complex analysis, and real analysis reminders (L^p spaces, Fourier analysis, probability, Green's theorem, Blaschke products), Simon proceeds to maximal functions, harmonic and subharmonic functions, classical Hardy space theory, with bonus chapters on wavelets, Calderón–Zygmund theory, Sobolev spaces, and Tauberian theorems.

Part 4 (*Operator Theory*) takes the reader from linear algebra to self-adjoint extensions of operators. As Simon points out in the introduction to this volume, there is some controversy here. His focus is on "mainly self-adjoint and/or compact operators on Hilbert space" and hence he omits, for example, the beautiful model theory of Sz.-Nagy and Foiaş and several other topics that concern non-normal operators. Simon's operator theory volume covers the basics of bounded operators on Hilbert/Banach spaces, including the development of compact operators. Included in this discussion is the Ringrose-West decomposition theorem for compact operators on a Hilbert space, a beautiful result not always covered in Hilbert space books. This volume also includes a chapter on orthogonal polynomials (Simon also has several imposing books on that topic) and a very thorough treatment of the spectral theorem, served three ways, for self-adjoint operators. Along the way, there is a discussion of perturbation theory for self-adjoint operators, where again, this important material is not always presented in well-known texts. Finally, as befitting a mathematical physicist, there is an excellent treatment of unbounded self-adjoint operators as well as operators that have self-adjoint extensions. The Schrödinger and Sturm-Liouville theory plays an important role here.

Though the title of these volumes is *A Comprehensive Course* in Analysis, Simon had to make some difficult choices of what to leave out. Some of your favorite topics, and maybe even your research area, might be missing from the main texts or perhaps relegated to the exercises—or maybe not appear at all. There are many reasons why topics do not appear. Maybe there are already some excellent and thorough texts of these subjects. Why try to re-write a perfectly good text when directing the reader to this text, via the references, will do? I couldn't resist and looked up several of my favorite operator theory topics to see if they "made the cut" and were mentioned in either the historical notes or the references. Some did while others did not. Maybe some of these excluded fields had their moment in the sun and Simon feels the need to focus elsewhere. Maybe some of these fields are struggling and need to re-

prove their worth to be reconsidered for readmission to the corpus of analysis. Either way, Simon maintains a Facebook page for *A Comprehensive Course in Analysis* to keep us up to date with revisions or updates.

Finally, perhaps I have been somewhat breezy and informal in this review and my language was a bit too friendly. However, I think this reflects the type of language Simon uses in his series. There are plenty of contractions ("we'll") and exclamation points to express excitement for analysis ("...this was proved by de Branges about 70 years after the conjecture was made!"). There are even catchy titles of sections ("The magic of maximal functions," "A warmup: the Euler product formula"). Indeed, Simon even admits "I have relied at times—horrors!—on information from the Internet." There is a certain amount of honesty, humility, genuineness, and dare I say, playfulness, in Simon's writing that draws readers in and welcomes them into the wonderful world of analysis.

Credits
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