# Algebraic, Geometric, and Topological Methods in Optimization

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The great mathematician Lobachevsky is attributed with saying "there is no branch of mathematics, however abstract, which may not some day be applied to phenomena of the real world." There are plenty of examples of how this is evidently true for algebra (broadly including number theory too) and geometry, and topology. In my talk, accessible to students and non-experts, I will stress this point further by showing how algebraic and geometric thinking strongly influences the vibrant field of optimization. Algebraic, geometric, and topological methods permeate every subfield of optimization and have a history (see e.g., [2–4]), but let me pique your curiosity with one recent exciting example:

The linear optimization problem, often called the linear programming problem, seeks to maximize a linear functional under linear inequality and equation constraints. In matrix form we can write it as

$$\text{Maximize } \mathbf{c}^T \mathbf{x} \text{ subject to } A\mathbf{x} = \mathbf{b} \text{ and } \mathbf{x} \geq 0.$$

Here $A$ is a real $m \times n$ matrix, $\mathbf{c}$,$\mathbf{b}$ are $n$ and $m$ vectors respectively. For a short introduction to linear optimization, see [6]. Linear optimization is a workhorse of optimization. Without reliable fast algorithms for linear optimization, other more sophisticated optimization challenges would simply not be possible, for instance, linear optimization is essential to the branch-and-bound methods necessary to solve non-linear mixed-integer optimization problems. Linear programming is crucial in several external scholarly domains, e.g., data science, combinatorics and graph theory, classical geometry, and others. Here is one way in which *algebraic geometry* has helped us understand linear optimization.

*Interior point methods* have had a profound impact in modern optimization, and in applications in engineering and science. They are among the most computationally successful algorithms for linear optimization. In practical computations, interior point methods follow a piecewise-linear approximation to the *central path*, using Newton methods steps (see e.g., [7]). But in reality, the central path is the quintessential algebraic-geometric object; the central path is the *algebraic curve* given exactly by the following system of quadratic and linear polynomial equations

$$A\mathbf{x} = \mathbf{b},\ A^T \mathbf{y} - \mathbf{s} = \mathbf{c} \text{ and } x_i s_i = \lambda \text{ for } i = 1, 2, \ldots, n.$$

Until recently optimizers had mostly looked at the central path as a discretized numeric approximation, but understanding the exact central path is important (after all, we are following it closely). For example, how "curvy" can the central path really be? The intuition is that curves with small curvature are easier to approximate with fewer line segments, with fewer Newton steps. Recently, using exciting new methods from *tropical* algebraic geometry [5], X. Allamigeon, P. Benchimol, S. Gaubert, and M. Joswig showed that the total curvature of the central path can grow *expo-*

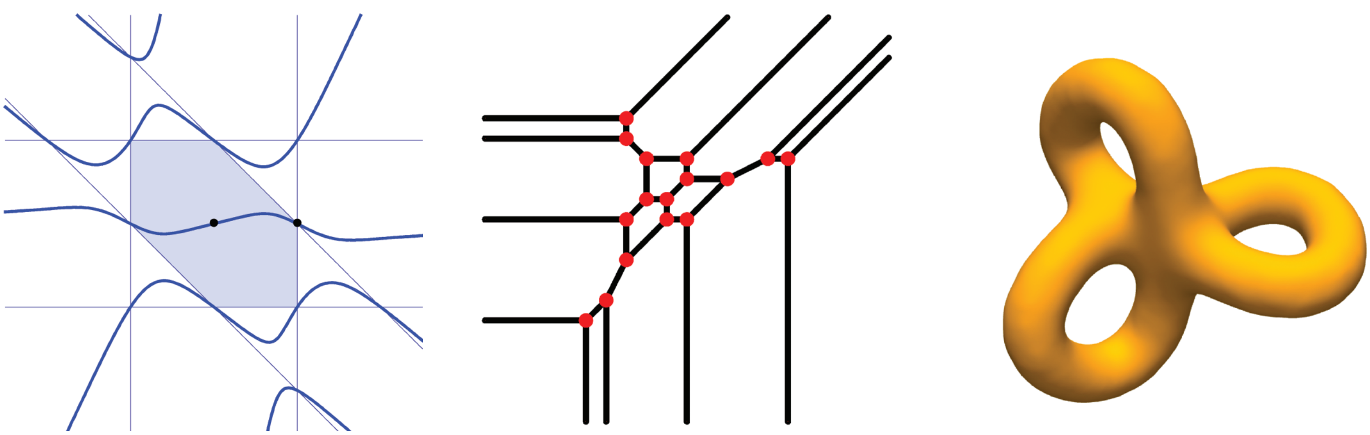

**Figure 1.** On the left a view of the entire central curve of a linear program projected to primal variable space. On the middle and right, a cartoon of the tropicalization of a genus three Riemann surface.

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*nentially* on the input data (see [1]). This implies that some interior-point methods cannot be strongly polynomial!

These techniques are fresh and unexpected, but there are many other examples like this with novel applications of algebraic and geometric techniques in optimization. I hope I tempted the reader to come and hear about opportunities for algebraists, topologists, and geometers to contribute to computational optimization. We will also organize a session on this topic. Join us in Baltimore! I promise my talk should be accessible to the non-expert and all students!

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Figure 1, left is courtesy of C. Vinzant; Figure 1, center and right are courtesy of M. Joswig.
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