Dynamics of Systems with Low Complexity

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The simplest bi-infinite sequences in $\{0,1\}^{\mathbb{Z}}$ are the periodic sequences, where a single pattern is concatenated with itself infinitely often. At the opposite extreme are bi-infinite sequences containing every possible configuration of 0's and 1's. For periodic sequences, the number of substrings of length n is bounded, while in the second case, there are 2^n substrings of length n. The growth rate of the possible patterns is a measurement of the complexity of the sequence, and can be used both to capture information about the sequence itself and to describe objects encoded by the sequence. Symbolic dynamics is the study of such sequences, associated dynamical systems, and their properties.

There is a simple relation between a local constraint on complexity and the global property of periodicity. If $x \in \mathcal{A}^{\mathbb{Z}}$ is a bi-infinite sequence over a finite *alphabet* $\mathcal{A}$, define the *complexity* $P_x(n)$ to be the number of *words* of length n in x, meaning the number of different substrings of n consecutive symbols appearing somewhere in x. A classical theorem of Morse and Hedlund [12] states that the sequence $x \in \mathcal{A}^{\mathbb{Z}}$ is periodic if and only if there exists an integer $n \geq 1$ such that $P_x(n) \leq n$. In other words, any sequence whose complexity grows very slowly actually has bounded complexity and is periodic.

A bi-infinite sequence naturally gives rise to an associated dynamical system. For $x = (x_n)_{n\in\mathbb{Z}} \in \mathcal{A}^{Z}$, define the *(left) shift* $\sigma\colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ to be the map given by $(\sigma x)_n = x_{n+1}$. Endowing $\mathcal{A}$ with the discrete topology and $\mathcal{A}^{\mathbb{Z}}$ with the product topology, σ is a homeomorphism from $\mathcal{A}^{\mathbb{Z}}$ to itself. If $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is a closed and shift invariant set, then the resulting dynamical system (X,σ) is a *subshift*. In particular, given some fixed $x \in \mathcal{A}^{\mathbb{Z}}$, by taking the closure X of its orbit under the shift σ, we obtain a subshift (X,σ). If we start with some periodic sequence $x \in \mathcal{A}^{\mathbb{Z}}$, then the resulting subshift is a periodic system. Generalizing the definition of complexity from a sequence to a system, we define the complexity $P_X(n)$ of the system (X,σ) to be the number of words of length n for any $x \in X$. The Morse-Hedlund Theorem can be reinterpreted in this context: if the complexity of the system (X,σ) satisfies $P_X(n) \leq n$ for some integer $n \geq 1$, then (X,σ) is a periodic system. At the opposite extreme, if we start with a bi-infinite sequence x that contains all words of all lengths and form the naturally associated dynamical system (X,σ), we obtain the *full shift*, meaning that $X = \mathcal{A}^{\mathbb{Z}}$ and $P_X(n) = |\mathcal{A}|^n$ for all $n \geq 1$.

The exponential growth rate of the complexity function $P_X(n)$ is one way to define the *entropy* of the system. This leads to a natural dichotomy between positive entropy, meaning exponential growth rate of the complexity, and zero entropy, meaning subexponential growth. Symbolic systems with positive entropy are often viewed as more random, typically exhibiting some sort of hyperbolic behavior, many invariant measures, and large automorphism groups (see for example [1, 11]). On the other hand, symbolic systems with zero entropy are viewed as deterministic and so simpler in some way. In spite of the restrictions imposed by zero entropy, many basic questions about such systems remain open. Even when placing strong constraints on the complexity of the system, easily formulated questions remain intractable.

It follows from the Morse-Hedlund Theorem that the first interesting case is a subshift (X,σ) satisfying
$P_X(n) \geq n+1$ for all $n \geq 1$.

If the complexity of the subshift has linear growth, we have a fairly complete picture of the constraints that this assumption places on the system. For example, consider the *automorphism group* $\mathrm{Aut}(X,\sigma)$, meaning the collection of all homeomorphisms $\phi\colon X \to X$ that commute with σ. If (X,σ) has linear complexity, then its automorphism group has a simple structure: every finitely generated subgroup of $\mathrm{Aut}(X,\sigma)$ is virtually $\mathbb{Z}^d$ for some d that depends on the linear growth rate of the complexity, and further constraints hold with stronger dynamical assumptions on the system (see [3, 10]). Linear growth also places constraints on the number of ergodic nonatomic measures that can be supported on a subshift, and in [8] we show that there can be at most finitely many such measures. These results have various applications outside of dynamics, leading to combinatorial and number theoretic results related to the complexity of particular systems.

However, by even raising the complexity to quadratic growth, the situation becomes more complicated, with only partial descriptions [2, 6, 7, 9] of the group of automorphisms, incomplete descriptions of the collection of invariant measures, and difficulties in applying the results. In an interesting twist, it turns out that some of these results can be approached by going beyond the one dimensional setting and considering $\mathcal{A}^{\mathbb{Z}^d}$ for some integer $d \geq 2$. Again, given a multi-dimensional configuration, there is a naturally associated dynamical system, but with d commuting shifts instead of a single one. The definitions of complexity and periodicity can be extended, but there are

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numerous ways in which one can do so, introducing new complications. As an example, it is not clear how to generalize the basic result of Morse and Hedlund to give a relation between complexity and periodicity. Partial results are known [4, 5, 13, 14], and while some of these can be used to further our understanding of one dimensional subshifts, we are far from having a complete understanding of these settings. We focus on the many open questions in this area, both within dynamics and in terms of applications.

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Credits

Author photo is by Jared Wunsch.

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