On Mathematical Problems in Geometric Optics

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Mathematical problems in geometric optics which deal with determining a surface that is capable of reshaping a light beam with a given illumination intensity into a prescribed intensity distribution on a target have in recent years received a lot of attention. One such problem of designing a surface which could redirect a beam of rays is the *Refractor Problem*.

In a version of this problem, we are given two media of propagation of light; medium I and medium II and a monochromatic light emanating from a point source located at some point, say the origin, $O \in \mathbb{R}^3$ in medium I, see Figure 1. The two media being considered have different optical properties that force the given light to travel with different velocities. As a result, bending of light rays, also known as *refraction* happens as a ray of light traverses an interface between medium I and medium II. From the point source, the light beam shines through an aperture $\Omega \subset S^2$, the unit sphere in $\mathbb{R}^3$, with input intensity given by a positive density function g which is integrable on Ω. We are also given $\Omega^* \subset S^2$ and positive density function $f \in L^1(\Omega^*)$ which models the output intensity.

The objective is then to find an interface (*lens*) $\mathcal{R}$ given by the graph of a radial function

ρ as $\mathcal{R} = \{\rho(x)x : x \in \Omega\}$

between media I and II, such that a ray emitted from the point O with a direction $x \in \Omega$ and input density g is refracted by the surface $\mathcal{R}$ into a direction $m \in \Omega^*$ as it proceeds into media II and the illumination intensity received is given by the density function f. It is natural to assume that g and f satisfy the energy conservation condition

$$\int_\Omega g\,dx = \int_{\Omega^*} f\,dx$$

with dx being the surface measure on S^2.

The input and output intensities and directions of radiation are prescribed a priori and so the problem of recovering the surface $\mathcal{R}$ is an inverse problem. The mathematical formulation of this problem is based on a systematic application of the laws of geometrical optics and laws of energy conservation.

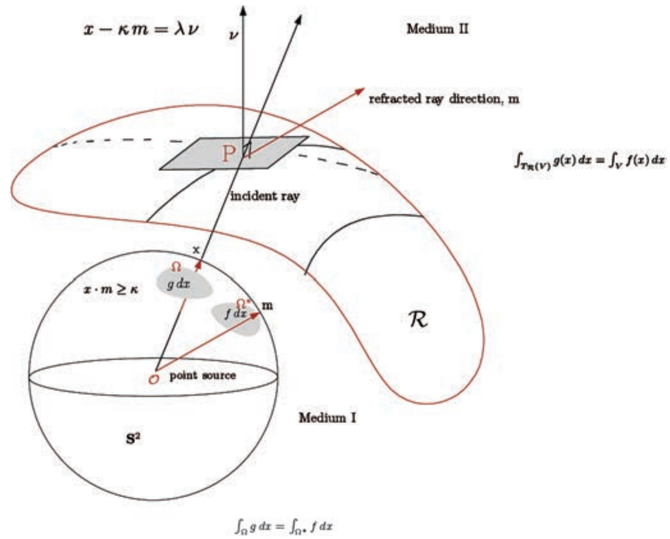

Figure 1. Figure 1. Far Field Refractor Problem

Indeed, if a ray of light traveling through medium I in the direction $x \in S^2$ traverses an interface $\mathcal{R}$ at point P and continues to propagate in medium II in the direction $m \in S^2$ (see Figure 1) then from the vector form of the law of refraction (*Snell's law*) we obtain

$$x - \kappa m = \lambda \nu$$

where ν is the normal to the surface $\mathcal{R}$ at P pointing into medium II, κ is the refractive index and $\lambda = F(x, \nu, \kappa)$. By this relation, if appropriate geometric conditions are imposed on Ω and Ω^*, the surface $\mathcal{R}$ generates a set valued map called the *tracing mapping*, $T_\mathcal{R} : \Omega^* \to \Omega$ which assigns to $m \in \Omega^*$ the set $T_\mathcal{R}(m)$ of directions $x \in \Omega$ which refract off $\mathcal{R}$ in the direction of m.

A solution $\mathcal{R}$ to the refractor design problem not only sends Ω to Ω^* via refraction, but also fulfills the energy redistribution condition. More precisely, we define a surface $\mathcal{R}$ to be a *weak* solution to the refractor problem with emitting illumination intensity g on Ω and prescribed refracted illumination intensity f on Ω^* if for any Borel set $V \subset \Omega^*$,

$$\begin{cases} \displaystyle\int_{T_\mathcal{R}(V)} g(x)\,dx = \int_V f(x)\,dx \\ T_\mathcal{R}(\Omega^*) = \Omega. \end{cases}$$

Interest to investigate several theoretical (existence and regularity of $\mathcal{R}$) and computational aspects of the above or similar problems in geometric optics has risen mainly because the techniques used in their analysis interweave

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ideas from the mathematics of optimal transportation theory, calculus of variations, and nonlinear partial differential equations of Monge-Ampère type.

In this talk I hope to present an overview of these problems and describe some recent results regarding the refractor problem.

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Credits

Figure 1 is courtesy of Henok Mawi.
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