Representation Theory and the Elliptic Frontier

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The authors of this piece are organizers of the AMS 2019 Mathematics Research Communities summer conference on **Geometric Representation Theory and Equivariant Elliptic Cohomology**, one of three topical research conferences offered this year that are focused on collaborative research and professional development for early career mathematicians.

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Equivariant elliptic cohomology is a topological invariant born out of deep work in algebraic topology and mathematical physics. Intriguingly, it both organizes previous work and points towards new terrain in the established but mysterious subject of elliptic representation theory. The central examples bridge a variety of subjects, including manifold invariants, integrable systems, algebraic combinatorics, and enumerative geometry. This article is meant to serve as a relaxed hike through this broad landscape.[1]

Functoriality in algebraic topology. The goal of algebraic topology is to associate algebraic invariants to topological spaces. In practice the most useful invariants are *functorial*: in addition to assigning an invariant to a space, continuous maps between spaces beget maps between invariants. A first example is cohomology, which assigns a graded abelian group $\mathrm{H}^\bullet(X)$ to a space X and a linear map $f^*\colon \mathrm{H}^\bullet(Y) \to \mathrm{H}^\bullet(X)$ (the *pullback*) to a continuous map $f\colon X \to Y$ between spaces. For nice maps $\pi\colon X \to B$ (e.g., with compact, oriented manifold fibers), we obtain a linear map $\pi_!\colon \mathrm{H}^\bullet(X) \to \mathrm{H}^{\bullet-d}(B)$ (the *pushforward*), where d is the fiber dimension of π. A more sophisticated invariant is K-theory, $\mathrm{K}(X)$, an abelian group generated by complex vector bundles on X. Once again, there are pullbacks $f^*\colon \mathrm{K}(Y) \to \mathrm{K}(X)$ associated to maps $f\colon X \to Y$ and pushforwards $\pi_!\colon \mathrm{K}(X) \to \mathrm{K}(B)$ for sufficiently nice maps $\pi\colon X \to B$ (e.g., when the fibers of π are compact complex manifolds).

Equivariant algebraic topology seeks to associate algebraic invariants to G-spaces, i.e., spaces endowed with the action of a Lie group G. For a G-space X, equivariant cohomology is $\mathrm{H}^\bullet_G(X) = \mathrm{H}^\bullet((X \times EG)/G)$ where EG is a

[1] *For simplicity, we make some technical elisions at various points, such as conflating homological and cohomological constructions or a compact Lie group and its complexification.*

$$Y \xleftarrow{\pi_1} X \xrightarrow{\pi_2} B \quad \rightsquigarrow \quad E(Y) \xrightarrow{\pi_1^*} E(X) \xrightarrow{(\pi_2)_!} E(B)$$

contractible space on which G acts freely, while equivariant K-theory, $K_G(X)$ is generated by G-equivariant vector bundles on X. Both of these define algebraic invariants of G-spaces that are (contravariantly) functorial with respect to equivariant maps, and both theories have pushforwards for appropriate maps $\pi : X \to B$.

Push-pull constructions in geometric representation theory. Applying functorial invariants to spaces built out of algebraic groups often results in rich representation-theoretic structures. [2] These can be loosely organized as *push-pull* constructions. Given a functorial invariant E and a pair of maps between manifolds on the left, we obtain the *push-pull map* $(\pi_2)_! \circ \pi_1^* : E(Y) \to E(B)$ as the composition on the right.

Example 1 (Convolution). Let X_1, X_2, X_3 be compact oriented smooth manifolds with dimension $\dim(X_i) = d_i$ and let $\pi_{ij} : X_1 \times X_2 \times X_3 \to X_i \times X_j$ be the projection. The *convolution product* is the map

$$-*- : H^k(X_1 \times X_2) \otimes H^l(X_2 \times X_3) \to H^{k+l-d_2}(X_1 \times X_3)$$

determined by the formula $\omega_{12} * \omega_{23} := (\pi_{13})_!(\pi_{12}^* \omega_{12} \smile \pi_{23}^* \omega_{23})$, where $\smile$ is the cup product. When X_1, X_2, X_3 are 0-dimensional (so just *finite sets*) $H^\bullet(X_i \times X_j)$ can be identified with the vector space of $d_i \times d_j$ matrices, and the convolution product is the matrix product.

Example 2 (Springer theory). Consider the *flag variety*, G/B, where $G = GL(n, \mathbb{C})$, B is the subgroup of upper triangular matrices and $\tilde{\mathcal{N}} = T^*(G/B)$ its cotangent bundle. The cotangent fiber over each flag is the space of nilpotent $n \times n$ matrices preserving that flag. Defining the *nilpotent cone* $\mathcal{N}$ as all nilpotent $n \times n$ matrices, there is a projection map $\tilde{\mathcal{N}} \to \mathcal{N}$ called the *Springer resolution*. The *Steinberg variety* is the fibered product $\tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$. Then the push-pull construction associated to $\tilde{\mathcal{N}} \xleftarrow{\pi_1} \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} \xrightarrow{\pi_2} \tilde{\mathcal{N}}$ defines a product on (compactly supported) cohomology $H_c^\bullet(\tilde{\mathcal{N}})$

$$w * v = (\pi_2)_!(\pi_1^* w \smile \pi_2^*(v)), \quad v, w \in H_c^\bullet(\tilde{\mathcal{N}}).$$

With this algebra structure, $H_c^{top}(\tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}})$ may be identified with $\mathbb{Z}[W]$, the group algebra of the Weyl group. In this case this is simply S_n, the symmetric group on n letters. By considering similar constructions for the fibers $\tilde{\mathcal{N}}_x$ of the Springer resolution, one may endow their cohomologies with actions of $\mathbb{Z}[W]$ and derive geometrically, for example, the standard classification of irreducible representations of S_n by partitions.

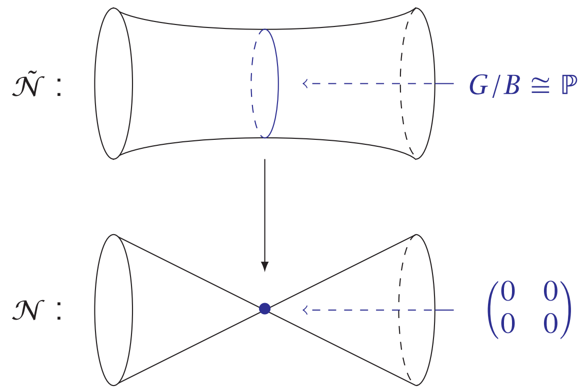

Figure 1. The Springer resolution for $G = GL(2, \mathbb{C})$.

Example 3 (Affine Hecke algebra). Continuing the previous example, let $T < G$ be the n-dimensional torus of diagonal matrices. The Springer resolution is naturally T-equivariant. Furthermore $\tilde{\mathcal{N}}$ has an additional $\mathbb{C}^\times$-action from scaling the cotangent fibers. These actions carry over to the Steinberg variety $\tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$, and so we can form the equivariant K-group $K_{T \times \mathbb{C}^\times}(\tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}})$. The analogous push-pull construction identifies this K-group with the affine Hecke algebra $\mathbb{H}^{aff}$, a deformation of the group algebra of $\Sigma_n \ltimes \mathbb{Z}^n$. This generalizes the Iwahori–Hecke algebra of p-adic representation theory, which controls (spherical) representations of the p-adic Lie group $G(\mathbb{Q}_p)$. As such, this yields a geometric construction of the Iwahori–Hecke algebra and a geometric classification of irreducible representations of $\mathbb{H}^{aff}$, first proved by Kazhdan and Lusztig.

Example 4 (Nakajima quiver varieties [6]). Consider the cotangent bundle $T^*G(k,n)$ of the Grassmannian $G(k,n)$ of k-planes in $\mathbb{C}^n$. A cotangent vector over a point $V_k \subset \mathbb{C}^n$ is an element $E \in \mathrm{Hom}(V_k, \mathbb{C}^n/V_k)$. The *Hecke correspondence* $Z_{k,k+1} \subset T^*G(k,n) \times T^*G(k+1,n)$ is the subspace of pairs (V_k, E_1) and (V_{k+1}, E_2) where $V_k \subset V_{k+1}$, (so there is a projection $p : \mathbb{C}^n/V_k \twoheadrightarrow \mathbb{C}^n/V_{k+1}$) and $E_1 = E_2 \circ p$. For fixed n, we consider the disjoint union over k of $T^*G(k,n)$ and the Hecke correspondences between them. Applying equivariant cohomology yields a representation of the *Yangian* $Y(\mathfrak{sl}_2)$, while applying equivariant K-theory yields a representation of the *quantum affine algebra* $U_q(\widehat{\mathfrak{sl}}_2)$. Both of these quantum algebras contain commutative subalgebras important to quantum integrable systems, and in both cases, the subalgebra is diagonalized by the fixed point basis. This diagonalization paradigm carries over to other well-known examples of quiver varieties, such as the Hilbert of scheme of points on the plane. Here, fixed point classes in equivariant K-theory have been associated to Macdonald polynomials in Haiman's celebrated proof of Macdonald positivity. On the other hand, Macdonald polynomials appear in integrable systems as eigenfunctions for the Macdonald operators, and the action of these operators can be realized geometrically through this kind of K-theoretic construction of a quantum group.

The elliptic frontier. The above examples prompt an obvious question: what happens when we apply ever more complicated functorial invariants to the spaces built out of algebraic groups? In algebraic topology, the chromatic filtration organizes generalized cohomology theories according to their *height*, which is a rough measure of complexity: ordinary cohomology with complex coefficients has height 0 while K-theory has height 1. The most interesting example of a height 2 theory is elliptic cohomology. *Equivariant* elliptic cohomology is still very much under development. The visionary insights of Grojnowski [3] defined the theory over $\mathbb{C}$ while Lurie's more recent work [5] gives indications of what one can expect from the full theory over $\mathbb{Z}$.

One mystery to be solved is the basic representation theoretic content of equivariant elliptic cohomology. To start with an analogy, the coefficients for equivariant K-theory are $K_G(\mathrm{pt}) = \mathrm{Rep}(G)$, the representation ring of G. The complexification of $\mathrm{Rep}(G)$ is the ring of class functions, where characters of representation take their values. Grojnowski's elliptic cohomology over $\mathbb{C}$ has as its coefficients a ring of θ-functions. One might therefore speculate that the coefficients for equivariant elliptic cohomology is some ring of *elliptic representations* whose characters are θ-functions. Positive energy loop group representations have θ-functions as characters, and so are expected to play a central role. However, this appears to only be one part of the story.

A second focal point comes from elliptic generalizations of constructions in geometric representation theory. The examples above point the way to the deep theories of elliptic Hecke algebras, elliptic quantum groups, and elliptic Macdonald polynomials. The recent work of Aganagic–Okounkov [1] suggests a presentation for elliptic cohomology of flag varieties in a precursor to a long-anticipated but unrealized elliptic Schubert calculus. Prominent work in related directions has been pursued by Etingof, Felder, Grojnowski, Rains, Schiffmann, Varchenko, Vasserot, Yang, Zhao, and Zhong among many others.

A third body of research aims to clarify the long-sought but still-mysterious connection between elliptic cohomology and mathematical physics. This was born out of Witten's string theory interpretation [8] of manifold invariants called *elliptic genera*. Conjectured properties of these invariants (e.g., rigidity) were proved using equivariant elliptic cohomology. Segal suggested [7] that these connections between string theory and topology could be deepened by constructing a cocycle model for elliptic cohomology in terms of a suitable space of 2-dimensional field theories. Although the full picture remains unrealized, many of the underlying ideas drove great advances in homotopy theory. Led by Hopkins and collaborators [4], a high point of these results is the string orientation of elliptic cohomology which gives a families version of elliptic genera. One of the more recent fruits of this labor has been the intriguing role of *categorical groups* in equivariant elliptic cohomology. In short, whereas groups are automorphisms of sets, categorical groups are automorphisms of categories. The *string group* is a particular 2-group that has a preferred role in elliptic cohomology and also appears to have an origin in physics as the automorphisms of a particular quantum field theory. Indeed, there are yet vast realms of representation theory to be explored along this elliptic frontier.

Want to learn more? Supported by the AMS, the authors are organizing a Mathematical Research Community on geometric representation theory and equivariant elliptic cohomology for PhD students and early career researchers. The event will be held June 2–8, 2019 in Rhode Island, during which time participants will work in small groups on research problems. Prior to the workshop, there will be an opportunity to learn relevant background material through a guided reading course. The AMS also provides support after the workshop for participants to continue to collaborate and to attend a special session at the JMM in January 2020.

To find out more, visit www.ams.org/programs/research-communities/2019MRC-Geometry, or email any of the organizers. Applications are due on February 15, 2019. We look forward to welcoming a diverse group of participants from different mathematical backgrounds—expertise in all areas is not at all expected!

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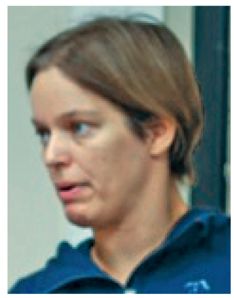
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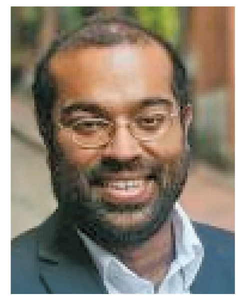
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