An Illustration in Number Theory

Katherine E. Stange

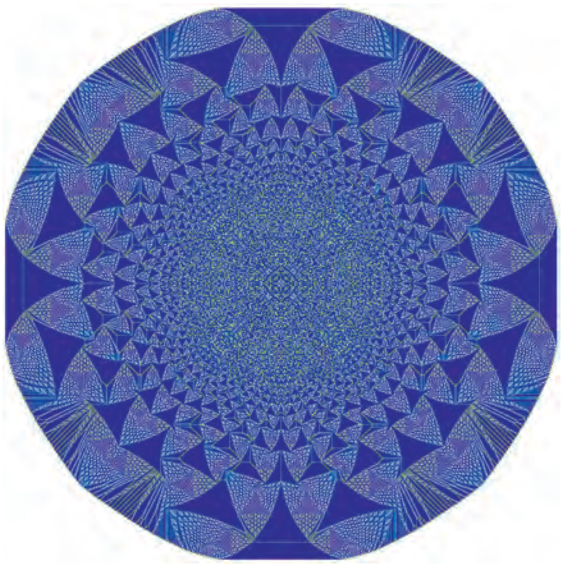

Figure 1. Stabilized sandpile for $500,000$ chips at the origin of $\mathbb{Z}^2$. Dark blue, light blue, green, and white represent 3, 2, 1, and 0 chips, respectively.

In the spirit of ICERM's upcoming semester on illustrating mathematics in fall 2019, my talk is a celebration of the symbiosis that can exist between research and illustration. I hereby invite you on a number-theoretic "explore" (as Pooh and Piglet would have it).

The story begins with a sandpile. A discrete sandpile on the square grid $\mathbb{Z}^2$ is a dynamical system in which each vertex contains some integer number of *chips*. Just as grains of sand will tumble and spread out, if these chips number four or greater, then the position will *topple*, sending one chip to each of its four neighbors. In this way, a stack of N chips initially placed at the origin will spread out across the grid until each position contains either 0, 1, 2, or 3 chips.

The result, for $N = 500,000$ chips is shown in Figure 1. Surprisingly, the patterns evident in Figure 1 do not fade as N tends to infinity. In fact, by rescaling to a constant width, one can define a scaling limit of this discrete sandpile, and its boundary is not even circular [11]. In evidence are curvilinear triangular regions of regularity in which periodic patterns of sand appear. This model, introduced by Bak, Tang, and Wiesenfeld, is an archetypical example of *self-organized criticality* [1]; see [10] for an overview.

Consider the following discrete Laplacian, which acts on functions $g : \mathbb{Z}^2 \to \mathbb{Z}$:

$$\Delta g(x) = \sum_{y \sim x} \left(g(y) - g(x)\right),$$

where $x \sim y$ indicates adjacency of vertices. If we start with a barren empty grid and think of $g : \mathbb{Z}^2 \to \mathbb{Z}$ as an *odometer*, that is, a local count of topples that occur, then Δg is the resulting distribution of sand (we must allow topplings that result in a negative number of chips).

This is the discrete analogue to the continuous Laplacian $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$, which has the property that $\Delta f(x,y) \equiv 0$ if and only if $f : \mathbb{R}^2 \to \mathbb{R}$ is harmonic. Levine, Pegden, and Smart therefore asked, *which odometers are superharmonic* [9]? More specifically, for which symmetric 2×2 real matrices A does there exist $g : \mathbb{Z}^2 \to \mathbb{Z}$ so that

$$g(x) = x^t A x + o(|x|^2), \quad \Delta g \leq 0?$$

They defined Γ to be the set of such A, as a subset of the parameter space $\mathbb{R}^3$ given by the entries of the matrix. It is not hard to show the set should be a union of cones (this property is downward closed), but when they approximated this set experimentally, they obtained a surprising fractal boundary; see Figure 2.

The cones were arranged according to an *Apollonian circle packing*. This is an iterative fractal generated from a quadruple of mutually tangent circles by filling in the triangular region between any three tangent circles with a *daughter* circle tangent to its ancestors (Figure 3). A remarkable

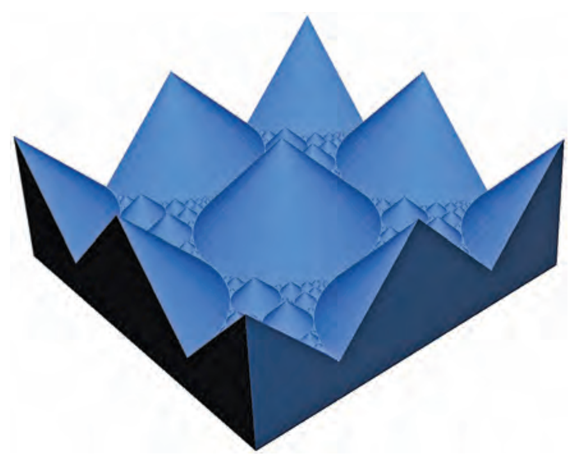

Figure 2. A part of the set Γ, which is a union of cones subtended by circles of an Apollonian circle packing.

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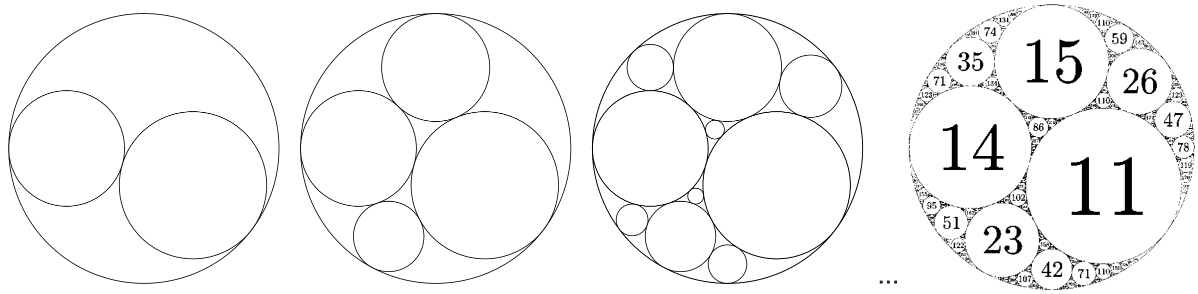

Figure 3. The iterative process generating an Apollonian circle packing, with curvatures shown in the final stage.

property of Apollonian circle packings, explained by René Descartes, is that if one begins with four mutually tangent circles with integral curvature (the curvature of a circle is 1 over its radius), then all circles in the packing will have integral curvature [5].

The collection of curvatures of an Apollonian circle packing has generated recent attention among number theorists. The collection is generated by the action of a thin matrix group. If G is an algebraic group, a subgroup H of $G(\mathbb{Z})$ is *thin* if it is Zariski dense but of infinite index; such groups are much less accessible than lattices such as congruence subgroups. The collection of curvatures is conjectured to include all but finitely many integers satisfying a congruence condition modulo 24 (a *local-to-global principle* for circle packings) [6, 7]. Bourgain and Fuchs applied the Hardy–Littlewood circle method to prove a density one result [2]. For an overview of this approach to thin groups, see [8].

Computer evidence for the shape of Γ was a surprise, and it turned the investigation toward the "Apollonian" peaks. These odometers g correspond to very special Laplacians Δg, which can be observed, experimentally, as periodic sandpiles (after stabilization) appearing in regions of Figure 1; see Figure 4. David Wilson of Microsoft Research

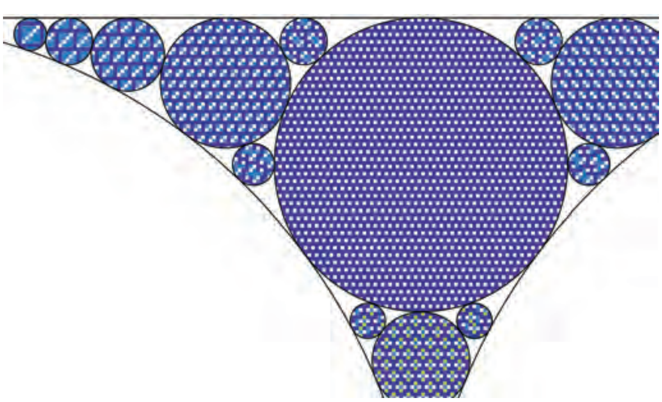

Figure 4. A selection of circles of the Apollonian circle packing filled with the corresponding stabilized sandpile patterns. Dark blue, light blue, green, and white represent 1, 0, −1, and −2 chips, respectively.

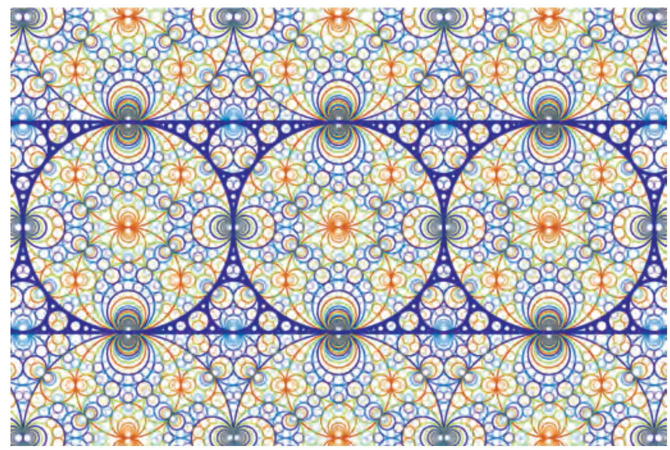

Figure 5. The Schmidt Arrangement of the Gaussian integers, with an Apollonian circle packing highlighted in dark blue. Circles fade as an increasing function of their curvatures. Light blue, dark blue, orange, and green represent curvatures 0, 1, 2, and 3 modulo 4, respectively. The picture has periodicity modulo $\mathbb{Z}[i]$ and is centered on $i/2$.

created an interactive computer program, wherein the user could mouse over a circle C in an Apollonian circle packing to obtain a basis for $\Lambda_C \subseteq \mathbb{Z}^2$, the lattice of periodicity of the corresponding sandpile. The covolume of Λ_C was observed to match the curvature of the circle.

From this computer data, one can glean a recursive rule for generating the lattices, from ancestors to daughters in the packing (just as one generates curvatures). The resulting theory is reminiscent of the *topograph*, which is a fractal tree demarcating $\mathbb{P}^1(\mathbb{Z})$, and which Conway and Fung use to elegant effect to classify quadratic forms [4]. For a fanciful paper comparing the two, see [13]. These ideas figure in the proof of Γ's shape by Levine, Pegden, and Smart [9].

The Apollonian circle packing is an orbit of circles under the Möbius action of a thin subgroup of $\mathrm{PSL}_2(\mathbb{Z}[i])$. If one computes the orbit of the entire Bianchi group $\mathrm{PSL}_2(\mathbb{Z}[i])$, the result is Figure 5. Each circle in this arrangement represents an element of the Bianchi group, and the lattice Λ_C is generated by $\mathbb{Z}$-linear combinations of the lower entries of the corresponding matrix [14]. This gives

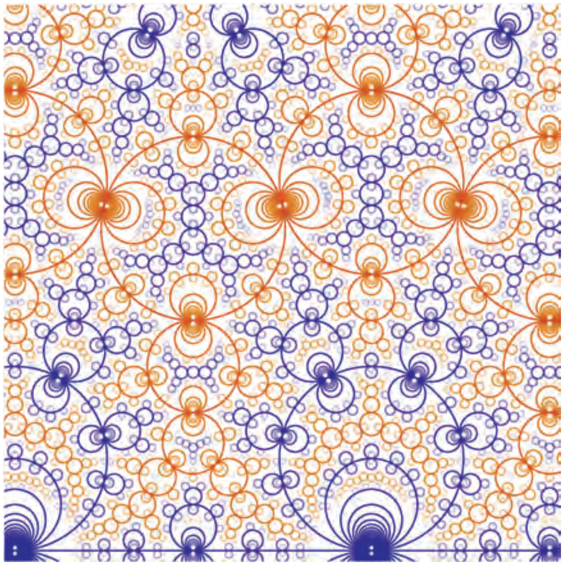

Figure 6. The Schmidt Arrangement of $\mathbb{Q}(\sqrt{-15})$ in blue, together with the remainder of the orbit of the extended Bianchi group, in yellow. The two orbits represent the two elements of the class group of $\mathbb{Q}(\sqrt{-15})$.

a bijection between sublattices of $\mathbb{Z}^2$ (or, if you prefer, certain ideals of orders in $\mathbb{Z}[i]$), and the circles of Figure 5 (up to certain translations and rotations). The geometric arrangement of the circles is reflected in algebraic relationships among the lattices.

Why not draw similar pictures for other Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_K)$, where K is an imaginary quadratic field? Each has its own iterative structure and contains Apollonian-like circle packings [15]. An example is shown in Figure 6. What aspects of the arithmetic of the field control the variation in corresponding geometry? One example is that the picture is connected if and only if the ring of integers is Euclidean [14]. These images are called *Schmidt Arrangements* for Asmus Schmidt, who used them to define complex continued fractions before the arrangements could be drawn by computer [12]; see [3] for another perspective on continued fractions and their relationship to Figure 5.

Aided by computer experimentation, we have now wandered from sandpiles, through Apollonian circle packings, to arithmetic geometry. In fact, a general theory connecting sandpiles and arithmetic geometry is now emerging by way of tropical geometry, where the odometers of our story will correspond to tropical theta functions.

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