



Notices

of the American Mathematical Society

April 2019

Volume 66, Number 4



Call for Nominations

AMS EXEMPLARY PROGRAM AWARD

The AMS Award for Exemplary Program or Achievement in a Mathematics Department is presented annually to a department that has distinguished itself by undertaking an unusual or particularly effective program of value to the mathematics community, internally or in relation to the rest of the society. Examples might include a department that runs a notable minority outreach program, a department that has instituted an unusually effective industrial mathematics internship program, a department that has promoted mathematics so successfully that a large fraction of its university's undergraduate population majors in mathematics, or a department that has made some form of innovation in its research support to faculty and/or graduate students, or which has created a special and innovative environment for some aspect of mathematics research.

The award amount is \$5,000. All departments in North America that offer at least a bachelor's degree in the mathematical sciences are eligible.

The Award Selection Committee requests nominations for this award, which will be announced in Spring 2020. Letters of nomination may be submitted by one or more individuals. Nomination of the writer's own institution is permitted. The letter should describe the specific program(s) for which the department is being nominated as well as the achievements that make the program(s) an outstanding success, and may include any ancillary documents which support the success of the program(s). The letter should not exceed two pages, with supporting documentation not to exceed an additional three pages.

Further information about AMS prizes can be found at the Prizes and Awards website: www.ams.org/prizes.

Further information and instructions for submitting a nomination can be found at the prize nomination website: www.ams.org/nominations.

For questions contact the AMS Secretary at secretary@ams.org.

Deadline for nominations is September 15, 2019.

William Benter Prize in Applied Mathematics 2020

Call for **NOMINATIONS**

The Liu Bie Ju Centre for Mathematical Sciences of City University of Hong Kong is inviting nominations of candidates for the William Benter Prize in Applied Mathematics, an international award.

The Prize

The Prize recognizes outstanding mathematical contributions that have had a direct and fundamental impact on scientific, business, financial, and engineering applications.

It will be awarded to a single person for a single contribution or for a body of related contributions of his/her research or for his/her lifetime achievement.

The Prize is presented every two years and the amount of the award is US\$100,000.

Nominations

Nomination is open to everyone. Nominations should not be disclosed to the nominees and self-nominations will not be accepted.

A nomination should include a covering letter with justifications, the CV of the nominee, and two supporting letters. Nominations should be submitted to:

Selection Committee

c/o Liu Bie Ju Centre for Mathematical Sciences
City University of Hong Kong
Tat Chee Avenue, Kowloon, Hong Kong

Or by email to: lbj@cityu.edu.hk

Deadline for nominations: 30 September 2019

Presentation of Prize

The recipient of the Prize will be announced at the **International Conference on Applied Mathematics 2020** to be held in summer 2020. The Prize Laureate is expected to attend the award ceremony and to present a lecture at the conference.

The Prize was set up in 2008 in honor of Mr William Benter for his dedication and generous support to the enhancement of the University's strength in mathematics. The inaugural winner in 2010 was George C Papanicolaou (Robert Grimmitt Professor of Mathematics at Stanford University), and the 2012 Prize went to James D Murray (Senior Scholar, Princeton University; Professor Emeritus of Mathematical Biology, University of Oxford; and Professor Emeritus of Applied Mathematics, University of Washington), the winner in 2014 was Vladimir Rokhlin (Professor of Mathematics and Arthur K. Watson Professor of Computer Science at Yale University). The winner in 2016 was Stanley Osher, Professor of Mathematics, Computer Science, Electrical Engineering, Chemical and Biomolecular Engineering at University of California (Los Angeles), and the 2018 Prize went to Ingrid Daubechies (James B. Duke Professor of Mathematics and Electrical and Computer Engineering, Professor of Mathematics and Electrical and Computer Engineering at Duke University).

The Liu Bie Ju Centre for Mathematical Sciences was established in 1995 with the aim of supporting world-class research in applied mathematics and in computational mathematics. As a leading research centre in the Asia-Pacific region, its basic objective is to strive for excellence in applied mathematical sciences. For more information about the Prize and the Centre, please visit <https://www.cityu.edu.hk/lbj/>





A WORD FROM...

Ken Ono, AMS Vice President

Photo is courtesy of Emory University.



Dear Members of the American Mathematical Society,

It is a great honor for me to serve as your Vice President. I have greatly benefited from the many programs offered by the AMS, and I hope to do my part to help continue its tradition of supporting the mathematics profession.

As the son of a mathematician, I have been aware of the AMS my entire life. Indeed, my father subscribed to *Mathematical Reviews* (replaced by *MathSciNet*®), and as a kid I marveled at the phone book sized orange volumes that seemed to reproduce and monopolize the floor space of our home. These monstrosities even found their way into my bedroom. Obviously, from an early age I could not escape the fact that people actually did research in mathematics. I simply had no idea what that meant.

Thirty years ago I became an AMS member thanks to the courtesy “nominee” memberships for graduate students. As a junior PhD student, I looked forward to finding the latest issues of the *Notices* and the *Bulletin* stuffed in my cubbyhole of a mailbox at UCLA. These publications offered me thrilling glimpses of cutting edge research (e.g., “One cannot hear the shape of a drum”), as well as a respite from the doldrums of preparing for the difficult qualifying exams that petrified me.

As a new PhD, I was grateful for the professional services that the AMS offered. I made great use of the AMS employment services. I published some of my first papers in the *Proceedings* and the *Transactions*. Many of my first conference talks were clumsy efforts at Special Sessions at AMS Sectional Meetings. As I look back now on my career, I find that most of its defining moments have somehow involved the AMS.

As a senior mathematician, I now have the important responsibility of paying forward my good fortune. To this end, I have enjoyed serving the AMS in a number of important capacities. I have been a member of the Council for over ten years, and I have been on a number of committees including the Editorial Board of the *Proceedings* (serving six years as the managing editor).

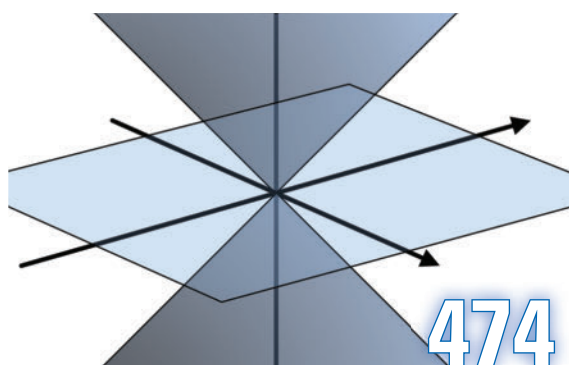
With this personal history in mind, I am humbled and honored to have been elected Vice President of this wonderful professional Society. I am committed to doing my best to serve the membership and the profession of mathematics.

Sincerely,

Ken Ono

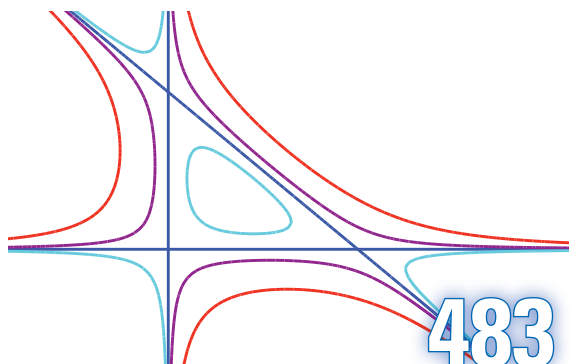
Ken Ono is Asa Griggs Candler Professor of Mathematics at Emory University and vice president of the American Mathematical Society. His email address is ken.ono@emory.edu.

FEATURED



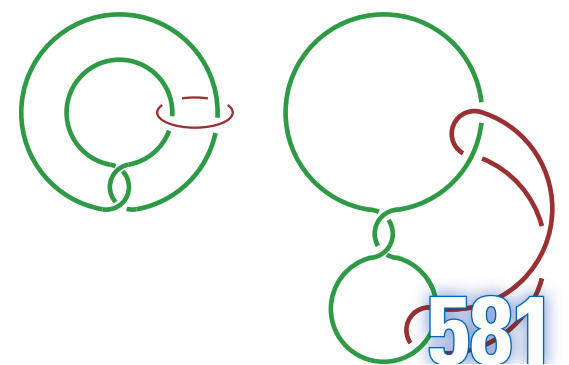
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NOTE OF CLARIFICATION

The March issue's cover was an artist's interpretation of original imagery that was provided courtesy of David Dumas.



A Letter from the AMS Committee on Human Rights of Mathematicians

American Mathematical Society Committee on Human Rights of Mathematicians Issues Statement of Concern about Turkish Mathematicians

The Turkish government has charged the mathematician Professor Ayse Berkman with the crime of “making propaganda for a terrorist organization,” based solely on her having signed a petition decrying military operations against civilians in Kurdish provinces. She appeared before the Heavy Penalty Court of Istanbul on January 10, 2019; a translation of her defense statement is available here <https://m.bianet.org/english/freedom-of-expression/204414-statement-of-academic-ayse-berkman> and the petition she signed may be seen here: <https://www.barisicinakademisyenler.net/node/63>.

Professor Berkman received her PhD at the University of Manchester Institute of Science and Technology in 1998. She has been teaching mathematics in Turkey since then, currently at Mimar Sinana University in Istanbul, and is a member of the American Mathematical Society.

The Committee on Human Rights of Mathematicians of the American Mathematical Society deplores these political charges against Professor Berkman, which are a clear violation of human rights and academic freedom. These charges are part of a disturbing pattern: hundreds of academics in Turkey have been charged, and scores sentenced, for similar expressions of opinion. We decry these assaults on academic freedom and urge the Turkish government to respect the political and human rights of Professor Berkman and her colleagues.

Dr. Arthur Ogus

*Univeristy of California at Berkeley
Committee Chair, AMS Committee on Human
Rights of Mathematicians*

February 8, 2019

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LETTERS TO THE EDITOR



From the AMS: The following letter was sent to members of Congress and to the President of the United States on February 11, 2019. It is shared here for our membership.

OPEN LETTER TO THE CONGRESS AND PRESIDENT OF THE UNITED STATES

Nobel Laureates and Science Community Leaders Comment on Harm to American Science from the Shutdown

Dear Mr. President and Members of Congress:

As American scientists—researchers, teachers, heads of major national scientific societies and institutes, and Nobel Laureates—we are writing to call attention to the harm done to the US scientific enterprise by the recently ended partial shutdown of the federal government. The disruptions caused by the shutdown have consequences that will extend well beyond the shutdown, with the potential to affect many aspects of our society, including our economy, security, health, and international competitiveness.

For decades, the US has led the world in basic scientific research. Our strength in fundamental research gave birth to the military technology that helped to end World War II and continues to safeguard us and our allies. Our past global scientific dominance fueled the technological innovations that have made our economy the strongest in the world. A critical component of that leadership was, and continues to be, sustained federal investment in basic research.

Today, in a trend starting long before the recent disruption, our scientific leadership is threatened by other countries whose investment in research is growing more rapidly than our own. The government shutdown closed some of the agencies most crucial to the maintenance of our leadership and of the health of American science. The National Science Foundation (NSF) funds much of the basic research in our universities. The National Oceanic and Atmospheric Administration (NOAA), the National Aeronautics and Space Administration (NASA), the National Institute of Standards and Technology (NIST), and others produce fundamental research leading to innovations that improve our daily lives, our security, and our economy.

Even the temporary loss of those activities has a profoundly disruptive effect on experimental work and the

functions of research teams at a time when American scientific leadership is challenged by China and other international competitors.

Make no mistake: although the shutdown's effect on science will not be as immediately evident as were the long airport security lines, flight delays, and missing paychecks for federal employees, the effects will be longer lasting and more widespread. Major science agencies like the National Institutes of Health (NIH), the Centers for Disease Control and Prevention (CDC), and the Department of Energy (DOE), which already had their funding approved and did not shut down, nevertheless felt the effects because important connections and collaborations with scientists supported by the shuttered agencies were put on hold. And scientists at non-government institutions, such as universities and research institutes, were impeded by the absence of staff at federal agencies that support their work.

Science is essential to our technological society. The development of advanced materials and devices, new medical treatments, worldwide communication technologies, new energy sources, GPS navigation with our smartphones—essentially all the technologies used by modern societies—were enabled by federal support for fundamental science. Future advances will depend on additional programs, such as the newly enacted National Quantum Initiative, designed to change the landscape of military and commercial capabilities. But during the shutdown much of the new quantum research could not even begin, while China and Europe continued to develop the new quantum technology at full speed. Similarly, while NASA had to suspend some of its efforts to explore space, other countries continued their programs to plant probes in previously unexplored parts of our universe. Of even greater long-term consequence, the interruption of the careers of young researchers has likely caused some to question their future involvement in our national scientific adventure.

We write to you now, at a time when another possible government shutdown looms, to draw your attention to the detrimental consequences of even short-term suspensions of federal funding on the nation's scientific enterprise. We are encouraged by discussions of proposals that would protect science, among other critical activities, from the

**We invite readers to submit letters to the editor at
notices-letters@ams.org.*

significant disruptions that occur during shutdowns of appreciable length, and we urge the avoidance of such lapses. Shutting down parts or all of the federally funded scientific enterprise, which enjoys support across the entire political spectrum, serves only our foreign competitors. Continued strong support for science benefits us all.

Yours respectfully,

Nobel Laureates

Frances H. Arnold

Nobel Laureate, Chemistry 2018

David Baltimore

Nobel Laureate, Physiology or Medicine, 1975

J. Michael Bishop

Nobel Laureate, Physiology or Medicine, 1989

Michael S. Brown

Nobel Laureate, Medicine or Physiology, 1985

Steven Chu

Nobel Laureate, Physics, 1997

President-elect, American Association for the Advancement of Science

Robert Curl

Nobel Laureate, Chemistry, 1996

Joseph Goldstein

Nobel Laureate, Physiology or Medicine, 1985

Carol Greider

Nobel Laureate, Physiology or Medicine, 2009

David Gross

Nobel Laureate, Physics, 2004 President, American Physical Society

Robert H. Grubbs

Nobel Laureate, Chemistry, 2005

Robert Horvitz

Nobel Laureate, Physiology or Medicine, 2002

Brian Kobilka

Nobel Laureate, Chemistry, 2012

Roger D. Kornberg

Nobel Laureate, Chemistry, 2006

W. E. Moerner

Nobel Laureate, Chemistry, 2014

William D. Phillips

Nobel Laureate, Physics, 1997

Randy Schekman

Nobel Laureate, Physiology or Medicine, 2013

Richard R. Schrock

Nobel Laureate, Chemistry, 2005

Harold E. Varmus

Nobel Laureate, Physiology or Medicine, 1989

David J. Wineland

Nobel Laureate, Physics, 2012

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Mathematics and Statistics Awareness Month

April marks a time to increase the understanding and appreciation of mathematics and statistics. Why? Both subjects play a significant role in addressing many real-world problems—climate change, disease, sustainability, the data deluge, internet security, and much more. Research in these and other areas is ongoing, revealing new results and applications every day in fields such as medicine, manufacturing, energy, biotechnology, and business. Mathematics and statistics are important drivers of innovation in our technological world, in which new systems and methodologies continue to become more complex.

Organize and host activities in April for Mathematics and Statistics Awareness Month! Past activities have included workshops, competitions, festivals, lectures, symposia, department open houses, math art exhibits, and math poetry readings. **Share your activities on social media.**



MathAware

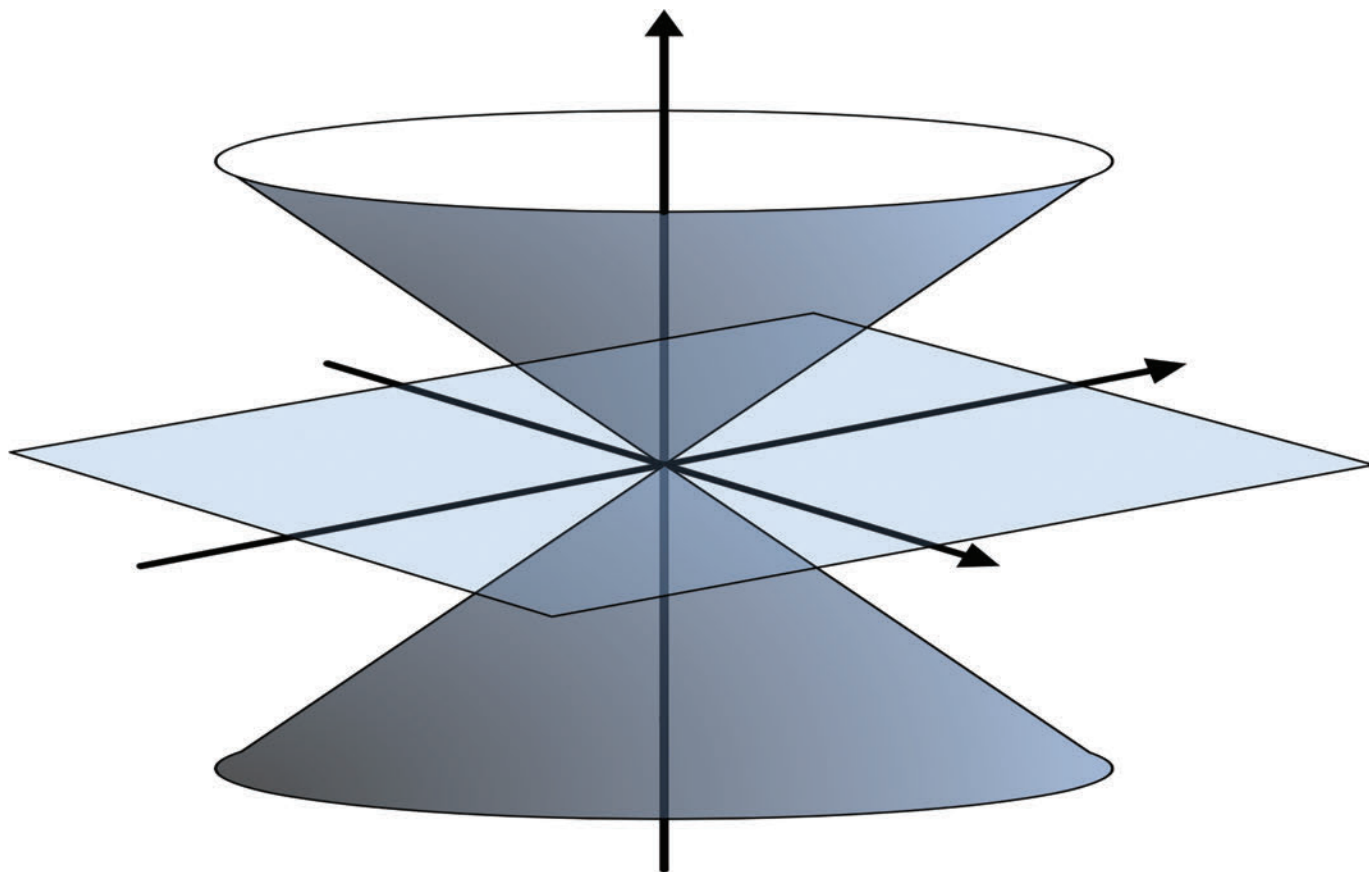


@MathAware

Mathematics and Statistics Awareness Month is a program of the Joint Policy Board for Mathematics (JPBM)—a collaborative effort of the American Mathematical Society, the American Statistical Association, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics.

Geometric Measure Theory

Recent Applications



Tatiana Toro

GMT Introduction

Geometric Measure Theory (GMT) provides a framework to address questions in very different areas of mathematics, including calculus of variations, geometric analysis, potential theory, free boundary regularity, harmonic analysis, and theoretical computer science. Progress in different branches of GMT has led to the emergence of new challenges, making it a very vibrant area of research. In this note we will provide a historic background to some of the

questions that gave rise to the field, briefly mention some of the milestones, and then focus on some of the recent developments at the intersection of GMT, potential theory, and harmonic analysis.

The origins of the field can be traced to the following question: *do the infinitesimal properties of a measure determine the structure of its support?*

In the late 1920s and early '30s Besicovitch was interested in understanding the structure of a set $E \subset \mathbb{R}^2$ satisfying $0 < \mathcal{H}^1(E) < \infty$ and such that for \mathcal{H}^1 -a.e. $x \in E$,

$$\lim_{r \rightarrow 0^+} \frac{\mathcal{H}^1(B(x, r) \cap E)}{2r} = 1, \quad (1)$$

where \mathcal{H}^1 denotes the 1-dimensional Hausdorff measure. (The formulation above is a modern version of the problem. Besicovitch, who most likely was unaware of the existence of the Hausdorff measure, formulated the question

Tatiana Toro is the Craig McKibben & Sarah Merner Professor in Mathematics at the University of Washington. She was partially supported by NSF grant DMS-1664867. Her email address is toro@uw.edu.

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in terms of the linear measure.) For $n \geq 1$, the n -dimensional Hausdorff measure \mathcal{H}^n in \mathbb{R}^m generalizes the notions of length of a curve ($n = 1$), surface area ($n = 2$), and volume ($n = 3$) to subsets of \mathbb{R}^m . Moreover $\mathcal{L}^n \ll \mathcal{H}^n \ll \mathcal{L}^n$

Besicovitch showed that if E satisfies the hypothesis (1), then E is 1-rectifiable; that is, E is contained in a countable union of Lipschitz images of \mathbb{R} union a set of 1-Hausdorff measure 0 (see [16], [17]). In GMT the notion of rectifiability is used to describe the structure (also the regularity) of a set or a measure in a way similar to how the degree of differentiability of charts is used to describe the smoothness of a manifold in differential geometry. A set $E \subset \mathbb{R}^m$ is n -rectifiable if

$$E \subset \bigcup_{i=1}^{\infty} f_j(\mathbb{R}^n) \cup E_0,$$

where $f_j : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a Lipschitz map and $\mathcal{H}^n(E_0) = 0$. Recall that f_j is Lipschitz if there exists $L_j > 0$ s.t. for $x, y \in \mathbb{R}^n$

$$|f_j(x) - f_j(y)| \leq L_j |x - y|.$$

In 1947, Federer [30] proved a general converse of Besicovitch's theorem: if $n < m$ and $E \subset \mathbb{R}^m$ is n -rectifiable, then for \mathcal{H}^n -a.e. $x \in E$

$$\lim_{r \rightarrow 0^+} \frac{\mathcal{H}^n(B(x, r) \cap E)}{\omega_n r^n} = 1, \quad (2)$$

where ω_n denotes the Lebesgue measure of the unit ball in \mathbb{R}^n .

We introduce some terminology that will help us set the framework. Let μ be a Radon measure in \mathbb{R}^m (i.e. a Borel regular measure that is finite on compact sets). The n -density of μ at x

$$\theta^n(\mu, x) := \lim_{r \rightarrow 0} \frac{\mu(B(x, r))}{\omega_n r^n} \quad (3)$$

exists if the limit exists and $\theta^n(\mu, x) \in (0, \infty)$

A locally finite measure μ on \mathbb{R}^m is n -rectifiable if μ is absolutely continuous with respect to \mathcal{H}^n ($\mu \ll \mathcal{H}^n$, i.e. $\mathcal{H}^n(F) = 0$ implies $\mu(F) = 0$) and

$$\mu(\mathbb{R}^m \setminus \bigcup_{j=1}^{\infty} f_j(\mathbb{R}^n)) = 0,$$

where each $f_j : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is Lipschitz. Recasting the results above in this light, we have in the case when $m = 2$ and $n = 1$, that by Besicovitch's work if $E \subset \mathbb{R}^m$ such that $0 < \mathcal{H}^1(E) < \infty$, and for \mathcal{H}^1 -a.e. $x \in E$ the density of $\mu = \mathcal{H}^1 \llcorner E$ exists and is 1, then E is 1-rectifiable. By Federer's work the converse in any dimension is true, that is, if $E \subset \mathbb{R}^m$ is n -rectifiable then the density of $\mu = \mathcal{H}^n \llcorner E$ exists and is 1 for \mathcal{H}^n -a.e. $x \in E$. Two natural questions arise at this point: 1) does Besicovitch's result hold for any $n, m \in \mathbb{N}$ with $n < m$? 2) what happens if we replace

$\mu = \mathcal{H}^n \llcorner E$ by a general Radon measure μ ? Initially progress on these questions was slow.

In 1944, Morse and Randolph [52] proved when $m = 2$ that if μ is a Radon measure on \mathbb{R}^m for which the 1-density exists μ -a.e., then μ is 1-rectifiable. In 1950, Moore [51] showed that this result holds for any m . In 1961, Marstrand [48] showed that if $E \subset \mathbb{R}^3$ and the 2-density exists for $\mathcal{H}^2 \llcorner E$, \mathcal{H}^2 -a.e. $x \in \mathbb{R}^3$, then E is 2-rectifiable. In 1975, Mattila [49] proved that if $E \subset \mathbb{R}^m$ and the n -density exists for $\mathcal{H}^n \llcorner E$, \mathcal{H}^n -a.e. $x \in \mathbb{R}^m$, then E is n -rectifiable, completing the study of the problem for measures that were defined as the restriction of Hausdorff measure to a subset of Euclidean space. This still left open the case of a general Radon measure.

In 1987, in a true masterpiece, Preiss [56] showed that if μ is a Radon measure on \mathbb{R}^m for which the n -density exists for μ -a.e. $x \in \mathbb{R}^m$, then μ is n -rectifiable. Preiss introduced a number of new tools and ideas whose applications are still being unraveled and play a central role in the results to be discussed later in this article. The question of rectifiability of a measure carries information about the fine structure of its measure-theoretic support. Motivated by this perspective, Preiss introduced the notion of tangent measures, which play the role that derivatives do when analyzing the regularity of a function. They are obtained by a limiting process of rescaled multiples of the initial measure. Preiss's argument includes a number of major steps, some of which have given rise to very interesting questions. Roughly speaking, a blow up procedure shows that when the n -density of μ exists μ -a.e., then at μ -a.e. point all tangent measures are n -uniform. A measure ν is n -uniform if there is a constant $C > 0$ so that for $r > 0$ and x in the support $\Lambda = \text{spt } \nu = \{y \in \mathbb{R}^m : \nu(B(y, s)) > 0 \text{ for every } s > 0\}$ of ν , we have

$$\nu(B(x, r)) = Cr^n. \quad (4)$$

His argument now requires a detailed understanding of the structure and geometry of the support of n -uniform measures. By work of Kirchheim and Preiss [43], the support of an n -uniform measure ν is an analytic variety. Thus, using the fact that at μ -a.e. point, tangent measures to μ are tangent measures to μ , he showed that at μ -a.e. point there is an n -flat tangent measure; that is, a measure that is a multiple of the n -dimensional Hausdorff measure restricted to an n -plane. Then he showed that either an n -uniform measure ν is an n -flat measure or its support is very far away from any n -plane. Using a deep result about the "cones" of measures, he proves that necessarily at μ -a.e. point all tangent measures are n -flat. Then modulo showing that this implies that the measure-theoretic support of μ satisfies the hypothesis of the Marstrand-Mattila rectifiability criterion, one concludes that μ is n -rectifiable.

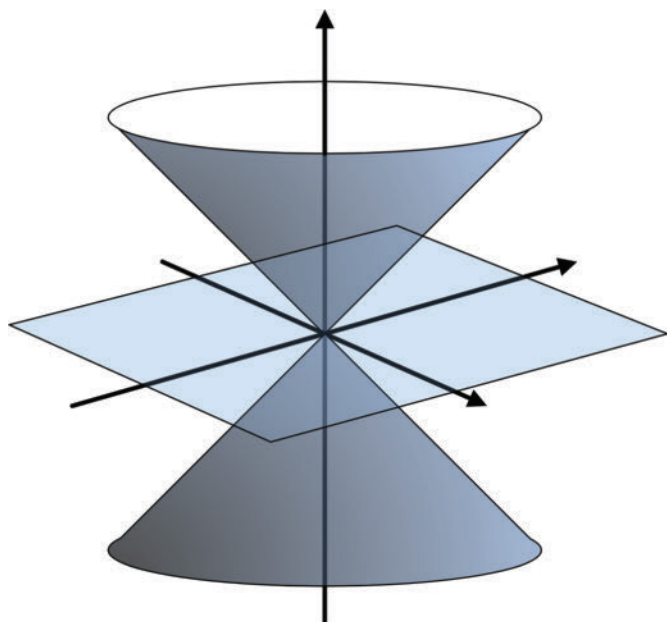


Figure 1. Kowalski-Preiss cone

A natural and easy-to-state question derived from this work is: what does the support of an n -uniform measure on \mathbb{R}^m really look like? Clearly the restriction of n -Hausdorff measure to an n -plane is n -uniform, but are there other examples? Work of Preiss [56] shows that if $n = 1, 2$ flat measures are the only examples of n -uniform measures. In 1987 Kowalski and Preiss [44] showed that if $m = n + 1$ and $n \geq 3$ then, modulo rotation and translation, Λ the support of an n -uniform measure ν is either:

- $\Lambda = \mathbb{R}^n \times \{0\}$

or

- $\Lambda = \{(x_1, x_2, x_3, x_4, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_4^2 = x_1^2 + x_2^2 + x_3^2\}$.

See figure above.

Thirty years later Nimer [55] produced the first examples of n -uniform measures in higher co-dimensions. His argument has an important combinatorial component and uses Archimedes' theorem, namely that, in \mathbb{R}^3 the surface measure of the intersection of the unit sphere with a ball of small radius r and centered on the sphere is exactly πr^2 . He classifies up to isometry all conical 3-uniform measures in \mathbb{R}^5 and produces families of examples in any co-dimension.

Much time has elapsed between the initial work of Preiss and collaborators and the next set of examples. This illustrates a trend in this area. Indeed for years Preiss's work was perceived as somewhat impenetrable. In the early-to-mid 2000s several successful attempts were made to understand and apply some segments of the paper (see [25, 39, 46, 57]). De Lellis [28] produced a more comprehensive version of a special case of the argument, which has made this work more accessible.

Harmonic Measure

To illustrate how ideas from one field can have a profound impact in another, we will focus on some of the recent applications of Preiss's work to harmonic analysis and potential theory. The harmonic measure is a canonical measure associated to the Laplacian (see definition below). It plays a fundamental role in potential theory, constitutes the main building block for the solutions of the classical Dirichlet problem, and in non-smooth domains is the object that allows us to describe boundary regularity of the solutions to Laplace's equation. We recall some of the background. Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded domain, let f be a continuous function on the boundary of Ω , i.e. $f \in C(\partial\Omega)$. The classical Dirichlet problem asks whether there exists a function $u \in C(\bar{\Omega}) \cap W^{1,2}(\Omega)$ such that

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases} \quad (5)$$

Here $u \in W^{1,2}(\Omega)$ means that u and its weak derivatives are in $L^2(\Omega)$ and $\Delta u = 0$ is interpreted in the weak sense; that is, for any $\zeta \in C_c^1(\Omega)$,

$$\int \langle \nabla u, \nabla \zeta \rangle = 0.$$

The questions here are whether a solution u of (5) exists, if so how regular it is, and whether there is a formula in terms of f to describe it. We say that Ω is *regular* if for all $f \in C(\partial\Omega)$, any solution u of (5) is in $C(\bar{\Omega}) \cap W^{1,2}(\Omega)$. In 1923 Wiener [60] provided a remarkable characterization of regular domains using capacity. If Ω is regular, then for $x \in \Omega$ and $f \in C(\partial\Omega)$ if $u \in C(\bar{\Omega})$ is the solution to (5), by the Maximum Principle $|u(x)| \leq \max_{\partial\Omega} |f|$. Thus for $x \in \Omega$, $T_x : C(\partial\Omega) \rightarrow \mathbb{R}$ defined by $T_x(f) = u(x)$ is a bounded linear operator, with $\|T_x\| \leq 1$. Moreover $T_x(1) = 1$. Hence, by the Riesz Representation Theorem, there is a probability measure ω^x , the *harmonic measure* with pole at x , satisfying

$$u(x) = \int_{\partial\Omega} f(q) d\omega^x(q). \quad (6)$$

If Ω is regular and connected, the Harnack Principle implies that for $x, y \in \Omega$, ω^x and ω^y are mutually absolutely continuous. Thus we will often drop the pole dependence and simply refer to the harmonic measure ω rather than ω^x .

The question of whether the behavior of the harmonic measure on a given domain yields information about the structure of the boundary of the domain has attracted considerable interest over the last century, with a period of intense activity over the last two decades. The initial results in \mathbb{R}^2 are very satisfactory. For a simply-connected

domain $\Omega \subset \mathbb{R}^2$, bounded by a Jordan curve, the boundary is a disjoint union, with the following properties:

$$\partial\Omega = G \cup S \cup N \quad (7)$$

1. In G , ω , and \mathcal{H}^1 are mutually absolutely continuous, which we denote by $\omega \ll \mathcal{H}^1 \ll \omega$,
2. Every point of G is the vertex of a cone in Ω . Moreover if C denotes the set of “cone points” of $\partial\Omega$, then $\mathcal{H}^1(C \setminus G) = 0$ and $\omega(C \setminus G) = 0$.
3. $\omega(N) = 0$ and $\mathcal{H}^1(S) = 0$.
4. S consists (ω a.e.) of “twist points” (see [31] for the definition).
5. For ω a.e. $q \in G$, the 1-density of ω exists and $\theta^1(\omega, q) \in (0, \infty)$ (see (3)).
6. At ω a.e. point $q \in S$ we have

$$\begin{aligned} \limsup_{r \rightarrow 0} \frac{\omega(B(q, r))}{r} &= +\infty, \\ \liminf_{r \rightarrow 0} \frac{\omega(B(q, r))}{r} &= 0. \end{aligned}$$

These results are a combination of work of Makarov, McMillan, Pommerenke, and Choi. See [31] for the precise references.

Recall that the Hausdorff dimension of ω (denoted by $\mathcal{H} - \dim \omega$) is defined by

$$\begin{aligned} \mathcal{H} - \dim \omega &= \inf \{k : \text{there exists } E \subset \partial\Omega \quad (8) \\ &\quad \text{with } \mathcal{H}^k(E) = 0 \text{ and} \\ &\quad \omega(E \cap K) = \omega(\partial\Omega \cap K) \\ &\quad \text{for all compact sets } K \subset \mathbb{R}^{n+1}\} \end{aligned}$$

Important work of Makarov [47] shows that for simply connected domains in \mathbb{R}^2 , $\mathcal{H} - \dim \omega = 1$, establishing Oksendal’s conjecture (i.e. for what type of domains in \mathbb{R}^{n+1} is $\mathcal{H} - \dim \omega = n$?) in dimension 2. Carleson [22], and Jones and Wolff [38] proved that, in general, for domains in \mathbb{R}^2 with a well defined harmonic measure ω , $\mathcal{H} - \dim \omega \leq 1$. Bourgain showed that there exists $\epsilon(n) > 0$ such that for domains in \mathbb{R}^{n+1} with a well defined harmonic measure ω , $\mathcal{H} - \dim \omega \leq n + 1 - \epsilon(n)$, see [21]. Finding the optimal bound for this Hausdorff dimension is an important open question in the area.

T. Wolff [61] showed, by a deep example, that, for $n \geq 2$, Oksendal’s conjecture (that $\mathcal{H} - \dim \omega = n$) fails. He constructed what are known as “Wolff snowflakes,” domains in \mathbb{R}^3 , for which $\mathcal{H} - \dim \omega > 2$ and others for which $\mathcal{H} - \dim \omega < 2$. In Wolff’s construction, the domains have a certain weak regularity property. They are non-tangentially accessible domains (NTA) in the sense of [37]. In fact, they are 2-sided NTA domains (i.e. $\Omega^+ = \Omega$ and $\Omega^- = \text{int}(\Omega^c)$ are both NTA) which plays an important role in his estimates. NTA domains are open, connected, and Wiener regular in a quantitative way. In [45],

Lewis, Verchota, and Vogel reexamined Wolff’s construction and were able to produce “Wolff snowflakes” in \mathbb{R}^{n+1} , $n \geq 2$ for which either $\mathcal{H} - \dim \omega^\pm < n$ or $\mathcal{H} - \dim \omega^\pm > n$, where ω^\pm denote the harmonic measure of Ω^\pm . They also observed, as a consequence of the monotonicity formula in [2], that if $\omega^+ \ll \omega^- \ll \omega^+$, then $\mathcal{H} - \dim \omega^\pm \geq n$.

Returning to the case of $n = 1$, when $\Omega \subset \mathbb{R}^2$ is again simply connected, and bounded by a Jordan curve, Bishop, Carleson, Garnett, and Jones [19] showed that if $E \subset \partial\Omega$, $\omega^\pm(E) > 0$, then ω^+ and ω^- are mutually singular (i.e. $\omega^+ \perp \omega^-$) on E if and only if $\mathcal{H}^1(Tn(\partial\Omega) \cap E) = 0$, where $Tn(\partial\Omega) \subset \partial\Omega$ is the set of points in $\partial\Omega$ where $\partial\Omega$ has a unique tangent line. Let $E \subset \partial\Omega$ be such that $\omega^+ \ll \omega^- \ll \omega^+$ on E and $\omega^\pm(E) > 0$. Then, because of [19], modulo sets of ω^\pm measure 0, $E \subset Tn(\partial\Omega)$. Using Beurling’s inequality, i.e., the fact that for $q \in \partial\Omega$ and $r > 0$, $\omega^+(B(q, r))\omega^-(B(q, r)) \leq Cr^2$, and the characterization above where $\partial\Omega = G^\pm \cup S^\pm \cup N^\pm$ (see (7)), we conclude that $\omega^+ \ll \mathcal{H}^1 \ll \omega^- \ll \omega^+$ on E . Thus, the set of mutual absolute continuity of ω^- , ω^+ is a subset of $G^+ \cap G^-$ and hence of Hausdorff dimension 1.

In [18], motivated by this last result, Bishop asked if in the case of \mathbb{R}^{n+1} , $n \geq 2$, the fact that ω^-, ω^+ are mutually absolutely continuous on a set $E \subset \partial\Omega$, with $\omega^\pm(E) > 0$, implies that ω^\pm are mutually absolutely continuous with respect to \mathcal{H}^n on E and hence $\dim_{\mathcal{H}}(E) = n$, where $\dim_{\mathcal{H}}$ denotes the Hausdorff dimension of a set.

Two Phase Case

While in [40] and [41] we had already used some of the properties of the tangent measures, Bishop’s question plus the desire to understand the Wolff snowflakes better led us to dig deep into Preiss’s work [56]. There we found the necessary tools to start tackling the problem of describing the boundary in terms of ω^\pm . In [39, Kenig, Preiss and the author] prove the following result:

Theorem 1 ([39]). *For $n \geq 3$, if $\Omega \subset \mathbb{R}^{n+1}$ is a 2-sided NTA domain, then*

$$\partial\Omega = \Gamma \cup S \cup N, \quad (9)$$

1. $\omega^+ \perp \omega^-$ on S and $\omega^\pm(N) = 0$.
2. On Γ , $\omega^+ \ll \omega^- \ll \omega^+$, $\dim_{\mathcal{H}} \Gamma \leq n$.
3. If $\omega^\pm(\Gamma) > 0$, $\dim_{\mathcal{H}} \Gamma = n$.
4. If $\mathcal{H}^n \ll \partial\Omega$ is a Radon measure then Γ is n -rectifiable, and $\omega^- \ll \mathcal{H}^n \ll \omega^+ \ll \omega^-$ on Γ .

As a consequence there can be no Wolff snowflake for which ω^+, ω^- are mutually absolutely continuous. Theorem 1 answered Bishop’s question under the assumption that $\mathcal{H}^n \ll \partial\Omega$ is a Radon measure. The general case was left open, and it was clear that a new idea was needed to deal with the main obstacle, namely the set of points for

which the n -density of the harmonic measure is 0. A noteworthy issue in this branch of GMT is that difficulties often arise from either a measure that is not locally finite or a measure whose appropriate density is zero.

The proof of Theorem 1 uses tools from the theory of non-tangentially accessible domains (NTA) introduced by Jerison and Kenig [37], the monotonicity formula of Alt, Caffarelli, and Friedman [2], the theory of tangent measures introduced by Preiss [56], and the blow up techniques for harmonic measures at infinity for unbounded NTA domains due to Kenig and Toro [40, 41]. For additional results along these lines see [11–14, 29].

We describe the main steps to emphasize the similarities with the train of thought present in Preiss's work. To accomplish our objective, we use the blow-up analysis developed in [41]. At ω^\pm a.e. point on the set where ω^+ and ω^- are mutually absolutely continuous, the tangent measures to ω^\pm (in the sense of [50], [56]) are harmonic measures associated to the Laplacian on domains where a harmonic polynomial is either positive or negative. The resulting harmonic measure is supported on the zero set of this harmonic polynomial. Using the fact that for almost every point a tangent measure to a tangent measure is a tangent measure (see [50]) and the fact that the zero set of a harmonic polynomial in \mathbb{R}^{n+1} is smooth except for a set of Hausdorff dimension $n - 1$ (see [33]), one shows that at ω^\pm a.e. point on this set, n -flat measures always arise as tangent measures to ω^\pm . They correspond to linear harmonic polynomials. We then show and this is the crucial step, that if one tangent measure is flat on the set of mutual absolute continuity, then all tangent measures are flat. To accomplish this we use a connectivity argument similar to the one from [56]. The key point is that if a tangent measure is not flat, being the harmonic measure supported on the zero set of a harmonic polynomial of degree higher than 1, its tangent measure at infinity is far from flat, and a connectivity argument, as in [56], gives a contradiction. Modulo a set of ω^\pm measure 0, Γ as in (10), is the set of mutual absolute continuity for which one (and hence all) tangent measures are n -flat. An easy argument then shows that $\dim_{\mathcal{H}} \Gamma \leq n$. To conclude that if $\omega^\pm(\Gamma) > 0$, $\dim_{\mathcal{H}} \Gamma = n$, one uses the Alt-Caffarelli-Friedman monotonicity formula of [2] as in [45]. This yields a version of Beurling's inequality in higher dimensions. If $\sigma = \mathcal{H}^n \llcorner \partial\Omega$, the surface measure to the boundary, is a Radon measure, we show that the n -density of σ is 1 a.e., which by Preiss's theorem ensures that Γ is n -rectifiable.

In a remarkable paper, Azzam, Mourougolou, and Tolsa [8] answer Bishop's question completely. Although their result holds in greater generality, we state it here in the context of the discussion above for simplicity.

Theorem 2 ([8]). *For $n \geq 2$, if $\Omega \subset \mathbb{R}^{n+1}$ is a 2-sided NTA domain, then*

$$\partial\Omega = G \cup S \cup N', \quad (10)$$

1. $\omega^+ \perp \omega^-$ on S and $\omega^\pm(N') = 0$.
2. G is n -rectifiable and $\omega^- \ll \mathcal{H}^n \ll \omega^+ \ll \omega^-$.

The main innovation in [8] is the introduction of a new set of ideas involving the n -dimensional Riesz transform. In particular they use a result by Girela-Sarrión and Tolsa [32] concerning the connection between Riesz transforms and quantitative rectifiability for general Radon measures. This allows them to deal with the set of points for which the n -density of the harmonic measure is 0. The connection between the Riesz transform and harmonic measure stems from the fact that the Riesz kernel is the gradient of the Newtonian potential. The relationship between the Riesz transform and rectifiability has been an important component in the development of quantitative geometric measure theory, a field initiated by David and Semmes (see [26], [27]) in the early 1990s, and embraced by a large community. Quantitative GMT has developed into a vibrant area in which several important milestones have been accomplished in recent years (e.g. the solution of the David-Semmes conjecture by Nazarov, Tolsa, and Volberg [53, 54]). In a subsequent paper, Azzam, Mourougolou, Tolsa, and Volberg significantly relax the hypothesis on the domains for which Theorem 2 holds [10].

We note that the narrative started with a question from potential analysis. The initial results were the product of a successful approach taking a GMT point of view. Once this work was in place, questions arose that required deep results in harmonic analysis and quantitative GMT to be tackled. The final outcome lies in the interface of potential theory and geometric measure theory. These results illustrate how the synergy between very distinct areas can produce truly unique and unexpected results.

One Phase Case

In the Harmonic Measure section, we started by asking whether the behavior of the harmonic measure of a domain yields information about the structure of its boundary (one phase case). The discussion very quickly turned to the situation where we consider the harmonic measures of a set and its complement (two phase case). The rationale was that the two phase case was more clearly related to GMT. We now return to the one phase case, where both the quantitative and qualitative questions have sparked an incredible amount of interest, generating lots of activity that has culminated in truly optimal results.

In 1916 F. and M. Riesz proved that for a simply connected domain in the complex plane with a rectifiable boundary, harmonic measure is absolutely continuous with respect to arc length measure on the boundary [58].

Bishop and Jones [20] have shown that in this type of domain, if only a portion of the boundary is rectifiable, then harmonic measure is absolutely continuous with respect to arc length on that portion. They also showed that the result of [58] may fail in the absence of some topological hypothesis (e.g., simple connectedness). Examples constructed in [62] and [63] show that, in higher dimensions, some topological restrictions, even stronger than those needed in the planar case, are required for the absolute continuity of ω with respect to surface measure to the boundary.

Higher dimensional analogues of this question have played a central role in the development of the study of partial differential equations in non-smooth domains. In 1982 Dahlberg [23] showed the harmonic measure of a Lipschitz domain and the surface measure to its boundary are mutually absolutely continuous (in a quantitative scale-invariant way, namely $\omega \in A_\infty(\sigma)$). Similar results hold on chord arc domains (these are NTA domains for which the surface measure to the boundary is Ahlfors regular; that is, the surface measure of a ball centered on the boundary and of radius r grows like r^n) (see [24], [59]). The relationship between quantitative absolute continuity properties of harmonic measure with respect to surface measure and the regularity of the boundary (also expressed in quantitative terms) is now very well understood, see for example [3, 6, 7, 15, 34–36].

Here we only focus on the optimal qualitative result that provides a complete answer to Bishop's question. In a very interesting piece of work, Azzam, Hofmann, Martell, Mayboroda, Mouroglou, Tolsa, and Volberg show a converse to the results in [20, 58] in all dimensions. See [4, 5]

Theorem 3 ([4], [5]). *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open connected set and let ω be the harmonic measure in Ω . Let $E \subset \partial\Omega$ with Hausdorff measure $\mathcal{H}^n(E) < \infty$.*

- *If ω is absolutely continuous with respect to \mathcal{H}^n on E , then ω restricted to E is an n -rectifiable measure.*
- *If \mathcal{H}^n is absolutely continuous with respect to ω on E , then E is an n -rectifiable set.*

This theorem can be understood as a free boundary regularity problem for the harmonic measure (an initial example of this type of question appears in [42]) where the goal is to obtain regularity of the boundary. The authors appeal to the *magic* of the Riesz transform, which for a measure μ in \mathbb{R}^{n+1} is defined as follows:

$$\mathcal{R}\mu(x) = \int \frac{x - y}{|x - y|^{n+1}} d\mu(y). \quad (11)$$

They manage to exploit the absolute continuity hypothesis to obtain a series of estimates on the harmonic measure and the corresponding Green function that allow them to show that the Riesz transform is a bounded operator. Then they appeal the work of Nazarov, Tolsa, and Volberg [53,

54] where the authors prove the David-Semmes conjecture in co-dimension 1; that is, they show that the boundedness of the Riesz transform of a measure implies its rectifiability.

Note that the description above of the results in the area does not include a qualitative version of the F. and M. Riesz type result in higher dimensions. At this stage, works of [1, 9, 15] indicate that obtaining an optimal condition on a domain to ensure that rectifiability of the boundary implies absolute continuity of harmonic measure with respect to the surface measure is challenging.

This field has been evolving in several interesting new directions. They all fit under the umbrella of understanding the structure of the support of a measure associated to a differential operator in a canonical way. One direction concerns understanding questions similar to those discussed in the three previous sections for the elliptic measure of a uniformly elliptic second order divergence form operator. Another one looks at the problems analogous to those appearing in the Harmonic Measure and One Phase Case sections for the elliptic measure corresponding to a degenerate elliptic operator on a domain in \mathbb{R}^{n+1} whose boundary has dimension strictly less than n . The unifying trait is the beautiful cross-pollination between geometric measure theory, potential theory, harmonic analysis, and partial differential equation. The expectation is that this synergy will continue to uncover unsuspected connections, leading to the development of the field in ways that are only possible thanks to the contributions from a diverse group of analysts.

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Credits

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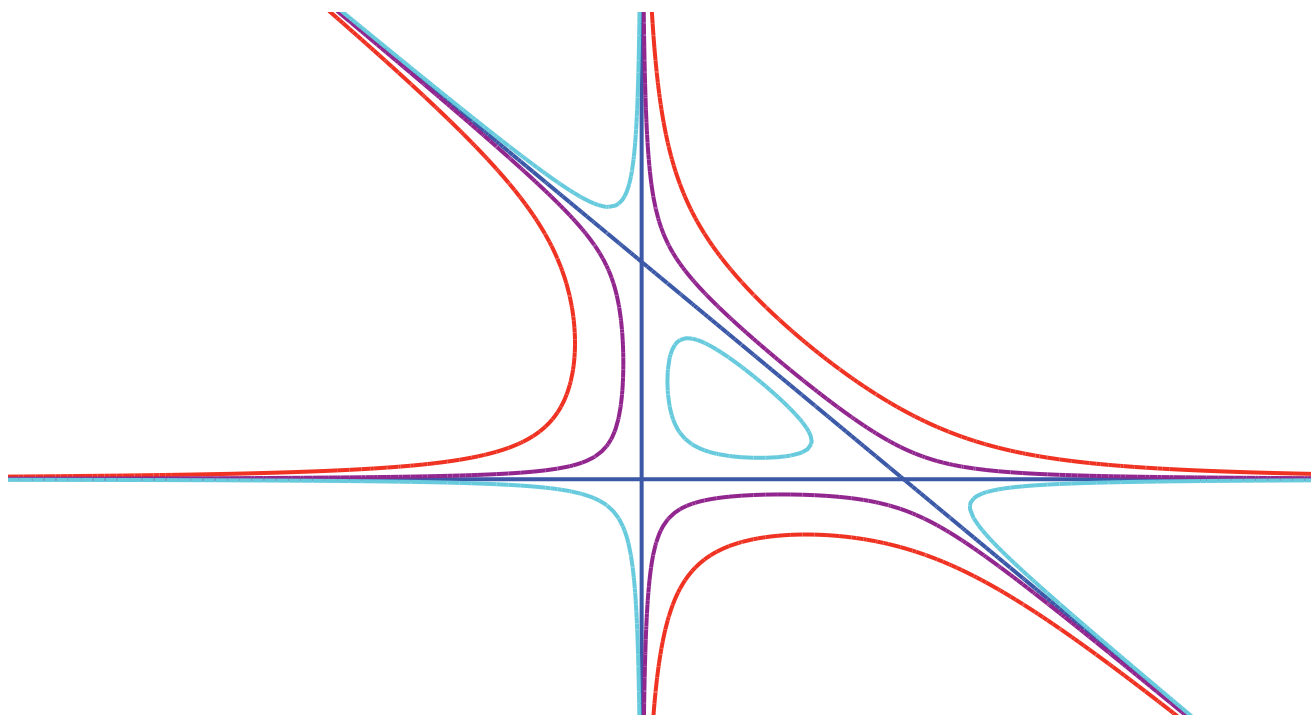
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THANK YOU

The Mirror's Magic Sights: An Update on Mirror Symmetry



Timothy Perutz

But in her web she still delights
To weave the mirror's magic sights
Tennyson, *The Lady of Shalott*

The Mirror Symmetry Mystery

Origins. The 1980s and '90s saw an astonishing entanglement of research in geometry and mathematical physics. String theorists, developing their candidate for a quantum theory incorporating gravity, not only drew on state-of-the-art mathematics, but introduced mathematical ideas of great power and prescience: none more so than mirror symmetry.

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In a 1989 paper [13], Lerche, Vafa, and Warner studied the algebraic structure of 2-dimensional $N = 2$ supersymmetric conformal field theories (SCFT). I will not define a 2-dimensional $N = 2$ SCFT, but only note that it is a type of quantum field theory—as such, involving operators on Hilbert spaces—in which the operators are associated with Riemann surfaces. The authors knew that a Calabi-Yau manifold gives rise to an $N = 2$ SCFT, the Riemann surfaces being traced out by the motions and interactions of closed strings, i.e., loops, inside the manifold. In such a theory, they wrote,

there are four types of rings arising from the various combinations of chiral and anti-chiral, and left and right. We will denote these rings by (a, c) , (a, a) , (c, a) , (c, c) There is a non-trivial relationship between (c, c) and (a, c) For superconformal models coming from compactification on Calabi-Yau manifolds, the (c, c) ring becomes isomorphic to the structure of the cohomology ring of the manifold in the large radius limit.

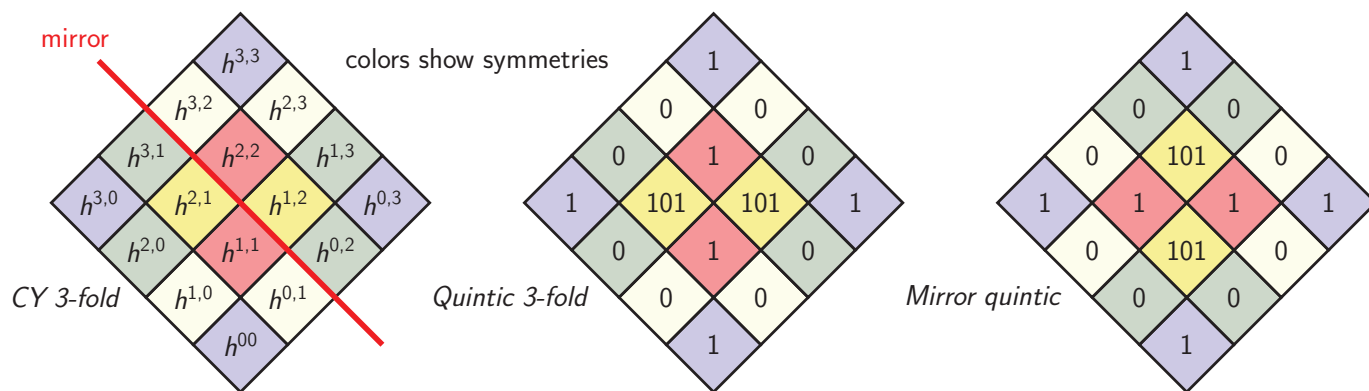


Figure 1. Mirrored Hodge diamonds for mirror pairs of CY 3-folds.

... One possibility might be that [the Poincaré series for the (a, c) ring] is the Poincaré series for (a deformation of) the cohomology ring of another manifold. If so, there must be another manifold \tilde{M} for which the Betti numbers satisfy $b_{p,q}^M = b_{d-p,q}^{\tilde{M}}$.

The possibility tentatively put forward in this passage¹ was soon enunciated with greater precision and certitude, and named *mirror symmetry* [9, 19].

Basic explanations: A *Kähler manifold* (M, ω) is a complex manifold M , together with a *Kähler form* ω : a \mathbb{C}^∞ real 2-form—i.e., a skew-symmetric bilinear form on the tangent bundle TM —which is closed ($d\omega = 0$), invariant under the complex structure, and positive on complex lines in TM . Being closed and non-degenerate, a Kähler form is an example of a *symplectic form*. Complex projective space \mathbb{P}^N has a unique Kähler form (up to a positive scalar factor) that is invariant under the transitive action of the projective unitary group of \mathbb{C}^{N+1} ; an embedding of M into \mathbb{P}^N determines a Kähler form on M by restriction.

A *Calabi–Yau (CY) manifold* (M, ω, Ω) is a compact Kähler manifold (M, ω) endowed with a *holomorphic volume form* Ω . In local holomorphic coordinates (z_1, \dots, z_d) , $\Omega = f(z) dz_1 \wedge \dots \wedge dz_d$, where f is holomorphic and nowhere-vanishing. Examples:

- When $d = 1$, the only CY manifolds are elliptic curves $\mathbb{C}/\text{lattice}$; one can take $\omega = i dz \wedge d\bar{z}$ and $\Omega = dz$.
- CY hypersurfaces $M \subset \mathbb{P}^{d+1}$, cut out from projective space by a homogeneous polynomial of degree $d + 2$. Elliptic curves arise as cubics in \mathbb{P}^2 .
- Complex tori $\mathbb{C}^d/\text{lattice}$.

The ‘Betti numbers’ $b_{p,q}^M$ in the quotation are really the Hodge numbers, $b_{p,q}^M = h^{p,q}(M) := \dim_{\mathbb{C}} H^q(M, \Omega_M^p)$: $h^{p,q}$ is the vector-space dimension of the q th cohomology of the sheaf Ω_M^p of holomorphic p -forms. The Betti number $b^i = \dim_{\mathbb{C}} H^i(M; \mathbb{C})$, the dimension of the i th singular cohomology, is the sum of the $h^{p,q}$ where $p + q =$

i . The ‘Poincaré series’ $P(t)$ of a graded ring is the generating function for the dimensions of its homogeneous parts, so for $H^*(M; \mathbb{C})$ the Poincaré series is the polynomial $P(t) = \sum b^i(M)t^i$.

The term ‘mirror symmetry’ refers to a literal mirroring of Hodge diamonds expressed by the relation $h^{p,q}(M) = h^{d-p,q}(\tilde{M})$ —the Hodge diamond is the conventional visualization of the array of Hodge numbers $h^{p,q}$ (Figure 1). But in retrospect, it seems mistaken to view that as a primary manifestation of mirror symmetry. I prefer to think of the term as a metaphor for the reciprocal relationship of M to \tilde{M} —the mirror of the mirror is the original.

The $N = 2$ SCFT which, string theorists argue, can be associated with a CY manifold M is a type of *sigma model*: it is based on maps $\Sigma \rightarrow M$ where Σ is a Riemann surface. There are two *topological twists* of the sigma model which are 2-dimensional *topological field theories*, called the A-model and the B-model. Formally they are on an equal footing, but their physical observables have quite different geometrical meanings, relating to holomorphic maps from Riemann surfaces to the CY in the A-model, and to period integrals of differential forms in the B-model. A statement of mirror symmetry, arising from string theory but congenial to mathematicians, is the following:

Mirror symmetry determines an isomorphism of 2-dimensional topological field theories between the A-model of M and the B-model of \tilde{M} , and vice versa.²

Readers familiar with topological field theory will know that the state space attached to the circle is a ring: these are the rings that appear in the quoted passage from [13].

Today, there is an ocean of literature on holomorphic maps from Riemann surfaces to Kähler, or more generally, symplectic manifolds, including Gromov–Witten invariants (the ‘closed string’ part of the story). The theory of Fukaya categories (the ‘open string’ part) is proceeding rapidly with respect to foundations and the development of tools. Laying down complete mathematical foundations

¹L. Dixon reportedly also put forward this idea.

²‘Topological field theory should be understood in an ‘extended’ or ‘open-closed’ sense; cf. [5].

for the A-model topological field theory appears to be within reach. We also have a good formulation of the parts of the B-model where Σ has genus 0, incorporating derived categories and variations of Hodge structure, and an emerging understanding of the higher genus part [5, 14].

Counting curves. It seems that the germinal ideas of mirror symmetry elicited little more than skeptical shrugs from geometers. But in 1991, Candelas, de la Ossa, Greene, and Parkes [4] made a prediction which geometers could not ignore, for it seemed magical yet the evidence was compelling.

Taking the example of a quintic 3-fold $X \subset \mathbb{P}^4$, and a mirror consisting of a certain holomorphic 1-parameter family \check{X}_q of CY 3-folds (the parameter q varies in a punctured disc $\Delta^* = \{q \in \mathbb{C} : 0 < |q| < 1\}$) they studied a facet of SCFT visible in the topologically twisted A- and B-models and expected to match under mirror symmetry: the *3-point Yukawa couplings*. For the A-model of X , the Yukawa coupling was identified as a generating function

$$Y_A(X) = 5 + \sum_{d=1}^{\infty} n_d d^3 q^d (1 + q^d + q^{2d} + \dots), \quad (1)$$

where n_d counts rational curves—the images of holomorphic maps $\mathbb{P}^1 \rightarrow X$ —meeting a hyperplane $H \subset \mathbb{P}^4$ with total multiplicity d . On the B-side, the Yukawa couplings are period integrals for the family $\{\check{X}_q\}$. Precisely, $Y_B(\check{X})$ is the Laurent series expansion of the holomorphic function on Δ^*

$$q \mapsto \int_{\check{X}_q} \check{\Omega} \wedge \left(q \frac{d}{dq} \right)^3 \check{\Omega},$$

where $\check{\Omega}$ is a holomorphic 3-form on the total space of the family, defining a volume form $\check{\Omega}_q$ on each fiber \check{X}_q ; it has to be correctly normalized as a function of q . Candelas *et al.* computed that

$$Y_B(\check{X}) = \frac{5}{(1 + 5^5 x)} \frac{1}{y(x)^2} \left(\frac{q dx}{x dq} \right)^3,$$

where

$$y(x) = \sum_{n \geq 0} \frac{(5n)!}{(n!)^5} (-1)^n x^n, \quad x(q) = -q + 770q^2 + \dots$$

The crucial change of coordinates $x = x(q)$, which they computed to all orders, is called the *mirror map*. Their prediction, then, was that

$$Y_A(X) = Y_B(\check{X}). \quad (2)$$

They wrote:

It is gratifying that [assuming (2)] we find that $n_1 = 2875$ which is indeed the number of lines (rational curves of degree one) and $n_2 = 609250$ which is known to be the number of conics (rational curves of degree 2).

Mathematicians soon proposed a precise definition for the coefficients N_d of the series

$$Y_A(X) = \sum N_d q^d$$

(so $N_0 = 5$, $N_1 = n_1$, $N_2 = 8n_2 + n_1$, etc.).

It is rooted in Gromov's notion of pseudo-holomorphic curves in symplectic manifolds. One defines N_d as a *genus-zero Gromov–Witten invariant*, a homological ‘count’ of holomorphic maps $u: \mathbb{P}^1 \rightarrow X$ of degree d , mapping three specified points $z_j \in \mathbb{P}^1$ ($j = 0, 1, 2$) to $H_j \cap X$, where $H_j \subset \mathbb{P}^4$ is a specified hyperplane (Figure 2).

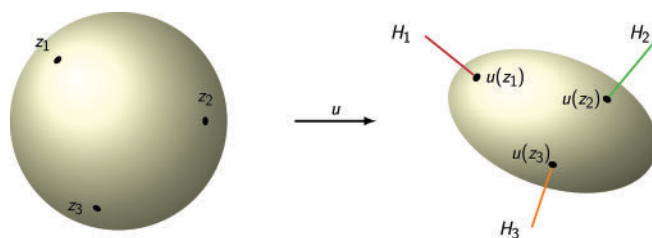


Figure 2. The A-side 3-point Yukawa coupling is a GW invariant enumerating holomorphic maps u .

GW invariants do not ultimately depend on the complex structure used on X used to define them, so any smooth quintic 3-fold will serve. Such maps u may factor through branched coverings $\mathbb{P}^1 \rightarrow \mathbb{P}^1$, and there is a qualified sense in which the n_d in (1) count the *images*, in X , of the maps u .

Principles. The intense activity inspired by the work of Candelas *et al.* made certain principles clear:

- The A-model of (X, ω, Ω) concerns the *symplectic* geometry of (X, ω) .

Gromov–Witten invariants—signed, weighted counts of holomorphic maps from Riemann surfaces into X invoke a complex structure on TX , but this should be viewed as an auxiliary choice not affecting the outcome.

- The mirror to a CY manifold is not a single CY manifold, but a *family* of CY manifolds. The B-model concerns the complex analytic geometry of this family.³

The next principle is that one cannot expect mirror symmetry to arise from a single CY manifold X , nor from an arbitrary family. Rather,

- X has a mirror when it undergoes a *maximal degeneration* to a singular variety, such as the degeneration of an elliptic curve to three projective lines (a degenerate cubic, Figure 3).⁴

³And, when these CY manifolds are projective varieties, their complex analytic geometry is interpretable as algebraic geometry.

⁴A maximal degeneration, parametrized by a small disc in \mathbb{C} , is one with maximally unipotent monodromy.

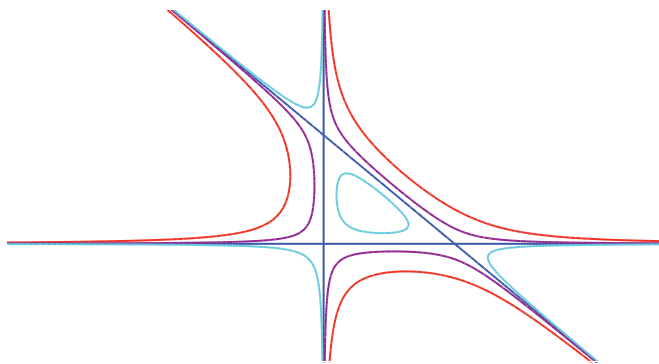


Figure 3. Plane cubic curves $xy(x + y - 1) = \epsilon$ degenerating to three lines as $\epsilon \rightarrow 0$.

Finally, there is Kontsevich's eagle-eyed conjecture from 1994 [11], today called *homological mirror symmetry* (HMS), connecting Lagrangian submanifolds of X to coherent sheaves on \check{X} . There are 'open string' topological field theories, governed by categorical structures called A_∞ -categories. In the A-model, one has the *Fukaya A_∞ -category* $\mathcal{F}(X, \omega)$ of the symplectic manifold (X, ω) —its objects are Lagrangian submanifolds of X —and in the B-model, the bounded *derived category* $D(\check{X})$, whose objects are those complexes of sheaves \mathcal{E}^\bullet of $\mathcal{O}_{\check{X}}$ -modules whose cohomology sheaves $\mathcal{H}^k(\mathcal{E})$ are coherent and of bounded degree k .⁵

We pause to define two of the terms:

Lagrangian submanifolds: A subspace Λ of a vector space V with a symplectic pairing ω_V is called *Lagrangian* if, for each $v \in \Lambda$, the linear form $\omega_V(v, \cdot)$ vanishes precisely on Λ ; this implies $\dim V = 2 \dim \Lambda$. A submanifold $L \subset M$ in a symplectic manifold (M, ω) is one whose tangent spaces $T_x L$ are Lagrangian in $T_x M$.

Coherence of sheaves: In algebraic geometry, and similarly in the rigid analytic geometry we shall discuss later, an algebraic variety Z comes with a sheaf of rings \mathcal{O}_Z , the *structure sheaf*, assigning a commutative ring $\mathcal{O}_Z(U)$ to each open set $U \subset Z$. A *sheaf of \mathcal{O}_Z -modules* \mathcal{E} assigns an $\mathcal{O}_Z(U)$ -module $\mathcal{E}(U)$ to each open U . Assuming for simplicity that $\mathcal{O}_Z(U)$ is a Noetherian ring for small neighborhoods U of an arbitrary point $z \in Z$, we say \mathcal{E} is *coherent* if each point of Z has an open neighborhood U such that (i) the $\mathcal{O}_Z(U)$ -module $\mathcal{E}(U)$ is finitely generated; and (ii), for all open sets $V \subset U$, the map $\mathcal{O}_Z(V) \otimes_{\mathcal{O}_Z(U)} \mathcal{E}(U) \rightarrow \mathcal{E}(V)$, $f \otimes s \mapsto f \cdot s|_V$, is an isomorphism.

- HMS: There is a functor $\mathcal{F}(X, \omega) \rightarrow D(\check{X})$ —mapping Lagrangian submanifolds of X to coherent complexes of sheaves on \check{X} —which is, in a certain sense, a categorical equivalence.⁶

⁵The derived category should here be treated not as a triangulated category, but its enhancement to a differential-graded (hence A_∞) category.

⁶Namely, it induces a quasi-equivalence of the associated A_∞ -categories of right modules. It may appear that HMS is incompatible with the notion that the mirror is a family. When HMS is formulated more precisely, this apparent disconnect proves illusory.

Kontsevich foresaw that HMS should be an *organizing principle*; that it should imply the isomorphism of topological field theories $A(X)$ and $B(\check{X})$, and thereby enumerative statements such as the prediction (2).

Verification, explanation. Some of mirror symmetry's predictions were soon verified. Candidate mirror partners were found for many CY manifolds. The Yukawa couplings $Y_A(X)$ were computed for a class of CY manifolds X including the quintic 3-fold [7] by showing that they satisfy the same differential equations as their B-side counterparts $Y_B(\check{X})$. Such work bore witness to the mirror symmetry phenomenon, but did not explain it.

Explanations gradually emerged [5, 12, 17]. The Gross–Siebert program [10] is a systematic and sophisticated construction of mirror pairs, for which several of the predictions of mirror symmetry have been proven. HMS has recently become tractable as basic tools for working with Fukaya categories have been developed. We now know [8] that HMS is an indeed an organizing principle, implying statements such as (essentially) (2). We know that HMS is true for (on the A-side) the quintic 3-fold [16], and we have a prototype for a truly explanatory proof of HMS [1, 2].

The Key Questions

- How do we construct a mirror \check{X} to a CY manifold X ?
- How can the symplectic geometry of X be read as analytic geometry of \check{X} —or vice versa?
- Why is HMS true?
- Why is mirror symmetry involutory?—Why is X the mirror of its mirror \check{X} ?

The germ of the answer to (a) and (d) was proposed by Strominger–Yau–Zaslow (SYZ) in 1996 [17]. The point is to find a smooth, surjective map

$$f: X^{2n} \rightarrow Q^n$$

to a middle-dimensional base Q such that the subspace $\ker D_x f \subset T_x X$ is Lagrangian for all regular points x ; so the regular fibers are Lagrangian submanifolds of X .

The regular fibers $F_q := f^{-1}(q)$ are necessarily tori: each fiber F_q has the structure of an n -dimensional affine vector space A modulo the action of a lattice L in its vector space V of translations. One then obtains the mirror \check{X} by replacing the non-singular fibers of that family by the *dual tori* $\check{F}_q := H^1(F_q; \mathbb{R}/\mathbb{Z}) \cong V^*/L^*$, the quotients of the dual vector spaces by the dual lattices. Provided one can find a way to handle the singular fibers, one obtains in this way a space \check{X} and a map $\check{f}: \check{X} \rightarrow Q$ with fibers \check{F}_q .

A CY manifold X admits an 'optimal' pair (ω, Ω) , one for which Ω is covariantly constant with respect to the Kähler metric: this is a famous theorem of S.-T. Yau. A Lagrangian $L \subset X$ is called *special* with respect to a CY

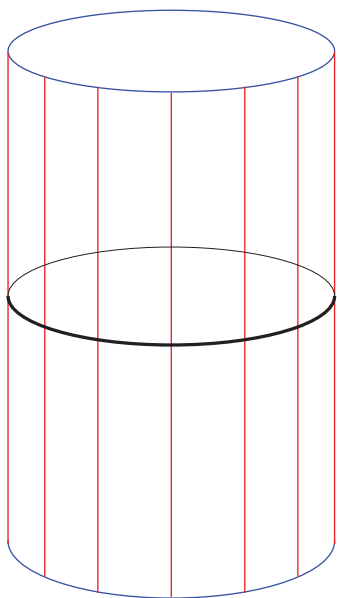


Figure 4. A schematic of T^*S^1 ; the top and bottom are glued together to form $(T^*S^1)/(T^*S^1)_{\mathbb{Z}}$. The red lines are the cotangent fibers, which become circles in the quotient.

metric if the imaginary part of Ω vanishes on $\Lambda^n TL$. SYZ proposed that the fibers F_q should be special Lagrangian. Orientable special Lagrangians admit *phase functions* ϕ : there is a non-vanishing section v_L of $\Lambda^n TL$ such that $\arg \Omega(v_L): X \rightarrow S^1$ admits a continuous logarithm ϕ , called a phase function. Lagrangians with phase functions will suffice for our needs in this article, and special Lagrangians will not reappear.

The basic model for the SYZ mirror—disregarding specialness—is as follows:

*Let Q be an integral affine n -manifold, that is, an n -manifold with an atlas of charts whose transition functions are affine transformations between open subsets of \mathbb{R}^n , of shape $x \mapsto Ax + b$ with $b \in \mathbb{R}^n$ and $A \in GL_n(\mathbb{Z})$. The cotangent bundle T^*Q is naturally a symplectic manifold, and there is a natural integer lattice $(T_q^*Q)_{\mathbb{Z}}$ in each cotangent space T_q^*Q . Let $X = (T^*Q)/(T_{\mathbb{Z}}^*Q)$ be the quotient of T^*Q by fiberwise-translations by lattice-vectors (Figure 4). Then X is symplectic, and is a bundle of Lagrangian n -tori over Q . The tangent bundle TQ contains a lattice dual to the one in T^*Q , and the quotient manifold $\check{X} = TQ/T_{\mathbb{Z}}Q$ is naturally complex. Then X and \check{X} are mirrors.*

SYZ's idea is at the heart of our current understanding of mirror symmetry, but the version I will outline in the section on rigid analytic mirrors is purely symplectic rather than Riemannian in nature, and, unlike the basic model just presented, it makes \check{X} a complex 1-parameter family.

Prototypes: Fourier Transforms, Classical and Geometric

Pontryagin duality. The most basic model for a duality such as mirror symmetry is the passage from a finite-dimensional vector space to its dual. A more instructive example is Pontryagin duality. The *characters* of a locally compact, abelian topological group G are the continuous homomorphisms $G \rightarrow \mathbb{T}$ to the circle-group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. In Fourier analysis, one takes $G = \mathbb{R}$ or \mathbb{T} , so that the respective characters are the maps $x \mapsto e^{2\pi i t x}$ for $t \in \mathbb{R}$ or $t \in \mathbb{Z}$. The set \hat{G} of characters is again a locally compact topological group, the Pontryagin dual of G . There is a 'universal character', which is the evaluation pairing $\chi: \hat{G} \times G \rightarrow \mathbb{T}$, $\chi(\xi, x) = \xi(x)$. A complex-valued function f on G has a Fourier transform \hat{f} , a function on \hat{G} : $\hat{f}(\xi) = \int_G \overline{\chi(\xi, x)} f(x) \mu_G$, where μ_G is a suitably-normalized left-invariant measure defined on the open sets.

The construction is a duality inasmuch as the evaluation map $\text{ev}: \hat{\hat{G}} \rightarrow G$ is an isomorphism, and $\hat{\hat{f}} = f \circ \text{inv} \circ \text{ev}$ (where $\text{inv}: G \rightarrow G$ is inversion).

Mirror symmetry, based on the SYZ idea, is roughly analogous to the formation of the Pontryagin dual group, with the Fourier transform a prototype for HMS.

Fourier–Mukai transforms for K3 surfaces. Fourier–Mukai transforms [15] bring us closer to mirror symmetry proper. Consider a simply connected, compact CY complex surface (S, ω, Ω) embedded in a projective space: a projective K3 surface.

Holomorphic vector bundles, or more generally, coherent sheaves \mathcal{F} , over S , have a discrete invariant, the Chern character, which is best packaged as the *Mukai vector* $v(\mathcal{F}) = v_0 + v_2 + v_4 \in H^0(S; \mathbb{Z}) \oplus H^2(S; \mathbb{Z}) \oplus H^4(S; \mathbb{Z})$.⁷ There is a moduli space $M_{S,v}$, parametrizing isomorphism classes of 'stable' coherent sheaves \mathcal{F} , with fixed Mukai vector v ; under assumptions that go unstated here, it is a compact complex manifold, projective, of dimension $2 + (v, v)$, where $(v, v) = \int_S (-2v_0v_4 + v_2^2)$. In the isotropic case $(v, v) = 0$, $M_{S,v}$ is again a surface, and is again CY.⁸ In the case that $v = 1 \in H^0(S; \mathbb{Z})$, one has $M_{S,v} = S$, the points of $M_{S,v}$ being merely the ideal sheaves for the points $s \in S$. But for other choices of Mukai vector, $M_{S,v}$ is a new K3 surface, and we can recover S as a moduli space of sheaves of $M_{S,v}$:

$$M_{S,v} \cong M_{M_{S,v}, v'}$$

for a certain Mukai vector v' for $M_{S,v}$. Thus a moduli space of geometric objects on a K3 surface gives rise to a new K3 surface, in a reciprocal relationship with the original.

⁷The Mukai vector is $\text{ch}(\mathcal{F}) \wedge (1 + \eta)$, where ch is the Chern character and η is the generator for $H^4(S; \mathbb{Z})$.

⁸The holomorphic volume form is the Serre duality pairing on $T_{\mathcal{F}} M_{S,v} = \text{Ext}_{\mathcal{O}_S}^1(\mathcal{F}, \mathcal{F})$.

There is a distinguished sheaf on $S \times M_{S,v}$, the *universal sheaf* $\mathcal{E}_{\text{univ}}$, whose restriction to the slice $S \times \{\mathcal{F}\} = S$ is isomorphic to \mathcal{F} .⁹ The Fourier–Mukai transform now inputs coherent sheaves \mathcal{E} on S , and outputs (complexes of) coherent sheaves on $M_{S,v}$:

$$\mathcal{F} \mapsto \hat{\mathcal{F}} = (\text{pr}_2)_*(\mathcal{E}_{\text{univ}} \otimes \text{pr}_1^* \mathcal{F}).$$

The Fourier–Mukai transform has a categorical manifestation, which is strongest when $(v, v) = 0$: it then defines an equivalence of derived categories of coherent sheaves on S and on $M_{S,v}$. This is the model for HMS.

Rigid Analytic Mirrors

The Novikov field and rigid analytic geometry. Fix a field F . The vector space $F^{\mathbb{R}}$ of all functions $\lambda: \mathbb{R} \rightarrow F$ has a subspace Λ_F of *Novikov series*: functions λ whose support is discrete and bounded below. One can multiply Novikov series, by convolution; thus we usually write Novikov series as formal series

$$\lambda = \sum_{j=1}^{\infty} \lambda_j q^{r_j}, \quad \lambda_j \in F, \\ r_j \in \mathbb{R}, \quad r_1 < r_2 < \dots, \quad r_j \rightarrow \infty.$$

(This series represents the function supported on $\{r_1, r_2, \dots\}$ given by $r_j \mapsto \lambda_j$.) In this way Λ_F becomes a field; the complex Novikov field $\Lambda_{\mathbb{C}}$ is algebraically closed.

The most important feature of Λ_F is that it comes with a *complete valuation*

$$\text{val}(\lambda) := \min \text{supp } \lambda. \quad (3)$$

A *valuation* on a field K is a map $\text{val}: K^{\times} \rightarrow \mathbb{R}$ (extended to K by setting $\text{val}(0) = +\infty$) such that $\text{val}(x + y) \geq \min(\text{val}(x), \text{val}(y))$ and $\text{val}(xy) = \text{val}(x) + \text{val}(y)$. There is an associated absolute value, $|x| = \exp(-\text{val}(x))$, and a metric $d(x, y) = |x - y|$. The valuation is *complete* if d -Cauchy sequences converge.

Rigid analytic geometry [18] is a variant of algebraic geometry, applicable over a complete valued field (K, val) : it builds in the internal geometry of the valuation.

In algebraic geometry over a field K —which, for brevity, we here assume algebraically closed—the basic objects are polynomial algebras $K[z_1, \dots, z_n]$. Maximal ideals therein correspond to points $x \in K^n$, as they take the form $(z_1 - x_1, \dots, z_n - x_n)$. In rigid analytic geometry, one instead studies the *Tate algebra* $T_n = K\langle z_1, \dots, z_n \rangle$, the algebra of power series $f(z) = \sum f_I z^I$, a sum over multi-indices $(i_1, \dots, i_n) \in (\mathbb{Z}_{\geq 0})^n$, with $f_I \in K$ and $z^I = \prod z_j^{i_j}$, such that $|f_I| \rightarrow 0$ as $\|I\| \rightarrow \infty$, where $\|I\| = \sum_j i_j$. If one has a point $x = (x_1, \dots, x_n)$ in the ‘unit polydisk’ $\mathbb{D}^n \subset K^n$, meaning $|x_j| \leq 1$ for all j , it defines a maximal ideal $\mathfrak{m}_x = (z_1 - x_1, \dots, z_n - x_n) \subset T_n$: there is

an isomorphism $T_n/\mathfrak{m}_x \rightarrow K$, given by $[f] \mapsto f(x) = \sum_{j \geq 0} \sum_{\|I\|=j} f_I x^I$ (convergent series). This construction accounts for all maximal ideals of T_n , and so one thinks of T_n geometrically as the polydisk \mathbb{D}^n .

A quotient $A = K[z_1, \dots, z_n]/(f_1, \dots, f_m)$ determines a topological space $X = \text{Spec } A$. The points of X are the prime ideals of A ; X has its Zariski topology, in which the maximal ideals are the closed points. One thinks of the closed points of X as the zero-set $f_1(x) = \dots = f_m(x) = 0$ inside K^n . There is a K -algebra of ‘functions’ \mathcal{O}_X on X , the maps $x \mapsto a(x) \in A/\mathfrak{m}_x$ where $a \in A$ and $x \in X$ labels a maximal ideal \mathfrak{m}_x . But actually, $\mathcal{O}_X \cong A$.

Likewise, a quotient $A = T_n/(f_1, \dots, f_m)$ determines a space $X = \text{Sp } A$ of *maximal* ideals, called an *affinoid space*. As before, it determines A as its ring of functions \mathcal{O}_X .

Certain subsets $U \subset X$ inside an affinoid space $X = \text{Sp } A$ are called *affinoid domains*. Take a (suitable) norm $\|\cdot\|$ on A , and the induced norms $\|\cdot\|_x$ on its quotients A/\mathfrak{m}_x : $\|a\|_x = \inf\{\|b\| : b - a \in \mathfrak{m}_x\}$. Then, for $f \in A$ and $c \in \mathbb{R}$, the set $X(f, c) = \{x \in X : \|f(x)\|_x \leq c\}$ is an affinoid domain. So too is a finite intersection $\bigcap X(f_j, c_j)$.

In algebraic geometry, spectra of K -algebras can be ‘glued’ together to form a global object, a K -scheme, which is a topological space Z equipped with a sheaf \mathcal{O}_Z of K -algebras, locally the spectrum of a K -algebra. Tate showed how affinoid subdomains of affinoid spaces can be glued together to form a global object—a *space Z with a sheaf of K -algebras \mathcal{O}_Z , which is locally the algebra of functions of an affinoid domain*.

Rigid analytic mirrors. Suppose we have a compact, convex polytope $P \subset \mathbb{R}^n$. To this we attach the set

$$\check{X}_P = \{x \in (\Lambda_{\mathbb{C}}^{\times})^n : (\text{val}(x_1), \dots, \text{val}(x_n)) \in P\}$$

(Figure 5). This subset is actually an affinoid subdomain of an affinoid space over the Novikov field $\Lambda_{\mathbb{C}}$. First, we can realize the annular domain $\{x \in \Lambda_{\mathbb{C}}^n : \epsilon \leq |x_j| \leq \epsilon^{-1}, j = 1, \dots, n\}$ as an affinoid space A_{ϵ}^n . The polytope P is cut out from \mathbb{R}^n by a finite list of inequalities, each of shape $\lambda \cdot x \geq c$, where $\lambda \in \mathbb{Z}^n$ and $c \in \mathbb{R}$. And \check{X}_P is cut out, inside A_{ϵ}^n for a suitably small ϵ , by inequalities $|x_1^{\lambda_1} \dots x_n^{\lambda_n}| \leq e^{-c}$; this identifies it as an affinoid subdomain of A_{ϵ}^n .

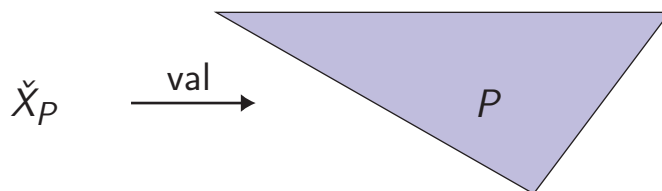


Figure 5. The values of the coordinates of the affinoid domain \check{X}_P form the polytope P .

⁹Mukai develops ‘quasi-universal sheaves’ in cases where automorphisms preclude a universal sheaf.

Suppose now that one has an n -manifold Q which is not merely smooth, but *integral affine* (cf. ‘The Key Questions’)—such as the base of a fibering of a symplectic manifold X by Lagrangian submanifolds $\{F_q\}_{q \in Q}$. ‘Triangulate’ Q by a collection of integral affine polytopes P_α . Each of them defines an affinoid domain \check{X}_{P_α} , and these glue together to form a rigid analytic space \check{X} over $\Lambda_{\mathbb{C}}$, which does not change when one subdivides the triangulation.

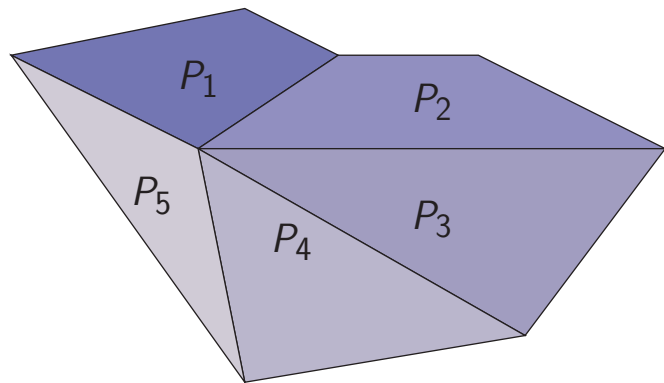


Figure 6. Fragment of a triangulation of Q , as it appears in an integral affine chart.

The set underlying \check{X} is the space of pairs (q, η) , where $q \in Q$ and $\eta \in H^1(F_q; U(\Lambda))$. Here $U(\Lambda) = \{\lambda \in \Lambda^\times : |\lambda| = 1\}$: so the mirror is a space of pairs of a torus-fiber F_q and a homomorphism from the first homology group $H_1(F_q) \cong \mathbb{Z}^n$ to the group of unit-norm Novikov series—made into a rigid analytic space.

For example, if $Q = \mathbb{R}/\mathbb{Z}$ is the circle—the base of a Lagrangian fibration on the 2-torus $\mathbb{R}^2/\mathbb{Z}^2$ viewed as a symplectic manifold—its affine integral structure is inherited from \mathbb{R} , and we can triangulate it by intervals $[a, b]$. The affinoid domain associated with an interval is an ‘annulus’ $\{z \in \Lambda^\times : e^{-b} \leq |z| \leq e^{-a}\}$, and these glue together to form an elliptic curve over Λ , the *Tate curve* $E_{\text{Tate}} = \Lambda^\times / q^\mathbb{Z}$.

Pseudo-holomorphic curves. Why should rigid analytic geometry over the Novikov field have anything whatsoever to do with symplectic topology? The brief answer is: *Gromov compactness*.

Symplectic topologists probe symplectic manifolds (X, ω) using *pseudo-holomorphic curves*: maps $u: \Sigma \rightarrow X$ from a Riemann surface Σ to X such that, for some specified complex structure J on TX , the derivative Du is complex linear. Thus, if j is the complex structure on $T\Sigma$, one has the ‘Cauchy–Riemann equation’ $J \circ Du = Du \circ j$. In the presence of a Lagrangian submanifold $L \subset X$, one

may suppose that Σ has boundary, and impose the boundary condition that ∂u (the restriction of u to the boundary $\partial\Sigma$) maps $\partial\Sigma$ to L .

Once one pins down the smooth surface underlying Σ , and the Lagrangian boundary conditions, there is a moduli space \mathcal{M} of pseudo-holomorphic curves in X , which one should think of as a smooth manifold. One can also allow pseudo-holomorphic curves with nodal domains, and from these one can construct a larger moduli space $\overline{\mathcal{M}}$. Gromov compactness says that the subspace $\overline{\mathcal{M}}_{\leq c}$, where the energy $E(u) = \int_\Sigma u^* \omega$ is at most c , is compact.

One typically imposes conditions on u so as to cut $\overline{\mathcal{M}}$ down to a zero-dimensional manifold N . Then the compact sub-level sets $N_{\leq c}$ for the energy function E are *finite*. Once one has a recipe for orienting N , one can ‘count’ its points with signs, and the result is a Novikov series, $\#N := \sum_{u \in N} \text{sign}(u) q^{E(u)} \in \Lambda_{\mathbb{C}}$.

From Lagrangians to coherent sheaves. Suppose that we have a compact CY manifold (X^{2n}, ω, Ω) and a non-singular fibering $f: X^{2n} \rightarrow Q^n$ by Lagrangian submanifolds—necessarily tori—which admit phase functions. Then Q acquires an integral affine structure. Suppose also that we have identified a section $\sigma: Q \rightarrow X$ of f whose image is Lagrangian; then $X = T^*Q/(T^*Q)_\mathbb{Z}$. As we discussed in the section on rigid analytic mirrors, we can use the integral affine structure of Q to define a rigid analytic Λ -space $\check{X} = \check{X}_{\text{rigid}}$. This is our mirror.¹⁰ It comes with a natural map $\tilde{f}: \check{X} \rightarrow Q$, and the fiber $\tilde{f}^{-1}(q)$ can be identified with $H^1(F_q; U_\Lambda)$, where $U_\Lambda = \text{val}^{-1}(0) \subset \Lambda^\times$ is the group of unit-norm Novikov series.

Now we come to the ‘Fourier transform’ underlying HMS, the process by which Lagrangians are converted into coherent sheaves on the mirror. Suppose $L \subset X$ is a compact Lagrangian submanifold, equipped with a phase function. One then defines sheaves $\mathcal{H}^k(\mathcal{E}_L)$ of $\mathcal{O}_{\check{X}}$ -modules on \check{X} : Cover Q by integral polytopes P_α , and let $q_\alpha \in P_\alpha$ be a reference point. For each α , we can perturb L to a new Lagrangian L_α such that $L_\alpha \cap F_q$ is a transverse intersection for every $q \in P_\alpha$. We define a module $\mathcal{E}_{L, \alpha}$ over the ring of functions $\mathcal{O}_\alpha := \mathcal{O}_{\check{X}_{P_\alpha}}$ of \check{X}_{P_α} by

$$\mathcal{E}_{L, \alpha} = (\mathcal{O}_\alpha)^{L_\alpha \cap F_{q_\alpha}} :$$

the free module on the set of intersection points. The module $\mathcal{E}_{L, \alpha}$ has a grading, defined via phase functions, and a differential δ —a square-zero endomorphism which increases the grading by 1. The construction of δ uses *family Floer cohomology*. It involves pseudo-holomorphic *bigons*, discs $\overline{\Delta} \rightarrow X$, with a boundary condition that requires the upper half of $\partial\overline{\Delta}$ to map to L_α , and the lower half to F_q

¹⁰An important and delicate issue is whether there are holomorphic discs in X whose boundary lies on a fiber of f , and if so, how properly to account for them in the construction of the mirror. For present purposes, assume there are none. This assumption is a major simplification of what is typically true.

for some $q \in P_\alpha$. For present purposes, we assume an absence of holomorphic discs whose entire boundary lies on F_q or L_α . This is vital; to make things work in generality, one will need to prove their absence rather than assuming it. The fact that δ makes sense expresses a compatibility between pseudo-holomorphic curves and rigid analytic geometry [2, 6].

We then pass to the cohomology module

$$H^*(\mathcal{E}_{L,\alpha}) = \ker \delta / \operatorname{im} \delta.$$

This is a finitely generated \mathcal{O}_α -module. While patterns of intersections change under perturbations of Lagrangians, $H^*(\mathcal{E}_{L,\alpha})$ does not depend on the perturbation $L \rightsquigarrow L_\alpha$. One can use that fact to assemble the modules $H^*(\mathcal{E}_{L,\alpha})$ into a sheaf $\mathcal{H}^*(\mathcal{E}_L)$ of $\mathcal{O}_{\check{X}}$ -modules. Locally, it is the sheaf associated with a finitely generated module over a Noetherian ring—so it is coherent.

The mapping $L \mapsto \mathcal{H}^(\mathcal{E}_L)$, sending a Lagrangian to a coherent sheaf on the rigid analytic mirror, is the ‘Fourier transform’ which explains HMS [2].*

Mirror Symmetry as an Operation on Holomorphic Families

We have just seen that the symplectic geometry of families of Lagrangian submanifolds, fibering X , gives rise to a rigid analytic mirror \check{X}_{rigid} over the complex Novikov field Λ , and that other Lagrangians in X then produce coherent analytic sheaves on \check{X}_{rigid} . But a rigid analytic space is not a symplectic manifold, so this cannot be an involutory process like Pontryagin duality or the Fourier–Mukai transform.

I want to outline, via an example, how the formation of rigid analytic mirrors should feed into an involutory procedure, not yet fully understood, the construction of the mirror partner to a *degenerating 1-parameter families of CY manifolds*, whereby the mirror of the mirror is the original.

The first point is that *degenerations should give rise to Lagrangian torus fibrations*. Start with projective space \mathbb{P}^d . This has a Lagrangian torus fibration $\mathbb{P}^d \rightarrow \Sigma_d$, of sorts, whose fibers are ‘Clifford tori,’ the points $(z_0 : \cdots : z_d)$ with $\sum |z_k|^2 = 1$ and $|z_j| = c_j$ (constant) for each j . The base Σ_d is a d -dimensional simplex. Some of the Clifford tori, those lying over the boundary of the simplex, are not Lagrangian, because they are tori of dimension less than d .

Now consider the ‘totally degenerate CY hypersurface’ $X_0 = \{z_0 \cdots z_d = 0\} \subset \mathbb{P}^{d+1}$. It is a union of $d + 1$ projective hyperplanes $x_k = 0$, and the Lagrangian torus fibrations over these hyperplanes assemble to give a map $\mu: X_0 \rightarrow P$ to a d -dimensional polyhedron formed by gluing the $d + 1$ simplices along faces (P actually just the boundary of a $(d + 1)$ -dimensional simplex). The fibers

of μ are Lagrangian tori over the interiors of the faces of P , and are lower-dimensional tori elsewhere (Figure 7).

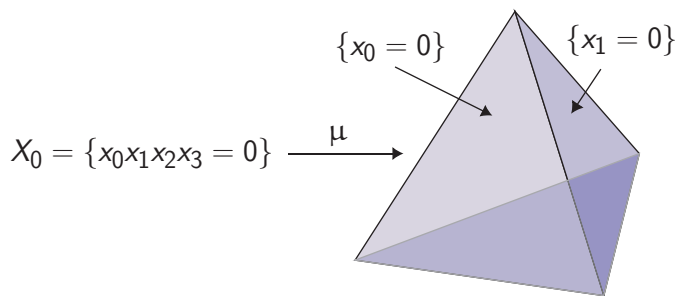


Figure 7. The map $\mu: X_0 \rightarrow P$ in the case $d = 2$.

Next, consider the family of CY hypersurfaces

$$X_t = \{(t, z) \in \mathbb{C} \times \mathbb{P}^{d+1} : tF(z) + z_0 \cdots z_d = 0\},$$

where F is a (generic) homogeneous polynomial of degree $d + 1$. Thus X_1 is a CY manifold, while X_0 is our singular, totally degenerate CY hypersurface. One can use the symplectic geometry of the family (with a Kähler form inherited from $\mathbb{C} \times \mathbb{P}^{d+1}$) to produce a map $\rho: X_1 \rightarrow X_0$ which is a symplectomorphism over the smooth locus in X_0 . The composite $f: X_1 \xrightarrow{\rho} X_0 \xrightarrow{\mu} P$ is then our candidate for a Lagrangian torus fibration. Over the interiors of the simplices of P , μ has Lagrangian fibers and ρ is a diffeomorphism; over a codimension k facet of P , the fibers of μ have dimension $d - k$, but those of ρ have dimension k , so f has fibers of dimension d , as we want. However, there is a ‘bad’ locus $B \subset X_0$ where the total space of the family is singular, and the mechanism breaks down; that is the source of singularities in the fibers of f (Figure 8).

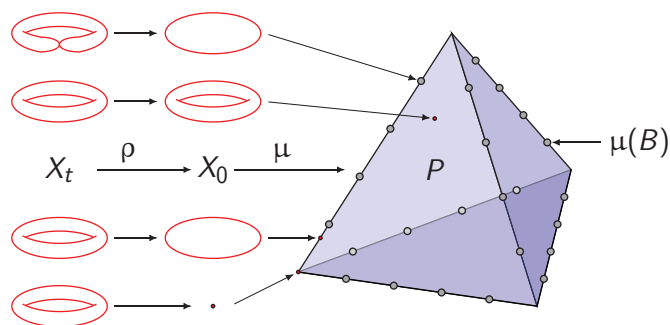


Figure 8. The map $f: X_t \rightarrow P$ in the case $d = 2$, showing some of its fibers in red. The 24 dots on the edges of the tetrahedron P are the images of the singular locus of the total space of the family.

This example illustrates a mechanism whereby *toric degenerations* of CY manifolds—roughly, degenerations to varieties each of whose irreducible components is a toric variety—should give rise to Lagrangian torus fibrations.¹¹

The fiber X_1 comes with a symplectic automorphism m , the monodromy around the unit circle, which—in a model situation, at any rate—preserves the fibers of f , and acts as translation of each of the non-singular fibers. This automorphism corresponds to extra structure on the mirror, a line bundle over \check{X}_{rigid} . One expects that this line bundle is *ample*, and therefore defines an embedding of \check{X}_{rigid} into rigid analytic projective space. Just as in complex analytic geometry, the image of an embedding into projective space is in fact cut out *algebraically* by polynomials—so the \check{X}_{rigid} becomes an algebraic *scheme* \check{X}_{alg} over $\Lambda_{\mathbb{C}}$.

Pause for a moment to observe that if we have a family Z_t of complex projective varieties, whose defining equations depend holomorphically on $t \in \Delta^*$ (the punctured disc), we can take the Laurent expansions of these equations to get a family \mathcal{Z} over the field $\mathbb{C}((t))$ of finite-tailed Laurent series, and therefore, by extending scalars, a variety over $\Lambda_{\mathbb{C}}$. One can ask whether \check{X}_{alg} arises in this way, from a family \check{X}_t of complex projective varieties. This is not the place to get into the details, but there are geometric reasons to expect that to be true. In this way, we end up with a new family $\{\check{X}_t\}$ of complex projective CY manifolds, mirror to the original family.

While the general picture described here has large gaps still to be filled, an algebro-geometric analogue of the composite process has been fully worked out by Gross–Siebert [10]. Their works centers on a part of the story called *wall-crossing* that I have not even hinted at.

Example. If one takes a degenerating family of elliptic curves $\mathcal{X} \rightarrow \Delta^*$, given as cubic curves in \mathbb{P}^2 , the generic fiber X is (symplectically) the 2-torus $\mathbb{R}^2/\mathbb{Z}^2$ and it has the Lagrangian fibration given by projection $f: \mathbb{R}^2/\mathbb{Z}^2 \rightarrow \mathbb{R}/\mathbb{Z}$. After choosing a section of f , one obtains the Tate curve as rigid analytic mirror, with a degree 1 line bundle over it. Section of powers of this line bundle define an embedding of the Tate curve into $\mathbb{P}^2(\Lambda)$ as a cubic curve

$$y^2 + xy = x^3 + a_4(q)x + a_6(q),$$

where a_4 and a_6 are certain power series in q . In particular, this curve is defined over $\mathbb{C}((q))$. Since a_4 and a_6 are convergent in the unit disc $|q| < 1$, it can also be viewed as a holomorphic family over Δ^* —the mirror to the original family.

¹¹This mechanism was first explored by W.-D. Ruan in 1999, but was recently revisited in R. Guadagni's 2017 University of Texas Ph.D. thesis.

Looking Ahead

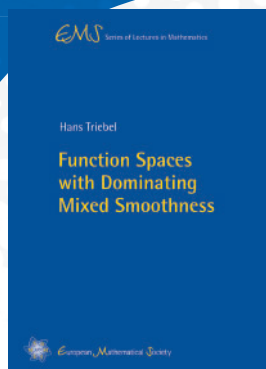
From this symplectic geometer's perspective, the most important task ahead is to fill the gaps in the picture just outlined—precisely how to construct Lagrangian fibrations with singularities from degenerations, and then, crucially, *how to construct their analytic mirrors*. The chief difficulty is with Floer theory for singular Lagrangians. The Gross–Siebert program provides an algebro-geometric solution, at the cost of losing the direct connection to symplectic topology and the natural construction of HMS as a Fourier transform. I hope and suspect that Gross–Siebert's work will be precisely linked to symplectic topology, perhaps even in the absence of a full understanding of the singular Lagrangians, and that a proof of HMS, valid in vastly more generality than we can currently manage, will thereby emerge.

I especially look forward to the weaving together of different threads of mirror symmetry, integrating the symplectic-analytic-algebraic picture with the Riemannian geometry of special Lagrangians; and the topological field theory of the A- and B-models with rigorous approaches to a quantum field theory on \check{X} [3, 14]. In this account I have not even touched on mirror symmetry for *Fano* manifolds—which is just as remarkable as for CY manifolds—nor on wall-crossing, applications of mirror symmetry in symplectic topology, or connections to the Langlands program. For mathematicians fascinated by hidden connections, mirror symmetry is a dazzling phenomenon.

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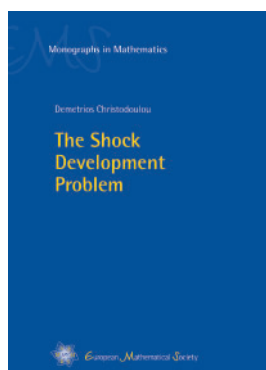
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Timothy Perutz

Credits

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CALL FOR APPLICATIONS & NOMINATIONS

Secretary of the American Mathematical Society

The American Mathematical Society is seeking candidates for the position of Secretary, one of the most important and influential positions within the Society. The Secretary participates in formulating policy for the Society, participates actively in governance activities, plays a key role in managing committee structures, oversees the scientific program of meetings, and helps to maintain institutional memory.

The first term of the new AMS Secretary will begin February 1, 2021, with initial appointment expected in January 2020 in order that the Secretary-designate may observe the conduct of Society business for a full year before taking office.

All necessary expenses incurred by the Secretary in performance of duties for the Society are reimbursed, including staff support. The Society is prepared to negotiate a financial arrangement with the successful candidate and the candidate's employer in order that the new Secretary be granted sufficient release time and provided staff support to carry out the many functions of the office.

QUALIFICATIONS

The Secretary should be a research mathematician and must have substantial knowledge of Society activities. Although the AMS Secretary is appointed by the Council for a term of two years, candidates should be willing to make a long-term commitment, as it is expected that the new Secretary will be reappointed for subsequent terms pending successful performance reviews.

DUTIES

- Organize and coordinate the Council and its committees.
- Serve as a member of the Council, Executive Committee, and several other committees, including the five policy committees, the Development Committee, and the Editorial Boards Committee.
- Work closely with the President to coordinate and administer the activities of committees.
- Oversee, together with the Associate Secretaries, the scientific programs of all AMS meetings.

APPLICATIONS & NOMINATIONS

A Search Committee, with Tara Holm as Chair, has been formed to seek and review applications. Persons wishing to apply should do so through MathPrograms.Org. Nominations and questions should be directed to the Chair of the Search Committee: ssc-chair@ams.org.

For full consideration, applications, nominations, and supporting documentation should be received by **May 15, 2019**.

Interview with Abel Laureate Robert P. Langlands

Robert P. Langlands is the recipient of the 2018 Abel Prize of the Norwegian Academy of Science and Letters.¹ The following interview originally appeared in the September 2018 issue of the *Newsletter of the European Mathematical Society*² and is reprinted here with permission of the EMS.

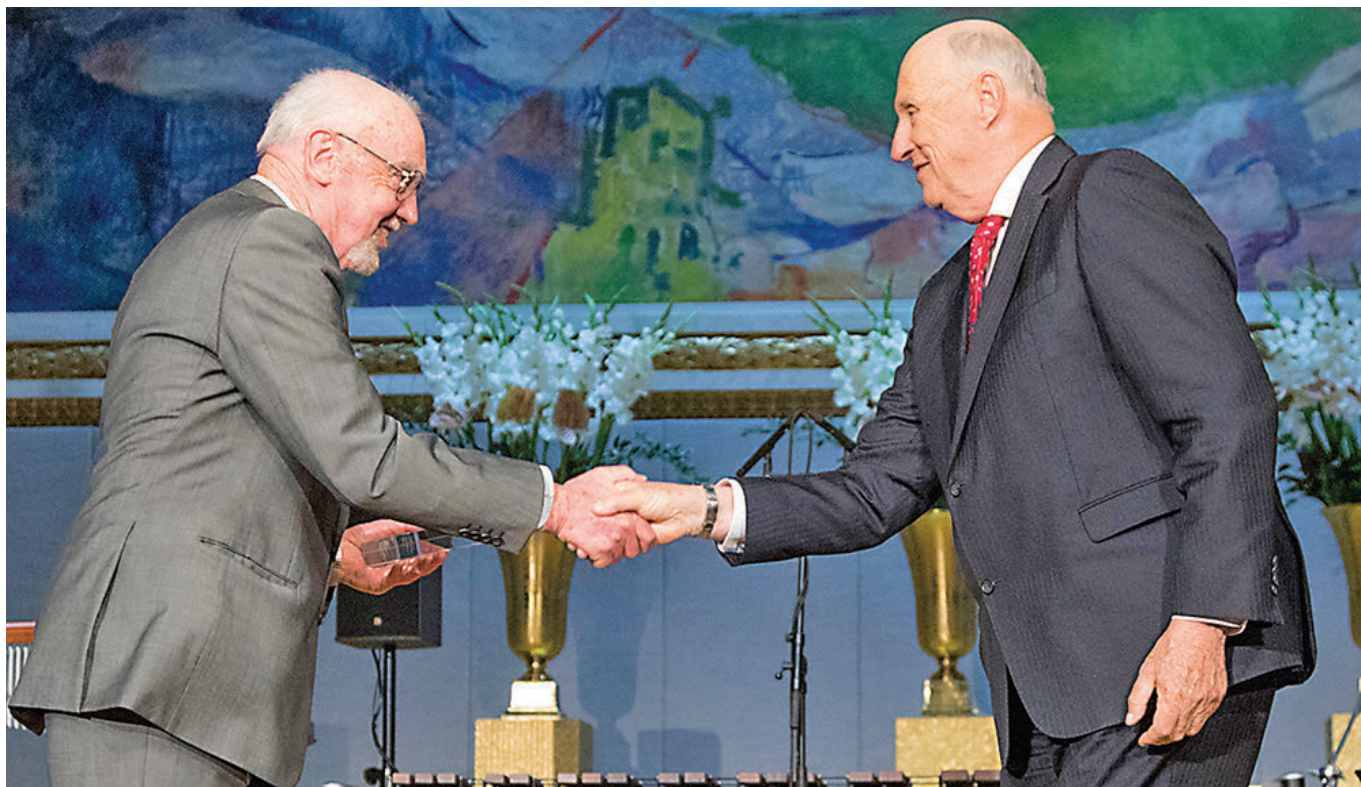


Figure 1. Robert Langlands (left) receives the Abel Prize from H. M. King Harald.

Bjørn Ian Dundas and Christian Skau

Bjørn Ian Dundas is a professor of mathematics at University of Bergen, Norway. His email address is Bjorn.Dundas@uib.no.

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¹See the June–July 2018 Notices <https://www.ams.org/journals/notices/201806/rnoti-p670.pdf>

²<http://www.ems-ph.org/journals/newsletter/pdf/2018-09-109.pdf#page=21>, pp.19–27

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Dundas and Skau: Professor Langlands, firstly we want to congratulate you on being awarded the Abel Prize for 2018. You will receive the prize tomorrow from His Majesty the King of Norway.

We would like to start by asking you a question about aesthetics and beauty in mathematics. You gave a talk in 2010 at the University of Notre Dame in the US with the intriguing title: *Is there beauty in mathematical theories?* The audience consisted mainly of philosophers—so non-mathematicians. The question can be expanded upon: Does one have to be a mathematician to appreciate the beauty of the proof of a major theorem or to

admire the edifice erected by mathematicians over thousands of years? What are your thoughts on this?

Langlands: Well, that's a difficult question. At the level of Euclid, why not? A non-mathematician could appreciate that.

I should say that the article was in a collection of essays on beauty. You will notice that I avoided that word; in the very first line, I said: "Basically, I do not know what beauty is!" I went on to other topics and I discussed the difference between theories and theorems.

I think my response to this is the same today. Beauty is not so clear for me; it is not so clear when you speak about beauty and mathematics at the same time. Mathematics is an attraction. If you want to call it beauty, that's fine. Even if you say you want to compare with the beauty of architecture. I think that architectural beauty is different from mathematical beauty. Unfortunately, as I said, I just avoided the question in the article and, if you forgive me, I will avoid it today.

Dundas and Skau: One other reason we ask this question is that, as you are well aware, Edward Frenkel, who you have worked with and who is going to give one of the Abel Lectures later this week about aspects of the Langlands programme, wrote a best-seller with the title *Love and Mathematics and the subtitle The Heart of Hidden Symmetry*. The Langlands programme features prominently in that book. He makes a valiant effort to try to explain to the layman what the Langlands programme is all about. We were very intrigued by the preface, where Frenkel writes: "There is a secret world out there, a hidden parallel universe of beauty and elegance, intricately intertwined with ours. It is the world of mathematics. And it is invisible to most of us." You have probably read the book. Do you have any comments?

Langlands: I have skimmed through the book but I have never read it. I am going to say something that is probably not relevant to your question. We are scientists: we ask about, we think about, we listen, at least, to what scientists say, in particular about the history of the Earth, the history of the creatures on it and the history of the Universe. And we even discuss sometimes the beginning of the Universe. Then, the question arises, something that puzzles me although I've seldom thought about it, except perhaps when I am taking a walk—how did it get started at all? It doesn't make any sense. Either something came out of nothing or there always was something. It seems to me that if I were a philosopher or you were a philosopher, we'd have to ask ourselves: how is it that something can be there? It's complicated; it's not irrelevant that the world is very complicated but the enigma is simply the fact that it is there. How did something come out of nothing? You may say with numbers it can happen but beyond that I don't know.

Dundas and Skau: You have your creative moments, where all of a sudden you have a revelation. Hasn't that been a feeling of intense beauty for you?

Langlands: You presumably mean when suddenly things fit together? This is not quite like looking at clouds or looking at the sea, or looking at a child. It is something else; it just works! It works and it didn't work before; it is very pleasant. The theories have to be structural and there has to be some sort of appealing structure in the theory.

But, you know, beauty...women are beautiful, men are beautiful, children are beautiful, dogs are beautiful, forests are beautiful and skies are beautiful; but numbers on the page or diagrams on the page? Beauty is not quite the right word. It is satisfying—it is intellectually satisfying—that things fit together, but beauty? I say it's a pleasure when things fit together.

As I said in the article, I avoided the word beauty because I don't know what it means to say that a mathematical theorem is beautiful. It is elegant, it is great, it is surprising—that I can understand, but beauty?!

Dundas and Skau: But we can at least agree that Frenkel's book was a valiant effort to explain to the layman what beauty in mathematics is and, in particular, that the Langlands programme is a beautiful thing.

Langlands: Well, yes, I would wish that Frenkel were here so I could present my views and he could present his. I have studied Frenkel because he explained the geometric theory but I wasn't interested that much in the beauty. I wanted to read his description of the geometric theory and I got quite a bit from it but I also had the feeling that it wasn't quite right. So, if I wanted to say more, I would want to say it in front of him so he could contradict me.

Dundas and Skau: You have an intriguing background from British Columbia in Canada. As we understand it, at school you had an almost total lack of academic ambition—at least, you say so. Unlike very many other Abel Laureates, mathematics meant nothing to you as a child?

Langlands: Well, except for the fact that I could add, subtract and multiply very quickly. There was an interview in Vancouver—actually, I was in New Jersey but the interviewer, he was in Vancouver—and he asked me a question along those lines and I answered rather frivolously. All the experience I had with mathematics was with arithmetic, apart from elementary school and so on, and I liked to count.

I worked in my father's lumberyard and those were the days when you piled everything on truck by hand and tallied it. And you counted the number of two by fours—is that a concept here? Two by fours: 10 feet, 12 feet, 8 feet, 16 feet...and then you multiply that and add it up with the number of 10s and multiply by 10, plus the number of 12-foot-lengths and multiply by 12, and so on and so forth and you get the number; convert it to board feet and you know how much it is worth. I would be loading the truck with some elderly carpenter or some elderly farmer from the vicinity. He would have one of these small carpenter pencils and he would very painfully be marking one, two, three, four, five; one, two, three, four, five; and so on. And

then you would have to add it all up. And me, I was 12, 13 or 14 and I could have told him the answer even before he started. But I waited patiently when he did that.

So, that was my only experience with mathematics except for one or two things, one or two tricks my father used when building window frames to guarantee that the angles are right angles and so on, but that was just a trick, right? The diagonals have to be of equal length if the rectangle is going to be right-angled.

Dundas and Skau: Then, why did you move toward mathematics? Why not languages or other things that you studied?

Langlands: Actually, when I went to university in the almost immediate post-World War II period, it was still regarded as necessary for mathematicians to learn several languages: French, English, Russian or maybe even Italian. Now, that fascinated me. The instruction of French in English-speaking Canada was rather formal; nobody paid too much attention to it. But learning languages rather fascinated me and the fascination has been with me all my life (but that was incidental to mathematics).

Dundas and Skau: Why did you start at university at all?

Langlands: Why did I start...? Here is my conjecture.

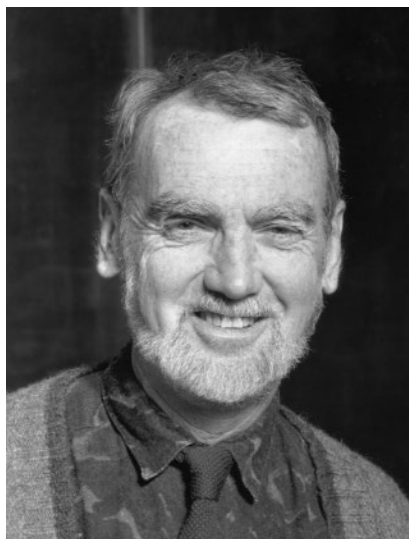


Figure 2. Robert P. Langlands, The Abel Prize Laureate 2018

There are two things (I will come back to the second thing in just a minute). I went to high school. There were children from the neighbourhood and from the surrounding country side, and they tested us. I was indifferent, you know. I didn't pay too much attention but they also used IQ tests and my conjecture has always been that I probably had an unusually high IQ—quite an unusually

high IQ—I don't know. It didn't mean much to me then but that is my conjecture in retrospect. Many of our teachers were just former members of the army in World War II, who were given positions as teachers more as a gratitude for their service in the army. This fellow—he was young, he probably had a university degree and he took an hour of class time to say that I absolutely must go to university. So, I noticed that.

And there was another reason: I had acquired a mild interest in science because I had a book or, rather, my future father-in-law had a book (it was rather a leftist book about eminent scientists; of course, Marx was included, Darwin was included, Einstein was included and so on—various

scientists from the 1600s, 1700s, 1800s, etc.) and he gave it to me. He himself had a childhood with basically no education and he learned to read aged about 37, during the Depression when the Labour parties were recruiting unemployed people. So, he learned to read but never very well and I think he never really could write. He always had a good memory so he remembered a number of things and he also had a library and, in particular, he had this book, which was very popular in the pre-war period. So, I began to read this book. My wife—my future wife—had a better idea of what one might do as an adult than I did and she influenced me. And I had this book, where I read about outstanding people like Darwin and so on, and that influenced me a little in the sense that it gave some ideas of what one might do.

And there was also the accident that I always wanted to leave school and hitch-hike across the country, but when I turned 15, which was the legal age when you can stop going to school—I had only one year left—my mother made a great effort and persuaded me to stay another year. During that last year, things were changing for various reasons, e.g. the lecture of that teacher and an introduction to one or two books, so I decided to go to university.

Dundas and Skau: You go on to obtain a Master's thesis at the University of British Columbia, you marry and then you go to Yale and start a PhD in mathematics. It is quite a journey that you were on there. How did you choose the thesis topic for your PhD at Yale?

Langlands: First of all, Hille had this book—you may know it—on semi-groups and I was an avid reader of that book, and I took a course from Felix Browder on differential equations. You may not know but Felix Browder was an abysmal lecturer and so you had to spend about two or three hours after each lecture sorting things out. He knew what he was talking about but it took him a long time to get to the point or to remember this or that detail of a proof. I went home and I wrote out everything he had talked about.

So, I had this background in partial differential equations from his course and I had read all of Hille's book on semi-groups and I just put the two together. I really liked to think about these things.

Dundas and Skau: In other words, you found your own PhD topic?

Langlands: Yes, I found my own PhD topic.

Dundas and Skau: But from there on, after your thesis, we have what we like to think of as a journey toward a discovery. Your work on Eisenstein series and your study of the theory of Harish-Chandra are crucial ingredients here. Would you care to explain to us what the background was that led to the Langlands programme?

Langlands: There was a Hungarian fellow, S. Gaal, who had immigrated to the US after the difficulties in Hungary and that was in the middle of the 1950s. The Norwegian mathematician Atle Selberg was a member of the Institute of Advanced Study (IAS) in Princeton. Selberg's wife was

Romanian and spoke Hungarian and I think Gaal (he and his wife and maybe their children too) was invited to the IAS by Selberg. He had come to the US sponsored more or less by Selberg and he was giving a graduate course at Yale, where he talked about Selberg's paper, basically at the time of Selberg's second so-called "Indian paper," a Tata publication from 1960. Selberg didn't write that many papers at the time but I think there were two and Gaal talked about that. Also, I have to mention that there was an important seminar on convexity in the theory of functions of several complex variables.

So, you hear about Selberg and you hear about Eisenstein series, and this theory about convexity, and then you want to prove things and you move more or less instantly to an analytic continuation of Eisenstein series in several variables. So, I had already thought about that but I thought about them in a rather restricted context—no algebraic numbers, for example.

And then I got a position at Princeton University, not because of anything I had done about Eisenstein series but because of my work on one-parameter semi-groups. So, I gave a lecture in one seminar; Bochner didn't run it but he kept an eye on it. I think he was impressed simply because I was talking about something that wasn't in my thesis. I talked about this work with Eisenstein series and I think he was impressed by me. Now, Bochner's family was from Berlin. He wasn't born there but he lived there as a child. He went to German universities and he had connections with Emmy Noether and Hasse, for example. So, he took an interest in anything that had to do with algebraic number theory and he encouraged me to think about Eisenstein theory in a more general context, not just for groups over rational numbers but also for groups over algebraic number fields.

Dundas and Skau: So Bochner was almost like a mentor for you for a while?

Langlands: Not a mentor but he was like a foster father, if you like. He encouraged me—more than an encouragement; he pushed me. Bochner encouraged me to work over algebraic number fields rather than just over the rational number field. Algebraic number fields I basically learned from Hecke and I read papers by Carl Ludwig Siegel (because there are ways to handle analytic continuation of series, which you can take from Siegel's papers). I started to read a little in the literature of these two, Hecke and Siegel, and I wrote about Eisenstein series basically using their very classical methods.

In any case, one year—just about a week before the classes were to start—I was going to give a course in class field theory. Emil Artin had been in Princeton and was the expert on class field theory; he had gone back to Germany in 1958 and there were one or two disappointed students who had come to Princeton to learn a little bit of class field theory. There was no real information on class field theory to be obtained from the courses offered. I had attended a

seminar that was arranged by these disappointed students but it wasn't such a good seminar, so I was quite ignorant. But Bochner said: "You are to give a course in class field theory." And I said: "How can I do it? I don't know anything about it and there is only one week left." But he insisted so I gave a course on class field theory from Chevalley's paper, which is the more modern view, and I got through it. There were three or four students, who said they learned something from it.

So, with that, I began to think about the fact that there was no non-abelian class field theory yet. Some people, like Artin, didn't expect there to be any. So, I was just aware of it, that's all. We are now in August of 1963 or something.

Dundas and Skau: You already had a position at Princeton University at the time?

Langlands: I had a comfortable position at the university and I went up the ladder reasonably rapidly. I think by 1967, I was an associate professor or something like that. Thanks to Selberg, I was at the IAS for a year, and I was at Berkeley, California, for a year. So, I was away two times.

Dundas and Skau: And all this while you were contemplating the trace formula, is that correct?

Langlands: Well, let me go back. I have forgotten something. I was concerned with the trace formula and I wanted to apply it. The obvious thing you want the trace formula for is to calculate the dimension of the space of automorphic forms; that is the simplest thing. So I wanted to do it. And, so, you plug in a matrix coefficient—as I understood it; it doesn't look like a matrix coefficient—of an infinite dimensional representation into the trace formula and you calculate.

I didn't quite know what to do with this and then I spoke to David Lowdenslager—he died very young—and he said: "Well, people are saying that this is really something you can find in Harish-Chandra." So, I started to read Harish-Chandra and what I observed very quickly, because of reading Harish-Chandra, was that the integral that was appearing in the trace formula was an orbital integral of a matrix coefficient. And that orbital integral of a matrix coefficient, we know from representations of finite groups, is a character and, basically, you learn from Harish-Chandra's paper that this is indeed the case. So, that meant that I had to start to read Harish-Chandra—as I did.

And once you start to read Harish-Chandra, of course, it goes on and on; but that was the crucial stage: this observation of Lowdenslager that people were beginning to think that Harish-Chandra was relevant. So, there we are, we have it all. And then I began to think about these things, slowly; and sometimes it worked out, and sometimes it didn't. I could actually apply the trace formula successfully.

In 1962, Gelfand gave a talk at the ICM in Stockholm and a year later his talk was circulating. Now, Gelfand gave his views of the matter. The point was that he introduced the notion of cusp forms explicitly. The cusp form is a critical notion and it is a notion that I think appears in rather

obscure papers by Harish-Chandra and Godement. But it is hard; you have to look for it. But with Gelfand it was clear why that was so fundamental. Now, an incidental question: I don't think Selberg ever really grasped the notion of a cusp form. Selberg, of course, didn't read other people's papers and I don't think he ever grasped the notion of a cusp form. I think that was an obstacle that he never overcame.

But as soon as you read Gelfand, you can do it—you can prove the general theory about Eisenstein series. You have to know something. In other words, you have to be someone who knows something about unbounded operators on Hilbert spaces. You have to be someone with this background or it doesn't mean anything to you. But if you had that background then you saw immediately what was to be done: take what Selberg had done in rank one to the general case.

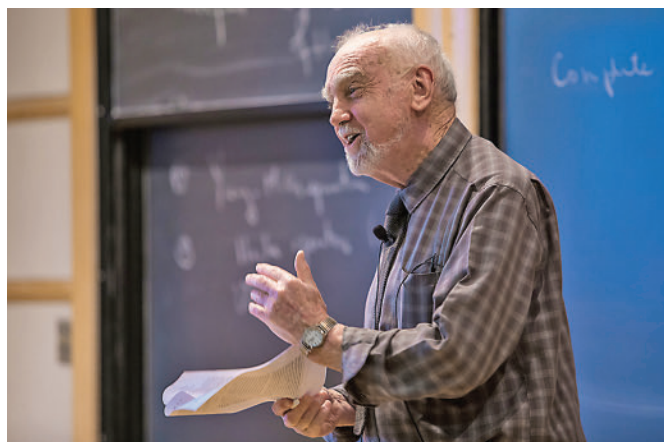


Figure 3. Robert P. Langlands giving a lecture.

Let me go back a little. I only talked mathematics with Selberg once in my life. That was in 1961, before I came to the Institute (IAS). It was at Bochner's instigation, I am sure. Selberg invited me over and he explained to me the proof of the analytic continuation in rank one. Now, of course, the proof of the analytic continuation in rank one is like Hermann Weyl's theory on differential equations of second-order on a half-line. I had read Coddington and Levinson's book *Theory of Ordinary Differential Equations* not too long before, so I could just sit there and listen to Selberg—listen to the kind of things he knew very well—and he explained it to me. Whether he regretted that afterwards I cannot say but he explained to me how it works in rank one.

Dundas and Skau: Were you impressed by his presentation?

Langlands: I had never spoken mathematics with a mathematician on that level before in my life. I had really never spoken mathematics with Bochner and he is the one that came closest.

Dundas and Skau: Even so, you didn't have further conversations with Selberg afterwards.

Langlands: No, he wasn't a talkative man. Well, I did have occasional conversations because then I was still continuing to try to prove the fundamental analytic contin-

uation of Eisenstein series. I approached him when I had done it in this or that case and he'd say: "Well, we don't care about this or that case. We want to do the general theory." So he didn't listen to me. While we were colleagues and our offices were basically side by side, we'd say hello but that's about it.

Dundas and Skau: So you spent many years together in virtually adjacent offices at the IAS and you never really talked mathematics?

Langlands: No. Selberg, you must know, didn't speak with very many people about mathematics. He spoke with one or two, I think, but not many. I am not sure how much he thought about mathematics in his later years. I just don't know.

Dundas and Skau: But even so, your work on Eisenstein series really had some consequences in hindsight, didn't it?

Langlands: Yes, it was critical in hindsight, right? So, that took me about a whole year and I think I was exhausted after that—it was one of the cases where you think you have it and then it slips away. There was, for example, an induction proof. In induction proofs, you have to know what to assume: if you assume too much it is not true and if you assume too little it doesn't work.

In fact, there was a problem; things were happening that I didn't recognise. In other words, it could be a second-order pole where you naturally assume that there is only a first-order pole. And it took me a long time to reach that stage. Specifically, it is the exceptional group G_2 of the Cartan classification. You think this is going to work; and you try and it doesn't work—it doesn't work in general. Then you think about what and where it can really go wrong and it turns out that it only goes wrong for G_2 . Then you make a calculation with G_2 and what do you see? You see this second-order pole or a new kind of first-order pole and that changes the game: you have a different kind of automorphic form.

It eventually worked; it was an exhausting year but it did eventually work.

Dundas and Skau: And in the Autumn of 1964, you went to Berkeley, is that correct?

Langlands: And then I went to Berkeley, pretty much exhausted by that particular adventure.

Dundas and Skau: Were you really so exhausted that you thought about quitting mathematics?

Langlands: Well, look, quitting mathematics is a rather strong statement. But I did decide to spend a year in Berkeley and got some things done in retrospect. I did more than I thought I had done. I was too demanding, you know. When you are younger, you are a little more demanding than when you are older. So, the next year I was really trying, I think, to do something with class field theory and I didn't see anything. I had a whole year where I don't feel I got anything done. In retrospect, in Berkeley, I did something but the year afterwards I didn't at first do anything and I was growing discouraged.

So, I decided on a little bit of foreign adventure. I pretty much decided that the time was right. I should just go away and maybe think of doing something else. I had a Turkish friend and he explained to me the possibility of going to Turkey. So, I decided to do that and, once I had decided to go to Turkey, there were various things to do; I wanted to learn some Turkish and then I went back to studying Russian. I had a very nice teacher. But I still had a little time to spare and I didn't know quite what to do and I began to calculate the constant terms of Eisenstein series.

Dundas and Skau: Just for the fun of it?

Langlands: Just for having something to do. And so I calculated them. I just calculated it for various groups and then I noticed that it was basically always of the form $f(x)/f(x+1)$ or something like that. But if you could continue the Eisenstein series you could continue the constant term and instead of $f(x)/f(x+1)$, you could continue $f(x)$. And these things are Euler products, so you have new Euler products. Of course, analytic number theorists just love Euler products. So you had it! You had something brand new: they had an analytic continuation and a functional equation. And you could basically do it for a lot of groups.

Dundas and Skau: You could even do it for reductive groups?

Langlands: You basically did it for split groups, i.e. those reductive groups with a split maximal torus, and then you have the classification. So, you had a whole bunch and, if you looked at them, you could see that somehow they were related to representations of Eisenstein series associated to parabolic groups of rank one. And they were somehow related to a representation; you have a parabolic group and you take the reductive subgroup—it is of rank one and you throw away the rank one part so you basically have some kind of L -function associated to the automorphic form on this reductive subgroup.

All right, so you have Euler products that are attached to a representation of a group. Euler products are Dirichlet series that number theorists love—and that is what you want. You have a large list of groups. And that already suggests something. You can formulate this—you can see somehow where this is coming from. You can see how to formulate it as a representation associated to an automorphic form and a particular representation of what I call the L -group, for L -series.

And there you are: you start to make a guess and you have this in general! For a particular reductive group, you have an Euler product with an analytic continuation, associated to a representation. But you think it works in general. So, once you have that—once you have something that might work in general—you have to think of how you are going to prove it in general.

Dundas and Skau: This must have been extremely exciting?

Langlands: Well, it was!

Dundas and Skau: Incidentally, did you continue with your classes in Russian or Turkish?

Langlands: I gave up both, even the Russian class where the teacher was this sweet woman; I think she liked me since I was an industrious student. She was very angry and wouldn't talk to me.

Dundas and Skau: Is it fair to say, then, that your discovery comes out of...well, you were extremely exhausted, you let your shoulders down, you play, you have some evidence and you make a major discovery?

Langlands: I think that's an apt description.

Dundas and Skau: When did you have this epiphany, if you like, where you saw the connection with the Artin conjecture about the analytic continuation to the whole complex plane of the Artin L -functions?

Langlands: It was during the Christmas vacation of 1966. Although I have forgotten the date the idea came to me, I still have a vivid recollection of the place. In the old Fine Hall at Princeton University, there was a small seminar room on the ground floor directly to the east of the entrance. The building itself, I recall, was of a Gothic style with leaded casement windows. I was looking through them into the ivy and the pines and across to the fence surrounding the gardens of the President's residency when I realised that the conjecture I was in the course of formulating implied, on taking $G = \{1\}$, the Artin conjecture. It was one of the major moments in my mathematical career.³

Dundas and Skau: Was this a so-called Poincaré moment for you? You know the story about Poincaré getting on a bus when all of a sudden he saw the solution to a problem he had been thinking about for months and then put aside.

Langlands: Except that somehow I was not searching. I had no idea I would stumble across a non-abelian class field theory.

Dundas and Skau: And this was right before you sent the 17-page, handwritten letter to André Weil outlining your theory?

Langlands: Yes. The letter to André Weil is somehow an accident. The point is, I went to a lecture by Chern. Weil went to the same lecture and we both arrived early. I knew him but not particularly well. We both arrived early and the door was closed so we couldn't go in. So, he was standing there in front of the door and I was standing there in front of the door. He wasn't saying anything so I thought I should say something. I started to talk about this business. And then he didn't understand anything, of course, and he probably behaved as you'd behave under those circumstances; I was this fellow talking to him and I just assumed he would walk away but he said "write me a letter." I wrote him a letter. He never read the letter so far as I know.

Dundas and Skau: He had your letter typed and distributed, didn't he?

Langlands: Yes, that's right.

³Langlands (2005): *The genesis and gestation of functoriality.* publications.ias.edu/sites/default/files/TheGenesis.pdf

Dundas and Skau: This is not the only moment you describe where you are making a discovery while not sitting behind your desk and working. On another occasion, you tell of how you are walking from here to there and suddenly you see something. Is that a pattern of yours? Is that how you find things?

Langlands: I have certainly seen these things very seldom in my life so I don't think one can speak about a pattern.

Dundas and Skau: Perhaps it is time that you actually tell us about what the Langlands programme is all about? Just in broad brush strokes.

Langlands: Okay, we sort of know what the quadratic reciprocity law is, right? There, two things that appear to be quite different are the same. Now, we also know that, after Weil, we can define zeta-functions (or L -functions would probably be better). You can define them over finite fields and you can also define them if you have a global field and you take the product of the ones over finite fields and you get some kind of an L -function associated to a variety or even, if you like, to a particular degree of the cohomology of that variety.

A basic problem in arithmetic for any kind of estimation of the number of solutions of Diophantine equations is reflected in the L -functions that you can formally associate—and you are in the half-plane—to the cohomology of the given degree of any kind of curve over a number field. They are there.

Presumably, if you can deal with these then you can, somehow or other, do more things about the estimation of the number of solutions and the nature of solutions. I think no one has a clear idea about this, except in very specific cases, i.e. what you can do with the knowledge of these global L -functions. But they are there, and you want to prove that they have analytic continuation. The only reasonable way, on the basis of evidence, is that they will be equal to automorphic L -functions.

Now, from the point of view of the variety and the cohomology of the variety, you have the Grothendieck formula. I don't know to what extent he actually had a complete theory—I don't think he had—but he had the notion of a motive, and a motive has certain multiplicative properties. So, you had a whole family of functions that behaved in a natural functorial manner. And you wanted to prove that they could be analytically continued. But he managed to associate a group; in other words, these motives were associated to representations of a group, whose nature had to be established. On the other hand, the group is there: you may never know its nature but you should be able to find out its relations to other groups.

Now, on the other hand, what you would like, normally, in order to establish the analytic properties of these things that are defined algebraically/geometrically is to associate them to something that is defined analytically because automorphic L -functions basically have analytic continuations. There are some questions about it, right, because you can do it if they are associated to GL_n and the standard

representations of GL_n (that is the theorem by Jacquet and Godement from 1972). But, in the end, you need to do two things that are more or less mixed, namely, for an automorphic form associated with a general group, you need to show that that automorphic form really sits on GL_n ; you push it toward GL_n and then you define the L -function. So, it is not just an automorphic form but it is an automorphic form that can be pushed toward GL_n .

Now, that will make you think that somehow an automorphic form is associated to a representation of a group, which has to be defined. In other words, there is a structure in the connection of all automorphic forms. You can pass it from one associated with G . (It is not true that if you associate it with G , you can pass it to another group G' if G goes to G' .)

This is the so-called L -group and you have to push it forward. If you have this motion and you can push, you could say you have the automorphic form here equal to one over there, and so the L -function is the same.

If the one you take over here is GL_n then you know, by Jacquet-Godement, that you can handle it. So, if you have a way of passing—whenever you have the form on one group—to other groups in the appropriate formalism then you can handle analytic continuation.

Dundas and Skau: This is what you call functoriality?

Langlands: Yes, this passing like that. So, this means that you can describe it by representations of a group. So, this is the same thing; something similar is happening over on the algebraic/geometric side. And there it is another group; it is the group defined in a similar way and that is the group of Grothendieck and its motive. And when you have the two, you can do all the analytic continuation you want and what you get is, of course, something for your great-grandchildren to discover.

Dundas and Skau: It seems like a very naive question, and it is, but let's ask it anyway. Why is it so crucial to analytically or meromorphically continue the L -functions?

Langlands: Why is that so crucial? That is a good question. Why is it so crucial to know anything about the zeta-function? Where do you go? In other words, you go for an estimate of the number of solutions and things like that. What do you do with the information you have about the zeta-function? And what would you do if you have all the possible information? Do you have an answer?

Dundas and Skau: No, we don't.

Langlands: Neither do I but I think, in both cases, it is that we haven't worked with it in the right area.

Dundas and Skau: Of course, we know that the classical zeta-function tells us something about prime numbers and their distribution. And Dirichlet's L -functions tell us something about prime numbers in arithmetic progressions.

Langlands: So, you get that kind of information but...

It is clear that it is what people are hoping for. But you can ask: why do they want it? Only God knows. So, you're

pushed by preconceptions and you're trapped in the way you think mathematics should work.

Dundas and Skau: In 2009, the so-called Fundamental Lemma, conjectured by you in 1983, was proved by Ngô. He was awarded the Fields Medal in 2010 for this. Time Magazine selected Ngô's proof as one of the Top Ten Scientific Discoveries of 2009.

Langlands: You can cancel your subscription to *Time Magazine*!

Dundas and Skau: In a joint paper from 2010 titled "*Formule des traces et fonctorialité*," the authors being you, Ngô and Frenkel, the very first sentence—translated into English—reads: "One of us, Langlands, encouraged by the work of one of us, Ngô, on the Fundamental Lemma, whose lack of proof during more than two decades was an obstacle for a number of reasons for making serious progress on the analytic theory of automorphic forms, has sketched a programme to establish functoriality—one of the two principal objects of this theory." Any comments?

Langlands: The Fundamental Lemma is needed to deal with a specific kind of technical question. Let's see if I can make it clear. This is not a good example but I'll try to explain something. Say you have something such as the group SL_n and you have SU_n . You know by Weyl's theory about the representations of SU_n . Those are basically the standard finite dimensional representations of this group. Now, look at the SL_n situation; SL_n has more representations than SU_n . SL_n is a non-compact group; it has a lot of representations. But, in particular, it has some things that are very much like those of SU_n ; the characters are basically the same. For example, you know the characters of SU_2 .

Now, let's go to SL_2 . By Harish-Chandra's theory—actually, SL_2 is prior to Harish-Chandra—you have corresponding representations. In this whole theory of representations of semi-simple groups or reductive groups, and therefore the theory of automorphic forms, and therefore the whole theory, what happens for SL_2 , for example—those things where there is only one? I mean, you know SU_2 , where there is only one representation in each dimension. Each one has basically something corresponding for SL_2 , the so-called discrete series, and at each end, it has two. It is just this one place where this unitary group becomes two for SL_2 .

These two are, for all practical purposes, the same; they're just two pieces. Now, considering the Fundamental Lemma and what you have to do if you are worrying about the trace formula: you want some part that is really useful for, say, SL_2 and that's the part where you put these two together so they look like SU_2 . Then, there is a supplementary part where you have to take into account the fact that they don't occur with the same multiplicity so you have this extra stuff. So, if you want to handle the trace formula, you have to see what you want to compare. You have to say that SU_2 is more or less like SL_2 . So, you can compare the trace formula of the two but the extra bit over here is causing you trouble. And the reason is that somehow the one representation here breaks up into two representations there, and some

of it doesn't have much to do with things and it is just there. You just take the difference of the characters rather than the sum.

If you are going to use the trace formula, you have to understand the part you don't really want. And there is some mysterious endoscopy. What is the so-called Fundamental Lemma? It is a fundamental lemma in the context of the specialised theory that was introduced for this special feature where things that should be the same could sometimes differ. What you do is that you treat them all as if they were the same and put them together and then you take the difference. You have to treat those differences separately so they look like something coming from the torus itself, the circle group that is sitting in there. So, it is a technical necessity; if you want to compare the representations of two groups you use the trace formula, but this stuff, this extra stuff, you have to get it out, put it aside and treat it separately, so you can compare what is left. And then what matters is just to understand what you can compare on its own. That means that you have to understand the differences—you have to look at just the circle group, which is all that matters, and for that you need the Fundamental Lemma, and that's all. The Fundamental Lemma is the fundamental lemma for these technical properties. It's a whole theory for this; it's rather complex but it takes care of that.

Dundas and Skau: Functoriality is the most important part of the Langlands programme. And to make progress on functoriality you have said you think that the crucial tool is going to be the Selberg–Arthur trace formula. Why is the trace formula going to be so important?

Langlands: Well, what do you want to show? You want to show that you can transfer everything to GL_n , basically. Let's put this somewhat differently. You want to show that you can move automorphic forms from one group to another. This is something you want to use the trace formula for: you compare the two trace formulae, right?

You want to move things from the group G to the group G' . You want to be able, in particular, to handle the L -function, so you want to be able to move to GL_n . These things work at the level of the L -group but let's just work with GL_n , so we don't have to worry about that. So, how are we going to do it?

You say here is this group; every time I have a homomorphism of the group—really of the L -group—from one to the other then I have a transfer representation. This means that every representation is obtained by transfer; it is a natural transfer. You can see this if you see the distribution of conjugacy classes. So, what would you do? There is, so to speak, a smallest place, a smallest group where it sits and then it propagates to the other groups.

For example, say, you have one group G that you want to understand. So, you say here is the smaller group, so it has to be the contribution of those things that sort of sit inside the bigger groups in that smaller group, so one-away, one-away you do it all along, moving from the larger to the



Figure 4. From left to right: Bjørn Ian Dundas, Christian Skau, and Robert P. Langlands.

smaller. You look at the trace formula here and you look at the trace formula there, and they cancel. In other words, you come from one place and you look to see what it cancels—it cancels something—and you go along and along and along and you know you understand it. Ultimately, the real building blocks are those things in the big group that come from the trivial group. So, the last stage is to analyse those. I take the small group and I want to send it to the big group and I just have to look: I take the trace formula up here and it cancels everything I know from this. It just cancels everything; I said it should be made up by pieces and each should come from smaller groups and this just comes from the smaller group, and this comes from the smaller group, and then I have to be careful because it can come from a bigger group and from a smaller group, and I have to be careful so I don't count it twice. So, I say they should be equal. I have to have a clear view of the combinatorics. Everything comes from a smaller group and some of it comes from two smaller groups and some is coming from three and so on. This depends on the image group. So, to show that this is really true, I just show that somehow the trace formula gives the same up here as it does for something in the selection of the various groups. This is pretty vague but in principle it is not so bad. And this is how it works but up until now at a very low level.

Dundas and Skau: So, that is at the forefront of your investigation?

Langlands: I mean, that is at the forefront of Arthur's investigation. I think if you want to hear what is available along these lines, you have to ask Arthur.

Dundas and Skau: We understand you are currently thinking in more differential geometric terms?

Langlands: I was thinking about the geometric theory and the geometric theory is not the trace formula, right? The geometric theory is basically Yang–Mills theory.

There are two papers—a brief one in English was premature and not entirely reliable. The other, which is longer and—so far as I know—reliable, is in Russian. This is already

an obstacle but it is also very difficult to understand, in part because very few people, perhaps no one, understands the connection with Yang–Mills as in the paper of Atiyah–Bott. I might be able to help you with further questions but I have had difficulties with one Russian speaker who, in spite of encouragement, still does not understand the basic idea of the paper. He is a well-regarded mathematician. So it appears that the paper is difficult. I am nonetheless confident that it is correct. You might ask around!

Dundas and Skau: That is a recent paper of yours that we can read?

Langlands: It can be found on the web.⁴

Dundas and Skau: In 1872, Felix Klein launched his famous Erlangen programme. To every geometry he associated an underlying group of symmetries. Klein stated in his autobiography that the Erlangen programme remained the greatest guiding principle, or “leitmotiv,” for his subsequent research. Do you see any analogy with the Langlands programme?

Langlands: I would hesitate to use the word programme but I think probably that leitmotiv is right. In other words, you have these two somewhat surprising structures on both sides: groups moving from one side to another. You have one side that is arithmetic and the other side that is analytic (or geometric, depending upon your view). So, you move around and you know that everything can go to GL_n , and with GL_n you have this one example of an Euler-product that you can analytically continue. These things give you a very clear focus—or leitmotiv if you like—on what one should try to achieve.

Dundas and Skau: In 2016, we interviewed Andrew Wiles, who was awarded the Abel Prize for his proof of the modularity theorem for semistable elliptic curves, from which the Fermat theorem follows. The modularity theorem fits into the Langlands programme and Wiles expressed the sentiment that its central importance in mathematics lent him courage: one simply could not ignore it—it would have to be solved!

You propose a theory of mathematics that is rather encompassing: it is not a particular thing; it is a structural thing. What are your comments on this?

Langlands: I think what one is looking for is a structural thing. All of the particular instances are of interest. Or something like that. There is so much you just can't do that I hesitate to answer really. But, if you like, you have this one structure on the one side, the Diophantine equation, which is sort of embedded in one of the automorphic forms. Automorphic forms have a lot of intricate structure on their own, so you have a lot of information about the L -functions there that moves back here, i.e. to the Diophantine side, and that is usually what you want. But I am not a specialist in those things.

⁴publications.ias.edu/rpl/section/2659

Dundas and Skau: Both Harish-Chandra and Grothendieck—two mathematicians we know you admired—were engaged in constructing theories, not being satisfied with partial insights and partial solutions. Do you feel a strong affinity with their attitude?

Langlands: I greatly admired both of them and, incidentally, do not feel that I am at their level. Their impulses were, however, different. Grothendieck himself has described his own impulses. Harish-Chandra never did. He just went where the material led him. He abandoned the mathematics of his youth, as a student in India, on which he wrote many papers, and turned to the topic of his thesis with Dirac: representation theory, a theory that was gaining in popularity and depth when he came to the IAS with his advisor Dirac. He just went where it led him. In retrospect, he just went where his strength and ambition took him. Incidentally, his thesis was, in contrast to what followed, not very impressive.

Dundas and Skau: To what extent has it been important to you to be around people and in an environment where new ideas circulate?

Langlands: There were two people who made an absolute difference to my mathematical life. The first was Edward Nelson, whom I met basically by accident as a graduate student—I had come as a graduate student with a friend, who was an instructor at Yale, to the IAS to visit some of my friend's friends from his graduate-student days at Chicago, one of whom was Nelson. An incidental consequence of an informal conversation that day, during which we discussed mathematical matters of common interest, was that Nelson suggested to the Princeton mathematics department, where he was to begin teaching the following year, that I be offered a position as an instructor—no application, no documents, nothing.

The second is Salomon Bochner, who, after hearing me talk in an informal Princeton seminar, urged me to move from the rational number field to arbitrary number fields and to study the work of Hecke. He also recommended me to Selberg. As a consequence, I had my one and only mathematical conversation with Selberg. It was, of course, he who talked.

Harish-Chandra, too, made an enormous difference, principally because of his papers (these I read on my own initiative, many years before meeting him) but also because my appointment to the IAS was made—I suspect—at his initiative. I should also observe that it was a young Princeton colleague (although they were older than me) who directed me to Harish-Chandra's papers. So the answer to your question is certainly 'yes.' I owe a great deal to my education at UBC, where a very innocent young man, a boy if you like, was introduced to intellectual possibilities to which he has been attached all his life, and to Yale, where for two years he followed his own whims and where there were mathematicians who supported his independence. Whatever reservations I have about Princeton and its two

academic establishments, it is clear from the preceding remarks that I am indebted in a serious way to specific individuals who were attached to them.

Dundas and Skau: Perhaps before we conclude the interview, it might be interesting to hear whether you have private, non-mathematical passions or interests of some sort, e.g. music, literature, language or poetry?

Langlands: Passions? I don't have any passions. But, you know, it is true that you want to take a look at other things, you know. History is fascinating: modern history, ancient history, the Earth's history, the Universe's history—these things are all fascinating. It is a shame to go through life and not have spent some time contemplating on that—certainly not everything of course but just to think about it a little bit.

Dundas and Skau: On behalf of the Norwegian Mathematical Society and the European Mathematical Society, and the two of us, we would like to thank you for this very interesting interview, and again congratulations on the Abel Prize.

Langlands: Thanks for inviting me.



Bjørn Ian Dundas



Christian Skau

Credits

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EARLY CAREER

The **Early Career Section** is a new community project, featured here in the *Notices*. This column will provide information and suggestions for graduate students, job seekers, junior academics of all types, and those who mentor them. Angela Gibney serves as the editor of this section.

This month's theme is jobs in Business, Industry, and Government. Next month's theme will be Plan for a productive summer.

Transitioning from Academia to Industry? Here is some advice.

Carol E. Fan



Mathematicians are not always well understood by hiring managers within industry, particularly within Silicon Valley technology companies. Once you've made the decision to make the transition from academia to industry, here are a few things to keep in mind.

First, identify the types of positions for which to apply.

Within many technology companies, roles with titles such as "data analyst" or "data scientist" are a good place to start. People in these positions analyze data and produce recommendations. The techniques used in analyzing data can range from simple analysis of patterns and statistical tests of proportions, to non-linear optimization and neural networks. The recommendations can run the gamut from changing the color of a button on a webpage, to implementing an artificial intelligence algorithm that returns translated text in near real time.

The skills required for these roles vary from Excel macro building and SQL programming, to writing production-ready code that is ready for implementation in Python, R, or Java. Having a strong mathematical background facilitates quickly picking up new coding languages and domain knowledge. The vast majority of employers prefer that you have these skills in hand at the time of application; a few hiring managers may train you on the job, typically only for entry-level positions.

There are many types of problems that data analysts and data scientists work on. At a typical Silicon Valley company there is often a website and a product to sell to customers.

- *Marketing analysts* model the effectiveness of different marketing channels, e.g., direct mail or Facebook ads, in attracting paying customers. Optimizing digital marketing spending on social media platforms such as Facebook is a difficult and evolving problem to solve.

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- *Product analysts* improve a company's website or mobile app to ensure that customers can easily and intuitively purchase its product. The most effective companies are continually testing their website to make minor improvements in what is called an "agile" workflow.
- *Operational analysts* improve a company's ability to fulfill customer orders in a timely and cost-effective manner, often using operations research techniques such as optimization, queuing, network analysis, and inventory theory.
- *Business intelligence (BI)* analysts curate the source of truth for data and provide company-wide reporting for the purposes of better decision making. Their work is typically seen at the highest levels of the company.

Second, signal to employers that you are the best person for the job.

A typical job interview will consist of a technical test of coding skills. Sometimes "homework" will be assigned with a time limit. Then there will be several interviews with potential coworkers to assess whether or not your experience and skill set are a good match for the open position.

From the employer's point of view, there are two potential drawbacks to hiring academic mathematicians: lack of urgency and lack of business sense.

The stereotype of academics is that they will provide you with a solution to a problem only after they have thought through all of the possibilities, which could take weeks or months. Most businesses cannot wait that long. Make sure that you can provide examples of where you were able to complete an assignment in a short period of time and can explain what short cuts or trade-offs you would have taken if the deadline were one day versus one week away.

The more troublesome stereotype to dispel is the lack of business sense. At a business, all problems are seen through the lens of return on investment. How much effort is required to achieve the estimated benefit? If the decision is a minor one, a machine learning algorithm requiring months to validate and implement is unlikely to be necessary. Make sure that you can explain why you would choose one technique over another, given the time frame and potential benefit.

Aside from these two points, employers are typically looking for someone who can communicate. The person who can explain their methodology and findings to stakeholders from marketing to engineering, from the CEO to the warehouse worker, is highly valued.

Finally, network, network, network.

If you are making a transition, you are less likely to have connections in your new chosen area. Take advantage of alumni networks, friends of friends, and meet-ups—social networks such as LinkedIn and Facebook can introduce you to new contacts. Ask for an informational interview, or take someone to coffee. You just need one person to take a chance on you; it is up to you to find that person.



Carol E. Fan

Credits

Author photo is courtesy of Carol E. Fan.

Industrial Research in Applied Statistics

Taylor Arnold

Introduction

For almost a year, I sat in Washington DC's National Airport every Sunday waiting for my flight to Houston. I was 22 years old with an undergraduate degree in mathematics, now working in consulting for IBM. I followed physicians in area hospitals each week collecting data. In the evenings, I taught myself R, the statistical programming language, in order to organize and analyze the data so we could build models to predict the flow of patients through the hospital system. Each week, I found that the theories of mathematics I learned in school were insufficient to address all of the tasks I was assigned. My work involved structuring datasets, making our code run faster, and understanding the specific details of how hospitals function. Data constructed from the social interactions was anything but straightforward.

I decided I needed to learn more and headed to Yale University for a PhD in statistics. I chose the program because I wanted to develop my mathematics background while blending it with computational statistics. My doctoral dissertation concerned the computational challenges of applying a certain class of statistical models to estimate forms of structural dependence in datasets with a large number of variables. As a result of this study, toward the end of my time in graduate school I became broadly interested in how methods from exploratory data analysis, data visualization, and statistical learning could be applied to very large datasets. My mentors in graduate school provided me with fundamental skills in statistical computing

for structuring and writing efficient code. I felt, however, a substantial disconnect between the data and problems I was working with in an academic setting and the data that statisticians typically work with in industry applications. Most of the work in academic statistics concerns the probabilistic modeling of data, with a particular focus on the estimation of unknown population parameters. In practice, often much more time is spent acquiring, structuring, and visualizing data.

With a desire to learn about the real challenges of applying computational statistics to large, messy data sets, I took a position as a research statistician at Travelers Insurance. In that role I applied machine learning algorithms to the task of predicting fraud and the price of future insurance claims. Two years later, I became a senior member of the technical staff at AT&T Labs Research in New York City where I focused on location analytics using cell-phone telemetry data.

In the sections that follow, I explain some of the research questions I worked on during my time in these two positions and how the skills I learned in graduate school prepared me to address them. I focus on what made these questions—and experiences in general—particularly interesting. I also include a discussion of some critical drawbacks of working in an industry lab. I conclude with my own vision for mutually beneficial partnerships between industry labs and academic researchers.

Travelers Research and Development

My first job out of graduate school was on the Travelers' research and development team responsible for personal automobile insurance. The large number of messy data sets and unsolved problems that required new innovative approaches made the position especially appealing to me. I found building models within the insurance industry

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particularly rewarding. I built predictive algorithms that Travelers actually used to make real decisions about what policies to write and how much to charge for them.

While I worked on a number of interesting problems, the majority of my time was spent constructing *pure premium* models. These models predict, using historic data, the expected amount of money that would be paid out on a particular automobile policy. Actuarial and sales teams incorporate overhead and market-based adjustments to pure premium models in order to arrive at the final rates that are actually charged to consumers for an insurance policy. Several machine learning competitions have featured anonymized datasets with the goal of predicting pure premium values [1, 11]. These competitions, however, obscure the most interesting features of building pricing models. Here, I describe three particularly challenging research questions that underly the construction of premium models within the insurance industry.

The distribution of observed pure premiums makes it difficult to apply many standard statistical techniques without some modifications. Most policies do not receive any claims and have an observed pure premium cost of zero. When claims are made, the amount of money requested has a property known in statistics as a “heavy tail”: a small number of insurance claims require an extremely large payout. These large costs primarily come from extensive medical expenses as a result of automobile accidents. The general distribution of premiums, therefore, should be modeled by a mixed distribution with a discrete mass at zero and a continuous distribution on the positive real numbers. It is possible to split a premium model into separate frequency (the discrete part, predicting whether the policy will have a claim) and severity (the continuous part, predicting how large a claim will be) components. However, splitting the model this way ignores important correlations between frequency and severity. A better alternative is to use a Tweedie distribution, which arises from assuming that the number of claims made on a policy follows a Poisson distribution and the amount of any given claim is distributed with a gamma distribution [12]. Software exists for fitting a generalized linear model where the dependent variable has a Tweedie distribution. (Figure 1 shows simulated values from three Tweedie models with varying dispersion parameters [6].) Interesting research questions arose whenever we wanted to use a new approach or statistical method in our pure premium models. For example, we wanted to incorporate constraints into our models to reduce the number of variables used in the final output. Implementing constrained models required new mathematical derivations and software implementations. Since estimating parameters in the Tweedie model can become numerically unstable, in addition to demanding a significant amount of computational power, these implementations required careful thought and nontrivial extensions of currently available algorithms.

Automobile pure premium models are typically constructed to estimate the cost of insuring a particular automobile. Variables used in this calculation may come from features of the automobile itself (e.g., cost, make, age, and safety features) or from details of the specific policy (e.g., zip code, deductibles, miles driven per year). Some particularly powerful features are also associated directly with the individual drivers on a policy. Examples of predictive driver-level features include credit histories, ages, number of prior claims, and the number of prior traffic violations. The challenge becomes how to summarize driver-level variables at the level of a particular automobile. Should we construct a variable equal to the average age of all drivers? Could we create variables for the minimum and maximum age of all drivers on a policy? Or should we count the number of drivers below some age threshold? Any of these new features could be computed for a policy and used in the pricing algorithm. A choice of how to create these aggregated features must be made for dozens of driver-level variables, with the typical trade-offs between variance and bias when including too many or too few correlated variables into a single model. The challenge of summarizing predictive variables at the level of an observed response, a particular example of feature engineering, is a frequent challenge in industry applications. I believe this is one of the single biggest challenges in applied machine learning that is largely overlooked within academic research.

Another important challenge in deriving pure premium models is ensuring that models conform to various government regulations. In the United States, automobile insurance is regulated at the state level, and each of the fifty states has its own set of rules. Credit information, for example, is not an allowed predictor variable for pricing policies in Massachusetts. In New Jersey, only a limited number of geographic regions can be defined for pricing and discount purposes. Many states allow insurers to use the age of drivers in pricing models but require that aging can only decrease prices and never increase them. Building models that follow these regulations, while retaining most of their predictive properties, was a constant challenge within the research and development group at Travelers.

The research problems I encountered at Travelers point to two takeaways about graduate education in statistics. We need more statisticians in industry who have the training and interest to conduct original, open-ended research. Many of the most interesting and beneficial projects could not be solved with off-the-shelf statistics tools. They require experience with graduate-level statistical theory as well as general skills in conducting original research. At the same time, we need graduate programs in statistics to include more training in computer science and the empirical social sciences. Computer science and engineering courses can provide skills for writing efficient code to deal with larger datasets, understanding how to implement new estimation algorithms, and knowing the principles of building data-

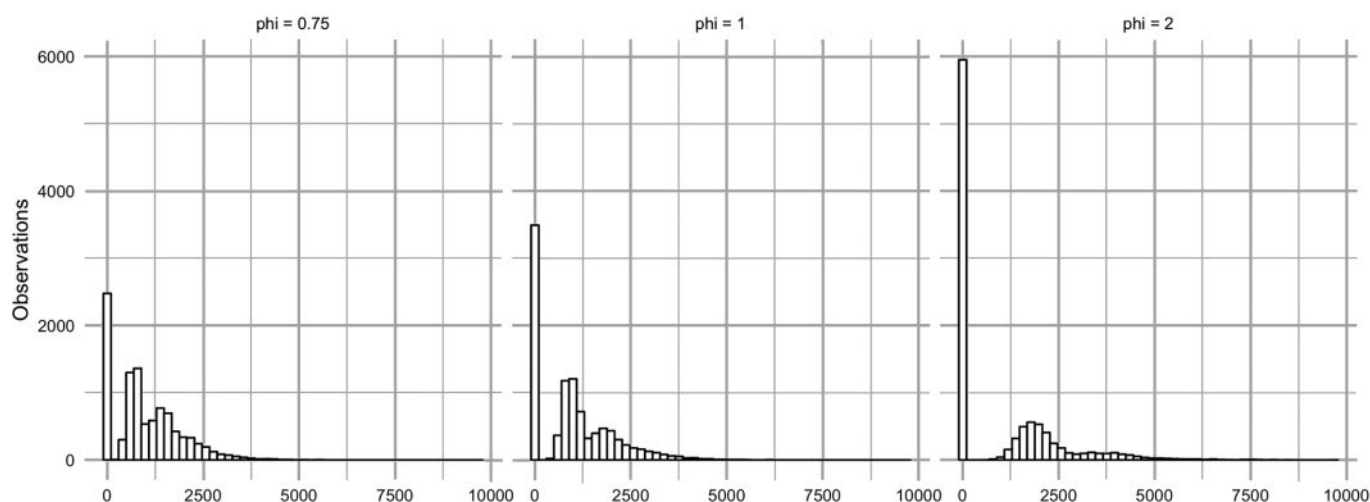


Figure 1. A histogram of simulated random draws from a Tweedie distribution for three different dispersion parameters. When modelling insurance premiums, high dispersion values are used to describe policies that incur claims on only a small percentage of policies.

bases, and experience writing and testing code that may be used in production. Social science applications give experience with the techniques and challenges of using data and models to understand human behavior. They also are more likely to explain the political and legal challenges that may underly the collection of data or deployment of empirically trained models.

AT&T Labs Research

In April 2014, I transitioned to the statistics department at AT&T Labs Research. The group has a long history of exceptional work in the field of applied and computational statistics and traces its roots back to the original Bell Labs [7]. Rick Becker was one of the three original authors of the S language, the precursor to the popular R programming language for statistical computing, which was developed at AT&T in the 1980s [4]. Simon Urbanek is one of the small set of core developers of the current R-Project. Chris Volinski and Robert Bell were both on the winning team for the million-dollar Netflix movie recommendation competition [5]. A large draw for my move to AT&T was the chance to work with these and other fantastic scholars in the field of computational statistics.

Another motivation for my interest in working at AT&T was the desire to work with extremely large datasets, a continuation of my graduate school research. My world-class colleagues at the labs in a range of fields gave me the opportunity to work collaboratively on new research questions and to keep learning about new areas. My group focused on cellphone location analytics, which required working with large data sources. Our primary dataset was built from observations known as call detail records, or CDRs. A CDR is generated whenever there is an interaction between a cellphone and cell tower. CDRs can include a cel-

lular voice call, a text message, or the transferring of generic data. With the widespread coverage of 4G networks and the proliferation of cellphone applications, most cellphones today are involved in a nearly constant stream of CDRs that cover the majority of the day [9]. By associating each cell tower in a CDR with its location, these records make it possible to determine approximately where a device is at any given moment in time (see Figure 2 for an example) [13]. This data has been widely used as legal evidence and was recently employed to assist aid workers helping with the West African Ebola virus epidemic from 2013–2016 [14].

The location analytics data that I worked with was so large that it needed to be distributed over hundreds of machines. Overall, the data I had access to amounted to several petabytes (1000 terabytes) and took days to process even over our large cluster. Given my expertise, I was tasked with building a data pipeline from scratch that ingested the raw CDR records and produced a normalized database of each observed device's location—a daunting but exciting challenge. In order to work with data stored over a large distributed system, I had to learn two new frameworks (Hadoop and HBase) and learn how to write code for them in a programming language I was not very familiar with (Java). Because data arrived hourly and needed to be processed immediately, my data pipeline needed to automatically run throughout the day. In applied statistics we are often reminded and taught how to interactively check whether there are potential issues in a data source. With the system I was building, it was important to build in automated tests that would check new data as it came in. This was necessary because there were frequent upstream data issues with the raw CDR files that were being delivered. For example, all of the data from a particular city for six hours in a day might go missing due to an internal networking

issue. Or the format of a field would occasionally change and cause some of the code to break. The completed data pipeline opened up many research questions for our team. Quick access to small selections of the corpus (through the distributed database) allowed for exploratory analysis that allowed us to start thinking critically about what the data was able to show. For example, we found that using the location data was great for detecting movement along highways and public transit routes. It was less useful, however, in the accurate detection of static devices.

Once the location data was cleaned and stored on our research servers, we created tools for modeling and visualizing the data. Mike Kane, Simon Urbanek, and I built a set of tools in R for working with large distributed datasets [3]. These functions focused on being able to process a fixed number of lines of data, allowing for chunk-wise operations on large datasets. Using these tools, we developed a distributed algorithm that allows for applying penalized regression to arbitrarily large datasets. Our work on this problem eventually led to a textbook focused on

the computational details of working at scale with large distributed datasets [2]. I also worked on integrating a new spatio-temporal visualization algorithm known as *nano-cubes* into an R package [10]. This allowed our researchers at AT&T Labs to easily explore small subsets of our data within their browsers.

During my time at AT&T Labs, I had the chance to develop new software and study approaches for working with extremely large datasets. Doing research at the Labs gave me expertise in the modeling and management of large datasets at a scale that would have been nearly impossible to work with in academia. My experience in an industry lab, in short, offered educational opportunities beyond what was available within a formal graduate program.

Drawbacks

Positions in industry labs are not without their own unique issues. For example, in an industry position there is a complete lack of personal ownership over ideas, work, and software. Projects that take months or years of work

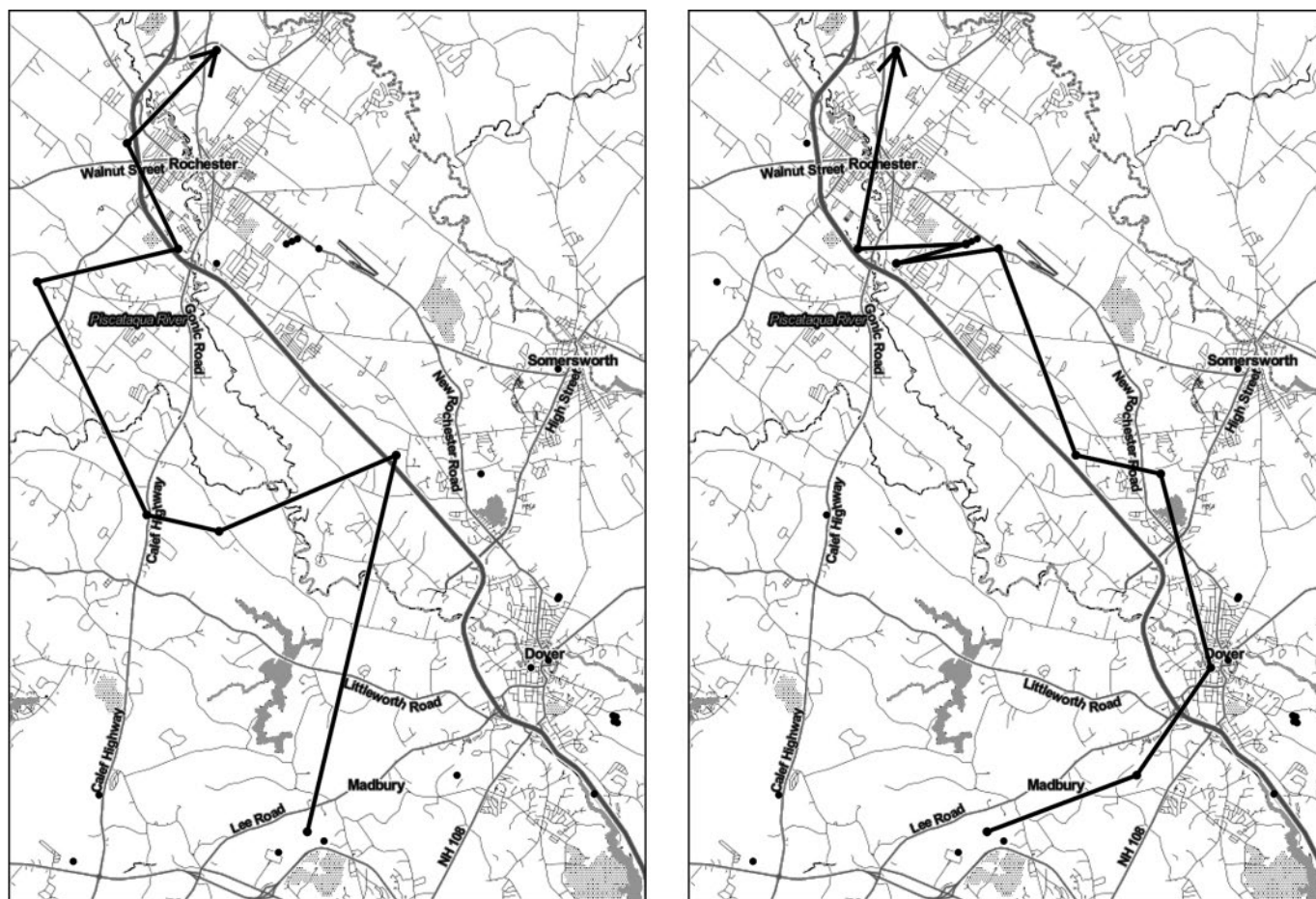


Figure 2. Maps show registered cell phone towers (solid dots) in the vicinity of Rochester, New Hampshire. Each path describes an artificial collection of towers that sequentially handle cell phone traffic for a fictional driver commuting from Madbury to Rochester. In the left panel, the driver takes a sequence of smaller roads—Littleworth, Calef Highway, and Gonic Road. The right shows an alternative path that travels by Route 16 (thick grey line).

often result in no tangible outcomes that are seen outside of the company. Business concerns may force researchers to abandon interesting lines of work in favor of other tasks.

I engaged in a wide array of interesting research projects at Travelers and AT&T. Unfortunately, almost none of this work is publicly available. Industry labs typically forbid the publishing of research that uses internal data; without the datasets as examples, most of the methodological innovations made in my work were hard to motivate or even explain.¹ At Travelers we were not even allowed to publish software that we had built. AT&T Labs, with its long tradition in computing, was more willing to allow the publication of software. The two papers I have from my time there both focus on specific software libraries we built. However, even this type of publication is increasingly rare.

Another concern I had while employed within an industry research lab was whether my work was being used in ethical and appropriate ways. Take, for example, the cellphone location analytics projects. All of the applications I directly worked on were either banal internal studies, such as testing network dead spots, or external consulting projects that made use of highly aggregated tabulations to show the general movement of people through space for urban planning purposes. However, there was no way for me to stop, or to even be aware of, my code being used for more objectionable applications. These concerns may also extend to all publicly available research. When publishing method papers or open source software, there is also no way to ensure that derivative work is being used responsibly. But, at least in the publicly available case, the research is not being internally motivated or funded by these applications. Also, I believe that the net benefit of publicly available research generally outweighs the concerns of misuse. The potential for abuse is harder to justify with research that is never made externally available.

After two and half years at AT&T, I left to join the faculty at the University of Richmond. The year prior to my departure, I taught two courses as a part-time lecturer at Yale University. This experience reignited my passion for teaching and convinced me that I wanted to make that a permanent part of my work. I also wanted the opportunity to make more of my research public in order to get external feedback and to see my methods and code made usable in other domains.

Academia and Future Directions

As I have transitioned back into academia, my experience in industry continues to shape my approach to research and teaching. For example, I no longer see a sharp divide between my work as a researcher and an educator. Teaching students how to work with messy, unstructured data makes

me reflect on how to turn my individual applied projects into larger-scale methodological frameworks. Similarly, watching students use and struggle with existing software libraries helps me to understand the shortcomings in available tools and how they can be addressed.

One challenge of leaving industry labs has been finding ways to continue my scholarship in large-scale statistical computing without access to industrial datasets. Currently, I have solved this problem by finding publicly available datasets that share common traits with those seen in industrial applications. For example, I have a current project that involves working with the entire corpus of page histories from Wikipedia, which amounts to several terabytes of textual data. The size of the corpus and complexities of dealing with the data address many of the same challenges I faced working with CDRs at AT&T Labs. I am involved in another project that uses computer vision techniques to extract features from video files. While the raw features are associated with individual frames, the predictive modeling tasks I am interested in—such as scene detection and character movement—require building models for sequences of images. The challenges here mirror the issues of aggregating driver-level data to a particular automobile that I faced at Travelers.

My experience in industry has impacted my own teaching philosophy. Across all of my classes, my ultimate goal is to help students develop the skills needed to engage in the ethical and insightful analysis of data. For me this means that I need to teach the entire pipeline of working with data instead of focusing only on probabilistic modeling. In my introductory courses we spend a lot of time talking about how to correctly structure data in a spreadsheet. We also spend several weeks working on how to interpret statistical visualizations in both written and oral formats. In my courses on data science, students learn how to fetch data through APIs and spend several weeks building interactive websites with Javascript. These experiences improve their ability to present useful data visualizations as well as make them comfortable working in new programming languages and approaching tasks outside their typical comfort zone.

I found my experiences in industrial research labs both rewarding and generally enjoyable. At the same time, I also understand the difficulties of life in an industry lab and appreciate the relative freedoms afforded by a position in academia. Some of the most influential scholars in my own work have had similar histories that intersect between industry and academic positions, including John Tukey (who split his time between Princeton and AT&T Labs), danah boyd (a researcher at Microsoft with an ongoing position at NYU), Yann LeCun (a computer science professor at NYU and director of research at Facebook), and Hadley Wickham (RStudio and Rice University). These scholars have produced some of the most important work in applied statistics. Hadley Wickham's triptych of papers and associated software for applied data analysis—"A

¹Prior to my time at AT&T there were more opportunities to publish data-driven research. See the paper [8] by my colleagues for a great example that illustrates the nature of the internal projects we worked on.

layered grammar of graphics” [15], “Tidy Data” [17], and “The split-apply-combine strategy for data analysis” [16]—have been highly influential, for example, in my own work. I hope to see more direct partnerships where academic faculty can participate in research with industry labs. These exchanges have the benefit of bringing to light many understudied problems in applied statistics. It also provides an external source for critically reviewing the ways data are being used in industry and its potential effects on society as a whole.

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Taylor Arnold

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Careers at National Laboratories

Emilie Purvine

It's common for math majors and professors to be aware of possible careers outside of teaching, especially jobs like data science, finance, actuarial science, and some government agencies. But from my experience talking with undergraduates, graduate students, and professors in mathematics, careers at National Laboratories are not as well known. It's my goal in this article to give you some background and history of the national lab system, a glimpse into the work mathematicians do, and, finally, some insight into getting a job at a national lab.

The national lab system grew out of a large investment by the US Government in scientific research during World War II. One of the more prominent efforts was the Manhattan Project, which established sites in Los Alamos (New Mexico), Oak Ridge (Tennessee), and Hanford (Washington), with the purpose of research and development for nuclear material and weapons manufacturing. These three sites ultimately led to the creation of Los Alamos National Laboratory and Sandia National Laboratory in New Mexico, Oak Ridge National Laboratory in Tennessee, and Pacific Northwest National Laboratory in Washington. Additional research in the Chicago, Illinois area in reactor technologies led to the creation of Argonne National Laboratory, and a push for competition created Lawrence Livermore National Laboratory in Livermore (California). There are now seventeen Department of Energy National Laboratories all across the country engaging in multidisciplinary research relevant to today's scientific challenges. At Pacific Northwest National Laboratory

(PNNL), for example, we focus on earth and biological science, physical and computational science, energy and environment, and national security.

There is not just one role for a mathematician in the national lab system. We do theoretical math research in areas like category theory, differential equations, graph theory, optimization, and operations research tackling applications in scalable algorithm development, sensor design, biology, chemistry, cyber security, power grid, machine learning, and more. We have the ability to participate in the full project life cycle, from proposal development, project planning, and execution all the way to software development and deployment. If that all sounds very overwhelming, don't worry, it's not like that from day one. As an early-career mathematician, you would be brought in on one or two projects and be expected to utilize your specific skills to contribute to project deliverables. As time goes on and your network develops, those skills will be recognized by others. You may be asked to join other projects, help write sections of proposals, and get more responsibility as you gain experience.

If a career at a national lab sounds interesting, there are some things you can do while still in school in order to be more successful. Because a lot of work at a lab deals with real data and implementation of algorithms, courses in statistics, data science, and programming are very helpful. Additionally, given that all of our research ultimately is applied, it's good to have a breadth of knowledge beyond mathematics. Take classes like physics, chemistry, biology, or economics in order to build up knowledge and vocabulary outside of math. Most of our teams are interdisciplinary and rely on communication with non-mathematician colleagues, including management. Classes in technical writing and presentation skills can give you a head start in these areas. Finally, the best way to really know what it's

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like to work at a national lab is to do an internship. Each lab has its own internship programs, which can typically be found in the jobs section on its website. For a complete listing of the national labs with links to their home pages, see <https://www.energy.gov/national-laboratories>.



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Author photo is courtesy of PNNL.

One Year in Washington Opens up a Whole New World of Possibility

Jennifer Pearl

Like most sectors, the federal government needs good mathematicians. Mathematicians and statisticians are in high demand in Washington—now more than ever.

I speak from experience: I am a mathematician who heads the AAAS Science & Technology Policy Fellowships (STPF) program and was a fellow myself. Mathematicians like us understand the difference between causation and correlation, and we bring a skeptic's mindset to the table, which helps ensure a sound basis for policy decisions.

The STPF program is the perfect opportunity for mathematicians to be on the front line of vital issues that impact society with fellowship assignments in federal agencies, on Capitol Hill, and in the judicial branch. Fellows are outstanding mathematicians, statisticians, scientists, and engineers at any career stage—from newly minted PhDs to seasoned professionals—who learn first-hand about policymaking while contributing their STEM mindset to American government.

The yearlong fellowship runs annually from September through August with a class of more than 250 fellows who represent a broad range of backgrounds and disciplines. AMS is among more than 30 partner scientific societies that sponsor fellowship placements in Congress. AAAS sponsors numerous placements in more than 18 executive branch agencies each year. Engaging with policymakers, adminis-

trators, and thought leaders, fellows make valuable contacts and broaden their career paths. The goals of the program are two-fold: (1) to provide hands-on professional development to fellows that shows them how policy works and how their scientific (mathematical) training can be used in the federal sector, and (2) to provide more analytical/scientific expertise to the government.

Across the federal government, fellows work on many issues and contribute in numerous ways. It is almost impossible to outline a typical fellowship experience. Karen Saxe, AMS Associate Executive Director, was a 2013–2014 AMS Congressional Fellow. She interviewed several fellows who served in the executive branch, highlighting how their math backgrounds were instrumental to their fellowships. Read about mathematicians from the STPF 2017–18 class—Tyler Kloefkorn (National Science Foundation), Kyle Novak (US Agency for International Development), Jessica M. Libertini (Department of Defense), and Chris Leary (US Agency for International Development)—at <https://bit.ly/2C9Yrej>.

Policy fellows with the STPF program, running strong since 1973, benefit greatly from the considerable expertise of its staff. These AAAS staff understand the needs of host offices, solicit position descriptions from agencies, find placements for fellows, and help manage fellows' relationships with their host offices. Applicants develop a personal statement describing their interests, and then finalists go through a matching process where placements are determined by mutual agreement between the fellow, agency host, and AAAS staff.

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The fellowship was an eye-opening pivot point in my career. It's one of the few things that will allow you to fully grasp and enter into the wide breadth of work that is done at the federal level.

After the fellowship, fellows become members of a strong corps of 3,000+ alumni who are policy-savvy STEM leaders in academia, government, industry, and the non-profit arena. STPF alumni with a mathematics background have leveraged their fellowship experience in many ways. Margaret Callahan, an applied mathematician, served as an AMS Congressional Fellow and is now an executive branch fellow at the Department of State. In an article in the January 2019 *Notices*, she stated, "Before starting the fellowship, in part because of the current political climate, many people expressed skepticism about the experience I was likely to have and what I would be able to realistically contribute. However, through this experience—in particular, in working with some of the smartest, most dedicated and hardworking people in my Senate office—I have become, if anything, less cynical about the work that is done on Capitol Hill. Most people I have met here are motivated by an honest desire to serve their country and to improve peoples' lives. The experience has been humbling and inspiring and I have learned more than I ever dared hope."

Other STPF mathematician alumni include Karoline Pershell, who directs strategy and evaluation at a tech company and is Executive Director of the Association for Women in Mathematics; Carla Cotwright-Williams, who is a scientist at the US Department of Defense; and Edgar Fuller, who recently took a position at the Florida International University as the associate director of the STEM Transformation Institute. Another STPF alumnus is D. J. Patil, a mathematician who went on to positions in academics and in industry and was appointed the nation's first-ever chief data scientist in 2015.

I can't think of another career move that you can make that parallels the depth and breadth of benefits you can gain from one short year as a AAAS policy fellow. Interested in jumpstarting your career? Want to contribute to the making of good policy? Learn more about STPF fellowships at <https://bit.ly/2Ahkop9>. Watch a video series on how to apply at <https://bit.ly/2LDB9zs>.



Jennifer Pearl

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Author photo is courtesy of Kat Song, AAAS.

It's a Big World

Jesse Johnson

Throughout undergraduate and graduate school, a postdoc, and my first few years in a tenure track position, I had never considered anything other than a career in academic mathematics. The reasons for my eventual decision to join Google as a software engineer are complex, but the process taught me a great deal about both the professional world outside academia and by contrast about the quirks of academia that were biasing how I approached it. Academia is a small world, and when you spend all your time in a small world, it's easy to ignore or forget just how big a big world can be. There are a number of things that are easy and seamless in a small world that you need to be much more intentional about in a big world. In this essay, I want to describe three particular ways in which academic math is a small world and how non-academic job seekers need to adjust to compensate for the larger world.

There's only one job in academic math

When I've talked to grad students and postdocs who are looking at non-academic careers, the biggest trap I've seen them fall into is to fixate on the first interesting-sounding direction, or the job that one friend does and seems happy doing. In academic math we don't think much about different career tracks because there's essentially just one track with different amounts of energy devoted to teaching, research, and service. There are differences between teaching at a community college, a liberal arts college, or a research university. There are differences between being a grad student, a lecturer, a postdoc, or a tenure-track professor. There are differences in how faculty members interact within different departments and varying degrees of function or disfunction. But these differences are all variations

on a single family of jobs, a family that looks very narrow when viewed from outside.

In the private sector, there is an unlimited variety of jobs, team/company dynamics, and career paths. Even the same job title at two different companies can mean vastly different things. Recognizing all the ways that jobs and careers can vary gives you the opportunity to evaluate what you really want, and recognize that what you want can change over time. What balance between work and personal/family time do you want? Do you want to be able to do more focused technical work or to influence the high-level direction of a team? How directly do you want to control the external impact of your work?

Questions like these are good to ask yourself regularly. There's flexibility along all these axes within an academic career, but the narrow range makes it easier to assess and compare different options. To do a comparable assessment outside of academia, you would need to investigate an order of magnitude more options. But coming from academic math where you're used to thinking about a narrow family of jobs, it's easy to think you've explored all the relevant options before you get to the one that best suits you. So while you can never explore all the options outside academia, it's important to be intentional about widening your search as much as possible before you begin to narrow it down.

Three degrees of separation

It is an exaggeration to say that everyone in academic mathematics knows everyone else, but only a slight exaggeration. By the time you've been working in a field for a few years, you will have met many of the mathematicians in that field at conferences, and read papers by most of the rest. There's probably someone in your own department who knows someone at any other university in the country, and their acquaintance will know all of the faculty and most of the graduate students there. So even if you have a high Erdős

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number, if you go by who knows whom you can probably get to any other academic mathematician in three jumps.

Many people inside and outside of academia find the idea of networking distasteful because they think of it as trading favors with people you only pretend to like. But it really means making personal connections that make it easier to communicate. Whenever you meet other mathematicians and tell them about your latest theorem, you're networking. All the lemmas you hear that haven't been published yet, the vague but valuable intuition that never will be published, come from networking. The academic math community is such a dense network that we barely recognize it as one.

Outside academia, social networks are similarly the most effective means of learning information that hasn't been or never will be written down. People outside academia enjoy sharing what they know just as much as we mathematicians do, particularly if they know it's helping someone else. The best way to learn about a particular career is to talk to someone doing it. The best way to learn about a company is to ask someone who works there. The best way to make a decision you've never faced before is to ask advice from someone who has. What you'll learn isn't secrets; it's information that large-scale communication is not suited to.

Social networks outside academia are large and sparse, which makes it harder to find the people with the information you need. In academic math, the very nature of doing mathematics puts you in contact with the people who know what you will need to learn. And the ones you haven't met have their email addresses on easily findable department web pages. Outside of such a tightly knit network, you have to be much more intentional about meeting and keeping in touch with people who can share the information you'll need, or connect you to others who can. That's why non-academics carry business cards and send LinkedIn invitations.

One factor that has been identified as contributing to a lack of diversity in many fields is that people tend to default to networking with others with similar backgrounds. Because these networks affect the flow of information about careers and job opportunities, as well as occasional favors, they reinforce disproportionate representation. By being intentional about how you network, you can fight these tendencies and look for more diverse connections.

It may feel odd asking an acquaintance to introduce you to someone they know, then asking that person to answer some questions about their experience. But outside academia those sorts of requests are not uncommon, because it's the best way to share and spread information. And as you gain experience, you can do the same for others.

In academia, you are your expertise

Academic departments typically hire for one or a small number of positions at a time, which means explicitly comparing applicants against one another. So to get an academic job, particularly a tenure-track job, it isn't enough

to be a competent, qualified mathematician; you need to stand out compared to all the other applicants based on your expertise in research and/or teaching. And because faculty jobs often have implicit or explicit restrictions on research field, the subject of your research can have as much or more impact than the quality of your papers.

This emphasis on expertise and direct competition makes it difficult to distinguish your work from yourself, which in turn makes work/life balance much harder to maintain. Outside academia there are some jobs that have similar dynamics, but there are also many that don't. In software engineering and data science, for example, the number of open positions typically far outpaces the number of applicants that meet the minimum bar that companies are looking for. This minimum bar is very high, but competing against even a high bar is very different from competing against other applicants.

Many non-academic jobs involve working on a team in which responsibilities rotate between multiple people who are all capable of doing the work. Anyone on the team can take a sick day or go on parental leave without shutting everything down, and being the sole expert on something isn't necessary, or even desired.

If you're switching to a non-academic job, you won't initially be an expert in that job since you've never done it before. Just like when you started graduate school, starting a non-academic job means being overwhelmed with a seemingly endless body of information that you need to learn in a hurry. Employers want to know that you're ready for this, so the ability to learn and adapt will be much more important than knowledge you already have.

This is one area where having gone to graduate school in any field is an advantage; it shows that you can handle the situation. However, you also have to fight your instinct to play the expert. You can't learn something new until you admit you don't know it, and you won't get hired if the employer thinks you haven't figured this out. You should learn as much as you can about a job before you apply, but there will always be things you can only learn from experience. Because you're competing against a bar rather than other applicants, you'll often be judged by your potential to learn rather than what you know today. So when you're applying for a non-academic job, it's important to be completely transparent about what you don't know, and how interested you are to learn.

From Research Mathematician to Quantitative Researcher

Ursula Gritsch and Melissa Yeung

Ursula Gritsch (UG) and Melissa Yeung (MY) are Quantitative Researchers on BlackRock's Systematic Active Equity team, which boasts a more than 30-year track record of combining human expertise and innovative technology in pursuit of broad market diversification and consistent, differentiated returns.

How did you end up in finance?

UG: After moving to Berkeley, California with my family, I decided that the loneliness of academic mathematics was not for me. I taught math courses at Cal for a year, then found a job at a tiny financial software company. It was hard at first. I had never really programmed, and suddenly, I was writing and maintaining professional code and implementing Black-Scholes style options pricing software. But I fell in love with writing high-quality mathematical software; it wasn't good enough that the code be right most of the time: it had to be right all the time because clients were making trades based on my calculations. I also had truly smart coworkers!

I eventually ended up at Barclays Global Investors, which is now a part of BlackRock. I have stayed at BlackRock for over ten years because BlackRock is first and foremost a fiduciary to our clients. Every math problem solved, and coding challenge surmounted helps us make better investment decisions for our clients. BlackRock also has a true

commitment to diversity and inclusion, which results in the variety of perspectives needed to be truly innovative.

MY: With the blessing of my graduate school advisor, on a whim, I spent a summer as a research intern on a trading and execution research team at a financial firm in Santa Monica. I was very fortunate; I found that I really enjoyed the work—I loved thinking about interesting, complex problems—I met wonderful mentors and sponsors who are very supportive and generous with their time and expertise, and I realized that I would be able to make a far greater impact in industry than I would in academia.

What kind of math do you do now?

UG: The first few years of my career, I priced energy derivatives and credit instruments using stochastic calculus. Now, I mostly work on numerical optimization and portfolio construction. Even though I don't prove theorems anymore, I am still guided by rigorous thinking and strive to understand whether a certain hypothesis or conjecture is backed by real world data.

MY: I don't prove theorems anymore either, but I still get to do all the things I love about math—rigorous problem solving, spirited collaboration, and wading through large, complex data to build a deeper understanding of our world. There's a richness, a complexity to the problems, and there's the same joy of discovery.

What does a day in the life look like?

MY: I have a hybrid researcher/portfolio manager role and devote about 75% of my time to research and 25% of my time to managing our Asia-Pacific portfolios.

The first 1–2 hours of most days are spent in research seminars or reading groups, where people share new work and receive feedback or discuss the latest academic research. After that, I work on research projects with my collaborators. Jointly, we draw on our backgrounds in mathematics, statistics, machine learning, computer science, economics,

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and finance as we seek to uncover novel insights and advance our understanding of financial markets. In the afternoon, I sometimes try to slip away for a quick workout before preparing for Asia's market open.

We manage portfolios collaboratively. When it is my week on the rotation, I work with our traders and a number of other teams at BlackRock to execute our investment strategies.

UG: Since part of our team is in London, I try to be in the office by 8:00 am so that we can meet via video conference before the London folks go home. Outside of team meetings, I mostly work on our optimization software or on individual research projects. These days, I exclusively code in Python. But earlier in my career, I wrote code in C++, Java, MATLAB, SAS—you name it!

Occasionally, I also attend company-wide events, such as talks or panel discussions organized by various business groups or employee networks, such as the women's network and the LGBT+ & allies network.

These days, I go home around 6:00 pm and have dinner with my family. When my kids were younger, I typically left the office at 5:00 pm, sharp. But now my kids are in high school, so I have a lot more flexibility.

What's some advice that you have found especially helpful?

MY: Cultivate a personal board of directors comprised of people whose counsel you respect and who have diverse experiences, values, and beliefs. Lean on them; let them help you navigate difficult decisions and challenging situations.

Do you have advice for young people who are interested in finance careers?

UG: Make sure this is your passion! You will be much more successful and willing to go above and beyond if you are truly passionate about what you are doing. Also make sure you can code well; knowing statistics and applied mathematics also helps. You may be valued for your quantitative skills, but you should be able to convince people that you are interested in investing!

MY: There are so many different kinds of careers in quantitative finance, and there are so many different paths to each one. You will encounter opportunities you never could have imagined as a student. Be open to those. Invest in yourself; build a strong quantitative foundation (math, statistics, machine learning, and computer science), develop unparalleled expertise in and become exceptional at what you are working on right now, hone your communication skills, and keep learning.



Ursula Gritsch



Melissa Yeung

Credits

Photo of Ursula Gritsch is courtesy of BlackRock.

Photo of Melissa Yeung is courtesy of the DOE Computational Science Graduate Fellowship Program.

Careers in Financial Engineering

Jim Gatheral and Dan Stefanica

It seems all areas of human activity are becoming increasingly mathematized, and the financial industry is no exception. In fact, finance has been very mathematical ever since investment banks started hiring so-called rocket scientists in the early 1980s.

According to the New York saying, “If they say it’s not about the money, it’s the money!” Whilst a career in the financial industry has many rewards, it is universally understood that the money is good, with starting salaries in the range of \$85–125K. As far as other benefits are concerned, you will find yourself in a fast-paced environment where your ideas are valued and have immediate impact, and your colleagues are very smart. Moreover, the finance industry is so closely linked with academia that some academics seem more like industry practitioners and some practitioners more like academics. Mathematical and technological innovation are often motivated by financial applications.

An incomplete list of potential employers would include investment and commercial banks on the so-called sell side and hedge funds and asset managers on the so-called buy side. PhDs in quantitative subjects are also increasingly being sought after for fintech (financial technology) and consulting roles.

There are a number of roles for quantitative PhD candidates. A quantitative researcher might work on topics ranging from option pricing to risk assessment to the identification of profitable trading opportunities. A desk

quant (short for “quantitative analyst” in finance industry jargon) helps the trading desk analyze trading problems, performing a function that is typically more tactical than that of the quantitative researcher. Traders trade (obviously), but the distinction between trader and desk quant is increasingly blurred. Indeed, nowadays most traders are quantitative and can code. In the case of algorithmic trading, all traders are programmers. A risk manager analyzes the risk of trading desk and/or firm positions, advising both traders and senior management. A model validator assesses the validity and correctness of models and strategies developed by quantitative researchers, desk quants, and traders. A data scientist specializes in extracting useful information from data, particularly large datasets using machine learning techniques. Finally, a consultant can be hired to provide external advice on any of the above aspects of the operations of a financial firm.

How does one get started? As a PhD in a mathematical subject from a school where Wall Street firms do regular recruiting, you may be attractive to a number of employers in the finance industry, notably hedge funds. If your PhD is from a non-target school, you may wish to consider obtaining a professional masters degree from a financial engineering program with an established pipeline for placing graduates in the financial industry.

Whereas the areas of technical expertise required obviously depend on the precise role, a well-rounded candidate will know stochastic calculus, time series analysis, and numerical methods. Programming skills are essential. Strong C++ is a prerequisite for many employers. On the other hand, Python appears to be the current language of choice for daily research, and thus strong Python skills can also be regarded as very important.

The ability to communicate well with others who do not necessarily share your technical background is of

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paramount importance. This includes the ability to read cues of all sorts—including nonverbal ones. If you have the ability to handle ambiguity, you will bring something valuable to the table, as assignments will typically not be as well defined as a typical mathematical problem. That is of course in the nature of the real world, and there is real pleasure to be had from formulating a vague observation or desire in such a way that the resulting problem may be solved mathematically.

For banks and large hedge funds, the time-honored route to obtaining permanent employment is to do a summer internship one or two years before completing your degree. This represents a “try before you buy” opportunity for both the employer and you. You can apply for such opportunities directly on company websites, typically in August or September. For smaller firms, personal referrals are often the preferred route.

While an in-depth knowledge of finance would be ideal, it is by no means a requirement for starting out in the financial industry. The kind of quantitative skills one obtains while working towards a PhD in mathematics, and innate personal talents, mean a great deal in this a career. As in any other field, an interest and passion for the work itself carries a lot of value for employers. It also makes it possible to thrive in an environment where the hours can be long and pressure can be high.



Jim Gatheral



Dan Stefanica

Credits

Photos of Jim Gatheral and Dan Stefanica are courtesy of the authors.

BIG Career Developments for Mathematics Graduate Students

Richard Laugesen, Rachel Levy, and Fadil Santosa

Graduate training in mathematics prepares excellent teachers and researchers. Students and postdocs thus may infer from their training that they are qualified only for jobs at colleges and universities, when their career opportunities actually are much broader. Options are a good thing, because while academic life has its attractions, it does not suit everyone. In addition, vastly increased PhD production nationwide means that the majority of new graduates in the mathematical sciences will spend their careers working in business, industry, or government (BIG).¹

The lack of exposure during graduate school to careers outside of academia leads many PhD graduates to take multiple short-term academic jobs, whereas a purposeful leap into BIG could be more rewarding both financially and personally. Graduate training in mathematics provides a solid foundation for that career move, once students and departments take some modest steps in advance. This article provides resources to help students prepare for BIG careers, and outlines steps faculty members and departments can take to open up career opportunities for graduates.

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¹PhD production is about 1900; see Figure A.2 in “Report on the 2015–2016 new doctoral recipients,” Notices of the American Mathematical Society 65(3) (2018), 350–364. The number of tenure track and tenured positions filled is around 750–800; see Figure F.2 in “Report on 2015–2016 academic recruitment, hiring, and attrition,” Notices of the American Mathematical Society 64(6) (2017), 584–588.

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So what are BIG careers? Business and industry roles for a mathematics graduate might range from business-oriented analytical and data science problem solving to industry-oriented technical R&D problems, with a vast range in between including policy analysis work at consulting firms and think tanks. By government careers, we mean jobs at local, state, and federal agencies such as national laboratories, the defense department, and medical research organizations.

How can students prepare for BIG careers? Inspirational stories and practical advice can be found at the BIG Math Network website (<https://bigmathnetwork.org>). The Network is an independent partnership that launched at the 2016 Joint Mathematics Meetings and is supported by the American Mathematical Society, the American Statistical Association, the Institute for Operations Research and the Management Sciences, the Mathematical Association of America, the MathWorks Math Modeling Challenge, and the Society for Industrial and Applied Mathematics.

The BIG Math Network website includes:

- Career transition stories by mathematical scientists who went on to BIG careers
- Links to resources for students and faculty
- Practical advice about seeking jobs

For further in-depth advice and career preparation strategies we recommend our recent book, the *BIG Jobs Guide: Business, Industry, and Government Careers for Mathematical Scientists, Statisticians, and Operations Researchers*, available from the online SIAM and AMS bookstores. The *BIG Jobs Guide* offers students and postdocs a practical how-to guide on topics such as:

- What skills can I offer employers?
- How do I write a high-impact résumé?
- Where can I find a rewarding internship?
- What kinds of jobs are out there for me?

The *Guide* helps students plan ahead for career pathways, right from the undergraduate years, through the early years in graduate school, to the final years and the job search, while being useful also for postdocs wanting to make a career transition.

Faculty members and department administrators are another audience for the *Guide*, with a chapter on low-cost activities by which departments can help students learn about and prepare for BIG jobs, and ways faculty members can build institutional relationships with internship mentors.

What comes next? Inspired by the vision of Philippe Tondeur, former director of mathematical sciences at NSF, the BIG Math Network aims to bring together the broad mathematical sciences community to:

- Communicate the value of mathematical sciences training to students, faculty members, and employers in BIG
- Facilitate connections between students, faculty members, and BIG employers
- Share knowledge on how to prepare for BIG internships and jobs
- Curate and create best practices and training material for preparing students for BIG jobs
- Collaborate with professional societies and BIG in connecting job opportunities with talent

Phillippe and Claire-Lise Tondeur have generously given over \$300,000 to further these goals. The AMS, MAA, and SIAM will use these funds in collaboration with the BIG Math Network to create new activities and programming over the coming three years. Please keep your eyes open for activities at MAA section meetings and at MAA MathFest, AMS sectional meetings, SIAM conferences, and the annual Joint Mathematics Meetings. Other efforts will take the form of products, services, and studies to help departments connect effectively with BIG employers and help students navigate the BIG job market.

If you are a faculty member, graduate director, or department chair, we hope you will actively encourage students to pursue careers in industry and government. The *BIG Jobs Guide* provides ideas for how to do it. If you are a student or postdoc, we hope you will explore **all** the career opportunities open to you. The BIG Math Network and *BIG Jobs Guide* explain how to prepare yourself, get an internship, and choose a career you find challenging and rewarding. Good luck on the journey.



Richard Laugesen



Rachel Levy



Fadil Santosa

Credits

Photo of Richard Laugesen is by Darrell Hoemann. Photo of Rachel Levy is courtesy of Harvey Mudd College. Photo of Fadil Santosa is courtesy of University of Minnesota.

POSITION AVAILABLE

DIRECTOR OF EDUCATION

AMERICAN MATHEMATICAL SOCIETY

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Vladimir Voevodsky

Charles Weibel

The mathematical community lost one of its brightest stars recently, when Vladimir Voevodsky passed away at the age of 51. In addition to being awarded the Fields Medal in 2002, he created two new areas of mathematics: *motivic homotopy theory* (a cross between topology and algebraic geometry), and an axiomatic formalization of mathematics, called *univalent foundations*.

The citation for his 2002 Fields Medal states that it was for his proof of the Milnor Conjectures, and for his concomitant development of motivic cohomology and motivic homotopy theory. He was also a member of the European Academy of Sciences, since 2003.



Vladimir Voevodsky (1966–2017).

His Life

Vladimir Alexandrovich Voevodsky was born in Moscow on June 4, 1966 and died in Princeton on September 30,

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2017, at the age of 51. For the next two pages, I will concentrate on his (colorful) early career.

Voevodsky's parents were both scientists. His father, Alexander, directed a laboratory in experimental physics at the Russian Academy of Sciences; his mother, Tatyana Voevodskaya, was a chemistry professor at Moscow University. As a youth, Vladimir was kicked out of high school three times, once for disagreeing with his teacher's assertion that the author Dostoyevsky, who died in 1881, was pro-Communist.

In 1983, at the age of 17, he enrolled at Moscow State University. He soon became bored by his classes and stopped attending them several times, taking what one might call several "gap semesters." Finally, in 1989, he was expelled for what he called "academic failure."

As was common, Voevodsky had a day job: working at the Lycee of Informational Technologies, as a technician responsible for running and fixing the printers at the Computer Center. There he met Professor George Shabat, who was also working in the Computer Center. When Voevodsky asked to be unofficially mentored by him, Shabat assigned a difficult problem in order to rid himself of what appeared to be just another student. After a few days, Voevodsky returned with a solution—and several examples worked out on the computer. This led to a collaboration [ShV, ShV1] about aspects of Grothendieck's *Dessins d'enfants*, with the English version of [ShV1] entitled "Drawing Curves over Number Fields." While still officially a student, Voevodsky wrote three more papers [V90, V91, V91a] in this vein.

Another collaboration sprung up during Voevodsky's undergraduate days, with Mikhael Kapranov. This resulted in the papers [KV1, KV2, KV3, KV4] on homotopy types and what Ronnie Brown called " ∞ -categories," structures defined using multi-simplicial sets.

At the same time, the Soviet Union was slowly collapsing. Gorbachev had introduced *perestroika* (the restructuring of the Soviet political and economic system) and *glasnost* (openness). The Berlin Wall fell in November 1989,



Voevodsky with his family (2001).

and the Soviet Union itself was in its last days. Travel abroad was becoming possible; in 1978, Margulis was not allowed to go to Helsinki to receive his Fields medal, but by 1989 it had become possible for Russians to enroll in a university abroad.

At this point, Voevodsky wanted to go to graduate school at the University of Wales, to work on category theory with Ronnie Brown. As an alternative, his friends Alexander Beilinson and Mikhail Kapranov arranged for him to be accepted at another university, one named Harvard, which Voevodsky had never heard of. After some persuasion, they convinced him to travel to the United States and attend Harvard.

Thus it was that Voevodsky became a graduate student at Harvard University in 1990, without a high school or college degree, and without even formally applying. Unsurprisingly, he had difficulty adjusting to life in Boston, and was even robbed at one point, so he temporarily went back to Russia. Upon returning, he lived in his office for a time. Nevertheless, he received his doctorate in 1992, under David Kazhdan. It is worth noting that this PhD was the only academic degree Voevodsky ever received!

In 1990, Vladimir met Nadia Shalaby, his life-long partner, at Harvard. They married in 1995 and had two children, Natalia Dalia Shalaby and Diana Yasmine Voevodsky, both of whom are now in college. Although the marriage ended in divorce in 2008, Vladimir and Nadia remained close friends all his life.

Upon graduating, Voevodsky spent a year at the Institute of Advanced Study (IAS), before returning to Harvard



He loved walking in the woods and taking photographs of nature.

in 1993–96 as a junior fellow in the Harvard Society of Fellows. According to Eric Friedlander, Voevodsky complained about having to go to dinner occasionally as a junior fellow at Harvard, as it took time away from his research.

After a year at the Max–Planck Institute, Voevodsky joined the faculty at Northwestern University in 1997. Although he was a gifted teacher, he did not enjoy teaching—again for the reason that it kept him from his research. In 1998, he accepted a position as a long-term member at IAS, becoming a Professor (permanent member) of IAS in 2002. When he was appointed to the IAS, he said with relief that he wouldn't have to teach any more!

He remained a Professor at the Institute of Advanced Study for the rest of his life. On September 30, 2017, after Nadia Shalaby had failed to reach him by telephone, friends found him in his Princeton home, having collapsed in his home from a sudden aneurysm.

The Institute for Advanced Study held a Memorial Service for him a week later (on October 8), and a funeral service was held for him in Moscow on December 27, 2017, followed by a Memorial Mathematical Conference on December 28, 2017 at the Steklov Mathematical Institute of the Russian Academy of Sciences.

A year later, two memorial conferences were held simultaneously. One was held September 11–14, 2018, at the Institute for Advanced Study in Princeton; the other was held September 10–14, 2018 at the Euler International Mathematical Institute in St. Petersburg, Russia.

Voevodsky's Mathematics

I now turn to Voevodsky's mathematical accomplishments. They are organized around several of his revolutionary ideas.

Motivic cohomology (1992–1995). Voevodsky's 1992 thesis, published as [V96], constructed a triangulated category $DM(S)$ for any noetherian base S , as an approximation to the hypothetical abelian category of "mixed motives" envisioned by Grothendieck, Deligne, Beilinson, Lichtenbaum, and others. Any scheme X over S is represented by an object $\mathbb{Z}(X)$ in $DM(S)$, and the motivic cohomology of X with coefficients in F is defined as the Ext-groups $H^n(X, F) = \text{Ext}^n(\mathbb{Z}(X), F)$.

When S is the scheme associated to a field, the category $DM(S)$ in his thesis differs only in sophistication from the subcategory of effective motives in the category DM that Voevodsky would use later; see [V00b]. Competing constructions were made by M. Hanaumra and M. Levine at about the same time.

The paper [V96] contained three revolutionary ideas, ideas which Voevodsky would continue to develop over the next few years. One was to divide the problem of constructing a category of motives into two parts: (a) constructing a triangulated category DM satisfying some basic properties, and (b) showing that DM is the derived category of an abelian one. (He accomplished (a), but (b) remains an open problem.)

Another new idea was to work with sheaves on schemes over S with respect to two new Grothendieck topologies: the h -topology and the quasi-finite h -topology (or qfh -topology). As Voevodsky remarked in [V96], the usual topologies (Zariski, étale, ...) "do not satisfy the properties we would expect from the 'theory of motives'."

A third idea was to construct DM from a derived category of sheaves D with respect to the thick subcategory generated by the "contractible" objects $F \otimes \mathbb{A}^1 \rightarrow F$, where (in this application) \mathbb{A}^1 is the affine line.

Almost immediately, he began working with Andrei Suslin in developing these ideas. In 1992, they used the qfh topology to construct a homology theory for schemes over an arbitrary field which, over \mathbb{C} and with finite coefficients, agrees with the singular homology of the underlying topological space. This was published in 1996, as [SV96].

In 1994, Voevodsky released a series of foundational papers on motivic cohomology with Suslin and Eric Friedlander: [V00a, V00b, FV, SV00a, S00]. These were later published in book form in 2000 [VSF]. Around this time, Suslin showed that motivic cohomology agrees with the higher Chow groups defined by Bloch in [B86] in characteristic 0; see [MVW]; the characteristic 0 assumption was later removed in [V02a].

In 1995, he wrote a related paper [V95] about correspondences between smooth projective varieties. The rational equivalence classes of such correspondences form the morphisms in Grothendieck's category of Chow motives. Generalizing the observation that a correspondence from X to itself that is algebraically equivalent to 0 is a nilpotent endomorphism of X , he formulated the notion of a correspondence $f : X \rightarrow Y$ being *smash nilpotent*: some $f^{\otimes n} : X^{\otimes n} \rightarrow Y^{\otimes n}$ is trivial. Then he stated the *Nilpotence conjecture*: A correspondence $f : X \rightarrow Y$ is smash nilpotent if and only if it is numerically equivalent to zero (that is, conjecturally, homologous to zero).

By 1995, Voevodsky was essentially finished with the framework of motivic cohomology. In 1999–2000, he gave a course on the subject at the Institute for Advanced Study, which my student Carlo Mazza and I attended. At the end of the course, Carlo and I were asked to write up the lecture notes. Whenever Carlo and I could not reconstruct an argument (often a proof that consisted of the word "obvious"), we would come to Voevodsky's office and he would explain it to us. This process uncovered a mistake in a key lemma [V00a, 4.23], which was corrected in [MVW, 22.10]. Thomas Geisser was also very helpful with some points. By 2004 we were finished, and the book [MVW] appeared in 2006.

\mathbb{A}^1 -homotopy theory (1996–2000). The notion of an \mathbb{A}^1 -homotopy theory for rings arose around 1970, largely due to the work of Steve Gersten [G]. The naive idea was that polynomials in t should be regarded as homotopies between $t = 0$ and $t = 1$. Thus if F is a functor from rings to sets, such as GL_n , two maps $f_0, f_1 : F(R) \rightarrow F(S)$ are considered "homotopic" if there is a map $f : F(R) \rightarrow F(S[t])$ so that f_i is the composite of f with the map $F(S[t]) \rightarrow F(S)$ induced by $t \mapsto i$. Any F has a universal homotopy invariant quotient: the coequalizer of $t = 0, 1 : F(R[t]) \rightrightarrows F(R)$. However, it was quickly discovered that this naive definition of homotopy theory was useful only in very limited contexts, and did not generalize well from rings to varieties.

In September 1995, Voevodsky heard about the work of Fabien Morel, who was just finishing his Habilitation in Paris, and was trying to define a natural homotopy theory on algebraic varieties using Quillen's model categories. Having had similar ideas, Voevodsky began an email correspondence with Morel. In May 1996, they met in person and Voevodsky proposed that they "should write a bit." The result was their joint paper \mathbb{A}^1 -homotopy theory of schemes [MV], released in 1998, which laid the foundations for what is (not surprisingly) called " \mathbb{A}^1 -homotopy theory." A few months later, Voevodsky gave a beautiful address at the 1998 Berlin ICM [V98], laying out the foundations of stable \mathbb{A}^1 -homotopy theory and motivic spectra.

In 2000, Voevodsky defined the *slice filtration* on the stable motivic homotopy category defined in [V98]. The slices $s_n E$ of a motivic spectrum E are again motivic spectra, and they form the motivic analogue of the Postnikov tower in classical stable homotopy theory. This tower yields a “slice” spectral sequence

$$E_2^{p,q} = H^p(X, s_{-q}^{p+q,n} E) \Rightarrow E^{p+q,n}(X).$$

Voevodsky discussed several aspects of this filtration in [V02b], ending with a list of open conjectures. Voevodsky settled one in [V04], proving that the 0^{th} slice of the motivic sphere spectrum $E = \mathbb{S}$ is the object $H\mathbb{Z}$ that represents motivic cohomology. All of the other conjectures have since been confirmed over fields of characteristic 0, but many remain open in finite characteristic.

The paper [V02c] studies the case when E is the motivic spectrum KGL representing algebraic K-theory. The resulting spectral sequence has $E_2^{p,q} = H^{p-q,-q}(X, \mathbb{Z})$ and converges to $K_{-p-q}(X)$. Suslin later showed that this spectral sequence may be identified with a spectral sequence first formulated by Bloch and Lichtenbaum.

A good introduction to stable \mathbb{A}^1 -homotopy theory, including the slice filtration, is given by Voevodsky’s notes [VRØ], based on a short course Voevodsky gave in 2002.

In 2000–1, Voevodsky gave a full-year course at IAS on equivariant motivic homotopy theory. The main application was in the action of the symmetric group Σ_n on the product X^n , and its resulting equivariant motive. This material was later used to analyze symmetric powers $\mathbf{Sym}^n X$ of a variety X , which occur in the proof of the Bloch–Kato conjecture. Each week, Deligne would give Voevodsky his notes on the lecture, and Voevodsky typed it up. The result was published as [D].

The Milnor conjectures (1995–2000). Let p be a prime, and let F be a field of characteristic not p , containing the p^{th} roots of unity for simplicity. Around 1969, Tate constructed the norm residue map from $K_2(F)$ to $H_{\text{et}}^2(F, \mathbb{Z}/p)$, and Milnor defined abelian groups $K_n^M(F)$ so that the norm residue map generalized to a map $K_n^M(F)/p \rightarrow H_{\text{et}}^n(F, \mathbb{Z}/p)$. (Milnor’s $K_2^M(F)$ agrees with $K_2(F)$, which is defined using the Steinberg group.) In 1970, focusing on the case $p = 2$, Milnor stated, “I do not know of any examples for which the [norm residue] homomorphism fails to be bijective.” [M70, p. 340]. The assertion that it is always an isomorphism became known as the *Milnor Conjecture*. It was established around 1982 by Merkurjev and Suslin in their celebrated paper [MS82] when $n = 2$, for all p . Some results for $n = 3$ were later established by Rost, Levine, and Merkurjev–Suslin.

In March 1996, Voevodsky announced a proof of the Milnor Conjecture; his preprint followed that December.

The following summer, he gave lectures on the proof at an AMS Summer Research Conference in Seattle. I was tasked with writing up the lecture notes, and they appeared in 1998 as [V99]. Preparing these notes let me really get to know Volodia, as he was very generous with his time in explaining the various aspects of the proof to me. The official proof, published in 2003 [V03a], differed in several places from the original preprint because Voevodsky had found a shorter method of proof.

In the same 1996 lecture that he announced a proof of the Milnor conjecture, Voevodsky also announced that he, Orlov, and Vishik had proven “Milnor’s conjecture for quadratic forms.” Formulated as “Question 4.3” in [M70], it asked if a certain map $s_n : K_n^M(F)/2 \rightarrow I^n/I^{n+1}$ is an isomorphism for all n and F . Here I is the augmentation ideal of the Witt ring $W(k)$ of quadratic forms over a field F . Milnor showed that s_2 was an isomorphism; s_3 and s_4 were shown to be isomorphisms in the late 1980s by Rost and Merkurjev–Suslin, but the general case remained open until the preprint based on this announcement appeared in 1997; Vishik was Voevodsky’s student and the preprint is cited in his thesis [Vsh]. The paper [OVV] was published in 2007.

The Bloch–Kato and Beilinson–Lichtenbaum conjectures.

The analogue of the Milnor conjecture when p is odd, dubbed the *Bloch–Kato conjecture* by Suslin, was first clearly formulated in 1980 by Kazuya Kato in [Kato, p. 608]:

Conjecture. *The [norm residue] homomorphism is bijective for any field k and any integer p which is invertible in k .*

Spencer Bloch’s version was: “I wonder whether the whole cohomology algebra $\oplus_r H_{\text{et}}^r(F, \mu_p^{\otimes r})$ might not be generated by H^1 ” [B80, p. 5.12].

In a 1995 preprint, Suslin and Voevodsky showed that the Bloch–Kato conjecture is equivalent to a conjecture originally made by Lichtenbaum and modified by Beilinson to connect motivic cohomology groups to étale cohomology. More precisely, if π is the usual morphism from the étale site to the Zariski site, the Beilinson–Lichtenbaum conjecture was that the motivic complex $\mathbb{Z}/p(n)$ should be quasi-isomorphic to the truncated complex $\tau_{\leq n} R\pi_*(\mu_p^{\otimes r})$. It is notable for its introduction and use of the *cdh* topology. After substantial rewriting, this result appeared in 2000 as [SV00b].

In 1998, Voevodsky announced a proof of the Bloch–Kato conjecture, assuming the existence of what we now call a *Rost variety*. Rost produced such a variety that same year, in [Ro], but the complete proof that Rost’s variety had the properties required by Voevodsky did not appear until 2007. An outline of Voevodsky’s proof appeared in a 2003 preprint, modulo the assumption that Rost varieties exist and two other assertions. The second assertion, that every



Voevodsky in 2010.

mod- p motivic cohomology operation is a polynomial in the P^a , turned out to be wrong.

In 2006–7, I was asked to run a seminar at IAS, explaining the state of affairs of the Bloch–Kato conjecture. Voevodsky generously shared his source files with me, and I spent a lot of time talking to him about them. Although his second assertion was false, I saw that it was not really needed, and patched his proof by using the notion of scalar weights for cohomology operations [W09]. This re-engaged Voevodsky in the project and he produced the final part of the proof in 2008. The next few years saw a flurry of long-delayed publications appear: [V10a, V10b, V10c, V10d, V10e, V10f] and the final piece of the puzzle in [V11]. The complete proof will be published soon in the book [HW].

Other interests. Voevodsky was briefly involved in several other projects. For example, in 1997 he spent several months thinking about artificial intelligence in robotic locomotion.

In 2001–2, Voevodsky gave a full-year course at IAS on the formalism of the two adjoint pairs of functors (f^*, f_*) and $(f_!, f^!)$ associated to a morphism f of schemes. Together with the adjunction of \otimes and Hom , this amounts to a generalization of Grothendieck’s “six-functor formalism.” Although Voevodsky never published his results, Joseph Ayoub figured out the details and published them as [Ayb].

In 2003–2008, Voevodsky became interested in inferring genetic history from current data. At each time $t \geq -T$, there is a set of (male) genomes X_t . As t varies we get a graph: it is a forest of valence 3, vertices being births and deaths. If one assumes a constant birth rate, we would like to know the most likely values of X_{-T} and death rates

which would produce a given population X_0 at time 0. Biologists refer to this process as “the coalescent.” As he explained it to me, Voevodsky had the idea that births and deaths could be described by a Markov branching process. Since time is totally ordered, we may regard it as a category, and one regards X as a stochastic category fibered over time. When he felt he was ready, he gave a lecture on this at Ohio State’s Math Biosciences Institute; the lecture was not well received. Looking back in 2013, he said, “I wasted two years, because I totally failed.” (See [Reh].)

In 2005–2006, Voevodsky was excited by what he called “homotopy λ -calculus” as a generalization of Church’s λ -calculus of types. Working in the homotopy category of topological spaces (in a fixed universe), he inductively defined a nested sequence of “levels.” Reindexing, X has level 0 if it is contractible, level 1 if it is either the empty set or a contractible space, and X has level 2 if it is homotopy equivalent to a discrete set; X has level $n + 1$ if the path spaces $P(X, x, x')$ have level n for all pairs x, x' in X . Homotopy λ -calculus would later evolve into *homotopy type theory*, and the levels in homotopy λ -calculus were the prototypes of the “ h -levels” in the Univalent Foundations.

Type theory. The inadvertent mistake in [V00a], mentioned above, led Voevodsky to become interested in using computers to verify proofs. Much later, in a public lecture in 2014, he would state the truism that “a technical argument by a trusted author, which is hard to check and looks similar to arguments known to be correct, is hardly ever checked in detail.”

Starting in 2004, he gradually focused on Per Martin-Löf’s formal language of “types” (and their elements) as a way to ensure grammatical correctness. The word *type* is an undefined term, much as the word *set* is an undefined term in Zermelo–Fraenkel set theory. One feature of this language is that if x, x' are elements of the same type then there is a new type: the type of “identifications” of x and x' . This led Awody and Bauer to define a *proposition* as a type whose elements are (pairwise) equal (see [AwB]); informally, all of its proofs are indistinguishable.

At this point, he started learning about “proof assistant” programs such as *Coq*. To learn how to use *Coq*, Voevodsky took an undergraduate course at Princeton in Fall 2009, from Andrew Appel, called “Programming Languages.” The course used *Coq* for doing machine-checked proofs in logic, with application to the semantics of programming languages, type systems, and verifying the correctness of algorithms and programs. Voevodsky completed the midterm exam, which was verification of the correctness of the “Binomial Queues” data structure. A few years later, he told Appel that the exam “convinced him that machine-checked proof in *Coq* could be a practical way to do mathematics.”

Voevodsky integrated the computer into the process of doing his own research, describing it in a 2013 interview [Reh] as a bit like a video game. “You tell the computer, ‘Try,’ and it tries, and it gives you back the result of its actions... Sometimes it’s unexpected what comes out of it. It’s fun.”

At this point he embarked on an enormous project to create proof-checking software so powerful and convenient that mathematicians could someday use it as part of their ordinary work and create a library of rock-solid mathematical knowledge that anyone in the world could access. This library, called *Foundations*, is described in [V15b].

The Univalence axiom. Already in 2006, Voevodsky had the notion that there were special maps which he called “univalent.” In late 2009, he wrote to Grayson that his ideas about a univalent homotopy-theoretic model of type systems had developed enough to “survive the verification stage, and I am in the process of writing things up.” Voevodsky announced his new axiom for type theory, the *univalence axiom*, in a lecture he gave at Carnegie Mellon, early in 2010. Since then, this axiom has had a dramatic impact on both mathematics and computer science.

The *univalence axiom* states that

$$(X = Y) \xrightarrow{\cong} (X \cong Y)$$

This needs a little explanation. The notation $f : X \xrightarrow{\cong} Y$ (read as “ f is an equivalence”) refers to a function such that for each y in Y the fiber $f^{-1}(y)$ is exactly one point. The notation $X \cong Y$ refers to the type T of all equivalences between X and Y . So the axiom refers to the natural function from the type S of all equalities to the type T of all equivalences.

There is a notion of *h-level* in univalent theory, which is the analogue of truncation level in homotopy theory, and is similar to the levels of homotopy λ -calculus. A type has *h-level* 0 if it has exactly one element; *h-level* 1 means that the type has at most one element, and is called a *proposition*. Informally, a proposition is a type which is “false” if it is empty, and “true” if it has one element, but this assumes the Law of the Excluded Middle; in an intuitionistic framework, a proposition need not be true or false. Inductively, a type X has *h-level* $n + 1$ if the type $x = x'$ in X has *h-level* n for all x, x' in X . Thus X has *h-level* 2 if the types $x = x'$ in X are propositions for all x, x' in X ; informally, a type of *h-level* 2 may be thought of as a (discrete) set, since $x = x'$ is either true or false for all x, x' in X —at least, assuming the Law of the Excluded Middle. The reader may enjoy interpreting *h-levels* of 3 or more.

In 2012–13, there was a special year at IAS on Univalent Foundations and type theory. This led to a resurgence in Voevodsky’s work, in which many results have been formalized in *Coq* using the UniMath library; see [V15b].



Voevodsky in 2013.

We mention two examples of papers using this technique. One is the paper [PVW], which formally constructs the p -adic numbers using a univalent approach, using *Coq* to verify the proof.

A second is the paper [ALV], where the authors compare three of the many algebraic structures that have been used for modeling type theory: categories with families, split type-categories, and representable maps of presheaves. If one assumes the univalence axiom, these notions can be meaningfully compared on the level of types; this contrasts with the situation in set theory, where one would need to resort to a comparison on the level of categories.

After 2013, Voevodsky worked primarily with contextual categories (also known as *C-systems*); the articles [V15, V16a, V16b, V17a, V17b] all deal with *C-systems*. One conjectures that *C-systems* can also be compared with the above structures.

When he passed away, Voevodsky left about 8 more preprints in various stages of completion, which begin to set up a formal framework in which one can prove the soundness of his univalent approach. Univalent foundations is now a subject in its own right, and Voevodsky’s creation of this subject is a monumental achievement.

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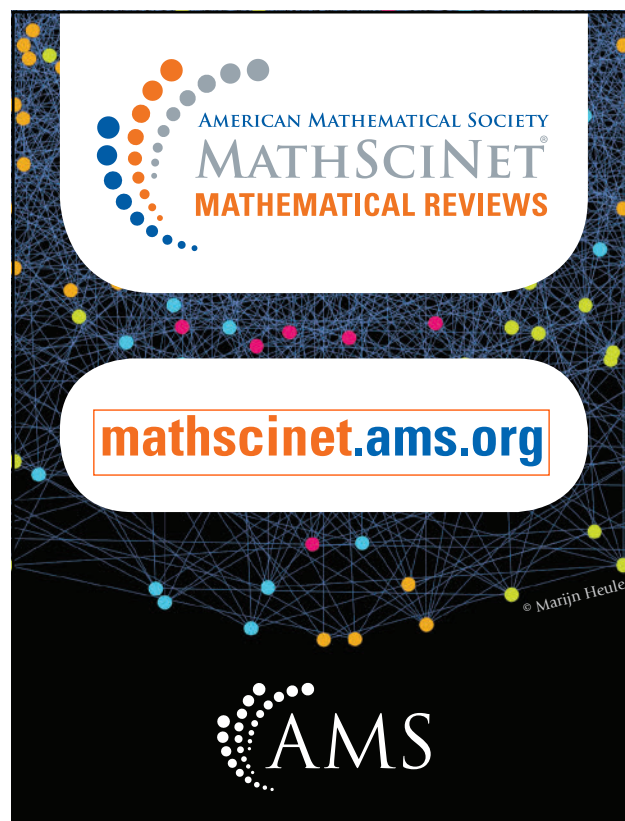
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Credits

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Addressing Mathematical Inequity in Indigenous Education:

Efforts by Mathematicians and Educators in Western Canada

Melania Alvarez



Government practices of discrimination and assimilation towards the Indigenous population in Canada have had lasting effects. In particular, the implementation of residential schools created by the federal government and mainly operated by Christian churches with the goal of educating and assimilating Indigenous children into Canadian society shaped people's whole life experience, including their

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employment opportunities and their interactions with governmental agencies. At the residential schools students experienced isolation by being separated from their parents and siblings (boys were usually separated from girls), and they were forbidden to speak their language. Upon arrival, children were stripped of their traditional clothes, given uniforms and Christian names. Boys hair was cut. At some schools students were forced to work, and many suffered from starvation due to the lack of funds to feed them. There were many cases where students were psychologically, physically, and sexually abused, and more than 6,000 students (incomplete records) died; their parents were never given an explanation as to how or why, or told where their children were buried. The residential schools ran for over 100 years and the last one closed in 1996 [4, 5].

The impact of the residential schools extended beyond the individuals who attended these schools; it also affected their families, communities, and culture. In general, members of Indigenous communities may feel that their lives are a constant battle against a system that does not work for them. There exists a justifiable distrust towards government agencies and towards the kind of education their children currently receive at schools. They feel that they are receiving an unequal educational experience, given that teachers usually have low expectations for them, which is translated in their practice [11]. There is also a lack of cultural support in that most children do not see themselves and their culture reflected in the curriculum. For example, in the province of British Columbia, Indigenous content only appears briefly in the history curriculum at the end of grade 10. There exists

a well-documented educational achievement gap between Indigenous and non-Indigenous Canadians [9].

The self-identified Indigenous population grew from 1.17 million in 2006 to 1.67 million in 2016, which represents an increase of 42.5% and a rate of growth four times faster than the rest of the Canadian population (Stat Canada). Within a few years, provinces such as Saskatchewan and Manitoba are expected to have more than 20% of their population self-identify as Indigenous.² Given the rapid growth of the Indigenous population, it is increasingly apparent that Canada must urgently address the great disparity in educational achievement; otherwise, the repercussions will be disastrous as Indigenous youth will not have equitable access to jobs and economic prosperity.

In order to positively narrow the educational gap between the Indigenous communities and the rest of the population, there needs to be a continuous and long-term intervention for change. In the case of schooling, we should be working with the Indigenous communities to look at a long-term continuum of choices and to present opportunities and positive interventions that provide students with a more affirmative outlook for life. What is required is a long-term commitment on behalf of the educational system to provide marginalized students with very much needed comprehensive support [3, 7].

Mathematics seems to be an excellent way to start this change as research has shown that taking advanced and rigorous math courses positively affects several other educational outcomes, including standardized test scores, high school completion, college performance, and post-secondary degree completion, as well as having an impact on earnings in adulthood [1, 8].

The Pacific Institute for the Mathematical Sciences (PIMS) is a research institute closely associated with mathematical science departments at universities and colleges across Western Canada. PIMS has recognized the challenges many students face if they lack the necessary prerequisites in math and science to pursue post-secondary studies (especially in STEM fields), specifically when it comes to Indigenous students and other students at risk. By leaving behind the philosophy of reduced expectations, mathematical scientists and educators associated with PIMS have introduced a variety of interesting and challenging programs

² First Nations, Inuit, and Métis are the groups that constitute the Indigenous people in Canada. First Nations are diverse bands of Indigenous peoples in Canada who are neither Inuit nor Métis. The Métis are people who descended from marriages between Europeans (mainly French) and First Nations/Inuit people going all the way back to the 17th century. According to anthropologists, the Inuit are the descendants of Thule culture, which originated in Alaska around 1,000 CE and later on spread towards the east through the Arctic. The 2016 census counted a total of 1,673,785 Indigenous people in Canada, about 4.9% of the national population, consisting of 977,230 First Nations people, 587,545 Métis, and 65,025 Inuit. There are more than 600 recognized First Nations governments or bands with distinctive cultures, languages, art, and music.

and exciting ways to learn mathematics seeking to address these issues. Our goal is to be able to provide Indigenous students with the tools they need to make career decisions of their choice, including a career in science.

The hypothesis that guides these outreach programs is that if we are able to teach students and provide them with a stronger academic background, they will feel more confident in school, and this confidence will empower them to feel better about themselves. However, real empowerment comes from within, and this change from within does not happen in one day; it is a long process where the educator can only provide the learner with opportunities. Until the students take them as their own, change will not happen [6].

Our first step has been to build partnerships with schools run by Indigenous communities, as well as with urban public schools with a high concentration of at-risk students. With their input and support, the PIMS outreach team has implemented a variety of programs, some of which are described in detail below. These activities have been funded by private donors, universities in the PIMS consortium; the Actuarial Foundation of Canada, the Vancouver Foundation, the government of Canada, and the governments of Alberta, British Columbia, and Saskatchewan.



High School Summer Camps: Having fun.

High school summer camps for Indigenous students in Vancouver: These camps run for five weeks and are designed for students in grades 9 to 12. Each year thirty or so students are selected to participate in these summer programs. Students take ninety minutes of both math and English every day from a master teacher. They also participate in an internship program where they will work for a professor at the university for three afternoons a week. They participate in a variety of academic activities throughout the internship and are also given the option to complete an independent study elective. Students also engage in a special topics series that include cultural activities and college access and success discussions. The math

program addresses the particular mathematical needs of each student with the goal of enhancing their learning for the upcoming school year. The English program is designed to strengthen skills that they will use across all coursework. Students who choose to complete the independent study project earn elective credit that can be used to meet high school graduation requirements. The internship provides an opportunity for students to explore academic areas of interest and possible career options while learning transferrable employment skills. Students come to the university campus to attend this camp, with the primary rationale being to expose them to a post-secondary environment.

The main objectives are to provide students with strong academic support skills in mathematics and writing, to explore the STEM fields through the internship and academic activities, to remove access barriers to a post-secondary education, and to develop relationships between participants and university faculty and student mentors. We also aim to create a sense of pride in the Indigenous culture by offering a variety of activities and lectures led by members of Indigenous communities. The original camp started in 2007, and it has been a source of inspiration for camps not only in Canada but also in the United States. The camp ran first at University of British Columbia (UBC) under my supervision, then at Simon Fraser University (GSU) under the supervision of Veselin Jungic, and it will be running next year at Langara College in partnership with UBC under the supervision of Richard Ouellet and myself. Our goal is to create paths for outreach where we connect universities to local schools and colleges and also bring in local industry to expose Indigenous high school students to careers, professions, and technical trade opportunities and help them find out how mathematics can open up opportunities in traditional and nontraditional careers.

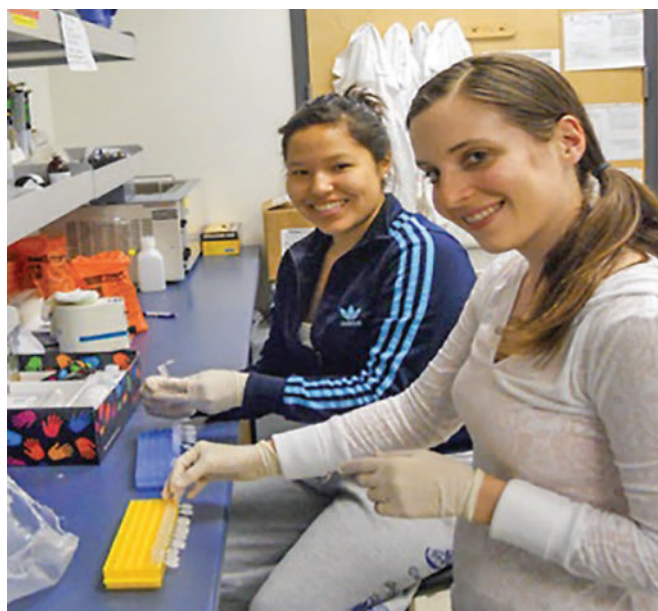
Alex³ attended this high school summer camp and, as an intern, he was able to do original research with professors in the chemistry department. This is what he thought of the camp:

I think it's the best summer school you can ever go to, and I highly recommend it for other students... My teacher is pretty cool, is kind of fun, a little challenging because I have to do a lot of things I haven't done before in the lab. I work with a couple of professors. The staff is kind, friendly, and helpful. They are hardworking people who I really respect and who are willing to take the time... Urban Aboriginal kids need an education for a really great future.

Alex is currently an undergraduate student at the UBC. He wants to become a teacher.

PIMS has also run **Transitional Summer Camps** (with a similar structure) for Indigenous children transitioning

³ Students' names have been changed to protect their privacy.



Internship Jobs: Chemistry lab.

from elementary school to high school.⁴ The transition from elementary to high school is a difficult one for many children, but for Indigenous students it seems to be particularly harsh. As a group, far too many Indigenous students are put into courses with low academic expectations, and as a result they often conclude that their coursework will not lead to a career path where academic knowledge is required. By grade 10 most students have stopped attending school on a regular basis. The goal of these camps is to provide a strong academic background as well as a sense of pride in the Indigenous cultures. Students who started high school with strong skills in mathematics and English showed a difference in their confidence, not only in their math classes, but also in other core science subjects. These summer camps take place at the high school most of the participants will be attending, allowing them to feel more comfortable as they start this new stage of their education.

Maria attended the first Transitional Summer Camp in 2008. She did well during the camp, though she was not at the top of the class. During her first year in high school, was unreliable with her attendance and she constantly tested the commitment of our support, but she always mentioned to us that what she had learned at the transitional camp helped her to feel that she had a chance. As time went by she realized the importance of consistent work and being able to ask for help when she needed it and how our program could help her maintain that consistency and provide the support she needed. She finished grade 10 with a good mark in math.

Later on, she attended the high school summer camp where she told us,

⁴ In Canada, elementary school covers grades 1–7 and high school grades 8–12.



Transitional Summer Camp.

I get to experience studying the rat brain and fetal alcohol syndrome... I'm getting ahead for my next year math and English and I get to study stuff that I wouldn't even learn in high school. It will be good for my future, especially because I want to be a nurse. I think this will give me a head start on everything.

Our goal for her was to finish high school with precalculus 12, and she surprised us when in her last year she took not only precalculus but also calculus. She has blossomed into a wonderful, feisty woman, full of confidence. She told us that being able to do well in math, by working very hard and with our support and confidence in her skills, made her realize that she has talents. She was one of the two students in the cohort who attended the first Transitional Summer Camp who were accepted at a top Canadian university but she chose to enroll at a college with a more personalized program.

Peer-mentorship program: PIMS recruits high-performing Indigenous students at a high school to become peer-mentors to four or five of their classmates. We provide the mentors with a stipend and academic support. This program was implemented at several schools in British Columbia with great success. Here is the testimonial from one of the teachers,

Wow, the second year of the Peer-Mentorship Program has come and gone in a flash. The organizational piece came together flawlessly and you could see that the three Aboriginal high school students truly enjoyed the opportunity they were given. This year we had two grade 12 students (one female, one male) and one grade 11 student (female). This opportunity gave them a chance to push their personal boundaries (show assertiveness, kindness, and other leadership skills) but also gave them a chance to demonstrate their math skills to younger Aboriginal and non-Aboriginal youth. These students are proud of their Aboriginal ancestry,

and this was revealed to the youth they worked with. From this opportunity one of our Math Mentors was requested to be a guest speaker for the All Nations Room at the school he mentored at. He often spoke proudly of his Metis ancestry and talked about his ability to speak the Michif language. The classroom teacher was thrilled about his ability to connect with the students on a personal level while supporting them with their studies.



End of Camps Celebration.

First Nations Math Education Workshops held at the Banff International Research Station (BIRS) have brought together a group of Elders, mathematicians, and math educators and teachers, with the goal of improving mathematics education among Aboriginals while at the same time acknowledging the importance of traditional culture. Members of this group worked together in creating resources to honour the spirit of each student as an individual and as part of a community. This way of thinking is an integral part of many aboriginal cultures as well as a successful way of learning mathematics in any culture. The reality is that most of the "Aboriginal resources in mathematics" are very simplistic and do not honour the similarities, differences, depth, and richness of First Nations cultures. One of the ideas developed by members from this group (Veselin Jungic from SFU, and Mark Maclean from UBC, together with the elder Rena Sinclair of the Siksika Nation), was the story and the movie of "Small Number counts to 100," with more stories that followed (www.math.sfu.ca/~vjungic/smaller.mov). This project has grown quite considerably thanks to the tireless and talented efforts of Jungic and MacLean, who have developed a slew of mathematics materials in Indigenous contexts.

So far, we have held five of these meetings, and, in doing so, BIRS and PIMS have shown their leadership in bringing various people, resources, and institutions together in working towards the improvement of Indigenous mathematics education.



First Nations Math Education, November 2009, at the Banff International Research Station (BIRS)

Other activities organized by PIMS that have an impact on the Indigenous population include the following:

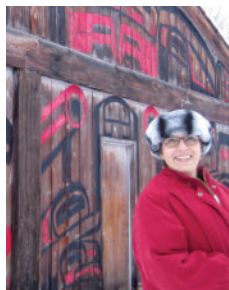
- A comprehensive teacher professional development upgrade program. This targeted initiative helps upgrade our teachers' mathematical knowledge and pedagogical skills as well as support their ability to help students see the relevance and connections of mathematics and science with a variety of career opportunities.
- Mentoring programs where we team undergraduate students from local universities/colleges with local teachers to support their students.
- PIMS has been providing assistance in choosing and implementing mathematics curricula at First Nations Schools.
- Math Mania and Parents Night. We bring a variety of hands-on math outreach activities such as puzzles and games at local schools and community centres with the participation of students, teachers, and parents.
- Working with Elders in order to emphasize the importance of storytelling as a traditional way of teaching, and to use it to teach mathematics and other subjects.
- Funding and facilitating the organization of a variety of month-long summer camps at First Nations schools. These camps can be full-day camps or half-day camps where math instruction is provided every day as well as instruction in other subjects. There is some flexibility in how these camps run given that we take into account specific community needs and resources.

Conclusions

The problem of providing equitable opportunities for Indigenous students is a daunting one and will require the ongoing, active involvement of mathematical scientists and educators. In this article we have summarized a variety of activities and programs organized by the PIMS community that have had an impact on the mathematical opportunities for students and teachers. These should be considered pilot programs that have the potential for broad implementation given the required funding and support. We are happy to share the outcomes of our efforts and experiences, and we invite you to contact us.

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Melania Alvarez

Credits


Photo of Melania Alvarez is courtesy of Sachiko Noguchi.
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Exploring Continued Fractions

From the Integers to Solar Eclipses

Andrew J. Simoson



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Exploring Continued Fractions


From the Integers to Solar Eclipses

Andrew J. Simoson, *King University,
Bristol, TN*

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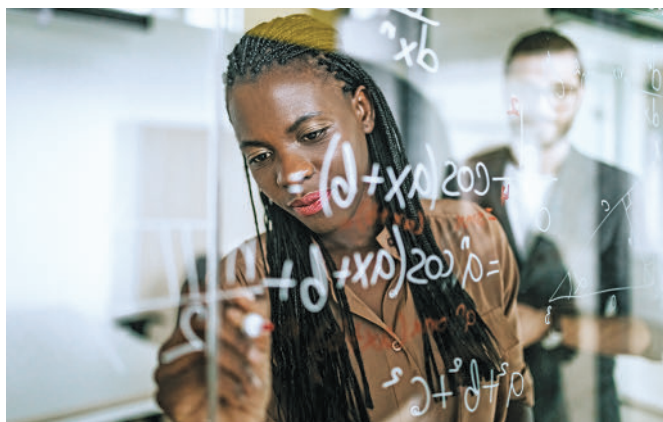


AMERICAN
MATHEMATICAL
SOCIETY

Advancing research. Creating connections.

Building the Alliance of Indigenous Math Circles

Bob Klein and Tatiana Shubin



To say that US Native Americans are underrepresented in mathematics is itself an understatement.

In 2010 the census recorded 5.2 million people in the United States who identified as American Indian or Alaskan Native, either alone or in combination with one or more other races. Of those, a mere twelve hold a mathematics PhD.¹

Compare this with Hungary, which has almost twice the population—9.8 million. From 1993 to the present, 460 mathematics PhDs were awarded there.² (We looked

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¹Dr. Robert Megginson, cf. <https://sciences.ucf.edu/news/math-and-native-american-culture/> for details.

²Országos Doktori Tanács (Hungarian Doctoral Council), <https://doktori.hu/index.php?menuid=110&lang=EN>

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for the total number of living people with PhDs, but these data were not available.)

Kazakhstan has just over three times the population, 17.8 million, with people in rural and remote areas. Four hundred Kazakhstanis hold PhDs in mathematics.³ Were we to regard the US Native American/Alaskan Native population as a nation, Hungary and Kazakhstan suggest that we would expect to see between 160 and 244 mathematics PhD holders in that nation. We clearly have a long road to go to match Hungary or Kazakhstan.

But why does this comparison matter? When a distinct group of people, in this case marked by their identification as American Indians or Alaskan Natives, do not participate in mathematics, the question that arises naturally is, “Why?” Biological explanations are both racist and scientifically unsupported, though science took some centuries to



Shiprock looms large over the New Mexico desert, not far from the site of the AIMC at Navajo Prep Summer Math Camp.

³Personal correspondence with Dr. Askar Dzhumadildayev, a Kazakh mathematician, doctor of physics and mathematics, professor, and Full Member of the Kazakhstan National Academy of Science. He was also member Supreme Council of Kazakh SSR and Republic of Kazakhstan.

get there.⁴ Cultural explanations would require assuming some key commonality to all American Indian or Alaskan Native peoples that contraindicated participation in mathematics. Alternatively, one could ask whether or not there exists a “culture of mathematics” that embodies certain norms that constitute a barrier for a group of people. Such questions are beyond the scope of this paper, though the related issue of how people construct their identity relative to tradition, affiliation, and intellectual pursuits and practices is not only relevant, but it informs our model for engaging indigenous communities and mathematicians in mathematical problem solving.

A further important question concerns the issue of whether or not mathematics itself should be an essential part of everyone’s education and which outcomes in particular matter most to the education of all. Readers of the *Notices* probably need little convincing of the importance of mathematics but ought to consider the outcomes and methods of a “school mathematics” (thought of here as typical K–12 curriculum in the United States) versus what they value most when thinking of mathematics. What we, the authors, see as “beautiful mathematics” seems largely to be absent from the encyclopedic techniques and formulas that constitute mathematics curricula in K–16 settings. To us, the “school mathematics,” necessary to some extent, is not only an insufficient exposure to mathematics, but by itself paints for the general populace a picture of mathematics incongruent with the beauty and joy that we find in it. As students shape their identities relative to abilities and affinities toward subjects, we want them to value mathematics as beautiful, intriguing, and powerful, and to see themselves as capable and creative practitioners of mathematics.

The Alliance of Indigenous Math Circles (AIMC) helps indigenous communities to provide experiences and infrastructure for students and teachers to realize the beauty and power of mathematics, recognizing that this involves the concomitant valuing of their budding mathematical and indigenous identities (not to mention other identities). By combining work with professional mathematicians and tribal elders, we covalue mathematics and culture so that an indigenous student sees no disjunction between being “indigenous” and a “mathematician”—we are working within culture and identity to try to stimulate a robust next generation of “indigenous mathematicians.”

Targeting mathematics has broad implications given the extent to which the “M” supports the “STE” of STEM. While it is true that mathematics is the basis for all STEM fields, it is arguably the most significant prerequisite for success in the world of post-secondary opportunities. And mathematics is far more than the checklist set of skills found in school

mathematics. It represents the purest subject for expanding the human mind’s capacity for critical thinking and problem solving. As such, mathematics prepares people with tools, mindsets, and techniques to fashion a successful and fulfilling life regardless of professional occupation.⁵

The Alliance of Indigenous Math Circles (AIMC <https://aimathcircles.org>) is devoted to bringing mathematicians and math professionals into direct contact with indigenous students and teachers throughout the United States and abroad in order to improve and strengthen their grasp of and attitude toward mathematics. And AIMC promotes the culture of problem solving within the framework of the indigenous culture, to both promote that culture in its own right and to bring more indigenous people into STEM fields.⁶

The AIMC is an initiative that grew out of the Navajo Nation Math Circles project (NNMC, launched in 2012). NNMC historically has included a number of components including mathematicians visiting schools and running math circle sessions for students, professional development workshops for teachers, and a summer math camp at Diné College in Tsaile, AZ. This work has demonstrated that math circles and summer math camps combining mathematics and indigenous culture led students who were not otherwise considering it to attend a college or university, many pursuing STEM-related degrees. Moreover, five years of the project gave ample opportunity to refine the model, confirming some elements as productive (e.g., mathematician visits to schools, summer camps) and others as more problematic (e.g., school-year pen-pal programs between mathematicians and K–12 students). In 2017, a group of directors from the NNMC project recognized that the model could be shared more broadly, requiring a new direction and adopting a new name that was more inclusive and representative of the mission. Thus, the Alliance of Indigenous Math Circles was born with the purpose of sharing the model with other indigenous communities and to provide a network of support for sustaining the work within and by those communities.

⁵Two excellent examples of this more general power of mathematics can be found in *Avoid Hard Work! ... and Other Encouraging Problem-Solving Tips for the Young, the Very Young, and the Young at Heart*, (Droujkova M, Tanton J, McManaman Y, *Natural Math*, 2016) and *The 5 Elements of Effective Thinking* (Burger E, Starbird M, Princeton University Press, 2012).

⁶While our immediate focus is on indigenous peoples living within the national boundaries of the United States, AIMC staff have also worked with indigenous peoples abroad. Available demographics use the terms “American Indian/Alaskan Native” (as in the case of the US Census) and sometimes “Native Americans” to refer to indigenous people living within US boundaries. We recognize that the term “American” is itself contested and is seen by many as broadly applicable to people living throughout North, Central, and South America. For this reason, and after conversation with our indigenous colleagues in the project, we favor the term “indigenous” and further specify “in the US” where applicable.

⁴See, for instance, Gould, Steven J. *The Mismeasure of Man*. New York: W.W. Norton & Co., 1991.

To date, the AIMC has had student and teacher participants representing Diné (Navajo), Hopi, and Apache tribes, as well as members of the nineteen Pueblo Tribes of New Mexico. We have had the guidance, support, and participation of elders from the Diné, Hopi, Chickasaw, Choctaw, and Pomo tribes/nations as well as the elders of the American Indian Science and Engineering Society (AISES). The AIMC is therefore a “circle” in its own right, a gathering of those who work to make the discipline and practice of mathematics as diverse and inclusive as possible.⁷

The contributions of mathematicians have been generous, and their impact significant. For instance, in March and April 2018, AIMC sponsored three mathematicians—Adnan Sabuwala and Maria Nogin of California State University, Fresno, and Tatiana Shubin of San Jose State University—to run a series of math circle sessions on Diné and Hopi reservations, at two Indian boarding schools (Navajo Preparatory School in Farmington, NM, and Santa Fe Indian School) as well as some rural schools in northern New Mexico. We visited sixteen schools, running two to four sessions in each. Altogether more than 500 students and forty-four teachers attended these sessions and enjoyed the beauty and challenge of doing math circle-style mathematics.

Besides school visits, Tatiana Shubin and Donna Fernandez, a Navajo Prep School math teacher serving as an AIMC Regional Coordinator, ran a workshop for teachers at Tuba City High School (AZ). We also helped to run Julia Robinson Math Festivals at Tuba City Boarding School and Many Farms Community Schools (AZ); hundreds of students from grades 1–8 visited these festivals and had fun sharing the joy of problem solving with one another. Other regional coordinators such as Craig Young (Diné reservation) and LaVerne Lomakema (Hopi reservation) are working within their communities to plan events.

For two years in a row, in 2017 and 2018, the AIMC Math Camp at Navajo Prep School has been attracting talented kids from the Four Corners states to participate in a residential camp for students nominated by their grades 7–12 teachers. The camp combines extensive and challenging math sessions and Native American cultural activities as well as various STEM and physical activities. This year, students enjoyed a field trip to the mine of a local Navajo-owned and operated company, the Navajo Transitional Energy Company (NTEC). In both years, the number of applications exceeded our capacity, and we have had to be selective and purposeful in building our summer cohorts.

Most of the thirty-five students at the camp were Diné (Navajo), but we also had students from Hopi and Apache

⁷A colleague recently shared the important distinction between diversity and inclusion: Diversity is being invited to the party, whereas inclusion is being asked to dance. Those who love and promote mathematics ought to be mindful of this distinction, to be sure that we are working beyond simple models of “outreach” toward true “engagement.”



AIMC Summer Math Campers display their results on a probability problem.

tribes. Among the facilitators of math sessions was Frederick Peck (University of Montana, Missoula). Fred has been a team member of the Montana Math Teachers’ Circle Network, a group that members of AIMC supported during their startup. Peck and colleagues are working with Montana tribal educators and community members to start math circles programs. This process involves patience and a respectful approach. We operate on a strict principle of only working where we are invited and recognizing that true and productive partnerships, while the longer-term goal, come only after the sequence of

Invitation → Cooperation → Collaboration → Partnership.

Given the historical traumas endured by generations of tribal members, it takes patience to build the trust and understanding that leads to the initial Invitation stage. As such, expanding the AIMC model requires patience, purpose, and a commitment to ongoing presence. Too often in these communities, initiatives have come and gone, generating the perception of “one-and-done” interventions—feel-good opportunities that ultimately make the work of developing trust and lasting relationships more difficult. Meaningful partnerships take time, commitment, and acknowledgment that partners come to the table as equals.

Recently, and in reflecting on this principle of Invitation outlined above, we recognized the potential of making the first invitation instead of waiting for an invitation. Rather than waiting to be invited into a community only to hope to share examples of math circles, we instead identify a *Champion* within a community showing potential interest in AIMC partnership. We invite that Champion to an AIMC event like the successful AIMC Summer Math Camp, and invite them to contribute as they like, to reflect, and to engage after hours in ongoing conversations about the approach and the “goodness of fit” of the AIMC model to the Champion’s community. It is an opportunity not unlike a working retreat to engage in conversations with the Cham-

pion and to determine if there is a workable path forward to an initial Invitation, hopefully leading to Partnership. In Summer 2018, one such Champion from Alaska was Ann Cherrier, a math teacher interested in building math circles within her community and with a focus on schools and teachers who primarily serve Alaskan Native students.

Yet another part of our vision for changing the culture of mathematics recognizes that there is important work to be done in the mentoring relationship between student participants (usually quite young) and math circle facilitators (often older in years though equally young at heart). Moreover, we recognized that the change we wanted to support within the mathematics community required engaging undergraduate students of mathematics directly. To address this gap, we have taken to including talented undergraduate students from universities across the United States to serve as *Junior Mentors*. Their job is to interact with the student participants and to share with them their love and enthusiasm for mathematics. Sierra Knavel, an undergraduate mathematics major from Ohio University and 2017 Junior Mentor, commented that “Being a peer mentor at the AIMC Math Camp shaped the way I see how accessibility of mathematics outside of school affects the amount of people who study it later on. Specifically, it reminded me of the critical moments in my youth where math was fun, exciting, and puzzling enough to retain my interest as I grew older. Even though undergraduate math is frustrating at times, the AIMC Math Camp reminded me of the positive reasons why I choose to continue with mathematics!”

At the 2018 AIMC Math Camp this role was played by Henry Austin, whose father David Austin is one of the handful of Native Americans holding PhDs in mathematics. Henry’s description of his role in the camp is featured below in the “Perspectives.” David himself served as our Guest of Honor and delivered a final talk at the camp; his perspective is similarly included below.

With the help and support of the entire mathematical community, AIMC will continue the work we have started. The AIMC represents a collection of interested and engaged

professionals and is not a formal nonprofit. We have been grateful for the financial support of the Carnegie Corporation of New York, and the administrative support of the Mathematical Sciences Research Institute, and we are excited to announce that, beginning in 2019, we will have the administrative support of the American Institute of Mathematics.

Future activity for the AIMC includes the following:

- Sustaining our flagship AIMC Math Camp at Navajo Preparatory School, inviting teams of Champions from up to three new sites to observe the camp each year. We anticipate having each team include a mathematics pro-

fessional to serve as a facilitator,⁸ one student camper, and one adult (a teacher, a parent, or a good organizer/administrator from the community). This is an ideal composition for a team of Champions, but even one highly motivated Champion from a site would be welcome.

- Sending mathematicians as visitors to various sites already in operation (e.g., Diné and Hopi schools) as well as prospective sites (e.g., Alaska or Oklahoma). Visitors run math circle sessions for kids and teachers, PD workshops for teachers, and math festivals for students or entire communities. AIMC helps to coordinate and support the visits and collects visitor information.

- Getting the word out and finding Champions. This requires active participation in a number of meetings, in-

cluding annual meetings of both AISES and SACNAS, running sessions, and distributing informational materials. (We have a great success rate with AISES meetings—when Bob attended in 2016, he met and recruited Donna; in 2017 Tatiana recruited the Montana team.)

- Providing limited financial support, as available, for up to three years to each new site. After the initial three years we will continue our support through *Regional Coordinators*—experienced math circle leaders who en-



Junior Mentor Sierra Knavel works with a camper on a problem.

⁸A PhD mathematician is not always available in some of the remote communities in which we work, though scientists, engineers, and others having significant background and training in mathematics content can often serve this purpose.

gage within the AIMC directly to help promote, extend, and support our efforts across regions. Beyond paying Regional Coordinators, AIMC cannot at this point provide monetary support after the initial three years. Each local Indigenous Math Circles (IMC) program would have to seek its own funding, and AIMC will support that effort as we are able, pointing out grant opportunities and helping to shape proposals.

- Maintaining a website with news, materials from the AIMC Math Camp at Navajo Preparatory School, and links to local IMCs. Moreover, starting in 2019, we will have a means for individuals to make financial contributions through the website, supporting our expansion efforts.
- Engaging a Board comprising national leaders with skills and accomplishments in relevant areas such as mathematics, education, indigenous issues, marketing, business, and STEM disciplines. The type of Board and model for meetings/membership was the subject of the meeting, “Founding Board,” which met at the 2018 annual AISES meeting in Oklahoma City to help establish a path for the AIMC’s future.

Short of seeing the camp, the school classroom visits, the teacher professional development workshops, or the festivals, we know of few better ways of understanding the impact of our approach than to hear directly from those who are engaged in this work. What follows is a series of perspectives, and they offer the reader a chance to listen to people’s experiences, to value their stories, and to understand that our stories and our efforts grow out of making meaningful connections with one another. Our hope is that their words motivate you, the reader, to think about your story and how it might connect to our efforts outlined here—to build a diverse and inclusive community of mathematical practice. Both authors invite you to share with us your story, your questions, or your interest in participating



Collaboration is key at Summer Math Camp. Participants huddle to debate a solution.

by using the contact information included in the footnotes about the authors.

PERSPECTIVES

Henry Austin (Junior Mentor. High School Student, Allendale, MI):

I found it quite remarkable to watch students’ eyes light up when they arrived at a solution as it was clear that they were truly interested and engaged in what they were learning. I have taken math for my entire school career...but the math that the students at AIMC learn is much different than a traditional classroom. Students are able to use math as a reasoning tool and understand its application in the natural world, rather than simply arriving at an answer. Upon completion of the camp, I found myself asking, “Why isn’t this math taught in school?”

Ann Cherrier (Champion. Middle School Math Teacher, Anchorage, AK):

I witnessed a group of young people from different communities [bound] together by their heritage and culture and mathematics blossom from scarcely speaking a single word to one another to problem-solving, sharing their thinking, trying new strategies, critiquing each other’s solutions, and getting excited about getting that much closer to a solution. Best of all was seeing all this occur with smiles, synergistic energy, and excitement about mathematics.

What I experienced has motivated me to begin a similar program here in Anchorage, Alaska. Starting on a smaller scale, I am planning to establish an indigenous math circle after-school program at two Title I middle schools in the Anchorage School District. Mathematical literacy is a powerful weapon in the fight against poverty and in promoting healthy families and good paying jobs.

Craig Young (Regional Coordinator. Middle School Math Teacher, Tuba City, AZ):

Currently, I am the Regional Coordinator for AIMC West, covering the Western Navajo Nation, roughly half the size of the state of West Virginia. I have been involved with the Navajo Math Circles Project as a college student and later as an educator. I run a Math Circle Program at Tuba City Boarding School in Tuba City, Arizona. Our Navajo youth needed a program that empowered them in a traditional holistic approach that complimented their way of life and way of thinking. All indigenous communities have a history of complex mathematics and

sciences through their cultures, and the Math Circles help our indigenous youth. Our students rediscover themselves and are confident after going through our program.

LaVerne Lomakema (Regional Coordinator. High School Math Teacher, Keams Canyon, AZ):

As a Native teacher on a Native American reservation, I see the crucial need to stress the importance of the mathematical field in our education system. Most Native communities are already economically disadvantaged, this creates a huge stress on many communities. If students could understand the role of mathematics in their communities and within mainstream society, their opportunities would be endless. The key is to get the kids excited about math and realize the many doors it could open for them in the future. The math circles model is a great opportunity for Native children to look at the field of mathematics in a different context. It gives the students a chance to explore and understand mathematical models. Math circles would be a great opportunity for any Native school.

Donna Fernandez (Regional Coordinator. IB Coordinator, Navajo Preparatory School, Farmington, NM):

I coordinate the four Corners Math Teachers' Circle (4CMTC) for the tribal communities. The teachers are in K–12 and primarily serve Native American students. Being a part of the Math Teachers' Circle has given me a renewed sense of the joy of math and built my confidence in problem solving. I get so excited to work with other teachers/professionals talking about the multiple ways to solve a problem. The camps and student circles show me how, when we give students open-problems and allow them to collaborate, they dig deep for patterns and understanding.

AIMC Math Circles values [my students'] intuitive nature of looking at patterns in our world for understanding. It allows students and teachers the opportunity to think about the math and not just computational and memory skills. Native American communities benefit from this model; teachers and students feel empowered by collaborating and struggling. Dr. Henry Fowler tells students at each camp that our Native communities need leaders and the best way to be a leader is to have a strong educational foundation, and math is the base of the foundation. It makes you feel confident to be that leader.

Natanii Yazzie (Participant. High School Student, Gallup, NM):

I have been a participant of the AIMC and NNMC for four years and recently I've been a team captain for three years representing my group and providing feedback for the math camp competition presentations. Through the years of attending the math circles, I've developed a passion for mathematics that truly changed my academic path and ignited an ambition I never knew I had inside of me. I've gained many valuable friendships and connections with professors across the nation with whom I could learn from to better prepare for college. It has been an absolutely eye opening and life changing experience for me, and I wish for other Native groups to attend this amazing program. I found the program to be effective at growing critical thinking skills essential for success in college, and for thriving in both modern and traditional Native worlds.

David Austin (Facilitator and Keynote Speaker. Professor, Grand Valley State University, MI):

For many members of underrepresented groups, education can appear to be a path leading away from one's family and culture and into some new and strange place. Particularly for indigenous students in the American southwest, who frequently deal with poverty, geographic isolation, and limited educational opportunities, education and the opportunities that come with it can often separate families physically or by experience and values.

The past two summers, I have had the pleasure of participating in summer math camps for indigenous 6–12th graders. The camps are structured to support students in their own culture. First, a group of mathematicians travels to their land, and, while some students may travel many hours to reach the camp, they share a common background with their fellow campers. Families, particularly parents, are welcomed into the camp as well and participate in the opening and closing ceremonies. Traditional meals served by tribal elders are sometimes on the menu. Every day includes cultural activities, led by Native mentors, that are both fun and authentic. All of this conveys a message to the campers and their families that they belong and are safe.

The mathematical content of these camps is similarly rich. In a typical day, students collaboratively explore problems that require little mathematical prerequisite knowledge; all

students, from grades 6 through 12, are able to work on the same problems. At the same time, these problems are often open-ended and naturally lend themselves to further exploration in considerable depth. At the end of the camp, students participate in a “math wrangle,” a friendly competition in which teams of students present their work to the entire group and are assessed on the quality of their presentation.

What’s more, the AIMC hosts workshops for teachers in schools with a large Native population and provides support to teachers leading math circles for students in their schools. In short, the goal of the AIMC is to embed meaningful mathematics within indigenous culture and provide support so that it can flourish on its own.

As a member of the Choctaw nation who grew up in Oklahoma, my experiences don’t perfectly overlap with those of the campers, but I do know the challenge of working to become a professional mathematician while feeling like “home” is far away. This is something that I’m able to share with the students, and I hope they hear in my story a message that it is possible to learn to live in two worlds at the same time, that there is great meaning to be found in accepting that challenge, and that there are resources to help.

On the final day of the camp, I gave a presentation to students and their families about sunflower seeds and continued fractions. What was particularly pleasing to me was that parents responded to questions that I asked as frequently as students and often with a look of surprise that they were able to contribute. I hope that this gave parents and families an understanding of what the students experienced at the camp and a taste of where they may be headed. While there are more opportunities to expand this work into other indigenous communities, these ideas may be useful as mathematicians reach out to welcome other groups into our discipline.

Perspectives such as these motivate our work and remind us of how much we, the authors and directors of AIMC, are learning as we engage in this effort. One important conclusion we have come to after working with indigenous youths for a number of years is that, while it is easy to look at the statistics on poverty, academic success, STEM participation, and the like, and to construct a model that casts “indigenous” as a deficit, this superficial approach fails to recognize the inherent assets or affordances that indigenous students

bring to mathematical problem solving and post-secondary participation in STEM fields. Many of these students are bilingual, speaking their indigenous languages at home and English outside of the home. Bilingual children have been shown to acquire third and fourth languages with easier facility than monolinguals. In our experience, the flexibility of mind required to approach new grammars and vocabularies constitutes a true asset in terms of learning mathematics. Moreover, many indigenous languages reflect embedded philosophies that are radically different from the Western philosophies embedded in English. Being fluent in English and an indigenous language therefore magnifies that flexibility of mind, fostering the kind of creativity that leads to great mathematical discoveries.

Indigenous peoples are bicultural (and often multicultural) by geography and history, navigating every day the norms, traditions, and world views of those cultures. We posit that with two cultures, they have not just average abilities, but extraordinary gifts for learning third and fourth cultures. Mathematics is just such a culture that they can absorb nimbly. The Alliance of Indigenous Math Circles is an effort to bring together many beautiful cultures, knowing that a culture of mathematics, as Craig Young says in his perspective above, always has been part of indigenous culture, and that as cultures mix, they change.



Bob Klein



Tatiana Shubin

Credits

Article photos are by Bob Klein.

Photo of Bob Klein is by Ben Siegel.

Photo of Tatiana Shubin is by George Csicsery.

CALL FOR APPLICATIONS & NOMINATIONS

Treasurer of the American Mathematical Society

The American Mathematical Society is seeking applications and nominations of candidates for the position of Treasurer. The Treasurer is an officer of the Society and is appointed by the Council for a two-year term. The first term of the new Treasurer will begin February 1, 2021, with initial appointment expected in January 2020 in order that the Treasurer-designate may observe the conduct of Society business for a full year before taking office.

All necessary expenses incurred by the Treasurer in performance of duties for the Society are reimbursed.

QUALIFICATIONS

The Treasurer should be a research mathematician and must have substantial knowledge of Society activities. Although the Treasurer is appointed by the Council for a term of two years, candidates should be willing to make a long-term commitment, as it is expected that the new Treasurer will be reappointed for subsequent terms pending successful performance reviews.

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- Monitor the receipt and expenditure of funds and the care of investments.
- Monitor budgets and trends of finance over periods of years.
- Review salary policy for AMS employees and its applications to individuals.
- Serve *ex officio* as a member of the Board of Trustees, Council, and several other committees; the Treasurer chairs the Audit and Risk, the Investment, and the Salary committees.

APPLICATIONS & NOMINATIONS

A Search Committee, with Alejandro Adem as Chair, has been formed to seek and review applications. Persons wishing to apply should do so through MathPrograms.Org. Nominations and questions should be directed to the Chair of the Search Committee: tsc-chair@ams.org.

For full consideration, applications, nominations, and supporting documentation should be received by **May 15, 2019**.



Broadening Math Outreach

Glen Whitney

Note: The opinions expressed here are not necessarily those of Notices.

As I think anyone reading these *Notices* is aware, mathematics suffers from a tremendous image problem in this country. For large segments of the population, math is the subject we love to hate. People proclaim almost proudly that they “are terrible at math,” something they would never do concerning other topics like reading or American history. We know the stereotypes: people who like math are out of touch, maladjusted, or just plain weird. People outside our profession are also confused about what mathematicians do; maybe work out the solutions to really tough calculus exercises?

The problem isn’t only anecdotal. Just to scratch the surface of quantitative investigation into attitudes toward mathematics, I’ll mention a couple of recent publications. In a 2017 national survey of teenagers by the Thomas B. Fordham Institute [3], one question asked respondents to choose their least favorite subject in school. Mathematics was the overwhelmingly most common answer, with 34% of the respondents selecting it. A 2016 survey article by Dowker, Sarkar, and Looi [1] cites several studies that suggest “attitudes to mathematics tend to deteriorate with age during childhood and adolescence,” as well as other studies that indicate that math anxiety affects variously some fraction of students ranging from 6% to 68% (depending on the definition of “math anxiety” and the population under consideration).

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Not only is this situation real, it’s a problem that we as mathematicians should be concerned about. Some reasons are obvious: As technology advances, mathematical skills, interpreted broadly, become ever more important in the workplace, and we need to help foster a productive workforce of the future. With the AMS Annual Survey showing the fraction of PhDs awarded to Americans flat at slightly under 50%, should the rate of foreign students leaving the country after graduate school rise, there could even be a problem replacing our profession’s own ranks. Other reasons may be more indirect, but nevertheless warrant concern: Can we assume a reliable supply of public or even private funding from a society that fails to appreciate or even understand what mathematicians do? Most of us are in this profession because of the thrill and beauty of mathematical discovery; isn’t there intrinsic value in sharing what we can of those experiences to as broad a population as possible?

Hence, there’s a need for institutions dedicated to improving the public perception of mathematics. The image problems aren’t going to solve themselves. Certainly, the commercial success of technology and financial firms based on their use of mathematics provides some good raw material for improving the status of mathematics, but someone has to be advocating on behalf of mathematics using that material. And I’m specifically interested in outreach directly to general audiences. Ideally, high-quality, engaging but demanding public education in mathematics would solve much of the problem. Working toward that goal is of course a good idea, and there exist many fine efforts in that direction, but there are entrenched reasons why public-education reform alone is unlikely to fix perception

problems plaguing mathematics in at least the short-to-medium term. These reasons include the limitations imposed by the elementary and secondary canon of math topics, the nature of standardized testing, generally less stringent math-background requirements for elementary educators, and the economic interests of businesses involved in school education.

In the early 2000s, well before I had contemplated most of the above issues consciously, I had the opportunity to visit the Goudreau Museum of Mathematics in Art and Science. (At the time, to visit at all, the Goudreau required at least ten people to come there together, and the visit had to be by appointment.) Whatever the popular image of mathematics might be, here was a place where math was thoroughly celebrated, played with, and enjoyed. The museum consisted of just two converted classrooms in a Herricks, Long Island junior-high-turned-community-center, densely packed with games and puzzles; its ceilings burgeoned with a stunning array of polyhedral models. I remember thinking when leaving, "What a great country this is – there can be a museum about anything, even mathematics!" Tacitly, I was acknowledging to myself the fringe status of mathematics in American culture. In the following years, however, I became the "coach" of the after-school mathematics club at my local elementary school, and I began to reflect more consciously on math's social status.

Thus in 2008, when I learned of the closure of the Goudreau Museum, I conceived of a new math museum project on a larger scale and with an explicit mission to improve the public perception of mathematics. By mid-2009, the project dubbed "MoMath" had opened a traveling exhibition, the *Math Midway*, and kicked off a capital campaign to fund the opening of a physical museum. In late 2012, that museum opened its doors in New York City. Along the way, MoMath organized or co-organized a long-running series of public lectures on mathematical topics, Math Encounters, a biannual conference on recreational mathematics, MOVES, and a biannual conference on math outreach, MATRIX (so far held just in Europe). My connection with MoMath ended in the fall of 2017, shortly before it celebrated its fifth anniversary open to the public. Despite its accomplishments and milestones, MoMath is not enough to resolve or even adequately address math's public perception problems. This is a big country, so naturally these are big problems, beyond the scope of any one institution.

So, what could improve math outreach further? There are numerous math blogs and video channels that undoubtedly reach a larger audience than it's possible for a single physical museum to do, so stronger ties and mutual support between those forms of communication and organizations focusing on math outreach could amplify the effectiveness of their efforts. Careful investigation of the impact of exhibits and programs, and creative experimentation with how they are presented, could help strengthen the link between what's presented and what the audience recognizes

as "mathematics." All too often, visitors to math exhibits respond along the lines that "this is just a playground with numbers thrown in," not really mathematics.

Even when someone does connect an activity or program with "math," we need to provide more compelling narratives about those mathematical topics to win the hearts and minds of participants. We need to do a better job of connecting mathematics to peoples' lives and the world around them, while continuing to celebrate the value and beauty of pursuing mathematical ideas for their own sake. We need to show ways that mathematics has improved the human condition, and highlight its role in other accomplishments of our civilization. And to broaden the audience that we can connect with, we need to do a better job of humanizing mathematics and mathematicians. The stories of a wide range of mathematicians, especially contemporary practitioners, with whom diverse audiences can identify and whose passions they can come to understand, can be a powerful tool for engagement with mathematics.

So what are some elements of a broader effort to solve mathematics' image problem? First, we need more institutions dedicated to the effort. Some of these are or will be existing organizations. There's not space here to mention all of the existing institutions and programs I'm aware of, and I know there are many others that I've yet to have the pleasure of learning about. To help strengthen the network of math outreach efforts in this country, and avoid fragmentation of the community involved in such efforts, I have established a list of such programs (at studioinfinity.org/outreach). If you know of or participate in a program not mentioned here, please visit and submit your program to the list!

So, what are some notable efforts out there already? The Mathematical Sciences Research Institute (MSRI) is strengthening its outreach activities: In Spring 2019, it is organizing the third National Math Festival. Modeled after the collection of science festivals that arose in the US in the past decade, this festival in Washington, DC, will consist of a day of public speakers, temporary exhibits, and group activities celebrating the beauty and diversity of mathematics. MSRI also established the National Association of Math Circles, which fosters groups of interested students who meet periodically to creatively explore wide-ranging mathematical topics with the guidance of math professionals. The American Institute of Mathematics administers the Julia Robinson Mathematics Festivals, instigated by Nancy Blachman. These JRM Festivals occur in many sites around the country each year, and present a buffet of facilitated tabletop math activities carefully chosen for their power to engage a broad audience. The Association of Women in Mathematics organizes and sponsors Sonia Kovalevsky days, consisting of "a program of workshops, talks, and problem-solving competitions for female high school and middle school students and their teachers, both women and men."

The US also needs more mathematics museums; if Germany can support ten, certainly there should be more than one in this country. Fortunately, some efforts are already underway. Frederic Mahieu is organizing a Math Cultural Center of Chicago (info@mathculturalcenter.org). His vision is to “approach math through its culture, history and applications and share fascinating stories about math,” but also to give visitors ample opportunities for “exploring all aspects of what math is: doing, thinking, observing, creating.” There is another math museum project in Boston at an even earlier stage in development, and plenty of room for more such efforts around the country.

Second, we need to strengthen the math outreach activities at related institutions. Just based on membership in the Association of Science and Technology Centers, there are over 600 science museums in the country. Currently, only a small percentage mount significant mathematics exhibitions or have robust permanent exhibits celebrating mathematics in its own right. (Notable contributors to public math content include the Oregon Museum of Science and Industry, the Science Museum of Minnesota, the Boston Museum of Science, the New York Hall of Science, and the North Carolina Museum of Life and Science, among others.) Providing attractive, low-cost, high-visitor-engagement exhibit and programming options, and advocating for their use at science centers, would go a long way toward geographically broadening math outreach.

Also, American universities, colleges, and math departments could do more. Numerous universities in the United Kingdom have offices and/or directors of mathematics outreach (just try searching the phrase “UK university director of math outreach”); imagine the impact if a similar percentage of American higher educational institutions followed suit. (In fact, as White and Pantano pointed out a couple of years ago in these *Notices* [4], there’s a great deal that the US could learn from—and contribute to—international math outreach, and certainly I applaud efforts toward such interchange.) There are already some efforts at US institutions along these lines: The STEAM Factory at OSU, organized by Jim Fowler and others, has created an interdisciplinary network that works together to make math and science accessible to general audiences in the Columbus, OH, area. The Arizona Math Road Show by Bruce Bayly et al. at the University of Arizona has brought an entertaining and participatory performance-based math program to numerous schools and other venues, literally driving it across the country in a reconditioned school bus. Dwyer and Schovanec report on a variety of math outreach initiatives at Texas Tech in their 2013 *Notices* article [3]. I know there are others, and I hope you’ll add them to the developing list on-line. But more such initiatives are needed, and they can make the “broader impact” efforts required by many granting agencies significantly more meaningful and effective.

Third, we need additional types of activities and delivery methods, to broaden the audience. Participants need to see people they identify with as role models presenting the material; they need to see peers excited about mathematics. Demographic studies show that visitors to science museums are significantly less racially and economically diverse than the general population. So we need to take math outreach to places where we can reach those who are not coming to science centers: to visit schools and after-school programs, or even to stadiums, malls, or basketball courts. (Certainly there already exist programs designed to reach diverse populations. One example is the Art of Problem Solving’s Bridge to Enter Advanced Mathematics headed by Dan Zaharopol, which identifies groups of promising middle schoolers from underrepresented populations in New York and Los Angeles and provides summer programs highlighting the thrill of problem-solving and the breadth of mathematics, along with ongoing support through college applications. But more programs that expand the population who can become engaged with mathematics are needed. It’s also important to broaden the age range we reach; for example, the Bedtime Math enterprise started by Laura Overdeck provides books, math club kits, and other materials that can foster positive attitudes toward math for 3–8 year olds. There’s definitely room for programs tailored to older Americans as well. And finally, we need effective methods to measure the impact of all these efforts; it’s nearly impossible to optimize what you can’t measure (or aren’t measuring).

All of these enterprises need the involvement of professional mathematicians. The science festival movement mentioned above provides many opportunities for practicing scientists to meet with general audiences and talk about or show off what they do. But in my conversations with science festival staff, they express disappointment at the number of mathematicians who respond to calls for participation, as compared to biologists, chemists, physicists, etc. Partly as a result, most science festivals have little mathematics content. And presuming that we can increase the number of math museums and other math-outreach organizations, all of those institutions will need advisors, idea generators and contributors, and fact-checkers. (Let’s make sure to get the math right when we’re using it for outreach!) So keep your eyes open for opportunities to help with math outreach and get involved—you’ll find, as I have, that it’s an exciting, challenging, and fundamentally rewarding pursuit.

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Glen Whitney

Credits

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Co-managed by the AMS and the MAA for many years, the Joint Mathematics Meetings (JMM) is the world's largest gathering of mathematicians.

Starting in 2022, the AMS will manage the JMM solely, while the MAA and other organizations will continue to participate.

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Falconer's Conjecture?

Alex Iosevich

The Statement of the Problem

Many problems in mathematics take the following form. Suppose that X, Y are sets and $f : X \rightarrow Y$ is a function. Suppose that X is sufficiently large and f is suitably non-trivial. Then $f(X)$ takes up a substantial portion of Y . A classical example of this phenomenon is Picard's Little Theorem, which says that any entire analytic function whose range omits two points must be a constant function.

Let $X = E \times E$, $Y = \mathbb{R}$, and $f(x, y) = |x - y|$, where E is a compact subset of \mathbb{R}^d , $d \geq 2$, and $|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}$. The Falconer distance problem asks how large does the Hausdorff dimension of E needs to be to ensure that the Lebesgue measure of the distance set

$$\Delta(E) = \{|x - y| : x, y \in E\}$$

is positive.

In this context, it is sufficient to think of Hausdorff dimension of a compact set E , denoted by $\dim_{\mathcal{H}}(E)$, in the following way. There exists a Borel measure supported on E such that for every $\alpha < \dim_{\mathcal{H}}(E)$, the α -energy integral

$$I_{\alpha}(\mu) = \iint |x - y|^{-\alpha} d\mu(x) d\mu(y) < \infty. \quad (1)$$

The background and the details pertaining to the Hausdorff dimension and energy integrals are beautifully described in Falconer's "Geometry of Fractal Sets" ([5]), and Mattila's "Fourier Analysis and Hausdorff Dimensions" ([12]).

This problem can be viewed as a more delicate variant of the celebrated Steinhaus Theorem, which says that if $E \subset$

\mathbb{R}^d is of positive Lebesgue measure, then $E - E = \{x - y : x \in E, y \in E\}$ contains an open ball centered at the origin.

The Falconer Distance Conjecture says that if the Hausdorff dimension of $E \subset \mathbb{R}^d$, $d \geq 2$, is greater than $\frac{d}{2}$, then the Lebesgue measure of $\Delta(E)$ is positive. This problem was formulated by Falconer in 1985 ([6]).

Connections with the Erdős Distance Problem

The Falconer Distance Conjecture is a continuous analog of the Erdős Distance Conjecture, which says that if $P \subset \mathbb{R}^d$, $d \geq 2$, is a finite set, then for every $\epsilon > 0$ there exists $C_{\epsilon} > 0$ such that

$$\#\Delta(P) \geq C_{\epsilon} (\#P)^{\frac{2}{d} - \epsilon}.$$

This problem was introduced by Erdős in 1945, and after 66 years of efforts by many of the most prominent experts in combinatorics and related fields, the problem was finally solved in two dimensions by Guth and Katz ([9]). In higher dimensions, the problem is still open, with the best exponents due to Jozsef Solymosi, Cszaba Toth, and Van Vu (see [15]).

Sharpness of the Erdős/Falconer Exponents

It is important to note that the conjectured exponent $\frac{d}{2}$ in the Falconer distance problem and the exponent $\frac{2}{d}$ in the Erdős distance problem are strongly linked. Let $P_q = \mathbb{Z}^d \cap [0, q]^d$. Then $\#P_q \approx q^d$. The size of $\Delta(P_q)$ does not exceed the number of values of the quadratic form $x_1^2 + x_2^2 + \cdots + x_d^2$, $x_j \in [0, q]$, which is bounded by $q^2 + q^2 + \cdots + q^2 = dq^2$. Setting $n = q^d$, we see that $\#\Delta(P_q) \leq dn^{\frac{2}{d}}$, and the sharpness of the $\frac{2}{d}$ exponent in the Erdős distance problem is established.

In order to establish the sharpness of the $\frac{d}{2}$ exponent in the Falconer distance conjecture, we bootstrap off the Erdős distance problem example above. Let $q_1 = 2$, $q_{i+1} =$

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q_i^s . Let E_i^s , $s \in (\frac{d}{2}, d)$ denote the $q_i^{-\frac{d}{s}}$ -neighborhood of $\frac{1}{q_i} P_{q_i}$. A result in Falconer's book ([5]), Chapter 8, shows that the Hausdorff dimension of $E^s = \cap_i E_i^s$ is equal to s . On the other hand,

$$|\Delta(E_i^s)| \leq C q_i^{-\frac{d}{s}} \cdot \# \Delta(P_{q_i}) \leq C' q_i^{2-\frac{d}{s}},$$

from which it follows that $|\Delta(E^s)|$, the Lebesgue measure of E^s , may be 0 if $s < \frac{d}{2}$, thus establishing the sharpness of the $\frac{d}{2}$ exponent up to the endpoint.

The L^∞ Theory

In order to understand how many distances a set $E \subset \mathbb{R}^d$, $d \geq 2$, determines, one cannot avoid studying the incidence function that counts how often a fixed distance occurs. In the discrete case this is simply a matter of counting the number of pairs of elements from E whose pairwise distance equals a given value. In the continuous case one must proceed a bit more carefully. Let σ_t denote the surface measure on the sphere of radius $t > 0$ centered at the origin. Let ρ be a smooth cut-off, $\equiv 1$ in the unit ball and vanishing outside a slightly larger ball. Let $\rho_\epsilon(x) = \epsilon^{-d} \rho(\frac{x}{\epsilon})$, and define $\sigma_t^\epsilon(x) = \sigma_t * \rho_\epsilon(x)$. Let

$$\nu^\epsilon(t) = \int \int \sigma_t^\epsilon(x-y) d\mu(x) d\mu(y),$$

where μ is a Borel measure supported on E . One should think of this quantity as the ϵ -approximation of the incidence function on $\Delta(E)$, which counts pairs of points in E separated by the distance t . Also, at least heuristically and this can be made quite precise, $\lim_{\epsilon \rightarrow 0^+} \nu^\epsilon(t)$ is the distance measure ν defined by the relation

$$\int f(t) d\nu(t) = \int \int f(|x-y|) d\mu(x) d\mu(y). \quad (2)$$

Falconer observed by a simple covering argument that if one can show that $\nu^\epsilon(t)$ is uniformly bounded, then the Lebesgue measure of $\Delta(E)$ is positive. More precisely, cover $\Delta(E)$ by the collection $\{(t_i - \epsilon_i, t_i + \epsilon_i)\}$. The following is a formal argument that can be made precise with a tiny bit of work. We have $1 = \mu \times \mu(E \times E)$

$$\begin{aligned} &\leq \sum_i \mu \times \mu\{(x, y) : t_i - \epsilon_i \leq |x-y| \leq t_i + \epsilon_i\} \\ &\leq C \sum_i \epsilon_i \nu^{\epsilon_i}(t_i) \\ &= C \sum_i \epsilon_i \int \int \sigma_{t_i}^{\epsilon_i}(x-y) d\mu(x) d\mu(y) \\ &= C \sum_i \epsilon_i \int |\hat{\mu}(\xi)|^2 \hat{\sigma}_{t_i}^{\epsilon_i}(\xi) d\xi. \end{aligned}$$

Using the method of stationary phase (see e.g. [14]), it is not difficult to see that

$$|\epsilon_i \hat{\sigma}_{t_i}^{\epsilon_i}(\xi)| \leq C |\xi|^{-\frac{d-1}{2}} \cdot \min\{|\xi|^{-1}, \epsilon_i\}. \quad (3)$$

Plugging the estimate (3) back in and tracing the inequalities backwards, we see that this quantity is bounded by $\sum_i \epsilon_i \cdot \int |\xi|^{-\frac{d-1}{2}} |\hat{\mu}(\xi)|^2 d\xi$.

By a simple Plancherel style argument, this expression equals

$$\begin{aligned} &\sum_i \epsilon_i \cdot \int \int |x-y|^{-\frac{d+1}{2}} d\mu(x) d\mu(y) \\ &= I_{\frac{d+1}{2}}(\mu) \cdot \sum_i \epsilon_i \leq C \sum_i \epsilon_i \end{aligned}$$

if the Hausdorff dimension of E is greater than $\frac{d+1}{2}$, as we explain in the paragraph preceding the formula (1). It follows that $\sum_i \epsilon_i \geq \frac{1}{C} > 0$, which implies that the Lebesgue measure of $\Delta(E)$ is positive.

The L^2 Theory: Setup

In the previous section we obtained a good exponent for the Falconer Distance Problem by obtaining an L^∞ estimate for the smoothed out measure on the distance set. In order to improve the exponent, we are going to describe the method that only relies on L^2 bounds for the distance measure ν . Observe that if $\nu \in L^2$, then

$$1 = \left(\int d\nu(t) \right)^2 \leq |\Delta(E)| \int \nu^2(t) dt \leq C |\Delta(E)|,$$

which would imply that $|\Delta(E)| \geq \frac{1}{C} > 0$.

The advantage of this point of view is two-fold. First, it is typically far easier to prove that something is in L^2 than to show that it is in L^∞ . Second, it turns out that the L^∞ bound on ν^ϵ , independent of ϵ , is not even true in general if the Hausdorff dimension of the underlying set is $< \frac{d+1}{2}$. This was shown by Mattila in two dimensions ([11]) and by the author and Senger ([10]) in three dimensions. In higher dimensions the question is still open, but the author and Senger ([10]) showed that ν^ϵ is not in L^∞ with constants independent of ϵ in dimensions four and higher if the Euclidean distance is replaced by a suitable variant of the parabolic metric.

Another advantage of L^2 norms is that Plancherel comes into play. Mattila proved that if the Hausdorff dimension of a compact set $E \subset \mathbb{R}^d$ is $> \frac{d}{2}$, μ is a Borel measure supported on E and

$$\mathcal{M}(\mu) = \int \left(\int_{S^{d-1}} |\hat{\mu}(r\omega)|^2 d\omega \right)^2 r^{d-1} dr < \infty, \quad (4)$$

then the distance measure ν introduced above has an L^2 density, and thus $|\Delta(E)| > 0$.

Mattila derived this result using the method of stationary phase and properties of Bessel functions. We are going to sketch a geometric derivation obtained by Greenleaf, the author, Liu, and Palsson ([7]) where more complicated geometric configurations are also studied.

Recalling the definition of the distance measure ν in (2), we see that in order to compute $\int \nu^2(t) dt$ we must come

to grips with quadruplets $x, y, x', y' \in E^4$ such that $|x - y| = |x' - y'|$. In reality we must consider quadruplets where distances are close to equal and then devise a careful limiting process, but let's keep going. If $|x - y| = |x' - y'|$, then there exists $g \in O_d(\mathbb{R})$ (the orthogonal group) such that $x - y = g(x' - y')$.

In the plane this g is unique. In higher dimensions, one must consider the appropriate stabilizer. Rewriting the equation we obtain $x - gx' = y - gy'$ and this has the L^2 norm of the natural measure on $E - gE$ written all over it. More precisely, define the measure ν_g by the relation

$$\int g(z) d\nu_g(z) = \int \int g(u - gv) d\mu(u) d\mu(v). \quad (5)$$

Arguing in this way we can show that

$$\int \nu^2(t) dt = \int \int \nu_g^2(z) dz dg,$$

where dg is the Haar measure on $O_d(\mathbb{R})$, provided that both sides make sense. The Fourier transform of ν_g is easy to compute using the formula (5). By Plancherel we conclude that

$$\begin{aligned} \int \int \nu_g^2(z) dz dg &= \int \left\{ \int |\hat{\mu}(g\xi)|^2 dg \right\} |\hat{\mu}(\xi)|^2 d\xi \\ &= c \int \left(\int_{S^{d-1}} |\hat{\mu}(r\omega)|^2 d\omega \right)^2 r^{d-1} dr \equiv \mathcal{M}(\mu). \end{aligned}$$

L^2 -theory: Wolff–Erdogan

Until very recently, the best known results on the Falconer distance problem were due to Wolff ([16]) in the plane and Erdogan (IMRN, 2006) in higher dimensions. They proved that the Lebesgue measure of the distance set is positive, provided that the Hausdorff dimension of the underlying set is $> \frac{d}{2} + \frac{1}{3}$. We shall briefly comment on the more recent efforts, but for now let us describe the $\frac{d}{2} + \frac{1}{3}$ theory that laid the foundation for further progress. The key estimate established by Wolff and Erdogan is the following.

$$\int_{S^{d-1}} |\hat{\mu}(t\omega)|^2 d\omega \leq C(d, s, \epsilon) t^{\epsilon - \left(\frac{d+2s-2}{4}\right)} I_s(\mu), \quad (6)$$

where $I_s(\mu) = \int \int |x - y|^{-s} d\mu(x) d\mu(y)$ is the energy integral of μ . Plugging this back into (4) yields

$$\begin{aligned} \mathcal{M}(\mu) &\leq C \int \int |\hat{\mu}(t\omega)|^2 t^{d-1} t^{\epsilon - \left(\frac{d+2s-2}{4}\right)} I_s(\mu) d\omega dt \\ &= C \int |\hat{\mu}(\xi)|^2 |\xi|^{\epsilon - \left(\frac{d+2s-2}{4}\right)} I_s(\mu) d\xi \\ &= C' I_{\epsilon + \frac{3d-2s+2}{4}}(\mu) I_s(\mu), \end{aligned}$$

which is bounded if $\dim_{\mathcal{H}}(E) > \frac{d}{2} + \frac{1}{3}$.

Recent Advances

After a long hiatus, the advances on the Falconer distance conjecture started coming again in recent months. X. Du,

L. Guth, H. Wang, B. Wilson, and R. Zhang ([2]) obtained the dimensional threshold $\frac{9}{5}$ in \mathbb{R}^3 and improved the threshold for $d \geq 4$ as well. Their higher dimensional threshold for $d \geq 4$ was further improved by X. Du and R. Zhang ([3]) to $\frac{d^2}{2d-1} = \frac{d}{2} + \frac{1}{4} + \frac{1}{8d-4}$.

What is behind all this activity? Several key recent advances in harmonic analysis come into play and perhaps the most important of these is the connection with the Schrodinger operator. Du and Zhang deduced their $\frac{d^2}{2d-1}$ threshold from the following Schrödinger estimate. Let $n \geq 1$, $\alpha \in (0, n+1]$ and μ a compactly supported Borel measure such that $\mu(B(x, r)) \leq Cr^\alpha$. Then

$$\|e^{it\Delta} f\|_{L^2(B(\vec{0}, R); d\mu_{R(x,t)})} \lesssim R^{\frac{\alpha}{2(n+1)}} \|f\|_2,$$


from which they deduced a good bound for the spherical average in (6).

In two dimensions, Guth, Iosevich, Ou, and Wang ([8]) improved the dimensional threshold to $\frac{5}{4}$, proved a pinned version of the result, and extended it to a variety of smooth metrics. In this setting, a completely new approach needed to be developed because the authors proved that for any $\alpha < \frac{4}{3}$ there exists a planar set of Hausdorff dimension α such that (4) is infinite. They solved this problem by considering $E_1, E_2 \subset E$ separated by distance ~ 1 , and letting μ_1 and μ_2 be Frostman measures on E_1, E_2 . They divided μ_1 into $\mu_1 = \mu_{1, \text{good}} + \mu_{1, \text{bad}}$, where $\mu_{1, \text{bad}}$, roughly speaking, comes from the example where the L^2 norm of the distance measure is infinite. They showed that the L^1 norm of $\mu_{1, \text{bad}}$ is not too large using a beautiful projection estimate due to Orponen ([13]). This reduced matters to obtaining an upper bound for the L^2 norm of $\mu_{2, \text{bad}}$, which was accomplished via a suitable Schrödinger type estimate partly based on the decoupling theorem of Bourgain and Demeter ([1]).

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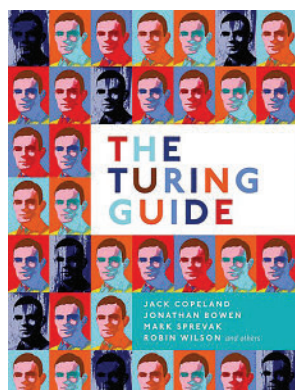
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Only Human

A review by Bjørn Kjos-Hanssen



The Turing Guide

Jack Copeland, Jonathan Bowen, Mark Sprevak, Robin Wilson, and others
Oxford University Press, 2017
ISBN 978-0-19-874783-3

Introduction

A victorious yet tragic hero. A genius but ominously an augur of the end of human dominance. Such characterizations

come to mind regarding Alan Mathison Turing (1912–1954) after reading the *The Turing Guide*.

It is a 500-page compilation of articles by many authors, written for “general readers,” which strikes a balance between focusing on Turing himself and on the collection of topics he was involved in. The driving force behind the book is philosopher Jack Copeland, who has written many books and articles about Turing and participates in sixteen of the forty-two chapters of the *Guide*. Officially, the author list is Copeland, Bowen, Sprevak, Wilson, “and others.”¹ The chapters are lightly cross-referenced but are largely independent. The book is solidly proofread: I got to page 55 before finding the first error (“during the did decades”).

Turing’s appeal in the popular imagination may stem from checking several boxes: he is viewed as a genius and

a hero, perhaps even a tragic hero. In support of the genius label, he defined a mathematical notion of “computer” that turned out to be the right one, proved some fundamental results (existence of the universal computer, unsolvability of its halting problem), and arguably founded mathematical biology (see Biological Growth below). As for heroism, he worked on cryptography during World War II and led a large team. However, the claim in the Preface that

It is no overstatement to say that, without Turing, the war [...] might even have been won by the Nazis.

is, in Chapter 9, modified to indicate that perhaps his work helped ensure that the war ended in 1945 rather than 1946. As for tragedy, he was convicted of homosexuality, ordered into female hormone therapy, and may have committed suicide.

Turing earned the standing to present to us all his thoughts on human and machine intelligence and, as discussed below, those thoughts now seem prophetic.

The *Guide* is divided into eight parts, each worth a section of this review.

Biography

In this part we learn many interesting facts about Turing. He thought that intellectual activity mainly consists of various search algorithms and that we should expect machines to take control. This is a possible counterpoint to the label of hero: perhaps he hastened the day of the “singularity” when machines take over and render humans irrelevant.

It is argued that he took his court-ordered female hormone therapy with an impressively resilient attitude, treating it almost as a case of freshman hazing. If true, that tends to make him less a tragic hero and more a simply mysterious hero. Turing died in a manner that involved cyanide and a lab next to his bedroom, but the jury is still out on whether it was suicide or some kind of experiment gone awry.

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¹Amusingly, as of March 8, 2018, MathSciNet gives “and others” the author ID 1227985 and reports that this book is his or her only publication so far.



Figure 1. Enigma encryption machine.

His work on morphogenesis [17] is here described as “even deeper” than the discovery of DNA molecules.

The Universal Machine and Beyond

Copeland writes about the move from electromagnetically controlled relays to blindingly fast digital electronics, first used for breaking German codes. These new devices would later mesh stunningly with the notion of a universal Turing machine. Turing’s Automatic Computing Engine (ACE) ran at 1 MHz, which outperformed the competition at the time (von Neumann’s design for a computer was less focused on speed). Turing’s device was essentially a Reduced Instruction Set Computer (RISC). Copeland gives us the impression that Turing was an engineer (-ing professor) as much as a mathematician (mathematics professor). It seems that Turing looked at Turing machines as idealized machines perhaps more than as purely mathematical concepts.

Codebreaker

Copeland argues convincingly, to me, that Turing’s contributions did not sway the outcome of World War II from the Axis powers to the Allies. On the other hand, it may have saved on the order of 10 million lives by helping to shorten the war.

We get a very detailed description of how the Germans’ secure communications machine “Enigma” worked (Figure 1). For a mathematician a more mathematical treatment would have been preferable; the given description of how some wheels are attached to others in certain ways and triples of letters associated with others was a bit bewildering.

Breaking the Germans’ codes was not a matter of solving a well-defined math problem but rather of thinking of lots of aspects of what the Germans were doing and finding a series of weak links: something to hack. Again we perhaps

see Turing’s engineering essence above his mathematical one.

We also learn about Turing’s cryptologic work. To decode German messages, one had to basically search through a huge space for some input whose output would be a recognizable German language message. Various heuristics and methods to reduce this search space were considered. Turing made extensive use of probability and used phrases like “cross” and “direct” while other less mathematically serious colleagues used “starfish” and “beetle.”

The *Bombes* were electromagnetic devices created to carry out the search that remained to be done after all heuristics and mathematical simplifications had been applied. Turing played a leading role in adapting these from Polish cryptanalysts’ *Bombas* (see page 560).

The Enigma encryption scheme is an elaboration of Vigenère ciphers, which themselves are elaborations of the simple Caesar ciphers. Turing wrote a manuscript on their decryption using the Bayes theorem; this was recently released on ArXiv [13].

Banburismus was a purely mechanical (not even electro-mechanical) means of reducing the search space before starting a Bombe run. It involved punch cards inspired by the loom industry (as also Lovelace and Babbage had been). It is explained that people of intermediate skill were not needed for the endeavor: there were the manual card-punchers and measurers, there were the cryptanalysts, who had a much more enjoyable job, and then there were Turing and his ilk, who designed the algorithms the cryptanalysts carried out. Sometimes the attempts to explain mathematical ideas in plain language become too vague (p. 139):

A two-letter sequence such as ‘en’ occurs more frequently in English than the combination of ‘e’ and ‘n’ counted separately.

A more advanced machine, Tunny, took over from Enigma, and we learn about the methods and computers (Colossus) used to decode Tunny messages. Encryption by vector addition modulo 2 is well explained. Doing it twice recovers the original message by associativity since

$$(A + B) + B = A.$$

Special tricks included waiting for Germans to send the same message again, but with some minor variation, because they thought the first message did not go through. Two similar messages could be more easily broken, and this is explained in some detail. The Colossus computer used electronic valves. These had at the time a status similar to that of quantum bits now. They were believed to be too unreliable to be used *en masse*; that is, to have many of them in one computer. It is claimed that had many Colossi not been destroyed after the war, things like the Internet

and social networking might have happened a decade earlier than they did. (The idea of Facebook starting as early as 1994 may not be universally viewed as a positive, however.)

A chapter by Eleanor Ireland details the secrecy and tedium of working on the Colossus machines. Global events during World War II and their relation to Bletchley Park are detailed. We hear what a large industrial-scale cyberwarfare operation it was. Turing, however, was “flicking the walls with his fingers as he walked,” an image that may feel familiar to mathematicians and children alike. We learn that when two messages are “in depth,” meaning encrypted by adding the same vector, we can add the encrypted versions

$$(A + B) + (C + B) = A + C$$

and cancel out the encrypting vector B . Next, we think of a piece of German phrase and add that to some consecutive entries of the vector $A + C$. If our German phrase was in A or C , we would be left with a fragment of C or A , respectively. Intimate knowledge of the German language, as it was used by the Tunny operators, was key.

Brian Randell writes about the revelation of some classified information about Ultra, the codename for the British efforts against German cryptography, in the 1970s. It reminds me that Turing via his Bletchley Park work becomes an almost unbelievable incarnation of the “nerd’s superhero”: someone who through mathematical work becomes a leader among thousands of regular people engaged in the largest war of all time.

Turing visited the United States to help with their Bombe making, and worked on a speech encryption device. Much work has been done to preserve Bletchley Park’s historical WWII buildings by increasing the public and funders’ interest with books, TV reports, special events, and publications.

Computers After the War

Baby, the first stored-program computer, was built in Manchester, England, but with inspiration from Princeton. Interestingly, von Neumann (at Princeton) pushed the idea of a CPU with an accumulator (familiar to those who have studied machine/assembly language), whereas Turing liked a more decentralized design.

Turing developed the ACE computer that rivaled Baby. It was fast but ultimately obsolete compared to competing designs. At the time, random-access memory had not been developed. Instead of scanning through memory until the desired memory location arrived, Turing’s design used something called “optimum programming” to lay out instructions in memory so that the desired information in memory tended to arrive quickly or, rather, at the exactly right time. Such programming suited Turing quite well, as

the architecture was similar to that of his own Turing machines.

Turing had a great deal of foresight with regard to the design of machine language. Brian E. Carpenter and Robert W. Doran give a beautifully simple description of recursion: a computer must keep track of where it is, so a stack is needed.

Copeland and composer Jason Long describe how Turing and colleagues made computer music. For someone growing up with Commodore machines in the 1980s, the similarity is striking and appealing.

We are also taken on a trip back to the time of Charles Babbage. Babbage was focused on arithmetic and algebra. He acknowledged that Ada Lovelace saw further and envisioned a machine that could make music and replicate the brain. The situation is summarized by saying that Babbage was focused on hardware (and algebra), Lovelace on applications, and Turing on theory (as he developed mathematical theory of what was needed to achieve Lovelace’s vision).

Artificial Intelligence and the Mind

Perhaps the most famous idea named after Turing is the Turing test. Turing proposed that to test whether a machine had achieved intelligence, it should be asked to try to fool a human into thinking it was human. More precisely, a human judge gets to interrogate both another human and the machine (via a neutral interface such as computer chat window) and is then asked to guess which is the human.

Turing hypothesized that around our current time, machines would be able to fool some people some of the time and that in another 50 years or so machines would be able to fully pass the Turing test. So far, so good for these predictions: for instance, Google’s artificial intelligence is able to play the role of someone booking an appointment with a hair stylist in such a way as to not be detected as a machine.

In this part we learn that Turing wanted to define intelligence subjectively, as behavior that we find mysterious and admirable but do not fully understand [16]. This way the judge in the Turing test becomes an important participant. Diane Proudfoot gives a delightful discussion of some of my favorite topics including consciousness zombies and solipsism. Turing imagined child machines that learned, a precursor to today’s machine learning. Proudfoot’s chapter includes an unnerving observation: robots must look like humans in order to build rapport with humans, in order to learn from humans.

The chapter on computer chess discusses heuristic search algorithms. Rather than searching through possible moves, one uses guiding rules such as “a rook is worth five points.” With machine learning one could even discover that it is better to value a rook at, say, 4.9 points. As in some other

chapters, however, there is a bit of historic trivia of little interest, such as who first lost a game of chess to a computer, who first won, and so forth. There is also some material that perhaps is of interest to lay-persons, such as a complete transcript of the first game of chess between a human and Turing acting as a computer. Some fascinating tidbits such as Mozart *Musikalische Würfelspiel* (randomly generated Mozart music) are also included.

The book does have a smattering of strange matters to a mathematician.

- The standard normal distribution is described as having mean 0, standard deviation 1, and also height 1 at the mean.
- The proof of the undecidability of the halting problem (pp. 410–411) seems to make no use of the crucial negation step whereby a computation halts if and only if it does *not*.
- The distinction between countable sets and computably enumerable sets is missing in Chapter 37. (Very nice though is that chapter's display of an explicit polynomial over \mathbb{Z} that produces the primes and no other positive integers.)

A chapter on WWII coding methods reads a bit tedious at times (imagine going through a detailed computation with repeated Bayes theorem usage in prose rather than equations), but there are some interesting things for me such as the use of the score $\log p$ of a probability p to simplify hand calculations so that the clerks at Bletchley Park could use addition rather than multiplication.

Extra-sensory perception (ESP) was credible to many scientists at Turing's time. He apparently worried that ESP used by the judge in the Turing test would lead the judge to falsely fail to attribute intelligence to the machine. Thus, Turing seems to sympathize, in theory, with intelligent machines.

Sprevak's chapter on cognitive science includes a discussion of the appropriateness of modeling the behavior of a human computation clerk by a Turing machine. Here it would be instructive to also compare clerks to *finite automata*, studied early on in cognitive science by McCulloch and Pitts [6]. Like a finite automaton but unlike a Turing machine, a human does not have unlimited memory.

There is a sense in which the language

$$(01)^* = \{\emptyset, 01, 0101, 010101, \dots\}$$

is understandable by humans and

$$\begin{aligned} \{0^n 1^n : n \geq 0\} \\ = \{\emptyset, 01, 0011, 000111, 00001111, \dots\} \end{aligned}$$

is not. For the former, we just have to scan the whole input, rejecting if we see 00 or 11. Only our lifespan or fatigue limits us in this regard. For the latter, we have to keep a

counter, and for large n that is beyond our memory capabilities whether in our brain or in hardware or paper.

Biological Growth

This interesting part introduces morphogenesis via the tale of the sweating grasshoppers and the fire. The basic idea is pretty clear even in the absence of any differential equations. While it is not mentioned in the *Guide*, Turing's work is related (see [2, 3]) to Schelling's [11] work on segregation. If individuals tend to prefer to live close to similar individuals, how do segregated neighborhoods form? In terms of a tolerance parameter, higher tolerance may lead individuals to be less likely to move, which can actually lead to more segregation: once individuals land in a rather homogeneous area they are likely to stay. Here, the neighborhoods (in economics) are analogous to the stripes on a zebra (in biology).

The chapter about radiolaria is amazing: suffice it to say that it concerns single-cell organisms shaped like Platonic solids with spikes!

Mathematics

Here we learn that Turing worked on the central limit theorem and on the Riemann ζ -function [15, 18]. Conveniently for this book, Turing worked on many fundamental topics.

Turing's work [14] on the *Entscheidungsproblem* (the decision problem for validity in first-order logic) is discussed in several chapters in the book. One chapter makes it seem like Turing did the most and Gödel a relatively minor amount, but Rod Downey's chapter gives the view that the *Entscheidungsproblem* had arguably already been solved before Turing. In any case, Gödel showed that any computable axiom system gives an incomplete set of theorems. Absent a procedure to determine which new axioms to add, it is clear that there can be no algorithm to decide which results are true and which are false in arithmetic.

Downey also discusses randomness and Turing's work on absolutely normal numbers and how they correspond to finite-state random sequences. He adds that it is not clear whether one can physically generate true randomness. One might add that it is not clear what that even means. Cornout argued that we need a principle, namely

events of very low probability simply *do not happen*,

in order to give a noncircular explanation of what probability is [12].

Finale

To me this was the most interesting part of the book. It deals with various arguments for how the time evolution of our physical universe may not be computable. Of course, if the universe is finite and discrete, then in some sense it

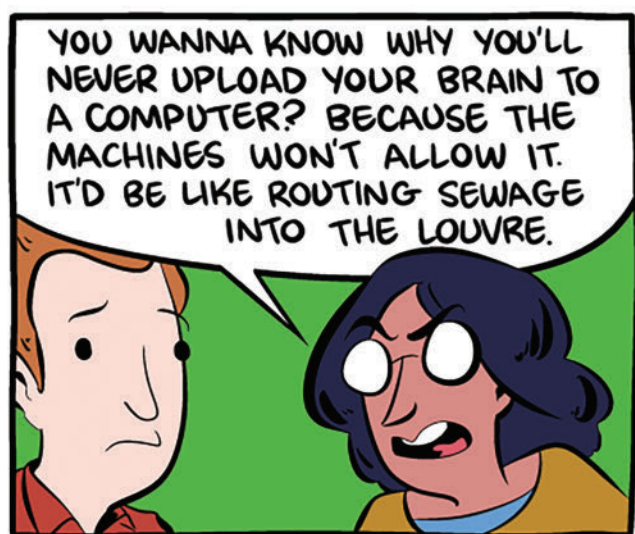


Figure 2. The impact of Turing's work on popular culture [20].

is computable. Since the universe is very large, however, it is conceivable that it is not well modeled as being computable but is better thought of as containing some random or noncomputable aspects.

Early thinking on this topic may have been motivated by the idea that, surely, human minds can do things that Turing machines cannot. In *Renewing Philosophy* [9], Hilary Putnam argues that artificial intelligence is impossible on the basis of the work of Pour-El and Richards [8] on noncomputability in classical physics. Next, there was the thought that physics, with its marvelous use of higher mathematics, may contain undecidability [4]. That argument has lost some of its shimmer [5]. Finally, at present it seems that a technological solution for achieving non-computability is all we are left to imagine. Now it seems that while perhaps *technology based on physical systems* can carry out non-Turing machine computations, that seems unlikely to mean *human minds* can do the same (see Figure 2).

Andréka, Némethi, and Székely [1] work on using time travel (closed timelike curves) to compensate for the lack of space (and time). In the so-called Malament–Hogarth (MH) spacetimes one can compute forever and thus solve the halting problem. Hogarth worked on this in the 1990s, and Welch [21] showed that even a larger class of problems than those solvable using the halting problem (all hyper-arithmetic problems) can be solved in MH-spacetimes.

Interestingly for our times, hypercomputation using closed timelike curves (CTCs) is a technological solution. Thus, researchers no longer claim that nature itself and certainly not humans are super Turing machines. For another example, consider Christina Perri's song "Human" [7] with the eerie lyrics,

But I'm only human
And I bleed when I fall down

I'm only human
And I crash and I break down

Could these words have been sung a century ago?

Polish Contribution

A good test of a biographical and historical book is how it holds up in light of new information. Sir Dermot Turing, Turing's nephew and author of a chapter of *The Turing Guide*, in 2018 published the book *X, Y & Z* [19] in which he argues that Polish mathematicians, including Marian Rejewski and Henryk Zygalski, should get more credit, and that they have not gotten it because of an exaggerated "Turing cult" [10]. To *The Turing Guide*'s credit, it is indeed mentioned in the book that the Polish mathematicians had addressed the Enigma problem more as a pure math problem than the British by the time the two groups compared notes. In particular, the Polish had the idea of using machinery to decrypt machine-produced codes, using what they called *bomba* (as opposed to Turing's *Bombe*). In my draft of this review, written before the article [10] appeared, I had already noted

The Polish were ahead of the British for a while, as the former realized right away that the code breaking was fundamentally a mathematical problem.

Conclusion

Overall, I found this to be a delightful book—it was even inspiring with, for instance, the mentions of seminar topics that turned into new research directions. Mathematicians should find a mixture of things they already knew, things they are glad to learn, and a couple of things they disagree with. I imagine a general well-educated audience, especially scientists and engineers who do not specialize in mathematics, may enjoy the book the most.

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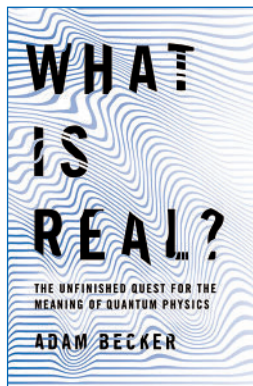
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BOOKSHELF

New and Noteworthy Titles on our Bookshelf
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What is Real? *The Unfinished Quest for the Meaning of Quantum Physics*

by Adam Becker (Basic Books, 2018, 384 pages).

Quantum mechanics is one of the most accurate and successful scientific theories ever discovered. However, deep questions about its implications and meaning have loomed over the theory since its inception. Where is the boundary between the quantum and classical worlds? Why does the observer

seem to enjoy a privileged status? What is the role of consciousness in observation? What does a wave function actually represent? Why does it collapse upon observation? Is quantum mechanics complete, or are there “hidden variables” that might explain away some of the apparent paradoxes?

In this book, Becker provides a nontechnical and highly readable account of some of the basic conundrums in the foundations of physics. He pays particular attention to the highly charged human dimension of the story. There were decades of political jousting and relentless posturing. Careers could be ruined for merely suggesting alternatives to the Copenhagen interpretation, the dominant paradigm. Papers could languish in obscurity or be suppressed for presenting novel ideas. The bitter conflict between Bohr and Einstein, the influence of the towering von Neumann, and the challenges faced by Bohm, Everett, and Bell all leap off the page. Mathematicians familiar with quantum mechanics from a theoretical perspective will find Becker’s historical and character-driven approach enlightening and engaging.

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Unsolved! *The History and Mystery of the World's Greatest Ciphers from Ancient Egypt to Online Secret Societies*

by Craig P. Bauer (Princeton, 2017, 624 pages).

This hefty book with an imposing title covers exactly what it claims to and does it with style and substance. *Unsolved!* is accessible to a wide audience. The most difficult mathematics discussed is the RSA algorithm, which appears toward the end of the book. A motivated

high-school student would find the material in this book both appropriate and interesting.

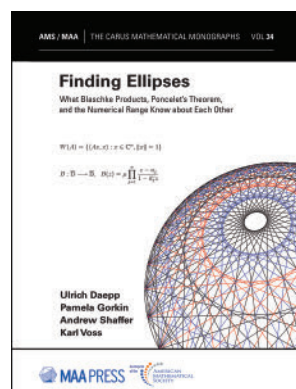
Well-known examples such as the Voynich manuscript, the Zodiac letters, and the Beale ciphers are discussed in great detail. Although these famous enigmas are frequently touched upon in other texts, Bauer delves deeply into each one. For example, he includes information about statistical analyses of the data, educated speculation about the encryption methods used, and a run-down of attempted decipherments. Even those who are familiar with some of the more sensational examples will learn something new.

Unsolved! provides enough detail and references to the code-breaking literature that motivated readers could plausibly begin their own attempts at cracking these codes. Relevant links to the actual data are often included and many dozens of illustrations and photographs grace this book.

The book particularly shines in its coverage of dozens of lesser-known ciphers, most of which have not received near the same level of attention as their more sensational cousins. Each of these tales is fascinating, and the coverage is deep. Many of these relate to unsolved murders, buried treasure, enigmatic manuscripts, espionage, and secret societies. There are even a few internet-era challenges that receive a robust treatment. This blend of intrigue and mystery makes for a surprisingly thrilling read.

Students and professional mathematicians alike will delight at the dozens of mysteries presented by the author. Even most instructors who regularly teach cryptography will pick up many anecdotes with which to spice up their classes.

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Finding Ellipses

What Blaschke Products, Poncelet's Theorem, and the Numerical Range Know about Each Other

Ulrich Daepf, Pamela Gorkin, Andrew Shaffer, Karl Voss

Finding Ellipses is the newest volume in the MAA's Carus Mathematical Monograph series. These books are intended to give tantalizing invitations to unfamiliar areas for profes-

sional non-specialists. This particular volume explores a surprising collection of connections between complex analysis, projective geometry and linear algebra. The complex analysis begins with the fact that every automorphism of the unit disk in the complex plane is of the form:

$$f(z) = \mu \frac{z - a}{1 - \bar{a}z},$$

where a is a point inside the disk and μ has norm one. So, up to an arbitrary rotation, the automorphism is completely determined by which point maps to zero. More is true, any map analytic on an open neighborhood of the closed disk that maps the disk to itself (and the boundary to the boundary) is a finite product of factors of this form. So, specify the pre-images of zero you want and, up to rotation, there is a unique analytic map with those zeroes that preserves the unit disk. These maps are called *finite Blaschke products*. They have many beautiful properties.

To take one, particularly apposite, example of such a property imagine an arbitrary Blaschke product of degree three, B , with zeroes at 0 , a , and b . It is, of course, a three-to-one map of the unit disk to itself. For each point w on the unit circle construct the triangle z_1, z_2, z_3 where $B(z_1) = B(z_2) = B(z_3) = w$. The envelope of this collection of line segments is an ellipse with foci at a and

b and common distance $|1 - \bar{a}b|$. It's called a *Blaschke ellipse*.

Suppose, to change the subject for a minute, that you have two nondegenerate, nonintersecting conics, C_1 and C_2 , in the plane. Suppose further that there exists a triangle with vertices on C_1 all of whose sides are tangent to C_2 . Then *every* point on C_1 is a vertex of such a triangle. This is *Poncelet's Theorem*. In the case that C_1 is the unit circle we'll call an ellipse that can be an associated C_2 a *Poncelet ellipse*. It is natural to wonder which ellipses can be Poncelet ellipses.

The surprising fact is that every Poncelet ellipse is a Blaschke ellipse, and vice versa. The connection is mediated by linear algebra, specifically by the *numerical range* of an $n \times n$ matrix, A , with complex entries. The numerical range of A is defined as:

$$W(A) = \{\langle Ax, x \rangle \mid x \in \mathbb{C}^n, \|x\| = 1\}.$$

First defined by Otto Toeplitz, the numerical range is a generalization, of a sort, of the set of eigenvalues. The triangles circumscribing a Blaschke ellipse are each the numerical range of a member of a certain family of related matrices. (The family consists of the unitary 1-dilations of a 2×2 matrix of a specific form.) And it's all constructive, given the ellipse, it is possible to construct the Blaschke product and the matrix. Given the matrix, one can construct the Blaschke product and the ellipse.

Finding Ellipses by Daepf, Gorkin, Shaffer, and Voss is full of one surprise after another. The connections are thoroughly and completely explored, of course. But there's more: There is a connection to dynamical systems and another to, of all things, Benford's Law. The middle part of the book extends the investigation to questions involving Blaschke products of degree higher than three. Poncelet-like properties are discovered and explored along with deep function-theoretic properties of Blaschke products.

The exposition throughout is crystalline, packed with illuminating examples and accessible. Reading it would be make for a great capstone project for an undergraduate.

The AMS Bookshelf is prepared bimonthly by AMS Acquisitions Specialist for MAA Press titles Stephen Kennedy. His email address is kennedy.maapress@ams.org.

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SPRING EASTERN SECTIONAL SAMPLER



Olivier Bernardi



Brian C. Hall



Christina Sormani

In this sampler, all three invited speakers at the AMS Spring Eastern Sectional taking place April 13–14 at University of Connecticut Hartford (Hartford Regional Campus), Hartford, CT, have kindly provided introductions to their Invited Addresses.

- Page 566 *Olivier Bernardi*, Brandeis University
Percolation on Triangulations, and a Bijective Path to Liouville Quantum Gravity
- Page 568 *Brian C. Hall*, Notre Dame University
Eigenvalues of Random Matrices in the General Linear Group in the Large- N Limit
- Page 570 *Christina Sormani*, Lehman College and CUNY Graduate Center
Compactness Theorems for Sequences of Riemannian Manifolds

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Percolation on Triangulations, and a Bijective Path to Liouville Quantum Gravity

Olivier Bernardi

ABSTRACT. We report on recent progress toward relating two notions of random surfaces introduced in theoretical physics. The first notion of random surface is *Liouville quantum gravity*, whose definition involves the Gaussian free field. The second notion is obtained by considering the scaling limit of random triangulations of the sphere. A key ingredient in proving an exact relation between these two notions is a bijective encoding of percolated triangulations by certain lattice paths.

This talk will attempt to achieve several goals:

1. Discuss the site-percolation model on random planar triangulations.
2. Provide an informal introduction to several probabilistic objects coming from theoretical physics: the *Gaussian free field*, *Schramm-Loewner evolutions*, and the *Brownian map*.
3. Present a bijective encoding of percolated triangulations, and explain its role in establishing exact relations between the above-mentioned random objects.

The results we will present are based on a collaboration between Nina Holden, Xin Sun, and me [1]. They build on a large body of work, among which a construction of Duplantier, Miller, and Sheffield [2] plays a particularly important role.

Random triangulations and site-percolation. A *planar triangulation* is a planar graph embedded in the sphere in such a way that each face is a triangle. A planar triangulation is represented in Figure 1. Triangulations are considered up to continuous deformation of the sphere, and are uniquely determined by the incidence relation between faces. Planar triangulations with n triangles are therefore obtained by taking a set of n triangles and “gluing” their edges in pairs, in such a way that the resulting surface has spherical topology.

Since the set of planar triangulations with n triangles is finite, one can consider the uniform probability mea-

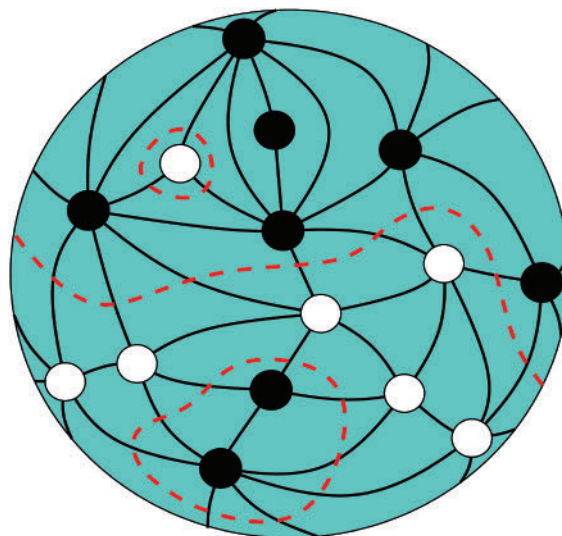


Figure 1. A planar triangulation endowed with a site-percolation configuration. The percolation interfaces are indicated by dashed lines.

sure on this set. Let us call *random triangulation of size n* a random planar triangulation with $2n$ triangles. A random triangulation can be thought of as a disorganized analogue of the triangular lattice. In fact, an important conjecture called *KPZ equation* relates the critical exponents of statistical mechanics models on the regular lattice to the same exponents for the disorganized lattice.

The *site-percolation* model on a triangulation is a random assignment of color (either *black* or *white*) to each vertex of the triangulation. Figure 1 shows a percolation configuration. In the *critical setting* each vertex is colored black with probability $1/2$ (independently of the other vertices). The clusters are the connected components of the unicolor subgraphs. The *percolation loops* are the set of closed curves on the sphere that separate the black clusters from the white clusters. Natural questions about the percolation model concern the size of the clusters and the distribution of the percolation interfaces.

A bijective tool. In [1] we establish a bijection which is key for our study of the percolation model on random triangulations. This bijection is represented in Figure 2 (see facing page). It relates (loopless) planar triangulations with $2n$ triangles endowed with a site-percolation configuration to lattice paths in PN^2 that start and end at $(0,0)$ and are made of $3n$ steps from the set $\{(0,1), (1,0), (-1,-1)\}$. This bijection is used to obtain the limiting distribution of several important observables of the percolation model (percolation loops, exploration tree, pivotal points measure).

Random surfaces and random curves. As discussed above, it is straightforward to define the notion of random

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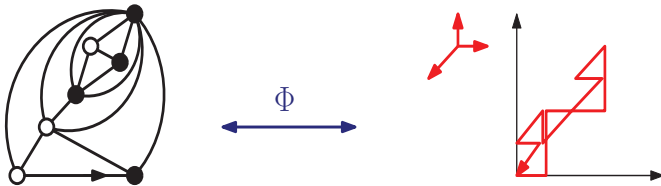


Figure 2. The bijection Φ between percolated triangulations and lattice walks.

triangulation of size n . This gives rise to an interesting notion of random metric space by considering the set of vertices of the triangulation endowed with the *graph distance* between them. In a major achievement, the *scaling limit* of this random metric space (when the number of triangles goes to infinity and their size goes to 0 at the appropriate rate) has been characterized in [5, 6]. It is a random metric space with spherical topology called the *Brownian map*, and it is a legitimate 2D analogue of the Brownian motion.

The *Gaussian free field* (GFF) is another 2D analogue of the Brownian motion, which is obtained by a completely different approach [8]. The GFF is a random distribution in (a domain of) the complex plane \mathbb{C} . The GFF can be used to define a random surface called *Liouville quantum gravity* (LQG) which was originally introduced by Polyakov [7] as a model for the random surface corresponding to the space-time evolution of a string. Heuristically, LQG_γ is a random surface which, when projected conformally on \mathbb{C} , gives rise to a density of area measure $e^{\gamma h}$, where h is the GFF and γ is a positive number.

The Brownian map and LQG have long been conjectured to be in some way related (although the proper definition of these objects and their possible relation is fairly recent). The proof of such a relation is under completion in a series of articles starting with [1] and culminating with [3]. Roughly speaking it is shown that, under a specific embedding of the random triangulations, the continuous limit of the vertex distribution has the law of the $\sqrt{8/3}$ -LQG area measure, while the continuous limit of the percolation interfaces has the law of the *conformal loop ensemble* CLE_6 (an infinite collection of random loops closely related to Schramm–Loewner evolutions [9]).

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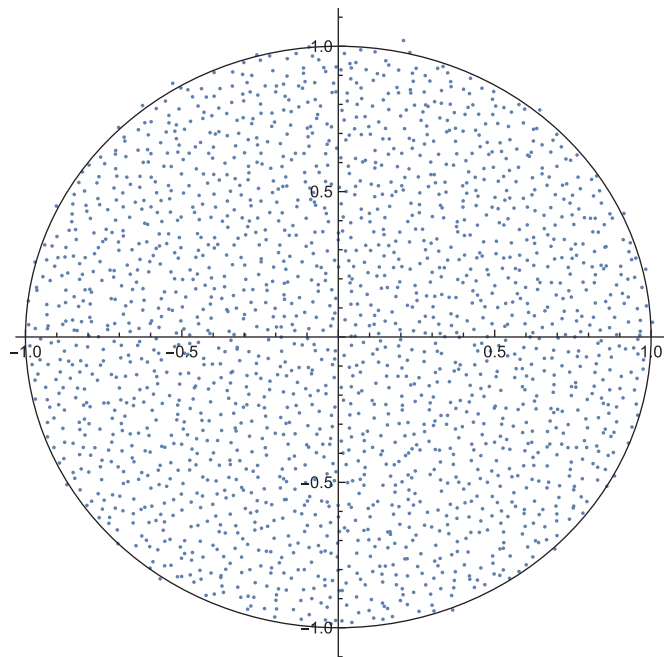
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Eigenvalues of Random Matrices in the General Linear Group in the Large- N Limit

Brian C. Hall

Random Matrices and the Circular Law

Random matrix theory was introduced by Eugene Wigner [3] in a 1955 paper modeling energy levels in atomic nuclei. The subject has now blossomed into a large industry with deep connections to physics. For my talk the most relevant sort of random matrix is the *Ginibre ensemble*. Here we consider $N \times N$ matrices with the entries chosen independently. Each entry X_{jk} is chosen to be a complex number, with the real and imaginary parts being independent normal random variables with mean zero and variance $1/(2N)$. We now consider the eigenvalues of this random matrix; these form a random collection of N points in the complex plane.



Eigenvalues of a random matrix from the Ginibre ensemble, with $N = 2,000$

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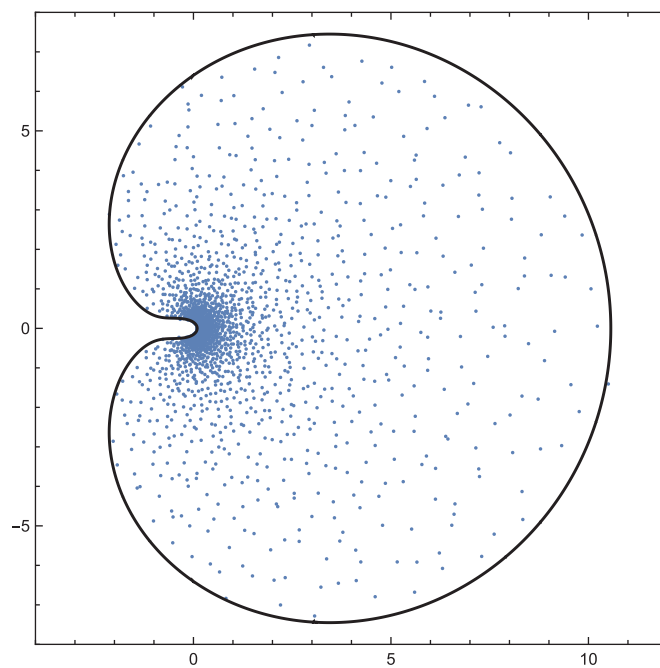
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Ginibre found that when N is large, these eigenvalues follow the *circular law*: with high probability, almost all of the eigenvalues lie in the unit disk and they are almost uniformly distributed there. This assertion exemplifies a general property of random matrices, that the bulk properties of the eigenvalues of randomly chosen $N \times N$ matrices tend to become nonrandom when N is large. Figure 1 shows a simulation of the Ginibre ensemble with $N = 2,000$.

Brownian Motion in the General Linear Group

I will discuss another random matrix model which is a deformation of the Ginibre ensemble. Let $GL(N; \mathbb{C})$ denote the general linear group, that is, the group of all $N \times N$ invertible matrices. Then consider Brownian motion b_t^N in $GL(N; \mathbb{C})$, starting from the identity. In a general Riemannian manifold, Brownian motion is a random path that can be obtained as the limit of random walks with very small step sizes. In the case of $GL(N; \mathbb{C})$, we should construct the approximating random walks *multiplicatively*; that is, at each stage, we *multiply* the current position by a matrix close to the identity.



A simulation of the eigenvalues of b_t^N with $N = 2,000$ and $t = 3.9$ shown on a plot of the domain Σ_t

The motivation for considering Brownian motion is this. If we consider Brownian motion in the space of all $N \times N$ matrices—but constructing the approximating random walks *additively*—the distribution at any time t is just the Ginibre ensemble scaled by a factor of \sqrt{t} . Thus, the distribution of Brownian motion in $GL(N; \mathbb{C})$ is a multiplicative analogue of the Ginibre ensemble.

In my talk, I will discuss results with Bruce Driver and Todd Kemp concerning the distribution of the eigenvalues of b_t^N in the large- N limit. Our first result [2] is that the eigenvalues cluster as $N \rightarrow \infty$ into a certain domain Σ_t in the plane identified by Biane [1]. We then have work in progress indicating a remarkably simple structure to the limiting distribution of eigenvalues within this domain. Figure 2 shows a simulation of the eigenvalues with $t = 3.9$ and $N = 2,000$.

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Brian C. Hall

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Compactness Theorems for Sequences of Riemannian Manifolds

Christina Sormani

Given a sequence of Riemannian manifolds, one may hope that there is a subsequence which converges to a possibly singular limit space. One may have sequences of spheres developing conical singularities (as in Figure 1), or thin deep wells (as in Figure 2), or perhaps even increasingly many wells (as in Figure 3). One may have a sequence of Riemannian manifolds which are not diffeomorphic to one another (as in Figure 4). Note that we consider sequences of distinct Riemannian manifolds that are not submanifolds of Euclidean space. In what sense might they converge? What are their limits?



Figure 1. Developing a cone tip



Figure 2. Thinner and thinner wells



Figure 3. Increasingly many wells



Figure 4. Increasing genus

In order to define a notion of convergence for distinct Riemannian manifolds, Gromov decided to view each Riemannian manifold, (M_i, g_i) , as a metric space, (X_i, d_i) . To define a distance between a pair of them, he embedded

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them into a common metric space, Z , via distance preserving maps, $\varphi_i : X_i \rightarrow Z$:

$$d_Z(\varphi_i(p), \varphi_i(q)) = d_i(p, q) \quad \forall p, q \in X_i.$$

He then took the Hausdorff distance, d_H^Z , between their images, $S_i = \varphi_i(X_i)$, as in Figure 5. Recall that $d_H^Z(S_1, S_2)$ is the smallest radius r such that

$$\forall z_i \in S_i \exists z_j \in S_j \text{ such that } d_Z(z_i, z_j) \leq r.$$

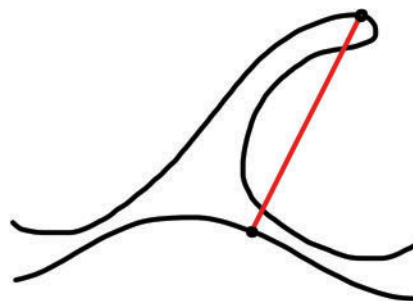


Figure 5. The Hausdorff distance between the black curves is the length of the red line.

More precisely the Gromov–Hausdorff distance between two metric spaces, $M_i = (X_i, d_i)$, is

$$d_{GH}(M_1, M_2) = \inf \left\{ d_H^Z(\varphi_1(X_1), \varphi_2(X_2)) \right\}$$

where the infimum is over all compact metric spaces Z and all distance preserving maps, $\varphi_i : X_i \rightarrow Z$. Gromov proved that $d_{GH}(M_1, M_2) = 0$ iff M_1 and M_2 are isometric.

The sequence in Figure 1 converges in the GH sense to a sphere with a conical singularity. The sequence in Figure 2 converges to a sphere with a line segment attached to it. In general GH limits are metric spaces with geodesics, but they have no smooth structure.

Gromov's compactness theorem states that any sequence of metric spaces which have a uniform upper bound on diameter, D , and a uniform maximal number, $N(r)$, of disjoint balls of radius r has a subsequence which converges in the GH sense. The sequence of manifolds with increasingly many wells in Figure 3 fails the hypothesis of Gromov's compactness theorem. The number of balls centered at the tips is increasing to infinity. In fact it has no subsequence converging in the GH sense. Ilmanen presented this example as a sequence that ought to converge to a sphere under some weak notion of convergence and asked what the notion could be.

In joint work with Wenger, we solved Ilmanen's question by introducing the intrinsic flat (\mathcal{F}) convergence. We viewed the oriented Riemannian manifolds as *integral current spaces*: metric spaces with oriented weighted biLipschitz charts, $M_i = (X_i, d_i, T_i)$. We defined the intrinsic

flat distance between two integral current spaces to be:

$$d_{\mathcal{F}}(M_1, M_2) = \inf \left\{ d_F^Z(\varphi_{1\#}(T_1), \varphi_{2\#}(T_2)) \right\} \quad (1)$$

where the infimum is over all complete metric spaces Z and all distance preserving maps, $\varphi_i : X_i \rightarrow Z$. Yet instead of taking the Hausdorff distance between the images we took the flat distance, d_F^Z , between them.

The flat distance was first defined by Whitney and Federer–Fleming on $Z = \mathbb{E}^N$ as the infimum of the weighted volumes:

$$d_F^Z(S_1, S_2) = \inf \{ M(A) + M(B) : A + \partial B = S_1 - S_2 \}$$

where the infimum is over all generalized submanifolds, A^m and B^{m+1} as in Figure 6:

$$\int_{S_1} \omega - \int_{S_2} \omega = \int_{\partial B} \omega + \int_A \omega = \int_B d\omega + \int_A \omega.$$

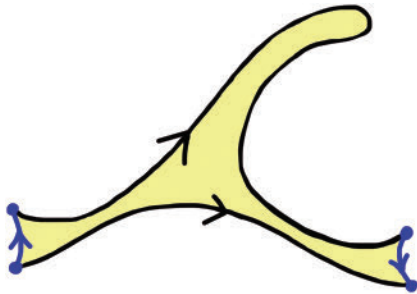


Figure 6. The flat distance between the black curves is the area of the yellow region plus the sum of the lengths of the blue curves.

We applied work of Ambrosio–Kirchheim to make this rigorous and prove that $d_{\mathcal{F}}(M_1, M_2) = 0$ iff there is an orientation preserving isometry from M_1 to M_2 .

Intuitively the \mathcal{F} distance measures the filling volume between the Riemannian manifolds. The sequence in Figure 3 converges in the \mathcal{F} sense to a standard sphere exactly as Ilmanen required. More generally, Wenger and I proved the limit spaces are metric spaces covered almost everywhere by biLipschitz charts that define an orientation and a weight (so they are rectifiable). We proved that whenever a sequence of manifolds has a GH limit, and a uniform upper bound on volume, then it has an \mathcal{F} limit which is a subset of the GH limit. For example, the \mathcal{F} limit of the sequence depicted in Figure 2 is only the sphere while the GH limit also has a line segment. See Figure 7.

Wenger’s compactness theorem states that a sequence of oriented Riemannian manifolds with a uniform upper bound on volume and diameter has a subsequence converging in the \mathcal{F} sense. However it is possible that the limit space is just the $\mathbf{0}$ space. This happens for example if the sequence of manifolds has volume converging to 0.

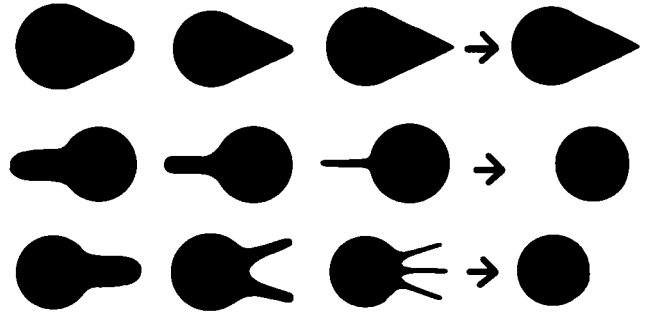


Figure 7. The intrinsic flat limits of our sequences.

Ideally one can show the GH and \mathcal{F} limits agree, thus proving the GH limit is rectifiable and the \mathcal{F} limit is not $\mathbf{0}$. For example the sequence in Figure 4 converges in the GH and \mathcal{F} sense to the same rectifiable limit space with infinitely many holes.

The Tetrahedral Compactness Theorem states that if a sequence of manifolds satisfies a uniform noncollapsing condition on tetrahedra lying in the spaces, then a subsequence converges in the GH and \mathcal{F} sense to the same rectifiable limit space. This theorem was proven jointly with Portegies, and improved by Nuñez-Zimbrón and Perales. This theorem and many more examples will be presented in the lecture.



Christina Sormani

Credits

All images are courtesy of the author.

Unfolding Humanity: Mathematics at Burning Man

Satyan L. Devadoss and Diane Hoffoss

Introduction

In August 2018, a two-ton metal, wood, and acrylic interactive sculpture showcasing unsolved problems of mathematics came to life in the middle of the Nevada desert. Rising 12 feet tall with an 18-foot wingspan, the unfolding dodecahedron was externally skinned with black panels containing 2240 acrylic windows, illuminated by more than 16,000 LEDs that were driven by 20 programmed controllers. The interior, large enough to hold 15 people, was fully lined with a massive mirror over each of the twelve pentagonal faces. With an estimated 6500 person-hours invested and over \$40,000 in grants and donations raised, the resulting artwork was displayed at Black Rock City, the desert location of Burning Man. This article outlines our journey, two mathematicians embracing the role of amateur sculpture artists.

Burning Man is an arts gathering, founded by Larry Harvey and Jerry James in 1986 on Baker Beach near San Francisco. The rapidly growing event moved to the Nevada desert in 1990, and now attracts more than 70,000 people. Its home is Black Rock City, a temporary metropolis that exists only during the week leading up to Labor Day. Today, this event has become the gold standard for large-scale sculpture exhibitions, with much of Silicon Valley in attendance to contemplate the cutting-edge technical and engineering feats [4]. A gathering once viewed as fringe is becoming a well-respected cultural phenomenon. Indeed, the Smithsonian devoted its entire Renwick Gallery to the exhibition *No Spectators: The Art of Burning Man*, which ran

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Figure 1. *Unfolding Humanity* at sunrise in Burning Man 2018, with LED animation along the edges and faces.

from March 30, 2018 to January 21, 2019. This show is now on a national tour, first to Cincinnati and then to Oakland.

Vision

Our sculpture, titled *Unfolding Humanity*, calls attention to unsolved problems in mathematics and physics. On one hand, it echoes the renaissance printmaker Albrecht Dürer's explorations nearly 500 years ago on *polyhedral nets*, providing the earliest known examples of polyhedra unfolded to lie flat for printing. Motivated by this, G. Shephard [9] asks: can every convex polyhedron be cut along some of its edges so that it unfolds into one flat piece without overlap? It is not too difficult to construct *nonconvex* polyhedra that offer counterexamples: negative (discrete Gaussian) curvature at certain vertices can obstruct the unfolding without overlap [1, Chapter 22].

For the convex case, however, this problem remains enticingly open. Ghomi recently showed that a convex polyhedron can be unfolded once it undergoes an affine transformation [3]. Horiyama and Shoji show that *every*



Figure 2. Standing inside the dust-covered *Unfolding Humanity* with LED animation along the edges.

edge-unfolding of the Platonic solids yields a valid net [5]. In particular, all 43,380 distinct edge-unfoldings of the dodecahedron lay flat. Our sculpture allows participants to unfold eight of the pentagonal panels of the dodecahedron (though not fully flat) to illustrate one such possible configuration.

On the other hand, the sculpture asks the observer to contemplate the shape of our universe. In 2003, data from WMAP, a satellite that measured cosmic microwave background radiation, revealed that the known universe might be modeled by the Poincaré dodecahedral space [7], obtained by identifying antipodal faces of the dodecahedron with a twist. The inside of our dodecahedron is covered with seven-foot-tall mirrors, becoming an illuminated mirrored room designed to allude to this space. While final refinements in 2012 of the WMAP data seem likely to fit with a Euclidean universe (with curvature within 0.4% of flat), Luminet claims that the data remains consistent with the Poincaré dodecahedral space as well as other finite topological spaces [7]. Of course, the reflections of the mirrors provide a vastly different geometry than the rotational twists needed for Poincaré space, but yet they allow

participants to enter a small finite universe, to witness the complex beauty where light folds back upon itself.

Origins

The past twenty years have brought about a strong intersection between mathematics and the visual arts. Venues for mathematically inspired art have included international conferences (such as Bridges), journals (such as MIT's *Leonardo*), gallery showcases (including those at the Joint Mathematics Meetings), and more. Although a tremendous amount of art has been based on mathematical ideas that are centuries old, a current trend has many contemporary mathematical artworks and exhibits striving to highlight recent and open mathematics. And so, our motivating question asks, what could it look like for vibrant and unsolved mathematics to be made embodied and physical, to engage the general public?

With Hoffoss's 2017 experience as a Burning Man installation lighting artist, our answer was to take the form of a large-scale sculpture. Fortunately, Devadoss was offering an upper-level mathematics geometry elective in Fall 2017 at the University of San Diego (USD), focusing on discrete and computational geometry. For the course's culminating

project, student teams submitted conceptual large-scale sculpture proposals satisfying the following constraints:

1. Make it interactive for participants.
2. Address an unsolved question in geometry.
3. Design around the 2018 Burning Man theme of Asimov's *I, Robot*.

Out of five team project proposals, we chose one with the most potential: a project based on an unfolding dodecahedron by USD students Jordan Abushahla, Nick Bail, and Eugene Wackerbarth. After minor alterations to this proposal, an initial *Letter of Intent* was put forward to Burning Man in November 2017. Passing the first round of approval, a full proposal was submitted at the end of January 2018, titled *Unfolding Humanity*. This version now included the iconic 'character rain' animation from *The Matrix*, an ambitious LED lighting infrastructure, a fully-lined mirrored interior, and a detailed budget.

This duality in the artwork, of technological framework on the outside, with reflections of humanity pondering vastness of space and time on the inside, opens the door to ask deep questions. In particular, the sculpture invites the participant to explore ways technology plays a role in illuminating and controlling our lives, and the quest to be truly human and free. Basing the artwork on unsolved problems in mathematics and physics gives greater weight to its voice.

Creation

With a conceptual design in mind, we reached out to students at USD for help. Soon afterwards, faculty members and the San Diego community at large were drawn in, to participate in the development, creation, and construction. There were roughly three global issues to consider:

1. the metal framework and its kinematics,
2. the pentagonal faces and lighting infrastructure, and
3. the transportation, build, and tear down at Burning Man.

During Spring and Summer 2018, engineering faculty, students, and alums took the lead in modeling, testing, and building the metal structure. The steel framing for the pentagonal faces was manufactured at USD, and the 20 vertices (with three edges emanating from a vertex, each at 108° from the other two) were commissioned to be custom welded. The structural design was further constrained by additional needs: for dismantling and reassembly, for the kinematic ability to unfold, and not least for rigidity

to handle enthusiastic interactivity from the Burning Man community. Added to this are the brutal desert conditions, with high heat, pervasive dust, and sustained winds of up to 100 mph.

The artistic features of the project were created at San Diego CoLab, a collaborative work-space that provided a dedicated build space, the use of many production quality tools and machinery, and access to skilled builders who offered expertise and advice. Although we orchestrated it, nearly all the work performed at CoLab was done by community drop-in volunteers. Each of the pentagonal faces were CNC routed, with 224 precisely spaced windows cut out to hold illuminated characters (which themselves were individual acrylic panes laser cut and etched on a large laser bed with one of 72 different characters). The faces were painted, acrylic panes glued, wiring installed, and LED strips cut from long spools and soldered into 235 special lengths strips. Microcontrollers were soldered with appropriate connectors, and then programmed to communicate with one another and to drive the LED animations on the faces and the edges.



Figure 3. Showcasing *Unfolding Humanity* at Balboa Park in San Diego, with eight panels open.

The construction work did not end in San Diego, however. A team of 18 volunteer builders was recruited to travel to Burning Man to unload, assemble, test, troubleshoot, support, disassemble, and repack the structure into a moving truck. During the week of Burning Man, considerable volunteer hours were invested in maintaining the sculpture, including troubleshooting and repairing the (three repeatedly failing) generators for the project, refueling the generators, cleaning and clearing the interactive chain hoists, replacing and repairing damaged LED strips, and cleaning mirrors of dust.

In all, the sculpture was conceived, designed, and built by more than 80 volunteers: five faculty members, 20 students and alums, and more than 55 members of the San Diego and Bay Area communities. An estimated total of 6500 person-hours was invested in this project by our volunteers. The cost for the project was over \$40,000, with \$10,000 from the San Diego Collaborative Arts Project, \$5000 from USD Humanities Center, and \$15,000 from the Fletcher Jones Applied Mathematics endowment. Most notably, community members donated more than \$10,000.

Conclusion

At Burning Man, outside of the hottest parts of the days, there was a constant stream of visitors to the artwork. Most

of them were curious about all aspects of the piece, and constantly engaged the build volunteers with follow-up questions, both about the unfolding mathematics and the unknown cosmology. Many of these interactions led to rather long discussions about either the nature of mathematics itself, or what it meant for the universe to have a shape and what other possible shapes might be. After Burning Man, *Unfolding Humanity* was also showcased in downtown San Diego in October 2018, in the famous Balboa Park, outside the Old Globe Theatre; see Figure 3. The final resting place for this artwork is currently being discussed.

Several mathematics research problems were also investigated around unfolding geometry. For instance, a visual algorithm was developed that proved that every unfolding of an n -cube is without overlap, resulting in a valid net [2]. Work is in progress to show that every unfolding of all regular polytopes is possible. These explorations also resulted in conversations with artists, some leading to gallery shows [8]. Overall, *Unfolding Humanity* was a remarkable success, showcasing not only the interplay between engineering, mathematics, and the arts, but the collaboration between academia and the arts community. A video showcasing this work can be found in [10].

Based on our experience, we offer interested readers pursuing similar projects of their own a few closing thoughts. First, get connected with your local art community. We have found most artists and makers to be very interested in hearing about mathematics, especially topics that avoid technical or computational issues. In particular, there is a great thirst for understanding ideas and questions that are open and accessible for exploration.

Second, try to introduce physical aspects of mathematics to students. There is now a national ‘maker space’ movement, led by Stanford’s *d.school*, that is trickling into the mathematics realm. There are versions of math laboratories at several colleges and universities (which can be traced back to the University of Minnesota’s famous Geometry Center) that try to bring an intersection of mathematics with other disciplines in a physical space.

And finally, be bold. In our experience as true amateurs in the sculpture world, we found tremendous encouragement and support. The art and maker communities are wonderfully enthusiastic and deeply gifted. Most likely, the help you need will be found, and those helping you will be grateful for the experience, for both the entry into mathematics and the building of community.

ACKNOWLEDGMENTS. This project was a deeply collaborative effort, with eighty volunteers and a faculty leadership team of Susie Babka, Gordon Hoople, and Nate Padre in addition to the authors. We are grateful for all involved, and especially thankful for the contributions from Max Elliott, Lee Hemingway, and Quinn Pratt, as well as Gilles Bonugli Kali (instagram@gbk.style) for his photography. We are also indebted to USD, San Diego Collaborative Arts Project, and San Diego CoLab for their trust and support of this endeavor. Finally, this project is dedicated to our students, who have shown deep sacrifice and love, from its conception to its creation.

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Credits

Figures 1 and 2 are by Gilles Bonugli Kali.
Figure 3 is by Satyan L. Devadoss.
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Photo of Diane Hoffoss is by Rand Larson, RandLarson.com.

Federal Funding for Mathematics Research

Reza Malek-Madani and Karen Saxe

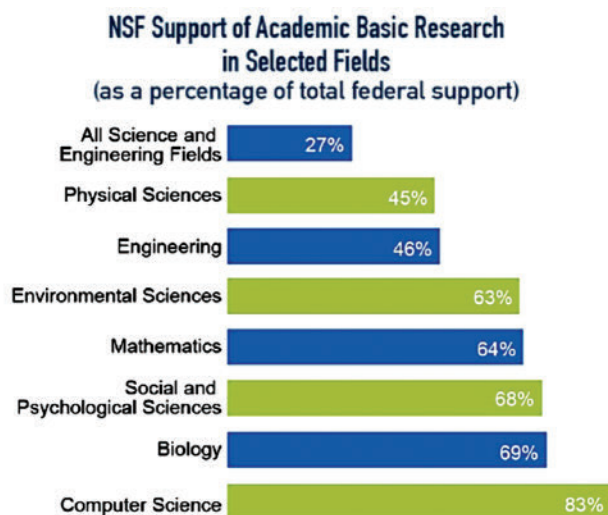
The AMS Office of Government Relations, located in Washington, DC, works with the federal government to try to make sure that the agencies that award money to mathematicians have the budget they need to ensure robust and stable funding for mathematics research. For example, the Office works with the Coalition for National Science Funding (CNSF) on annual appropriations for the National Science Foundation (NSF), and with the Coalition for National Security Research (CNSR) for Department of Defense (DoD) appropriations. Fair warning: this article contains (too) many acronyms.

Mathematicians working in academia often seek federal funding for their research from the NSF. This makes sense, as about 64% of federal support for basic research in the mathematical sciences—and done at colleges and universities—comes from the NSF (Figure 1).¹ NSF funding is available for a wide variety of projects, including for individual investigator awards.

The NSF is the only agency that supports mathematics research broadly across all fields. The goal of this article is

to explore other federal agency sources of funding for mathematics. The chart below shows that mathematicians are more dependent on the NSF for support than are scientific researchers in some other disciplines. It is worthwhile to note that other agencies contribute funds for mathematics research, but only to projects that contribute to their respective missions.

While 64% is clearly the majority of funding, this leaves a sizable chunk. Where does the remainder come from? The vast majority of the remainder comes from DoD and the Department of Health and Human Services (HHS) at 18% and 12% respectively. The Department of Energy (DoE) provides a smaller portion of funds for mathemati-



Notes: Biology includes Biological Sciences and Environmental Biology. Biology and Psychological Sciences exclude National Institutes of Health funding from the total amount of federal support.

Source: NSF/National Center for Science and Engineering Statistics, Survey of Federal Funds for Research and Development, FY 2015.

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<https://www.nsf.gov/homepagefundingandsupport.jsp>

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Figure 1: About 64% of federal support for basic research in the mathematical sciences—and done at colleges and universities—comes from the NSF.

cal sciences research; this amounts to just under 5% of all federal funding for basic research done at universities and colleges in mathematics.²

The rest of this article will focus on DoD and HHS opportunities for mathematicians. The NSF is the only agency of the federal government that does not do research in its own labs (“in house” or intramural research). DoD and HHS opportunities are thus in the form of grants for faculty members (extramural) and also include options to work at the agency doing intramural research. In this article, we focus on extramural opportunities.

At DoD there are several agencies that contribute significantly to research funding—the Air Force Office of Scientific Research, Army Research Office, Office of Naval Research, Defense Advanced Research Projects Agency, and the National Security Agency.

At HHS, mathematics research is supported by several of the National Institutes of Health’s (NIH) 21 institutes, including the National Heart, Lung, and Blood Institute (established 1948), the National Institute of General Medical Sciences (established 1962), and the National Institute of Biomedical Imaging and Bioengineering (established 2000).³

Department of Defense funding for mathematics

Industry is a vital partner for DoD in technology development; it is perhaps not as well known that academic researchers also play a significant role in furthering DoD’s mission. According to NSF data, nearly one in every four dollars of DoD awards for scientific research goes to colleges, universities, and non-profits. Further, over 57% of DoD-sponsored basic research takes place at non-profits, and on college and university campuses.⁴

You may wonder why DoD supports basic research. The mission of the DoD is to provide the military forces needed to deter war and to protect the security of our nation, and DoD’s capability to do so depends on technology. Technology is the fruit of science, and basic research produces the new, transcendent ideas that will enable our future technologies. It is the role of DoD Basic Research Program Managers to make informed, but not guaranteed, investments in particular directions that they deem have the best chance of helping provide DoD with the new scientific understanding that will enable unprecedented future technologies.

We highlight five agencies within the DoD. Three are the “services”—Air Force, Army, and Navy. Each of the services has a Young Investigator Program (YIP), and

their own Multidisciplinary University Research Initiative (MURI). The first of these supports early career scientists who show exceptional promise in the specific service’s priority research areas. The awards are generous, and may be budgeted against any reasonable costs related to the conduct of the proposed research, including salary for the young investigator, graduate student support, supplies and operating expenses. Recent YIP awardees include:

- SHAYAN OVEIS GHARAN, University Of Washington, “Applications of Algebraic Techniques in Algorithm Design” (Navy 2018);
- JONATHAN HAUENSTEIN, University of Notre Dame, “The Geometry of Multiscale Models: Identifiability, Reparameterization, Comparisons, and Parameter Space Exploration” (Army 2014);
- LAURA BALZANO, University of Michigan, “Non-convex Optimization Algorithms and Theory for Matrix Factorization with Dynamic Massive Data” (Air Force 2018);
- SANJAM GARG, University of California, Berkeley, “Cryptography for Big Data” (Air Force 2017).

Each year there are twenty-four MURI topics, with eight provided by each of the Air Force, Army, and Navy basic research offices. Many of the 2019 topics across all three services seek mathematical models and computational approaches to provide theory and understanding to complex physical and biological processes. As an example, one topic offered by Navy is “Fundamental Limits on Information Latency,” and this MURI will require researchers with backgrounds in signal processing, optimization, and game theory. “Advanced Analytical and Computational Modeling of Arctic Sea Ice” is another Navy topic, and the Army has a call out for proposals related to “Multi-layer Network Modeling of Plant and Pollen Distribution across Space and Time.” Watch for the 2020 topics announcement.

The strategy of the Air Force Office of Scientific Research (AFOSR) is to invest in basic research with the goal of transferring its fruits to industry, to the academic community to further scientific knowledge, and to the various branches of the Air Force Research Laboratory, for further research and development of technology. Investment in pure mathematics research has been a key ingredient in AFOSR’s strategy where funding has been provided primarily in three programs: Dynamics & Control, Information Science, and Cybersecurity. The research topics being funded include: applied category theory, applied algebraic topology, and analysis. In this context, the word “applied” means seeking new mathematical concepts and theories that have potential applications in scientific disciplines such as dynamics of abstract systems, homotopy type theory, univalent foundations, and formalized proofs.

Since 2011, AFOSR has sponsored a number of single-investigator efforts, including foreign researchers, and a MURI project on Homotopy Type Theory and Univalent Foundations. The latter has the participation of a number of

²Funding percentages for FY2015. See https://ncesdata.nsf.gov/fedfunds/2015/html/FFS2015_DST_080.html

³<https://www.nih.gov/institutes-nih/list-nih-institutes-centers-offices>

⁴<https://ncesdata.nsf.gov/fedfunds/2015/>

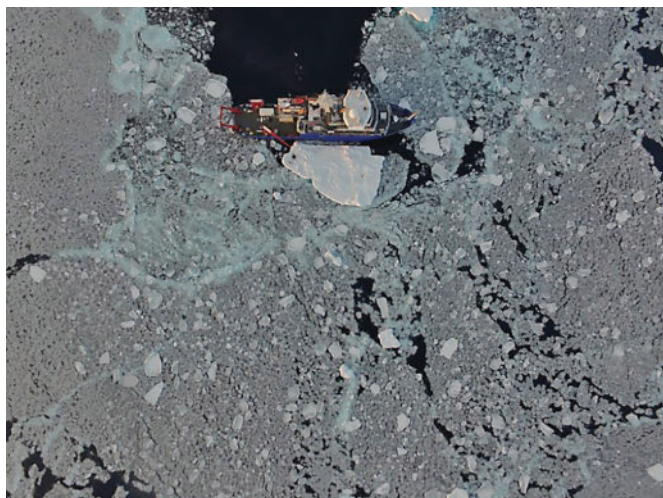


Figure 2: A snapshot of the state of sea ice in the Arctic, displaying its complex multiscale nature.

mathematicians, including Vladimir Voevodsky (deceased; formerly at the Institute for Advanced Study), Michael Shulman (University of San Diego), and Steve Awodey and Jeremy Avigad (both at Carnegie Mellon University). AFOSR has also sponsored work on data interpolation since 2011. In this effort, mathematicians are collaborating with control theorists to formulate a new framework for modeling and learning dynamical systems using only small amounts of data. Sponsorship of research on using category theory and operads to study abstract complex systems and their dynamics is leading to new discoveries and directions in this field. A group of mathematicians are using Wasserstein geometry and optimal transport for computer vision problems. AFOSR has sponsored research at the intersection of systems biology and differential equations. Another sponsored effort is investigating robotic navigation via sheaf-theoretic techniques.

In applied and computational mathematics, AFOSR has had several MURIs in areas such as Uncertainty Quantification (combination of applied probability and computational tools), Game Theory on Networks, and most recently in 2018 a MURI on Mean-Field Game Theory.

For more information: <https://www.wpafb.af.mil/afrl/afosr>.

The Army Research Office supports any sort of mathematics that may help enable a stronger future Army, and is the only funding agency within the DoD with a Mathematical Sciences Division. This division is currently organized into four programs: Modeling of Complex Systems (interests involving geometry, topology, non-smooth differential analysis), Probability and Statistics (interests involving SDEs, QSDEs, multivariate heavy-tailed phenomena), Biomathematics, and Computational Mathematics.

In addition to the YIP and MURIs described above, Army funding can be obtained through several programs, including: Single Investigator program, Short-Term Innovative

Research (STIR) program, Presidential Early Career Award for Scientists and Engineers (PECASE), Historically Black College and Universities/Minority Institutions (HBCU/MI) program, Conference Grants, and the High School Apprenticeship Program (HSAP)/Undergraduate Research Apprenticeship Program (URAP). This agency supported John Tukey's work in statistical analysis, and his famous Fast Fourier Transform paper with Cooley acknowledges the Army Research Office as sole sponsor.⁵ They also supported Lotfi Zadeh's work in fuzzy mathematics, which was honored posthumously with a 2017 Golden Goose Award.⁶

For more information: <https://www.arl.army.mil>.

The Office of Naval Research (ONR) funding opportunities are organized according to technology needs; mathematics falls in the department of "Command, Control, Communications, Computers, Intelligence, Surveillance, and Reconnaissance" (or, "Code 31").⁷ Code 31 is further broken down into three divisions, one of which is "Mathematics, Computer and Information Systems" (Code 311). Code 311 funds basic research in areas including network theory, decision-making, cybersecurity, mathematical data science, and computational analysis. It includes three programs that have significant mathematical content.

- The Applied and Computational Mathematics Program focuses on developing analytical and computational tools for solving PDEs arising from various physical problems, such as ocean and atmospheric dynamics, inverse problems in acoustics, imaging targets in cluttered media, and multi-scale/multi-physics problems arising in the modeling of fatigue, fractures, and shocks. The mathematical modeling of sea ice dynamics and its multiscale nature is a current focus of research and several applied mathematicians are working on this topic (see Figure 2).
- The Mathematical Data Science Program focuses on developing mathematical tools for efficiently representing data, understanding relationships in data, and extracting information from data. This program draws upon basic research in mathematics, probability, statistics, signal processing, machine learning, data engineering, and information theory.
- The Mathematical Optimization Program focuses on developing theory and algorithms for solving large-scale optimization problems. This includes, but is not limited to, cutting plane and polyhedral techniques for mixed-integer programming, decomposition ap-

⁵ Cooley, J. W. and Tukey, J. W. "An Algorithm for the Machine Calculation of Complex Fourier Series." *Math. Comput.* **19**, 297–301, 1965.

⁶ <https://www.goldengooseaward.org/awardees/fuzzy-logic>

⁷ <https://www.onr.navy.mil/en/Science-Technology/Departments/Code-31>

proaches for large convex and non-convex problems, first- and second-order algorithms for convex optimization, and distributionally robust methods for stochastic optimization. Also of interest is research in combinatorial optimization and discrete structures associated with these problems.

There are additional programs within the Division in which mathematics plays a supporting role, such as the Machine Learning, Reasoning, and Intelligence Program, and the Cyber Security and Complex Software Systems Program.

ONR-supported research has led to the proofs of the Four Color Theorem, the Strong Perfect Graph Theorem in graph theory, and Rota's Conjecture in matroid theory.

The Naval Research Enterprise (which includes ONR and the Naval Research Laboratories) has a variety of opportunities for students (undergraduate and graduate) and faculty.⁸

For more information: <https://www.onr.navy.mil>.

The Defense Advanced Research Projects Agency (DARPA) funds projects focused on improving our national security. Many efforts across the agency may have a significant mathematical component, and within the Defense Sciences Office (DSO) of DARPA, several programs have had a strong mathematical focus. Mathematics-focused programs are generally aimed at developing new mathematical approaches and algorithms to solve problems spanning Artificial Intelligence, optimization, uncertainty quantification, dynamical systems modeling, control theory and design. Indeed the current DSO Broad Agency Announcements (BAAs) list "Frontiers in Math, Computation and Design" as one of four research topics of interest and will support basic research that fits with the call for proposals.⁹ It also offers a Young Faculty Award program to identify and engage rising research stars in junior faculty positions at US academic institutions.¹⁰ Mechanical engineer Mark Fuge of the University of Maryland is a Young Faculty awardee for a project entitled "Topology and Synthesis of Design Manifolds," studying how mathematical tools from topology, group theory, and machine learning can aid human exploration of large, complex design spaces.

For more information: <https://www.darpa.mil>.

The National Security Agency's Mathematical Sciences Program (MSP) was "started in 1987 in response to an increasingly urgent need to support mathematics in the United States."¹¹ The MSP supports conferences and

workshops in the five subject areas of Algebra, Discrete Mathematics, Number Theory, Probability, and Statistics. They also support Research Experiences for Undergraduates programs in any area of mathematics or computer science. In the past, MSP has funded individual researchers. For budgetary reasons, individual grants as well as Young Investigator Grants and the NSA Sabbatical Program are not currently available. It is possible that we will see these opportunities re-emerge in the future.

For more information: <https://www.nsa.gov>.

In addition to the opportunities described above, there are joint programs between the NSF and other agencies. The "Algorithms for Threat Detection" program is just one example, and is jointly run by NSF and DoD. Harvard mathematician Shing-Tung Yau currently holds an award and is using tools from random graph theory and differential geometry to focus on finding patterns in large graphs that may be hidden and that could potentially be indicative of emerging threats of various kinds (internets, critical infrastructure networks, financial networks, social networks, etc.).

Department of Health and Human Services funding for mathematics

As with all other federal agencies outside of the NSF, research supported by HHS is mission driven. The NIH sits in HHS and, as you probably know, it works to prevent disease and improve health. Mathematicians have an array of opportunities to collaborate through NIH funding. Reinhard Laubenbacher of the University of Connecticut is working with a medical colleague on algorithms that could lead to new therapies to treat invasive Aspergillosis, a fungal disease of the lungs that poses serious health dangers to patients with weakened immune systems, including organ transplant patients and cancer patients undergoing chemotherapy.

Within the National Institute of General Medical Sciences at NIH, the Division of Biophysics, Biomedical Technology, and Computational Biosciences supports research in mathematical biology through a program run jointly with the NSF.¹² Both agencies recognize the need for promoting research at the interface between the mathematical sciences and the life sciences. This program is designed to encourage new collaborations, as well as to support existing ones. Mathematician Christine Heitsch (Georgia Tech), whose research interests lie at the interface between discrete mathematics and molecular biology, has been supported by this program.

The National Institute of Biomedical Imaging and

⁸<https://www.onr.navy.mil/Education-Outreach>

⁹https://www.fbo.gov/index?s=opportunity&mode=form&id=36a2f123679010655f4930a24cc8a750&tab=core&_cview=0

¹⁰<https://www.darpa.mil/work-with-us/for-universities/young-faculty-award>

¹¹<https://www.nsa.gov/what-we-do/research/math-sciences-program>

¹²<https://www.nigms.nih.gov/Research/specificareas/mathbio/Pages/default.aspx>

Bioengineering funds research in modeling, simulation, and analysis.¹³ As with DoD, the NIH also collaborates with the NSF and there are inter-agency programs such as the Collaborative Research in Computational Neuroscience.¹⁴ For more information: <https://www.nih.gov>.

Summary

Federal funding for mathematical research is available from agencies other than the NSF. The DoD and NIH granting agencies, in tandem with the NSF, have had a remarkable record of focused investments in almost all areas of mathematics, but especially in applied and computational mathematics, that have resulted in rapid acceleration of several areas with game-changing effects on science and technology. Among these achievements are algorithm developments ranging from the Simplex method to the Fast Multipole method, the development of wavelets and its huge impact on JPEG 2000, to compressive sensing, uncertainty quantification, and now the mathematics of mean fields. All of these developments have benefited from investments by the NSF and the DoD and NIH granting agencies.

This article has attempted to give an overview of funding opportunities at the DoD and the NIH, in particular. Key advice for the potential PI:

(a) Keep an eye on Broad Agency Announcements, the announcements that the DoD and NIH granting offices routinely publish to communicate launching of new initiatives.

(b) Have a conversation with the program officer, who often has a large degree of autonomy in developing their research portfolio, prior to submitting a proposal or a white paper to determine whether the proposed research is a good fit for the program.

(c) Remember that both the DoD and NIH are charged with developing a portfolio of research investments that contribute to their respective missions.



Reza Malek-Madani



Karen Saxe

Credits

Figure 2 is courtesy of Guy Williams (University of Tasmania/ACE CRC) and Ted Maksym (WHOI).

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¹⁴https://www.nsf.gov/funding/pgm_summ.jsp?pims_id=5147



Wild Wild Whitehead

Danny Calegari

Take a knot. Take a tube around the knot. Put a new knot in the tube, twisted around and clasping itself as in Figure 1. The new knot goes once around the tube, and then “doubles back” and clasps itself. The new knot is the *Whitehead double* of the old.

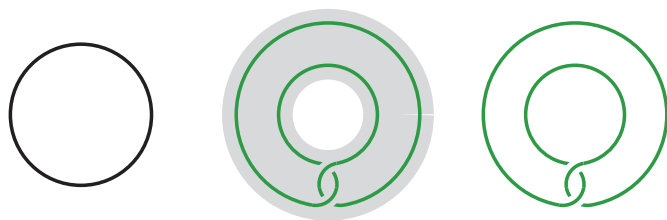


Figure 1. The green knot is the Whitehead double of the black knot.

Let’s call the first knot K_0 and its Whitehead double K_1 , and let’s call the tube around K_0 (actually a solid torus) N_0 . It makes sense to take the Whitehead double of any knot, but in Figure 1, K_0 is a trivial knot; i.e., it bounds an embedded disk in S^3 . In this case, K_1 is trivial too: it bounds an embedded disk in S^3 . But K_1 is *knotted* in N_0 . Any embedded disk that K_1 bounds must go outside N_0 .

On the other hand, K_1 is *homotopically* trivial in N_0 ; i.e., it bounds an *immersed* disk, one that crosses itself, but does not cross N_0 . To see this, just push one of the clasps of K_1 through the other one; this undoes the knotting, and the

result can be shrunk down to a point. The track of the knot K_1 during this process sweeps out an immersed disk.

In terms of the fundamental group, a knot K in a space X determines a conjugacy class $[K]$ in the fundamental group $\pi_1(X)$. Now, $\pi_1(N_0) = \mathbb{Z}$, and since K_1 bounds an immersed disk, $[K_1]$ is trivial in $\pi_1(N_0)$.

We can keep going. Let N_1 be a tube around K_1 , thin enough to fit in N_0 , and let $K_2 \subset N_1$ be the Whitehead double of K_1 . And so on. Each K_i bounds an embedded disk in S^3 , but each of these disks must go (many times!) all the way outside N_0 .

The tubes get thinner and thinner as we go, and longer and longer. Consequently, the knots must get longer and longer too: Each K_n must wind back and forth at least 2^n times around N_0 , clasping itself in a complicated way at the end. The infinite intersection $\bigcap_i N_i$ is called the *Whitehead continuum*, which we write Wh , see Figure 2. The Whitehead continuum is connected but not path connected. It has an entangled dyadic Cantor set of “strands” that wind around N_0 .

The complement $S^3 - Wh$ is an open 3-manifold called the *Whitehead manifold*. It turns out that $S^3 - Wh$ is contractible but not homeomorphic to a 3-ball. Let’s see why.

The outside of N_0 in S^3 is another solid torus N'_0 , whose core is a knot K'_0 linking K_0 in a Hopf link. $[K'_0]$ is the generator of $\pi_1(S^3 - N_0) = \pi_1(N'_0) = \mathbb{Z}$. The knots K'_0 and K_1 together form a 2-component link called the *Whitehead link*. This link is symmetric: we can isotope it around and interchange K'_0 and K_1 ; see Figure 3.

Since $[K_1]$ is trivial in $\pi_1(N_0) = \pi_1(S^3 - N'_0)$, it follows by symmetry that $[K'_0]$ is trivial in $\pi_1(S^3 - N_1)$. Consequently the inclusion map $S^3 - N_0 \rightarrow S^3 - N_1$

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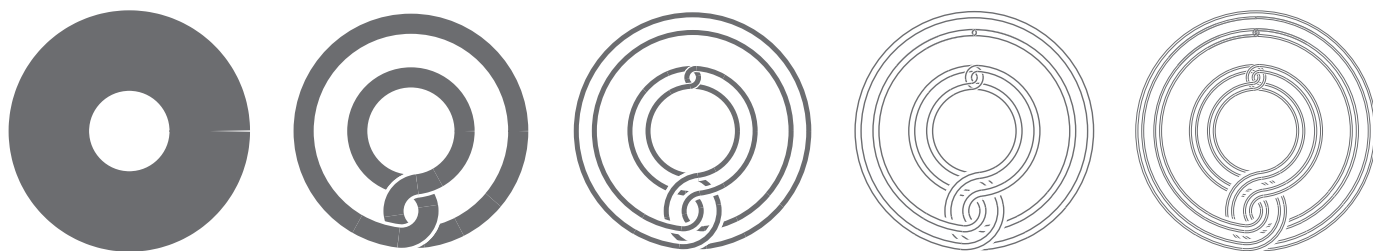


Figure 2. Successive approximations converge to Wh .

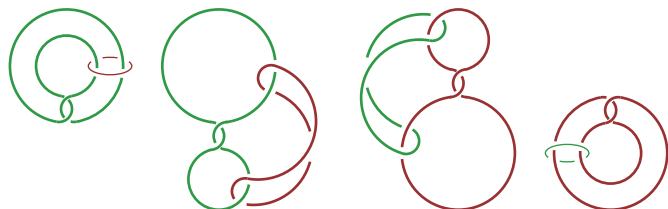


Figure 3. The Whitehead link is symmetric.

induces the zero map on π_1 . Each N_i is unknotted in S^3 , and each N_{i+1} sits in N_i the same way that N_1 sits in N_0 . So each $\pi_1(S^3 - N_i) = \mathbb{Z}$, and each inclusion $S^3 - N_n \rightarrow S^3 - N_{n+1}$ induces the zero map on π_1 . Taking a direct limit, $\pi_1(S^3 - Wh)$ itself is trivial. A similar argument shows that all the homotopy groups of $S^3 - Wh$ vanish, and it is contractible.

On the other hand, the complement $S^3 - Wh - N'_0 = N_0 - Wh$ has an *infinitely generated* fundamental group; each $\pi_1(N_i - N_{i+1})$ is complicated (it contains free groups of every rank), and all of them include as subgroups of $\pi_1(N_0 - Wh)$. This shows that $S^3 - Wh$ is not a ball.

Now let's compactify $S^3 - Wh$ by adding a single point at infinity. This compact space can also be thought of as the quotient space S^3/Wh that we get by crushing Wh to a single point. Because $S^3 - Wh$ is not a ball, S^3/Wh is not a manifold. However—remarkably—it is a *manifold factor*: the product $(S^3/Wh) \times \mathbb{R}$ is homeomorphic to $S^3 \times \mathbb{R}$!

How can this possibly be???

First, each N_1 slice can be unknotted by a tiny perturbation in $N_0 \times \mathbb{R}$. To distinguish the \mathbb{R} factor and, for the sake of brevity, we refer to it as the “time” coordinate (this is purely a notational convenience). In this language, we unclasp N_1 from itself by nudging one clasp very slightly forward into the future, and the other very slightly back into the past. After the nudge, N_1 will not clasp itself, but it will clasp a “future” N_1 on one side, and a “past” N_1 on the other. Instead of N_1 clasping itself in a circle, we get a chain of successive N_1 's, each clasping the next, in a slowly ascending *spiral*. Let's let $\epsilon/4$ be the size of the perturbations of each clasp in the time direction, so that the projection of each N_1 to the time coordinate after it's been nudged has total length $\epsilon/2$.

Nudging adjusts points in $N_1 \times \mathbb{R}$ by sliding each point $\times \mathbb{R}$ slightly backward or forward in time. Nudging extends to a self-homeomorphism ν of $N_0 \times \mathbb{R}$, fixed on the boundary.

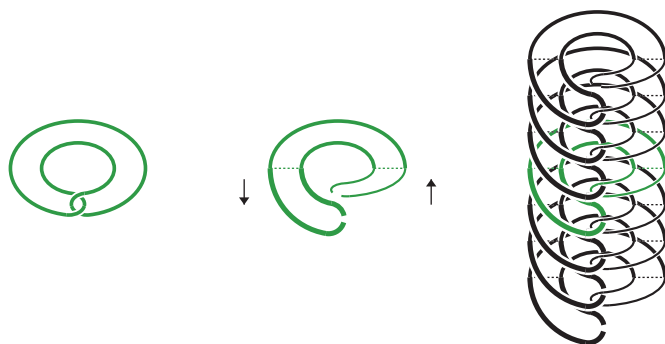


Figure 4. Folding the clasps of each N_1 back and forth in time nudges the union of all N_1 's into a collection of spirals.

By the way, there's not just one spiral—there's a circle's worth of them, filling the whole of $N_1 \times \mathbb{R}$. Two slices $\nu(N_1 \times t)$, $\nu(N_1 \times s)$ are in the same spiral if and only if $t - s$ is an integer multiple of $\epsilon/2$.

After nudging, the next move will straighten out this and every other spiral so that its projection to the S^3 factor is small (let's say for concreteness it has diameter $< \epsilon/2$) *without* affecting the projection to the \mathbb{R} factor.

The cylinder $K_0 \times \mathbb{R} \subset S^3 \times \mathbb{R}$ has polar coordinates (θ, t) where $\theta \in \mathbb{R}/\mathbb{Z}$. Extend these polar coordinates to a small tubular neighborhood of $K_0 \times \mathbb{R}$ containing $N_1 \times \mathbb{R}$, with closure contained in the interior of $N_0 \times \mathbb{R}$.

We can “untwist” every spiral simultaneously by the map

$$(\theta, t) \rightarrow (\theta - 2t/\epsilon, t)$$

on our small tubular neighborhood. Twisting extends to a self-homeomorphism τ of $N_0 \times \mathbb{R}$, once again fixed on the boundary.

In summary, first we nudge, then we twist. After doing this, every $\tau\nu(N_1)$ slice projects to subsets of diameter at most $\epsilon/2$ in *both* the \mathbb{R} and the S^3 directions. So $\tau\nu(N_1)$ has diameter at most ϵ .

In other words, $h_1 := \tau\nu$ simultaneously *shrinks* all the N_1 slices in $S^3 \times \mathbb{R}$ as small as we like, while keeping $(S^3 - N_0) \times \mathbb{R}$ fixed pointwise.

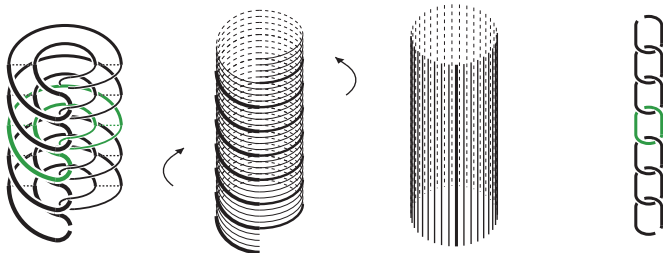


Figure 5. Screw the top and bottom in opposite directions like you're taking the lid off a pickle jar.

Take a sequence $\epsilon_i \rightarrow 0$, and repeat this operation for each $i > 1$ in place of 1 with ϵ_i in place of ϵ . We get a sequence of self-homeomorphisms $h_i : S^3 \times \mathbb{R} \rightarrow S^3 \times \mathbb{R}$, each supported in $N_{i-1} \times \mathbb{R}$, as a composition of a nudge-and-twist $h_i := \tau_i \nu_i$. Each N_i slice gets smaller and smaller in diameter as we apply consecutive h_i 's. Under application of successive h_i 's, the orbit of every point is a Cauchy sequence, and the infinite composition

$$h := \lim_{i \rightarrow \infty} h_i h_{i-1} \cdots h_1 : S^3 \times \mathbb{R} \rightarrow S^3 \times \mathbb{R}$$

is well-defined and continuous.

For any compact subset X of $S^3 - Wh$ the restriction of h to X is the composition of finitely many homeomorphisms, so the restriction of h to $(S^3 - Wh) \times \mathbb{R}$ is a homeomorphism. On the other hand, each Wh slice is successively shrunk smaller and smaller by successive h_i , so in the end h crushes each Wh slice to a point, and h factors as $h = g\pi$,

$$S^3 \times \mathbb{R} \xrightarrow{\pi} (S^3/Wh) \times \mathbb{R} \xrightarrow{g} S^3 \times \mathbb{R},$$

where $\pi : S^3 \times \mathbb{R} \rightarrow (S^3/Wh) \times \mathbb{R}$ is the quotient map, and g is the homeomorphism we've been looking for.

AUTHOR'S NOTE. The main theorem in this article and its proof are both well-known, and not due to me! They are due to J. Andrews and L. Rubin, *Bull. Amer. Math. Soc.* **71**(1965), 675-677.



Danny Calegari

Credits

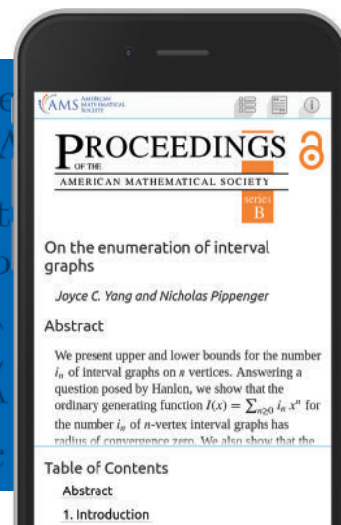
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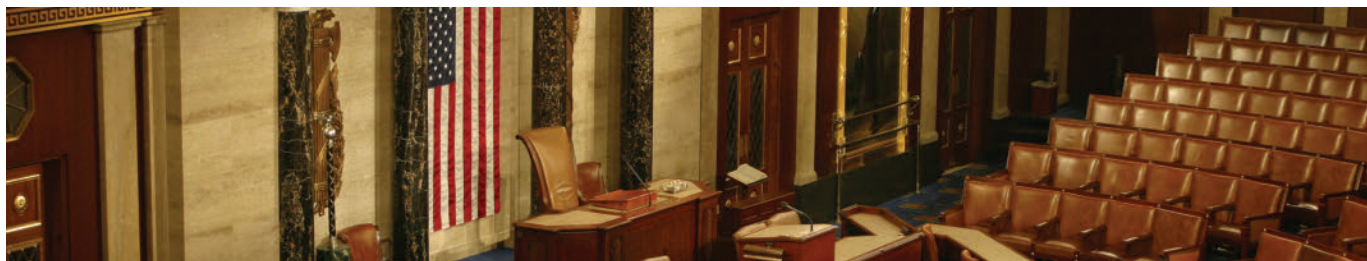


Image courtesy of House.gov.
Photo cropped on the top and bottom.

Math Societies to Lawmakers: Help Us Make the Master Key

Sophia D. Merow

Biannual Briefings

The American Mathematical Society (AMS) Office of Government Relations (<https://www.ams.org/government/government>)—jointly with the Mathematical Sciences Research Institute (MSRI)—sponsors two congressional briefings per calendar year.¹ These briefings provide an opportunity for the mathematics community to impart information to policymakers and, in particular, to tell compelling stories of how federal investment in basic research in mathematics pays off for the American taxpayer.

At a lunch briefing in the Dirksen Senate Office Building on December 4, 2018, Rodolfo H. Torres gave the Hill staffers in attendance what former US Representative Bart Gordon (D-TN) called “ammunition” for their science advocacy. In his half hour at the lectern, the University of Kansas mathematician (and AMS Fellow) made the case for federal funding of basic research by showing how investigation of foundational questions can launch lines of inquiry that produce, sometimes decades later, societally beneficial applications galore.

“This is the way, many times, research takes place,” Torres explained. “We move from fundamental research in maybe math, physics, or biology to applied research in which we

want to apply these tools to solve specific problems, and eventually we get—sometimes—to translational research. We find cures for diseases and solve problems that affect our society and the way in which we live.”

Titled “From the Color of Birds to Nanomaterials and New Technologies,” Torres’s talk juxtaposed what he acknowledged to be an unlikely trio of topics, “things you won’t normally think to find in the same sentence.” Before Torres discussed the blue hue of birds, however, he used analogies to give his audience a feel for Fourier analysis.

Fourier analysis decomposes a signal into a combination of oscillating waves of different frequencies and amplitudes, he said, much as a prism separates a beam of light into a spectrum of colors of different wavelengths. Representing a signal using its Fourier coefficients is like writing a recipe for a cake when the ingredients—here the waves of different frequencies—are always the same. The amounts—the amplitudes—suffice to encode the information. Similarly, Torres explained, a two-dimensional image can be represented as a superposition of planar waves.

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¹For coverage of the first congressional briefing of 2018, see the January 2019 Notices <https://www.ams.org/journals/notices/201901/rnoti-p102.pdf>.

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Torres briefly displayed a slide bearing (along with two recent *Notices* covers) a list of the “arsenal of tools” that deploy Fourier analysis for signal analysis and image processing: wavelets, curvelets, diffusion maps, compressed sensing... “By treating signals and images as mathematical objects, they can be quantitatively and algorithmically



Rodolfo Torres mentions the arsenal of tools for signal analysis and image processing that are based on Fourier analysis.

manipulated to extract, enhance, filter, and compress the information they represent,” the slide read.

Wavelets afforded Torres the opportunity to both discuss return on federal investment in mathematics and provide a first sampling of useful applications. A. P. Calderón’s 1964 paper “Intermediate spaces and interpolation, the complex method” includes the following acknowledgement: “This work was partly supported by N.S.F. grant GP.-574.”

“So this paper of Calderón, which is the birth of wavelets [within mathematics], was funded by the National Science Foundation in 1964,” Torres said. “I would be very curious to know how much was awarded to Calderón.”

From Calderón’s likely modest seed grant has grown a field of research with ramifications for everything from national security to agriculture. While the FBI can’t match crime-scene fingerprints to those of known criminals *quite* as quickly as detectives do in television procedurals, wavelet-based compression of fingerprint images does speed up the process. Statistical analysis using wavelets of computed tomography (CT) images enables more accurate determination of cancer prognosis. Applying Fourier analysis to satellite imagery gives soybean producers in the American heartland projected crop yields for their competitors in Brazil.

But back to the titular birds. It turns out that the blue facial skin sported by birds such as the asities of Madagascar derives its color not from a pigment but from the structure of the tissue itself. The skin is composed of long, thin collagen fibers. Look at it with an electron microscope and you see something like the cross section of a bundle of angel hair

spaghetti. Torres and his biologist colleagues saw a certain order in these cross sections, and they adapted the two-dimensional Fourier transform to analyze the periodicity and optical properties of the tissue. They demonstrated that the nanostructure of the collagen arrays accounts for that brilliant blue.

“Only certain wavelengths that resonate with the physical distance present in the material get scattered,” Torres explained, “and those are the ones we see.”

Such structural color exists not just in birds but also in mammals (a mandrill’s nose, for instance), butterflies, and dragonflies. And photonic structures, structures that selectively interfere with light, don’t possess only optical properties. They can also render tissues antibacterial or self-cleaning or superhydrophobic (extremely difficult to wet). Study of tissues with desirable properties such as these has “really led to what I would call the bioinspired nanomaterial revolution,” said Torres.

Research in the field of evolutionary photonics promises applications in renewable energy, nanomedicine, and advanced material engineering. Torres encouraged audience members to visit the website of the NSF-funded National Nanotechnology Coordinated Infrastructure (<https://www.nnci.net>) to learn how nanostructures might enable more efficient light harvesting or advance energy storage. As an example of a medical application of nanofabrication, he highlighted a flexible membrane hundreds of times thinner than aluminum foil that alleviates glaucoma by distributing pressure and facilitating absorption of fluid buildup.

Torres closed his talk by reiterating the role of mathematics in science and technology and underscoring the potential of basic mathematics to unlock future advances. “When we start talking about order and symmetry and geometry, this is the field of mathematics,” Torres said. Mathematicians can quantify this order (or lack thereof), thereby explaining



MSRI Director David Eisenbud (standing right) solicited from audience members their reasons for attending the briefing.

physical phenomena found in nature. These newly understood natural phenomena then serve as inspiration for cutting-edge human-developed technologies.

Torres suggested thinking of applied research as building a key to a particular door we need to open. Fundamental research, in contrast, is like creating a master key, one that will open many different doors. "In fact, it's going to be opening doors that you still don't know that you need to open," he said. "But you will have it ready when the need arises."

Of course the purpose of congressional briefings such as the one on December 4 is to publicize among policy-makers and keepers of the federal purse-strings instances of sometimes long-unused master keys of mathematics opening societally fruitful doors. It's a service that Joel Creswell from the office of Congressman Daniel Lipinski (D-IL) appreciates as he supports Lipinski's work on the Committee of Science, Space, and Technology.

"These aren't stories that you can come up with from looking at the research literature," he said in response to MSRI Director David Eisenbud's post-presentation question about what prompted audience members to attend. "You really need experts in the field to be able to say, 'Here's a discovery that traces back to this basic research.'"

What You Can Do

You, too, can help get the word out about the importance of funding basic research:

- Write a letter to your senator or representative.
- Write an op-ed for a local publication.
- Organize a talk at your institution about the unexpected utility of basic math. Invite your representative and work with your school's government relations office.
- See the AMS Office of Government Relations' "Tools to take action" page <https://www.ams.org/government/dc-prepare> to get started.



Sophia D. Merow

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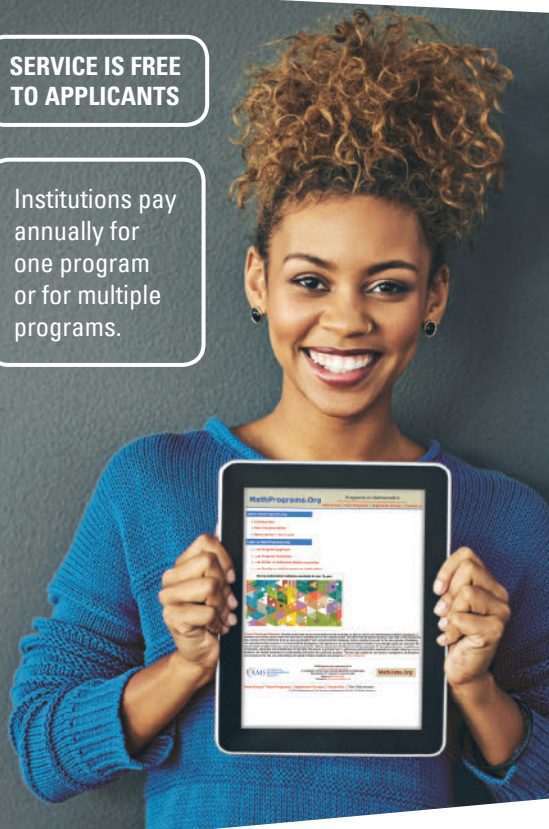
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An Interview with Jill C. Pipher

Evelyn Lamb

Every other year, when a new AMS president takes office, the *Notices* publishes interviews with the outgoing and incoming presidents. What follows is an edited version of an interview with Jill C. Pipher, whose two-year term as president began on February 1, 2019. Pipher is a professor of mathematics and vice president for research at Brown University, and was founding director of the Institute for Computational and Experimental Research in Mathematics (ICERM). The interview was conducted in fall 2018 by freelance writer Evelyn Lamb. An interview with Immediate Past President Kenneth A. Ribet appeared in the March 2019 issue of the *Notices*.

Notices: *How do you see the mathematics profession changing, and what kind of challenges do those changes pose?*

Pipher: Over time, mathematics has become more collaborative. There's an increase in social and team approaches. I think that, on the whole, this is very healthy for the profession. It should entail that mathematics as a profession will be able to attract people with different work styles and ways of contributing, resulting in more diversity. An increase in team approaches and collaboration results in a comparable increase in output and in publications per person. An individual researcher is able to be part of many more projects.

One of the more slightly worrisome aspects of this is the escalation of what a good graduate student resume looks like. Those graduate students who are lucky enough to have really collaborative advisors, or to be in environments where there are active research seminars and visitors flowing through, or who have the good fortune to participate in institute programs or opportunities like the AMS Math Research Communities are going to have resumes that look more like the resumes of postdocs a generation ago. The worrisome aspect is that it could create a starker division of

"haves" and "have nots" when it comes to graduate student job opportunities.

Another thing I want to say about the profession is that I think that mathematics has been steadily broadening its scope over the last few decades with increased interaction with computer science, with the social sciences, with biology and other sciences. This is not brand new; it's been expanding in scope for a while now. I think this expansion has really blurred boundaries between applied and pure math, or at least inspired more connections between pure and applied math. That's also a healthy development for the field.

Notices: *And your work kind of spans applied and pure mathematics, correct?*

Pipher: Yes, most of my work is in analysis, and that aspect is pure math, proving theorems. But I also do research in cryptography, and that is not about proving theorems. It's really done with certain applications in mind.

Notices: *You mentioned that you see the diversity of the profession increasing.*

Pipher: Let me just say the profession *should* entail an increase in diversity as you're able to attract people with different work styles. But this whole subject of diversity in the profession is a larger one.

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Notices: What do you see as the role of the AMS in promoting diversity? What tangible things do you think the AMS can do to help really increase diversity and inclusion in the profession?

I'd like the AMS to foster a broader view of what it means to be a mathematician.

community is really one of the four or five priorities that I have identified for my presidency.

First of all, I'd like to see a new committee—hopefully, ideally, from my point of view, a policy committee—devoted to diversity and inclusion. This is not to isolate the efforts, but to help to distribute them and to help the Society focus on this mission. I also want to see the AMS reach out to people in our community who are already working very hard on diversity and inclusion efforts in their institutions, in their particular communities, to bring their expertise and ideas to the AMS and to help us with our efforts. That's something I think that we can do a better job with. I'd like to examine the policies and the processes at AMS that encourage or discourage diversity in everything from our publications to our prizes.

I think that transparency and communication are central to establishing that the AMS truly takes this issue seriously. And finally, I think that the AMS could take a leadership position in taking climate issues seriously. For example, the APS, the American Physical Society, has a program where it facilitates climate reviews of departments at the request of the chairs. I think it's a good model for us. It's something that's being discussed within AMS now.

The second priority I want to mention is advocacy. Advocacy for mathematics and communication about mathematics to a broad constituency is very important. Advocating for research with federal funding agencies, foundations and congressional offices is already part of my job as Vice President for Research at Brown. I've also for several years been a member of the SIAM [Society for Industrial and Applied Mathematics] Science Policy Committee, which is something I'd like to continue. It offers me a somewhat different perspective than I get from membership in the similar AMS committee.

As director of ICERM, I've had a lot of interaction with DMS [Department of Mathematical Science] and other NSF [National Science Foundation] directorates. And I look forward to more work in this capacity. I really intend to support the work of the Washington office of the AMS,

Pipher: I'd like to talk about that in the context of my priorities for AMS. Continuing to expand upon the work of past presidents in diversity and inclusion in the math

and I look forward to working with Karen Saxe, who has been a great leader of that office.

Notices: When you're talking about advocacy for mathematics, apart from research funding, what does that involve?

Pipher: It involves communicating mathematics to a lay audience. What is a mathematician? What is mathematics? Why is it important? For example, the congressional hearings that AMS and MSRI [Mathematical Sciences Research Institute] collaborate on twice a year in Washington are great examples. Twice a year, a mathematician is invited to give a general-audience talk about mathematics to an audience which includes congressional staffers and members of Congress themselves. It's a great opportunity to show what the mathematical community is doing for the good of society. This is the kind of advocacy that is foundational to everything else, to getting people to understand why mathematics is so important and exciting.



AMS President Jill C. Pipher.

Notices: Are there any other priorities you want to mention?

Pipher: I'd like the AMS to work to foster a broader view of what it means to be a mathematician. Our profession is much bigger than academia. Clearly academic research mathematics is a central part of what the AMS should be supporting. But there are more careers and more opportunities in mathematics beyond academia. And if we start with this perspective, then the AMS might be able to take a leadership role in professional development opportunities for students—both undergraduates and graduates—who will, of necessity, be pursuing non-academic careers. These careers will be mathematically rewarding, but they won't be careers as faculty members or professors. I want the AMS to be their organization too.

I think that the partnerships that AMS has developed, like those with the NSF and with the Simons Foundation, are important. Two terrific programs have resulted: the Mathematics Research Communities [MRC] and the Travel Awards. I've served on the selection committee for the MRC and there are many more excellent proposals than can be funded for a program that is doing an outstanding job of getting graduate students involved in cutting edge research. I hope to help identify and develop more such partnerships.

And speaking of the MRC, I am thrilled with the increased focus of AMS on the "next generation" of mathematicians. I will pay close attention to the Society's role in providing opportunities for students and in helping recent graduates find rewarding careers in the mathematical sciences.

Notices: You are I think somewhat unique in terms of other past presidents in having a background in industry. You co-founded a company, so you spent some time working outside of academia.

Pipher: I have. I think that helps me appreciate the ways in which mathematicians can contribute to many different endeavors in the world and how interesting and valuable it is to have these experiences [outside of academia].

I would like to mention another priority, which may seem smaller by comparison with others, but there is a larger context for it. Within the AMS, I'd like to revisit the list of committees that we have and do some real consolidation. The larger context is this: I want service on an AMS committee to be truly impactful and rewarding, and result in outcomes that the very busy people we are asking to serve on these committees can point to with great satisfaction. I want to be sure that organizationally our committees are high-functioning, running well, and having an impact.

And then finally, I'm still learning a lot about the AMS; I'm listening to what's important to members of the AMS. I'm still forming priorities and developing ideas for what I'd like to do.

Notices: Going back to your point about broadening the profession, it's something that outgoing president Ken Ribet also mentioned, when I talked with him, really acknowledging that a lot of people who get degrees in math won't be doing jobs within academia. Are there particular priorities or perspectives you feel like you've gotten from being outside of academia that might help academics in their work?

Pipher: The thing about an academic research career is that in many ways, an individual researcher drives their own research agenda. And of course, one's career is more successful when aligned with general trends in the profession, where the funding is going, and so forth. So we can't say that as academics we're totally independent. The level of independence of an academic career is not there in an industry career. I talk to people who have worked as mathematicians in the NSA [National Security Agency]. And of course, they can't discuss the kinds of prob-



Jill C. Pipher lecturing at Brown University.

lems that they're working on. But I've heard many of them say, "Well, I'm working on some really fascinating things, some great problems, and it's very challenging and rewarding. And then it's done at five o'clock, and I go home and do other things. I don't take my work home." Many industry jobs have a more defined and perhaps more balanced day than we have as academics. Many of the people I know working in academia are working all the time.

In industry, you probably have to like working with teams and working on group projects, which many more mathematicians seem to be gravitating towards in any case. It's a different kind of career, but it's also a mathematically rewarding one. When a graduate student graduates and goes into a career in industry or government or whatever, if it's not academic, I just don't want to hear somebody say, "Oh, they left math." Because they're not leaving math. They are going on to a non-academic mathematical career. It has to be part of our culture to appreciate and value multiple career paths.

I believe that valuing multiple career paths has to simply be part of the culture of our mathematical community. In fact, this attitude is a good example of what I mean by "inclusion" as I think about committees or initiatives focused on "diversity and inclusion". These two concepts—diversity and inclusion—are typically lumped together, but they are distinct notions, and equally important. By now, most people understand what diversity means. However, "inclusion" seems to be a little more elusive. I want to stress that inclusion is not an afterthought—it's about creating a welcoming environment, one where people are comfortable to be themselves.

Notices: How have you been involved in the AMS in the past, and what led you to get involved in the first place?

Pipher: I've been a member of AMS in order to support all the ways the Society advances the profession and individual careers—through its publications, conferences and meetings, advocacy, and special programs for students and researchers at all levels, like the Math Research Communities, the travel awards, and the fellowships. The benefits of membership in a professional society are twofold: what you get, and what you can give. At this moment in my career, membership in the AMS gives me the further opportunity to actively contribute to and help shape the priorities of the Society. These include supporting the next generation of mathematicians, advocating for the importance of mathematics in science and society, and recognizing and promoting mathematical research.

Notices: Can you talk a little bit about what it was like to be the founding director of ICERM? It sounds very intimidating to found an entire new research center.

Pipher: Sure! Got a couple hours? [laughter] Just kidding. But I could talk about that a lot.

First of all, the Institute was founded by a remarkable team of people in math and applied math, who all contributed huge amounts of time and ideas to the proposal and its implementation. It was an incredibly collaborative effort from our team, the Brown administration, and the departments. Really, the start of this whole process was very auspicious, and there was a lot of support at all levels of the university. I was very energized to do this job and to do it well.

It's a lot like a startup company that's gotten some venture capital. Suddenly you have some money, and you have to build something. The first year was hiring staff. The key staff positions—the assistant director, the IT director, and so on—were key to making everything else work. Then we worked weekly with the architects who were planning the renovation, and with the weekly construction management team meetings after that. That was the opportunity to tell the architects and the construction team what a math institute should look like, what did we really need. One of the things we needed was to be able to write everywhere, and that was realized at ICERM.

In our proposal, we had formed our science advisory board and our Board of Trustees, so those boards were in place, but we still had to jump in (with the help of those board members) and build two years of semester programs all at once, plus our summer undergraduate research programs and independent workshops. There were lots of days at the very beginning that I just felt like my head was spinning at the end of the day. I might even have gotten a little cranky here and there, but I did feel like I had a lot of help, and that was key. After the proposal was in, the co-PIs on the grant continued continued to work alongside me to support all the efforts of the Institute. That was Björn Sandstede, Jeff Hoffstein, Jeff Brock, and Jan Hesthaven. Without the help of our boards and their advice, it would not have gone as smoothly as it did.

Being the director was a very satisfying professional experience. I came to know and appreciate a lot of mathematics that was new for me. I met a lot of incredibly talented and dedicated people whose research I was thrilled to support.

I believe that our programs have been, and continue to be, tremendously beneficial to the grad students and postdocs that we encourage to participate and that the Institute supports. They come, and they spend the three months of the program enjoying not only the benefits of the scientific aspects of the Institute—the mathematical activities and events, the lectures and so forth—but we also provide a professional seminar series for them with information about the profession, about how to apply for jobs, about how to create resumes, about ethics training in research. I think that is very beneficial for young people, and that was something I was especially proud of. Finally,

the special events that ICERM hosted while I was director, like the AWM [Association for Women in Mathematics] 40th anniversary Research Symposium, which was one of the very first things we did in 2011, and the Blackwell–Tapia Prize conference, and CAARMS, the Conference for African American Researchers in the Mathematical Sciences, were all very meaningful events for me personally.

All in all, it was a tremendously exciting and deeply significant professional experience.

Notices: *I'd imagine getting to know researchers in a lot of really different fields of math has to be a benefit as you're moving into the presidency of the AMS.*

Pipher: It did certainly broaden my perspective in the field. I think having that background means there are certain things that I won't have to learn on the job that are relevant to my objectives in this position at AMS, and also to the expectations of the position. I'm excited to bring that experience to this job.

Notices: *You mentioned that you might be interested in expanding the number of prizes that the AMS offers, or prizes in mathematics in general. Can you talk a little bit more about that?*

Pipher: I can only speak to what I think is important, but of course, I can't plan to do something like this on my own. First, I'd like to get a sense of the interest of the community and the appropriate governance committees of the AMS in such an enterprise. When I compare mathematics to the other physical sciences, I think that mathematicians are under-recognized. There are just too few awards and prizes to adequately recognize all the great contributions to research and to the profession. At the Association for Women in Mathematics, we started these research prizes during my presidency. Right from the very start, the research prize nominations were incredibly competitive and anguishing—so many good people for each individual prize! It helped underscore for me the need for more recognition in our community. So I would like to start this conversation about prizes in the appropriate committees in the organization and see what other people think and if we can take some steps to improve the situation.

Notices: *We touched a little bit on public understanding of math and public communication of math. How do you feel that the public perception of math has changed in the past few years or decades?*

Pipher: I think it's slowly changing to reflect a greater appreciation of the ubiquity and power of mathematics. That's partly a function of the expansion of the scope of mathematics in fields like computer science, and biology, and neuroscience, and so forth. This is just anecdotal, but

it used to be that every time I had a casual conversation with a stranger, like on a plane, and the word math was mentioned, I would hear, "Oh, it was my worst subject," and there would often be the word 'hate'. But I've noticed a shift. I'm discovering more people who are responding positively, who find or who found math interesting, and especially more parents who want to support their children's skills and interest in math.

I had a great experience connecting with parents who wanted to support their daughters' interest in math at an ICERM program called Girls Get Math; the program I founded about five years ago with the help of a private donation from the Phoebe Snow foundation. It's a one-week summer day camp for rising 10th and 11th grade girls who express an interest in math, not self-identified-definitely-going-to-be-math-majors necessarily, but who have expressed enough of an interest to get a letter of recommendation from their high school teacher. This program serves about twenty-five or thirty high school girls in our community. What we wanted to do was create a curriculum and a model that could go nationwide, so that anybody who wanted to run a Girls Get Math program in their community could do that with materials and computer labs that ICERM provided. At the end of the week, we have an awards ceremony. The parents are invited, sometimes their high school teachers come, and it's really meaningful to meet the parents of these girls, about half of whom are on full scholarships for this program, and see how much it means to them that their daughters are expressing an interest in math...how much they want to support that interest. Maybe I don't really have more than anecdotal impressions of a changing public perception, but I feel positive and optimistic.

Notices: *I believe you are only the third woman to be President of AMS, and I think also maybe only the third person who isn't a white man. I wonder, do you feel a lot of pressure from that? In general, in your career, you've probably often been one of the only women, or the highest-ranking woman, in various settings. How do you feel about that?*

Pipher: It's a complicated question. Yes, I'm very keenly aware that there have been only three women who have been President of the AMS in 130 years. That's a burden that minorities in any profession face, the burden of representing not just your own best efforts, but the best efforts of your entire gender or your entire ethnic group. I have to avoid thinking like that because it's distracting. I feel that I have a great opportunity to help an organization that really matters, that's important, that has a tremendous impact. And I view it as an opportunity to be of service, and so I'm trying not to think about myself personally in this role, but rather what I can do or what impact I can have on the Society itself.

Notices: *I'd imagine, though, there might be the flip side where you do feel happy to be a role model for a large number of people, having a leadership roles like this.*

Pipher: In the last decade or so, there has been a fair amount of research on the importance of role models. Some studies have concluded that implicit stereotypes can be reduced in the presence of positive role models, and others suggest that exposure to female role models in STEM can help close the gender gap. All of this reinforces my personal view of the tremendous importance of role models. I was a graduate student in the early '80s at UCLA, working in analysis, when S.-Y. Alice Chang arrived as a newly tenured professor. She was impressive and confident, and was also the first female professor in science that I ever talked to or got to know. Her mathematics and her career path has always been an inspiration to me.

I understand that my own career path may be inspiring to girls and early-career women in mathematics—a fact that is both gratifying and daunting. I am grateful to have opportunities to support and encourage the next generation of women mathematical scientists. At the same time, it is both challenging and disconcerting to feel that one “represents” an entire group while trying to pursue a research career in science. When I graduated in 1985, there were very few women in my field (besides Alice). At conferences, I would rarely see other women. Sometimes, I was the only female speaker. Today, young women studying science in college report that they carry the burden of feeling that their performances represent their gender, not just themselves. I understand and remember that feeling.

So, I would say that the issue of being a role model is mixed. It's both a privilege and a burden. From my present vantage point, I most keenly feel the privilege of this role. But I still remember the burdens from an earlier stage in my career.

Notices: *When you're not working, what kind of things do you like to spend your time on?*

Pipher: Thinking about math. [laughter] Which is not so much of a joke. Given my current job and how much administration I have to do, when I get a chance to go to a conference or block out days to work with a collaborator, I feel like I'm on math vacation. But outside of math and work, I like spending time with my family. I like playing the piano. And I enjoy traveling.

Notices: *What are some of the math questions that you're currently thinking about when you get to go on math vacation?*

Pipher: I'm continuing to work on solvability of boundary value problems for elliptic and parabolic equations with non-smooth coefficients. This research, at the interface of harmonic analysis and PDE, is part of a large program

aimed at quantifying how the properties of coefficients of the equation, particularly smoothness, affect the behavior of the solutions. In addition, we want to have a sharp understanding of the interaction between the geometric properties of the boundary of the domain on which the solution is defined, and the regularity of solutions. For example, if a function is harmonic in the upper half-space or the ball, and it vanishes on a portion of the boundary, then near that piece of the boundary the solution decays like the distance to the boundary. But if the boundary has corners or Lipschitz singularities, then the rate of vanishing is merely a Hölder continuous function of the distance. More specifically, my research concerns questions of solvability of a range of boundary value problems—Dirichlet, Neumann, Regularity—for linear elliptic and parabolic divergence form equations with non-smooth coefficients in domains with singularities on the boundary. There is a large, active community of analysts working on these problems and I've benefited from some great collaborations over the years, and most recently with Martin Dindos, Steve Hofmann, Carlos Kenig, Linhan Li, and Svitlana Mayboroda.

I'm also continuing to think about some problems in post-quantum cryptography and homomorphic encryption, in collaboration with Jeff Hoffstein, Joe Silverman, William Whyte and Zhenfei Zhang.

Notices: *Post-quantum cryptography definitely sounds very scary to me. I guess I have these sci-fi dystopia things in mind, computer pirate hackers.*

Pipher: Public key cryptography is the tool that makes it possible to have secure on-line financial transactions, but the public-key cryptosystems that are currently in wide use would be broken by a quantum computer. This is thanks to quantum algorithms, such as Shor's algorithm, that would run in polynomial time on a quantum computer. When will we be able to overcome the serious obstacles in building a scalable quantum computer? That's a question I won't try to answer. Let me just say that there is great pressure from government agencies like NIST [National Institute of Standards and Technology], from government intelligence agencies [and offices] like NSA and GCHQ [Government Communications Headquarters], and from other bodies such as the National Academy of Sciences, to identify algorithms for public key cryptography and secure key exchange that are resistant to quantum speedups.

The NTRU public key encryption system, that I am a co-inventor of, uses an algorithm that remains resistant to the speed-ups afforded by quantum computing. That is, no one has yet found a quantum algorithm that can find the shortest vector in an integer lattice faster than a classical computer. So I have a vested interest in the creation of a quantum computer—I'm all for that. [laughter]

Beyond cryptography, people are working hard to understand what a quantum computer can do more efficiently than a classical computer. The big question is: When can we find quantum algorithms that realize exponential speedups over their classical counterparts?

Notices: [The creation of a quantum computer] would make your work more valuable.

Pipher: Absolutely. That's my conflict of interest disclosure for the article. [laughter]

Notices: Thanks for taking the time to talk with me, Jill.

Pipher: Thank you.



Evelyn Lamb

Credits

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2019 Leroy P. Steele Prizes

The 2019 Leroy P. Steele Prizes were presented at the 125th Annual Meeting of the AMS in Baltimore, Maryland, in January 2019. The Steele Prizes were awarded to HARUZO HIDA for Seminal Contribution to Research, to PHILIPPE FLAJOLET and ROBERT SEDGEWICK for Mathematical Exposition, and to JEFF CHEEGER for Lifetime Achievement.



Haruzo Hida



Philippe Flajolet



Robert Sedgewick



Jeff Cheeger

Citation for Seminal Contribution to Research: Haruzo Hida

The 2019 Leroy P. Steele Prize for Seminal Contribution to Research is awarded to Haruzo Hida of the University of California, Los Angeles, for his highly original paper “Galois representations into $GL_2(\mathbb{Z}_p[[X]])$ attached to ordinary cusp forms,” published in 1986 in *Inventiones Mathematicae*.

In this paper, Hida made the fundamental discovery that ordinary cusp forms occur in p -adic analytic families. J.-P. Serre had observed this for Eisenstein series, but there the situation is completely explicit. The methods and perspectives that Hida introduced have been used in the past three decades to solve fundamental problems in the theory of p -adic Galois representations and p -adic L -functions, and they have led to progress on p -adic analogues of the conjecture of Birch and Swinnerton-Dyer. Hida families are now ubiquitous in the arithmetic theory of automorphic forms, and his research has changed the way we view the subject.

Biographical Note: Haruzo Hida

Haruzo Hida is a Distinguished Professor of mathematics at UCLA. Born in 1952 in the beach resort town of

Hamadera (presently, Sakai West-ward), Japan, he received an MA (1977) and Doctor of Science (1980) from Kyoto University. He did not have a thesis advisor. He held positions at Hokkaido University (Japan) from 1977–1987 up to an associate professorship. He visited the Institute for Advanced Study for two years (1979–1981), though he did not have a doctoral degree in the first year there, and the Institut des Hautes Études Scientifiques and Université de Paris Sud from 1984–1986. Since 1987, he has held a full professorship at UCLA (and was promoted to Distinguished Professor in 1998).

Hida’s main research interests lie in arithmetic geometry, both Archimedean and Henselian, through the automorphic approach (initiated by Erich Hecke). He was an invited speaker at the ICM in Berkeley (1986), a Guggenheim fellow (1991–1992), a recipient of the Spring Prize from the Mathematical Society of Japan (1992), a senior scholar at the Clay Mathematics Institute (2010–2011), an inaugural fellow of the American Mathematical Society (2012), and a recipient of a Docteur Honoris Causa, Université de Paris XIII (2015). He is the author of seven research books and monographs on his own results.

Response from Haruzo Hida

It is a great honor (and also a big surprise) to receive the Leroy P. Steele Prize for Seminal Contribution to Research from the AMS. Why a surprise? The name of the town Hamadera appears (as “Takashi-no-Hama”) in the sixth–eighth-century Japanese “Tan-ka/Chou-ka” poem anthology “Manyou-shu” (Ten thousand leaves), and by a tradition of the town, I was familiar with the ancient poems at an early age (as they are all written in Japanese phonetic symbols, so, easy to read). Starting with the poems, I enjoyed Japanese and Chinese classics. Chinese poems by a decadent Japanese Zen monk of the fifteenth century greatly impressed me; they could be interpreted (with a question mark) to suggest the purpose of one’s life could be found only in an enjoyable pastime (or more precisely, a way to kill time), lasting until one’s demise. From that time on, I tried in earnest to find such a way to kill time. I finally found one accidentally in the mid-1970s and, after that, became totally addicted to math. Therefore, I am hardly professional nor academic in mathematical work, and I often create mathematics without tangible reference to contemporaries. It seems unfair that such a person would be chosen for a prestigious AMS prize. Nevertheless, my work has found some deep applications. This hopefully legitimizes the award.

A seventeenth-century Japanese playwright told a Confucianist that creating a play is to walk the boundary of imaginary and real (or dream and truth) without stepping out of the narrow path. When in 1975 I started to study (with Koji Doi) the relation between congruence and L -values, Doi told me that a Hecke eigenform appears to have siblings having eigenvalues (of Hecke operators) congruent modulo a (parent) prime with the eigenvalues of the initial form. While at IAS, I felt that pathwise connectedness of the Archimedean topology forces the core cuspidal spectrum of Hecke operators to be discrete; so, under a Henselian topology, totally disconnected, I imagined that the spectrum is prevalently continuous. After having returned to Japan in the fall of 1981, I started making progress in proving this guess and succeeded (partially) in getting a proof via arithmetic geometry by the end of January 1982. Since the result seemed too strong, I sought one more proof. I got another via Betti Étale cohomology of modular curves within a couple of months. Afterwards, I sent out preprints to senior number theorists I’d encountered at Princeton. The second proof is in the paper published in 1986 for the award (and now there are more than two proofs).

Since I enjoy finding results independent of my fellow mathematicians, I did not make too much effort to find applications to classical questions posed by others, but a handful of excellent number theorists became interested in later years, and found good applications for my result.

Citation for Mathematical Exposition: Philippe Flajolet and Robert Sedgewick

The 2019 Leroy P. Steele Prize for Mathematical Exposition is awarded to Philippe Flajolet (posthumously) of the Institut National de Recherche en Informatique et en Automatique (INRIA) and Robert Sedgewick of Princeton University for their book *Analytic Combinatorics* (Cambridge University Press, Cambridge, 2009), an authoritative and highly accessible compendium of its subject, which demonstrates the deep interface between combinatorial mathematics and classical analysis. It is a rare work, one that defines the relatively young subject in its title, mixing equal parts of complex analysis and combinatorial structure. The authors have combined their extraordinary analytical and expository skills to organize the entire subject into a well-developed and fascinating story. Its publication in 2009 was a major event, and as a result, analytic combinatorics is now a thriving subdiscipline of combinatorial and stochastic mathematics, as well as a key component of the analysis of algorithms.

Quoting Robin Pemantle’s 2010 review of *Analytic Combinatorics*, published in *SIAM Review*, “This is one of those books that marks the emergence of a subfield.” The book magically summarizes a vast amount of information. It identifies and expounds key techniques that have never been explained so well before, while consistently paying proper attention to the historical context. It features world-class graphics and typesetting and a definitive bibliography. The book is largely self-contained and a pleasure to read—any mathematician can use it as the basis for teaching a course on analytic combinatorics as an undergraduate elective in mathematics.

Biographical Note: Philippe Flajolet

Philippe Flajolet (1948–2011) was an extraordinary French mathematician and computer scientist. He graduated from École Polytechnique in Paris in 1970, obtained a PhD from Université Paris 7 with Maurice Nivat in 1973 and a Doctorate in Sciences from the University of Paris at Orsay in 1979. He spent his career at INRIA in Rocquencourt, France, where he eventually led the ALGO research group, which produced numerous outstanding young scientists and attracted visiting researchers from all over the world.

He held numerous visiting positions: at Waterloo, Stanford, Princeton, Wien, Barcelona, IBM, and Bell Laboratories. He received several prizes, including the Grand Science Prize of UAP (1986), the Computer Science Prize of the French Academy of Sciences (1994), and the Silver Medal of CNRS (2004). He was elected a Corresponding Member (Junior Fellow) of the French Academy of Sciences in 1994, a Member of the Academia Europaea in 1995, and a Member (Fellow) of the French Academy of Sciences in 2003. He was made a knight of the Légion d’Honneur in 2010.

Flajolet's extensive and far-reaching research in mathematics and computer science spanned formal languages, computer algebra, combinatorics, number theory, and analysis, all oriented toward the study of algorithms and discrete structures. During his forty years of research, he contributed nearly 200 publications. An important proportion of these are foundational contributions or represent uncommon breadth and depth. Highlights range from pioneering work in computer algebra in the 1980s to theorems in asymptotic analysis in the 1990s that inspired decades of later research to a probabilistic algorithm that is widely used in modern cloud computing. Much of his research laid the foundation for the development, with Sedgewick, of the subfield of mathematics that is now known as analytic combinatorics, a calculus for the study of discrete structures.

These research contributions will have impact for generations. Flajolet's approach to research, based on endless curiosity, discriminating taste, deep knowledge, relentless computational experimentation, broad interest, intellectual integrity, and genuine camaraderie, will serve as an inspiration for years to come to those who knew him.

Biographical Note: Robert Sedgewick

Robert Sedgewick is the William O. Baker Professor in the Department of Computer Science at Princeton University. Born in 1946 in Willimantic, Connecticut, he graduated from Brown University in 1968 and did his doctoral work with Donald E. Knuth at Stanford University, receiving his PhD in 1975. After ten years on the faculty at Brown, he left to be the founding chair of Princeton's Department of Computer Science in 1985. He served for twenty-six years as a member of the board of directors of Adobe Systems and has held visiting research positions at Xerox PARC, IDA, INRIA, and Bell Laboratories.

Sedgewick is the author of twenty books. He is best known for *Algorithms*, which has been a best-selling textbook since the early 1980s and is now in its fourth edition. His other current textbooks include *An Introduction to the Analysis of Algorithms and Analytic Combinatorics* (with Philippe Flajolet) and *Computer Science: An Interdisciplinary Approach* (with Kevin Wayne).

Beyond his work with Flajolet on analytic combinatorics, Sedgewick's research is characterized by a scientific approach to the study of algorithms and data structures, where careful implementations and appropriate mathematical models are validated by experimentation and then used to understand performance and develop improved versions. Many of his research results are expressed in his *Algorithms* books, and his implementations routinely serve as reference and are featured throughout our global computational infrastructure.

In recent years, Sedgewick has been a pioneer in developing modern approaches to disseminating knowledge, from introductory to graduate level. He has developed six

massive open online courses (MOOCs) and published extensive online content on analysis of algorithms and analytic combinatorics and, with Kevin Wayne, algorithms and computer science. These materials have made it possible and convenient for millions of people around the world to teach and learn these subjects, particularly in regions where access to higher education is difficult.

Response from Robert Sedgewick

This award is thrilling and humbling for me, but also bittersweet, because Philippe is not here to share it. But all of us who were there vividly remember his excitement at our event in Paris on the occasion of his sixtieth birthday when we presented him with the first printed copy of *Analytic Combinatorics*. I keep the look on his face at that moment fresh in my mind and know that the same look would grace us now.

Philippe and I (and many others) were students of the work of Don Knuth in the 1970s, and inspired by the idea that it was possible to develop precise information about the performance of computer programs through classical analysis. When we first began working together in 1980, our goal was just to organize models and methods that we could use to teach our students what they needed to know. As we traveled between Paris and Princeton, producing conference papers, journal articles, and INRIA research reports, we began to understand that something more general was at work, and *Analytic Combinatorics* began to emerge. It is particularly gratifying to see citations of the book by researchers in physics, chemistry, genomics, and many other fields of science, not just mathematicians and computer scientists.

Analyzing algorithms is challenging—at the outset, known results were often either excessively detailed or rough, questionably useful approximations. Thus, what fun it was to consider the idea that maybe (despite the formidable barrier of the Halting Problem) one could develop a black box that could take a program as input and produce as output an asymptotic estimate of its running time. How challenging it was to develop a rigorous calculus that takes us from simple formal descriptions of combinatorial objects through properties of generating functions in the complex plane to precise information about the objects. How exciting it was to build on this work to develop theorems of sweeping generality that encompass whole families of combinatorial classes. As Philippe said, developing new theorems like these “constitutes the very essence of analytic combinatorics.”

With a vibrant community of researchers working on developing and applying such theorems, I suspect and hope that the story of analytic combinatorics is just in its infancy.

I am particularly heartened by the statement in the citation that any mathematician could use our book to teach an undergraduate course on the subject. Having the

broadest possible reach was indeed our hope when, with the support of our editor, we provided free access to the book on the web. For the past several years, I have been working hard to apply twenty-first-century tools to develop a unique resource for teaching this material. Anyone can now teach and learn *Analytic Combinatorics* using the studio-produced lecture videos, new problems with solutions, and other online content found at ac.cs.princeton.edu. Philippe, who always embraced technology, would be particularly pleased with the idea that it now makes analytic combinatorics accessible to large numbers of people around the world.

Citation for Lifetime Achievement: Jeff Cheeger

The 2019 Leroy P. Steele Prize for Lifetime Achievement is awarded to Jeff Cheeger of the Courant Institute, New York University, for his fundamental contributions to geometric analysis and their far-reaching influence on related areas of mathematics. For more than half a century, Jeff Cheeger has been a central figure in differential geometry and, more broadly, geometric analysis. His work on the profound and subtle effects of curvature on the topology and geometry of manifolds, often under very weak regularity conditions, has laid and continues to lay foundations for much of the progress in these areas ever since his 1967 dissertation.

His work, both alone and in collaboration with others, has yielded such spectacular results as the Soul and Splitting Theorems (with Detlef Gromoll) and the Compactness and Collapsing Theories (with Kenji Fukaya and Misha Gromov), which have been among the most important developments in geometry in the past three decades. These fundamental theories have had far-reaching consequences, for instance, playing an essential role in Perelman's resolution of the Poincaré conjecture. Cheeger's inequality bounding from below the first nonzero eigenvalue of the Laplacian in terms of a certain isoperimetric constant, known as Cheeger's constant, has had numerous applications, as has his work on the Hodge theory and spectral geometry of singular spaces, the structure theory of spaces with bounds on Ricci curvature, his resolution of the Ray-Singer Conjecture, the theory of differential characters (with James Simons), his work on differentiability of Lipschitz functions on metric measure spaces, and many others have been the fundamental tools that enabled major advances in geometry and analysis that continue to bear fruit and shape the field.

Biographical Note: Jeff Cheeger

Jeff Cheeger was born in Brooklyn, New York, in 1943. He graduated from Erasmus Hall High School in 1960 and from Harvard College in 1964. He received his PhD from Princeton under Salomon Bochner and James Simons in 1967. After a year in Berkeley as an NSF Postdoctoral Fellow and a year at the University of Michigan as an assistant professor, he moved to Stony Brook, where he remained for the next twenty years, rising to the rank of Distinguished Professor. Since 1989, he has been a member of the Courant Institute, where since 2003 he has been Silver Professor of Mathematics.

Cheeger has given invited addresses at the International Congress of Mathematicians in 1974 and 1986. He was awarded the Max Planck Research Prize of the Alexander von Humboldt Society in 1996 and the Oswald Veblen Prize of the AMS in 2001. He was elected to the National Academy of Sciences in 1997, the Finnish Academy of Science and Letters in 1998, and the American Academy of Arts and Sciences in 2006. He was elected a Fellow of the AMS in 2012.

Response from Jeff Cheeger

It is a great honor to have been awarded the Leroy P. Steele Prize for Lifetime Achievement. It is especially gratifying to have received an award for research done over my whole career and for which the citation includes work with a number of remarkable mathematicians, the interactions with whom have enriched my life. I would particularly like to thank my collaborators Paul Baum, Detlef Gromoll, Jim Simons, S.-T. Yau, Michael Taylor, Werner Muller, Robert Schrader, Misha Gromov, Jean-Michel Bismut, Mike Anderson, Gang Tian, Xiaochun Rong, Xianzhe Dai, Kenji Fukaya, Toby Colding, Bruce Kleiner, Assaf Naor, and Aaron Naber. I would also like to acknowledge the influence of my friends Blaine Lawson, Dennis Sullivan, and Is Singer.

I was introduced to mathematics by my father, Thomas Cheeger, a structural engineer. He could not have given me a better gift. My mother, Pauline, stressed to me the benefits of hard work.

In junior high school, I made a very good friend, Mel Hochster, with whom I could share my interest in mathematics. It was exciting and fun. When I was an undergraduate at Harvard, two professors, Shlomo Sternberg and Raoul Bott, made a big impression. They introduced me to differential geometry and algebraic topology. Beyond that, they conveyed the feeling that being a mathematician was something like being a member of a special order, an order into which one could hope to one day be initiated. During my last year, I took a PDE course from Jim Simons. In graduate school at Princeton, along with my official advisor, Salomon Bochner, Jim became my teacher and then my friend. I owe him a lot.

I was very lucky to have found my way into differential geometry which, I have come to believe, was the right area for my particular turn of mind. When I started, it was a bit out of fashion, underdeveloped, not overly competitive, but poised to take off. For me, this was ideal. Later, I learned some analysis, which opened up new vistas.

As researchers, our job is to produce new mathematics. Still, looking back over a whole career, it is somewhat mind blowing to realize how little we understood when I began, as compared to what has since been discovered.

From the time I was young, I was struck by the fact that in mathematics, questions have a right or wrong answer. This has a consequence. With small exceptions, mathematicians tend to genuinely admire each other's achievements. Another thing, as mathematicians we have quite direct access to some of the most original minds of the past and of the present. From such people, if you keep your ears open, you can really learn something. Finally, we are lucky in that we get to think about what we want and to interact with brilliant young people. I feel very fortunate to have had a life in mathematics.

About the Prizes

The Leroy P. Steele Prizes were established in 1970 in honor of George David Birkhoff, William Fogg Osgood, and William Caspar Graustein. Osgood was president of the AMS during 1905–1906, and Birkhoff served in that capacity during 1925–1926. The prizes are endowed under the terms of a bequest from Leroy P. Steele. Up to three prizes are awarded each year in the following categories: (1) Lifetime Achievement: for the cumulative influence of the total mathematical work of the recipient, high level of research over a period of time, particular influence on the development of a field, and influence on mathematics through PhD students; (2) Mathematical Exposition: for a book or substantial survey or expository research paper; (3) Seminal Contribution to Research: for a paper, whether recent or not, that has proved to be of fundamental or lasting importance in its field or a model of important research. The Prize for Seminal Contribution to Research is awarded on a six-year cycle of subject areas. The 2019 prize was open; the 2020 prize will be given in analysis/probability; the 2021 prize in algebra/number theory; the 2022 prize in applied mathematics; the 2023 prize in geometry/topology; and the 2024 prize in discrete mathematics/logic.

The Leroy P. Steele Prizes for Mathematical Exposition and Seminal Contribution to Research carry a cash award of US\$5,000; the Prize for Lifetime Achievement, a cash award of US\$10,000.

The Steele Prizes are awarded by the AMS Council acting on the recommendation of a selection committee. The members of the committee for the 2019 Steele Prizes were:

- Robert L. Bryant,
- Tobias H. Colding,
- Eric M. Friedlander,
- Mark L. Green,
- B. H. Gross (Chair),
- Carlos E. Kenig,
- Dusa McDuff,
- Victor Reiner,
- Thomas Warren Scanlon

The list of previous recipients of the Leroy P. Steele Prizes may be found on the AMS website at <https://www.ams.org/profession/prizes-awards/ams-prizes/steele-prize>.

Credits

Photos of the winners were provided by each of them.

2019 Mary P. Dolciani Prize for Excellence in Research

STEPHAN RAMON GARCIA was awarded the inaugural Mary P. Dolciani Prize for Excellence in Research of the AMS at the 125th Annual Meeting of the AMS in Baltimore, Maryland, in January 2019.



Stephan Ramon Garcia

Citation

The Mary P. Dolciani Prize for Excellence in Research is awarded to Stephan Ramon Garcia, W. M. Keck Distinguished Service Professor and Professor of Mathematics at Pomona College, for his outstanding record of research in operator theory, complex analysis, matrix theory, and number theory, for high-quality scholarship with a diverse set of undergraduates, and for

his service to the profession.

Garcia received his PhD in Mathematics in 2003 from the University of California at Berkeley. He is the author of eighty-nine research papers in several areas, including operator theory, linear algebra, complex analysis, mathematical physics, and number theory. His work has appeared in top research journals, as well as top expository journals, and he has been the Principal Investigator on four NSF research grants. He has co-authored four books and is currently writing two more. Garcia has also co-authored over twenty-nine articles with undergraduates, with papers appearing in the *American Mathematical Monthly*, the *Notices of the American Mathematical Society*, *Proceedings of the American Mathematical Society*, and the *Journal of Number Theory*, among others. His paper "G. H. Hardy: Mathematical Biologist," written with a student, was included in the 2016 book series *The Best Writing on Mathematics*, published by Princeton University Press. Garcia currently serves as an editor of the *Notices of the American Mathematical Society*, the *American Mathematical Monthly*, *Proceedings of the Amer-*

ican Mathematical Society, *Annals of Functional Analysis*, and the undergraduate research journal *Involve*. He serves on the Human Resources Board of the American Institute of Mathematics (AIM), whose goal is to foster diversity in the activities of AIM. He is also a member of the advisory board of Research Experiences for Undergraduate Faculty (REUF), an NSF-funded program for faculty who are interested in conducting research with underrepresented minority students, students with disabilities, and first-generation college students.

Garcia's research began with complex analysis and H^p spaces and now includes, among several other topics, operator theory on Hilbert spaces. One of his objectives is to develop models for various classes of operators. In a series of highly cited papers published in *Transactions of the AMS* and the *Journal of Functional Analysis*, he and his coauthors pioneered the study of complex symmetric operators. Specifically, the theory behind linear transformations T that are "almost" self-adjoint by means of a conjugate-linear, isometric involution C ; that is, $T = CT^*C$. Thus, the conjugation C works to express an operator in terms of its adjoint. These almost self-adjoint operators are called complex symmetric operators. Many unexpected and highly non-normal operators have been shown to be complex symmetric, as have several classes of familiar operators. Garcia and his colleagues have developed a structure theory for this important (and large) class of operators. They are currently developing the machinery to connect truncated Toeplitz operators and complex symmetric operators. They conjecture that every complex symmetric operator on a Hilbert space can be concretely represented in terms of truncated Toeplitz operators.

Garcia has also made significant contributions to number theory. His work in number theory has been primarily

in four areas: geometric lattice theory, exponential sums, arithmetic quotient sets, and the behavior of the Euler totient near prime arguments. Exponential sums, such as Gauss sums, Kloosterman sums, Ramanujan sums, and others, are classical objects of study in analytic number theory. Garcia's novel approach was to view these sums from the standpoint of supercharacter theory. From this perspective, classical exponential sums can be viewed as orthogonal functions on certain abelian groups. Garcia and his co-authors (many of whom were undergraduate students) used this approach to visualize exponential sums, exhibiting some rather remarkable and visually stunning graphical features of these objects. An arithmetic quotient set is a set of fractions a/b , where a and b are elements of an infinite arithmetically defined set. Garcia and his co-authors explored the relationship between the arithmetic properties of a set and the analytic properties of its corresponding quotient set, for example its density in the positive reals or in p -adic completions of the field of rational numbers. Concerning the Euler totient, one striking recent result of Garcia, his student Elvis Kahoro, and Florian Luca (subject to the Bateman–Horn conjecture) is that for an overwhelming majority of twin prime pairs $(p, p + 2)$, the first prime p has more primitive roots than the second, $p + 2$. Moreover, this is reversed for a small positive proportion of the twin primes.

Again, in these rich and deep subject areas, Garcia has been able to involve undergraduates in this work.

Biographical Note

Stephan Ramon Garcia is W. M. Keck Distinguished Service Professor and Professor of Mathematics at Pomona College. He earned his BA and PhD in mathematics from UC Berkeley and was a postdoc at UC Santa Barbara. Since 2006 he has been on the faculty of Pomona College. He was recently elected a Fellow of the AMS (2019).

He is the author of over eighty-nine research articles in operator theory, complex analysis, matrix analysis, number theory, discrete geometry, and other fields. Several dozen of these papers were co-authored with students, many of whom are from underrepresented groups in the mathematical sciences. Garcia has also written four books: *Introduction to Model Spaces and Their Operators* (with W. T. Ross and J. Mashreghi, Cambridge, 2016), *A Second Course in Linear Algebra* (with R. A. Horn, Cambridge, 2017), *Finite Blaschke Products and Their Connections* (with W. T. Ross and J. Mashreghi, Springer, 2018), and *100 Years of Math Milestones: The Pi Mu Epsilon Centennial Collection* (with S. J. Miller, AMS, forthcoming).

Garcia has received four NSF research grants as principal investigator and five teaching awards. He serves on the editorial boards of the *Notices of the American Mathematical Society* (2019–), *Proceedings of the American Mathematical Society* (2016–), *Involve* (2011–), the *American Mathemat-*

ical Monthly (2017–), and *Annals of Functional Analysis* (2013–). He has served on the Human Resources Board of the American Institute of Mathematics since 2008.

Response from Stephan Ramon Garcia

I am deeply honored to receive the inaugural Mary P. Dolciani Prize for Excellence in Research. Thanks go to the American Mathematical Society and the Mary P. Dolciani Halloran Foundation for initiating this award. Although I am the first recipient of this prize, there are many vibrant researchers at non-PhD-granting institutions who are also worthy. I look forward to celebrating the achievements of future prizewinners in the years to come.

This would not have been possible without the advice and support of my many colleagues in the profession and the members of my department. I owe a great deal of thanks to those mathematicians who mentored me during my formative years. My advisor, Donald Sarason, and my postdoctoral mentor, Mihai Putinar, are due special consideration. I also thank my innumerable co-authors, from whom I learned a great deal, and my many research students throughout the years. Finally, I wish to thank my wife, Gizem Karaali, and our children, Reyhan and Altay, for their constant support and affection.

About the Prize

The Mary P. Dolciani Prize for Excellence in Research is awarded by the AMS Council acting on the recommendation of a selection committee. The members of the committee to select the inaugural winner of the Mary P. Dolciani Prize were:

- Linda Chen,
- Pamela Gorkin (Chair),
- Jeremy T. Teitelbaum.

The AMS Mary P. Dolciani Prize for Excellence in Research recognizes a mathematician from a department that does not grant a PhD who has an active research program in mathematics and a distinguished record of scholarship. It is funded by a grant from the Mary P. Dolciani Halloran Foundation. Mary P. Dolciani Halloran (1923–1985) was a gifted mathematician, educator, and author. She devoted her life to developing excellence in mathematics education and was a leading author in the field of mathematical textbooks at the college and secondary school levels.

Credits

Photo of Stephan Ramon Garcia by Gizem Karaali.

2019 Ruth Lyttle Satter Prize in Mathematics

MARYNA VIAZOVSKA was awarded the 2019 Ruth Lyttle Satter Prize in Mathematics at the 125th Annual Meeting of the AMS in Baltimore, Maryland, in January 2019.



Maryna Viazovska

Citation

The 2019 Ruth Lyttle Satter Prize in Mathematics is awarded to Maryna Viazovska of École Polytechnique Fédérale de Lausanne for her groundbreaking work in discrete geometry and her spectacular solution to the sphere-packing problem in dimension eight.

In his 1900 list of outstanding mathematical problems, David Hilbert asked, “How can one arrange most densely in space an infinite number of equal solids of a given form, e.g., spheres with given radii...?” Viazovska’s work is a major advance in addressing this question. Her 2017 paper in *Annals of Mathematics* shows that the E_8 root lattice is the densest sphere packing in eight dimensions. Shortly after this much heralded breakthrough, Dr. Viazovska, in collaboration with Henry Cohn, Abhinav Kumar, Stephen D. Miller, and Danylo Radchenko, adapted her methods to prove that the optimal sphere-packing density in dimension twenty-four is achieved by the Leech lattice. Prior to these results, the sphere-packing problem had not been solved beyond dimension three.

Maryna Viazovska’s work has been described as “simply magical,” “very beautiful,” and “extremely unexpected.” Her solution to the sphere-packing problem in dimension eight, while conceptually simple, has a deep structure based on certain functions that she explicitly constructs in terms of modular forms. It establishes a new, unanticipated connection between modular forms and discrete geometry.

Dr. Viazovska’s earlier results on spherical designs are fundamental contributions to the topic. Her 2013 *Annals of Mathematics* paper with Andriy Bondarenko and Danylo Radchenko solved a conjecture of J. Korevaar and J. L. H. Meyers by showing for $N > C_d t^d$, where C_d is a positive constant depending only on d , that spherical t -designs with N points exist in the unit sphere S^d . Spherical designs have been essential tools of practical importance in the statistical design of experiments and in both combinatorics and geometry. Most recently, spherical t -designs have appeared in the guise of quantum t -designs with applications to quantum information theory and quantum computing.

For more about the proof and background on the sphere-packing problem, see “A conceptual breakthrough in sphere packing,” by Henry Cohn, *Notices of the AMS*, 64 (2017), no. 2; 102–115.

Biographical Sketch

Maryna Viazovska was born in Ukraine and received her doctorate from the University of Bonn in 2013. She was a postdoctoral researcher at Berlin Mathematical School and Humboldt University of Berlin, as well as a Minerva Distinguished Visitor at Princeton University, before joining the faculty at Lausanne as a full professor in 2018. She has been awarded the Salem Prize (2016), a Clay Research Award (2017), the SASTRA Ramanujan Prize (2017), a European Prize in Combinatorics (2017), and a New Horizons Prize in Mathematics (2018). She was an invited speaker at the 2018 International Congress of Mathematicians in Rio de Janeiro.

About the Prize

The Ruth Lyttle Satter Prize is awarded every two years to recognize an outstanding contribution to mathematics research by a woman in the previous six years. Established in 1990 with funds donated by Joan S. Birman, the prize honors the memory of Birman's sister, Ruth Lyttle Satter. Satter earned a bachelor's degree in mathematics and then joined the research staff at AT&T Bell Laboratories during World War II. After raising a family, she received a PhD in botany at the age of forty-three from the University of Connecticut at Storrs, where she later became a faculty member. Her research on the biological clocks in plants earned her recognition in the United States and abroad. Birman requested that the prize be established to honor her sister's commitment to research and to encourage women in science. The prize carries a cash award of US\$5,000.

The Satter Prize is awarded by the AMS Council acting on the recommendation of a selection committee. For the 2019 prize, the following individuals served as members of the selection committee:

- Georgia Benkart,
- Estelle Basor (Chair),
- Richard Taylor.

A list of previous recipients of the Ruth Lyttle Satter Prize in Mathematics may be found on the AMS website at <https://www.ams.org/profession/prizes-awards/pabrowse?purl=satter-prize>.

Credits

Photo of Maryna Viazovska is courtesy of Maryna Viazovska.

2019 Levi L. Conant Prize

ALEX WRIGHT was awarded the 2019 Levi L. Conant Prize at the 125th Annual Meeting of the AMS in Baltimore, Maryland, in January 2019.



Alex Wright

Citation

The 2019 Levi L. Conant Prize is awarded to Alex Wright for his article “From rational billiards to dynamics on moduli spaces,” published in 2016 in the *Bulletin (New Series) of the American Mathematical Society*; 53 (2016), no.1, 41–56.

In only sixteen pages, the article gives a panoramic view of the theory of translation surfaces and of the recent breakthrough by Alex Eskin,

Maryam Mirzakhani, and Amir Mohammadi on the structure of the orbit closure of a translation surface. Wright’s account combines brevity with clarity. It is a considerable feat: this active and highly technical research area comprises the work of many. The article gives nonspecialists a good entry point and a guide to further reading.

The article starts with motivation from billiards inside planar polygons. The billiard dynamical system describes the motion of a particle in a domain, subject to specular reflections off the boundary. Many mechanical systems with elastic collisions, that is, collisions in which the energy and momentum are preserved, are described as billiard systems. Little is known about billiards in general polygons (for example, we still do not know whether every obtuse triangle has a periodic billiard trajectory!); the situation is considerably better understood when the angles of the polygon are π -rational, because of their relation to translation surfaces. A translation surface is a surface that is presented as a finite collection of planar polygons, glued together along pairings of parallel edges. Reflected copies of rational polygons are special examples of translation surfaces.

Through ample figures and examples, Wright gives a simple definition of translation surfaces and their moduli

space, clearly explains the relation to rational billiards, and describes an action of the general linear group $GL(2, \mathbf{R})$ on the moduli space. He provides a brief survey of seminal work by Kerckhoff, Masur, Smillie, and Veech (in the 1980–1990s), including the surprising result by Veech that billiards in a regular polygon share a familiar property with billiards in a square: in countably many directions, every billiard trajectory is periodic, but in every other direction, trajectories are equidistributed.

The second half of the article is devoted to the recent breakthrough by Eskin, Mirzakhani, and Mohammadi: *the closure of the $GL(2, \mathbf{R})$ orbit of a translation surface is always a manifold, defined locally by linear equations in (the standard) period coordinates*.

Wright outlines the proof and describes the relation of this theorem to other fundamental results, such as Ratner’s orbit closure theorem and the high and low entropy methods of Einsiedler, Lindenstrauss, and Katok in homogeneous space dynamics. Wright also describes an intimate connection between moduli of translation surfaces and Teichmüller theory.

Several applications of the theorem are presented. For example, given a polygon and two points x and y inside it, the illumination problem asks whether there exists a billiard trajectory in the polygon from x to y . Recently, Lelièvre, Monteil, and Weiss proved that if the polygon is rational, then for every x there are at most finitely many y not illuminated by x ; this work relies heavily on the theorem of Eskin, Mirzakhani, and Mohammadi.

Over the years, a number of surveys of the theory of translation surfaces and related topics have appeared, from lengthy and detailed ones to short overviews of the subject. Wright’s article is based on his talk in the *Current Events Bulletin* at the Joint Mathematics Meetings in January of 2015. It is a tribute to the work of Maryam Mirzakhani, who passed away in 2017.

Biographical Note

Alex Wright received his BMath at the University of Waterloo in 2008 and his PhD at the University of Chicago in 2014. He was then awarded a five-year Clay Research Fellowship, which he held primarily at Stanford University. He is now at the University of Michigan. His research interests include Teichmüller theory, geometry, and dynamical systems, including special families of algebraic curves that arise in this context. In 2018 he received the Michael Brin Dynamical Systems Prize for Young Mathematicians.

Response from Alex Wright

I'm honored to receive this recognition for my expository article on the breakthrough work of Eskin, Mirzakhani, and Mohammadi. This work lies in Teichmüller dynamics, and yet it has remarkable connections to toy models in physics, other dynamical systems, ergodic theory on homogeneous spaces, and special families of algebraic curves. I am especially thankful to Alex Eskin and Maryam Mirzakhani for teaching me so much about the field. I'm also grateful to David Eisenbud for inviting me to speak on this topic at the *Current Events Bulletin*, and to Susan Friedlander for encouraging me to publish an article based on that talk.

About the Prize

The Levi L. Conant Prize is awarded by the AMS Council acting on the recommendation of a selection committee. For the 2019 prize, the selection committee consisted of the following individuals:

- Thomas C. Hales (Chair),
- Izabella Joanna Laba,
- Serge L. Tabachnikov.

The Levi L. Conant Prize is awarded annually to recognize an outstanding expository paper published in either the *Notices of the AMS* or the *Bulletin of the AMS* in the preceding five years.

Established in 2001, the prize honors the memory of Levi L. Conant (1857–1916), who was a mathematician at Worcester Polytechnic Institute. The prize carries a cash award of US\$1,000.

A list of previous recipients of the Levi L. Conant Prize may be found on the AMS website at <https://www.ams.org/profession/prizes-awards/pabrowse?purl=conant-prize>.

Credits

Photo of Alex Wright is courtesy of Alex Wright.

2019 E.H. Moore Research Article Prize

CIPRIAN MANOLESCU was awarded the 2019 E. H. Moore Research Article Prize at the 125th Annual Meeting of the AMS in Baltimore, Maryland, in January 2019.



Ciprian Manolescu

Citation

The 2019 E. H. Moore Research Article Prize is awarded to the paper “Pin(2)-equivariant Seiberg–Witten Floer homology and the triangulation conjecture” by Ciprian Manolescu, published in the *Journal of the American Mathematical Society*; 29 (2016), no.1, 147–176.

This paper resolves the Triangulation Conjecture, showing that there are topological manifolds that do not admit a simplicial triangulation in each dimension greater than 4. This is achieved by introducing Pin(2)-equivariant Seiberg–Witten Floer homology to give homology cobordism invariants of oriented homology 3-spheres, including an integral lift of the Rokhlin invariant which is negated by taking the mirror image (reverse orientation). The new invariants are powerful enough to show that there does not exist a homology 3-sphere with Rokhlin invariant 1 which is homology cobordant to its mirror image. In turn, this implies the existence of non triangulable manifolds in dimensions 5 and higher by the work of D. E. Galewski and R. J. Stern and of T. Matumoto. Note that it was known before that 2- and 3-dimensional manifolds are triangulable, and there are 4-manifolds which do not admit a triangulation, thus resolving the triangulation question in all dimensions.

One expert referred to this as a “landmark article.” Moreover, the techniques from the paper are already being applied to answer other questions in low-dimensional topology, for example regarding the homology cobordism

groups, and inspired a related theory of involutive Heegaard Floer homology.

Biographical Sketch

Ciprian Manolescu was born in Romania in 1978. He received his BA in 2001 and his doctorate in 2004, both from Harvard University. After appointments at Princeton University, Columbia University, and the University of Cambridge, he joined the University of California, Los Angeles, where he is now a professor. He was previously awarded the Frank and Brennie Morgan Prize, a Clay Research Fellowship, and a European Mathematical Society Prize. In 2017 he became a Fellow of the American Mathematical Society, and in 2018 he gave an invited talk at the International Congress of Mathematicians.

Response from Ciprian Manolescu

I feel very honored to receive the E. H. Moore Research Article Prize from the AMS. The main result of the paper is the existence of non triangulable manifolds in dimensions at least 5. In principle, a low-dimensional topologist like me could have no hope of proving such a result. Luckily, in the 1970s, David Galewski, Ron Stern, and Takao Matumoto managed to reduce this statement to a conjecture about the homology cobordism group in dimension 3, and this is the conjecture I proved. They deserve more than half of the credit for the final theorem. I would like to thank my mentors Peter Kronheimer, Mike Hopkins, and Lars Hesselholt. With their help, during my student years at Harvard I developed a stable homotopy version of Seiberg–Witten Floer homology. I found a few applications for this construction back then, but the theory lay more or less dormant for the next decade. In 2012 I started thinking about homology cobordism, and I then realized that by

incorporating an extra symmetry into my old construction I could get new information. The result was the article cited for this award. I am happy to see that, in the past few years, several young mathematicians have further developed the techniques from my paper to yield even more insight into homology cobordism. I would particularly like to acknowledge the contributions of Irving Dai, Kristen Hendricks, Jennifer Hom, Tye Lidman, Francesco Lin, Jianfeng Lin, Matt Stoffregen, Linh Truong, and Ian Zemke. It was a pleasure having some of them as collaborators and students. Finally, I want to thank my colleagues at UCLA for making the department a great place to do research.

About the Prize

The E.H. Moore Research Article Prize is awarded every three years for an outstanding research article that appeared in one of the primary AMS research journals: *Journal of the AMS*, *Proceedings of the AMS*, *Transactions of the AMS*, *AMS Memoirs*, *Mathematics of Computation*, *Electronic Journal of Conformal Geometry and Dynamics*, or *Electronic Journal of Representation Theory*. The article must have appeared during the six calendar years ending a full year before the meeting at which the prize is awarded. The prize carries a cash award of US\$5,000.

The prize honors the extensive contributions of E. H. Moore (1862–1932) to the AMS. Moore founded the Chicago section of the AMS, served as the Society's sixth president (1901–1902), delivered the Colloquium Lectures in 1906, and founded and nurtured the *Transactions of the AMS*.

The E.H. Moore Research Article Prize is awarded by the AMS Council acting on the recommendation of a selection committee. For the 2019 prize, the members of the selection committee were:

- Ian Agol (Chair),
- F. Michael Christ,
- Sergio Roberto Fenley,
- Nets H. Katz,
- Claire Marie Voisin.

A list of previous recipient of the E.H. Moore Research Article Prize may be found on the AMS website at <https://www.ams.org/profession/prizes-awards/pabrowse?purl=moore-prize>.

Credits

Photo of Ciprian Manolescu is courtesy of Reed Hutchinson.

2019 David P. Robbins Prize

ROGER BEHREND, ILSE FISCHER, and MATJAŽ KONVALINKA were awarded the 2019 David P. Robbins Prize at the 125th Annual Meeting of the AMS in Baltimore, Maryland, in January 2019.



Roger Behrend



Ilse Fischer



Matjaž Konvalinka

can bear the pain of living without understanding why they are true."

All had been proven by 2006, with the exception of the conjecture for diagonally and antidiagonally symmetric ASMs, which had resisted proof until the present paper.

The Robbins conjectures have led to the development of new methods of enumeration, as well as to the discovery of deep connections to statistical physics. The first breakthrough came in 1996, with the proof by Doron Zeilberger that $n \times n$

Citation

The David P. Robbins Prize is awarded to Roger Behrend, Ilse Fischer, and Matjaž Konvalinka for the paper "Diagonally and antidiagonally symmetric alternating sign matrices of odd order," published in 2017 in *Advances in Mathematics*.

In this work, Behrend, Fischer, and Konvalinka prove, after more than thirty years, the conjectured formula for the number of odd-order diagonally and antidiagonally symmetric alternating sign matrices, the last remaining of David Robbins's conjectures on alternating sign matrices.

An alternating sign matrix (ASM) is a square matrix in which every entry is 0, 1, or -1 , and along each row and column the nonzero entries alternate in sign and have a sum of 1. They were introduced by David Robbins and Howard Rumsey in work on a certain generalization of the determinant where these matrices surfaced naturally. Robbins, in the mid-1980s, initiated a program of counting symmetry classes of ASMs of a given size and conjectured remarkably simple product formulae for most of these symmetry classes. The quote from his 1991 survey paper reads: "These conjectures are of such compelling simplicity that it is hard to understand how any mathematician

ASMs are equinumerous with totally symmetric, self-complementary plane partitions in a $2n \times 2n \times 2n$ box, for which George Andrews had derived a simple product formula. In the same year, Greg Kuperberg made the connection to statistical physics by deriving the same ASM enumeration from the Izergin–Korepin determinant for a partition function for the six-vertex model on a square grid with domain wall boundary conditions. Kuperberg subsequently used this approach to enumerate three other symmetry classes of ASMs, and the enumeration by Roger Behrend, Ilse Fischer, and Matjaž Konvalinka builds on his work.

The main technical tool introduced by Kuperberg is a set of determinants and Pfaffian formulae for ASM partition functions, and it is these formulae that explain why the enumeration formulae are products of small factors. Behrend, Fischer, and Konvalinka arrive at a partition function with a compact formula by introducing vertex weights, depending on many parameters, into the model. Through computational experiments, they were able to guess the form of the partition function, which of course depends fundamentally on the choice of weights. To arrive at the compact formula, they took advantage of the observation by Soichi Okada, and by Alexander Razumov and Yuri Stroganov, that parti-

tion functions associated with ASM enumeration can often be written in terms of determinants which are associated with characters of irreducible representations of classical groups. In this case the partition function turns out to be a sum of two determinants, and each determinant reduces at some point to a Schur function.

The David P. Robbins Prize is awarded to a novel research paper in algebra, combinatorics, or discrete mathematics with a significant experimental component. This proof of the last remaining of David Robbins's conjectures on alternating sign matrices is a shining example. Not only is it a crowning achievement that makes use of deep methods developed by a community of researchers over more than twenty-five years, it is also a paper that makes new problems accessible.

Biographical Sketch: Roger Behrend

Roger Behrend was born in Melbourne, Australia. He studied mathematics and physics at the University of Melbourne and Imperial College London, receiving a PhD in mathematical physics from the University of Melbourne in 1997. Between 1997 and 2000, he held postdoctoral positions at the Physics Institute of the University of Bonn and the C. N. Yang Institute for Theoretical Physics at Stony Brook University. He has worked in the School of Mathematics at Cardiff University since 2001 and held a visiting position in the Faculty of Mathematics at the University of Vienna during 2017–2018. His research throughout the past decade has been in combinatorics. Much of his spare time is spent listening to classical music.

Response from Roger Behrend

I feel deeply honored to receive the David P. Robbins Prize together with my collaborators Ilse Fischer and Matjaž Konvalinka. It is fitting that in the research recognized by this award, we proved a conjecture of Robbins himself, and that this conjecture involved alternating sign matrices, which were first encountered by David Robbins and Howard Rumsey.

I believe that our construction of a proof of Robbins's conjecture for the number of odd-order diagonally and antidiagonally symmetric alternating sign matrices lies some distance from both the beginning and the end of the overall story of alternating sign matrices. Looking back, the proof depended on a significant body of earlier work, including that of Mills, Robbins, Rumsey, Izergin, Korepin, Zeilberger, Kuperberg, Okada, Razumov, and Stroganov. Looking forward, there remain many intriguing mysteries still to be resolved. As an important example, bijective proofs are currently lacking for known equalities between numbers of alternating sign matrices and numbers of certain plane partitions.

I am thankful to my wife Rachael and to my colleagues, family, and friends for their support throughout my exploration of the fascinating world of alternating sign matrices.

Biographical Sketch: Ilse Fischer

Ilse Fischer received her doctoral degree in 2000 from the University of Vienna under the direction of Christian Krattenthaler. After some years as a postdoctoral researcher at the University of Klagenfurt, she returned to a faculty position at the University of Vienna in 2004. In 2009 she was awarded the START prize of the Austrian Federal Ministry for Science, the most prestigious award for young researchers in Austria, and a 1.1 million € research grant endowment. In 2017 she was promoted to full professor. Her research is devoted to enumerative and algebraic combinatorics, and its connections to statistical physics and other fields.

Response from Ilse Fischer

The idea of working on Robbins's last open conjecture on alternating sign matrices slowly manifested in my mind as I was writing a grant proposal about ten years ago, when I identified it as an ultimate, albeit unrealistic, goal. In the beginning I hardly dared spend much time on it, but every now and then I discussed it with other combinatorialists. Roger Behrend and Matjaž Konvalinka were obviously among them, but I also had a particularly fruitful exchange with Arvind Ayyer back in 2012, which led us to several conjectures on the enumeration of extreme diagonally and antidiagonally symmetric alternating sign matrices of odd order. About three years later, Arvind, Roger, and I were able to prove these conjectures, and to some extent also this work paved the way for the eventual proof of Robbins's conjecture. I feel deeply honored and moved to now receive, together with Matjaž and Roger, the David P. Robbins Prize.

I would like to express my appreciation for the initiative to support mathematical research with an experimental component. Results discovered through experiment rather than intuition have the potential to be particularly surprising, and proving them can present a challenge because initially one may have no clue as to the reason why they are true. The area of enumerative combinatorics Robbins and several others originated serves as a good example: They introduced objects such as alternating sign matrices, plane partitions, and lozenge tilings, and while for most enumerations no explicit formula exists containing, say, only the basic arithmetic operations, certain enumerations of those objects are expressible by simple product formulas, which were usually discovered through computer experiments. Although all of Robbins's conjectures have now been proven, the proofs are complicated and we still lack thorough understanding just in what situations to expect a simple enumeration formula, nor are we able to explain phenomena such as the same enumeration formula ap-

pearing in the context of two very different combinatorial objects. Much of my past and current research has been driven by these questions.

Biographical Sketch: Matjaž Konvalinka

Matjaž Konvalinka was born in Ljubljana, Slovenia. He obtained his bachelor's and master's degrees at the University of Ljubljana, and his PhD at the Massachusetts Institute of Technology in 2008 under Igor Pak. He held a postdoctoral position at Vanderbilt University until 2010, and has been a professor at the Faculty of Mathematics and Physics, University of Ljubljana, since then. In 2012, he received a University award for excellent teaching and research. He mostly works in enumerative and algebraic combinatorics, and particularly enjoys bijective proofs, Schur functions, and tableaux combinatorics.

Response from Matjaž Konvalinka

I am deeply honored to be one of the recipients of the AMS David P. Robbins Prize. One of the reasons I love combinatorics is that many of its problems can be explained to a child, even when they are fiendishly hard to solve, and they inspire deep new tools and theorems. Problems involving alternating sign matrices are a prime example of this. Combinatorialists will forever be grateful to David Robbins and his coauthors for introducing them to the community and for the conjectures related to their enumeration.

I owe a debt of gratitude to many people. First and foremost I have to thank Ilse and Roger, my coauthors, both amazing mathematicians and people. They are truly worthy recipients of this prize. I am also deeply grateful to Marko Petkovšek for my first combinatorics courses; to my PhD advisor Igor Pak for everything he taught me and for always knowing what problems I will like; to Richard Stanley for his wonderful lectures, papers, and books; and to Sara Billey for being the best collaborator and friend one could imagine. My colleagues and students at the University of Ljubljana are a big part of why I enjoy my job. Many thanks also go to my husband Danijel and our daughter Ana, to the rest of my family, and to my friends, not least for seeming less surprised by this prize than I am.

About the Prize

The David P. Robbins Prize was established in 2005 in memory of David P. Robbins by members of his family. Robbins, who died in 2003, received his PhD in 1970 from the Massachusetts Institute of Technology. He was a long-time member of the Institute for Defense Analysis Center for Communications Research and a prolific mathematician whose work (much of it classified) was in discrete mathematics. The prize is given for a paper published during the preceding six calendar years that (1) reports on novel research in algebra, combinatorics, or discrete mathematics, (2) has a significant experimental component, (3)

is on a topic broadly accessible, and (4) provides a simple statement of the problem and clear exposition of the work. The US\$5,000 prize is awarded every three years.

The David P. Robbins Prize is awarded by the AMS Council acting on the recommendation of a selection committee. The members of the 2016 David P. Robbins Prize Committee were:

- Nola Alon
- Robert Calderbank (Chair)
- Timothy Chow
- Sylve Corteel
- Avi Wigderson

A list of previous recipients of the David P. Robbins Prize can be found on the AMS website at: <http://www.ams.org/profession/prizes-awards/ams-prizes/robbins-prize>.

Credits

Photo of Roger Behrend is courtesy of Roger Behrend.
Photo of Ilse Fischer is by Barbara Mair ©University of Vienna.
Photo of Matjaž Konvalinka is by Peter Legiša.

2019 Oswald Veblen Prize in Geometry

The 2019 Oswald Veblen Prize in Geometry was presented at the 125th Annual Meeting of the AMS in Baltimore, Maryland, in January 2019. The prize was awarded to XIUXIONG CHEN, SIMON DONALDSON, and SONG SUN.



Xiuxiong Chen



Simon Donaldson



Song Sun

necessarily involve an algebro-geometric notion of stability. Seminal work of Gang Tian and then Donaldson clarified and generalized this idea. The resulting conjecture—that a Fano manifold admits a Kähler–Einstein metric if and only if it is K -stable—became one of the most active topics in geometry. In 1997 Tian introduced the notion of K -stability used in the cited papers, and used this to demonstrate that there are Fano manifolds with trivial au-

tomorphism group which do not admit Kähler–Einstein metrics.

Proving this conjecture had long been understood to involve a vast combination of ideas from symplectic and complex geometry, infinite-dimensional Hamiltonian reduction, and geometric analysis. All methods involved some kind of continuity method; in 2011 Donaldson proposed one involving Kähler–Einstein metrics with cone singularities (published by Springer in *Essays in Mathematics and Its Applications* in 2012).

One of the main technical obstacles then was how to control certain limits of sequences of Kähler metrics on Fano manifolds (equivalently, how to obtain the “partial C^0 -estimate”). One can take the so-called Gromov–Hausdorff limit, but a priori this could be a metric space with no algebro-geometric description.

It was a huge breakthrough when, in 2012, Donaldson and Sun managed to use Bergman kernels to put the structure of a normal projective algebraic variety on the Gromov–Hausdorff limit of a noncollapsing sequence of

Citation

The 2019 Oswald Veblen Prize in Geometry is awarded to Xiuxiong Chen, Simon Donaldson, and Song Sun for the three-part series entitled “Kähler–Einstein Metrics on Fano Manifolds, I, II and III” published in 2015 in the *Journal of the American Mathematical Society*, in which Chen, Donaldson, and Sun proved a remarkable nonlinear Fredholm alternative for the Kähler–Einstein equations on Fano manifolds. They show that this fully nonlinear PDE can be solved if and only if a certain stability condition involving only finite-dimensional algebro-geometric data holds.

In 1982 Shing-Tung Yau received the Fields Medal in part for his 1978 proof of the so-called Calabi Conjecture. In particular Yau proved that if the first Chern class of a compact Kähler manifold vanishes (respectively, is negative), then it admits a Kähler–Einstein metric, i.e., there is a unique Kähler metric in the same class with vanishing (respectively, constant negative) Ricci curvature.

Yau later conjectured that a solution in the case of Fano manifolds, i.e., those with positive first Chern class, would

polarized Kähler manifolds with bounded Ricci curvature (published in *Acta Mathematica* in 2014).

Chen, Donaldson, and Sun gave a complete solution of the conjecture for Fano manifolds a few months later. The announcement was published in *International Mathematics Research Notices* in 2014, and full proofs followed in “Kähler–Einstein metrics on Fano manifolds. I: Approximation of metrics with cone singularities,” “Kähler–Einstein metrics on Fano manifolds. II: Limits with cone angle less than 2π ,” and “Kähler–Einstein metrics on Fano manifolds. III: Limits as cone angle approaches 2π and completion of the main proof,” all published in 2015 in the *Journal of the AMS*.

As one nominator put it, “This is perhaps the biggest breakthrough in differential geometry since Perelman’s work on the Poincaré conjecture. It is certainly the biggest result in Kähler geometry since Yau’s solution of the Calabi conjecture thirty-five years earlier. It is already having a huge impact that will only grow with time.”

Biographical Note: Xiuxiong Chen

Xiuxiong Chen received his undergraduate degree in 1987 from the University of Science and Technology of China (USTC) and a master’s degree from the graduate school of USTC and the Academia Sinica in 1989, supervised by JiaGui Peng in geometry and Weiyue Ding in analysis. He then moved to the University of Pennsylvania in 1989 for his doctoral degree under the supervision of E. Calabi. He held positions at McMaster University (1994–1996), Stanford University (1996–1998), Princeton University (1998–2002), and the University of Wisconsin–Madison (2002–2009). Since 2009 he has been a professor of mathematics at Stony Brook University. He was an invited speaker at ICM 2002 in Beijing and is a 2015 Fellow of the American Mathematical Society and a 2016 Simons Fellow in mathematics. Over his career, he has supervised around twenty PhD students in mathematics.

Biographical Note: Simon Donaldson

Simon Donaldson received his undergraduate degree in 1978 from Cambridge University and moved to Oxford for his doctorate, supervised by Michael Atiyah and Nigel Hitchin. He held positions in Oxford and Stanford before moving to Imperial College, London, in 1998. At present he is a permanent member of the Simons Center for Geometry and Physics, Stony Brook. Over his career he has supervised about forty-five doctoral students, many of whom are now leading figures in mathematical research. Donaldson was awarded a Fields Medal in 1986 for his work on gauge theory and four-dimensional manifolds, and he has made contributions to several other branches of differential geometry. He was an invited speaker at ICMs in 1983, 1986, 1998, and 2018. He has held a number of editorial positions (including, currently, the *Journal of the*

AMS), and served on a variety of committees, including the Executive Committee of the International Mathematical Union (1994–2002).

Biographical Note: Song Sun

Song Sun was born in 1987 in Huaining, Anhui province, China. He received a BS from the University of Science and Technology of China in 2006 and a PhD from the University of Wisconsin–Madison in 2010, supervised by Xiuxiong Chen. He held a postdoctoral position at Imperial College London from 2010–2013, and then became an assistant professor at Stony Brook University. In 2018, he joined the faculty at University of California, Berkeley. Sun received an Alfred P. Sloan Research Fellowship in 2014, and was an invited speaker at ICM 2018 in Rio de Janeiro.

Response from Xiuxiong Chen, Simon Donaldson, and Song Sun

It is a great honor to be awarded the 2019 Oswald Veblen Prize for our work on Kähler–Einstein metrics. Our work builds on that of many others. In 1954, Calabi proposed his vision of far-reaching existence theorems for canonical metrics on Kähler manifolds—a vast extension of the classical theory for Riemann surfaces. The foundation for this vision came from the developments of complex differential geometry over the preceding decades by Kähler, Hodge, Chern, and others. In its general formulation, involving “extremal” Kähler metrics, Calabi’s problem remains to a large extent open, but in the case of Kähler–Einstein metrics the existence theory is now in a relatively satisfactory state. A crucial breakthrough by S.-T. Yau, which famously dealt with the cases of negative or zero first Chern class, was recognized in the 1981 Veblen Prize. Many mathematicians have contributed to the understanding of the remaining “positive” case over the four decades since Yau’s work. We feel very fortunate and privileged to have had the opportunity to play a part in this long story.

Our cited work interweaves strands from several different fields. One is the theory of the complex Monge–Ampère equation, with estimates in the style going back to Calabi and Yau, but also with modern developments which extend the theory to singular varieties. Another is the convergence theory of Riemannian manifolds with Ricci curvature bounds: our work blends these ideas with complex geometry through the L^2 or “Hörmander” method. A third strand brings in the circle of ideas linking geometric invariant theory in algebraic geometry, and notions of “stability,” to symplectic geometry. In the few years following our cited work, several other proofs of the main result have appeared, but all sharing a similar diversity of techniques. This diversity is an intrinsic feature of the problem, which seeks a bridge between differential and algebraic geometry. While our work provides an answer to one long-standing question, these recent developments open up wonderful

new vistas, for example in the study of moduli spaces and singularities, within this grand theme.

We are very glad to have this opportunity to thank our wives—Holly, Nora, and Jiajia—for their wonderful support, which was crucial for us in completing this work. Xiuxiong Chen wishes to take this opportunity to thank his advisor, E. Calabi, for his mathematical guidance and inspiration.

About the Prize

The Oswald Veblen Prize in Geometry is awarded every three years for a notable research memoir in geometry or topology that has appeared during the previous five years in a recognized North American journal (until 2001 the prize was usually awarded every five years). Established in 1964, the prize honors the memory of Oswald Veblen (1880–1960), who served as president of the AMS during 1923–1924. It was established in 1961 in memory of Veblen through a fund contributed by former students and colleagues and later doubled by Veblen's widow. In 2013, Cathleen Synge and Herbert Morawetz made a major donation that substantially increased the prize fund. Cathleen S. Morawetz served as president of the AMS in 1995–1996. The Veblen Prize carries a cash award of US\$5,000.

The Veblen Prize is awarded by the AMS Council acting on the recommendation of a selection committee. For the 2019 prize, the members of the selection committee were:

- Danny C. Calegari,
- Albert Marden (Chair),
- Ulrike Tillmann.

A list of previous recipients of the Oswald Veblen Prize in Geometry may be found on the AMS website at <https://www.ams.org/profession/prizes-awards/pabrowse?url=veblen-prize>.

Credits

Photo of Xiuxiong Chen is by Holly Chen.

Photo of Simon Donaldson is courtesy of Simon Donaldson.

Photo of Song Sun is by Jia Jia He.

2019 Norbert Wiener Prizes in Applied Mathematics

The 2019 Norbert Wiener Prizes in Applied Mathematics were presented at the 125th Annual Meeting of the AMS in Baltimore, Maryland, in January 2019. The prizes were awarded to MARSHA BERGER and to ARKADI NEMIROVSKI.



Marsha Berger

Citation: Marsha Berger

The 2019 Norbert Wiener Prize in Applied Mathematics is awarded to Marsha Berger for her fundamental contributions to Adaptive Mesh Refinement and to Cartesian mesh techniques for automating the simulation of compressible flows in complex geometry.

In solving partial differential equations, Adaptive Mesh Refinement (AMR) algorithms can improve the accuracy of a solution by locally and dynamically resolving complex features of a simulation. Marsha Berger is one of the inventors of AMR. The block-structured approach to AMR was introduced by Berger in her 1982 thesis, and, from this, the Berger–Oliger algorithm and the Berger–Colella algorithm were developed by Berger, Joseph Oliger, and Phillip Colella. Berger provided the mathematical foundations, algorithms, and software that made it possible to solve many otherwise intractable simulation problems, including those related to blood flow, climate modeling, and galaxy simulation. Her mathematical contributions include local error estimators to identify where refinement is needed, stable and conservative grid interface conditions, and embedded boundary and cut-cell methods. She is part of the team that created Cart3D, a NASA code based on her AMR algorithms that is used extensively for aerodynamic simulations and which was instrumental in understanding the Columbia Space Shuttle disaster. She also helped build GeoClaw, an open source software project for ocean-scale wave modeling. It is used

to simulate tsunamis, debris flows, and dam breaks, among other applications.

Biographical Note: Marsha Berger

Marsha Berger received her PhD in computer science from Stanford in 1982. She started as a postdoc at the Courant Institute of Mathematical Sciences at NYU, and is currently a Silver Professor in the computer science department, where she has been since 1985.

She is a frequent visitor to NASA Ames, where she has spent every summer since 1990 and several sabbaticals. Her honors include membership in the National Academy of Sciences, the National Academy of Engineering, and the American Academy of Arts and Science. She is a fellow of the Society for Industrial and Applied Mathematics. Berger was a recipient of the Institute of Electrical and Electronics Engineers Fernbach Award and was part of the team that won the 2002 Software of the Year Award from NASA for their Cart3D software.

Response from Marsha Berger

What a thrill to learn that I will be one of the recipients of the 2019 Norbert Wiener Prizes! One of the main enjoyments I get from my research is developing tools to solve real problems in aerodynamics, tsunami modeling, etc., that others can use. This has been possible because of collaborators I have been fortunate to meet, starting with Phil Colella and Antony Jameson, and later Randy LeVeque and Michael Aftosmis, along with a number of postdocs.

I am particularly pleased that this kind of research is being recognized. The Adaptive Mesh Refinement (AMR) and Cartesian grid projects have both required the creation of new techniques in mathematics and computer science. They were decade-long efforts where I and my collaborators developed theory and algorithms, while paying attention

to important practical aspects of their use in realistic geometries. Complicated algorithms have complicated implementations, and accuracy, robustness, and performance are all essential parts of the research.

About the Prize

The AMS-SIAM Norbert Wiener Prize in Applied Mathematics is awarded every three years to recognize outstanding contributions to applied mathematics in the highest and broadest sense. Established in 1967 in honor of Norbert Wiener (1894–1964), the prize was endowed by the Department of Mathematics of the Massachusetts Institute of Technology. The prize is given jointly by the AMS and the Society for Industrial and Applied Mathematics (SIAM). The recipient must be a member of one of these societies. The prize carries a cash award of US\$5,000.

For the 2019 prize, the members of the AMS-SIAM selection committee were:

- Emmanuel Candes (Chair),
- James Weldon Demmel,
- Charles R. Doering.

A list of the previous recipients of the Norbert Wiener Prize in Applied Mathematics may be found on the AMS website at <https://www.ams.org/profession/prizes-awards/pabrowse?purl=wiener-prize>.

Credits

Photo of Marsha Berger is courtesy of Marsha Berger.

2019 Norbert Wiener Prizes in Applied Mathematics

The 2019 Norbert Wiener Prizes in Applied Mathematics were presented at the 125th Annual Meeting of the AMS in Baltimore, Maryland, in January 2019. The prizes were awarded to MARSHA BERGER and to ARKADI NEMIROVSKI.



Arkadi Nemirovski

Citation: Arkadi Nemirovski

The 2019 Norbert Wiener Prize in Applied Mathematics is awarded to Arkadi Nemirovski for his fundamental contributions to high-dimensional optimization and for his discovery of key phenomena in the theory of signal estimation and recovery.

A powerful and original developer of the mathematics of high-dimensional optimization, Nemirovski, with D.

Yudin, invented the ellipsoid method used by Leonid Khachiyan to show for the first time that linear programs can be solved in polynomial time. With Yurii Nesterov, he extended interior point methods in the style of Narendra Karmarkar to general nonlinear convex optimization. This foundational work established that a rich class of convex problems, called semidefinite programs, are solvable in polynomial time; semidefinite programs are nowadays routinely used to model concrete applied problems or to study deep problems in theoretical computational complexity. A third breakthrough, with Aharon Ben-Tal, was the invention of methods of robust optimization to address problems in which the solution may be very sensitive to problem data. Nemirovski also, and rather amazingly, made seminal contributions in mathematical statistics, establishing the optimal rates at which certain classes of nonparametric signals can be recovered from noisy data and investigating limits of performance for estimation of nonlinear functionals from noisy measurements. All in all, Nemirovski's contri-

butions have become bedrock standards with tremendous theoretical and practical impact on the field of continuous optimization and beyond.

Biographical Note: Arkadi Nemirovski

Arkadi Nemirovski was born in 1947 in Moscow, Russia. He earned his PhD (1974) from Moscow State University, under the supervision of Georgi Evgen'evich Shilov. His research areas are convex optimization (information-based complexity of convex optimization, design of efficient first order and interior point algorithms, robust optimization) and nonparametric statistics. He held research associate positions at the Moscow Research Institute for Automatic Equipment (1973–1987) and the Central Economic Mathematical Institute of USSR/Russian Academy of Sciences (1987–1993) and was professor at the Faculty of Industrial Engineering and Management, Technion, Israel (1993–2005). Since 2005, he has held a professorship at the H. Milton Stewart School of Industrial and Systems Engineering at Georgia Institute of Technology.

Arkadi Nemirovski was elected to the US National Academy of Engineering (2017) and the American Academy of Arts and Sciences (2018). He is a recipient of the Fulkerson Prize of the Mathematical Programming Society (MPS) and the AMS (1982, joint with L. Khachiyan and D. Yudin), the Dantzig Prize of MPS and SIAM (1991, joint with M. Grötschel), and the John von Neumann Theory Prize of the Institute for Operations Research and the Management Sciences (INFORMS) (2003, joint with M. Todd).

Response from Arkadi Nemirovski

I am deeply honored and grateful to receive the 2019 Norbert Wiener Prize in Applied Mathematics—a distinction I never dreamt of. As a student, I have been fortunate to be taught by brilliant mathematicians at the Mechanical

and Mathematical Faculty of Moscow University, where I was mentored by Georgi Shilov. During my professional life, I had the honor and privilege to collaborate with outstanding colleagues, first and foremost, with Yuri Nesterov, Aharon Ben-Tal, and Anatoli Iouditski, to whom I am extremely grateful for their indispensable roles in our joint research and for decades of friendship. I owe a lot to the excellent working conditions I enjoyed at the Central Economic Mathematical Institute in Moscow, at Technion—the Israel Institute of Technology, and at Georgia Institute of Technology.

I always thought that the key word in “applied mathematics” is “mathematics”—even when all we need at the end of the day is a number, I believe that what matters most are rigorous results on how fast this number could be found and how accurate it is, which poses challenging and difficult mathematical problems. I am happy to observe how my research area—convex optimization—thrives due to the effort of new generations of researchers, and how rapidly extends the scope of its applications.

About the Prize

The AMS-SIAM Norbert Wiener Prize in Applied Mathematics is awarded every three years to recognize outstanding contributions to applied mathematics in the highest and broadest sense. Established in 1967 in honor of Norbert Wiener (1894–1964), the prize was endowed by the Department of Mathematics of the Massachusetts Institute of Technology. The prize is given jointly by the AMS and the Society for Industrial and Applied Mathematics (SIAM). The recipient must be a member of one of these societies. The prize carries a cash award of US\$5,000.

For the 2019 prize, the members of the AMS-SIAM selection committee were:

- Emmanuel Candes (Chair)
- James Weldon Demmel
- Charles R. Doering.

A list of the previous recipients of the Norbert Wiener Prize in Applied Mathematics may be found on the AMS website at <https://www.ams.org/profession/prizes-awards/pabrowse?pur1=wiener-prize>.

Credits

Photo of Arkadi Nemirovski is courtesy of Arkadi Nemirovski.

2019 Frank and Brennie Morgan Prize for Outstanding Research in Mathematics by an Undergraduate Student

RAVI JAGADEESAN was awarded the 2019 Frank and Brennie Morgan Prize for Outstanding Research in Mathematics by an Undergraduate Student at the 125th Annual Meeting of the AMS in Baltimore, Maryland, in January 2019.



Ravi Jagadeesan

Citation

The recipient of the 2019 AMS-MAA-SIAM Frank and Brennie Morgan Prize for Outstanding Research in Mathematics by an Undergraduate Student is Ravi Jagadeesan of Harvard University. Jagadeesan was selected as the winner of the Prize for “[his] fundamental contributions across several topics in pure and applied mathematics, including algebraic geometry, statistical theory, mathematical

economics, number theory, and combinatorics” from a pool with outstanding candidates who impressed the selection committee. His papers have been published or accepted for publication in journals such as *Proceedings of the London Mathematical Society*, *Electronic Journal of Combinatorics*, *Research in Number Theory*, *American Economic Journal: Microeconomics*, and *Games and Economic Behavior*. Additionally, he has presented three papers at the Association for Computing Machinery Conference on Economics and Computation.

Jagadeesan’s research in mathematics began early, when he published combinatorics papers on pattern avoidance for permutations in the context of (i) alternating permutations and (ii) Young’s diagrams and tableaux (joint with

Nihal Gowravaram). Then he went on to derive a new invariant for the action of the absolute Galois group of \mathbb{Q} on the set of isomorphism classes of the so-called *dessins d’enfants* (children’s drawings). In another paper, he gave a new proof of Serre’s characterization of regular local rings (joint with Aaron Landesman). At Harvard, he has worked on the birational geometry of elliptic fibrations and its connections to the combinatorics of hyperplane arrangements. His resulting award-winning senior thesis and three related papers (joint with Mboyo Esole, Steven Jackson, Monica Kang, and Alfred Noël) lie at the interface of algebraic geometry, combinatorics, and string theory.

Jagadeesan’s work in mathematical economics is in the fields of matching theory, market design, and public finance. In the view of his references, he brings deep mathematical insights and connections from multiple areas to the table. His papers in matching theory (joint with Tamás Fleiner, Zsuzsanna Jankó, Scott Kominers, Ross Rheingans-Yoo, and Alex Teytelboym) leverage topological fixed-point theorems and ideas from general equilibrium to yield insights into the structure of equilibria in markets with frictions. His work in market design streamlined the analysis of proposed market-clearing mechanisms and clarified the role of key mathematical assumptions. His paper on optimal taxation with an endogenous growth rate is described as being an important contribution to theoretical public finance.

In addition to the above work, Jagadeesan has extended Ramsey theory via quasi-colorings to write a paper on causal statistical inference in the presence of an underlying graph or a network. Regarding this contribution, a reference letter writer states that they were most satisfied by Jagadeesan's "harnessing the beauty and power of mathematics to find structure in a messy real-world problem...making fundamental progress on an important problem of our times." Indeed, the committee members felt that this statement could be applied as well to much of Jagadeesan's work in economics and other areas. Case in point: he has used ideas from category theory to coauthor a Python library, *Matriarch*, for biomaterials architecture (joint with Tristan Giesa, David Spivak, and Markus Buehler).

Biographical Note: Ravi Jagadeesan

Ravi Jagadeesan grew up in Naperville, Illinois. His interest in mathematics was inspired at a young age by his grandparents—all four of them mathematicians—and his parents—who are both computer scientists. He attended Phillips Exeter Academy in Exeter, New Hampshire, for high school, where he had the opportunity to take advanced courses in mathematics and develop his problem-solving skills. He graduated from Harvard with an AB *summa cum laude* in mathematics (with a minor in economics) and with an AM in statistics.

He had the opportunity to work in several different areas of pure and applied mathematics—including algebraic geometry, combinatorics, number theory, statistical theory, and mathematical economics—under a host of advisers. His first experience with mathematical research was during high school, when he was a student in the MIT math department's Program for Research in Mathematics, Engineering, and Science (PRIMES). He then became interested in exploring applied work and spent summers working on research in applied mathematics at the Center for Excellence in Education's Research Science Institute (RSI) at MIT and as an Economic Design Fellow at the Harvard Center of Mathematical Sciences and Applications (CMSA). He is currently a student in Harvard's PhD program in Business Economics, where he is a National Science Foundation Graduate Research Fellow.

Jagadeesan earned a gold medal at the International Mathematical Olympiad in 2012 and was named a Putnam Fellow in 2014. He received Harvard's Jacob Wendell Scholarship Prize, and his senior thesis on "Crepan resolutions of \mathbb{Q} -factorial threefolds with compound Du Val singularities" was awarded the Thomas Temple Hoopes Prize.

Outside of mathematics and economics, he enjoys dancing and is a member of the Harvard Ballroom Dance Team.

Response from Ravi Jagadeesan

It is a great honor to receive the 2019 AMS-MAA-SIAM Frank and Brennie Morgan Prize for Outstanding Research

in Mathematics by an Undergraduate Student. I would like to thank Mrs. Morgan, as well as the AMS, MAA, and SIAM, for establishing this prize and for recognizing me.

I would also like to thank my many mentors—Markus Buehler, Noam Elkies, Mboyo Esole, Pavel Etingof, Zuming Feng, John Geanakoplos, Tristan Giesa, Jerry Green, Joel Lewis, Akhil Mathew, Natesh Pillai, John Rickert, David Spivak, Stefanie Stantcheva, Alex Teytelboym, Alex Volfovsky, Shing-Tung Yau, and, especially, Scott Kominers—for their advice and support over the years.

I am grateful to the MIT Program for Research in Mathematics, Engineering and Science, the Research Science Institute, and the Harvard Center of Mathematical Sciences and Applications for providing excellent work environments.

I am also grateful for research and travel grants from Harvard Business School, the Harvard College Research Program, and the Harvard math department. Most of all, I would like to thank my family—including my wonderful grandparents, parents, and sister—for their love and support.

Citation for Honorable Mention: Evan Chen

Evan Chen is recognized with an Honorable Mention for the 2019 Frank and Brennie Morgan Prize for Outstanding Research in Mathematics by an Undergraduate Student. He has authored many papers in combinatorics and number theory, some as a single author and some in collaboration. He has had papers accepted to the *Proceedings of the AMS*, the *Electronic Journal of Combinatorics*, *Research in Number Theory*, and the *International Journal of Number Theory*. In joint work, he proved an elliptic curve version of Linnik's theorem. He answered an open question on balance constants of posets and, in joint work, made progress on the long-studied problem of classification of Wilf-equivalence classes of patterns. He is currently a PhD student at the Massachusetts Institute of Technology, where he is supported by an NSF Graduate Fellowship.

Biographical Sketch: Evan Chen

Evan Chen was born and raised in California and completed his undergraduate degree in Cambridge, Massachusetts. He is currently pursuing a PhD in mathematics at the Massachusetts Institute of Technology, supported by an NSF fellowship.

Besides research, Evan is deeply involved in the training of the USA team for the International Math Olympiad (IMO), after having won a gold medal himself in high school. Among other roles, he is the assistant academic director for the USA's training camp and the coordinator for the USA team selection tests. He is also the current chief of staff for the Harvard-MIT math tournament and the author of a popular MAA-published book in competitive geometry. Outside of math and teaching, Evan enjoys board games and Korean pop dance.

Response from Evan Chen

It is a wonderful privilege to receive an Honorable Mention for the 2019 Frank and Brennie Morgan Prize. I would like to thank Mrs. Morgan and the AMS, MAA, and SIAM for their generosity and support of undergraduate research.

I would like to acknowledge and thank Joe Gallian and Ken Ono for their mentorship and support during my undergraduate years. The three summers I spent at these REU programs were immensely productive learning and research experiences and contributed greatly to my development. I am also deeply grateful for their encouragement and advice.

I would also like to extend thanks to my professors and teachers from the past several years, with particular thanks to Zuming Feng, Po-Shen Loh, Zvezda Stankova, and Yan Zhang. Finally I would like to thank my family and friends for their constant care and support.

Citation for Honorable Mention: Huy Tuan Pham

Huy Tuan Pham is recognized with an Honorable Mention for the 2019 Frank and Brennie Morgan Prize for Outstanding Research in Mathematics by an Undergraduate Student. He has jointly authored several papers in additive combinatorics. These papers compose his undergraduate thesis, for which he won the Kennedy Thesis Prize at Stanford University. Two of his papers have been accepted to *International Mathematical Research Notices* and to *Discrete Analysis*. His work uses tools from combinatorics, number theory, and analysis to show that tower-type bounds are needed in some natural applications of Szemerédi's regularity method, including Green's generalization of Roth's theorem for popular difference. He is currently at the University of Cambridge supported by a Trinity Studentship and will start his PhD studies at Stanford this fall.

Biographical Note: Huy Tuan Pham

Huy Tuan Pham was born and raised in Ho Chi Minh City, Vietnam. After finishing high school at High School for the Gifted at Vietnam National University, Ho Chi Minh City, he attended Stanford University, where he received a BS in Mathematics with Honors and a minor in Computer Science, and an MS in Statistics. He is now at Cambridge University pursuing Part III of the Mathematical Tripos and will return to Stanford University for his PhD.

Huy's initial interest in combinatorics was developed during International Math Olympiad trainings in Vietnam. Since his sophomore year, he has been working with Jacob Fox on probabilistic and additive combinatorics. He plans to continue his study of combinatorics and probability in his PhD.

Response from Huy Tuan Pham

I am honored to receive an Honorable Mention for the 2019 Frank and Brennie Morgan Prize. I would like to thank Mrs. Frank Morgan and the AMS, MAA, and SIAM for spon-

soring this meaningful award. I am extremely thankful to my advisor Jacob Fox for his help and support throughout my undergraduate years, which has shaped my passion and understanding of combinatorics. I am also grateful to Yufei Zhao, who has given me useful advice throughout our collaboration. I am fortunate to have learned great mathematics from Stanford math professors, particularly Amir Dembo, Persi Diaconis, Andrea Montanari, Lenya Ryzhik, Ravi Vakil, and Jan Vondrak. Last but not least, I would like to thank my family and friends for their support, especially to my friend Phan-Minh Nguyen, who has provided me with tremendous encouragement and insights through our endless conversations in mathematics and statistics.

About the Prize

The Frank and Bernie Morgan Prize is awarded annually for outstanding research in mathematics by an undergraduate student (or students having submitted joint work). Students in Canada, Mexico, or the United States or its possessions are eligible for consideration for the prize. Established in 1995, the prize was endowed by Mrs. Frank (Brennie) Morgan of Allentown, Pennsylvania, and carries the name of her late husband. The prize is given jointly by the AMS, the Mathematical Association of America (MAA), and the Society for Industrial and Applied Mathematics (SIAM) and carries a cash award of US\$1,200.

Recipients of the Frank and Bernie Morgan Prize are chosen by a joint AMS-MAA-SIAM selection committee. For the 2019 prize, the members of the selection committee were:

- Nathan Louis Gibson,
- Anant P. Godbole (Chair),
- V. Kumar Murty,
- Ken Ono,
- Catherine Sulem,
- Melanie Matchett Wood.

A list of previous recipients of the Frank and Brennie Morgan Prize for Outstanding Research in Mathematics by an Undergraduate Student may be found on the AMS website at <https://www.ams.org/profession/prizes-awards/pabrowse?url=morgan-prize>

Credits

Photo of Ravi Jagadeesan is by Ross Campbell Photography.

2019 Joint Policy Board for Mathematics Communications Award

MARGOT LEE SHETTERLY was awarded the 2019 Joint Policy Board for Mathematics Communications Award at the Joint Mathematics Meetings in Baltimore, Maryland, in January 2019.



Margot Lee Shetterly

Citation

The 2019 JPBM Communications Award is presented to Margot Lee Shetterly for her book and subsequent movie *Hidden Figures*, which opened science and mathematics to a new generation of women and people of color by bringing into the light the stories of the African American women who made significant contributions to aeronautics and astronautics and, ultimately, to America's victory in the Space Race.

Biographical Note

Margot Lee Shetterly is a writer, researcher, and entrepreneur. She is the author of *Hidden Figures: The American Dream and the Untold Story of the Black Women Mathematicians Who Helped Win the Space Race*, which was a top book of 2016 for both *TIME* and *Publisher's Weekly* (William Morrow and Company, New York, 2016), a *USA Today* best-seller, and a no. 1 *New York Times* bestseller. Shetterly is also the founder of the Human Computer Project, a digital archive of the stories of NASA's African American "Human Computers," whose work tipped the balance in favor of the United States in World War II, the Cold War, and the Space Race. According to the *New York Times*, the 2017 film adaptation of her book introduces viewers to "real people you might wish you had known more about earlier...[who]

can fill you with an outrage at the persistence of injustice and gratitude towards those who had the grit to stand up against it."

About the Prize

The JPBM Communications Award is presented annually to reward and encourage journalists and other communicators who, on a sustained basis, bring mathematical ideas and information to nonmathematical audiences. JPBM represents the American Mathematical Society, the American Statistical Association, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics. The award carries a cash prize of US\$2,000.

A list of previous recipients of the JPBM Communications Award may be found on the AMS website at <https://www.ams.org/profession/prizes-awards/pabrowse?purl=jpbm-comm-award>.

Credits

Photo of Margot Lee Shetterly is by Aran Shetterly.

Community Updates

A community-wide effort builds The Next Generation Fund



Early career mathematicians at JMM 2019.

Thanks to generous donors, The Next Generation Fund (NextGen) will provide dedicated funding to AMS programs for doctoral students and early career mathematicians, with the goal of issuing modest support to as many people as possible.

Fundraising for NextGen began in 2017, with a forward-thinking philanthropist's offer to match up to \$1.5 million in donations. Numerous donors made gifts toward this goal and the fund has received support from across the community. Visit www.ams.org/nextgen to see our progress to date in making the match.

The Maryam Mirzakhani Fund for The Next Generation is part of NextGen. Gifts to this fund support rising mathematicians while also commemorating Maryam Mirzakhani and her mathematics contributions.

A brochure about The Next Generation Fund with information about how to give is being mailed to AMS members and friends this spring.

From the AMS Public Awareness Office



Luke Robitaille, winner of the 2019 Who Wants to Be a Mathematician Championship.

2019 Who Wants to Be a Mathematician Champion. LUKE ROBITAILLE a homeschooled freshman from Texas, won the 2019 *Who Wants to Be a Mathematician Championship*, which took place at the Joint Mathematics Meetings in Baltimore. Luke, pictured here with AMS President Ken Ribet and AMS President-Elect Jill Pipher, won \$5,000 for himself and \$5,000 for the Mid-Cities Math Circle at the University of Texas at Arlington. See Luke just after his victory at <https://www.youtube.com/watch?v=AMrqIXuDUpQ> and see a webcast of the game at <https://livestream.com/psav/wwtbam2019/videos/186039084>.



"DT-MSH/TC Klein Bottle," by Elizabeth Paley.

2019 Mathematical Art Awards. The 2019 Mathematical Art Exhibition Awards were made at the Joint Mathematics Meetings last week "for aesthetically pleasing works that combine mathematics and art." The three chosen works were selected from the exhibition of juried works in various media by eighty mathematicians and artists from around the world.

Best photograph, painting, or print: "Roundabout" (pair of prints), by JAMES MAI

Best textile, sculpture, or other medium: "DT-MSH/TC Klein Bottle," by ELIZABETH PALEY

Honorable Mention: "Breaking the Ruled," by MATT ENLOW

The Mathematical Art Exhibition Award "for aesthetically pleasing works that combine mathematics and art" was established in 2008 through an endowment provided to the AMS by an anonymous donor who wishes to acknowledge those whose works demonstrate the beauty and elegance of mathematics expressed in a visual art form.



From left to right: Johnston, Xia, Torres-Rodas, and contest judge JoAnne Growney.

AMS Math Poetry Contest Winners. The AMS conducted a math poetry contest for Maryland students—middle school, high school, and undergraduate students—as part of the 2019 Joint Mathematics Meetings. The winners read their poems:

"Math is Me," by BROOKE C. JOHNSTON, Notre Dame Preparatory School

"A Love Letter to My X," by TINA XIA, Walt Whitman High School

"Coalition," by KELIN TORRES-RODAS, Prince George's Community College

To read the winning poems, see www.ams.org/math-poetry.

Credits

Photos from the AMS Public Awareness Office by Annette Emerson.

Photos by Kate Awtrey, Atlanta Convention Photography.

Mathematics People

Ghoussoub Awarded 2019 CRM-Fields-PIMS Prize



Nassif
Ghoussoub

NASSIF GHOUSSOUB of the University of British Columbia has been awarded the 2019 CRM-Fields-PIMS Prize. The citation reads: "Nassif Ghoussoub has a remarkable record of deep, original, and influential contributions to the theory and applications of functional analysis, the calculus of variations, and partial differential equations. His pioneering work on the resolution of De Giorgi's

conjecture, on the PDE of microelectromechanical systems, and on the theory of self-dual PDE have all had a lasting impact on mathematical analysis. This is in addition to his extraordinary contributions to Canadian mathematics in general."

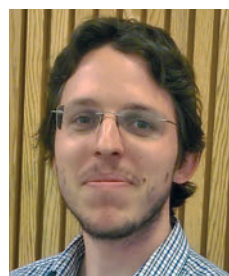
Nassif Ghoussoub was born in Mali and obtained his PhD from Université Pierre et Marie Curie in 1975. He did a postdoctoral fellowship at the Ohio State University (1976–1977), then joined the mathematics department at the University of British Columbia, where he is currently Distinguished University Professor. His honors include the Coxeter-James Prize (1990), the Jeffery-Williams Prize (2007), and the David Borwein Distinguished Career Award (2010), all from the Canadian Mathematical Society (CMS). He was awarded the Queen Elizabeth II Diamond Jubilee Medal in 2012. He served as vice president of the CMS from 1994 to 1996. He is founder and scientific director of the Banff International Research Station and has also been founding director of PIMS and co-editor-in-chief of the *Canadian Journal of Mathematics*. He is a Fellow of the Royal Society of Canada (1994), of the American Mathematical Society (2012), of the Fields Institute (2017), and of the CMS (2018). He is an Officer of the Order of Canada.

The prize is awarded by the Centre de recherches mathématiques (CRM), the Fields Institute, and the Pacific Institute for the Mathematical Sciences (PIMS) to recognize exceptional research achievement in the mathematical sciences. The candidate's research should have been conducted primarily in Canada or in affiliation with a Cana-

dian university. The prize consists of a monetary award and an invitation to present a lecture at each institute.

—From a CRM-Fields-PIMS announcement

Vasey and Harrison-Trainor Awarded Sacks Prizes



Sebastien Vasey

MATTHEW HARRISON-TRAINOR of Victoria University, Wellington, and SEBASTIEN VASEY of Harvard University were awarded 2017 Gerald Sacks Prize of the Association for Symbolic Logic (ASL).

In his thesis, "The Complexity of Countable Structures," Harrison-Trainor established many very strong theorems in computable structure theory. Of these results, two stand out. His full description of the Scott spectrum of a theory was a very surprising general result whose proof settled several open problems, including ones raised by Marker, Sacks, and Montalbán. The second provides a thorough analysis of degree spectra and degrees of categoricity on cones. It shows that the behaviors of these notions are natural in the sense of relativizing to all degrees above some fixed one. Harrison-Trainor received his PhD in 2017 from the University of California, Berkeley, under the direction of Antonio Montalbán.

Vasey, in his thesis "Superstability and categoricity in abstract elementary classes," undertook a deep and sustained study of classification theory for abstract elementary classes. Among the many theorems he proved, his eventual categoricity theorem for universal classes is recognized as a landmark achievement towards Shelah's conjecture generalizing Morley's theorem on uncountable categoricity to abstract elementary classes. A second remarkable result is his classification of the stability spectrum for tame AECs, which may well pave the way for connections with, and applications to, other areas of mathematics. Vasey received his PhD in 2017 from Carnegie Mellon University under the direction of Rami Grossberg. He tells the *Notices*: "I started out as an engineer: my undergraduate studies (at EPFL, in Lausanne) were in communication systems engineering.

My encounter with mathematical logic in 2010, and an exchange year at Carnegie Mellon University in 2011–2012, made me decide to change fields and turn to pure mathematics. I also have a solid background in computer science, especially in programming languages and the theory of computation. I know bits and pieces about GNU Linux/Unix system administration.”

The Sacks Prize is awarded annually for the most outstanding doctoral dissertation in mathematical logic.

—From an ASL announcement

Churchill Scholarships Awarded

Three students in the mathematical sciences have received scholarships from the Winston Churchill Foundation of the United States for the 2019–2020 academic year. The Churchill Scholars are RYAN CHEN of Princeton University (pure mathematics), ANTHONY CONIGLIO of Indiana University (applied mathematics), and BRIAN SEYMOUR of the University of Virginia (applied mathematics). The scholarships cover one year of master’s study at Churchill College in the University of Cambridge. The awards cover full tuition, a stipend, travel costs, and the chance to apply for a US\$2,000 special research grant.

—From a Churchill Foundation announcement

Sjöstrand Awarded 2018 Bergman Prize



Johannes Sjöstrand

JOHANNES SJÖSTRAND of Université de Bourgogne has been awarded the 2018 Bergman Prize. Established in 1988, the prize recognizes mathematical accomplishments in the areas of research in which Stefan Bergman worked. Sjöstrand will receive a cash award of US\$26,000, the 2018 income from the Stefan Bergman Trust.

Citation

Johannes Sjöstrand is awarded the Bergman Prize for his fundamental work on the Bergman and Szegő kernels, as well as for his numerous fundamental contributions to microlocal analysis, spectral theory, and partial differential equations (PDEs). He is especially being recognized for his groundbreaking work with L. Boutet de Monvel on describing the singularities and asymptotics of the Bergman and Szegő kernels in strictly pseudoconvex domains in \mathbb{C}^n . This work has been highly influential in subsequent developments on these and related topics. Sjöstrand is also being recognized for his contributions to microlocal analysis, spectral theory, and PDEs. Together with A. Melin, he has developed the theory of Fourier integral operators with complex-valued phase functions, with applications to the oblique derivative problem. In joint work with R. B. Melrose, he has obtained fundamental results on the propagation of singularities for boundary value problems. Sjöstrand has created the powerful and highly influential approach to analytic microlocal analysis, based on the theory of Fourier-Bros-Iagolnitzer (FBI) transforms and on the use of exponentially weighted spaces of holomorphic functions on the transform side. This approach was shown to be crucial in the study of regularity and propagation of singularities for PDEs (including boundary value problems) in the real analytic category. In joint work with B. Helffer, Sjöstrand has developed an incisive and far-reaching analysis of the tunnel effect for semiclassical Schrödinger operators, including a study of the Witten complex, and has contributed significantly to the understanding of the fine spectral properties of the Harper operator. The work of Johannes Sjöstrand in the theory of scattering resonances, including joint work with M. Zworski, has had a truly revolutionary impact on the subject. Among the many groundbreaking results obtained by Sjöstrand in this direction, we mention a microlocal version of the method of complex scaling and a local trace formula for resonances. Sjöstrand has given numerous decisive contributions to the spectral theory of non-self-adjoint operators, including operators of Kramers-Fokker-Planck type (joint work with

F. Hérau and C. Stolk) and analytic non-self-adjoint operators in dimension two (joint work with A. Melin and with M. Hitrik). More recently, Sjöstrand has completed a deep and fundamental analysis of the Weyl asymptotics for the eigenvalues of non-self-adjoint differential operators in the presence of small random perturbations.

Sjöstrand received his PhD from Lund University in 1972 under the direction of Lars Hörmander. He has been affiliated with the University of Paris XI as well as Bourgne. He is a member of the Royal Swedish Academy of Sciences and was elected to the American Academy of Arts and Sciences in 2017.

Sjöstrand says: "I was asked by the *Notices* to give some interesting facts. Here is one: My thesis advisor, Professor Lars Hörmander, was away to spend the academic year 1970–1971 at Courant Institute, and since there was not very much for me to do in Lund, I asked Professor Lars Gårding about the possibility of travelling to some other place in the Spring semester. He advocated Aarhus, Paris, or Cambridge (England). After some thought, I decided on Paris as the most exciting place. This was a slightly random decision about a mathematical excursion at an unstable equilibrium point of my life; the following events included less choice, and I ended up in France for good. Maybe there was also some fascination for French culture that started with Babar that my parents read to me in Swedish translation. It takes more than a generation to become fully French and to be accepted as such, but with my wife we are happy to see that all our children and grandchildren live in France and are quite well settled."

About the Prize

The Bergman Prize honors the memory of Stefan Bergman, best known for his research in several complex variables, as well as the Bergman projection and the Bergman kernel function that bear his name. A native of Poland, he taught at Stanford University for many years and died in 1977 at the age of eighty-two. He was an AMS member for thirty-five years. When his wife died, the terms of her will stipulated that funds should go toward a special prize in her husband's honor.

The AMS was asked by Wells Fargo Bank of California, the managers of the Bergman Trust, to assemble a committee to select recipients of the prize. In addition, the Society assisted Wells Fargo in interpreting the terms of the will to ensure sufficient breadth in the mathematical areas in which the prize may be given. Awards are made every one or two years in the following areas: (1) the theory of the kernel function and its applications in real and complex analysis and (2) function-theoretic methods in the theory of partial differential equations of elliptic type with attention to Bergman's operator method.

The members of the selection committee for the 2018 Bergman Prize were:

- Donatella Danielli
- Peter Ebenfelt
- Anna Mazzucato (Chair)

—Elaine Kehoe

A list of the past recipients of the Bergman Prize can be found at www.ams.org/profession/prizes-awards/pabrowse?purl=bergman-prize.

Credits

Photo of Nassif Ghoussoub is courtesy of the Pacific Institute for the Mathematical Sciences.

Photo of Sebastien Vasey is courtesy of Samaneh Mesbahi-Vasey.

Photo of Johannes Sjöstrand is courtesy of Johannes Sjöstrand.

Advertisement

Call for Nominations for the Ostrowski Prize, 2019

The aim of the Ostrowski Foundation is to promote the mathematical sciences.

Every second year it provides a prize for recent outstanding achievements in pure mathematics and in the foundations of numerical mathematics. The value of the prize for 2019 is 100,000 Swiss francs.

The prize has been awarded every two years since 1989. The most recent winners are Oded Schramm in 2007, Sorin Popa in 2009, Ib Madsen, David Preiss and Kannan Soundararajan in 2011, Yitang Zhang in 2013, Peter Scholze in 2015, and Akshay Venkatesh in 2017.

See https://www.ostrowski.ch/index_e.php for the complete list and further details.

The jury invites nominations for candidates for the 2019 Ostrowski Prize.

Nominations should include a CV of the candidate, a letter of nomination and 2-3 letters of reference.

The Chair of the jury for 2019 is Marcus Grote of the University of Basel, Switzerland.

Nominations should be sent to marcus.grote@unibas.ch by May 31, 2019.

Mathematics Opportunities

Listings for upcoming mathematics opportunities to appear in Notices may be submitted to notices@ams.org.

Early Career

NSF-CBMS Regional Conferences 2019

With National Science Foundation (NSF) support, the Conference Board of the Mathematical Sciences (CBMS) will hold five Regional Research Conferences during the summer of 2019. Each five-day conference features a distinguished lecturer who delivers ten lectures on a topic of important current research. Support for about thirty participants is provided for each conference.

May 13–17, 2019: Mathematical Molecular Bioscience and Biophysics. Guowei Wei, Michigan State University, lecturer. University of Alabama, Tuscaloosa. Organizer: Shan Zhao, szhao@ua.edu. Website: cbms.ua.edu.

May 13–17, 2019: Topological Methods in Machine Learning and Artificial Intelligence. Gunnar Carlsson, Stanford University, lecturer. College of Charleston. Organizer: Ben Cox, coxbl@cofc.edu. Website: blogs.cofc.edu/cbms-tda2019/.

May 20–24, 2019: L-functions and Multiplicative Number Theory. Kannan Soundararajan, Stanford University, lecturer. University of Mississippi. Organizer: Micah Milinovich, mbmilino@olemiss.edu. Website: math.olemiss.edu/cbms2019/.

May 20–24, 2019: The Cahn-Hilliard Equation: Recent Advances and Applications. Alain Miranville, University of Poitiers, lecturer. Montgomery Bell State Park Conference Center and Inn, Burns, Tennessee, sponsored by the University of Memphis. Organizer: Gisele Goldstein, cahnhilliardsfcbms@gmail.com. Website: memphis.edu/msci/cahn-hilliard-nsf-cbms2019.

August 5–9, 2019: Fitting Smooth Functions to Data. Charles L. Fefferman, Princeton University, lecturer.

The most up-to-date listing of NSF funding opportunities from the Division of Mathematical Sciences can be found online at: www.nsf.gov/dms and for the Directorate of Education and Human Resources at www.nsf.gov/dir/index.jsp?org=ehr. To receive periodic updates, subscribe to the DMSNEWS listserv by following the directions at www.nsf.gov/mps/dms/about.jsp.

University of Texas, Austin. Organizer: Arie Israel, arie@math.utexas.edu. Website: <https://web.ma.utexas.edu/conferences/cbms>.

For more information, see the individual conference websites or cbmsweb.org/regional-conferences/2019-conferences.

—CBMS announcement

Early Career

2019 Clay Research Conference and Workshops

The 2019 Clay Research Conference and Workshops will be held at the Clay Mathematical Institute, University of Oxford, from September 29 to October 4, 2019. The Research Conference will be held on October 2 and associated workshops during the week. For more information see the website.org/events/2019-clay-research-conference-and-workshops.

—From a Clay Mathematics Institute announcement

Call for Proposals for 2020 NSF-CBMS Regional Research Conferences

The NSF-CBMS Regional Research Conferences in the Mathematical Sciences are a series of five-day conferences that usually feature a distinguished lecturer delivering ten lectures on a topic of important current research in one sharply focused area of the mathematical sciences. The Conference Board of the Mathematical Sciences (CBMS) publicizes the conferences and disseminates the resulting conference materials. Support is provided for about thirty participants at each conference. Proposals should address the unique characteristics of the NSF-CBMS conferences, outlined in the Program Description found at <https://>

nsf.gov/pubs/2019/nsf19539/nsf19539.htm

The deadline for proposals is April 26, 2019.

—From an NSF announcement

AWM Gweneth Humphreys Award

The Association for Women in Mathematics awards the Gweneth Humphreys Award annually to a mathematics teacher who has encouraged female undergraduates to pursue mathematical careers and/or the study of mathematics at the graduate level. The deadline for nominations is April 30, 2019. See <https://awm-math.org/humphreys-award> or email awm@awm-math.org.

—From an AWM announcement

Call for Applications for the 2019 Rosenthal Prize

The National Museum of Mathematics (MoMath) awards the Rosenthal Prize annually. Designed to recognize and promote hands-on math teaching in upper elementary and middle school classrooms, the prize carries a cash award of US\$25,000 for the single best activity, plus up to five additional monetary awards for other innovative activities. The deadline for applications is May 15, 2019. See <https://rpz.momath.org>.

—From a MoMath announcement

Sabbatical Opportunities at the National Security Agency

The Mathematical Sciences Program at the National Security Agency (NSA) invites mathematicians, statisticians, and other researchers in the mathematical sciences to apply for a sabbatical at the NSA. Successful applicants will join a team of technical experts to collaborate on operational problems at NSA's headquarters in Maryland. Since employment at NSA requires a security clearance, applications for sabbatical positions need to be submitted well in advance of the date on which the sabbatical is scheduled to commence. For the same reason, these positions are open to US citizens only. Individuals selected for these sabbatical positions will receive salary supplement, travel expenses, and a housing allowance. Further information can be obtained on the webpage (<https://www.nsa.gov/What-We-Do/Research/Math-Sciences-Program/Sabbaticals>) or by email from Charles Toll or Barbara Johnson at msp-grants.nsa.gov.

—Charles Toll



MATHJOBS.ORG

The automated job application database
sponsored by the AMS.

MathJobs.Org offers a paperless application process for applicants and employers in mathematics.

Registered Applicants Can:

- Create their own portfolio of application documents
- Make applications online to participating employers
- Choose to make a cover sheet viewable by all registered employers

FREE FOR APPLICANTS

Registered Employers Can:

- Post up to seven job ads
- Set all criteria for required documents, and add specific questions
- Receive and upload reference letters
- Manage applicant information and correspondence quickly and easily
- Set limited access permissions for faculty and EOE administrators
- Search for and sort additional applicants in the database
- Choose an advertising-only account, or a discounted single ad account

Visit **mathjobs.org** for pricing information.



CONTACT:

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800.321.4267, ext. 4096 | mathjobs@ams.org



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Employment Opportunities

OHIO

Ohio Northern University Visiting Assistant Professor of Mathematics

Posting Number: F000070

Position Type: Full-Time Faculty

Department: Mathematics & Statistics

Benefits Summary:

Benefit package includes: Medical, Dental, Vision and Prescription insurance, Life insurance, Workers' Compensation insurance, Unemployment insurance, and Total Disability insurance. Retirement: The University contributes 4% of the regular salary with up to 3% of additional matched contributions into the TIAA Retirement Program. Other benefits include tuition remission for employee, spouse, and employee's dependent children under the age of 25 (this does not include the last two year of the PharmD program or the JD), and twenty days of paid medical leave per year.

Job Summary:

The successful candidate will show excellence in teaching a variety of lower and upper-level mathematics courses. The applicant will be pursuing a solid research agenda, will advise and involve in research undergraduate students, and will serve on department, college or university committees. The rank is commensurate with experience and credentials.

Scope:

One year Visiting Assistant Professor of Mathematics and Statistics.

Principal Responsibilities:

Teaching (including advising/mentoring students and engaging them in high impact learning practices), maintaining an effective scholarly activity.

Required Skills:

Excellence in teaching and the ability to pursue a solid research agenda. Good interpersonal skills and an ability to mentor students both inside and outside the classroom. Women and minorities are encouraged to apply.

Minimum Qualifications:

PhD in Mathematics or Applied Mathematics.

Appointment Length: 9-months

Closing Date: 04/01/2019

Open Until Filled: No

Status: Full-Time

To view full description and apply online go to:
<https://jobs.onu.edu/postings/5933>.

ONU is an equal employment opportunity employer. Accordingly, no person shall be discriminated against on the basis of race, color, sex, gender identity, transgender status, religion, national origin, age, disability, sexual orientation, marital status, military or veteran status, genetic information or any other category protected by federal, state, or local law. This policy applies to all areas of employment including recruitment, hiring, training and development, promotion, transfer, compensation, benefits, discipline, separation and other terms, condition and privileges of employment.

07

The *Notices Classified Advertising* section is devoted to listings of current employment opportunities. The publisher reserves the right to reject any listing not in keeping with the Society's standards. Acceptance shall not be construed as approval of the accuracy or the legality of any information therein. Advertisers are neither screened nor recommended by the publisher. The publisher is not responsible for agreements or transactions executed in part or in full based on classified advertisements.

The 2019 rate is \$3.50 per word. Advertisements will be set with a minimum one-line headline, consisting of the institution name above body copy, unless additional headline copy is specified by the advertiser. Headlines will be centered in boldface at no extra charge. Ads will appear in the language in which they are submitted. There are no member discounts for classified ads. Dictation over the telephone will not be accepted for classified ads.

Upcoming deadlines for classified advertising are as follows: May 2019—February 18, 2019; June/July 2019—April 18, 2019; August 2019—May 16, 2019; September 2019—June 17, 2019; October 2019—July 17, 2019; November 2019—August 5, 2019; December 2019—September 17, 2019.

US laws prohibit discrimination in employment on the basis of color, age, sex, race, religion, or national origin. Advertisements from institutions outside the US cannot be published unless they are accompanied by a statement that the institution does not discriminate on these grounds whether or not it is subject to US laws.

Submission: Send email to classads@ams.org.

CHINA

Tianjin University, China
Tenured/Tenure-Track/Postdoctoral Positions at
the Center for Applied Mathematics

Dozens of positions at all levels are available at the recently founded Center for Applied Mathematics, Tianjin University, China. We welcome applicants with backgrounds in pure mathematics, applied mathematics, statistics, computer science, bioinformatics, and other related fields. We also welcome applicants who are interested in practical projects with industries. Despite its name attached with an accent of applied mathematics, we also aim to create a strong presence of pure mathematics. Chinese citizenship is not required.

Light or no teaching load, adequate facilities, spacious office environment and strong research support. We are prepared to make quick and competitive offers to self-motivated hard workers, and to potential stars, rising stars, as well as shining stars.

The Center for Applied Mathematics, also known as the Tianjin Center for Applied Mathematics (TCAM), located by a lake in the central campus in a building protected as historical architecture, is jointly sponsored by the Tianjin municipal government and the university. The initiative to establish this center was taken by Professor S. S. Chern. Professor Molin Ge is the Honorary Director, Professor Zhiming Ma is the Director of the Advisory Board. Professor William Y. C. Chen serves as the Director.

TCAM plans to fill in fifty or more permanent faculty positions in the next few years. In addition, there are a number of temporary and visiting positions. We look forward to receiving your application or inquiry at any time. There are no deadlines.

Please send your resume to mathjobs@tju.edu.cn.

For more information, please visit cam.tju.edu.cn or contact Ms. Erica Liu at mathjobs@tju.edu.cn, telephone: 86-22-2740-6039.

01

REPUBLIC OF KOREA

Korean Institute for Advanced Study (KIAS)
Assistant Professor & Research Fellow
in Pure and Applied Mathematics

The School of Mathematics at the Korea Institute for Advanced Study (KIAS) invites applicants for the positions at the level of KIAS Assistant Professor and Postdoctoral Research Fellow in pure and applied mathematics. KIAS, founded in 1996, is committed to the excellence of research in basic sciences (mathematics, theoretical physics, and computational sciences) through high-quality research programs and a strong faculty body consisting of distinguished scientists and visiting scholars.

Applicants are expected to have demonstrated exceptional research potential, including major contributions beyond or through the doctoral dissertation. The annual salary starts from 50,500,000 Korean Won (approximately US\$45,500 at current exchange rate) for Research Fellows, and 57,500,000 Korean Won for KIAS Assistant Professors, respectively. In addition, individual research funds of 10,000,000 ~ 13,000,000 Korean Won are available per year. The initial appointment for the position is for two years and is renewable for up to two additional years, depending on research performance and the needs of the research program at KIAS.

Applications will be reviewed twice a year, May 20 and November 20, and selected applicants will be notified in a month after the review. In exceptional cases, applications can be reviewed other times based on the availability of positions. The starting date of the appointment is negotiable. Application materials must include a standard cover sheet which is posted on the website (www.kias.re.kr), a cover letter, a curriculum vitae including a list of publications, a research plan, and three letters of recommendation. All documents, prepared in English, should be submitted by post or e-mail to:

Ms. Sojung Bae (mathkias@kias.re.kr)

School of Mathematics

Korea Institute for Advanced Study (KIAS)

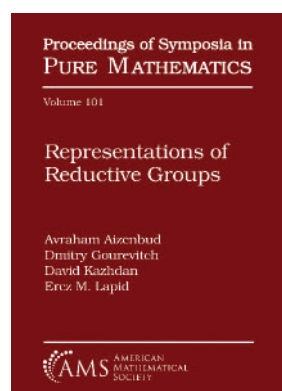
85 Hoegiro (Cheongnyangni-dong 207-43), Dongdaemun-gu,

Seoul 02455, Republic of Korea

08

New Books Offered by the AMS

Algebra and Algebraic Geometry



Representations of Reductive Groups

Avraham Aizenbud, *Weizmann Institute of Science, Rehovot, Israel*,
Dmitry Gourevitch, *Weizmann Institute of Science, Rehovot, Israel*,
David Kazhdan, *Hebrew University of Jerusalem, Israel*, and Erez M. Lapid, *Weizmann Institute of Science, Rehovot, Israel*, Editors

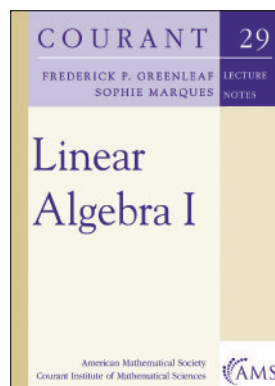
This volume contains the proceedings of the Conference on Representation Theory and Algebraic Geometry, held in honor of

Joseph Bernstein, from June 11–16, 2017, at the Weizmann Institute of Science and The Hebrew University of Jerusalem. The topics reflect the decisive and diverse impact of Bernstein on representation theory in its broadest scope.

Proceedings of Symposia in Pure Mathematics, Volume 101

April 2019, 450 pages, Hardcover, ISBN: 978-1-4704-4284-2, LC 2018041948, 2010 *Mathematics Subject Classification*: 11F27, 11F70, 20C08, 20C33, 20G05, 20G20, 20G25, 22E47, List US\$133, **AMS members US\$106.40**, **MAA members US\$119.70**, Order code PSPUM/101

bookstore.ams.org/pspum-101



Linear Algebra I

Frederick P. Greenleaf, *Courant Institute, New York University, NY*, and Sophie Marques, *University of Cape Town, South Africa*

This book is the first of two volumes on linear algebra for graduate students in mathematics, the sciences, and economics, who have: a prior undergraduate course in the subject; a basic understanding of matrix algebra; and some proficiency with mathematical proofs. Proofs are

emphasized and the overall objective is to understand the structure of linear operators as the key to solving problems in which they arise.

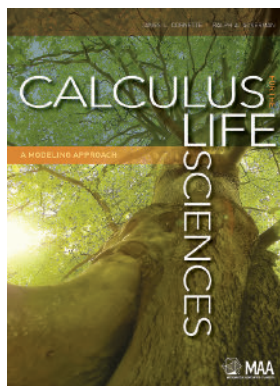
Titles in this series are co-published with the Courant Institute of Mathematical Sciences at New York University.

Courant Lecture Notes, Volume 29

March 2019, 261 pages, Softcover, ISBN: 978-1-4704-4871-4, LC 2018047584, 2010 *Mathematics Subject Classification*: 97H60, 15-01, 15A04, 15A15, 15A16, 15A18, 15A21, 40A05, 40A25, 20B30, List US\$51, **AMS members US\$40.80**, **MAA members US\$45.90**, Order code CLN/29

bookstore.ams.org/cln-29

Calculus



Calculus for the Life Sciences

A Modeling Approach

James L. Cornette and Ralph A. Ackerman

Freshman and sophomore life sciences students respond well to the modeling approach to calculus, difference equations, and differential equations presented in this book. Examples of population dynamics, pharmacokinetics, and biologically

relevant physical processes are introduced in Chapter 1, and these and other life sciences topics are developed throughout the text.

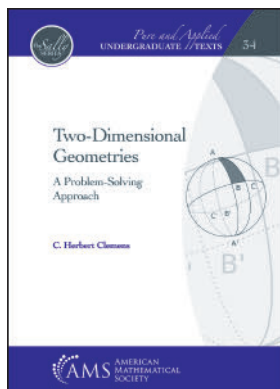
This title will be available in print version for the first time. This item will also be of interest to those working in applications.

AMS/MAA Textbooks, Volume 29

December 2015, 713 pages, Softcover, ISBN: 978-1-4704-5142-4, List US\$99, Order code TEXT/29

bookstore.ams.org/text-29

Geometry and Topology



Two-Dimensional Geometries

A Problem-Solving Approach

C. Herbert Clemens, *Ohio State University, Columbus, OH*

This book on two-dimensional geometry uses a problem-solving approach to actively engage students in the learning process. The aim is to guide readers through the story of the subject,

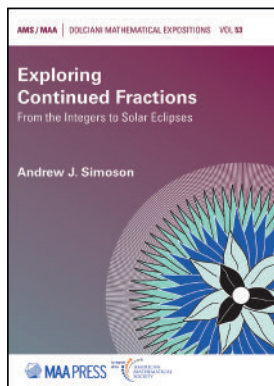
while giving them room to discover and partially construct the story themselves. The book bridges the study of plane geometry and the study of curves and surfaces of non-constant curvature in three-dimensional Euclidean space.

Pure and Applied Undergraduate Texts, Volume 34

April 2019, 142 pages, Hardcover, ISBN: 978-1-4704-4760-1, LC 2018045838, 2010 *Mathematics Subject Classification*: 51F10, 51F20, 51F25, 51-01, 51K10, List US\$89, AMS members US\$71.20, MAA members US\$80.10, Order code AMSTEXT/34

bookstore.ams.org/amstext-34

Number Theory



Exploring Continued Fractions

From the Integers to Solar Eclipses

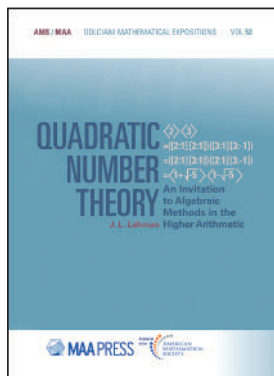
Andrew J. Simoson, *King University, Bristol, TN*

This text explores recurrent phenomena, including astronomical transits and conjunctions, lifecycles of cicadas, and eclipses, by way of continued fraction expansions. The deeper purpose is to find patterns, solve puzzles, and discover some appealing number theory.

Dolciani Mathematical Expositions, Volume 53

May 2019, approximately 371 pages, Hardcover, ISBN: 978-1-4704-4795-3, 2010 *Mathematics Subject Classification*: 11J70, 00A05, 70F15, List US\$52, AMS Individual member US\$39, AMS Institutional member US\$41.60, MAA members US\$39, Order code DOL/53

bookstore.ams.org/dol-53



Quadratic Number Theory

An Invitation to Algebraic Methods in the Higher Arithmetic

J. L. Lehman, *University of Mary Washington, Fredericksburg, VA*

This text is an introduction to algebraic number theory for readers with a moderate knowledge of elementary number theory and some familiarity with the terminology of abstract algebra.

With its exceptionally clear prose, hundreds of exercises, and historical motivation, it would make an excellent textbook for a second undergraduate course in number theory or a terrific choice for independent reading.

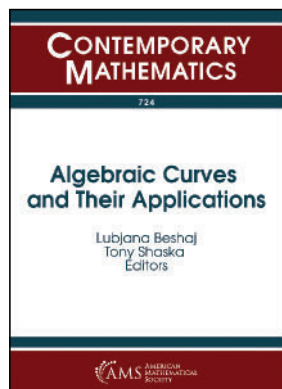
Dolciani Mathematical Expositions, Volume 52

March 2019, 394 pages, Hardcover, ISBN: 978-1-4704-4737-3, LC 2018040720, 2010 *Mathematics Subject Classification*: 11R11, 11R29, 11R27, 11E25, 11E16, 11A55, 11B50, 11Y40, List US\$52, AMS Individual member US\$39, AMS Institutional member US\$41.60, MAA members US\$39, Order code DOL/52

bookstore.ams.org/dol-52

New in Contemporary Mathematics

Algebra and Algebraic Geometry



Algebraic Curves and Their Applications

Ljubana Beshaj, *West Point Military Academy, NY*, and Tony Shaska, *Oakland University, Rochester, MI*, Editors

This volume contains a collection of papers on algebraic curves and their applications. While algebraic curves traditionally have provided a path toward modern algebraic geometry, they also provide many applications in number theory, computer

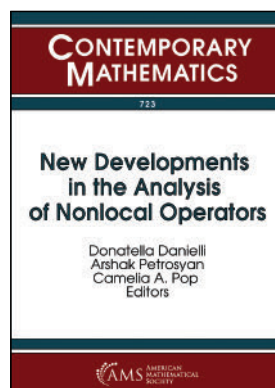
security and cryptography, coding theory, differential equations, and more.

Contemporary Mathematics, Volume 724

March 2019, 344 pages, Softcover, ISBN: 978-1-4704-4247-7, LC 2018040058, 2010 *Mathematics Subject Classification*: 11G30, 11G50, 11G32, 11T71, 11T06, 14H37, 14H40, 14H45, 14H52, 14H55, List US\$117, **AMS members US\$93.60**, **MAA members US\$105.30**, Order code CONM/724

bookstore.ams.org/conm-724

Differential Equations



New Developments in the Analysis of Nonlocal Operators

Donatella Danielli, *Purdue University, West Lafayette, IN*, Arshak Petrosyan, *Purdue University, West Lafayette, IN*, and Camelia A. Pop, *University of Minnesota, Minneapolis, MN*, Editors

This volume contains the proceedings of the AMS Special Session on New Developments in the Analysis of Nonlocal Operators, held from October

28–30, 2016, at the University of St. Thomas, Minneapolis, Minnesota. Problems represented in this volume include uniqueness for weak solutions to abstract parabolic equations with fractional time derivatives and the behavior of the one-phase Bernoulli-type free boundary near a fixed boundary and its relation to a Signorini-type problem.

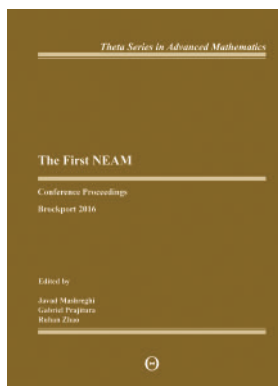
Contemporary Mathematics, Volume 723

March 2019, 214 pages, Softcover, ISBN: 978-1-4704-4110-4, LC 2018039664, 2010 *Mathematics Subject Classification*: 35R09, 35R11, 35R35; 11M41, 26A33, 60G51, 91G80, List US\$117, **AMS members US\$93.60**, **MAA members US\$105.30**, Order code CONM/723

bookstore.ams.org/conm-723

New AMS-Distributed Publications

Analysis



The First NEAM Conference Proceedings, Brockport 2016

Javad Mashreghi, *Laval University, Quebec, Canada*, Gabriel Prajitura, *SUNY Brockport, NY*, and Ruhan Zhao, *SUNY Brockport, NY*, Editors

The volume contains the proceedings of the First Northeastern Analysis Meeting, held in Brockport between October 14 and 16, 2016. It consists of a

careful selection of papers covering a large range of subjects in mathematical analysis.

Among the topics discussed are: (1) classical complex function theory; (2) differential operators on trees; (3) integral operators; (4) operator theory on function spaces; (5) Fourier analysis; and (6) geometry of Banach spaces.

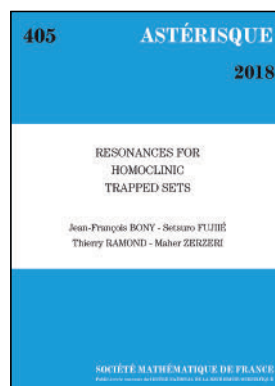
A publication of the Theta Foundation. Distributed worldwide, except in Romania, by the AMS.

International Book Series of Mathematical Texts

December 2018, 166 pages, Hardcover, ISBN: 978-606-8443-11-9, 2010 *Mathematics Subject Classification*: 00B25, 30-06, 42-06, 46-06, 47-06, List US\$48, **AMS members US\$38.40**, Order code THETA/25

bookstore.ams.org/theta-25

Differential Equations



Resonances for Homoclinic Trapped Sets

Jean-François Bony, *IMB, CNRS, Université de Bordeaux, Talence, France*, Setsuro Fujiié, *Ritsumeikan University, Kusatsu, Japan*, Thierry Ramond, *Université Paris-Sud, CNRS, Orsay, France*, and Maher Zerzeri, *Université Paris 13, Sorbonne Paris Cité, CNRS, Villetaneuse, France*

The authors study semiclassical resonances generated by homoclinic trapped sets. First, under some general assumptions, they prove that there is no resonance in a region below the real axis. Then, they obtain a quantization rule and the asymptotic expansion of the resonances when there is a finite number of homoclinic trajectories. The same kind of result is proved for homoclinic sets of maximal dimension.

Next, the authors generalize to the case of homoclinic/heteroclinic trajectories and study the three bump cases. In all of these settings, the resonances may either accumulate on curves or form clouds. The authors also describe the corresponding resonant states.

This item will also be of interest to those working in analysis.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the U.S., Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Astérisque, Number 405

December 2018, 314 pages, Softcover, ISBN: 978-2-85629-894-7, 2010 *Mathematics Subject Classification*: 35B34, 35P20, 37C29, 37C25, 35C20, 81Q20, 35S10, 35J10, List US\$90, **AMS members US\$72**, Order code AST/405

bookstore.ams.org/ast-405

Meetings & Conferences of the AMS

April Table of Contents

The Meetings and Conferences section of the *Notices* gives information on all AMS meetings and conferences approved by press time for this issue. Please refer to the page numbers cited on this page for more detailed information on each event.

Invited Speakers and Special Sessions are listed as soon as they are approved by the cognizant program committee; the codes listed are needed for electronic abstract submission. For some meetings the list may be incomplete. Information in this issue may be dated.

The most up-to-date meeting and conference information can be found online at: www.ams.org/meetings.

Important Information About AMS Meetings: Potential organizers, speakers, and hosts should refer to page 127 in the January 2019 issue of the *Notices* for general information regarding participation in AMS meetings and conferences.

Abstracts: Speakers should submit abstracts on the easy-to-use interactive Web form. No knowledge of \LaTeX is necessary to submit an electronic form, although those who use \LaTeX may submit abstracts with such coding, and all math displays and similarly coded material (such as accent marks in text) must be typeset in \LaTeX . Visit www.ams.org/cgi-bin/abstracts/abstract.pl. Questions about abstracts may be sent to abs-info@ams.org. Close attention should be paid to specified deadlines in this issue. Unfortunately, late abstracts cannot be accommodated.

Associate Secretaries of the AMS

Central Section: Georgia Benkart, University of Wisconsin–Madison, Department of Mathematics, 480 Lincoln Drive, Madison, WI 53706-1388; email: benkart@math.wisc.edu; telephone: 608-263-4283.

Eastern Section: Steven H. Weintraub, Department of Mathematics, Lehigh University, Bethlehem, PA 18015-3174; email: steve.weintraub@lehigh.edu; telephone: 610-758-3717.

Southeastern Section: Brian D. Boe, Department of Mathematics, University of Georgia, 220 D W Brooks Drive, Athens, GA 30602-7403, email: brian@math.uga.edu; telephone: 706-542-2547.

Western Section: Michel L. Lapidus, Department of Mathematics, University of California, Surge Bldg., Riverside, CA 92521-0135; email: lapidus@math.ucr.edu; telephone: 951-827-5910.

Meetings in this Issue

2019

March 15–17	Auburn, Alabama	p. 635
March 22–24	Honolulu, Hawai'i	p. 637
April 13–14	Hartford, Connecticut	p. 639
June 10–13	Quy Nhon, Vietnam	p. 640
September 14–15	Madison, Wisconsin	p. 644
October 12–13	Binghamton, New York	p. 646
November 2–3	Gainesville, Florida	p. 647
November 9–10	Riverside, California	p. 647

2020

January 15–18	Denver, Colorado	p. 648
March 13–15	Charlottesville, Virginia	p. 648
March 21–22	Medford, Massachusetts	p. 649
April 4–5	West Lafayette, Indiana	p. 649
May 2–3	Fresno, California	p. 649
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January 6–9	Washington, DC	p. 650
July 5–9	Grenoble, France	p. 650
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October 9–10	Omaha, Nebraska	p. 651

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January 5–8	Seattle, Washington	p. 651
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2023

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See <https://www.ams.org/meetings> for the most up-to-date information on the meetings and conferences that we offer.

Meetings & Conferences of the AMS

IMPORTANT INFORMATION REGARDING MEETINGS PROGRAMS: AMS Sectional Meeting programs do not appear in the print version of the *Notices*. However, comprehensive and continually updated meeting and program information with links to the abstract for each talk can be found on the AMS website. See www.ams.org/meetings/.

Final programs for Sectional Meetings will be archived on the AMS website accessible from the stated URL .

Auburn, Alabama

Auburn University

March 15–17, 2019

Friday – Sunday

Meeting #1146

Southeastern Section

Associate secretary: Brian D. Boe

Announcement issue of *Notices*: January 2019

Program first available on AMS website: January 31, 2019

Issue of *Abstracts*: Volume 40, Issue 2

Deadlines

For organizers: Expired

For abstracts: Expired

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtg/sectional.html.

Invited Addresses

Grigoriy Blekherman, Georgia Institute of Technology, *Do Sums of Squares Dream of Free Resolutions?*.

Carina Pamela Curto, Pennsylvania State University, *Graphs, network motifs, and threshold-linear algebra in the brain*.

Ming Liao, Auburn University, *Invariant Markov processes under actions of Lie groups*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at www.ams.org/cgi-bin/abstracts/abstract.pl.

Algebraic and Discrete Methods in Mathematical Biology, **Carina Curto**, The Pennsylvania State University, **Katherine Morrison**, University of Northern Colorado, and **Nora Youngs**, Colby College.

Applications of Algebraic Geometry, **Greg Blekherman**, Georgia Institute of Technology, **Michael Burr**, Clemson University, and **Tianran Chen**, Auburn University at Montgomery.

Clustering Methods and Applications, **Benjamin McLaughlin**, Naval Surface Warfare Center Panama City Division (NSWCPCD), and **Sung Ha Kang**, Georgia Institute of Technology.

Combinatorial Matrix Theory, **Zhongshan Li**, Georgia State University, and **Xavier Martínez-Rivera**, Auburn University.

Commutative and Combinatorial Algebra, **Selvi Kara Beyarslan**, University of South Alabama, and **Alessandra Costantini**, Purdue University.

Developments in Commutative Algebra, **Eloísa Grifo**, University of Michigan, and **Patricia Klein**, University of Kentucky.

Differential Equations in Mathematical Biology, **Guihong Fan**, Columbus State University, **Zhongwei Shen**, University of Alberta, and **Xiaoxia Xie**, Idaho State University.

MEETINGS & CONFERENCES

Discrete and Convex Geometry, **Andras Bezdek**, Auburn University, **Ferenc Fodor**, University of Szeged, and **Włodzimierz Kuperberg**, Auburn University.

Evolution Equations and Applications, **Dmitry Glotov**, **Wenxian Shen**, and **Paul G. Schmidt**, Auburn University.

Experimental Mathematics in Number Theory, Analysis, and Combinatorics, **Amita Malik**, Rutgers University, and **Armin Straub**, University of South Alabama.

Geometric Flows and Minimal Surfaces, **Theodora Bourni**, University of Tennessee, and **Giuseppe Tinaglia**, King's College London and University of Tennessee.

Geometric Methods in Representation Theory, **Jiuzu Hong** and **Shrawan Kumar**, University of North Carolina, Chapel Hill, and **Yiqiang Li**, University at Buffalo, the State University of New York.

Geometric and Combinatorial Aspects of Representation Theory, **Mark Colarusso**, University of South Alabama, and **Jonas Hartwig**, Iowa State University.

Geometry and Topology of Low Dimensional Manifolds, and Their Invariants, **John Etnyre**, Georgia Institute of Technology, **Bulent Tosun**, University of Alabama, and **Shea Vela-Vick**, Louisiana State University.

Graph Theory in Honor of Robert E. Jamison's 70th Birthday, **Robert A Beeler**, East Tennessee State University, **Gretchen Matthews**, Virginia Tech, and **Beth Novick**, Clemson University.

Hopf Algebras and Their Applications, **Robert Underwood**, Auburn University at Montgomery, and **Alan Koch**, Agnes Scott College.

Mapping Class Groups, **Joan Birman**, Columbia University, and **Kevin Kordek** and **Dan Margalit**, Georgia Institute of Technology.

Mathematical Analysis and Control Theory of Coupled Partial Differential Equation Models, **George Avalos** and **Pelin Gu'ven Geredeli**, University of Nebraska-Lincoln, and **László Kindrat**, University of New Hampshire.

Nonlinear Reaction-Diffusion Equations and Their Applications, **Jerome Goddard,II**, Auburn University at Montgomery, **Nsoki Mavinga**, Swarthmore College, **Quinn Morris**, Appalachian State University, and **R. Shivaji**, University of North Carolina at Greensboro.

Probability and Stochastic Processes, **Ming Liao**, **Erkan Nane**, and **Jerzy Szulga**, Auburn University.

Random Discrete Structures, **Lutz P Warnke**, Georgia Institute of Technology, and **Xavier Pérez-Giménez**, University of Nebraska-Lincoln.

Recent Advances in Coarse Geometry, **Jerzy Dydak**, University of Tennessee.

Recent Advances in Numerical Methods for PDEs and PDE-constrained Optimization, **Yanzhao Cao**, **Thi-Thao-Phuong Hoang**, and **Junshan Lin**, Auburn University.

Recent Developments in Graph Theory, **Xiaofeng Gu**, **Jeong-Hyun Kang**, **David Leach**, and **Rui Xu**, University of West Georgia.

Representations of Lie Algebras, Algebraic Groups, and Quantum Groups, **Joerg Feldvoss**, University of South Alabama, **Lauren Grimley**, Spring Hill College, and **Cornelius Pillen**, University of South Alabama.

The Modeling and Analysis of Spatially Extended Structures, **Shibin Dai**, University of Alabama, **Keith Promislow**, Michigan State University, and **Qiliang Wu**, Ohio University.

Topological Data Analysis, Statistics and Applications, **Yu-Min Chung**, University of North Carolina at Greensboro, and **Vasileios Maroulas**, University of Tennessee.

Honolulu, Hawai'i

University of Hawai'i at Mānoa

March 22–24, 2019

Friday – Sunday

Meeting #1147

Central Section & Western Section

Associate secretaries: Georgia Benkart
and Michel L. Lapidus

Announcement issue of *Notices*: January 2019

Program first available on AMS website: February 7, 2019

Issue of *Abstracts*: Volume 40, Issue 2

Deadlines

For organizers: Expired

For abstracts: Expired

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtg/sectional.html.

Invited Addresses

Barry C. Mazur, Harvard University, *On the arithmetic of curves* (Einstein Public Lecture in Mathematics).

Aaron Naber, Northwestern University, *Analysis of geometric nonlinear partial differential equations*.

Deanna Needell, UCLA, *Simple approaches to complicated data analysis*.

Katherine E Stange, University of Colorado, Boulder, *An illustration in number theory*.

Andrew Suk, University of California San Diego, *On the Erdos-Szekeres convex polygon problem*.

Special Sessions

Advances in Iwasawa Theory, **Frauke Bleher**, University of Iowa, **Ted Chinburg**, University of Pennsylvania, and **Robert Harron**, University of Hawai'i at Mānoa.

Advances in Mathematical Fluid Mechanics, **Kazuo Yamazaki**, University of Rochester, and **Adam Larios**, University of Nebraska - Lincoln.

Algebraic Groups, Galois Cohomology, and Local-Global Principles, **Raman Parimala**, Emory University, **Andrei Rapinchuk**, University of Virginia, and **Igor Rapinchuk**, Michigan State University.

Algebraic Number Theory and Diophantine Equations, **Claude Levesque**, University of Laval.

Algebraic Points, **Barry Mazur** and **Hector Pasten**, Harvard University.

Algebraic and Combinatorial Structures in Knot Theory, **Sam Nelson**, Claremont McKenna College, **Natsumi Oyamauchi**, Shumei University, and **Kanako Oshiro**, Sophia University.

Algebraic and Geometric Combinatorics, **Andrew Berget**, Western Washington University, and **Steven Klee**, Seattle University.

Analysis of Nonlinear Geometric Equations, **Aaron Naber**, Northwestern University, and **Richard Bamler**, University of California Berkeley.

Analytic and Probabilistic Methods in Convex Geometry, **Alexander Koldobsky**, University of Missouri, **Alexander Litvak**, University of Alberta, **Dmitry Ryabogin**, Kent State University, **Vladyslav Yaskin**, University of Alberta, and **Artem Zvavitch**, Kent State University.

Applications of Ultrafilters and Nonstandard Methods, **Isaac Goldbring**, University of California, Irvine, and **Steven Leth**, University of Northern Colorado.

Arithmetic Dynamics, **Andrew Bridy**, Yale University, **Michelle Manes**, University of Hawai'i at Mānoa, and **Bianca Thompson**, Harvey Mudd College.

Arithmetic Geometry and Its Connections, **Laura Capuano**, University of Oxford, and **Amos Turchet**, University of Washington.

Arithmetic and Transcendence of Special Functions and Special Values, **Matthew A. Papanikolas**, Texas A&M University, and **Federico Pellarin**, Université Jean Monnet, St. Étienne.

Coarse Geometry, Index Theory, and Operator Algebras: Around the Mathematics of John Roe, **Erik Guentner**, University of Hawai'i at Mānoa, **Nigel Higson**, Penn State University, and **Rufus Willett**, University of Hawai'i at Mānoa.

Coding Theory and Information Theory, **Manabu Hagiwara**, Chiba University, and **James B. Nation**, University of Hawai'i.

Combinatorial and Experimental Methods in Mathematical Phylogeny, **Sean Cleary**, City College of New York and the CUNY Graduate Center, and **Katherine St. John**, Hunter College and the American Museum of Natural History.

Commutative Algebra and its Environs, **Olgur Celikbas** and **Ela Celikbas**, West Virginia University, and **Ryo Takahashi**, Nagoya University.

MEETINGS & CONFERENCES

Computability, Complexity, and Learning, **Achilles A. Beros** and **Bjørn Kjos-Hanssen**, University of Hawai'i at Mānoa.
Computational and Data-Enabled Sciences, **Roummel Marcia**, **Boaz Ilan**, and **Suzanne Sindi**, University of California, Merced.

Constructive Aspects of Complex Analysis, **Ilia Binder** and **Michael Yampolsky**, University of Toronto, and **Malik Younsi**, University of Hawai'i at Mānoa.

Differential Geometry, **Vincent B. Bonini**, Cal Poly San Luis Obispo, **Jie Qing**, University of California, Santa Cruz, and **Bogdan D. Suceava**, California State University, Fullerton.

Dynamical Systems and Algebraic Combinatorics, **Maxim Arnold**, University of Texas at Dallas, **Jessica Striker**, North Dakota State University, and **Nathan Williams**, University of Texas at Dallas.

Emerging Connections with Number Theory, I, **Katherine Stange**, University of Colorado, Boulder, and **Renate Scheidler**, University of Calgary.

Equivariant Homotopy Theory and Trace Methods, **Andrew Blumberg**, University of Texas, **Teena Gerhardt**, Michigan State University, **Michael Hill**, UCLA, and **Michael Mandell**, Indiana University.

Factorization and Arithmetic Properties of Integral Domains and Monoids, **Scott Chapman**, Sam Houston State University, **Jim Coykendall**, Clemson University, and **Christopher O'Neill**, University of California, Davis.

Generalizations of Symmetric Spaces, **Aloysius Helminck**, University of Hawai'i at Mānoa, **Vicky Klima**, Appalachian State University, **Jennifer Schaefer**, Dickinson College, and **Carmen Wright**, Jackson State University.

Geometric Approaches to Mechanics and Control, **Monique Chyba**, University of Hawai'i at Mānoa, **Tomoki Ohsawa**, The University of Texas at Dallas, and **Vakhtang Putkaradze**, University of Alberta.

Geometry, Analysis, Dynamics and Mathematical Physics on Fractal Spaces, **Joe P. Chen**, Colgate University, **Lü(Tim) Hùng**, Hawai'i Pacific University, **Machiel van Frankenhuysen**, Utah Valley University, and **Robert G. Niemeyer**, University of the Incarnate Word.

Homotopy Theory, **Kyle Ormsby** and **Angélica Osorno**, Reed College.

Interactions between Geometric Measure Theory, PDE, and Harmonic Analysis, **Mark Allen**, Brigham Young University, **Spencer Becker-Kahn**, University of Washington, **Max Engelstein**, Massachusetts Institute of Technology, and **Mariana Smit Vega Garcia**, University of Washington.

Interactions between Noncommutative Algebra and Noncommutative Algebraic Geometry, **Garrett Johnson**, North Carolina Central University, **Bach Nguyen** and **Xingting Wang**, Temple University, and **Daniel Yee**, Bradley University.

Lie Theory in the Representations of Groups and Related Structures - dedicated to the memory of Kay Magaard, **Christopher Drupieski**, DePaul University, and **Julia Pevtsova**, University of Washington.

Mapping Class Groups, **Asaf Hadari**, University of Hawai'i, and **Jing Tao**, University of Oklahoma.

Mathematical Analysis of Nonlinear Phenomena, **Mimi Dai**, University of Illinois at Chicago.

Mathematical Methods and Models in Medicine, **Monique Chyba**, University of Hawai'i, and **Jakob Kotas**, University of Hawai'i and University of Portland.

New Trends in Geometric Measure Theory, **Antonio De Rosa**, Courant Institute of Mathematical Sciences, New York University, and **Luca Spolaor**, Massachusetts Institute of Technology.

New Trends on Variational Calculus and Non-Linear Partial Differential Equations, **Craig Cowan**, University of Manitoba, **Michinori Ishiwata**, Osaka University, **Abbas Moameni**, Carleton University, and **Futoshi Takahashi**, Osaka City University.

Nonlinear Wave Equations and Applications, **Boaz Ilan**, University of California, Merced, and **Barbara Prinari**, University of Colorado, Colorado Springs.

Numerical Methods for Partial Differential Equations, **Evan Gawlik**, **Michael Holst**, and **Martin Licht**, University of California, San Diego.

Real and Complex Singularities, **Leslie Charles Wilson**, University of Hawai'i, Mānoa, **Goo Ishikawa**, Hokkaido University, and **David Trotman**, Aix-Marseille University.

Recent Advances and Applications of Modular Forms, **Amanda Folsom**, Amherst College, **Pavel Guerzhoy**, University of Hawai'i at Mānoa, **Masanobu Kaneko**, Kyushu University, and **Ken Ono**, Emory University.

Recent Advances in Lie and Related Algebras and their Representations, **Brian D. Boe**, University of Georgia, and **Jonathan Kujawa**, University of Oklahoma.

Recent Advances in Numerical Methods for PDEs, **Hengguang Li**, Wayne State University, and **Sara Pollock**, University of Florida.

Recent Developments in Automorphic Forms, **Solomon Friedberg**, Boston College, and **Jayce Getz**, Duke University.

Recent Trends in Algebraic Graph Theory, **Sebastian Cioaba**, University of Delaware, and **Shaun Fallat**, University of Regina.

SYZ Mirror Symmetry and Enumerative Geometry, **Siu Cheong Lau**, Boston University, **Naichung Leung**, The Chinese University of Hong Kong, and **Hsian-Hua Tseng**, Ohio State University.

Several Complex Variables, **Peter Ebenfelt**, University of California, San Diego, **John Erik Fornaess**, University of Michigan and Norwegian University of Science and Technology, **Ming Xiao**, University of California, San Diego, and **Yuan Yuan**, Syracuse University.

Spaces of Holomorphic Functions and Their Operators, **Mirjana Jovovic** and **Wayne Smith**, University of Hawai'i.

Sparsity, Randomness, and Optimization, **Deanna Needell**, University of California, Los Angeles.

Spectral Geometry: The Length and Laplace Spectra of Riemannian Manifolds, **Benjamin Linowitz**, Oberlin College, and **Jeffrey S. Meyer**, California State University at San Bernardino.

Stability and Singularity in Fluid Dynamics, **Tristan Buckmaster**, Princeton University, **Steve Shkoller**, University of California, Davis, and **Vlad Vicol**, Princeton University.

Structural Graph Theory, **Zixia Song**, University of Central Florida, **Martin Rolek**, College of William and Mary, and **Yue Zhao**, University of Central Florida.

The Mathematics of Cryptography, **Shahed Sharif**, California State University, San Marcos, and **Alice Silverberg**, University of California, Irvine.

Three-dimensional Floer Theory, Contact Geometry, and Foliations, **Joan Licata**, Australian National University, and **Robert Lipshitz**, University of Oregon.

Topics at the Interface of Analysis and Geometry, **Alex Austin** and **Sylvester Eriksson-Bique**, University of California, Los Angeles.

Valuations on Algebraic Function Fields and Their Subrings, **Ron Brown**, University of Hawai'i, **Steven Dale Cutkosky**, University of Missouri, and **Franz-Viktor Kuhlmann**, University of Szczecin.

What is Happening in Mathematical Epidemiology? Current Theory, New Methods, and Open Questions, **Olivia Prosper**, University of Kentucky.

Hartford, Connecticut

University of Connecticut Hartford (Hartford Regional Campus)

April 13–14, 2019

Saturday – Sunday

Meeting #1148

Eastern Section

Associate secretary: Steven H. Weintraub

Announcement issue of *Notices*: February 2019

Program first available on AMS website: February 21, 2019

Issue of *Abstracts*: Volume 40, Issue 2

Deadlines

For organizers: Expired

For abstracts: Expired

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtg/sectional.html.

Invited Addresses

Brian C Hall, University of Notre Dame, *Eigenvalues of random matrices in the general linear group in the large- N limit*.

Christina Sormani, Lehman College and CUNYGC, *Compactness theorems for sequences of Riemannian Manifolds*.

Olivier Bernardi, Brandeis University, *Percolation on triangulations, and a bijective path to Liouville quantum gravity*.

Special Sessions

Algebraic Number Theory, **Harris Daniels**, Amherst College, and **Alvaro Lozano-Robledo** and **Erik Wallace**, University of Connecticut.

Analysis, Geometry, and PDEs in Non-smooth Metric Spaces, **Vyron Vellis**, University of Connecticut, **Xiaodan Zhou**, Worcester Polytechnic Institute, and **Scott Zimmerman**, University of Connecticut.

Banach Space Theory and Metric Embeddings, **Mikhail Ostrovskii**, St. John's University, and **Beata Randrianantoanina**, Miami University.

Chip-firing and Divisor Theory, **Caroline Klivans**, Brown University, and **David Perkinson**, Reed College.

Cluster Algebras and Related Topics, **Emily Gunawan** and **Ralf Schiffler**, University of Connecticut.

Combinatorial Commutative Algebra and Polyhedral Geometry, **Elie Alhajjar**, US Military Academy, and **McCabe Olsen**, Ohio State University.

Computability Theory, **Damir Dzhafarov** and **Reed Solomon**, University of Connecticut, and **Linda Brown Westrick**, Pennsylvania State University.

Convergence of Riemannian Manifolds, **Lan-Hsuan Huang** and **Maree Jaramillo**, University of Connecticut, and **Christina Sormani**, City University of New York Graduate Center and Lehman College.

Discrete Dynamical Systems and Applications, **Elliott J. Bertrand**, Sacred Heart University, and **David McArdle**, University of Connecticut.

Invariants of Knots, Links, and Low-dimensional Manifolds, **Patricia Cahn**, Smith College, and **Moshe Cohen** and **Adam Lowrance**, Vassar College.

Knot Theory, the Colored Jones Polynomial, and Khovanov Homology, **Adam Giambrone**, Elmira College, and **Katherine Hall**, University of Connecticut.

Mathematical Cryptology, **Lubjana Beshaj**, United States Military Academy, and **Jaime Gutierrez**, University of Cantabria, Santander, Spain.

Mathematical Finance, **Oleksii Mostovyi**, University of Connecticut, **Gu Wang**, Worcester Polytechnic Institute, and **Bin Zhou**, University of Connecticut.

Modeling and Qualitative Study of PDEs from Materials Science and Geometry., **Yung-Sze Choi**, **Changfeng Gui**, and **Xiaodong Yan**, University of Connecticut.

Recent Advances in Structured Matrices and Their Applications, **Maxim Derevyagin**, University of Connecticut, **Olga Holz**, University of California, Berkeley, and **Vadim Olshevsky**, University of Connecticut.

Recent Development of Geometric Analysis and Nonlinear PDEs, **Ovidiu Munteanu**, **Lihan Wang**, and **Ling Xiao**, University of Connecticut.

Representation Theory of Quantum Algebras and Related Topics, **Drew Jaramillo**, University of Connecticut, **Garrett Johnson**, North Carolina Central University, and **Margaret Rahmoeller**, Roanoke College.

Special Session on Regularity Theory of PDEs and Calculus of Variations on Domains with Rough Boundaries, **Murat Akman**, University of Connecticut, and **Zihui Zhao**, University of Washington.

Special Values of L-functions and Arithmetic Invariants in Families, **Ellen Eischen**, University of Oregon, **Yifeng Liu**, Yale University, **Liang Xiao**, University of Connecticut, and **Wei Zhang**, Massachusetts Institute of Technology.

Stochastic Analysis and Related Fields, **Fabrice Baudoin**, University of Connecticut, and **Cheng Ouyang**, University of Illinois at Chicago.

Stochastic Processes, Random Walks, and Heat Kernels, **Patricia Alonso Ruiz**, University of Connecticut, and **Phanuel Mariano**, Purdue University.

Sub-Riemannian and CR Geometric Analysis, **Fabrice Baudoin**, University of Connecticut, and **Luca Capogna**, Worcester Polytechnic Institute.

Quy Nhon, Vietnam

International Centre for Interdisciplinary Science and Education (ICISE)

June 10–13, 2019

Monday – Thursday

Meeting #1149

Associate secretary: Brian D. Boe
Announcement issue of *Notices*: April 2019

Deadlines

For organizers: Expired
For abstracts: April 16, 2019

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/internmtgs.html.

Invited Addresses

Henry Cohn, Microsoft Research, *Sphere packing, Fourier interpolation, and ground states in 8 and 24 dimensions.*

Robert Guralnick, University of Southern California, *Fixed point free permutations and applications.*

Le Tuan Hoa, Institute of Mathematics, Vietnam Academy of Science and Technology, *On the complexity of polynomial systems.*

Nguyen Dong Yen, Institute of Mathematics, Vietnam Academy of Science and Technology, Hanoi, Vietnam, *Second-Order Tangent Sets and Second-Order Subdifferentials.*

Zhiwei Yun, Massachusetts Institute of Technology, *Endoscopy for Hecke categories and character sheaves.*

Nguyen Tien Zung, Toulouse Mathematics Institute, *On a universal conservation law in dynamics*.

Special Sessions

Algebraic Topology, **Nguyen Huu Viet Hung**, Vietnam National University, Hanoi (Vietnam), **Le Minh Ha**, Vietnam Institute for Advanced Study in Mathematics (Vietnam), and **Dev Sinha**, University of Oregon (USA).

Applied and Industrial Mathematics, **John Birge**, University of Chicago (USA), and **Vu Hoang Linh**, Vietnam National University, Hanoi (Vietnam).

Arithmetic Algebraic Geometry and related topics, **Phung Ho Hai**, Institute of Mathematics, Vietnam Academy of Science and Technology (Vietnam), **Le Hung Viet Bao**, Northwestern University (USA), and **Ngo Duc Tuan**, CNRS and University of Caen Normandy (France).

Commutative Algebra and Its Interactions to Combinatorics, **Le Tuan Hoa**, Vietnam Academy of Science & Technology (Vietnam), and **Ha Huy Tai**, Tulane University (USA).

Complex Geometry and Dynamical Systems, **Pham Hoang Hiep**, Vietnam Academy of Science and Technology (Vietnam), **Daniel Burns**, University of Michigan (USA), **Tien-Cuong Dinh**, National University of Singapore (Singapore), and **Thai Thuan Quang**, Quy Nhon University (Vietnam).

Discrete Mathematics, **Phan Thi Ha Duong**, Vietnam Academy of Science & Technology (Vietnam), and **Vu Ha Van**, Yale University (USA).

Formal Mathematics, **Thomas C. Hales**, University of Pittsburgh (USA), and **Tran Nam Trung**, Vietnam Academy of Science & Technology (Vietnam).

Geometry and Physics, **Nguyen Tien Zung**, Toulouse Mathematics Institute (France), and **Tudor Ratiu**, Shanghai Jiao Tong University (China).

Groups, Representations and Applications, **Pham Huu Tiep**, Rutgers University (USA/Vietnam), and **Robert M. Guralnick**, University of Southern California (USA).

Homological Methods in the Representation Theory of Groups and Algebras, **Daniel K. Nakano**, University of Georgia (USA), **Jon Carlson**, University of Georgia (USA), and **Ngo Vo Nham**, University of North Georgia (USA).

Optimization and Variational Analysis, **Nguyen Dong Yen**, Vietnam Academy of Science & Technology (Vietnam), **Phan Quoc Khanh**, Vietnam National University, Ho Chi Minh City (Vietnam), and **Huynh Van Ngai**, Quy Nhon University (Vietnam).

Singularities and Algebraic Geometry, **Le Quy Thuong**, Vietnam National University, Hanoi (Vietnam), **Nero Budur**, KU Leuven (Belgium), and **Pho Duc Tai**, Vietnam National University, Hanoi (Vietnam).

Value Distribution Theory, Complex Geometry, Diophantine Approximation, and Related Topics, **William Cherry**, University of North Texas (USA), **Ta Thi Hoai An**, Vietnam Academy of Science & Technology (Vietnam), and **Min Ru**, University of Houston (USA).

This announcement was composed with information taken from the website maintained by the local organizers at vnus2019.viasm.edu.vn. Please watch this website for the most up-to-date information.

Abstract Submissions

Abstracts must be submitted online at <https://vnus2019.vasm.edu.vn>. The deadline for abstract submissions is April 16, 2019.

Accommodations

The Vietnam Mathematical Society has suggested some hotels for participants of the meeting while they are in Quy Nhon. Should participants choose to utilize a room at one of these properties, all arrangements should be made via the local organizer's website. A list of suggested properties can be found here: vnus2019.viasm.edu.vn/article/accommodation-11.

Please note, hotel reservations are only valid after a meeting registration fee has been collected via the online registration form located here: vnus2019.viasm.edu.vn.

Khách Sạn Hải Âu (Seagull) Hotel (**)** 489 An Duong Vuong Street, Quy Nhon City, Binh Dinh Province, Vietnam, Tel : (84-56) 3 846 377; seagullhotel.com.vn. (7km from ICISE, 12 km from Dieu Tri Station, and 37 km from Phu Cat Airport)

A limited number of rooms at Khách Sạn Hải Âu (Seagull) Hotel (****) will be reserved for meeting participants at a discounted rate. Please check the Vietnam-US Joint Meeting website for more details. Conveniently located 7 km away from the ICISE, Khách Sạn Hải Âu (Seagull) Hotel is a beachfront property offering contemporary rooms with sea views.

MEETINGS & CONFERENCES

The hotel features an outdoor pool and two on-site dining options. Guests can enjoy free WiFi access in all areas of the hotel as well as free private parking.

All rooms have air conditioning, a personal safe, cable TV and a mini-bar. Beautiful views of either the city or of the sea can be enjoyed from private balconies. Amenities also include a spa and fitness center. The hotel's front desk operates 24-hours a day and has a tour desk to assist guests with currency exchange, day tour arrangements and airport transfer services. Khách Sạn Hải Âu (Seagull) Hotel's Salagane Restaurant serves an array of Asian and European dishes, while Royal Restaurant features international cuisine.

One room can be reserved per person. Each room has either one double bed or two single beds.

A round trip shuttle bus will be arranged between Khách Sạn Hải Âu (Seagull) Hotel and ICISE for each day of the conference.

The local organizers for the Vietnam–US Joint Meeting also suggest the following hotels:

Royal Hotel and Healthcare Resort Quy Nhon, 01 Han Mac Tu Street, Ghenh Rang Ward, Quy Nhon, Vietnam, Tel: +84 256 3747 100; royalquynhon.com. The Royal Hotel and Healthcare Resort Quy Nhon is 7km from the ICISE.

Sai Gon Quy Nhon Hotel, 24 Nguyen Hue Street, Le Loi Ward, Quy Nhon City, Binh Dinh Province, Tel +84 2563 829922; saigonquynhonhotel.com.

For more information regarding hotels near ICISE, please visit vnus2019.viasm.edu.vn.

Local Information /Tourism

The Vietnam-US Joint Meeting will take place at the International Centre for Interdisciplinary Science and Education (ICISE). Just a few kilometers away from the city centre of Quy Nhon, capital of Binh Dinh province, ICISE is located on 50 acres of park land between the mountains and the sea. <https://www.icisequynhon.com/icise-about-us>.

The city of Quy Nhon is located on the east coast of Vietnam, just 45 minutes north, by air, from Ho-Chi-Minh City (Saigon) and 1.5 hours south of Hanoi. This economically growing city, still preserved from mass tourism, has retained its traditions including martial arts and reveals an authentic face of Vietnam.

Local tourism information and maps can be found at <https://vnus2019.viasm.edu.vn>. This site offers details on general travel information, accommodations, restaurants and transportation in Quy Nhon.

Vietnamese currency is called Vietnamese Dong, often abbreviated as VND. The denominations of paper notes include 500,000; 200,000; 100,000; 50,000; 20,000; 10,000; 5,000 and 2,000 Dong. At the time of publication of this announcement, the exchange rate was US\$1 equal to 23,204.50 Dong. Cash (foreign currency) can be exchanged upon arrival at the international airports in Hanoi and Ho Chi Minh City (Saigon). Banks will exchange money and travelers cheques. Banking hours are typically Mondays through Fridays, 9 am to 5 pm.

Plan to bring a number of payment options on your Vietnamese trip for peace of mind. Credit cards are good for bigger purchases, but cash is what you'll need the most. Major credit cards can be accepted at most hotels, tourist shops and some department stores. You should notify your bank of your international travel, and the potential legitimate use of your card abroad, prior to leaving your home country. It is highly recommended that you exchange your money before you arrive in Vietnam.

Vietnam's electrical current is supplied at 220 volts, 50 cycles. The most common plug types are the dual and three-pointed prongs, which are different from the western plugs. Please note that Vietnam runs on 220 volts, which will burn 110-volt appliances.

Please visit the Vietnam–US Joint Meeting website vnus2019.viasm.edu.vn/article/useful-information-15 for information on the area's dining and attractions as well as other useful information such as emergency contacts while traveling.

Registration and Meeting Information

Please visit vnus2019.viasm.edu.vn for information regarding registration for the Vietnam–US Joint Meeting, as well as the conference schedule and specific locations at ICISE.

Special Needs

It is the goal of the AMS to ensure that its conferences are accessible to all, regardless of disability. The AMS will strive, unless it is not practicable, to choose venues that are fully accessible to the physically handicapped.

If special needs accommodations are necessary in order for you to participate in an AMS international meeting, please communicate your needs in advance to the local organizers.

AMS Policy on a Welcoming Environment

The AMS strives to ensure that participants in its activities enjoy a welcoming environment. In all its activities, the AMS seeks to foster an atmosphere that encourages the free expression and exchange of ideas. The AMS supports equality of opportunity and treatment for all participants, regardless of gender, gender identity, or expression, race, color, national or ethnic origin, religion or religious belief, age, marital status, sexual orientation, disabilities, or veteran status.

The AMS thanks our hosts for their gracious hospitality.

Travel

In order to welcome you to Quy Nhon and to organize your transfer from Phu Cat Airport, train or bus station to the conference hotel, please provide your travel details to the conference secretariat. The travel details that you will need to provide include your flight, train, or bus number as well as the date and the arrival time in Quy Nhon.

By Air: Quy Nhon can be reached by plane from Hanoi and from Ho Chi Minh City (Saigon). Arriving from abroad, the second alternative is recommended because of the fare and the time duration of the journey (1 hour from Ho Chi Minh City instead of 1.5 hours from Hanoi).

Phu Cat Airport is situated in Cat Tan commune, Phu Cat district, Binh Dinh province, about 35 km far from the northwest of Quy Nhon city center. Flights to Quy Nhon are currently operated by the domestic airlines: Vietnam Airlines, Vietjet, Jetstar, and Bamboo Airways with the frequency of up to 10 flights per day from Ho Chi Minh City, and up to 3 flights per day from Hanoi.

From Phu Cat Airport, travellers can take either a taxi or the shuttle bus to Quy Nhon City Center. The price for a 4-seat taxi is approximately US\$12 and the price for the shuttle bus is approximately US\$2 per person.

Vietnam Airlines <https://www.vietnamairlines.com/vn/en/Home>;

Vietjet air <https://www.vietjetair.com/Sites/Web/en-US/Home>;

Jetstar Pacific <https://www.jetstar.com/us/en/home>;

Bamboo Airways <https://www.bambooairways.com/en>.

By Train: The railway from Ho Chi Minh (Saigon) station to Dieu Tri (Binh Dinh province) station is about 630 Kilometers. There are 4 Reunification Express trains including SE2, SE4, SE6 and SE8 on this route. These depart daily and take about 11 to 12 hours one-way from Saigon to Dieu Tri Dieu Tri station (Binh Dinh Province). There are several seat types to purchase ranging from US\$35–\$50, one-way.

The railway from Hanoi station to Dieu Tri station (Binh Dinh Province) is 1096 Kilometers. There are 4 Reunification Express trains including SE1, SE3, SE5 and SE7 on this route departing daily. This trip is 20–23 hours, one-way, and requires changing trains. There are several seat types to purchase ranging US\$45–\$80, one way. <https://vietnam-railway.com>.

By Bus: It is possible to travel by bus from Ho Chi Minh City (12–13 hours, one-way) for US\$11–\$14 person and from Hanoi (22–25 hours, one-way) for US\$30–\$60 per person.

There will be a complimentary shuttle service from Quy Nhon City Center to ICISE during the conference.

Local Transportation

Taxi Service: Taxis in Quy Nhon and other provinces in Vietnam are charged by the number of kilometers on the meter. The taxi call-center operators do not speak English and most likely will not understand the pronunciation of the street names of a non-Vietnamese speaking customer. Please ask a receptionist at your hotel to place the call on your behalf.

Sun Taxi: +84 56 368 6868

Taxi Mai Linh: +84. 3838 3838

Taxi Chí Thành: +84 56 3827888

Taxi Dân: +84 56 3818881

Taxi Hoàng Anh: +84 56 3525525

Local Bus Service: Public buses in Quy Nhon are often crowded and time consuming, but serve as an inexpensive way to travel. Bus hours are: 5:30 am–7:00 pm. Buses that stop near ICISE are bus numbers: T2; T5; T7; T8; T9; T12.

Weather

Quy Nhon is quite warm in June with high temperatures around 93°F (34°C) and lows around 80°F (26°C). June is one of the less humid months in Quy Nhon where humidity is usually about 63 percent. June is typically one of the months in Quy Nhon with least amount of precipitation although visitors should be prepared for inclement weather and check weather forecasts in advance of their arrival.

Information for International Participants:

Visitors to Vietnam should obtain a visa prior to their travel unless they are bearing a nationality which has visa exemption agreements with Vietnam. A Visa page will be issued at the Vietnamese consulates in US or at the entry airport in Vietnam.

For some countries eligible for electronic visas, foreigners can apply for a tourist visa online via an official website of the Vietnam Immigration Department.

Local organizers recommend that all participants apply for a business visa and will provide an approval letter for the application process.

Please visit the Vietnam-USA Joint Mathematical Meeting website for more information regarding obtaining a VISA for the meeting. vnus2019.viasm.edu.vn/article/visa-useful-information-8.

Madison, Wisconsin

University of Wisconsin–Madison

September 14–15, 2019

Saturday – Sunday

Meeting #1150

Central Section

Associate secretary: Georgia Benkart

Announcement issue of *Notices*: June 2019

Program first available on AMS website: July 23, 2019

Issue of *Abstracts*: Volume 40, Issue 3

Deadlines

For organizers: Expired

For abstracts: July 16, 2019

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtg/sectional.html.

Invited Addresses

Nathan Dunfield, University of Illinois, Urbana-Champaign, *Title to be announced*.

Teena Gerhardt, Michigan State University, *Title to be announced*.

Lauren Williams, University of California, Berkeley, *Title to be announced* (Erdős Memorial Lecture).

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at www.ams.org/cgi-bin/abstracts/abstract.pl.

Algebraic and Geometric Combinatorics (Code: SS 12A), **Benjamin Braun**, University of Kentucky, **Marie Meyer**, Lewis University, and **McCabe Olsen**, Ohio State University.

Analysis and Probability on Metric Spaces and Fractals (Code: SS 10A), **Guy C. David**, Ball State University, and **John Dever**, Bowling Green State University.

Arithmetic of Shimura Varieties (Code: SS 26A), **Chao Li**, Columbia University, and **Solly Parenti** and **Tonghai Yang**, University of Wisconsin–Madison.

Association Schemes and Related Topics – in Celebration of J.D.H. Smith's 70th Birthday (Code: SS 8A), **Kenneth W. Johnson**, Penn State University Abington, and **Sung Y. Song**, Iowa State University.

Automorphic Forms and L-Functions (Code: SS 16A), **Simon Marshall** and **Ruixiang Zhang**, University of Wisconsin–Madison.

Classical and Geophysical Fluid Dynamics: Modeling, Reduction and Simulation (Code: SS 17A), **Nan Chen**, University of Wisconsin–Madison, and **Honghu Liu**, Virginia Tech University.

Combinatorial Algebraic Geometry (Code: SS 21A), **Juliette Bruce** and **Daniel Erman**, University of Wisconsin–Madison, **Chris Eur**, University of California Berkeley, and **Lily Silverstein**, University of California Davis.

Commutative Algebra: in Celebration of the 150th Birthday of Roger and Sylvia Wiegand (Code: SS 22A), **Nicholas Baeth**, Franklin & Marshall College, and **Graham Leuschke**, Syracuse University.

Computability Theory in honor of Steffen Lempert's 60th birthday (Code: SS 6A), **Joseph S. Miller**, Noah D. Schweber, and **Mariya I. Soskova**, University of Wisconsin–Madison.

Connections between Noncommutative Algebra and Algebraic Geometry (Code: SS 15A), **Jason Gaddis** and **Dennis Keeler**, Miami University.

Extremal Graph Theory (Code: SS 14A), **Józef Balogh**, University of Illinois, and **Bernard Lidický**, Iowa State University.

Fully Nonlinear Elliptic and Parabolic Partial Differential Equations, Local and Nonlocal (Code: SS 25A), **Fernando Charro**, Wayne State University, **Stefania Patrizi**, The University of Texas at Austin, and **Peiyong Wang**, Wayne State University.

Geometry and Topology of Singularities (Code: SS 13A), **Laurentiu Maxim**, University of Wisconsin–Madison.

Hodge Theory in Honor of Donu Arapura's 60th Birthday (Code: SS 11A), **Ajneet Dhillon**, University of Western Ontario, **Kenji Matsuki** and **Deepam Patel**, Purdue University, and **Botong Wang**, University of Wisconsin–Madison.

Homological and Characteristic $p > 0$ Methods in Commutative Algebra (Code: SS 1A), **Michael Brown**, University of Wisconsin–Madison, and **Eric Canton**, University of Michigan.

Lie Representation Theory (Code: SS 19A), **Mark Colarusso**, University of South Alabama, **Michael Lau**, Université Laval, and **Matt Ondrus**, Weber State University.

Model Theory (Code: SS 5A), **Uri Andrews** and **Omer Mermelstein**, University of Wisconsin–Madison.

Recent Developments in Harmonic Analysis (Code: SS 3A), **Theresa Anderson**, Purdue University, and **Joris Roos**, University of Wisconsin–Madison.

Recent Work in the Philosophy of Mathematics (Code: SS 4A), **Thomas Drucker**, University of Wisconsin–Whitewater, and **Dan Slougher**, Furman University.

Several Complex Variables (Code: SS 7A), **Hanlong Fang** and **Xianghong Gong**, University of Wisconsin–Madison.

Special Functions and Orthogonal Polynomials (Code: SS 2A), **Sarah Post**, University of Hawai'i at Mānoa, and **Paul Terwilliger**, University of Wisconsin–Madison.

Topics in Graph Theory and Combinatorics (Code: SS 20A), **Songling Shan** and **Papa Sissokho**, Illinois State University.

Topology and Descriptive Set Theory (Code: SS 18A), **Tetsuya Ishi** and **Paul B. Larson**, Miami University.

Uncertainty Quantification Strategies for Physics Applications (Code: SS 9A), **Qin Li**, University of Wisconsin–Madison, and **Tulin Kaman**, University of Arkansas.

Wave Phenomena in Fluids and Relativity (Code: SS 24A), **Sohrab Shahshahani**, University of Massachusetts, and **Willie W.Y. Wong**, Michigan State University.

Zero Forcing, Propagation, and Throttling (Code: SS 23A), **Josh Carlson**, Iowa State University, and **Nathan Warnberg**, University of Wisconsin–La Crosse.

Fully Nonlinear Elliptic and Parabolic Partial Differential Equations, Local and Nonlocal (Code: SS 25A), **Fernando Charro**, Wayne State University, **Stefania Patrizi**, The University of Texas at Austin, and **Peiyong Wang**, Wayne State University.

Arithmetic of Shimura Varieties (Code: SS 26A), **Chao Li**, Columbia University, **Solly Parenti**, University of Wisconsin–Madison, and **Tonghai Yang**, University of Wisconsin–Madison.

Hall Algebras, Cluster Algebras and Representation Theory (Code: SS 27A), **Xueqing Chen**, UW–Whitewater and **Yiqiang Li**, SUNY at Buffalo.

Stochastic Partial Differential Equations and Related Fields (Code: SS 28A), **Igor Cialenco**, Illinois Institute of Technology, **Yu Gu**, Carnegie Mellon University, and **Hyun-Jung Kim**, Illinois Institute of Technology.

Floer Homology in Dimensions 3 and 4 (Code: SS 29A), **Jianfeng Lin**, UC San Diego and **Christopher Scaduto**, University of Miami.

Functional Analysis and Its Applications (Code: SS 30A), **Clement Boateng Ampadu**, Boston, MA and **Waleed Al-Rawashdeh**, Montana Tech University.

Connecting Network Structure and Behavior of Biological Interaction Systems (Code: SS 31A), **David Anderson**, University of Wisconsin–Madison, **Gheorghe Craciun**, University of Wisconsin–Madison, and **Abhishek Deshpande**, University of Wisconsin–Madison.

Relations Between the History and Pedagogy of Mathematics (Code: SS 32A), **Emily Redman**, University of Massachusetts, Amherst, **Brit Shields**, University of Pennsylvania, and **Rebecca Vinsonhaler**, University of Texas, Austin.

Large Scale Properties of Interacting Stochastic Systems (Code: SS 33A), **Timo Seppäläinen**, University of Wisconsin–Madison, **Hao Shen**, University of Wisconsin–Madison, and **Benedek Valko**, University of Wisconsin–Madison.

Homotopy Theory, Gabe Angelini-Knoll (Code: SS 34A), Michigan State University, **Teena Gerhardt**, Michigan State University, and **Bertrand Guillou**, University of Kentucky.

Quasigroups and Loops—in honor of J.D.H. Smith's 70th birthday (Code: SS 35A), **J.D. Phillips**, Northern Michigan University and **Petr Vojtechovsky**, University of Denver.

Supergeometry, Poisson Brackets, and Homotopy Structures (Code: SS 36A), **Ekaterina Shemyakova**, University of Toledo and **Theodore Voronov**, University of Manchester.

Nonlinear Dispersive Equations and Water Waves (Code: SS 37A), **Mihaela Ifrim**, University of Wisconsin–Madison and **Daniel Tataru**, University of California, Berkeley.

Applications of Algebra and Geometry (Code: SS 38A), **Shamgar Gurevich** and **Jose Israel Rodriguez**, University of Wisconsin–Madison.

Recent Trends in the Mathematics of Data (Code: SS 39A), **Sebastien Roch**, University of Wisconsin–Madison, **David Sivakoff**, Ohio State University, and **Joseph Watkins**, University of Arizona.

Number Theory and Cryptography (Code: SS 40A), **Eric Bach**, University of Wisconsin–Madison and Jon Sorenson, Butler University.

Geometry and Topology in Arithmetic (Code: SS 41A), **Rachel Davis**, University of Wisconsin–Madison.

Binghamton, New York

Binghamton University

October 12–13, 2019

Saturday – Sunday

Meeting #1151

Eastern Section

Associate secretary: Steven H. Weintraub

Announcement issue of *Notices*: August 2019

Program first available on AMS website: August 29, 2019

Issue of *Abstracts*: Volume 40, Issue 3

Deadlines

For organizers: Expired

For abstracts: August 20, 2019

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtg/sectional.html.

Invited Addresses

Richard Kenyon, Brown University, *Title to be announced*.

Tony Pantev, University of Pennsylvania, *Title to be announced*.

Lai-Sang Young, New York University, *Title to be announced*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at www.ams.org/cgi-bin/abstracts/abstract.pl.

Groups and Their Representations (Code: SS 1A), **Jamison Barsotti** and **Rob Carman**, College of William and Mary, and **Daniel Rossi** and **Hung P. Tong-Viet**, Binghamton University.

Representations of Lie algebras, Vertex Operators, and Related Topics (Code: SS 2A), **Alex Feingold**, Binghamton University, and **Christopher Sadowski**, Ursinus College.

Gainesville, Florida

University of Florida

November 2–3, 2019

Saturday – Sunday

Meeting #1152

Southeastern Section

Associate secretary: Brian D. Boe

Announcement issue of *Notices*: September 2019

Program first available on AMS website: September 19, 2019

Issue of *Abstracts*: Volume 40, Issue 4

Deadlines

For organizers: April 2, 2019

For abstracts: September 10, 2019

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Jonathan Mattingly, Duke University, *Title to be announced.*

Isabella Novik, University of Washington, *Title to be announced.*

Eduardo Teixeira, University of Central Florida, *Title to be announced.*

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at www.ams.org/cgi-bin/abstracts/abstract.pl.

Combinatorial Lie Theory (Code: SS 3A), **Erik Insko**, Florida Gulf Coast University, **Martha Precup**, Washington University in St. Louis, and **Edward Richmond**, Oklahoma State University.

Fractal Geometry and Dynamical Systems (Code: SS 2A), **Mrinal Kanti Roychowdhury**, University of Texas Rio Grande Valley.

Geometric and Topological Combinatorics (Code: SS 1A), **Bruno Benedetti**, University of Miami, **Steve Klee**, Seattle University, and **Isabella Novik**, University of Washington.

Riverside, California

University of California, Riverside

November 9–10, 2019

Saturday – Sunday

Meeting #1153

Western Section

Associate secretary: Michel L. Lapidus

Announcement issue of *Notices*: September 2019

Program first available on AMS website: September 12, 2019

Issue of *Abstracts*: Volume 40, Issue 4

Deadlines

For organizers: April 9, 2019

For abstracts: September 3, 2019

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Mohsen Aliabadi, University of Illinois at Chicago, Chicago, IL, *A connection between matchings in field extensions and the fundamental theorem of algebra.*

Jonathan Novak, University of California, San Diego, *Title to be announced.*

Anna Skripka, University of New Mexico, Albuquerque, *Title to be announced.*

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at <http://www.ams.org/cgi-bin/abstracts/abstract.pl>.

MEETINGS & CONFERENCES

Geometric Partial Differential Equations and Variational Methods (Code: SS 4A), **Longzhai Lin**, University of California, Santa Cruz, **Xiangwen Zhang**, University of California, Irvine, and **Xin Zhou**, University of California, Santa Barbara.

Inverse Problems (Code: SS 3A), **Hanna Makaruk**, Los Alamos National Laboratory, and **Robert Owczarek**, University of New Mexico, Albuquerque and University of New Mexico, Los Alamos.

Random Matrices and Related Structures (Code: SS 2A), **Jonathan Novak**, University of California, San Diego, and **Karl Liechty**, De Paul University.

Topics in Operator Theory (Code: SS 1A), **Anna Skripka** and **Maxim Zinchenko**, University of New Mexico.

Denver, Colorado

Colorado Convention Center

January 15–18, 2020

Wednesday – Saturday

Meeting #1154

Joint Mathematics Meetings, including the 126th Annual Meeting of the AMS, 103rd Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with

sessions contributed by the Society for Industrial and Applied Mathematics (SIAM)

Associate secretary: Michel L. Lapidus

Announcement issue of *Notices*: October 2019

Program first available on AMS website: November 1, 2019

Issue of *Abstracts*: To be announced

Deadlines

For organizers: April 2, 2019

For abstracts: To be announced

Charlottesville, Virginia

University of Virginia

March 13–15, 2020

Friday – Sunday

Southeastern Section

Associate secretary: Brian D. Boe

Announcement issue of *Notices*: To be announced

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced

For abstracts: To be announced

The scientific information listed below may be dated. For the latest information, see www.ams.org/amsmtg/sectional.html.

Invited Addresses

Moon Duchin, Tufts University, *Title to be announced*. (Einstein Public Lecture in Mathematics)

Isabella Novik, University of Washington, *Title to be announced*

Eduardo Teixeira, University of Central Florida, *Title to be announced*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at <http://www.ams.org/cgi-bin/abstracts/abstract.pl>.

Curves, Jacobians, and Abelian Varieties (Code: SS 1A), **Andrew Obus**, Baruch College (CUNY), **Tony Shaska**, Oakland University, and **Padmavathi Srinivasan**, Georgia Institute of Technology.

Medford, Massachusetts

Tufts University

March 21–22, 2020

Saturday – Sunday

Eastern Section

Associate secretary: Steven H. Weintraub

Announcement issue of *Notices*: To be announced

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced

For abstracts: To be announced

West Lafayette, Indiana

Purdue University

April 4–5, 2020

Saturday – Sunday

Central Section

Associate secretary: Georgia Benkart

Announcement issue of *Notices*: To be announced

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced

For abstracts: To be announced

Fresno, California

California State University, Fresno

May 2–3, 2020

Saturday – Sunday

Western Section

Associate secretary: Michel L. Lapidus

Announcement issue of *Notices*: To be announced

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced

For abstracts: To be announced

El Paso, Texas

University of Texas at El Paso

September 12–13, 2020

Saturday – Sunday

Central Section

Associate secretary: Georgia Benkart

Announcement issue of *Notices*: To be announced

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced

For abstracts: To be announced

State College, Pennsylvania

Pennsylvania State University, University Park Campus

October 3–4, 2020

Saturday – Sunday

Eastern Section

Associate secretary: Steven H. Weintraub

Announcement issue of *Notices*: To be announced

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced

For abstracts: To be announced

Salt Lake City, Utah

University of Utah

October 24–25, 2020

Saturday – Sunday

Western Section

Associate secretary: Michel L. Lapidus

Announcement issue of *Notices*: To be announced

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced

For abstracts: To be announced

Washington, District of Columbia

Walter E. Washington Convention Center

January 6–9, 2021

Wednesday – Saturday

Joint Mathematics Meetings, including the 127th Annual Meeting of the AMS, 104th Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).

Associate secretary: Brian D. Boe

Announcement issue of *Notices*: October 2020

Program first available on AMS website: November 1, 2020

Issue of *Abstracts*: To be announced

Deadlines

For organizers: April 1, 2020

For abstracts: To be announced

Grenoble, France

Université Grenoble Alpes

July 5–9, 2021

Monday – Friday

Associate secretary: Michel L. Lapidus

Announcement issue of *Notices*: To be announced

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced

For abstracts: To be announced

Buenos Aires, Argentina

The University of Buenos Aires

July 19–23, 2021

Monday – Friday

Associate secretary: Steven H. Weintraub

Announcement issue of *Notices*: To be announced

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced

For abstracts: To be announced

Omaha, Nebraska

Creighton University

October 9–10, 2021

Saturday – Sunday

Central Section

Associate secretary: Georgia Benkart

Announcement issue of *Notices*: To be announced

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced

For abstracts: To be announced

Seattle, Washington

Washington State Convention Center and the Sheraton Seattle Hotel

January 5–8, 2022

Wednesday – Saturday

Associate secretary: Georgia Benkart

Announcement issue of *Notices*: October 2021

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced

For abstracts: To be announced

Boston, Massachusetts

John B. Hynes Veterans Memorial Convention Center, Boston Marriott Hotel, and Boston Sheraton Hotel

January 4–7, 2023

Wednesday – Saturday

Associate secretary: Steven H. Weintraub

Announcement issue of *Notices*: October 2022

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced

For abstracts: To be announced

Upcoming Features and Memorial Tributes

The Mathematics of Quantum-Enabled Applications on the D-Wave Quantum Computer

by Jesse J. Berwald

Memorial Tribute: L. Gaunce Lewis Jr. (1949–2006) and Mark Steinberger (1950–2018)

by J. Peter May

Algebraic, Geometric, and Topological Methods in Linear Optimization

by Jesús A. De Loera

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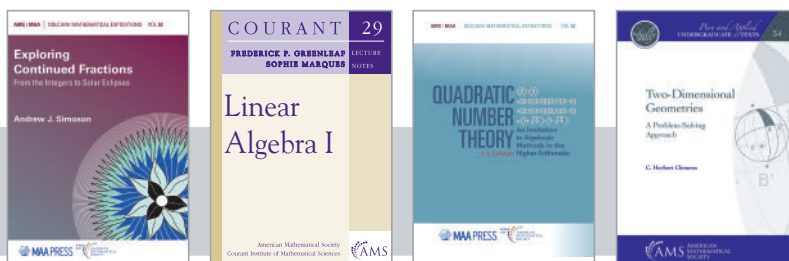
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Andrew J. Simoson, *King University, Bristol, TN*

This text explores recurrent phenomena, including astronomical transits and conjunctions, lifecycles of cicadas, and eclipses, by way of continued fraction expansions.

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J. L. Lehman, *University of Mary Washington, Fredericksburg, VA*

This text is an introduction to algebraic number theory for readers with a moderate knowledge of elementary number theory and some familiarity with the terminology of abstract algebra.

Dolciani Mathematical Expositions, Volume 52; 2019; 394 pages; Hardcover; ISBN: 978-1-4704-4737-3; List US\$52; AMS members US\$39; MAA members US\$39; Order code DOL/52

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