WHAT IS...

Falconer's Conjecture?

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The Statement of the Problem

Many problems in mathematics take the following form. Suppose that X, Y are sets and $f : X \to Y$ is a function. Suppose that X is sufficiently large and f is suitably nontrivial. Then $f(X)$ takes up a substantial portion of Y. A classical example of this phenomenon is Picard's Little Theorem, which says that any entire analytic function whose range omits two points must be a constant function.

Let $X = E \times E$, $Y = \mathbb{R}$, and $f(x,y) = |x-y|$, where E is a compact subset of $\mathbb{R}^d$, $d \geq 2$, and $|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}$. The Falconer distance problem asks how large does the Hausdorff dimension of E needs to be to ensure that the Lebesgue measure of the distance set

$$\Delta(E) = \{|x-y| : x, y \in E\}$$

is positive.

In this context, it is sufficient to think of Hausdorff dimension of a compact set E, denoted by $dim_{\mathcal{H}}(E)$, in the following way. There exists a Borel measure supported on E such that for every $\alpha < dim_{\mathcal{H}}(E)$, the α-energy integral

$$I_\alpha(\mu) = \int \int |x-y|^{-\alpha} d\mu(x) d\mu(y) < \infty. \quad (1)$$

The background and the details pertaining to the Hausdorff dimension and energy integrals are beautifully described in Falconer's "Geometry of Fractal Sets" ([5]), and Mattila's "Fourier Analysis and Hausdorff Dimensions" ([12]).

This problem can be viewed as a more delicate variant of the celebrated Steinhaus Theorem, which says that if $E \subset \mathbb{R}^d$ is of positive Lebesgue measure, then $E - E = \{x - y : x \in E, y \in E\}$ contains an open ball centered at the origin.

The Falconer Distance Conjecture says that if the Hausdorff dimension of $E \subset \mathbb{R}^d$, $d \geq 2$, is greater than $\frac{d}{2}$, then the Lebesgue measure of $\Delta(E)$ is positive. This problem was formulated by Falconer in 1985 ([6]).

Connections with the Erdős Distance Problem

The Falconer Distance Conjecture is a continuous analog of the Erdős Distance Conjecture, which says that if $P \subset \mathbb{R}^d$, $d \geq 2$, is a finite set, then for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$\#\Delta(P) \geq C_\epsilon (\#P)^{\frac{2}{d}-\epsilon}.$$

This problem was introduced by Erdős in 1945, and after 66 years of efforts by many of the most prominent experts in combinatorics and related fields, the problem was finally solved in two dimensions by Guth and Katz ([9]). In higher dimensions, the problem is still open, with the best exponents due to Jozsef Solymosi, Cszaba Toth, and Van Vu (see [15]).

Sharpness of the Erdős/Falconer Exponents

It is important to note that the conjectured exponent $\frac{d}{2}$ in the Falconer distance problem and the exponent $\frac{2}{d}$ in the Erdős distance problem are strongly linked. Let $P_q = \mathbb{Z}^d \cap [0,q]^d$. Then $\#P_q \approx q^d$. The size of $\Delta(P_q)$ does not exceed the number of values of the quadratic form $x_1^2 + x_2^2 + \cdots + x_d^2$, $x_j \in [0,q]$, which is bounded by $q^2 + q^2 + \cdots + q^2 = dq^2$. Setting $n = q^d$, we see that $\#\Delta(P_q) \leq dn^{\frac{2}{d}}$, and the sharpness of the $\frac{2}{d}$ exponent in the Erdős distance problem is established.

In order to establish the sharpness of the $\frac{d}{2}$ exponent in the Falconer distance conjecture, we bootstrap off the Erdős distance problem example above. Let $q_1 = 2$, $q_{i+1} =$

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q_i^i. Let E_i^s, $s \in \left(\frac{d}{2}, d\right)$ denote the $q_i^{-\frac{d}{s}}$-neighborhood of $\frac{1}{q_i} P_{q_i}$. A result in Falconer's book ([5]), Chapter 8, shows that the Hausdorff dimension of $E^s = \cap_i E_i^s$ is equal to s. On the other hand,

$$|\Delta(E_i^s)| \leq C q_i^{-\frac{d}{s}} \cdot \#\Delta(P_{q_i}) \leq C' q_i^{2-\frac{d}{s}},$$

from which it follows that $|\Delta(E^s)|$, the Lebesgue measure of E^s, may be 0 if $s < \frac{d}{2}$, thus establishing the sharpness of the $\frac{d}{2}$ exponent up to the endpoint.

The L^∞ Theory

In order to understand how many distances a set $E \subset \mathbb{R}^d$, $d \geq 2$, determines, one cannot avoid studying the incidence function that counts how often a fixed distance occurs. In the discrete case this is simply a matter of counting the number of pairs of elements from E whose pairwise distance equals a given value. In the continuous case one must proceed a bit more carefully. Let σ_t denote the surface measure on the sphere of radius $t > 0$ centered at the origin. Let ρ be a smooth cut-off, $\equiv 1$ in the unit ball and vanishing outside a slightly larger ball. Let $\rho_\epsilon(x) = \epsilon^{-d}\rho\left(\frac{x}{\epsilon}\right)$, and define $\sigma_t^\epsilon(x) = \sigma_t * \rho_\epsilon(x)$. Let

$$\nu^\epsilon(t) = \int\int \sigma_t^\epsilon(x-y)d\mu(x)d\mu(y),$$

where μ is a Borel measure supported on E. One should think of this quantity as the ϵ-approximation of the incidence function on $\Delta(E)$, which counts pairs of points in E separated by the distance t. Also, at least heuristically and this can be made quite precise, $\lim_{\epsilon\to 0^+} \nu^\epsilon(t)$ is the distance measure ν defined by the relation

$$\int f(t)d\nu(t) = \int\int f(|x-y|)d\mu(x)d\mu(y). \quad (2)$$

Falconer observed by a simple covering argument that if one can show that $\nu^\epsilon(t)$ is uniformly bounded, then the Lebesgue measure of $\Delta(E)$ is positive. More precisely, cover $\Delta(E)$ by the collection $\{(t_i - \epsilon_i, t_i + \epsilon_i)\}$. The following is a formal argument that can be made precise with a tiny bit of work. We have $1 = \mu \times \mu(E \times E)$

$$\begin{aligned}
&\leq \sum_i \mu \times \mu\{(x,y) : t_i - \epsilon_i \leq |x-y| \leq t_i + \epsilon_i\} \\
&\leq C \sum_i \epsilon_i \nu^{\epsilon_i}(t_i) \\
&= C \sum_i \epsilon_i \int\int \sigma_{t_i}^{\epsilon_i}(x-y)d\mu(x)d\mu(y) \\
&= C \sum_i \epsilon_i \int |\widehat{\mu}(\xi)|^2 \widehat{\sigma}_{t_i}^{\epsilon_i}(\xi)d\xi.
\end{aligned}$$

Using the method of stationary phase (see e.g. [14]), it is not difficult to see that

$$|\epsilon_i \widehat{\sigma}_{t_i}^{\epsilon_i}(\xi)| \leq C|\xi|^{-\frac{d-1}{2}} \cdot \min\{|\xi|^{-1}, \epsilon_i\}. \quad (3)$$

Plugging the estimate (3) back in and tracing the inequalities backwards, we see that this quantity is bounded by $\sum_i \epsilon_i \cdot \int |\xi|^{-\frac{d-1}{2}} |\widehat{\mu}(\xi)|^2 d\xi$.

By a simple Plancherel style argument, this expression equals

$$\begin{aligned}
&\sum_i \epsilon_i \cdot \int\int |x-y|^{-\frac{d+1}{2}} d\mu(x)d\mu(y) \\
&= I_{\frac{d+1}{2}}(\mu) \cdot \sum_i \epsilon_i \leq C \sum_i \epsilon_i
\end{aligned}$$

if the Hausdorff dimension of E is greater than $\frac{d+1}{2}$, as we explain in the paragraph preceding the formula (1). It follows that $\sum_i \epsilon_i \geq \frac{1}{C} > 0$, which implies that the Lebesgue measure of $\Delta(E)$ is positive.

The L^2 Theory: Setup

In the previous section we obtained a good exponent for the Falconer Distance Problem by obtaining an L^∞ estimate for the smoothed out measure on the distance set. In order to improve the exponent, we are going to describe the method that only relies on L^2 bounds for the distance measure ν. Observe that if $\nu \in L^2$, then

$$1 = \left(\int d\nu(t)\right)^2 \leq |\Delta(E)| \int \nu^2(t)dt \leq C|\Delta(E)|,$$

which would imply that $|\Delta(E)| \geq \frac{1}{C} > 0$.

The advantage of this point of view is two-fold. First, it is typically far easier to prove that something is in L^2 than to show that it is in L^∞. Second, it turns out that the L^∞ bound on ν^ϵ, independent of ϵ, is not even true in general if the Hausdorff dimension of the underlying set is $< \frac{d+1}{2}$. This was shown by Mattila in two dimensions ([11]) and by the author and Senger ([10]) in three dimensions. In higher dimensions the question is still open, but the author and Senger ([10]) showed that ν^ϵ is not in L^∞ with constants independent of ϵ in dimensions four and higher if the Euclidean distance is replaced by a suitable variant of the parabolic metric.

Another advantage of L^2 norms is that Plancherel comes into play. Mattila proved that if the Hausdorff dimension of a compact set $E \subset \mathbb{R}^d$ is $> \frac{d}{2}$, μ is a Borel measure supported on E and

$$\mathcal{M}(\mu) = \int \left(\int_{S^{d-1}} |\widehat{\mu}(r\omega)|^2 d\omega\right)^2 r^{d-1}dr < \infty, \quad (4)$$

then the distance measure ν introduced above has an L^2 density, and thus $|\Delta(E)| > 0$.

Mattila derived this result using the method of stationary phase and properties of Bessel functions. We are going to sketch a geometric derivation obtained by Greenleaf, the author, Liu, and Palsson ([7]) where more complicated geometric configurations are also studied.

Recalling the definition of the distance measure ν in (2), we see that in order to compute $\int \nu^2(t)dt$ we must come to grips with quadruplets $x, y, x', y' \in E^4$ such that $|x - y| = |x' - y'|$. In reality we must consider quadruplets where distances are close to equal and then devise a careful limiting process, but let's keep going. If $|x-y| = |x'-y'|$, then there exists $g \in O_d(\mathbb{R})$ (the orthogonal group) such that $x - y = g(x' - y')$.

In the plane this g is unique. In higher dimensions, one must consider the appropriate stabilizer. Rewriting the equation we obtain $x - gx' = y - gy'$ and this has the L^2 norm of the natural measure on $E - gE$ written all over it. More precisely, define the measure ν_g by the relation

$$\int g(z)d\nu_g(z) = \int\int g(u - gv)d\mu(u)d\mu(v). \quad (5)$$

Arguing in this way we can show that

$$\int \nu^2(t)dt = \int\int \nu_g^2(z)dzdg,$$

where dg is the Haar measure on $O_d(\mathbb{R})$, provided that both sides make sense. The Fourier transform of ν_g is easy to compute using the formula (5). By Plancherel we conclude that

$$\int\int \nu_g^2(z)dzdg = \int\left\{\int |\widehat{\mu}(g\xi)|^2 dg\right\} |\widehat{\mu}(\xi)|^2 d\xi$$
$$= c\int \left(\int_{S^{d-1}} |\widehat{\mu}(r\omega)|^2 d\omega\right)^2 r^{d-1}dr \equiv \mathcal{M}(\mu).$$

L^2-theory: Wolff–Erdogan

Until very recently, the best known results on the Falconer distance problem were due to Wolff ([16]) in the plane and Erdogan (IMRN, 2006) in higher dimensions. They proved that the Lebesgue measure of the distance set is positive, provided that the Hausdorff dimension of the underlying set is $> \frac{d}{2} + \frac{1}{3}$. We shall briefly comment on the more recent efforts, but for now let us describe the $\frac{d}{2} + \frac{1}{3}$ theory that laid the foundation for further progress. The key estimate established by Wolff and Erdogan is the following.

$$\int_{S^{d-1}} |\widehat{\mu}(t\omega)|^2 d\omega \le C(d, s, \epsilon)t^{\epsilon - \left(\frac{d+2s-2}{4}\right)} I_s(\mu), \quad (6)$$

where $I_s(\mu) = \int\int |x - y|^{-s}d\mu(x)d\mu(y)$ is the energy integral of μ. Plugging this back into (4) yields

$$\mathcal{M}(\mu) \le C\int\int |\widehat{\mu}(t\omega)|^2 t^{d-1} t^{\epsilon - \left(\frac{d+2s-2}{4}\right)} I_s(\mu)d\omega dt$$
$$= C\int |\widehat{\mu}(\xi)|^2 |\xi|^{\epsilon - \left(\frac{d+2s-2}{4}\right)} I_s(\mu)d\xi$$
$$= C' I_{\epsilon + \frac{3d-2s+2}{4}}(\mu) I_s(\mu),$$

which is bounded if $dim_{\mathcal{H}}(E) > \frac{d}{2} + \frac{1}{3}$.

Recent Advances

After a long hiatus, the advances on the Falconer distance conjecture started coming again in recent months. X. Du, L. Guth, H. Wang, B. Wilson, and R. Zhang ([2]) obtained the dimensional threshold $\frac{9}{5}$ in $\mathbb{R}^3$ and improved the threshold for $d \ge 4$ as well. Their higher dimensional threshold for $d \ge 4$ was further improved by X. Du and R. Zhang ([3]) to $\frac{d^2}{2d-1} = \frac{d}{2} + \frac{1}{4} + \frac{1}{8d-4}$.

What is behind all this activity? Several key recent advances in harmonic analysis come into play and perhaps the most important of these is the connection with the Schrodinger operator. Du and Zhang deduced their $\frac{d^2}{2d-1}$ threshold from the following Schrödinger estimate. Let $n \ge 1$, $\alpha \in (0, n+1]$ and μ a compactly supported Borel measure such that $\mu(B(x,r)) \le Cr^\alpha$. Then

$$||e^{it\Delta}f||_{L^2(B(\vec{0},R);d\mu_R(x,t)} \lessapprox R^{\frac{\alpha}{2(n+1)}}||f||_2,$$

from which they deduced a good bound for the spherical average in (6).

In two dimensions, Guth, Iosevich, Ou, and Wang ([8]) improved the dimensional threshold to $\frac{5}{4}$, proved a pinned version of the result, and extended it to a variety of smooth metrics. In this setting, a completely new approach needed to be developed because the authors proved that for any $\alpha < \frac{4}{3}$ there exists a planar set of Hausdorff dimension α such that (4) is infinite. They solved this problem by considering $E_1, E_2 \subset E$ separated by distance ~ 1, and letting μ_1 and μ_2 be Frostman measures on E_1, E_2. They divided μ_1 into $\mu_1 = \mu_{1,good} + \mu_{1,bad}$, where $\mu_{1,bad}$, roughly speaking, comes from the example where the L^2 norm of the distance measure is infinite. They showed that the L^1 norm of $\mu_{1,bad}$ is not too large using a beautiful projection estimate due to Orponen ([13]). This reduced matters to obtaining an upper bound for the L^2 norm of $\mu_{2,bad}$, which was accomplished via a suitable Schrödinger type estimate partly based on the decoupling theorem of Bourgain and Demeter ([1]).

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