2019 Oswald Veblen Prize in Geometry

The 2019 Oswald Veblen Prize in Geometry was presented at the 125th Annual Meeting of the AMS in Baltimore, Maryland, in January 2019. The prize was awarded to Xiuxiong Chen, Simon Donaldson, and Song Sun.


Xiuxiong Chen


Simon Donaldson


Song Sun

Citation

The 2019 Oswald Veblen Prize in Geometry is awarded to Xiuxiong Chen, Simon Donaldson, and Song Sun for the three-part series entitled "Kähler–Einstein Metrics on Fano Manifolds, I, II and III" published in 2015 in the *Journal of the American Mathematical Society*, in which Chen, Donaldson, and Sun proved a remarkable nonlinear Fredholm alternative for the Kähler–Einstein equations on Fano manifolds. They show that this fully nonlinear PDE can be solved if and only if a certain stability condition involving only finite-dimensional algebro-geometric data holds.

In 1982 Shing-Tung Yau received the Fields Medal in part for his 1978 proof of the so-called Calabi Conjecture. In particular Yau proved that if the first Chern class of a compact Kähler manifold vanishes (respectively, is negative), then it admits a Kähler–Einstein metric, i.e., there is a unique Kähler metric in the same class with vanishing (respectively, constant negative) Ricci curvature.

Yau later conjectured that a solution in the case of Fano manifolds, i.e., those with positive first Chern class, would necessarily involve an algebro-geometric notion of stability. Seminal work of Gang Tian and then Donaldson clarified and generalized this idea. The resulting conjecture—that a Fano manifold admits a Kähler–Einstein metric if and only if it is K-stable—became one of the most active topics in geometry. In 1997 Tian introduced the notion of K-stability used in the cited papers, and used this to demonstrate that there are Fano manifolds with trivial automorphism group which do not admit Kähler–Einstein metrics.

Proving this conjecture had long been understood to involve a vast combination of ideas from symplectic and complex geometry, infinite-dimensional Hamiltonian reduction, and geometric analysis. All methods involved some kind of continuity method; in 2011 Donaldson proposed one involving Kähler–Einstein metrics with cone singularities (published by Springer in *Essays in Mathematics and Its Applications* in 2012).

One of the main technical obstacles then was how to control certain limits of sequences of Kähler metrics on Fano manifolds (equivalently, how to obtain the "partial C^0-estimate"). One can take the so-called Gromov–Hausdorff limit, but a priori this could be a metric space with no algebro-geometric description.

It was a huge breakthrough when, in 2012, Donaldson and Sun managed to use Bergman kernels to put the structure of a normal projective algebraic variety on the Gromov–Hausdorff limit of a noncollapsing sequence of

polarized Kähler manifolds with bounded Ricci curvature (published in *Acta Mathematica* in 2014).

Chen, Donaldson, and Sun gave a complete solution of the conjecture for Fano manifolds a few months later. The announcement was published in *International Mathematics Research Notices* in 2014, and full proofs followed in "Kähler–Einstein metrics on Fano manifolds. I: Approximation of metrics with cone singularities," "Kähler–Einstein metrics on Fano manifolds. II: Limits with cone angle less than 2π," and "Kähler–Einstein metrics on Fano manifolds. III: Limits as cone angle approaches 2π and completion of the main proof," all published in 2015 in the *Journal of the AMS*.

As one nominator put it, "This is perhaps the biggest breakthrough in differential geometry since Perelman's work on the Poincaré conjecture. It is certainly the biggest result in Kähler geometry since Yau's solution of the Calabi conjecture thirty-five years earlier. It is already having a huge impact that will only grow with time."

Biographical Note: Xiuxiong Chen

Xiuxiong Chen received his undergraduate degree in 1987 from the University of Science and Technology of China (USTC) and a master's degree from the graduate school of USTC and the Academia Sinica in 1989, supervised by JiaGui Peng in geometry and Weiyue Ding in analysis. He then moved to the University of Pennsylvania in 1989 for his doctoral degree under the supervision of E. Calabi. He held positions at McMaster University (1994–1996), Stanford University (1996–1998), Princeton University (1998–2002), and the University of Wisconsin–Madison (2002–2009). Since 2009 he has been a professor of mathematics at Stony Brook University. He was an invited speaker at ICM 2002 in Beijing and is a 2015 Fellow of the American Mathematical Society and a 2016 Simons Fellow in mathematics. Over his career, he has supervised around twenty PhD students in mathematics.

Biographical Note: Simon Donaldson

Simon Donaldson received his undergraduate degree in 1978 from Cambridge University and moved to Oxford for his doctorate, supervised by Michael Atiyah and Nigel Hitchin. He held positions in Oxford and Stanford before moving to Imperial College, London, in 1998. At present he is a permanent member of the Simons Center for Geometry and Physics, Stony Brook. Over his career he has supervised about forty-five doctoral students, many of whom are now leading figures in mathematical research. Donaldson was awarded a Fields Medal in 1986 for his work on gauge theory and four-dimensional manifolds, and he has made contributions to several other branches of differential geometry. He was an invited speaker at ICMs in 1983, 1986, 1998, and 2018. He has held a number of editorial positions (including, currently, the *Journal of the AMS*), and served on a variety of committees, including the Executive Committee of the International Mathematical Union (1994–2002).

Biographical Note: Song Son

Song Sun was born in 1987 in Huaining, Anhui province, China. He received a BS from the University of Science and Technology of China in 2006 and a PhD from the University of Wisconsin–Madison in 2010, supervised by Xiuxiong Chen. He held a postdoctoral position at Imperial College London from 2010–2013, and then became an assistant professor at Stony Brook University. In 2018, he joined the faculty at University of California, Berkeley. Sun received an Alfred P. Sloan Research Fellowship in 2014, and was an invited speaker at ICM 2018 in Rio de Janeiro.

Response from Xiuxiong Chen, Simon Donaldson, and Song Sun

It is a great honor to be awarded the 2019 Oswald Veblen Prize for our work on Kähler–Einstein metrics. Our work builds on that of many others. In 1954, Calabi proposed his vision of far-reaching existence theorems for canonical metrics on Kähler manifolds—a vast extension of the classical theory for Riemann surfaces. The foundation for this vision came from the developments of complex differential geometry over the preceding decades by Kähler, Hodge, Chern, and others. In its general formulation, involving "extremal" Kähler metrics, Calabi's problem remains to a large extent open, but in the case of Kähler–Einstein metrics the existence theory is now in a relatively satisfactory state. A crucial breakthrough by S.-T. Yau, which famously dealt with the cases of negative or zero first Chern class, was recognized in the 1981 Veblen Prize. Many mathematicians have contributed to the understanding of the remaining "positive" case over the four decades since Yau's work. We feel very fortunate and privileged to have had the opportunity to play a part in this long story.

Our cited work interweaves strands from several different fields. One is the theory of the complex Monge–Ampère equation, with estimates in the style going back to Calabi and Yau, but also with modern developments which extend the theory to singular varieties. Another is the convergence theory of Riemannian manifolds with Ricci curvature bounds: our work blends these ideas with complex geometry through the L^2 or "Hörmander" method. A third strand brings in the circle of ideas linking geometric invariant theory in algebraic geometry, and notions of "stability," to symplectic geometry. In the few years following our cited work, several other proofs of the main result have appeared, but all sharing a similar diversity of techniques. This diversity is an intrinsic feature of the problem, which seeks a bridge between differential and algebraic geometry. While our work provides an answer to one long-standing question, these recent developments open up wonderful

new vistas, for example in the study of moduli spaces and singularities, within this grand theme.

We are very glad to have this opportunity to thank our wives—Holly, Nora, and Jiajia—for their wonderful support, which was crucial for us in completing this work. Xiuxiong Chen wishes to take this opportunity to thank his advisor, E. Calabi, for his mathematical guidance and inspiration.

About the Prize

The Oswald Veblen Prize in Geometry is awarded every three years for a notable research memoir in geometry or topology that has appeared during the previous five years in a recognized North American journal (until 2001 the prize was usually awarded every five years). Established in 1964, the prize honors the memory of Oswald Veblen (1880–1960), who served as president of the AMS during 1923–1924. It was established in 1961 in memory of Veblen through a fund contributed by former students and colleagues and later doubled by Veblen's widow. In 2013, Cathleen Synge and Herbert Morawetz made a major donation that substantially increased the prize fund. Cathleen S. Morawetz served as president of the AMS in 1995–1996. The Veblen Prize carries a cash award of US$5,000.

The Veblen Prize is awarded by the AMS Council acting on the recommendation of a selection committee. For the 2019 prize, the members of the selection committee were:

- Danny C. Calegari,
- Albert Marden (Chair),
- Ulrike Tillmann.

A list of previous recipients of the Oswald Veblen Prize in Geometry may be found on the AMS website at `https://www.ams.org/profession/prizes-awards/pabrowse?purl=veblen-prize`.

Credits

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