Has Carmichael's Totient Conjecture Been Proven? No, No, It Has Not.

Sophia D. Merow

Carmichael's totient conjecture lacks the name recognition of such media darlings as the twin prime conjecture or the Riemann hypothesis, but in 2018 this open question in number theory blipped briefly into the public consciousness. It cropped up in, of all places, a DriveTime® commercial.

Besides bearing a name beginning with the syllable *car,* Carmichael's totient conjecture (CTC) has nothing to do with DriveTime's stock-in-trade, which is the sale and financing of used automobiles. CTC concerns the multiplicity of values of Euler's totient function. Euler's totient function $\phi(n)$ returns for a positive integer n the number of positive integers at most n that are relatively prime to n (where 1 is counted as being relatively prime to all numbers). CTC posits that for every $n \in \mathbb{Z}+$ there is at least one $m \in \mathbb{Z}+$, $m \neq n$ such that $\phi(m)=\phi(n)$.

The DriveTime spot risks leaving credulous viewers under the same misapprehension the conjecture's eponym harbored for over a decade: namely, that the CTC has been proven (or "solved" as the ad's copywriters might inaptly put it). Robert Daniel Carmichael (1879–1967) published a paper [1] purporting to establish the CTC in 1907. Deeming the proof trivial enough even for students, Carmichael included it as an exercise—Chapter 2, #8—in his 1914 textbook *The Theory of Numbers.*[1] By 1922, however, several readers had pointed out a gap in Carmichael's argument, a gap he could not fix.

"So far I have been unable to supply a proof of the theorem," Carmichael conceded in a note [2] in the *Bulletin*, "though it seems probable that it is correct." He felt "compelled to allow it to stand in the status of a conjectured or empirical theorem."

Now if any conjecture warrants being designated an "empirical theorem," CTC is it. The statement remains unproven, but mathematicians have over the decades increased by impressive orders of magnitude the lower bound on a counterexample (see Table 1). Carmichael himself began the push, establishing in his 1922 note that if there exists an n such that the value $\phi(n)$ is attained only by that unique n, that n exceeds 10^{37}. In the 1994 paper [10] in which they raised the bar to $10^{10,000,000}$, Aaron Schlafly and Stan Wagon remarked, "We do not know of another unsolved problem

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[1] *Paul T. Bateman told Kevin Ford that when, as an undergraduate, he was learning number theory from Carmichael's book, Chapter 2 #8 was, to his frustration, the only exercise he couldn't complete. He was relieved to learn that the problem was, in fact, an open one.*

Lower bound	Reference	Year
$n > 10^{37}$	R. D. Carmichael [2]	1922
$n > 10^{400}$	V. L. Klee, Jr. [6]	1947
$n > 10^{10,000}$	P. Masai & A. Valette [8]	1982
$n > 10^{10,000,000}$	A. Schlafly & S. Wagon [10]	1994
$n > 10^{10,000,000,000}$	K. Ford [3]	1998

Table 1. The lower bound on a counterexample to CTC

in mathematics for which a lower bound on a counterexample is so high."

Kevin Ford, who in 1998 [3] bumped that lower bound up to a staggering $10^{10,000,000,000}$, calls CTC his "favorite problem" and has puzzled over it on and off since his grad school days.

In 1999, Ford proved a result related to an alternate formulation of CTC. If $A(f)$ denotes the number of positive integers n for which $\phi(n)=f$, CTC says that $A(f)$ can never equal 1. Ford showed [4] that every positive integer other than 1 occurs as a value of $A(f)$.

In 2014, Ford shed some light on the difficulty of a particular strategy for resolving the CTC. One approach to the problem is based on showing that any counterexample must be divisible by many primes; if the set of such primes were infinite, CTC would follow. Ford showed [5] that the set S of such primes is very "thin" (more precisely, if $P(x)=\#\{p \in S : p \le x\}$, then $P(x)=O(x^{1-c})$ for some $c>0$). Proving the infinitude of this set, Ford says, "seems to be very hard."

In 2018, Kannan Soundararajan sent Ford a link to the DriveTime commercial.

"I was pleasantly surprised," Ford recalls, "as this conjecture is not that widely known."

The DriveTime commercial (`https://bit.ly/2sNH6B7`), unfortunately, does little more than name-drop CTC and bolster well-worn stereotypes.

"This is Dr. Gunter Zoolof," the spot opens, panning upward from a pocket protector to a drooping bowtie to a bespectacled face framed in an Einstein-esque mane.

"Did he solve Carmichael's totient conjecture?" intones Will Lyman, the narrator of PBS's *Frontline*, as the camera cuts to Zoolof atop a ladder at a multi-story blackboard crammed with the integrals, derivatives, and summations that to the lay public pass for cutting-edge mathematics.[2] "Yes, yes he did."

[2] *"The board work seems to have nothing to do with any coherent thoughts," says Carl Pomerance, whose first published paper [9] was on CTC. "I saw the quadratic discriminant formula, some calculus, some other unrecognizable stuff."*

"But did he buy his car at DriveTime, using the industry's smartest online tools?" continues the voiceover as a man in coveralls enters stage left and Zoolof's satisfied smile transforms into a wide-eyed look of alarm. "No, no he did not," narrates Lyman as the janitor's wet cloth begins to obliterate the mathematician's chalkings. "Dr. Gunter Zoolof was almost a genius."

That last line underscores what Carmen Latterell, who has written [7] about pop-culture's portrayal of those who do mathematics, sees as the DriveTime spot's most deleterious implication. Yes it's unfortunate in Latterell's eyes that the ad depicts a mathematician stereotypical in his demographics—male, white—and appearance. But she finds it even worse that the commercial perpetuates the myth that a mathematician is brilliant in mathematics but stupid in all other areas.

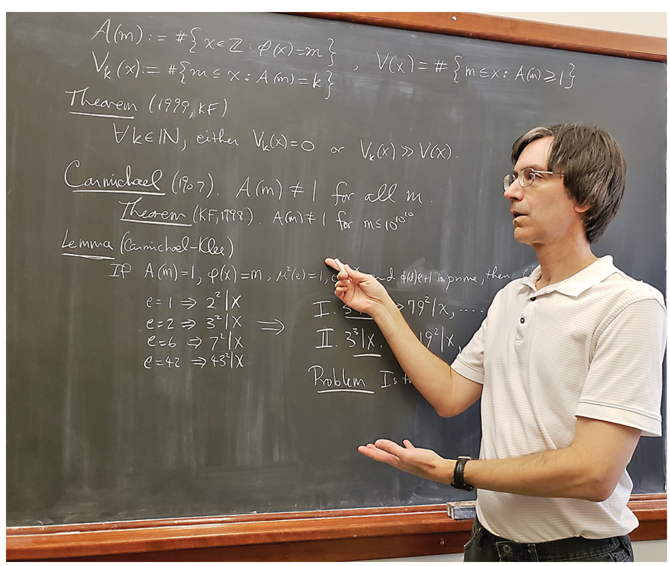
"This is the harmful stereotype that makes students not want to study math," she says.

Not all ads featuring mathematicians reinforce all stereotypes, however. A 2013 commercial for Beautyrest's ComforPedic mattress (`https://bit.ly/2T88sxw`), for instance, shows graph theorist Maria Chudnovsky exercising sound judgment in choosing a sleep surface. And in a 2016 TurboTax spot (`https://bit.ly/2RPGKcg`) a smartly clad Chudnovsky (one mathematician, two commercials? surely a record!) deftly deploys an on-screen help function to demonstrate that "it doesn't take a genius to do your taxes."

Many mathematics educators find the equation of "mathematician" and "genius" problematic, but *were* someone to resolve CTC at last, would s/he have claim to the "genius" mantle?

Stan Wagon thinks so.

"Whenever a big conjecture, around for over 100 years, that is 'obviously true' by various heuristic considerations is actually proved, that is a big deal," he says. "Any such



Kevin Ford (`https://faculty.math.illinois.edu/~ford`) practices for a lecture on CTC.

person would have gained a measure of immortality, and yes, 'genius' would be a reasonable description."

"Of course," Wagon hedges, "it depends a bit on how the proof goes."

References

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Credits

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