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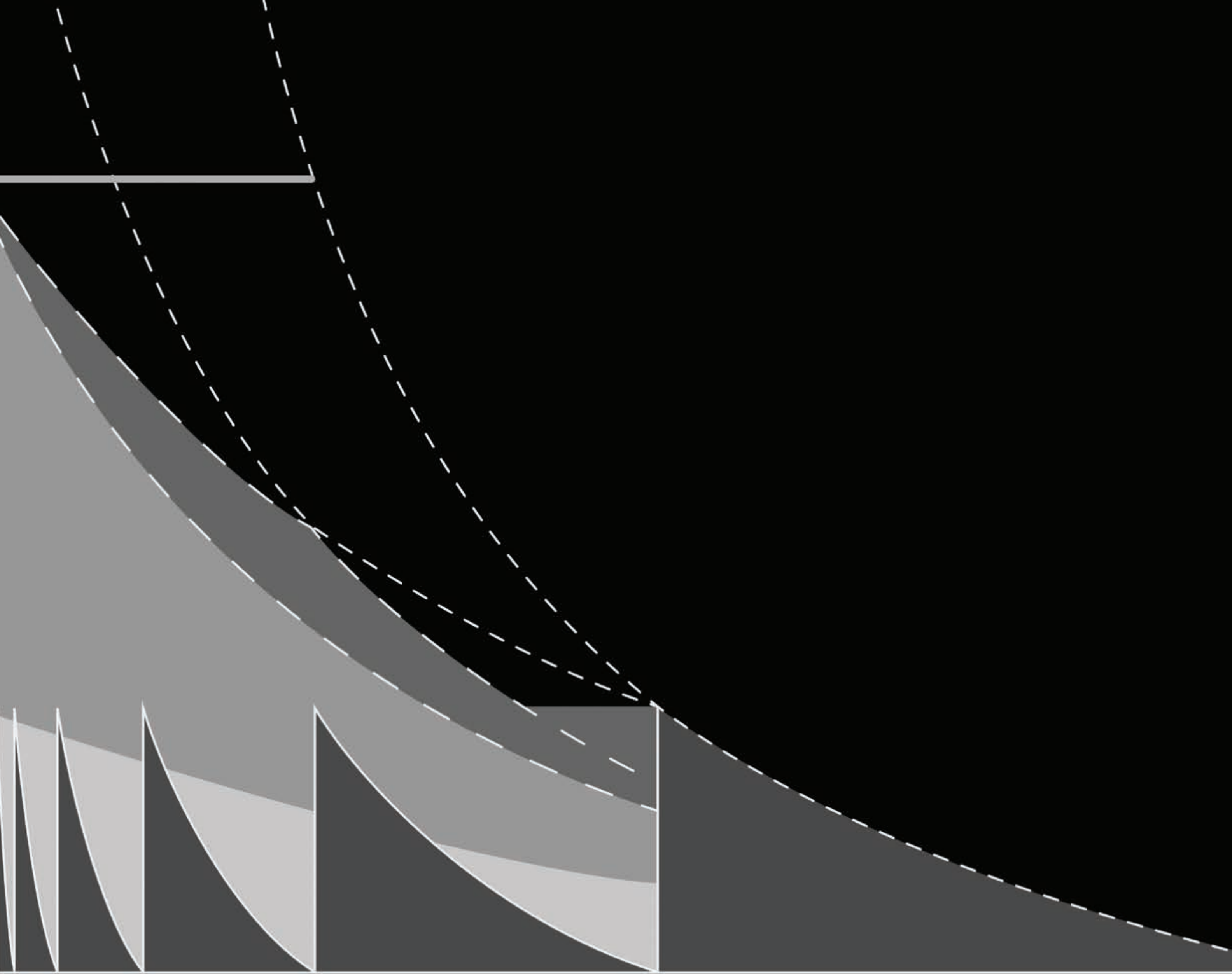
# Notices

## of the American Mathematical Society

June/July 2019

Volume 66, Number 6



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*The cover design is based on imagery from An Invitation to Gabor Analysis, page 808.*

# Call for Nominations

The selection committees for these prizes request nominations for consideration for the 2020 awards, which will be presented at the Joint Mathematics Meetings in Denver, CO, in January 2020. Information about past recipients of these prizes may be found at **www.ams.org/prizes-awards**.

## BÔCHER MEMORIAL PRIZE

The Bôcher Prize is awarded for a notable paper in analysis published during the preceding six years. The work must be published in a recognized, peer-reviewed venue.

## CHEVALLEY PRIZE IN LIE THEORY

The Chevalley Prize is awarded for notable work in Lie Theory published during the preceding six years; a recipient should be at most twenty-five years past the PhD.

## LEONARD EISENBUD PRIZE FOR MATHEMATICS AND PHYSICS

The Eisenbud Prize honors a work or group of works, published in the preceding six years, that brings mathematics and physics closer together.

## FRANK NELSON COLE PRIZE IN NUMBER THEORY

This Prize recognizes a notable research work in number theory that has appeared in the last six years. The work must be published in a recognized, peer-reviewed venue.

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*Nominations that reflect the diversity of our profession are encouraged.*

## LEVI L. CONANT PRIZE

The Levi L. Conant Prize, first awarded in January 2001, is presented annually for an outstanding expository paper published in either the *Notices of the AMS* or the *Bulletin of the AMS* during the preceding five years.

## JOSEPH L. DOOB PRIZE

The Doob Prize recognizes a single, relatively recent, outstanding research book that makes a seminal contribution to the research literature, reflects the highest standards of research exposition, and promises to have a deep and long-term impact in its area. The book must have been published within the six calendar years preceding the year in which it is nominated. Books may be nominated by members of the Society, by members of the selection committee, by members of AMS editorial committees, or by publishers. The prize is awarded every three years.

## AWARD FOR DISTINGUISHED PUBLIC SERVICE

The Award for Distinguished Public Service recognizes a research mathematician who has made recent or sustained distinguished contributions to the mathematics profession through public service. The award is given every other year.

Further information about AMS prizes can be found at the Prizes and Awards website: **www.ams.org/prizes-awards**.

Further information and instructions for submitting a nomination can be found at the prize nomination website: **www.ams.org/nominations**.

For questions contact the AMS Secretary at secretary@ams.org.

**The nomination period is March 1 through June 30, 2019.**

# A WORD FROM...

Jill Pipher, AMS President

Photo is courtesy of Brown University.

My term as President is just days old (as of this writing), and I am excited to get started on the hard work of advancing my highest priority goals. These include advocacy for mathematics research and education, and increasing public awareness of the importance of mathematics; identifying processes that support, or hinder, diversity and inclusion in AMS programs and activities and in the profession more broadly; strengthening partnerships with other mathematical professional societies; and ensuring support for programs that develop a full range of professional opportunities for the next generation of mathematicians.

In this note, I'd like to highlight a few of the ways that AMS has been working to support students and early career mathematicians, and then look ahead to future initiatives and challenges.

Recently, one of my graduate students had the good fortune to participate in a Mathematics Research Community (MRC). This extraordinary program, partially funded by a grant from the National Science Foundation, offers early career mathematicians an intensive one-week collaborative research retreat led by faculty experts. Several MRCs are chosen and developed each year from a very competitive pool of high-quality proposals. The graduate student research teams typically find results leading to publications and long-lasting collaborations, and they are also given funds to attend the next Joint Mathematics Meetings (JMM).

It is really important for students and recent graduates to be able to travel for research, and attend mathematics conferences like JMM and the sectional meetings. In another partnership, this time with the Simons Foundation, the AMS launched a travel program for research-related travel for early career mathematicians, with awards of $2000 per year. A second travel award program, made possible through the support of a private gift, provides funds for graduate students to attend AMS meetings.

Professional development, interacting with the experts in one's field, and creating connections with peers are some of the important benefits of going to conferences. Some of these goals can be accomplished closer to home, through activities generated by student chapters. At my home institution, Brown University, our AMS graduate student chapter has held several annual one-day conferences featuring student talks and posters. I was happy to learn that there are now more than sixty AMS graduate student chapters, engaging students with our Society while building mathematical communities. AMS provides a small annual fund to support chapter activities like the one at Brown.

Looking ahead, there are great opportunities for AMS to continue its important work in these directions: The Campaign for The Next Generation, new commitments to mathematics education, and Joint Mathematics Meetings Reimagined.

Historically, the most successful AMS programs for early career scholars have existed on temporary funding. The public phase of The Campaign for The Next Generation (`www.ams.org/giving/nextgen`) was launched at the 2019 JMM. A generous benefactor is providing matching funds of up to $1.5 million to help establish this endowment. At present, the AMS is 85% of the way to the initial goal of a $3 million endowed fund. Funds from this endowment will be used to provide small yet impactful grants to support early career mathematicians in multiple ways, such as with travel and child care grants. And in the near future, the AMS will bring on a Director of Education located in the Office of Government Relations in Washington, DC, thus expanding its commitment to mathematics education.

*Jill Pipher is Elisha Benjamin Andrews Professor of Mathematics at Brown University and president of the American Mathematical Society. Her email address is* `Jill_Pipher@Brown.edu`.

Finally, and importantly, the leadership of AMS is inviting the community to help in reimagining our future JMM (https://www.ams.org/about-us/jmm-reimagined); starting in 2022, MAA will focus its administrative efforts on MAA MathFest, and management of JMM will be handled entirely by AMS. This creates both a challenge and an opportunity—to ensure that JMM contains all the programming attractive to a wide constituency of mathematicians, from undergraduate students to senior mathematicians. I believe that the success of this annual meeting lies in its amazing breadth, in programming and events that reflect the wide-ranging interests in teaching and research of its attendees. We welcome your ideas as the process of remaking JMM unfolds.

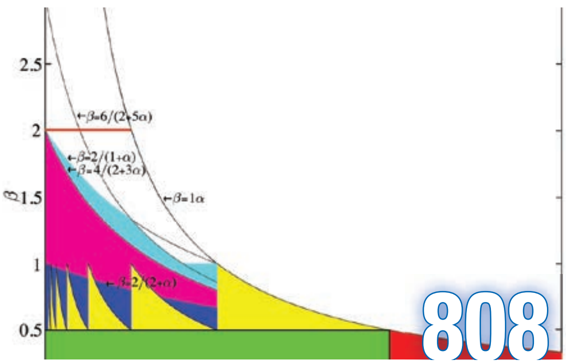
Jill Pipher
AMS President

Notices
of the American Mathematical Society

June/July 2019

*Cover Credit: The image used in the cover design was generated by Kasso A. Okoudjou with G. Atindehou and F. Ndjakou Njeunje.*

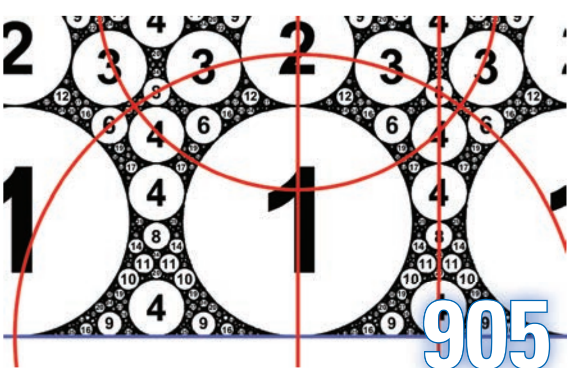
## FEATURED











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Notices
of the American Mathematical Society



## IN EVERY ISSUE



## FROM THE AMS SECRETARY



## CURRENT INVITATIONS FROM THE AMS



**CORRIGENDUM**

The Spring Lecture Sampler from the March 2019 issue contained an attribution error. On page 412 of Katherine E. Stange's article "An Illustration in Number Theory," a density one result concerning curvatures in an Apollonian circle packing was attributed to Bourgain and Fuchs (cited as [2]). In fact, the result is due to Jean Bourgain and Alex Kontorovich, "On the local-global conjecture for integral Apollonian gaskets, with an appendix by Péter Varjú," in *Invent. Math.*, 196 (2014), no. 3, 589–650.

# Notices
of the American Mathematical Society




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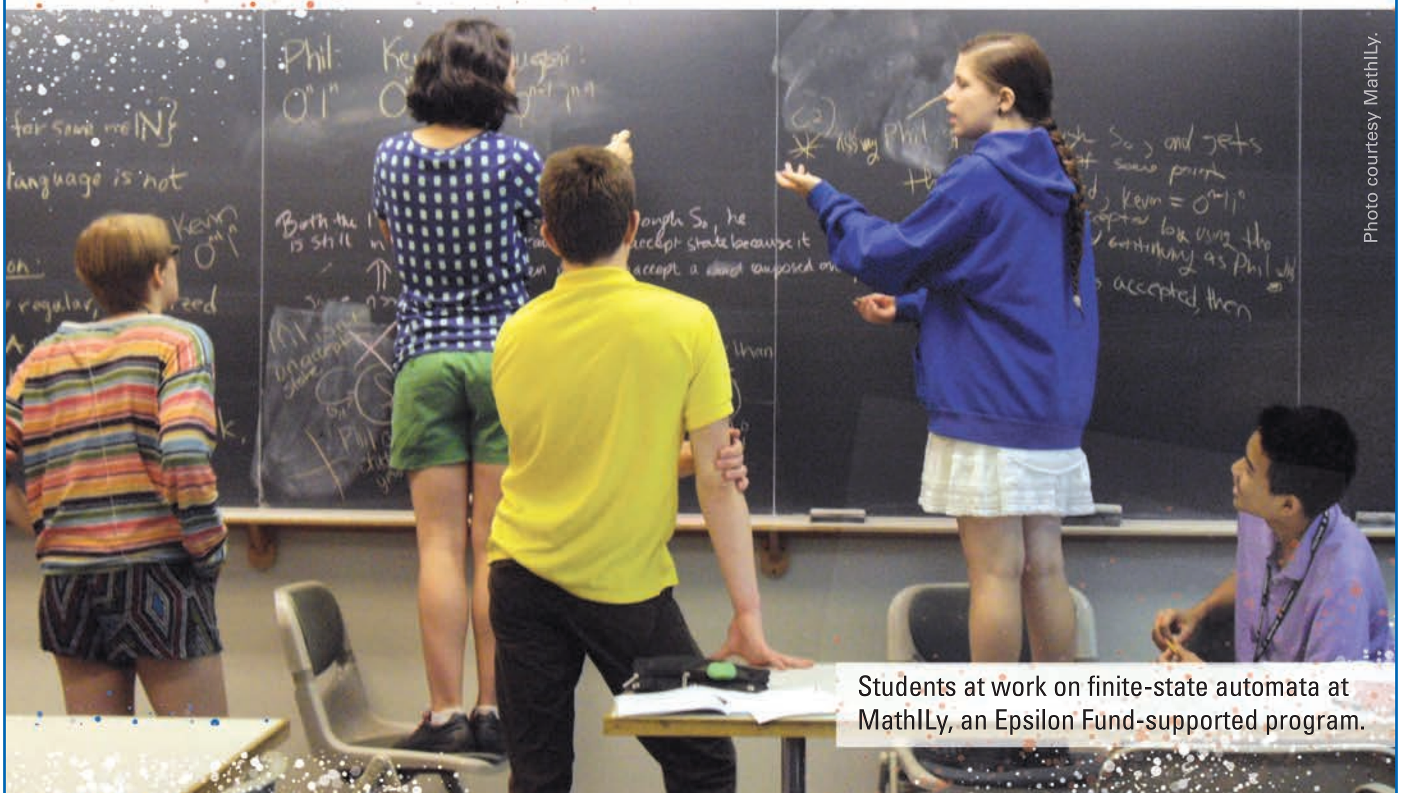

Photo courtesy MathILy.

Students at work on finite-state automata at MathILy, an Epsilon Fund-supported program.

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# LETTERS TO THE EDITOR

## New Online *Notices* Format

Dear editors,

I found the new *Notices of the AMS* webpage to be a step down from the previous version. One of the most annoying things is that I could no longer select a past issue I want from a pull-down menu and see a webpage with links to all the articles in that past issue. Clicking on the "back issue" link takes me to a page with only links to the entire PDF of the full volume of earlier issues. When would anyone ever want to download the entire volume of a journal? Readers are always looking for a specific article.

Please revert to the earlier format.

*Sincerely,*
*Izube Reitung*

(Received March 21, 2019)

A NOTE FROM THE AMS. Thank you for sharing your comments; *Notices* online users have provided us with valuable feedback on the new webpage. This feedback has led to updates that include submenus to make navigation of the new format more user-friendly. These updates were released at the end of April, 2019.

## On Cryptography of "The Ubiquity of Elliptic Curves"

Dear Prof. Goins and Editors of the *Notices of the AMS*,

The article in the February issue of the *Notices* titled "The Ubiquity of Elliptic Curves" by Edray Herber Goins is an engaging introduction to the fascinating world of elliptic curves. The article mentions several important applications of elliptic curves in cryptography. As a cryptographer myself, I feel compelled to correct the record regarding several claims in the article.

The article sketches the elliptic-curve Diffie-Hellman protocol, which allows two parties to compute a shared secret from their public shares. Variants of this protocol secure much of today's Internet, and its creators, Whitfield Diffie and Martin E. Hellman, were recognized for their contribution with the prestigious ACM A. M. Turing Award in 2015.

Using the cast of characters and notation of the article, Shuri and T'Challa negotiate a key using the Diffie-Hellman protocol by exchanging public shares $[s]P$ and $[t]P$. Their shared secret, which only they can compute, is $[st]P$. (The article conflates authenticity, i.e., "Shuri and T'Challa can feel confident that they are indeed who they say they are" and secrecy, which is what the Diffie-Hellman key agreement offers, but this can be repaired with some modifications to the protocol.) The more substantive issue is that the article seems to imply that the security of the protocol rests on the hardness of recovering secret $s$ and $t$ from the public shares, known as the (elliptic curve) discrete logarithm problem. Hardness of this problem is necessary but not sufficient. The actual problem faced by the eavesdropper is to compute $[st]P$ from $[s]P$ and $[t]P$, which is known as the (Computational) Diffie-Hellman problem.

Although the distinction may appear to be subtle or even trivial (we do not know any candidate groups where the Computational Diffie-Hellman problem is easy but the discrete logarithm problem is hard), the grand program of precisely relating hardness assumptions and security of protocols such as this one has played a central role in development of cryptography as a rigorous discipline. A common misconception, which the article echoes, is that the attacker's goal in breaking a cryptographic protocol is to recover secret keys of the honest participants. This is rarely the case; relatively few cryptographic definitions require the adversary to produce a secret key in its entirety. Instead, definitions typically focus on the damage that the

*We invite readers to submit letters to the editor at `notices-letters@ams.org`.

adversary seeks to inflict, such as reading a sent message or forging a signature.

As an example of elliptic curve cryptography in the real world, the article uses HomeKit, Apple's smart appliances framework. HomeKit-compliant devices are expected to use a particular set of protocols and parameters, such as Curve25519 for encryption or 3072-bit prime modulus for password authentication.

The article claims inaccurately that the discrete logarithm problem in a 3072-bit prime field is as hard as AES-256. In reality, the (conjectured) security of 3072-bit discrete logarithm is a rough equivalent of AES-128. By comparison, to achieve parity with AES-256, one would have to use a modulus that is approximately five times as long. (See, for instance, NIST Special Publication 800-57 "Recommendation for Key Management.")

The article concludes with an anecdote that somehow explains lukewarm adoption of the HomeKit specification with the software engineers' unfamiliarity (due to their assumed unsophistication) when it comes to elliptic curve arithmetic. While I cannot offer any insight into HomeKit's market strategy, I do disagree with the statement that software developers are intimidated by elliptic curve cryptography. In fact, quite the opposite is true.

After more than a decade of cautious experimentation by the industry and efforts by standardization bodies such as NIST, ANSI, or IETF, elliptic curve cryptography adoption kicked into high gear when Suite B was released by NSA in 2005. In its announcement NSA strongly endorsed several elliptic curve-based protocols. Transport Layer Security (TLS), the protocol behind the secure version of HTTP, incorporated support for elliptic-curve key agreement starting in 2006. The Bitcoin protocol, as described in the 2008 paper, is based on a (Koblitz) elliptic curve. Following Bitcoin, virtually all blockchains use some elliptic curves for their public-key operations, and, thanks to the recent boom in cryptocurrencies, elliptic curve arithmetic has been implemented by enthusiasts on every conceivable hardware platform and in a myriad of programming languages.

To the best of my knowledge, the primary source linking HomeKit's woes with its Curve25519 requirement was a *Forbes* article from July 2015 (`https://www.forbes.com/sites/aarontilley/2015/07/21/whats-the-hold-up-for-apples-homekit`). *Forbes*'s story is, however, more nuanced and positive. The first—unoptimized—ports of elliptic curve cryptography mandated by HomeKit were indeed unacceptably slow, such as needing 40 seconds to open a smartlock. Later implementations, assisted by specialized hardware and additional memory, turned out to be much more efficient. If anything, we can conclude that engineers successfully internalized necessary concepts from computational number theory, refactored their implementations, and achieved their performance targets.

I'd like to conclude my letter on an optimistic and welcoming note. Modern cryptography is a remarkably rich and diverse field, with dozens of subareas with their own priorities and skillsets. It greatly benefits from contributions by experts from other subjects, with the only prerequisite being an open mind and intellectual curiosity.

—*Ilya Mironov,*
*San Francisco, CA*

(Received March 14, 2019)

# How do AMS Graduate Student Chapters support the mathematical community and beyond?

## AMS Chapter at Clemson University

*Photo courtesy of Clemson University*

"Our Clemson University AMS student chapter recently brought back one of our alumni, Drew Lipman, who is now the lead data scientist at Hypergiant. We had a great career discussion with lots of good questions from graduate students and good insights from Drew about how to be ready for the transition into an industry career. Over lunch we learned more about the culture of working in the fast-paced world of startups, giving grad students a perspective that will help them in their upcoming career choices. We also enjoyed Drew's Math Club talk on game theory. Thanks for coming back Drew!"

*Photo courtesy of University of Georgia*

## AMS Chapter at University of Georgia

"On November 10th, as the Georgia Bulldogs prepared to face off against Auburn, graduate students from STEM departments across campus came together to participate in STEMzone, an outreach event made up of STEM-themed tailgating booths. Our booth, MathZone, aimed to provide audience members with a fun, colorful, and hands-on mathematical experience, in an effort to combat the math negativity and fear prevalent among the general population. Among other things, our booth activities included bubbles, math toys, an art project, stickers, puzzles, and a Mobius strip cutting activity."

## AMS Chapter at University of Mississippi

*Photo courtesy of University of Mississippi*

"In honor of AMS Day, our second year graduate student Moriah Gibson presented Algebraic Properties and Geometric Applications of Fibonacci Numbers in our University of Mississippi AMS Graduate Student seminar."

*Photo courtesy of Washington State University*

## AMS Chapter at Washington State University

"We held an Introduction to LaTeX workshop. Graduate and senior undergraduate students from different departments attended the workshop that was led by Dr. Sergey Lapin who introduced the audience to the basics of LaTeX, and shared some of his go-to resources."

## AMS Chapter at Sam Houston State University

*Photo courtesy of Sam Houston State University*

"The math and statistics department came together as a family and had a memorable night full of delicious food and fun games! We could not have asked for a better department and community to be a part of! We hope you all had a wonderful time and we are thankful for the bond developed amongst all of us. #mathisfun #amsshsu"

## AMS Chapter at Southern Illinois University, Carbondale

*Photo courtesy of Southern Illinois University, Carbondale*

"We organized a mathematics workshop in the 28th annual "Expanding Your Horizons" conference. The main purpose of the conference is to give girls from the Southern Illinois region a chance to envision the career possibilities open to them in Science, Engineering, Technology and Mathematics."

## AMS Chapter at Central Michigan University

"This year's AMS Chapters' Integration Bee was another huge success. Undergraduate and graduate students competed for title of grand integrator. In the undergraduate bracket, the integrator was freshman Shashwat Maharjan and the runner-up was Austin Konkel. In the graduate bracket, the grand integrator was Kati Moug and the runner-up was Panakaj Kumar Singh."

*Photo courtesy of Central Michigan University*

## AMS Chapter at Michigan State University

*Photo courtesy of Michigan State University*

"We have had a lot of fun with three very successful events so far. We have had two faculty colloquiums (one of which was jointly organized with the AWM Student Chapter), and one potluck trivia night."

## AMS Chapter at University of Toledo

"Our "Student Seminar" was geared toward undergraduates/early graduate students interested in mathematics research. Dr. Trieu Le gave a talk titled, "How I think about my 4th grader's math homework". In this talk, Dr. Le selected some problems from his son's math work from school, and explained how he as a mathematician thinks about these problems and how to generalize them. He eventually generalized the problems enough to teach the audience some new and useful mathematical tools!"

*Photo courtesy of University of Toledo*

**For information about starting an AMS Graduate Student Chapter, please visit:**

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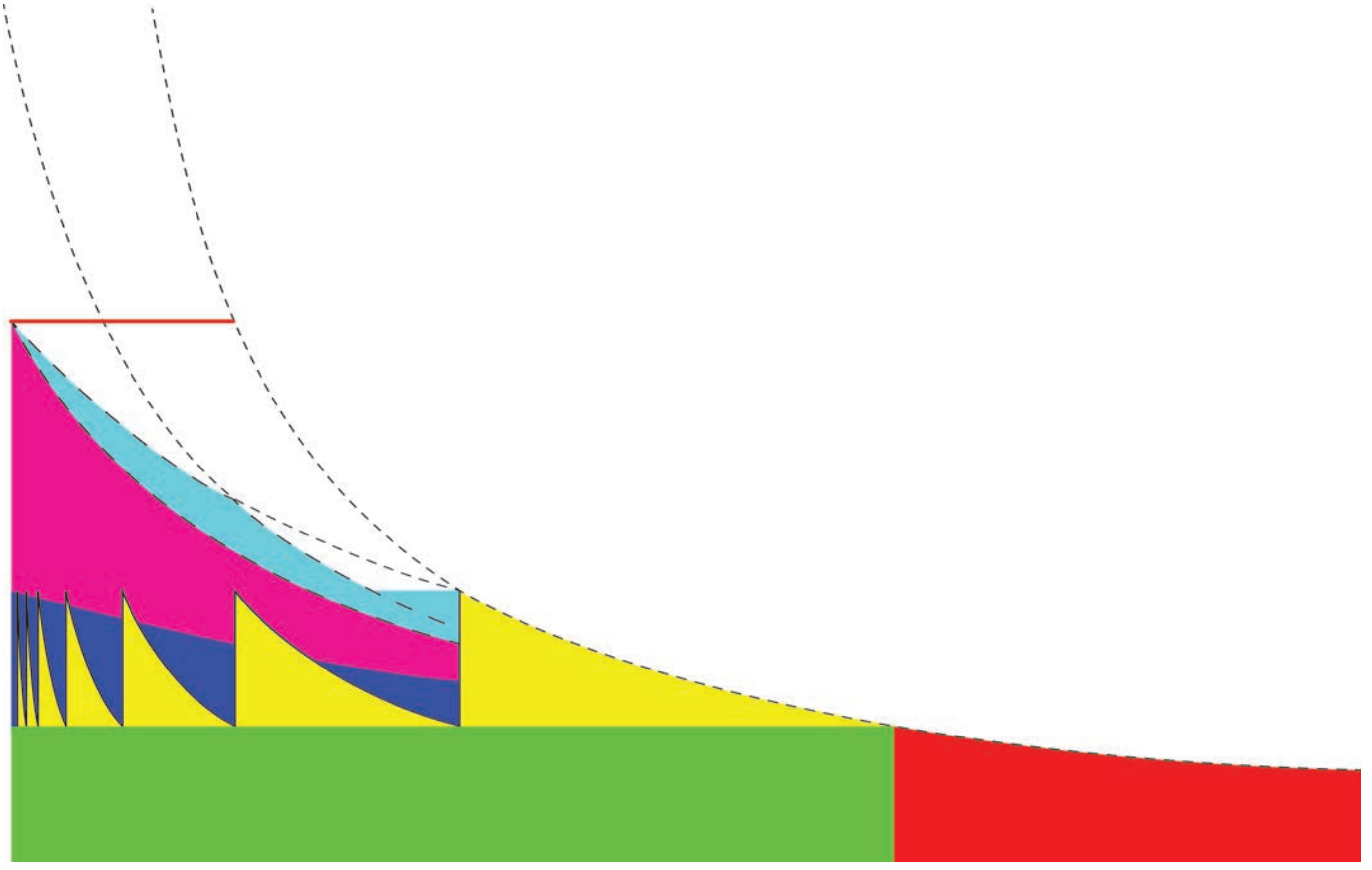
# An Invitation to Gabor Analysis



*Kasso A. Okoudjou*

## Introduction

Using the ubiquitous theory of Fourier series, one can decompose and reconstruct any 1-periodic and square integrable function in terms of complex exponential functions with frequencies at the integers. More specifically, for any such function $f$ we have

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi int}$$

where the coefficients $\{c_n\}_{n\in\mathbb{Z}}$ are square summable and the series converges in mean square, that is

$$\lim_{N\to\infty} \int_0^1 \left| f(t) - \sum_{n=-N}^{N} c_n e^{2\pi int} \right|^2 dt = 0.$$

The significance of this simple fact is that $f$ is completely determined by the coefficients $\{c_n\}$, and, conversely, each square summable sequence gives rise to a unique 1-periodic and square integrable function. This fact is equivalent to saying that the set $\{e_n(t) := e^{2\pi int}\}_{n=-\infty}^{\infty}$ forms an orthonormal basis (ONB) for $L^2([0,1))$. We shall consider these functions as the building blocks of Fourier analysis on the space of 1-periodic square integrable functions.

In his celebrated work [12], Dennis Gabor sought to decompose any square integrable function on the real line in a similar manner. To this end, he proposed to "localize" the Fourier series decomposition of such a function, by first using translates of an appropriate window function to restrict the function to time intervals that cover the real

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*Communicated by* Notices *Associate Editor Reza Malek-Madani.*



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line. The next step in the process is to write the Fourier series of each of the "localized functions," and finally, one superimposes all these local Fourier series. Putting this into practice, Gabor chose the Gaussian as a window and claimed that every square integrable function $f$ on $\mathbb{R}$ has the following (nonorthogonal) expansion

$$f(x) = \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} c_{nk}\, e^{-\frac{\pi(x-n\alpha)^2}{2\alpha^2}}\, e^{2\pi ikx/\alpha} \tag{1}$$

where $\alpha > 0$. Furthermore, he argued on how to find the coefficients $(c_{nk})_{n,k\in\mathbb{Z}} \in \mathbb{C}$ using successive local approximations by Fourier series. In fact, in 1932, John von Neumann already made a related claim, when he stipulated that the system of functions

$$\mathcal{G}(\varphi, 1, 1) = \{\varphi_{nk}(\cdot) := e^{2\pi ik\cdot}\varphi(\cdot - n) : n, k \in \mathbb{Z}\} \tag{2}$$

where $\varphi(x) = e^{-\pi x^2}$ spans a dense subspace of $L^2(\mathbb{R})$, see [16] for details.

Both of these claims were positively established in the 70s, and it follows that both statements hint at the fact that any square integrable function $f$ is completely determined in the time-frequency plane by the coefficients $\{c_{nk}\}_{k,n\in\mathbb{Z}}$, see [16] and the references therein for a historical account. In contrast to the theory of Fourier series, the building blocks in this process are the time and frequency shifts of a function such as the Gaussian: $\{\varphi_{nk}(x) = e^{2\pi ik\cdot}\varphi(\cdot - n) : n, k \in \mathbb{Z}\}$. But as we shall see later, we could consider time-frequency shifts of other square integrable functions along a lattice $\alpha\mathbb{Z} \times \beta\mathbb{Z}$ leading to $\{e^{2\pi i\beta k\cdot} g(\cdot - \alpha n) : k, n \in \mathbb{Z}\}$. The main point here is that the building blocks can depend on three parameters: $\alpha > 0$ corresponding to shifts in time/space, $\beta > 0$ representing shifts in frequency, and a square integrable window function $g$.

In some sense, both Gabor and von Neumann's statements can also be thought of as the foundations of what is known today as Gabor analysis, an active research field at the intersection of (quantum) physics, signal processing, mathematics, and engineering. In broad terms, Gabor analysis seeks to develop (discrete) joint time/space-frequency representations of functions (distributions, or signals) initially defined only in time or frequency, and it re-emerged with the advent of wavelets [7]. For a more complete introduction to the theory and applications of Gabor analysis we refer to [11, 13].

The goal of this paper is to give an overview of some interesting open problems in Gabor analysis that are in need of solutions. But first, in "Gabor Frame Theory" we review some fundamental results in Gabor analysis. In "The Frame Set Problem for Gabor Frames," we consider the problem of characterizing the set of all "good" parameters $\alpha, \beta$ for a fixed window function $g$. In "Wilson Bases" we consider the problem of constructing orthonormal bases for $L^2(\mathbb{R})$ by taking appropriate (finitely many) linear combinations of time-frequency shifts of $g$ along a lattice $\alpha\mathbb{Z}\times\beta\mathbb{Z}$. Finally, in "HRT" we elaborate on a conjecture that asks whether any finite set of time-frequency shifts of a square integrable function is linearly independent.

## Gabor Frame Theory

We start with a motivating example based on the $L^2$ theory of Fourier series. In particular, we would like to exhibit a set of building blocks $\{g_{nk}\}_{k,n\in\mathbb{Z}}$ that can be used to decompose every square integrable function. To this end, let $g(x) = \chi_{[0,1)}(x)$, where $\chi_I$ denotes the indicator function of the measurable set $I$. Any $f \in L^2(\mathbb{R})$ can be localized to the interval $[n, n+1)$ by considering its restriction $f(\cdot)\, g(\cdot - n)$ to this interval. By superimposing all these restrictions over all integers $n \in \mathbb{Z}$, we recover the function $f$. That is, we can write

$$f(x) = \sum_{n=-\infty}^{\infty} f(x)g(x-n) \tag{3}$$

with convergence $L^2$. But since the restriction of $f$ to $[n, n+1)$ is square integrable, it can be expanded into its $L^2$ convergent Fourier series leading to

$$f(x)\, g(x-n) = \sum_{k\in\mathbb{Z}} c_{nk} e^{2\pi ixk} \tag{4}$$

where for each $k \in \mathbb{Z}$,

$$\begin{aligned} c_{nk} &= \langle f(\cdot)g(\cdot - n), e^{2\pi ik\cdot}\rangle_{L^2([n,n+1))} \\ &= \int_{-\infty}^{\infty} f(x)\, g(x-n)\, e^{-2\pi ikx}\, dx = \langle f, g_{nk}\rangle \end{aligned}$$

with $g_{nk}(x) = g(x-n)\, e^{2\pi ikx}$. Here and in the sequel, $\langle\cdot,\cdot\rangle$ denotes the inner product on either $L^2(\mathbb{R})$, the space of Lebesgue measurable square integrable functions on $\mathbb{R}$, or $\ell^2(\mathbb{Z}^2)$ the space of square summable sequences on $\mathbb{Z}^2$. In addition, we use the notation $\|\cdot\| := \|\cdot\|_2$ to denote the corresponding norm. The context will make it clear which of the two spaces we are dealing with.

Substituting this in (4) and (3) leads to

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} f(x)g(x-n) \\ &= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_{nk} e^{2\pi ikx} g(x-n) \\ &= \sum_{k,n=-\infty}^{\infty} \langle f, g_{nk}\rangle g_{nk}(x). \end{aligned} \tag{5}$$

This expansion of $f$ is similar to Gabor's claim (1), with the following key differences:

- The coefficients in (5) are explicitly given and are linear in $f$.
- (1) is based on the Gaussian while (5) is based on the indicator function of $[0, 1)$.

• Finally, the expansion given in (5) is an orthonormal decomposition while the one given by (1) is not.

One of the goals of this section is to elucidate the difference in behavior between the two building blocks appearing in (1) and (5). In addition, we shall elaborate on the existence of orthonormal bases of the form $\{e^{2\pi i\beta k\cdot}g(\cdot - \alpha n) : k, n \in \mathbb{Z}\}$.

The two systems of functions in (1) and (2) are examples of Gabor (or Weyl-Heisenberg) systems. More specifically, for $a, b \in \mathbb{R}$ and a function $f$ defined on $\mathbb{R}$, let $M_b f(x) = e^{2\pi i b x} f(x)$ and $T_a f(x) = f(x-a)$ be, respectively, the modulation operator and the translation operator. The *Gabor system* generated by a function $g \in L^2(\mathbb{R})$, and parameters $\alpha, \beta > 0$, is the set of functions [13]

$$\begin{aligned}\mathcal{G}(g,\alpha,\beta) = \{M_{k\beta}T_{n\alpha}g(\cdot) \\ = e^{2\pi i k\beta\cdot}g(\cdot - n\alpha) : k, n \in \mathbb{Z}\}.\end{aligned}$$

Given $g \in L^2(\mathbb{R})$, and $\alpha, \beta > 0$, the Gabor system $\mathcal{G}(g,\alpha,\beta)$ is called a *frame* for $L^2(\mathbb{R})$ if there exist constants $0 < A \le B < \infty$ such that

$$\begin{aligned}A\|f\|^2 \le \sum_{k,n\in\mathbb{Z}} |\langle f, M_{k\beta}T_{n\alpha}g\rangle|^2 \\ \le B\|f\|^2 \qquad \forall f \in L^2(\mathbb{R}). \qquad (6)\end{aligned}$$

The constant $A$ is called a lower frame bound, while $B$ is called an upper frame bound. When $A = B$ we say that the Gabor frame is *tight*. In this case, the frame bound $A$ is referred to as the *redundancy* of the frame. Loosely speaking, the redundancy $A$ measures by how much the Gabor tight frame is overcomplete. A tight Gabor frame for which $A = B = 1$ is called a *Parseval* frame. Clearly, if $\mathcal{G}(g,\alpha,\beta)$ is an ONB then it is a Parseval frame, and conversely, if $\mathcal{G}(g,\alpha,\beta)$ is a Parseval frame and $\|g\| = 1$, then it is a Gabor ONB.

More generally, a Gabor frame is a "basis-like" system that can be used to decompose and/or reconstruct any square integrable function. As such, it will not come as a surprise that generalizations of certain tools from linear algebra might be useful in analyzing Gabor frames. We refer to [7, 13] for more background on Gabor frames, and summarize below some results needed in the sequel.

Suppose we would like to analyze $f$ using the Gabor system $\mathcal{G}(g,\alpha,\beta)$. We are then led to consider the correspondence that takes any square integrable function $f$ into the sequence $\{\langle f, M_{k\beta}T_{n\alpha}g\rangle\}_{k,n\in\mathbb{Z}}$. This correspondence is sometimes called the *analysis or decomposition operator* and denoted by

$$C_g : f \to \{\langle f, M_{k\beta}T_{n\alpha}g\rangle\}_{k,n\in\mathbb{Z}}.$$

Its (formal) adjoint $C_g^*$, called the *synthesis or reconstruction operator*, maps sequences $c = \{c_{kn}\}_{k,n\in\mathbb{Z}}$ to

$$C_g^* c = \sum_{k,n\in\mathbb{Z}} c_{kn} M_{k\beta}T_{n\alpha}g.$$

The composition of these two operators is called the *(Gabor) frame operator* associated to the Gabor system $\mathcal{G}(g,\alpha,\beta)$ and is defined by

$$Sf := S_{g,\alpha,\beta}f = C_g^* C_g(f) = \sum_{n,k\in\mathbb{Z}} \langle f, M_{k\beta}T_{n\alpha}g\rangle M_{k\beta}T_{n\alpha}g. \qquad (7)$$

It follows that, given $f \in L^2(\mathbb{R})$, we can (formally) write that

$$\begin{aligned}\langle Sf, f\rangle &= \langle C_g^* C_g f, f\rangle \\ &= \langle C_g f, C_g f\rangle = \sum_{k,n\in\mathbb{Z}} |\langle f, M_{k\beta}T_{n\alpha}g\rangle|^2.\end{aligned}$$

Therefore, $\mathcal{G}(g,\alpha,\beta)$ is a frame for $L^2$ if and only if there exist constants $0 < A \le B < \infty$ such that

$$A\|f\|_2^2 \le \langle Sf, f\rangle \le B\|f\|_2^2 \qquad \forall f \in L^2(\mathbb{R}).$$

In particular, $\mathcal{G}(g,\alpha,\beta)$ is a frame for $L^2$ if and only if the self-adjoint frame operator $S$ is bounded and positive definite. Furthermore, the optimal upper frame bound $B$ is the largest eigenvalue of $S$ while the optimal lower bound $A$ is its smallest eigenvalue. In addition, $\mathcal{G}(g,\alpha,\beta)$ is a tight frame for $L^2$ if and only if $S$ is a multiple of the identity.

Viewing a Gabor frame as an overcomplete "basis-like" object suggests that any square integrable function can be written in a non-unique way as a linear combination of the Gabor atoms $\{M_{k\beta}T_{n\alpha}g\}_{k,n\in\mathbb{Z}}$. Akin to the role of the pseudo-inverse in linear algebra, we single out one expansion that results in a somehow canonical representation of $f$ as a linear combination of $\{M_{k\beta}T_{n\alpha}g\}_{k,n\in\mathbb{Z}}$. To obtain this decomposition we need a few basic facts about the frame operator.

Suppose that $\mathcal{G}(g,\alpha,\beta)$ is a Gabor frame for $L^2$, and let $f \in L^2$. For all $(\ell, m) \in \mathbb{Z}^2$ the frame operator $S$ and $M_{\ell\beta}T_{m\alpha}$ commute. That is

$$S(M_{\ell\beta}T_{m\alpha}f) = M_{\ell\beta}T_{m\alpha}(S(f)) \text{ for all } (\ell, m) \in \mathbb{Z}^2.$$

It follows that $S^{-1}$ and $M_{\ell\beta}T_{m\alpha}$ also commute for all $(\ell, m) \in \mathbb{Z}^2$. As a consequence, given $f \in L^2(\mathbb{R})$ we have

$$\begin{aligned}f = S(S^{-1}f) &= \sum_{k,n\in\mathbb{Z}} \langle S^{-1}f, M_{k\beta}T_{n\alpha}g\rangle M_{k\beta}T_{n\alpha}g \\ &= \sum_{k,n\in\mathbb{Z}} \langle f, S^{-1}M_{k\beta}T_{n\alpha}g\rangle M_{k\beta}T_{n\alpha}g \\ &= \sum_{k,n\in\mathbb{Z}} \langle f, M_{k\beta}T_{n\alpha}\tilde{g}\rangle M_{k\beta}T_{n\alpha}g\end{aligned}$$

where $\tilde{g} = S^{-1}g \in L^2(\mathbb{R})$ is called the *canonical dual* of $g$. Similarly, by writing $f = S^{-1}(Sf)$ we get that

$$f = \sum_{k,n\in\mathbb{Z}} \langle f, M_{k\beta}T_{n\alpha}g\rangle M_{k\beta}T_{n\alpha}\tilde{g}.$$

The coefficients $\{\langle f, M_{k\beta}T_{n\alpha}\tilde{g}\rangle\}_{k,n\in\mathbb{Z}}$ give the least square approximation of $f$. Indeed, for $f \in L^2$, let $\tilde{c} = (\langle f, M_{k\beta}T_{n\alpha}\tilde{g}\rangle)_{k,n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}^2)$. Given any (other) sequence $(c_{k,n})_{k,n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}^2)$ such that

$$f = \sum_{k,n\in\mathbb{Z}} \tilde{c}_{k,n} M_{n\beta}T_{k\alpha}g = \sum_{k,n\in\mathbb{Z}} c_{k,n}M_{k\beta}T_{n\alpha}g,$$

we have

$$\begin{aligned}\|\tilde{c}\|_2^2 &= \sum_{k,n\in\mathbb{Z}} |\langle f, M_{k\beta}T_{n\alpha}\tilde{g}\rangle|^2 = \langle S^{-1}f, f\rangle \\ &= \sum_{k,n\in\mathbb{Z}} c_{k,n}\langle S^{-1}M_{k\beta}T_{n\alpha}g, f\rangle \\ &= \sum_{k,n\in\mathbb{Z}} c_{k,n}\overline{\tilde{c}_{k,n}} = \langle c, \tilde{c}\rangle.\end{aligned}$$

Consequently, $\langle c - \tilde{c}, \tilde{c}\rangle = 0$, leading to

$$\|c\|_2^2 = \|c - \tilde{c}\|_2^2 + \|\tilde{c}\|_2^2 \geq \|\tilde{c}\|_2^2$$

with equality if and only if $c = \tilde{c}$. In other words, for a Gabor frame $\mathcal{G}(g, \alpha, \beta)$, and given $f \in L^2$, among all expansions $f = \sum_{k,n\in\mathbb{Z}} c_{k,n}M_{k\beta}T_{n\alpha}g$, with $c = (c_{k,n})_{k,n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}^2)$, the coefficient $\tilde{c} = (\langle f, M_{k\beta}T_{n\alpha}\tilde{g}\rangle)_{k,n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}^2)$ has the least norm.

Because the frame operator $S$ is positive definite, $S^{1/2}$ is well defined and positive definite as well. Thus, we can write

$$\begin{aligned}f &= S^{-1/2}SS^{-1/2}f \\ &= \sum_{k,n}\langle f, S^{-1/2}M_{k\beta}T_{n\alpha}g\rangle S^{-1/2}M_{k\beta}T_{n\alpha}g \\ &= \sum_{k,n}\langle f, M_{k\beta}T_{n\alpha}g^\dagger\rangle M_{k\beta}T_{n\alpha}g^\dagger\end{aligned}$$

where $g^\dagger = S^{-1/2}g \in L^2$. In other words, $\mathcal{G}(g^\dagger, \alpha, \beta)$ is a Parseval frame.

Finally, assume that $A, B$ are the optimal frame bounds for $\mathcal{G}(g, \alpha, \beta)$. Then, for all $f \in L^2$, we have

$$\begin{aligned}\sum_{k,n\in Z} |\langle f, M_{k\beta}T_{n\alpha}\tilde{g}\rangle|^2 &= \langle S^{-1}f, f\rangle = \langle S^{-1}f, S(S^{-1}f)\rangle \\ &\leq B\|S^{-1}f\|^2 \leq \tilde{B}\|f\|^2\end{aligned}$$

and similarly, we have the lower bound

$$\langle S^{-1}f, f\rangle = \langle S^{-1}f, S(S^{-1}f)\rangle \geq A\|S^{-1}f\|^2 \geq \tilde{A}\|f\|^2.$$

Therefore, if $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame for $L^2(\mathbb{R})$, then so is $\mathcal{G}(\tilde{g}, \alpha, \beta)$ where $\tilde{g} = S^{-1}g \in L^2(\mathbb{R})$. We summarize all these facts in the following result.

**Proposition 1** (Reconstruction formulas for Gabor frame). *Let $g \in L^2(\mathbb{R})$ and $\alpha, \beta > 0$. Suppose that $\mathcal{G}(g, \alpha, \beta)$ is a frame for $L^2(\mathbb{R})$ with frame bounds $A, B$. Then the following statements hold.*

(a) *The Gabor system $\mathcal{G}(\tilde{g}, \alpha, \beta)$ with $\tilde{g} = S^{-1}g \in L^2$ is also a frame for $L^2$ with frame bounds $1/B, 1/A$. Furthermore, for each $f \in L^2$ we have the following reconstruction formulas:*

$$\begin{aligned}f &= \sum_{k,n\in\mathbb{Z}}\langle f, M_{k\beta}T_{n\alpha}\tilde{g}\rangle M_{k\beta}T_{n\alpha}g \\ &= \sum_{k,n\in\mathbb{Z}}\langle f, M_{k\beta}T_{n\alpha}g\rangle M_{k\beta}T_{n\alpha}\tilde{g}.\end{aligned}$$

*In addition, among all sequences $c = (c_{k,n})_{k,n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}^2)$ such that $f = \sum_{k,n\in\mathbb{Z}} c_{k,n}M_{k\beta}T_{n\alpha}g$, the sequence $\tilde{c} = (\langle f, M_{k\beta}T_{n\alpha}\tilde{g}\rangle)_{k,n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}^2)$ satisfies*

$$\|\tilde{c}\|_2^2 = \sum_{k,n\in\mathbb{Z}} |\langle f, M_{k\beta}T_{n\alpha}\tilde{g}\rangle|^2 \geq \sum_{k,n\in\mathbb{Z}} |c_{k,n}|^2 = \|c\|_2^2$$

*with equality if and only if $c = \tilde{c}$.*

(b) *The Gabor system $\mathcal{G}(g^\dagger, \alpha, \beta)$, where $g^\dagger = S^{-1/2}g \in L^2$, is a Parseval frame. In particular, each $f \in L^2$ has the following expansion:*

$$f = \sum_{k,n}\langle f, M_{k\beta}T_{n\alpha}g^\dagger\rangle M_{k\beta}T_{n\alpha}g^\dagger.$$

It is worth pointing out that the coefficients $(\langle f, M_{k\beta}T_{n\alpha}g\rangle)_{k,n}$ appearing in (6) and in (7) are samples of the *Short-Time Fourier Transform* (STFT) of $f$ with respect to $g$. This is the function $V_g$ defined on $\mathbb{R}^2$ by

$$V_gf(x, \xi) = \langle f, M_\xi T_x g\rangle = \int_{\mathbb{R}} f(t)\overline{g(t-x)}e^{-2\pi i t\xi}\,dt.$$

When $g \in L^2(\mathbb{R})$ is chosen such that $\|g\| = 1$, then $V_g$ is an isometry from $L^2(\mathbb{R})$ onto a closed subspace of $L^2(\mathbb{R}^2)$ and for all $f \in L^2(\mathbb{R})$

$$\int_{\mathbb{R}} |f(t)|^2\,dt = \iint_{\mathbb{R}^2} |V_gf(x, \xi)|^2\,dx\,d\xi. \tag{8}$$

Furthermore, for any $h \in L^2$ such that $\langle g, h\rangle \neq 0$

$$f(t) = \tfrac{1}{\langle g,h\rangle}\iint_{\mathbb{R}^2} V_gf(x, \xi)M_\xi T_x h(t)\,dx\,d\xi \tag{9}$$

where the integral is interpreted in the weak sense. We refer to [13, Chapter 3] for more on the STFT and related phase-space or time-frequency transformations.

The reconstruction formulas in Proposition 1 can be viewed as discretizations of the inversion formula for the STFT (9). In particular, sampling the STFT on the lattice $\alpha\mathbb{Z}\times\beta\mathbb{Z}$ and using the weights $\tilde{c} = (\langle f, M_{k\beta}T_{n\alpha}\tilde{g}\rangle)_{k,n\in\mathbb{Z}} = (V_{\tilde{g}}f(\alpha k, \beta n))_{k,n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}^2)$ perfectly reconstructs $f$. As such one can expect that in addition to the quality of the window $g$ (and hence $\tilde{g}$), the density of the lattice must play a role in establishing these formulas. Thus, it must not come as a surprise that the following results hold.

**Proposition 2** (Density theorems for Gabor frames). *Let $g \in L^2(\mathbb{R})$ and $\alpha, \beta > 0$.*

(a) *If $\mathcal{G}(g,\alpha,\beta)$ is a Gabor frame for $L^2(\mathbb{R})$ then $0 < \alpha\beta \leq 1$.*
(b) *If $\alpha\beta > 1$, then $\mathcal{G}(g,\alpha,\beta)$ is incomplete in $L^2(\mathbb{R})$.*
(c) *$\mathcal{G}(g,\alpha,\beta)$ is an orthonormal basis for $L^2(\mathbb{R})$ if and only if $\mathcal{G}(g,\alpha,\beta)$ is a tight frame for $L^2(\mathbb{R})$, $\|g\| = 1$, and $\alpha\beta = 1$.*

These results were proved using various techniques ranging from operator theory to signal analysis illustrating the multi-origin of Gabor frame theory. For a complete historical perspective on these density results we refer to [16].

At this point some questions arise naturally. For example, can one classify $g \in L^2(\mathbb{R})$ and the parameters $\alpha, \beta > 0$ such that $\mathcal{G}(g,\alpha,\beta)$ generates a frame or an ONB for $L^2(\mathbb{R})$? Despite some spectacular results both in the theory and the applications of Gabor frames [11], these problems have not been completely resolved. "The Frame Set Problem for Gabor Frames" will be devoted to addressing the *frame set problem for Gabor frames*. That is, given $g \in L^2(\mathbb{R})$, characterize the set of all $(\alpha,\beta) \in \mathbb{R}_+^2$ such that $\mathcal{G}(g,\alpha,\beta)$ is a frame. On the other hand, and as seen from part (c) of Proposition 2, Gabor ONB can only occur when $\alpha\beta = 1$. In addition to this restriction, there does not exist a Gabor ONB with $g \in L^2$ such that

$$\int_{-\infty}^{\infty} |x|^2\,|g(x)|^2\,dx \int_{-\infty}^{\infty} |\xi|^2\,|\hat{g}(\xi)|^2\,d\xi < \infty$$

where

$$\hat{g}(\xi) = \int_{-\infty}^{\infty} g(t)e^{-2\pi i t\xi}dt$$

is the Fourier transform of $g$. This uncertainty principle-type result known as the Balian-Low Theorem (BLT) precludes the existence of Gabor ONBs with well-localized windows. We refer to [2] and the references therein for a complete overview and the history of the BLT. We use the term *well-localized window* to describe functions $g$ that behave well in both time/space and frequency. For example, functions in certain Sobolev spaces, and more generally in the so-called modulation spaces, can be thought of as well-localized [13, Chapter 11]. With this in mind, the following result holds.

**Proposition 3** (The Balian-Low Theorem). *Let $g \in L^2(\mathbb{R})$ and $\alpha > 0$. If $\mathcal{G}(g,\alpha,1/\alpha)$ is an orthonormal basis for $L^2(\mathbb{R})$ then*

$$\int_{-\infty}^{\infty} |x|^2\,|g(x)|^2\,dx \int_{-\infty}^{\infty} |\xi|^2\,|\hat{g}(\xi)|^2\,d\xi = \infty.$$

In "Wilson Bases" we will introduce a modification of Gabor frames that will result in an ONB called *Wilson* basis with well-localized (or regular) window functions $g$. These ONBs were introduced by K. G. Wilson [20] under the name of generalized Wannier functions. The fact that these are indeed ONBs was later established by Daubechies, Jaffard, and Journé [9] who developed a systematic construction method for these kinds of systems. The method starts with constructing a tight Gabor frame of redundancy $A = 2$ and a well-localized window $g$. By then taking appropriate linear combinations of at most two Gabor atoms from this tight Gabor frame, the authors removed the original redundancy and obtained an ONB. While it is clear that tight Gabor frames with well-localized generators and arbitrary redundancy can be constructed, it remains an open question how or if one can get ONBs from these systems. We survey this question in "Wilson Bases," and mention that an interesting application involving the Wilson bases is the recent detection of the gravitational waves [4].

## The Frame Set Problem for Gabor Frames

As mentioned in the Introduction, a Gabor system is determined by three parameters: the shift parameters $\alpha, \beta$, and the window function $g$. Ideally, one would like to classify the set of all these three parameters for which the resulting system is a frame. However, and in general, this is a difficult question and we shall only consider the special case in which the window function $g$ is fixed and one seeks the set of all parameters $\alpha, \beta > 0$ for which the resulting system is a frame.

In this setting, the *frame set* of a function $g \in L^2(\mathbb{R})$ is defined as

$$\mathcal{F}(g) = \left\{(\alpha,\beta) \in \mathbb{R}_+^2 \,:\, \mathcal{G}(g,\alpha,\beta) \text{ is a frame}\right\}.$$

In general, determining $\mathcal{F}(g)$ for a given function $g$ is also an open problem. However, it is known that $\mathcal{F}(g)$ is an open subset of $\mathbb{R}_+^2$ if $g \in L^2(\mathbb{R})$ belongs to the modulation space $M^1(\mathbb{R})$ ([13]), i.e.,

$$\iint_{\mathbb{R}^2} |V_g g(x,\xi)|\,dxd\xi < \infty.$$

Examples of functions in this space include $g(x) = e^{-\pi|x|^2}$ or $g(x) = \frac{1}{\cosh x}$. In fact, for these specific functions more is known. Indeed,

$$\mathcal{F}(g) = \left\{(\alpha,\beta) \in \mathbb{R}_+^2 \,:\, \alpha\beta < 1\right\}$$

if $g \in \{e^{-\pi x^2}, \frac{1}{\cosh x}, e^{-x}\chi_{[0,+\infty]}(x), e^{-|x|}\}$, [14]. On the other hand when $g(x) = \chi_{[0,c]}(x), c > 0$, $\mathcal{F}(g)$ is a rather complicated set that has only been fully described in recent years by Dai and Sun [6].

Let $g(x) = e^{-|x|}$ and observe that $\hat{g}(\xi) = \frac{2}{1+4\pi^2\xi^2}$, which makes $g(x) = e^{-|x|}$ an example of a totally positive function of type 2. More generally, $g \in L^2(\mathbb{R})$ is *a totally positive function of type $M$*, where $M$ is a natural number, if its Fourier transform has the form $\hat{g}(\xi) = \prod_{k=1}^{M}(1 + 2\pi i\delta_k\xi)^{-1}$ where $\delta_k \neq \delta_\ell \in \mathbb{R}$ for $k \neq \ell$. It was proved that for all such functions $g$,

$$\mathcal{F}(g) = \left\{(\alpha,\beta) \in \mathbb{R}_+^2 \,:\, \alpha\beta < 1\right\}.$$

A similar result holds for the class of totally positive functions of Gaussian type, which are functions whose Fourier transforms have the form $\hat{g}(\xi) = \prod_{k=1}^{M}(1 + 2\pi i\delta_k\xi)^{-1} \times e^{-c\xi^2}$ where $\delta_1 \neq \delta_2 \neq \ldots \neq \delta_M \in \mathbb{R}$ and $c > 0$. We refer to [14] for a survey of the structure of $\mathcal{F}(g)$ not only for the rectangular lattices we consider here, but more general Gabor frames on discrete (countable) sets $\Lambda \subset \mathbb{R}^2$.

However, there are other "simple" functions $g$ for which determining $\mathcal{F}(g)$ remains largely a mystery. In the rest of this section we consider the frame set for the $B$ splines $g_N$ given by

$$\begin{cases} g_1(x) = \chi_{[-1/2,1/2]}, & \text{and} \\ g_N(x) = g_1 * g_{N-1}(x) & \text{for} \quad N \geq 2. \end{cases}$$

The characterization of $\mathcal{F}(g_N)$ for $N \geq 2$ is considered as one of the six main problems in frame theory. Due to the fact that $g_N \in M^1(\mathbb{R})$ for $N \geq 2$, we know that $\mathcal{F}(g_N)$ is an open subset of $\mathbb{R}^2_+$. The current description of points in this set can be found in [1, 5, 18].

For example, consider the case $N = 2$ where

$$\begin{aligned} g_2(x) &= \chi_{[-1/2,1/2]} * \chi_{[-1/2,1/2]}(x) = \max(1 - |x|, 0) \\ &= \begin{cases} 1 + x & \text{if} \quad x \in [-1, 0] \\ 1 - x & \text{if} \quad x \in [0, 1]. \end{cases} \end{aligned}$$

The known results on $\mathcal{F}(g_2)$ can be summarized as follows.

**Proposition 4** (Frame set of the 2-spline, $g_2$). *The following statements hold.*

(a) *If $(\alpha, \beta) \in \mathcal{F}(g_2)$, then $\alpha\beta < 1$ and $\alpha < 2$* [8]. *This is illustrated by the green region in Figure 1.*

(b) *Assume that $1 \leq \alpha < 2$ and $0 < \beta < \frac{1}{\alpha}$. Then, $(\alpha, \beta) \in \mathcal{F}(g_2)$* [5]. *This is illustrated by part of the yellow region in Figure 1.*

(c) *Assume that $0 < \alpha < 2$, and $0 < \beta \leq \frac{2}{2+\alpha}$. Then, $(\alpha, \beta) \in \mathcal{F}(g_2)$, and there is a unique dual $h \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that $\operatorname{supp} h \subseteq \left[-\frac{\alpha}{2}, \frac{\alpha}{2}\right]$* [5]. *This is illustrated by the blue region in Figure 1.*

(d) *Assume that $0 < \alpha < 2$, and $\frac{2}{2+\alpha} < \beta \leq \frac{4}{2+3\alpha}$. Then, $(\alpha, \beta) \in \mathcal{F}(g_2)$, and there is a unique dual $h \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that $\operatorname{supp} h \subseteq \left[-\frac{3\alpha}{2}, \frac{3\alpha}{2}\right]$* [18]. *This is illustrated by the magenta region in Figure 1.*

(e) *Assume that $0 < \alpha < 1/2$, and $\frac{4}{2+3\alpha} < \beta \leq \frac{2}{1+\alpha}$. Then, $(\alpha, \beta) \in \mathcal{F}(g_2)$, and there is a unique dual $h \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that $\operatorname{supp} h \subseteq \left[-\frac{5\alpha}{2}, \frac{5\alpha}{2}\right]$* [1]. *This is illustrated by the cyan region in Figure 1.*

(f) *Assume that $\frac{1}{2} \leq \alpha \leq \frac{4}{5}$, and $\frac{4}{2+3\alpha} < \beta \leq \frac{6}{2+5\alpha}$, with $\beta > 1$. Then, $(\alpha, \beta) \in \mathcal{F}(g_2)$, and there is a unique dual $h \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that $\operatorname{supp} h \subseteq \left[-\frac{5\alpha}{2}, \frac{5\alpha}{2}\right]$* [1]. *This is illustrated by the cyan region in Figure 1.*

(g) *Assume that $\frac{2}{3} \leq \alpha \leq 1$, and $\frac{4}{2+3\alpha} < \beta < 1$. Then, $(\alpha, \beta) \in \mathcal{F}(g_2)$, and there is a unique compactly supported dual $h \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$* [1]. *This is illustrated by the cyan region in Figure 1.*

(h) *If $0 < \alpha < 2$, $\beta = 2, 3, \ldots$, and $\alpha\beta < 1$, then $(\alpha, \beta) \notin \mathcal{F}(g_2)$* [14]. *This is illustrated by the red horizontal lines in Figure 1.*

These results are illustrated in Figure 1, where except for the red regions, all other regions are contained in $\mathcal{F}(g_2)$. For the proofs we refer to [1, 5, 14, 18], and the references therein. But we point out that the main idea in establishing parts (c–g) is based on the following result. Before stating it we recall that for $\alpha, \beta > 0$ and $g \in L^2(\mathbb{R})$, the Gabor system $\mathcal{G}(g, \alpha, \beta)$ is called a *Bessel sequence* if only the upper bound in (6) is satisfied for some $B > 0$.

**Proposition 5** (Sufficient and necessary condition for dual Gabor frames). *Let $\alpha, \beta > 0$ and $g, h \in L^2(\mathbb{R})$. The Bessel sequences $\mathcal{G}(g, \alpha, \beta)$ and $\mathcal{G}(h, \alpha, \beta)$ are dual Gabor frames if and only if*

$$\sum_{k\in\mathbb{Z}} \overline{g(x - n/\beta - k\alpha)}\, h(x - k\alpha) = \beta\delta_{n,0} \qquad a.e.\ x \in [0, \alpha].$$

Using this result with $g = g_N$ and imposing that $h$ is also compactly supported leads one to seek an appropriate (finite) square matrix from the (infinite) linear system

$$\sum_{k\in\mathbb{Z}} g_N(x - \tfrac{\ell}{\beta} + k\alpha) h(x + k\alpha) = \beta\delta_\ell \qquad \text{for almost every } x \in [-\tfrac{\alpha}{2}, \tfrac{\alpha}{2}].$$

In particular, the region $\{(\alpha, \beta) \in \mathbb{R}^2_+ : 0 < \alpha\beta < 1\}$ can be partitioned into subregions $R_m$, $m \geq 1$, such that a $(2m - 1) \times (2m - 1)$ matrix $G_m$ can be extracted from the above system leading to

$$G_m(x) \begin{bmatrix} h(x + (1 - m)\alpha) \\ \vdots \\ h(x) \\ \vdots \\ h(x + (m - 1)\alpha) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ \beta \\ \vdots \\ 0 \end{bmatrix} \qquad \text{for almost every } x \in [-\alpha/2, \alpha/2].$$

Choosing $N = 2$ results in parts (c–g) of Proposition 4, for the cases $m = 1, 2$, and 3. For these cases, one proves that the matrix $G_m(x)$ is invertible for almost every $x \in [-\alpha/2, \alpha/2]$. However, only a subregion for the case $m = 3$ has been settled in [1]. It is also known that the remaining part of this subregion contains some obstruction points, for example the line $\beta = 2$ in Figure 1. Nonetheless, it seems that one should be able to prove that the region

$$\{(\alpha, \beta) : \tfrac{1}{2} \leq \alpha < 1,\ \tfrac{6}{2+5\alpha} \leq \beta < \tfrac{2}{1+\alpha},\ \beta > 1\}$$

is also contained in $\mathcal{F}(g_2)$. But this is still open.

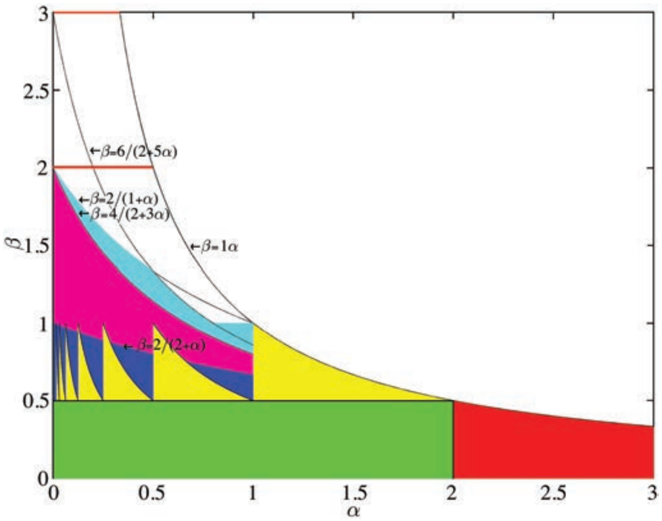


**Figure 1.** A sketch of $\mathcal{F}(g_2)$. The red region contains points $(\alpha, \beta)$ for which $\mathcal{G}(g_2, \alpha, \beta)$ is not a frame. All other colors indicate the frame property. The green region is the classical: "painless expansions" [8]. For the yellow and magenta regions see [5]. The blue and the cyan regions are respectively from [18] and [1].

We end this section by observing that the frame set problem is a special case of the more general question of characterizing the *full frame set* $\mathcal{F}_{\text{full}}(g)$ of a function $g$, where

$$\mathcal{F}_{\text{full}}(g) = \{\Lambda \subset \mathbb{R}^2 : \mathcal{G}(g, \Lambda) \text{ is a frame}\}$$

where $\Lambda$ is the lattice $\Lambda = A\mathbb{Z}^2 \subset \mathbb{R}^2$ with $\det A \neq 0$. The only general result known in this case is for $g(x) = e^{-a|x|^2}$ with $a > 0$ in which case

$$\mathcal{F}_{\text{full}}(g) = \{\Lambda \subset \mathbb{R}^2 : \text{Vol}\Lambda < 1\},$$

where the volume of $\Lambda$ is defined by $\text{Vol}(\Lambda) = |\det A|$, see [14].

## Wilson Bases

By the BLT (Proposition 3) and Proposition 2(c), we know that $\mathcal{G}(g, \alpha, 1/\alpha)$ cannot be an ONB if $g$ is well-localized in the time-frequency plane. To overcome the BLT, K. G. Wilson introduced an ONB $\{\psi_{n,\ell}, n \in \mathbb{N}_0, \ell \in \mathbb{Z}\}$, where $\psi_{0,\ell}(x) = \psi_\ell(x)$ and for $n \geq 1$, $\psi_{n,\ell}(x) = \psi_\ell(x - n)$, and such that $\hat{\psi}_{n,\ell}$ is localized around $\pm n$, that is, $\psi_{n,\ell}$ is a bimodal function. Wilson presented numerical evidence that this system of functions is an ONB for $L^2(\mathbb{R})$. In 1992, Daubechies, Jaffard, and Journé formalized Wilson's ideas and constructed examples of bimodal Wilson bases generated by smooth functions. To be specific, the Wilson system associated with a given function $g \in L^2$ is $\mathcal{W}(g) = \{\psi_{j,m} : j \in \mathbb{Z}, m \in \mathbb{N}_0\}$ where

$$\psi_{j,m}(x) = \begin{cases} g(x-j) & \text{if } j \in \mathbb{Z} \\ \frac{1}{\sqrt{2}} T_{\frac{j}{2}}(M_m + (-1)^{j+m} M_{-m}) g(x) & \\ & \text{if } (j,m) \in \mathbb{Z} \times \mathbb{N}, \end{cases} \tag{10}$$

which can simply be rewritten as

$$\psi_{j,m}(x) = \begin{cases} \sqrt{2} \cos 2\pi m x \, g(x - \frac{j}{2}), & \text{if } j+m \text{ is even} \\ \sqrt{2} \sin 2\pi m x \, g(x - \frac{j}{2}), & \text{if } j+m \text{ is odd.} \end{cases}$$

It is not hard to see $\{\psi_{j,m}\}$ is an ONB for $L^2(\mathbb{R})$ if and only if

$$\begin{cases} \|\psi_{j,m}\| = 1 \text{ for all } (j,m) \in \mathbb{N}_0 \times \mathbb{Z} \\ \langle f, h \rangle = \sum_{j,m} \langle f, \psi_{j,m} \rangle \overline{\langle h, \psi_{j,m} \rangle} \text{ for all } f, h \in L^2. \end{cases}$$

Assuming that $g$ and $\hat{g}$ are smooth enough, $\hat{g}$ real-valued, one can show that this is equivalent to

$$\sum_{m \in \mathbb{Z}} \hat{g}(\xi - m) \hat{g}(\xi - m + 2j) = \delta_{j,0}$$
$$\text{for almost every } \xi, \text{ and for each } j \in \mathbb{Z}.$$

It follows that one can construct compactly supported $\hat{g}$ that will solve this system of equations. On the other hand, one can convert these equations into a single one by using another time-frequency analysis tool, the Zak transform which we now define. For $f \in L^2(\mathbb{R})$ we let $Zf : [0,1) \times [0,1) \to \mathbb{C}$ be given by

$$Zf(x, \xi) = \sqrt{2} \sum_{j \in \mathbb{Z}} f(2(x-j)) e^{2\pi i j \xi}.$$

$Z$ is a unitary map from $L^2(\mathbb{R})$ onto $L^2([0,1)^2)$ and enjoys some periodicity-like properties [13, Chapter 8]. Using the Zak transform, and under suitable regularity assumptions on $g$ and $\hat{g}$, one can show that $\{\psi_{j,m}\}$ is an ONB if and only if

$$|Z\hat{g}(x,\xi)|^2 + |Z\hat{g}(x, \xi + \tfrac{1}{2})|^2 = 2$$
$$\text{for almost every } (x,\xi) \in [0,1]^2.$$

Real-valued functions $g$ solving this equation can be constructed with the additional requirement that both $g$ and $\hat{g}$ have exponential decay.

To connect this Wilson system to Gabor frames, we use once again the Zak transform, and observe that the frame operator of the Gabor system $\mathcal{G}(g, 1, 1/2)$ is a multiplication operator in the Zak transform domain, that is

$$ZS_g f(x, \xi) = M(x, \xi) Zf(x, \xi)$$

where $M(x,\xi) = |Zg(x,\xi)|^2 + |Zg(x,\xi-\frac{1}{2})|^2$. Consequently, $\mathcal{G}(g,1,1/2)$ is a tight frame if and only if

$$M(x,\xi) = |Zg(x,\xi)|^2 + |Zg(x,\xi-\tfrac{1}{2})|^2 = A \quad \text{for almost every } (x,\xi) \in [0,1]^2,$$

where $A$ is a constant. These ideas were used in [9] resulting in the following.

**Proposition 6** ([9]). *There exist unit-norm real-valued functions $g \in L^2(\mathbb{R})$ with the property that both $g$ and $\hat{g}$ have exponential decay and such that the Gabor system $\mathcal{G}(g,1,1/2)$ is a tight frame for $L^2(\mathbb{R})$ if and only if the associated Wilson system $\mathcal{W}(g)$ is an orthonormal basis for $L^2(\mathbb{R})$.*

Proposition 6 also provides an alternate view of the Wilson ONB. Indeed, each function in (10) is a linear combination of at most two Gabor functions from a tight Gabor frame $\mathcal{G}(g,1,1/2)$ of redundancy 2. Furthermore, such Gabor systems can be constructed so that the generators are well-localized in the time-frequency plane. Suppose now that we are given a tight Gabor system $\mathcal{G}(g,\alpha,\beta)$ where $(\alpha\beta)^{-1} = N \in \mathbb{N}$ where $N > 2$. Hence, the redundancy of this tight frame is $N$. Can a Wilson-type ONB (generated by well-localized window) be constructed from this system by taking appropriate linear combinations? This problem was posed by Gröchenig for the case $\alpha = 1$ and $\beta = 1/3$ [13, Section 8.5], and to the best of our knowledge it is still open. If one is willing to give up on the orthogonality, one can prove the existence of Parseval Wilson-type frames for $L^2(\mathbb{R})$ from Gabor tight frames of redundancy 3. More recently, explicit examples have been constructed starting from Gabor tight frames of redundancy $\frac{1}{\beta} \in \mathbb{N}$ where $N \geq 3$.

**Proposition 7.** [3] *For any $\beta \in [1/4, 1/2)$ there exists a real-valued function $g \in S(\mathbb{R})$ such that the following equivalent statements hold.*

(i) *$\mathcal{G}(g,1,\beta)$ is a tight Gabor frame of redundancy $\beta^{-1}$.*

(ii) *The associated Wilson system given by*

$$\mathcal{W}(g,\beta) = \{\psi_{j,m} : j \in \mathbb{Z}, m \in \mathbb{N}_0\} \tag{11}$$

*where*

$$\psi_{j,m}(x) = \begin{cases} \sqrt{2\beta}\, g_{2j,0}(x) = \sqrt{2\beta}\, g(x - 2\beta j) & \text{if } j \in \mathbb{Z}, m = 0, \\ \sqrt{\beta}\left[e^{-2\pi i\beta jm} g_{j,m}(x) + (-1)^{j+m} e^{2\pi i\beta jm} g_{j,-m}(x)\right] & \text{if } (j,m) \in \mathbb{Z} \times \mathbb{N} \end{cases} \tag{12}$$

*is a Parseval frame for $L^2(\mathbb{R})$.*

*If in addition $\beta = \frac{1}{2n}$ where $n$ is any odd natural number, then we can choose $g$ to be real-valued such that both $g$ and $\hat{g}$ have exponential decay.*

To turn these Parseval (Wilson) systems into ONBs, one needs to ensure that $\|\psi_{j,m}\|_2 = 1$ for all $j,m$. This requires in particular that $\|g\| = \frac{1}{\sqrt{2\beta}}$, which seems to be incompatible with all the other conditions imposed on $g$. It has then been suggested in [3] that to obtain a Wilson ONB with redundancy different from 2, one must modify in a fundamental way (12). For example, if we want to have a Wilson ONB with $\alpha = 1, \beta = 1/3$, it seems that one should take linear combinations of three Gabor atoms instead of the two in Proposition 7. While we have no proof of this claim, it seems to be supported by a recent construction of multivariate Wilson ONBs that is not a tensor product on 1-Wilson ONBs. In this new approach a relationship between these bases and the theory of Generalized Shift Invariant Spaces (GSIS) was used to construct (non-separable) well-localized Wilson ONBs for $L^2(\mathbb{R}^d)$ starting from tight Gabor frames of redundancy $2^k$ where $k = 0,1,2,\ldots d-1$. In particular, the functions in the corresponding Wilson systems are linear combinations of $2^k$ elements from the tight Gabor frame.

## HRT

In any application involving Gabor frames, a truncation is needed, and one considers only a finite number of Gabor atoms. As such, and from a numerical point of view, determining the condition number of the projection matrix

$$P_{N,K} = \sum_{n=-N}^{N} \sum_{k=-K}^{K} \langle \cdot, M_{k\beta}T_{n\alpha}g\rangle M_{k\beta}T_{n\alpha}g$$

for $N, K \geq 1$ is useful. In fact, and beyond any numerical considerations, one could ask if this operator is invertible, which will be the case if $\{M_{k\beta}T_{n\alpha}g,\ |n| \leq N, |k| \leq K\}$ was linearly independent. Clearly this is the case if the starting Gabor frame was an ONB. However, and in general, this is not known. In fact, this is a special case of a broader problem that we consider in this last section. This fascinating (due in part to the simplicity of its statement) open problem was posed in 1990 by C. Heil, J. Ramanathan, and P. Topiwala, and is now referred to as the HRT conjecture [17].

**Conjecture 1** (The HRT Conjecture). *Given any $0 \neq g \in L^2(\mathbb{R})$ and $\Lambda = \{(a_k,b_k)\}_{k=1}^N \subset \mathbb{R}^2$, $\mathcal{G}(g,\Lambda)$ is a linearly independent set in $L^2(\mathbb{R})$, where*

$$\mathcal{G}(g,\Lambda) = \{e^{2\pi i b_k \cdot} g(\cdot - a_k),\ k = 1,2,\ldots,N\}.$$

To be more explicit, the conjecture claims the following:

Given $c_1, c_2, \dots, c_N \in \mathbb{C}$ such that

$$\sum_{k=1}^{N} c_k M_{b_k} T_{a_k} g(x) = \sum_{k=1}^{N} c_k e^{2\pi i b_k x} g(x - a_k) = 0$$
$$\text{for almost every } x \in \mathbb{R} \implies c_1 = c_2 = \dots = c_N = 0. \tag{13}$$

The conjecture is still generally open even if one assumes that $g \in S(\mathbb{R})$, the space of $C^\infty$ functions that decay faster than any polynomial.

Observe that for a given $\Lambda = \{(a_k, b_k)\}_{k=1}^N \subset \mathbb{R}^2$, and $g \in L^2(\mathbb{R})$, we can always assume that $(a_1, b_1) = (0, 0)$; if not, applying $M_{-b_1} T_{-a_1}$ to $\mathcal{G}(g, \Lambda)$ results in $\mathcal{G}(M_{-b_1} T_{-a_1} g, \Lambda')$ where $\Lambda'$ will include the origin. In addition, by rotating and scaling if necessary, we may also assume that $\Lambda$ contains $(0, 1)$. This will result in unitarily changing $g$. Finally, by applying a shear matrix, we may assume that $\Lambda$ contains $(a, 0)$ for some $a \neq 0$. Consequently, given $\Lambda = \{(a_k, b_k)\}_{k=1}^N \subset \mathbb{R}^2$ with $N \geq 3$, we shall assume that $\{(0,0), (0,1), (a,0)\} \subseteq \Lambda$, for some $a \neq 0$.

To illustrate some of the difficulties arising in investigating this problem, we would like to give some ideas of the proof of the conjecture when $N \leq 3$ and $0 \neq g \in L^2(\mathbb{R})$. Let us first consider the case $N = 2$, and from the above observations we can assume that $\Lambda = \{(0,0), (0,1)\}$. Suppose that $c_1, c_2 \in \mathbb{C}$ such that $c_1 g + c_2 M_1 g = 0$. This is equivalent to

$$(c_1 + c_2 e^{2\pi i x}) g(x) = 0.$$

Since $g \neq 0$ and $c_1 + c_2 e^{2\pi i x}$ is a trigonometric polynomial, we see that $c_1 = c_2 = 0$.

Now consider the case $N = 3$, and assume that $\Lambda = \{(0,0), (0,1), (a,0)\}$ where $a > 0$ is such that $\mathcal{G}(g, \Lambda)$ is linearly dependent. Thus there are non-zero complex numbers $c_1, c_2$ such that

$$g(x-a) = (c_1 + c_2 e^{2\pi i x}) g(x) = P(x) g(x) \qquad \forall\, x \in S$$

where $S \subset \operatorname{supp}(g) \cap (0,1)$ has positive Lebesgue measure. Note that $P(x)$ is a 1-periodic trigonometric polynomial that is nonzero almost everywhere. We can now iterate this last equation along $\pm na$ for $n > 0$ to obtain

$$\begin{cases} g(x - na) = g(x) \prod_{j=0}^{n-1} P(x - ja) = g(x) P_n(x) \\ g(x + na) = g(x - a) \prod_{j=0}^{n} P(x + ja)^{-1} \\ \qquad\qquad\quad = g(x) Q_n(x). \end{cases}$$

Consequently, $g(x + na) = g(x) Q_n(x) = g(x) P_n(x + na)^{-1}$ implying that

$$Q_n(x) = P_n(x + na)^{-1} \qquad x \in S. \tag{14}$$

In addition, using the fact that $g \in L^2(\mathbb{R})$ one can conclude that

$$\lim_{n \to \infty} P_n(x) = \lim_{n \to \infty} Q_n(x) = 0 \qquad a.e.\, x \in S. \tag{15}$$

However, one can show that (14) and (15) cannot hold simultaneously by distinguishing the case $a \in \mathbb{Q}$ and the case $a$ is irrational. Hence, the HRT conjecture holds when $\#\Lambda = 3$. We refer to [17] for details.

In addition to the fact that the HRT conjecture is true for any set of three distinct points, the known results generally fall into the following categories, see [15] and [19, Proposition 1] for details.

**Proposition 8** (HRT for arbitrary set $\Lambda \subset \mathbb{R}^2$). *Suppose that $\Lambda \subset \mathbb{R}^2$ is a finite subset of distinct points. Then the HRT conjecture holds in each of the following cases.*

(a) *$g$ is compactly supported, or just supported within a half-interval $(-\infty, a]$, or $[a, \infty)$.*
(b) *$g(x) = p(x) e^{-\pi x^2}$ where $p$ is a polynomial.*
(c) *$g$ is such that $\lim_{x \to \infty} |g(x)| e^{cx^2} = 0$ for all $c > 0$.*
(d) *$g$ is such that $\lim_{x \to \infty} |g(x)| e^{cx \log x} = 0$ for all $c > 0$.*

**Proposition 9** (HRT for arbitrary $g \in L^2(\mathbb{R})$). *Suppose that $0 \neq g \in L^2(\mathbb{R})$ is arbitrary. Then the HRT conjecture holds in each of the following cases.*

(a) *$\Lambda$ is a finite set with $\Lambda \subset A(\mathbb{Z}^2) + z$ where $A$ is a full rank $2 \times 2$ matrix and $z \in \mathbb{R}^2$. In particular, Conjecture 1 holds when $\#\Lambda \leq 3$.*
(b) *$\#\Lambda = 4$ where two of the four points in $\Lambda$ lie on a line and the remaining two points lie on a second parallel line. Such a set $\Lambda$ is called a $(2,2)$ configuration, see Figure 2 for an illustrative example.*
(c) *$\Lambda$ consists of collinear points.*
(d) *$\Lambda$ consists of $N - 1$ collinear and equi-spaced points, with the last point located off this line.*

**Figure 2.** Example of a $(2,2)$ configuration.

We observe that when $\Lambda$ consists of collinear points, the

HRT conjecture reduces to the question of linear independence of (finite) translates of $L^2$ functions.

To date and to the best of our knowledge Proposition 8 and Proposition 9 are the most general known results on the HRT conjecture. Nonetheless, we give a partial list of known results when one makes restrictions on both the function $g$ and the set $\Lambda$.

**Proposition 10** (HRT in special cases). *The HRT conjecture holds in each of the following cases.*

(a) $g \in S(\mathbb{R})$, *and* $\#\Lambda = 4$ *where three of the four points in* $\Lambda$ *lie on a line and the fourth point is off this line. Such a set* $\Lambda$ *is called a* $(1,3)$ *configuration, see Figure 3 for an illustrative example.*

(b) $g \in L^2(\mathbb{R})$ *is ultimately positive, and* $\Lambda = \{(a_k, b_k)\}_{k=1}^N \subset \mathbb{R}^2$ *is such that* $\{b_k\}_{k=1}^N$ *are independent over the rationals* $\mathbb{Q}$.

(c) $\#\Lambda = 4$, *when* $g \in L^2(\mathbb{R})$ *is ultimately positive,* $g(x)$ *and* $g(-x)$ *are ultimately decreasing.*

(d) $g \in L^2(\mathbb{R})$ *is real-valued, and* $\#\Lambda = 4$ *is a* $(1,3)$ *configuration.*

(e) $g \in S(\mathbb{R})$ *is a real-valued function in* $S(R)$ *and* $\#\Lambda = 4$.

**Figure 3.** Example of a $(1,3)$ configuration.

Recently, some of the techniques used to establish the HRT for $(2,2)$ configurations were extended to deal with some special $(3,2)$ configurations [19]. From these results, and when restricting to real-valued functions, it was concluded that the HRT holds for certain sets of four points. We briefly describe this method here.

Let $\Lambda = \{(0,0),(0,1),(a_0,0),(a,b)\}$, and assume that $\Lambda$ is neither a $(1,3)$ nor a $(2,2)$ configuration. Let $0 \neq g \in L^2(\mathbb{R})$ be a real-valued function. Suppose that $\mathcal{G}(g,\Lambda)$ is linearly dependent. Then there exist $0 \neq c_k \in \mathbb{C}$, $k = 1,2,3$ such that

$$T_{a_0}g = c_1 g + c_2 M_1 g + c_3 M_b T_a g.$$

Taking the complex conjugate of this equation leads to

$$T_{a_0}g = \overline{c}_1 g + \overline{c}_2 M_{-1} g + \overline{c}_3 M_{-b} T_a g.$$

Taking the difference of these two equations gives

$$(c_1 - \overline{c}_1)g + c_2 M_1 g - \overline{c}_2 g + c_3 M_b T_a g - \overline{c}_3 M_{-b} T_a g = 0.$$

Since $c_2, c_3 \neq 0$ we conclude that $\mathcal{G}(g,\Lambda')$, where $\Lambda' = \{(0,0),(0,1),(0,-1),(a,b),(a,-b)\}$ is a (symmetric) $(3,2)$ configuration, is linearly dependent. Consequently, we have proved the following result.

**Proposition 11.** *Let* $0 \neq g \in L^2(\mathbb{R})$ *be a real-valued function. Suppose that* $(a,b) \in \mathbb{R}^2$ *is such that* $\mathcal{G}(g,\Lambda_0)$ *is linearly independent where* $\Lambda_0 = \{(0,0),(0,1),(0,-1),(a,b),(a,-b)\}$. *Then for all* $0 \neq c \in \mathbb{R}$, $\mathcal{G}(g,\Lambda)$ *is linearly independent where* $\Lambda = \{(0,0),(0,1),(c,0),(a,b)\}$.

In [19, Theorem 6, Theorem 7] it was proved that the hypothesis of Proposition 11 is satisfied when $g \in L^2(\mathbb{R})$ (not necessarily real-valued) for certain values of $a$ and $b$. These results were viewed as a *restriction principle* for the HRT, whereby proving the conjecture for special sets of $N+1$ points one can establish it for certain related sets of $N$ points. In addition, a related *extension principle* that can be viewed as an induction-like technique was introduced. The premise of this principle is based on the following question. Suppose that the HRT conjecture holds for all $g \in L^2(\mathbb{R})$ and a set $\Lambda = \{(a_k,b_k)\}_{k=1}^N \subset \mathbb{R}^2$. For which points $(a,b) \in \mathbb{R}^2 \backslash \Lambda$ will the conjecture remain true for the same function $g$ and the new set $\Lambda' = \Lambda \cup \{(a,b)\}$?

We elaborate on this method for $\#\Lambda = 3$. Let $g \in L^2(\mathbb{R})$ with $\|g\|_2 = 1$ and suppose that $\Lambda = \{(0,0),(0,1),(a_0,0)\}$. We denote $\Lambda' = \Lambda \cup \{(a,b)\} = \{(0,0),(0,1),(a_0,0),(a,b)\}$. Since $\mathcal{G}(g,\Lambda)$ is linearly independent, the Gramian of this set of functions is a positive definite matrix. We recall that the Gramian of a set of $N$ vectors $\{f_k\}_{k=1}^N \subset L^2(\mathbb{R})$ is the (positive semi-definite) $N \times N$ matrix $(\langle f_k, f_\ell\rangle)_{k,\ell=1}^N$. In the case at hand, the $4\times 4$ Gramian matrix $G := G_g(a,b)$ of $\mathcal{G}(g,\Lambda')$ can be written in the following block structure:

$$G = \begin{bmatrix} A & u(a,b) \\ u(a,b)^* & 1 \end{bmatrix} \tag{16}$$

where $A$ is the $3\times 3$ Gramian of $\mathcal{G}(g,\Lambda)$ and

$$u(a,b) = \begin{bmatrix} V_g g(a,b) \\ V_g g(a,b-1) \\ e^{-2\pi i a_0 b} V_g g(a-a_0,b) \end{bmatrix}$$

and $u(a,b)^*$ is the adjoint of $u(a,b)$. By construction $G$ is positive semi-definite for all $(a,b) \in \mathbb{R}^2$, and we seek the set of points $(a,b) \in \mathbb{R}^2 \backslash \Lambda$ such that $G$ is positive definite. We can encode this information into the determinant of this matrix, or into a related function $F : \mathbb{R}^2 \to [0, \infty)$ given by

$$F(a,b) = \langle A^{-1}u(a,b), u(a,b)\rangle. \tag{17}$$

The following was proved in [19].

**Proposition 12** (The HRT Extension function). *Given the above notations the function $F$ satisfies the following properties.*

(i) $0 \leq F(a,b) \leq 1$ *for all* $(a,b) \in \mathbb{R}^2$, *and moreover,* $F(a,b) = 1$ *if* $(a,b) \in \Lambda$.
(ii) $F$ *is uniformly continuous and* $\lim_{|(a,b)|\to\infty} F(a,b) = 0$.
(iii) $\iint_{\mathbb{R}^2} F(a,b)dadb = 3$.
(iv) $\det G_g(a,b) = (1 - F(a,b)) \det A$.

*Consequently, there exists $R > 0$ such that the HRT conjecture holds for $g$ and $\Lambda' = \Lambda \cup \{(a,b)\} = \{(0,0),(0,1),(a_0,0),(a,b)\}$ whenever $|(a,b)| > R$.*

We conclude the paper by elaborating on the case $\Lambda = 4$. Let $\Lambda \subset \mathbb{R}^2$ contain four distinct points, and assume without loss of generality that $\Lambda = \{(0,0),(0,1),(a_0,0),(a,b)\}$.

When $b = 0$ and $a = -a_0$ or $a = 2a_0$, then $\Lambda$ is a $(1,3)$ configuration with the additional fact that its three collinear points are equi-spaced. This case is handled by Fourier methods as was done in [17]; see Proposition 9 (d). But, for general $(1,3)$ configurations, the Fourier methods are ineffective. Nonetheless, this case was considered by Demeter [10], who proved that the HRT conjecture holds for all $(1,3)$ configurations when $g \in S(\mathbb{R})$, and for a family of $(1,3)$ configurations when $g \in L^2(\mathbb{R})$. It was later proved that in fact, the HRT holds for all functions $g \in L^2(\mathbb{R})$ and for almost all (in the sense of Lebesgue measure) $(1,3)$ configurations. In fact, more is true, in the sense that for $g \in L^2(\mathbb{R})$, there exists at most one (equivalence class of) $(1,3)$ configuration $\Lambda_0$ such that $\mathcal{G}(g,\Lambda_0)$ is linearly dependent [19]. Here, we say that two sets $\Lambda_1$ and $\Lambda_2$ are equivalent if there exists a symplectic matrix $A \in SL(2,\mathbb{R})$ (the determinant of $A$ is 1) such that $\Lambda_2 = A\Lambda_1$. However, it is still not known if the HRT holds for all $(1,3)$ configurations when $g \in L^2$.

Next if $b = 1$ with $a \notin \{0, a_0\}$, or if $a = a_0$ with $b \neq 0$ then $\Lambda$ is a $(2,2)$ configuration, for which the HRT was established, see [10].

Consequently, to establish the HRT conjecture for all sets of four distinct points and all $L^2$ functions, one needs to focus on

- showing that there is no equivalence class of $(1,3)$ configurations for which the HRT fails; and
- proving the HRT for sets of four points that are neither $(1,3)$ configurations nor $(2,2)$ configurations.

For illustrative purposes we pose the following question.

**Question 1.** *Let $0 \neq g \in L^2(\mathbb{R})$. Prove that $\mathcal{G}(G,\Lambda)$ is linearly independent in each of the following cases:*

(a) $\Lambda = \{(0,0),(0,1),(1,0),(\sqrt{2},\sqrt{2})\}$.
(b) $\Lambda = \{(0,0),(0,1),(1,0),(\sqrt{2},\sqrt{3})\}$.

*To be more explicit, the question is to prove that each of the following two sets are linearly independent:*

$$\{g(x), g(x-1), e^{2\pi ix}g(x), e^{2\pi i\sqrt{2}x}g(x-\sqrt{2})\}$$

*and*

$$\{g(x), g(x-1), e^{2\pi ix}g(x), e^{2\pi i\sqrt{3}x}g(x-\sqrt{2})\}.$$

When $g$ is real-valued, then part (a) was proved in [19], but nothing can be said for part (b). On the other hand, [19, Theorem 7] establishes part (b) when $g \in S(\mathbb{R})$.

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Kasso A. Okoudjou

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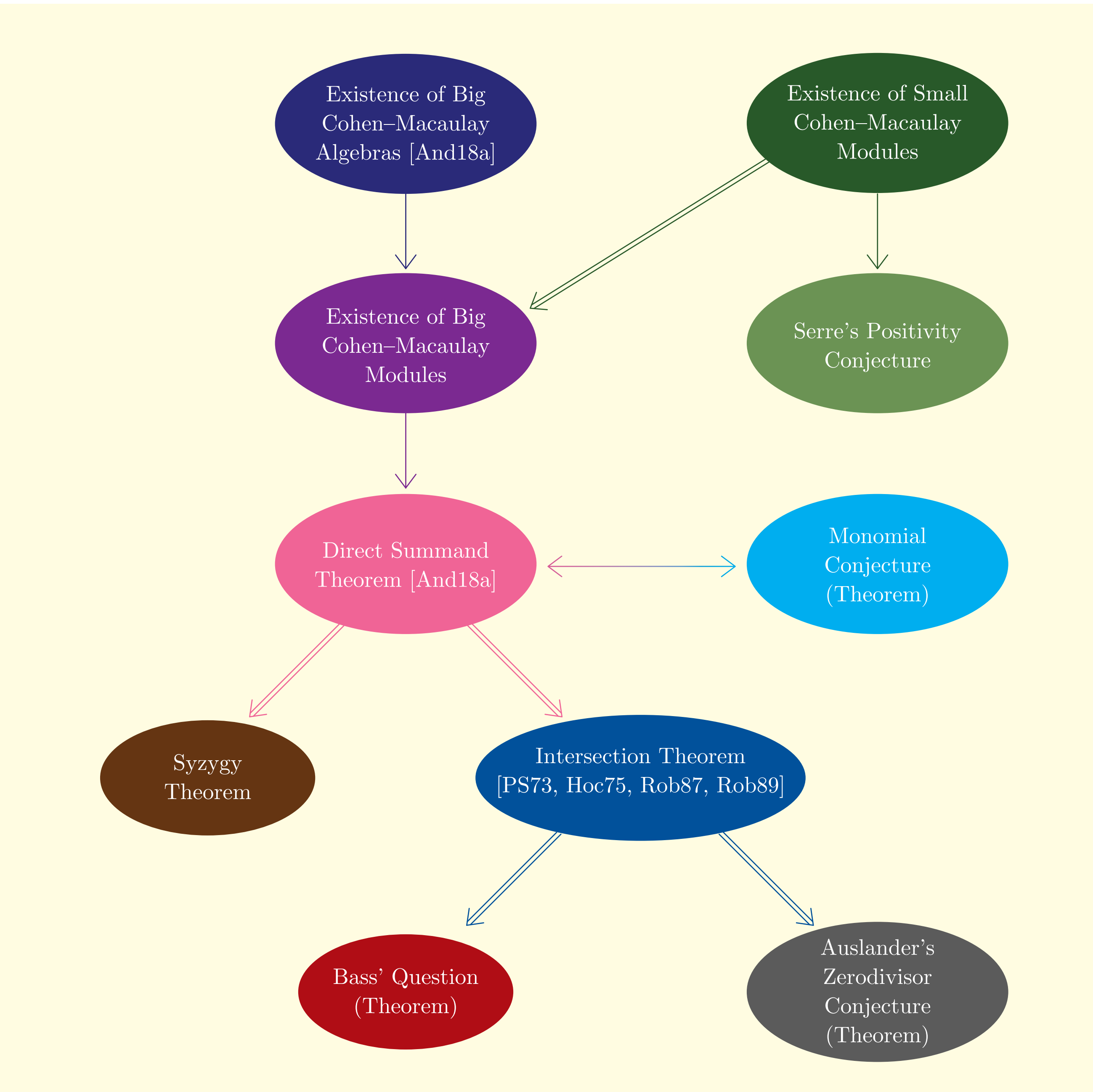
# Recent Applications of $p$-adic Methods to Commutative Algebra




*Linquan Ma and Karl Schwede*

Suppose that $k$ is a field and $R = k[x_1, \dots, x_n]/I$ is a commutative polynomial ring over $k$ modulo an ideal $I$. In other words, it is a finitely generated commutative $k$-algebra. Additionally assume that $R$ is also an integral domain. Emmy Noether's celebrated Normalization Theorem proves that inside $R$, there always exists a polynomial subring

$$A = k[t_1, \dots, t_d] \subseteq R$$

where the $t_i$ are algebraically independent (have no relations between them) satisfying the following property: the extension of rings $A \subseteq R$ makes $R$ into a *finitely generated module* over the polynomial ring $A = k[t_1, \dots, t_d]$. For example, in $R = k[x, y]/(y^2 - x^3)$ we have the subring $A = k[x]$ (or $B = k[y]$).

Consider the induced *finite* extension of fraction fields

$$k(t_1, \dots, t_d) = K(A) \subseteq K(R).$$

By viewing $K(R)$ as a vector space over $K(A)$, each element $u$ of $K(R)$ acts via multiplication

$$\times u : K(R) \longrightarrow K(R)$$

and so we can take its trace, which is then an element of $K(A)$. This induces a map

$$\mathrm{Tr} : K(R) \longrightarrow K(A).$$

It is not difficult to verify $\mathrm{Tr}(R) \subseteq A$, and so one obtains a map

$$\mathrm{Tr} : R \longrightarrow A = k[t_1, \dots, t_d].$$

Since the trace is a sum of the diagonal matrix entries, the composition

$$A \subseteq R \xrightarrow{\mathrm{Tr}} A$$

is multiplication by the extension degree $n := [K(R) : K(A)]$. If $k$ has characteristic zero (or more generally if the characteristic does not divide the extension degree $n = [K(R) : K(A)]$), then the composition

$$A \subseteq R \xrightarrow{\frac{1}{n}\cdot\mathrm{Tr}} A$$

is the identity. Hence, if $M = \ker(\frac{1}{n} \cdot \mathrm{Tr})$, then $R \cong A \oplus M$ as an $A$-module. In this case, we say that $R$ *has an $A$-summand* and that *$A \subseteq R$ splits*.

Consider now a more general setup. Suppose that

$$(*) \qquad A \subseteq R$$

is a finite extension of Noetherian integral domains. We ask when $R$ has an $A$-summand, i.e., when is $R \cong A \oplus M$ for some $M$? In [Hoc73] Hochster conjectured that if $A$ is a Noetherian regular ring[1] and $R \supseteq A$ is any extension ring that is finite as an $A$-module (henceforth, a *finite* extension), then $R$ has an $A$-summand.

Melvin Hochster.

This was the direct summand conjecture, now André's theorem [And18a], and it was one of the central and guiding questions of commutative algebra over the past half-century.

**Theorem 1** (Direct Summand Theorem). *Suppose $A$ is a Noetherian regular ring and $R \supseteq A$ is a finite extension, then $A \hookrightarrow R$ splits as a map of $A$-modules. In other words, $R$ has an $A$-summand.*

We observed above that the theorem holds if $A$ is a polynomial ring over a field of characteristic zero. In fact, the same argument works if $A$ is any regular domain (or even normal[2] domain) containing the rational numbers $\mathbb{Q}$. Hochster proved in [Hoc73] that Theorem 1 also holds if $A$ is a regular ring containing the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ (e.g., $A = \mathbb{F}_p[x_1, \dots, x_d]$). The methods that go into this and the areas of research they spawned are the topic of "The Direct Summand Conjecture and Singularities in Characteristic $p$."

**Example 2.** The finite extension $A = \mathbb{Q}[t^2, t^3] \hookrightarrow R = \mathbb{Q}[t]$ does not split. If there was a splitting $\phi : R \longrightarrow A$, it must send $1 \mapsto 1$ and therefore it must also send $t^2$ and $t^3$ to themselves. But $t^3 = \phi(t^3) = \phi(t^2 \cdot t) = t^2\phi(t)$ and so $\phi(t) = t$, which does not exist in $A$. Note $A$ is *not* normal.

In a recent breakthrough [And18b, And18a], André solved the conjecture in the *mixed characteristic*[3] case, using Scholze's theory of perfectoid algebras and spaces [Sch12]. This will be the topic of the section "Perfectoid Algebras and Ingredients in the Mixed Characteristic Proof."

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[1] *Regular rings are natural generalizations of polynomial rings over fields and also include rings such as $\mathbb{Z}[x_1, \dots, x_d]$.*

[2] *Meaning $A$ is integrally closed in its fraction field $K(A)$. In particular, if $f \in K(A)$ satisfies a monic polynomial with coefficients in $A$, then $f \in A$.*

[3] *A ring $A$ has mixed characteristic if it contains the integers $\mathbb{Z}$ as a subring, and there is some prime $p \in \mathbb{Z} \subseteq A$ that is not invertible in $A$.*

Yves André.

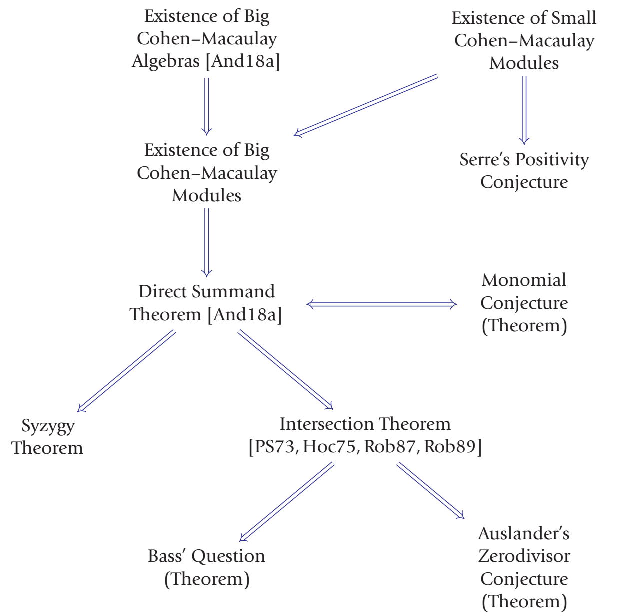
Previously, the best case that was known was the case when $A$ is a regular ring of dimension $\leq 3$, which is due to Heitmann [Hei02]. In the mixed characteristic setting, Bhargav Bhatt and Ofer Gabber also made substantial contributions to these circles of ideas [Bha14a, Bha18, Gab18].

The methods of André's proof have also been used to prove generalizations of the direct summand theorem, notably the existence of big Cohen–Macaulay algebras and the derived direct summand theorem, see [And18a, And18c, Bha18, Gab18, HM18, Shi17]. We expect that the existence of big Cohen–Macaulay algebras will stimulate further study of $f$ in mixed characteristic: In fact, they can be thought of as a tool that replaces certain aspects of Hironaka's resolution of singularities from characteristic zero algebraic geometry, as explained in [MS18b]. We will discuss big Cohen–Macaulay algebras and singularities in mixed characteristic in the section "Big Cohen–Macaulay Algebras and Singularities in Mixed Characteristic." As an application of these ideas, in our final section, "An Application to Symbolic Powers," we discuss a result on uniform growth of symbolic powers of ideals [MS18a].

**Homological Conjectures.** The *Homological Conjectures* in commutative algebra are a network of conjectures relating various homological properties of a commutative ring with its internal ring structure. They have generated a tremendous amount of activity over the last fifty years. The following is a diagram of homological conjectures, which is part of Hochster's 2004 diagram [Hoc04] (one sees that the Direct Summand Conjecture/Theorem lies in the heart). Most of these conjectures are now completely resolved thanks to the work of André and others.

Ofer Gabber.

Some of these implications are highly nontrivial: For example, the fact that the Direct Summand Theorem implies the Syzygy Theorem and the Intersection Theorem was due to Hochster [Hoc83],[4] and that the Intersection Theorem implies Bass' Question and Auslander's Conjec-




ture was proved by Peskine–Szpiro [PS73]. We note that many of the early homological conjectures are solved in mixed characteristic, thanks to Roberts' proof of the Intersection Theorem using localized Chern characters [Rob87, Rob89]. We also mention that there are various stronger forms of some of these conjectures that are proved based on André's work, see for example [And18c, AIN18, Gab18, HM18].

We want to highlight that, despite the recent breakthroughs in mixed characteristic, Serre's Positivity Conjecture on intersection multiplicity is still wide open in the ramified mixed characteristic case. To this date, the most important progress towards Serre's Conjecture is due to Gabber, see [Hoc97]. We refer the reader to [Hoc17] for a recent extensive survey on Serre's Conjecture and other (old and new) homological conjectures and theorems.

Paul C. Roberts.

We end the introduction by briefly discussing one of the homological theorems in the diagram above.

**Theorem 3** (The Syzygy Theorem). *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay (or even regular) local domain and let $M$ be a nonfree finitely generated $R$-module. If $M$ is a $k$-th syzygy module of finite projective dimension,*[5] *then the rank of $M$ is at least $k$.*

For instance, the first syzygy module is a submodule of

[4] *See also* [Dut87] *for other interesting connections between the homological conjectures.*

[5] *This means $M \cong \mathrm{Image}(\delta_k)$ in a finite free resolution $0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_k \xrightarrow{\delta_k} \cdots \longrightarrow F_0 \longrightarrow N \longrightarrow 0$ of a finitely generated $R$-module $N$.*

a free module $F_0$, so its rank is at least one (i.e., it cannot be torsion). Theorem 3 is a huge generalization to higher syzygy modules.

Theorem 3 was first proved by Evans–Griffith when $R$ contains a field [EG81] based on earlier work of Hochster [Hoc75]. The fact that it follows from Theorem 1 in mixed characteristic is a result of Hochster [Hoc83]. So this is now a theorem by André's work.

Theorem 3 itself has many unexpected consequences. For instance, it has connections to Horrock and Hartshorne's question on the cohomology of vector bundles of small rank on $\mathbb{P}^n$ [Har79].

## The Direct Summand Conjecture and Singularities in Characteristic $p$

Suppose now that $A$ is a regular Noetherian domain and $R \supseteq A$ is a finite extension that is also a domain. The fraction field extension

$$K(A) \subseteq K(R)$$

is separable if and only if the map $\mathrm{Tr} : R \longrightarrow A$ is nonzero. So to solve the direct summand conjecture, we cannot expect to use the field trace as we did when $A$ contains $\mathbb{Q}$.

For an arbitrary ring $A$ that contains $\mathbb{F}_p$, we have the Frobenius map (which is a ring homomorphism):

$$F : A \longrightarrow A \qquad a \longmapsto a^p.$$

We can iterate the Frobenius map: we label the $e$-fold self-composition $F^e : A \longrightarrow A$ and observe it sends $a \mapsto a^{p^e}$. One of the first issues one runs into when working with the Frobenius is that it can be difficult to distinguish the source and target of the map as they are the *same* ring. We explain one way to handle this issue. In the case that $A$ is an integral domain, we let $A^{1/p^e}$ denote the ring of $p^e$th roots of elements of $A$ embedded inside the algebraic closure of the fraction field, in other words:

$$A^{1/p^e} := \{x \in \overline{K(A)} \mid x^{p^e} \in A\}.$$

The ring $A^{1/p^e}$ is *abstractly* isomorphic to $A$ via the map $A^{1/p^e} \longrightarrow A$ which sends $b \mapsto b^{p^e}$. This isomorphism identifies the inclusion

$$A \subseteq A^{1/p^e}$$

with Frobenius, and it provides us with a convenient way of distinguishing the source and target of the Frobenius map.

The proof of the direct summand conjecture in characteristic $p > 0$ we present here follows from [Hoc73] in spirit. We begin by proving the following lemma (we use formal power series instead of the polynomial ring, but the idea is the same); this lemma also motivates further investigations on characteristic $p > 0$ singularities.

**Lemma 4.** *Suppose $k$ is a perfect field of positive characteristic and $A = k[\![x_1, \ldots, x_d]\!]$ is the formal power series ring. Then for every nonzero $c \in A$, there exists an $e > 0$ such that the $A$-module map $A \xrightarrow{1 \mapsto c^{1/p^e}} A^{1/p^e}$ splits.*

*Proof.* For any nonzero $c \in A$, by looking at terms of $c$ of minimal degree, there exists an $e$ so that

$$c \notin (x_1^{p^e}, \ldots, x_d^{p^e}).$$

In other words, $c^{1/p^e} \notin (x_1, \ldots, x_d) \cdot A^{1/p^e}$. Thus the image of $c^{1/p^e}$ is nonzero in the $k$-vector space

$$\frac{A^{1/p^e}}{(x_1, \ldots, x_d) \cdot A^{1/p^e}}.$$

By Nakayama's lemma, $c$ can be chosen as part of a basis for the free[6] $A$-module $A^{1/p^e}$. Hence there is a map $\psi : A^{1/p^e} \longrightarrow A$ such that $\psi(c^{1/p^e}) = 1$, which proves the lemma. □

Rings that satisfy the conclusion of Lemma 4 are called *strongly $F$-regular* (see Definition 7). We will discuss them more in what follows.

**Theorem 5.** *If $A$ is a Noetherian regular ring of characteristic $p > 0$, then any finite ring extension $A \subseteq R$ splits.*

*Proof.* By standard commutative algebra techniques, we may assume that $A \cong k[\![x_1, \ldots, x_d]\!]$ where $k$ is perfect and that $R$ is an integral domain.

For any integer $e > 0$ and any $A$-linear map $\phi : A^{1/p^e} \longrightarrow A$ we consider the commutative diagram:

$$\begin{tikzcd} \mathrm{Hom}_{A^{1/p^e}}(R^{1/p^e}, A^{1/p^e}) \arrow[r] \arrow[d, "\mathrm{eval@1}"'] & \mathrm{Hom}_A(R, A) \arrow[d, "\mathrm{eval@1}"] \\ A^{1/p^e} \arrow[r, "\phi"] & A. \end{tikzcd}$$

The top horizontal map is obtained by restricting to $R$ the domain of an element $\psi \in \mathrm{Hom}_{A^{1/p^e}}(R^{1/p^e}, A^{1/p^e})$ and then post-composing with $\phi$. Since the vertical maps are evaluation at 1, to show that $A \longrightarrow R$ splits it is enough to show that the right vertical map surjects (then there exists $\psi \in \mathrm{Hom}_A(R, A)$ so that $\psi(1) = 1$). Because $\mathrm{Hom}_A(R, A) \xrightarrow{\mathrm{eval@1}} A$ surjects generically, i.e., it is surjective if we tensor with the fraction field of $A$, we can choose a nonzero $c$ in the image of $\mathrm{Hom}_A(R, A) \xrightarrow{\mathrm{eval@1}} A$. It follows that $c^{1/p^e}$ is in the image of the left vertical map (for any $e > 0$). Next, using Lemma 4, we choose $e > 0$ and an $A$-linear map $\phi : A^{1/p^e} \longrightarrow A$ sending $c^{1/p^e} \longrightarrow 1$. The commutative diagram implies that the composition from the upper left to the lower right surjects, and hence so does $\mathrm{Hom}_A(R, A) \xrightarrow{\mathrm{eval@1}} A$. □

[6] *One common choice of basis is that made up of all elements $x_1^{j_1/p^e} \cdots x_d^{j_d/p^e}$ where each $j_i$ varies between $0$ and $p^e - 1$.*

**Philosophy.** We take a step back to think through this proof. In characteristic $p > 0$, the trace map does not always lead to splittings, even for separable extensions due to the presence of what is called *wild* ramification. Roughly speaking, we get around this by choosing an element $c \in A$ over which $A \subseteq R$ is ramified. By taking $1/p^e$th roots of $c$ (or in other words using Frobenius), we minimize the ramification until it almost goes away and so that extension $A \subseteq R$ splits.

We also give a new example of a ring with a non-split extension.

**Example 6.** Consider $k = \overline{\mathbb{F}_p}$, the algebraic closure of the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. We form the ring $A = k[x, y, z]/(zy^2 - x(x - z)(x + \lambda z))$, defining the affine cone over a (projective) elliptic curve $E$ (for example, one could take $\lambda = 1$). If the characteristic $p > 0$ is such that the curve is supersingular, then the (absolute) Frobenius map $F : A \longrightarrow A$ (which sends $r \mapsto r^p$ for all $r \in R$) does not split. If $p > 0$ is such that the curve is ordinary, then there exists a degree $p$ étale map $E' \longrightarrow E$ between elliptic curves [Sil09, Chapter V, Theorem 3.1]. This gives a finite extension $A \hookrightarrow A'$ of the rings corresponding to the cones. This extension is not étale at the cone point(s) and in fact is not split. In either case $A$ has a finite extension $A \hookrightarrow R$ that is not split.

**Singularities in characteristic $p > 0$.** The techniques we used to prove the direct summand conjecture in characteristic $p > 0$ have led to a vigorous study of singularities in characteristic $p > 0$ (typically under the names tight closure theory and Frobenius splitting theory).

**Definition 7** (Strongly $F$-regular singularities). A Noetherian domain $S$ containing $\mathbb{F}_p$ such that $S^{1/p}$ is a finitely generated $S$-module is called *strongly $F$-regular* if for any nonzero $c \in S$, there is an $e > 0$ and $\phi : S^{1/p^e} \longrightarrow S$ such that $\phi(c^{1/p^e}) = 1$. In this case, the singularities of $\operatorname{Spec} S$ are called *strongly $F$-regular singularities*.

The proof we gave in Theorem 5 shows that strongly $F$-regular rings are direct summands of all their finite extensions. In fact, a Noetherian domain that is a direct summand of every finite extension is called a *splinter*. Thus we have:

$$(\text{Strongly } F\text{-regular ring}) \Rightarrow (\text{Splinter})$$

The converse is open except in the case that the ring is (close to) Gorenstein [HH94, Sin99]. In fact, the converse would imply arguably the most studied question in characteristic $p > 0$ commutative algebra: that a weakly $F$-regular ring[7] is strongly $F$-regular.

[7] *A ring is called* weakly $F$-regular *if all of its ideals are tightly closed* [HH90]; *we will not delve into these definitions here, however.*

**Example 8.** In Example 6, we saw that the affine cone over an elliptic curve is not a splinter. The ring $\overline{\mathbb{F}_p}[x, y, z]/(xy - z^2)$ is strongly $F$-regular, and hence it is a splinter. The fact that it is strongly $F$-regular most easily follows from the fact that it is a direct summand of a regular ring.

Karen E. Smith.

Strongly $F$-regular singularities are intimately tied to singularities that appear in complex algebraic geometry. Roughly speaking, a complex algebraic variety $X$ has *rational singularities* if its line bundles have the same sheaf cohomology as the pullbacks of those line bundles to a resolution of singularities.[8]

A refinement of rational singularities is *log terminal singularities* (rational singularities whose finite étale[9] in codimension 1 covers also have rational singularities). Rational singularities are exactly the same as log terminal singularities on hypersurfaces (and more generally, on Gorenstein varieties).

What is really surprising, given their completely disjoint definitions, is that log terminal singularities are essentially the same as strongly $F$-regular singularities, modulo reduction to characteristic $p > 0$. Specifically, suppose that a chart $U$ on $X$ is given as the spectrum of $R_{\mathbb{C}} = \mathbb{C}[x_1, \dots, x_d]/I_{\mathbb{C}}$. Suppose for simplicity that all the coefficients of the generators of the ideal $I_{\mathbb{C}}$ live in $\mathbb{Z}$. We can reduce $R_{\mathbb{C}}$ to characteristic $p > 0$ by taking the coefficients of the generators of the ideal $I_{\mathbb{C}}$ modulo some $p > 0$. For example:

$$f = x^2 + 101xy - 7 \overset{\text{mod } 5}{\rightsquigarrow} x^2 + xy + 3.$$

For each $p > 0$, this gives us a ring $R_p = \mathbb{F}_p[x_1, \dots, x_d]/I_p$.

**Theorem 9** ([HW02, Har98, Smi97, MS97]). *$U = \operatorname{Spec} R_{\mathbb{C}}$ has log terminal singularities if and only if for all $p \gg 0$, $R_p$ has strongly $F$-regular singularities.*

In fact, this connection is a small part of a large dictionary where notions from higher dimensional complex algebraic geometry correspond to concepts involving the Frobenius map.

**Definition 10** (Derived splinters). A Noetherian domain $S$ is called a *derived splinter* if for every proper surjective map $\pi : Y \longrightarrow X = \operatorname{Spec} S$, the induced map $\mathcal{O}_X \longrightarrow R\pi_* \mathcal{O}_Y$ splits in the derived category of $\mathcal{O}_X$-modules.

[8] *A resolution of singularities is a proper map $\pi : Y \longrightarrow X$ of varieties such that $Y$ is nonsingular and $\pi$ is birational, which means it is an isomorphism "almost everywhere" (i.e., on a Zariski open and dense subset).*

[9] *essentially covering spaces*

Kei-ichi Watanabe.

It is a theorem of Bhatt that for rings of finite type over fields of characteristic zero, rational singularities are exactly derived splinters [Bha12] (and also see [Kov00]). But, in [Bha12], Bhatt also shows that derived splinters in characteristic $p > 0$ are exactly the same as splinters. Both these results strongly suggest that regular rings should be derived splinters in general. This derived statement is an extension of the direct summand conjecture (since any finite extension of ring $A \subseteq R$ induces a proper surjective map $\operatorname{Spec} R \longrightarrow \operatorname{Spec} A$). In [Bha18], applying some of André's ideas, Bhatt gives a simplified proof of Theorem 1 and also proves this derived version.

**Theorem 11** (The Derived Direct Summand Theorem). *Any Noetherian regular ring is a derived splinter.*

## Perfectoid Algebras and Ingredients in the Mixed Characteristic Proof

Bhargav Bhatt.

Now we move to the discussion of the proof of the Direct Summand Theorem (Theorem 1) in mixed characteristic, which uses perfectoid techniques. Consider a Noetherian local ring $A$ with maximal ideal $\mathfrak{m}$. As before, we say that $A$ has mixed characteristic $(0, p)$ if $A$ has characteristic 0 and $A/\mathfrak{m}$ has prime characteristic $p > 0$. For example, the ring of $p$-adic integers $\mathbb{Z}_p$ (the ring of formal power series in $p$) has maximal ideal generated by $p$, its residue field is $\mathbb{F}_p$, while its fraction field $\mathbb{Q}_p$ has characteristic 0.

We use $\mathbb{Q}_p$ to construct our first example of a perfectoid ring. Begin by adjoining all $p^e$th roots of $p$ to $\mathbb{Q}_p$ to form $\mathbb{Q}_p(p^{1/p^\infty})$. We $p$-adically complete and call the resulting ring $K = \widehat{\mathbb{Q}_p(p^{1/p^\infty})}$. This field $K$ is a typical example of a *perfectoid field*. The field $K$ contains a subring $K^\circ = \widehat{\mathbb{Z}_p[p^{1/p^\infty}]}$, which is its "ring of integers." $K^\circ$ is a typical example of an *integral perfectoid ring*. For the rest of this section, we fix $K$ and $K^\circ$ as above. We now give a definition of a perfectoid algebra.

**Definition 12** (Perfectoid algebras [Sch12, BMS18, And18c]). A *perfectoid $K$-algebra* is a Banach $K$-algebra $R$ such that the set of power-bounded elements[10] $R^\circ \subseteq R$ is bounded and the Frobenius is surjective on $R^\circ/p$. A $K^\circ$-algebra $S$ is called *integral perfectoid* if it is $p$-adically complete, $p$-torsion free, and the Frobenius induces an isomorphism $S/p^{1/p} \longrightarrow S/p$.

If $R$ is a perfectoid $K$-algebra, then the ring of power-bounded elements $R^\circ$ is integral perfectoid, and if $S$ is integral perfectoid, then $S[1/p]$ is a perfectoid $K$-algebra, see [Sch12, Theorem 5.2].

We give examples of integral perfectoid algebras. It is important to note that these algebras are *never* Noetherian. As before, $\widehat{\bullet}$ denotes the $p$-adic completion of $\bullet$.

**Example 13.** (a) $K^\circ = \widehat{\mathbb{Z}_p[p^{1/p^\infty}]}$.
(b)
$$K^\circ\langle x_2^{1/p^\infty}, \dots, x_d^{1/p^\infty}\rangle := \widehat{\mathbb{Z}_p[p^{1/p^\infty}][x_2^{1/p^\infty}, \dots, x_d^{1/p^\infty}]}.$$
(c) $\widehat{R^+}$, where $(R, \mathfrak{m})$ is a Noetherian complete local domain of mixed characteristic $(0, p)$ and $R^+$ is the integral closure of $R$ in an algebraic closure of its fraction field.

Although we will not use it in this survey, a key idea in the theory of perfectoid algebras (and spaces) is that numerous questions can be studied via *tilting*. This can turn a mixed characteristic question (or ring) into one in positive characteristic, and vice versa. This principle is used extensively (behind the scenes) in what follows. For more discussion, see for instance [Bha14b, Sch12].

A key part of the general theory of perfectoid rings is that we can talk about "almost mathematics" (appearing originally in the work of Faltings, [Fal88], also see the work of Gabber and Ramero [GR03]). Roughly speaking, we treat modules that are annihilated by the ideal $(p, p^{1/p}, p^{1/p^2}, \dots) =: (p^{1/p^\infty}) \subseteq K^\circ$ as if they were zero (since $p^{1/p^e}$ is almost 1 for $e \gg 0$).

**Definition 14.** Let $S$ be an integral perfectoid $K^\circ$-algebra.

(a) An $S$-module $M$ is *almost zero* if $(p^{1/p^\infty})M = 0$.
(b) An $S$-module $M$ is *almost flat* if
$$(p^{1/p^\infty})\operatorname{Tor}_1^S(M, N) = 0$$
for all $S$-modules $N$.
(c) A short exact sequence of $S$-modules $0 \longrightarrow M \longrightarrow N \longrightarrow N/M \longrightarrow 0$ represented by a class $\eta \in \operatorname{Ext}_S^1(N/M, M)$ is *almost split* if
$$(p^{1/p^\infty})\eta = 0 \text{ in } \operatorname{Ext}_S^1(N/M, M).$$

[10] *Elements $x$ such that the norm of $x^n$ is bounded independent of $n$.*

The most crucial result that is relevant to the proof of Theorem 1 is the following theorem, proved by Scholze [Sch12] and independently by Kedlaya–Liu [KL15]. Special cases were first obtained by Faltings [Fal88, Fal02] and Gabber–Ramero [GR04].

**Theorem 15** (The Almost Purity Theorem). *Suppose $S$ is integral perfectoid and $S[1/p] \longrightarrow T$ is a finite étale extension. Then the integral closure $S'$ of $S$ in $T$ is an almost finite étale extension of $S$. In particular, $S \longrightarrow S'$ is almost split.*

Here, *almost finite étale* roughly means that the obstructions to being finite étale are annihilated by $(p^{1/p^\infty})$. For our purposes however, we will only need the weaker fact that the map $S \longrightarrow S'$ is almost split.

We sketch the proof of the Direct Summand Theorem (Theorem 1) in mixed characteristic. For simplicity, we set $A = \mathbb{Z}_p[\widehat{x_2, \dots, x_d}]$. Our goal is to show that every finite extension $A \subseteq R$ splits.

**The case $A[1/p] \longrightarrow R[1/p]$ is finite étale.** We let

$$A_\infty = \mathbb{Z}_p[p^{1/p^\infty}][\widehat{x_2^{1/p^\infty}, \dots, x_d^{1/p^\infty}}];$$

this is an integral perfectoid algebra. Consider the following diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & R \arrow[d] \arrow[dr] & \\ A_\infty \arrow[r] & A_\infty \otimes R \arrow[r] & (A_\infty \otimes R)' \end{tikzcd}$$

where $(A_\infty \otimes R)'$ denotes the normalization of $A_\infty \otimes R$ in $A_\infty[1/p] \otimes R$. Since $A_\infty[1/p] \longrightarrow A_\infty[1/p] \otimes R$ is a finite étale extension by base change, Theorem 15 says that the composition

$$A_\infty \longrightarrow A_\infty \otimes R \longrightarrow (A_\infty \otimes R)'$$

is almost split and hence $A_\infty \longrightarrow A_\infty \otimes R$ is almost split. This implies $A \longrightarrow R$ is split, since $A$ is Noetherian and $A \longrightarrow A_\infty$ is faithfully flat (we omit the details here, this follows from a standard commutative algebra argument).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The argument above was first observed by Bhatt [Bha14a]. Notice that we only used Theorem 15 for the integral perfectoid algebra $A_\infty = \mathbb{Z}_p[p^{1/p^\infty}][\widehat{x_2^{1/p^\infty}, \dots, x_d^{1/p^\infty}}]$. This version is due to Faltings [Fal02].

**General case.** We now assume that $A[1/p] \longrightarrow R[1/p]$ is not necessarily étale. But since $A \longrightarrow R$ is "generically étale" (i.e., the extension of the fraction field $K(A) \longrightarrow K(R)$ is finite étale, i.e., separable), we can invert some other element to make $A \longrightarrow R$ étale. Thus we suppose that $A[1/pg] \longrightarrow R[1/pg]$ is finite étale for some nonzero element $g$. The main obstruction to "running" the above argument is that Theorem 15 no longer works: $A_\infty[1/p] \otimes R$ is no longer finite étale over $A_\infty[1/p]$. We only know that it becomes finite étale when we further invert $g$. To overcome this difficulty, André proved two remarkable theorems using Scholze's theory of perfectoid spaces:

**Theorem 16** ([And18a]). *Suppose $S$ is an integral perfectoid algebra and $g \in S$. Then there exists a map $S \longrightarrow \widetilde{S}$ of integral perfectoid algebras, such that $g$ admits a compatible system of $p$-power roots $\{g^{1/p^e}\}_{e=1}^\infty$ in $\widetilde{S}$ (i.e., $(g^{1/p^{e+1}})^p = g^{1/p^e}$ for all $e$) and that $S \longrightarrow \widetilde{S}$ is almost faithfully flat modulo powers of $p$.*

**Theorem 17** ([And18b]). *Suppose $S$ is integral perfectoid such that $g \in S$ has a compatible system of $p$-power roots in $S$, and $S[1/pg] \longrightarrow T$ is a finite étale extension. Then the integral closure of $S$ in $T$ is $(pg)^{1/p^\infty}$-almost finite étale over $S$ modulo powers of $p$.*

We return to the proof of Theorem 1. As before, we set $A_\infty = \mathbb{Z}_p[p^{1/p^\infty}][\widehat{x_2^{1/p^\infty}, \dots, x_d^{1/p^\infty}}]$. We apply Theorem 16 to construct $A_\infty \longrightarrow A_{\infty,\infty} := \widetilde{S}$ such that $g$ has a compatible system of $p$-power roots in $A_{\infty,\infty}$ and such that $A_{\infty,\infty}$ is almost faithfully flat over $A_\infty$ mod powers of $p$. Consider the following diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & R \arrow[d] \arrow[dr] & \\ A_{\infty,\infty} \arrow[r] & A_{\infty,\infty} \otimes R \arrow[r] & (A_{\infty,\infty} \otimes R)' \end{tikzcd}$$

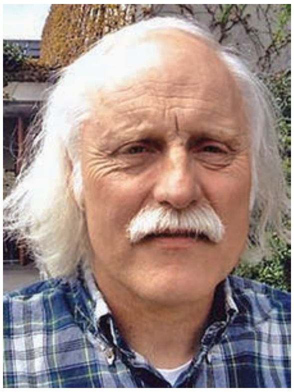
where $(A_{\infty,\infty} \otimes R)'$ denotes the normalization of $A_{\infty,\infty} \otimes R$ in $(A_{\infty,\infty} \otimes R)[1/pg]$. Applying Theorem 17 we find that the composition map

$$A_{\infty,\infty} \longrightarrow A_{\infty,\infty} \otimes R \longrightarrow (A_{\infty,\infty} \otimes R)'$$

is $(pg)^{1/p^\infty}$-almost split modulo powers of $p$. It follows that $A_{\infty,\infty} \longrightarrow A_{\infty,\infty} \otimes R$ is $(pg)^{1/p^\infty}$-almost split mod powers of $p$. This is enough to conclude that $A \longrightarrow R$ is split by the Noetherianity and $p$-adic completeness of $A$, the faithful flatness of $A_\infty$ over $A$, and the almost faithful flatness of $A_{\infty,\infty}$ over $A_\infty$ mod powers of $p$ (again some work is required here, but we omit the details).

## Big Cohen–Macaulay Algebras and Singularities in Mixed Characteristic

Let $(R, \mathfrak{m})$ be a Noetherian local ring. Recall that a system of parameters $x_1, \dots, x_d$ in $(R, \mathfrak{m})$ is a collection of $d = \dim R$ elements that generate the maximal ideal up to radical, i.e., $\mathfrak{m} = \sqrt{(x_1, \dots, x_d)}$. An $R$-algebra $B$ is called *big Cohen–Macaulay* if every system of parameters $\underline{x} = x_1, \dots, x_d$ of $R$ is a regular sequence on $B$. This means $x_{i+1}$ is a nonzero divisor on $B/(x_1, \dots, x_i)$ and that $\mathfrak{m}B \neq B$.



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Now suppose that $A \longrightarrow R$ is a module-finite extension of Noetherian local rings such that $A$ is regular and $R$ admits a big Cohen–Macaulay algebra. We claim that the map $A \longrightarrow R$ splits. So suppose that $B$ is a big Cohen–Macaulay $R$-algebra; it is easy to see that $B$ is also a big Cohen–Macaulay $A$-algebra (since every system of parameters of $A$ becomes a system of parameters of $R$). Because $B$ is big Cohen–Macaulay and $A$ is regular, $B$ is faithfully flat over $A$ (see [HH92, p.77]). It then follows from the factorization $A \longrightarrow R \longrightarrow B$ that $A \longrightarrow R$ is split (this is similar to the final arguments of the last section).

Therefore the existence of big Cohen–Macaulay algebras implies Theorem 1 for local rings. The general case of Theorem 1 follows from the local case: the evaluation at 1 map $\mathrm{Hom}_A(R, A) \longrightarrow A$ is surjective if and only if it is surjective locally. This explains that the existence of big Cohen–Macaulay algebras sits at the top of the diagram at the end of the introduction.

In the case that $R$ contains a field $\mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$, the existence of big Cohen–Macaulay algebras was established by Hochster–Huneke [HH92, HH95]. Hochster had also shown that big Cohen–Macaulay algebras exist in mixed characteristic in dimension three [Hoc02], using ideas of Heitmann's proof of the direct summand conjecture in dimension three [Hei02]. André [And18a] completed this program and showed they exist in mixed characteristic in all dimensions.

We roughly sketch the strategy of the construction of big Cohen–Macaulay algebras following [Hoc94, Hoc02, And18a, HM18]. Suppose that $T$ is an $R$-algebra and that $x_1, \dots, x_{k+1}$ is part of a system of parameters for $R$. Further assume that $t_1, \dots, t_{k+1}$ are elements of $T$ satisfying

$$(\star) \qquad x_{k+1}t_{k+1} = \sum_{i=1}^{k} x_i t_i \text{ but } t_{k+1} \notin (x_1, \dots, x_k)T.$$

In other words, $x_{k+1}$ is a zero divisor in $T/(x_1, \dots, x_k)T$ and so $T$ is not (big) Cohen–Macaulay over $R$. We therefore call $(\star)$ a *bad relation*.

Let $X_1, \dots, X_k$ be indeterminates over $T$. We consider the extension

$$T \longrightarrow T' = T[X_1, \dots, X_k] \Big/ \left(t_{k+1} - \sum_{i=1}^{k} x_i X_i\right).$$

We have forced $t_{k+1}$ to be inside $(x_1, \dots, x_k)T'$, and so the bad relation is trivialized. Such a $T'$ is called an algebra modification of $T$. Repeat this process to trivialize all bad relations on $T$ and we obtain a *total algebra modification* of $T$; call it $T_1$. Now we might have new bad relations on $T_1$, but we can repeat the whole process above and take a (huge) direct limit. More precisely, we set

$$B := \varinjlim(R = T \longrightarrow T_1 \longrightarrow T_2 \longrightarrow \cdots).$$

The above construction guarantees that for every system of parameters $x_1, \dots, x_d$ of $R$, $x_{i+1}$ is a nonzero divisor on $B/(x_1, \dots, x_{i-1})$. However, one must show that $\mathfrak{m}B \neq B$. In characteristic $p > 0$, this can be proved using the Frobenius map. If $R$ contains $\mathbb{Q}$, a reduction to characteristic $p > 0$ technique can be applied (basically by noticing that if $\mathfrak{m}B = B$ then this must happen at a *finite* level). Later Hochster [Hoc02] essentially observed that $\mathfrak{m}B \neq B$ as long as we can map $T$ to a certain "almost Cohen–Macaulay algebra"[11] (e.g., in characteristic $p > 0$, we have $R \longrightarrow R^{1/p^\infty}$). Finally in mixed characteristic, André replaced $R^{1/p^\infty}$ by $(A_{\infty,\infty} \otimes R)'$, which is the object that appears in the argument in the proof of Theorem 1, to prove $\mathfrak{m}B \neq B$.

It turns out that big Cohen–Macaulay algebras have deep connections with singularities. In fact, as we mentioned at the start of the section, if $A$ is regular, then a big Cohen–Macaulay $A$-algebra $B$ is faithfully flat over $A$. From one perspective the role of $B$ is analogous to a resolution of singularities in equal characteristic zero. Suppose that $S$ is (essentially) of finite type over a field $k$ of characteristic zero. Let $\pi : Y \longrightarrow X = \mathrm{Spec}\, S$ be a resolution of singularities. Grauert–Riemenschneider vanishing [GR70] (a relative version of Kodaira or Kawamata–Viehweg vanishing [Kaw82, Vie82, EV92]) tells us that the higher direct images of the canonical sheaf

$$R^i \pi_* \omega_Y = 0$$

vanish for $i > 0$. By local duality [Har66], this vanishing is equivalent to the following vanishing of local cohomology,

$$H^j_x(R\pi_* \mathcal{O}_Y) = 0$$

where $j < \dim X$, $x \in X$ is any closed point, and $\mathbb{H}^j_x$ denotes sheaf cohomology with support at $x$. In other words, the local cohomology of the complex $R\pi_*\mathcal{O}_Y$ vanishes except in the top degree. For finitely generated modules, this property of vanishing local cohomology is equivalent to

[11] *This means systems of parameters on R are "almost regular sequences"; we omit the detailed definition.*

the Cohen–Macaulay property. In other words, the complex

$$R\pi_*\mathcal{O}_Y$$

is a Cohen–Macaulay algebra, *except* that it is not an algebra: it *lives in the derived category*! This was first observed by Roberts [Rob80] (and we omit the non-triviality condition analogous to $\mathfrak{m}B \neq B$ for simplicity). Many common local applications of Grauert–Riemenschneider vanishing can be proved using the vanishing of local cohomology of $R\pi_*\mathcal{O}_Y$. Tied up closely with this vanishing are log terminal singularities, which we define in a special case:

**Definition 18.** A Gorenstein variety $X$ in characteristic zero is called *log terminal* if the canonical map

$$H^d_x(\mathcal{O}_X) \longrightarrow H^d_x(R\pi_*\mathcal{O}_Y)$$

injects for every $x \in X$ and for some (equivalently every) resolution of singularities $\pi : Y \longrightarrow X$.

Of course, for a big Cohen–Macaulay algebra $B$ over a local ring $S$, we also have the vanishing

$$H^i_{\mathfrak{m}}(B) = 0$$

for the maximal $\mathfrak{m} \in \operatorname{Spec} S$, and all $i < \dim S$. Furthermore, it follows from work of Smith [Smi94] that in characteristic $p > 0$, a Gorenstein local ring $(S, \mathfrak{m})$ is strongly $F$-regular if and only if

$$H^d_{\mathfrak{m}}(S) \longrightarrow H^d_{\mathfrak{m}}(B)$$

is injective for every big Cohen–Macaulay $R$-algebra $B$.

Inspired by this, we introduce the following definition in [MS18b].

**Definition 19.** Let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension $d$ and let $B$ be a big Cohen–Macaulay $R$-algebra. Then we say $R$ is *big-Cohen–Macaulay-regular with respect to* $B$ (or more compactly is $\mathrm{BCM}_B$*-regular*) if the natural map $H^d_{\mathfrak{m}}(R) \longrightarrow H^d_{\mathfrak{m}}(B)$ is injective.

It turns out that $\mathrm{BCM}_B$-regular singularities share many analogous properties of log terminal singularities in equal characteristic 0 or strongly $F$-regular singularities in equal characteristic $p > 0$ (again, all based on this vanishing). Furthermore, we apply this to study singularities when the characteristic varies, e.g., families of singularities defined over $\operatorname{Spec} \mathbb{Z}$. We refer to [MS18b] for more results in this direction.

## An Application to Symbolic Powers

We discuss another commutative algebraic application of integral perfectoid big Cohen–Macaulay algebras in mixed characteristic [MS18a]. In fact, our proof strategy is directly inspired by the connection between big Cohen–Macaulay algebras and resolution of singularities and the vanishing theorems discussed above. Let us state the problem.

Suppose $A$ is a Noetherian regular ring (e.g., a polynomial ring over a field or over $\mathbb{Z}$). Suppose $Q \subseteq A$ is a prime ideal. For any integer $n > 0$ we define the *$n$th symbolic power* of $Q$ to be

$$Q^{(n)} := (Q^n A_Q) \cap A.$$

In other words, $Q^{(n)}$ is the set of elements of $A$ (or functions on $\operatorname{Spec} A$) that vanish to order $n$ at the generic point of $V(Q) \subseteq \operatorname{Spec} A$.

Evidently, $Q^n \subseteq Q^{(n)}$ but they are not always equal. A very extensively explored question in commutative algebra studies the difference between $Q^{(n)}$ and $Q^n$. For example, when $Q$ is generated by (part of) a regular sequence, a classical result in commutative algebra says that $Q^n = Q^{(n)}$ for all $n$. However $Q^{(n)}$ can be much bigger than $Q^n$.

**Example 20.** Let $R = k[x, y, z]/(xy - z^m)$. Then $Q = (x, z)$ is a prime ideal of height one and $x \notin Q^m$ (in fact, $x$ is not even in $Q^2$). However, we see that $x \in Q^{(m)}$ because $x \in Q^m R_Q$ ($y$ is a unit in $R_Q$).

Although the above example shows that $Q^n$ and $Q^{(n)}$ can be quite different, a surprising result was obtained by Swanson [Swa00] (see also [HKV09]), who proved essentially that, for any complete local domain $(R, \mathfrak{m})$ and any prime ideal $Q \subseteq R$, there is a constant $k$ (depending on $R$ and $Q$) such that $Q^{(kn)} \subseteq Q^n$ for all positive integers $n$. In other words, the difference between $Q^n$ and $Q^{(n)}$ is bounded "linearly."

For complete regular local rings, we have an even stronger result. The following theorem was proved when our ring contains a field, by Hochster–Huneke [HH02] (see also Ein–Lazarsfeld–Smith [ELS01]), and recently it was extended to mixed characteristic in [MS18a]. Compared with Swanson's result mentioned above, the theorem shows that for regular rings the constant $k$ can be chosen to be the dimension of the ring. In particular it is independent of the ideal $Q$!

**Theorem 21.** *Let $A$ be a complete regular local ring of dimension $d$. Then for every prime ideal $Q \subseteq A$ and every $n$, we have $Q^{(dn)} \subseteq Q^n$.*

We briefly explain the strategy of the proof of Theorem 21 in mixed characteristic. The idea is to construct a multiplier ideal like object in mixed characteristic and then use the same strategy as in Ein–Lazarsfeld–Smith [ELS01].[12]

For the moment suppose that $A$ is a regular ring of finite type over a field of characteristic 0 (e.g., $A=\mathbb{Q}[x_1, \ldots, x_d]$). Suppose that $\mathfrak{a} \subseteq A$ is an ideal and $t \in \mathbb{R}_{\geq 0}$ a *formal* exponent for $\mathfrak{a}$. In this setting we can take a log resolution

[12] *A similar object exists in characteristic $p > 0$ and is called the test ideal.*

$\pi : Y \longrightarrow \operatorname{Spec} A$[13] with $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-G)$ and define the multiplier ideal at a maximal ideal $\mathfrak{m} \subseteq A$[14]:

$$\mathcal{J}(A_\mathfrak{m}, \mathfrak{a}^t) = \operatorname{Ann}_{A_\mathfrak{m}}\{\eta \in H^d_\mathfrak{m}(A) \mid \eta \mapsto 0 \in \mathbb{H}^d_\mathfrak{m}(R\pi_*\mathcal{O}_Y(\lfloor tG \rfloor))\}.$$

This is an ideal of $A_\mathfrak{m}$ that measures the singularities of $V(\mathfrak{a}) \subseteq \operatorname{Spec} A$, scaled by $t$, at $\mathfrak{m} \in \operatorname{Spec} A$. Crucially, for the applications to the result on symbolic powers, the multiplier ideal satisfies the following "subadditivity" property [DEL00]:

(†) $\mathcal{J}(A_\mathfrak{m}, \mathfrak{a}^{tn}) \subseteq \mathcal{J}(A_\mathfrak{m}, \mathfrak{a}^t)^n$ for all positive integers $n$.

This essentially follows from the Kawamata–Viehweg type vanishing result that accompanies the multiplier ideals [Laz04], which can be stated dually as:

$$\mathbb{H}^i_\mathfrak{m}(R\pi_*\mathcal{O}_Y(\lfloor tG \rfloor)) = 0 \text{ for } i < d.$$

**Philosophy.** We examine the definition again. *Roughly speaking*, the multiplier ideals associated to the pair $(A, \mathfrak{a}^t)$ are elements of $A_\mathfrak{m}$ that annihilate all elements in the top local cohomology module $H^d_\mathfrak{m}(A)$ whose image in

$$H^d_\mathfrak{m}(R\pi_*\mathcal{O}_Y)$$

is "almost annihilated" by $\mathfrak{a}^t$ (this is made precise in the above definition as $\mathfrak{a}$ in $\mathcal{O}_Y$ is just $\mathcal{O}_Y(-G)$).

Therefore, in order to extend the definition to mixed characteristic and still have the nice properties such as the subadditivity, one needs an object $B$ like $R\pi_*\mathcal{O}_Y$, which has good vanishing properties and such that one can make sense of, or at least approximate, $\mathfrak{a}^t$ in $B$. It turns out that a sufficiently large integral perfectoid big Cohen–Macaulay (or almost big Cohen–Macaulay) algebra will do the job! Below we give a definition of perfectoid multiplier ideal [MS18a] for $A = \mathbb{Z}_p[\![x_2, \ldots, x_d]\!]$.

Using the fact that ideals are made up of principal ideals, one can essentially reduce the definition to the case where $\mathfrak{a} = (g)$ is principal (we are absolutely hiding subtleties here to keep the definitions cleaner). Let $A_\infty$ denote the $p$-adic completion of $A[p^{1/p^\infty}, x_2^{1/p^\infty}, \ldots, x_d^{1/p^\infty}]$; this is the same as the definition of $A_\infty$ we used in the proof of Theorem 1 in the second section. We then form $A_{\infty,\infty}$ using Theorem 16 for our fixed element $g$ (or in the non-principal case, to the generators of $\mathfrak{a}$). Then for a fixed real number $t > 0$, we can approximate $t$ (from above) by rational numbers of the form $a/p^e$, and observe we can identify elements $g^{a/p^e} \in A_{\infty,\infty}$ by construction. These $g^{a/p^e}$ *approximate* $g^t$.

[13] *This means* $\pi : Y \longrightarrow \operatorname{Spec} A$ *is proper birational,* $Y$ *is regular, and* $\mathfrak{a} \cdot \mathcal{O}_Y$ *defines a SNC divisor.*

[14] *This is not the usual definition of multiplier ideals as in* [Laz04], *but it is equivalent to the usual definition via local and Grothendieck's duality* [Har66], *see* [MS18b, Section 2] *for a detailed explanation.*

In the multiplier ideal definition, we extend $\mathfrak{a}$ to $\mathcal{O}_Y$ and then multiply the associated divisor by $t$ (and round as appropriate). We do the same thing here: we define

$$\tau(A, (g)^t) = \operatorname{Ann}_A\left\{\eta \in H^d_\mathfrak{m}(A) \mid p^{1/p^\infty} g^{a/p^e}\eta = 0 \text{ in } H^d_\mathfrak{m}(A_{\infty,\infty})\right\}$$

where $a/p^e > t$ approximates $t$ from above.

Although this definition looks a bit technical, it turns out that it satisfies many properties (including the subadditivity property (†)) similar to the multiplier ideal $\mathcal{J}(A, \mathfrak{a}^t)$ in characteristic 0. This allows us to prove Theorem 21. Moreover, the crucial reason that the subadditivity holds for $\tau(A, \mathfrak{a}^t)$ is because $A_{\infty,\infty}$ is almost big Cohen–Macaulay. We refer the interested reader to [MS18a, Section 4] for details.

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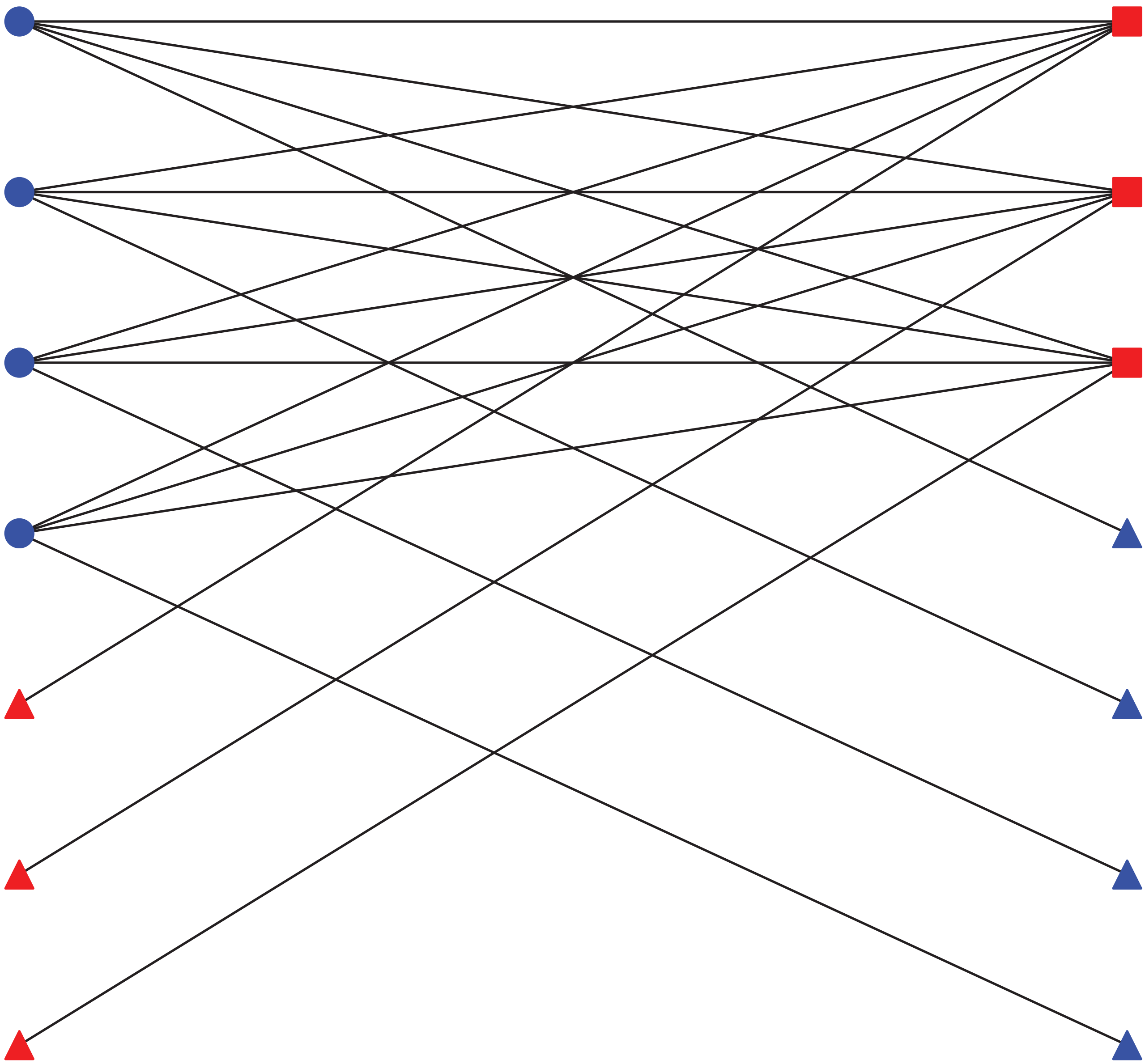
# The Mathematics of Quantum-Enabled Applications on the D-Wave Quantum Computer



*Jesse J. Berwald*

## Introduction

Over half a century ago, a groundbreaking technology, the microchip, started appearing in computers and research facilities around the world. Today there is no question of its importance. Yet in 1968, ten years after its invention, it was still a novelty to some: An IBM engineer famously asked, "But what... is it good for?"[1] Recent advances in the development of quantum computers in some ways mirror this evolution, though time, experience, and feverish media coverage ensure that few will ask the same naive question. The similarity comes from the observation that quantum computers are on a similar cusp, that of having broad societal impact, as the microchip was in the last century.

After some reflection, mathematicians and scientists may find themselves asking related questions. For instance, *What are quantum computers good for today? As a mathematician, what's in it for me?* and of course, *How do they do work?* Other than the last question, there are few definitive answers available. This article attempts to guide the reader towards her own intuition regarding the first two questions, but limits the "how" to a cursory glance and a host of references.

This article covers quantum computing from the angle of *adiabatic quantum computing* [7, 13], which has proven to have the shortest horizon to real-world applications, partly due to a slightly easier path to development[2] than alternative approaches such as *gate-model* quantum computers.

In this article we cover background on quantum annealing computing generally, the canonical problem formulation necessary to program the D-Wave quantum processing unit (QPU), and discuss how such a problem is compiled onto the QPU. We also cover recent joint work solving a problem from topological data analysis on the D-Wave quantum computer. The goal of the article is to cover the above from a mathematical viewpoint, accessible to a wide range of levels, and introduce as many people as possible to a small portion of the mathematics encountered in this industry.

## Quantum Computing Background

**Historical background.** Richard Feynman is credited with the initial ideas for computing with quantum mechanics, presented in a seminal talk and subsequent paper from 1982, titled "Simulating Physics with Computers" [8]. Significant progress over the past decade has brought the quantum computer industry into what some term the *noisy intermediate-scale quantum* (NISQ) era [19]. While quantum computers have yet to show an undeniable advantage over classical systems, their theoretical advantages are well documented. Particularly noteworthy are Shor's algorithm [20] and Grover's search algorithm [10] for gate-model quantum computers. Quantum annealing, the model adopted by D-Wave Systems, also promises quantum speedup [22]. Already, in a number of narrowly defined use cases, improvements over classical computers have been observed on the D-Wave quantum computer [16, and references therein; 22].

Recalling Feynman's famous quote from [8], "Nature isn't classical... and if you want to make a simulation of Nature, you'd better make it quantum mechanical," precise control over annealing properties, as exists on the D-Wave quantum computer, allows for novel quantum material simulation. Recent work by Harris, et al. [11, and references therein] on phase transitions in spin glasses; and by King, et al. [15] on Kosterlitz-Thoulless phase transitions in exotic forms of matter show manifestations of the central thesis of Feynman's original paper.

**Technical background.** We absorb the majority of the technical description of quantum annealing into this section, saving a mathematical reformulation of certain aspects for the next two sections. The discussion below is general to quantum annealing.

To lay the groundwork, we briefly describe D-Wave's specific implementation of a QPU, though much of the technology described here is used by others exploring quantum computers using superconductivity. The D-Wave quantum computer is a programmable quantum computer whose QPU is composed of a network of superconducting flux qubits, each of which acts as a programmable Ising spin. Electrical current may travel in either direction in the qubit bodies, corresponding to up and down spins. Tunable mutual inductances between pairs of qubit bodies allow for in situ adjustment of magnetic coupling energy between these pairs. On the current model, the D-Wave 2000Q, the grid of qubits is arranged in tiles of $K_{4,4}$ bipartite graphs, known as unit cells. Qubits are connected across unit cells, giving each qubit a degree of at most six on the current hardware.

The D-Wave quantum computer implements a process known as quantum annealing [7, 13], which is independent of the chip architecture. The goal of a quantum annealing computer is to find a low-energy state of a *problem Hamiltonian*, $H_P$. The key is to initialize the system in a ground state of a driver Hamiltonian, $H_D$, that is computationally trivial to obtain, then evolve the system from that known state to the unknown ground state of $H_P$. The quantum adiabatic theorem guarantees that if the time evo-

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[1] *Attributed to a particularly myopic engineer at the Advanced Computing Systems Division of IBM, 1968, commenting on the microchip.*

[2] *But by no means trivial, as D-Wave's nineteen-year journey can attest to.*

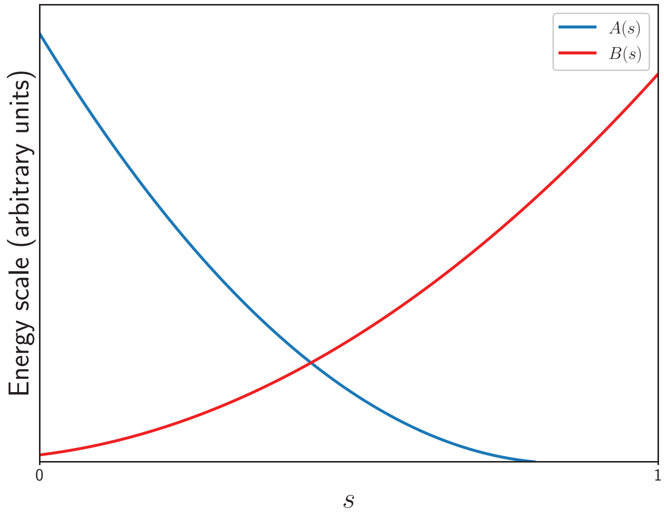


**Figure 1.** A typical annealing schedule. Scales are arbitrary. The parameter $s$ evolves from 0 to 1, typically on the scale of microseconds. At $s = 0$ the system is in a ground state of the driver Hamiltonian, $H_D$, in which $A(0) \gg B(0)$ and is then evolved to a classical state such that $B(1) \gg A(1)$.

lution of a closed quantum system is slow enough, then the system will remain in its ground state throughout the process. Thus, at the end of a slow anneal process the ground state of the quantum state will also be the global minimum of the classical problem Hamiltonian. (See [7] for more details.) Note that the D-Wave computer actually runs the quantum annealing algorithm on an open quantum system, that is, one that is coupled to a thermal environment. Even with a slow annealing process, thermal excitations can result in a distribution of states with energies above the ground state. Sampling the system many times helps to mitigate such effects, which are fundamental to any open quantum system.

While not critical to understanding the mathematics in the following sections, we state the adiabatic theorem more precisely and describe its use in computation as this provides important context for the goal of a quantum annealing computer in general. Suppose the evolution of our quantum system is governed by the time-dependent Schrödinger equation [7],

$$i\frac{d}{dt}\left|\psi(t)\right\rangle = H(t)\left|\psi(t)\right\rangle,$$

where $|\psi(t)\rangle$ is a *state vector* in a complex $n$-dimensional Hilbert space. Studying the ground state of $|\psi(t)\rangle$ boils down to an eigenstate problem. We consider certain instantaneous eigenstates of $H$, $E_0(t) \le E_1(t) \le \cdots \le E_{n-1}(t)$, for fixed $0 \le t \le T$.[3] In the language of eigenstates, the adiabatic theorem guarantees that if $|\psi(0)\rangle$ is the ground state of $H(0)$, with eigenvalue $E_0(0)$, and if the spectral gap between the ground state and first excited state is positive for all $t \in [0, T]$, that is, $|E_0(t) - E_1(t)| > 0$, then the probability that $|\psi(T)\rangle$ is in the ground state is arbitrarily close to one. For technical caveats, see [7].

Consider the Hamiltonian,

$$H(s) = \frac{1}{2}A(s)H_D + \frac{1}{2}B(s)H_P, \qquad (1)$$

that contains the problem Hamiltonian, $H_P$, a driver Hamiltonian, $H_D$, whose ground state is relatively easy to construct,[4] and $s \in [0,1]$ (units are arbitrary). Typical curves, $A(s)$ and $B(s)$, governing the evolution, or annealing schedule, of the system are shown in Fig. 1. Eq. (1) allows us to leverage the adiabatic theorem for computation: At $s = 0$ the ground state is a global superposition of all computational basis states, obtained through application of a precise transverse magnetic field. From there the system is evolved to the ground state of the classical system, defined by $H_P$, at $s = 1$.

We now describe the problem Hamiltonian in greater detail. The structure of the operator

$$H_P = \sum_i h_i \sigma_i^z + \sum_{i,j} J_{i,j}\sigma_i^z\sigma_j^z \qquad (2)$$

is defined physically by manipulating local, real-valued fields $h_i$ and $J_{i,j}$ on the QPU; the $\sigma^z$ are Pauli spin matrices. In quantum computation, the $i$th bit $z_i$ is replaced by a *qubit*, $|z_i\rangle$. Each $|z_i\rangle$ represents the eigenstate of the $\sigma_i^z$ operator, an observable state of the $i$th physical flux qubit in a D-Wave quantum computer. The eigenstates take values $|\uparrow\rangle$ or $|\downarrow\rangle$, with eigenvalues $+1$ and $-1$, indicating the "spin" of the quantum system to be either "up" or "down." Hence, a problem space with $n$ qubits is spanned by a $2^n$-dimensional Hilbert space.

## Some Examples, Broadly Described

We now divorce ourselves from the physics and focus solely on the problem Hamiltonian, $H_P$, going forward. In the following subsections we discuss $H_P$ from three distinct viewpoints. These short sections are meant to motivate mathematicians by highlighting areas of deeper mathematical formalism lurking behind quantum algorithms in general, and quantum annealing formulations in particular. In the first subsection, we consider $H_P$ as a polynomial and point out some of the interesting consequences of this perspective. Next, we discuss one of the fundamental questions that arises with a limited, and rather sparse, chip topology: How does one fit the graphical structure of a general problem Hamiltonian onto the fixed graphical structure of the QPU? Lastly, we briefly follow up on the statement made in the subsection about sampling, and argue

[3] *The exact value of $T$ is critical to the success of quantum annealing. A good derivation can be found in* [7] *and references therein.*

[4] *This in no way implies that construction of a quantum computing device is trivial, only that obtaining the ground state for $H_D$ is easier than finding the ground state of $H_P$ through non-adiabatic means.*

that there are possible benefits to this form of error correction.

**Example 1: A polynomial viewpoint.** Assume we have a quantum annealing computer, such as the D-Wave quantum computer, that is designed to seek a minimum energy solution to an Ising problem, Eq. (2), or equivalently a combinatorial optimization problem known as a *quadratic unconstrained binary optimization* (QUBO) problem. A simple transformation takes the variables in an Ising formulation to binary variables in a QUBO setting, using $y \mapsto \frac{y+1}{2}$, where $y$ is a spin variable in $H_P$ taking values in $\{-1, 1\}$.

We now describe the QUBO formulation of the problem Hamiltonian in more detail. We begin with a couple of definitions.

**Definition 3.** *Let $\mathbb{B}^n := \mathbb{Z}_2^n$, the $n$-dimensional hypercube of binary vectors.*

After a transformation of variables, we can define the problem Hamiltonian as a QUBO taking arguments from $\mathbb{B}^n$.

**Definition 4.** *Let*

$$H_P(\mathbf{x}) := \sum_{i=1}^{n} h_i x_i + \sum_{i=1}^{n} \sum_{j=1}^{n} J_{i,j} x_i x_j$$

*be a real-valued polynomial with arguments $\mathbf{x} \in \mathbb{B}^n$.*

The polynomial $H_P$ is an interesting mathematical object: The coefficients of $H_P$ live in $\mathbb{Q}$, yet the variables are restricted to $\mathbb{B}^n$. To deal with this, at least notationally, we define a restriction to the polynomials over the rationals.[5]

**Definition 5.** *Define the set of polynomials with rational coefficient and binary variables, $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{B}^n$, as $\mathbb{Q}[\mathbf{x}|\mathbb{B}^n] \subset \mathbb{Q}[\mathbf{x}]$.*

**Remark 6.** *We can regard $\mathbb{Q}[\mathbf{x}|\mathbb{B}^n]$ as a quotient ring, $\mathbb{Q}[x_1, \ldots, x_n]/\langle x_1^2 - x_1, \ldots, x_n^2 - x_n\rangle$. The ideal $\langle x_1^2 - x_1, \ldots, x_n^2 - x_n\rangle$ absorbs all polynomials in $\mathbb{Q}[\mathbf{x}]$ for which the binary constraint, $x_i^2 = x_i$, holds.*

This remark points to an elegant area of research. Dridi et al. [4] have leveraged the algebraic properties of $H_P$ in developing a number of applications using the D-Wave quantum computer. For instance, in [4] they leverage Groebner bases to reduce the size of the problem prior to sending it to the quantum computer. In [5], Dridi et al. leverage computational algebraic geometry for the important problem of embedding the problem Hamiltonian onto the QPU, a topic described in the subsection "Technical background."

With Definition 3 and Definition 5, we can now state the problem solved by the quantum computer more precisely.

[5] *We could also take coefficients over $\mathbb{R}$.*

**Definition 7.** *Suppose we are given a Hamiltonian $H_P \in \mathbb{Q}[\mathbf{x}|\mathbb{B}^n]$ and a quantum computer, $\mathcal{Q}$, designed to implement the adiabatic theorem using the time-dependent Schrödinger equation Eq. (1). Then $\mathcal{Q}$ solves the combinatorial optimization problem*

$$\mathcal{H} \equiv \underset{\mathbf{x}\in\mathbb{B}^n}{\arg\min}\, H_P(\mathbf{x}), \tag{8}$$

*given proper assumptions on the evolution of the system $H$.*

The combinatorial optimization problem defined by $\mathcal{H}$ represents, abstractly, the problem to be solved. These are, in general, NP-hard problems, making the prospect of a quantum annealing computer that can solve the class of problems described by $\mathcal{H}$ enticing. Lucas [17] provides a thorough overview of methods for formulating a number of NP-hard problems as QUBOs.

**Example 2: Compiling $H_P$ – a graph minor embedding problem.** Much like a classical computer converts high-level, abstract, and human-readable languages to machine instructions, Eq. (8) must be converted to a *quantum machine instruction* (QMI) that will run on the quantum computer. There are numerous steps in this process, one of which, *embedding*, we touch on briefly in this section. It is convenient to view the problem Hamiltonian, $H_P$, as a weighted graph. In the subsection "The Wasserstein Graph as a QUBO," we construct a specific problem Hamiltonian to make this connection more clear. Define $G = \langle V, E\rangle$, where the node set

$$V = \{(x_1, h_1), \ldots, (x_n, h_n) \mid h_i \neq 0\}$$

is composed of nodes that are in direct correspondence with each binary variable $x_i$, where $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{B}^n$. Each node is weighted by the bias on the qubit, $h_i$. Similarly, the edge set is composed of weighted edges defined by the coupling terms in $H_P$, so that

$$E = \{(x_i, x_j, J_{i,j}) \mid x_i, x_j \in V \text{ and } J_{i,j} \neq 0\}.$$

This definition of $E$ encodes the variable coupling in $H_P$.

The graph $G$ must be embedded onto the hardware to solve $H_P$. Embedding the *logical graph*, $G$, onto the *hardware graph*, $K$, amounts to finding a *minor embedding*. A *minor* of a graph $K$ is a subgraph of $K$ obtained by contracting or deleting edges, and omitting isolated vertices. A minor embedding is a function that maps the vertices of $G$ to the power set of the vertices of $K$,

$$\psi : V_G \to 2^{V_K},$$

such that for each $u \in V_G$, the graph induced in $K$ by $\psi(u) \in V_K$ is connected. These connected components within the hardware graph are termed *chains*. Embeddings for which $\psi(u)$ is a singleton for all $u$ are called *native* embeddings. Lastly, there exists an edge between $\psi(u)$ and $\psi(v)$ whenever $u$ and $v$ are adjacent in the logical graph. The map $\psi$ is the minor embedding we seek. Whether

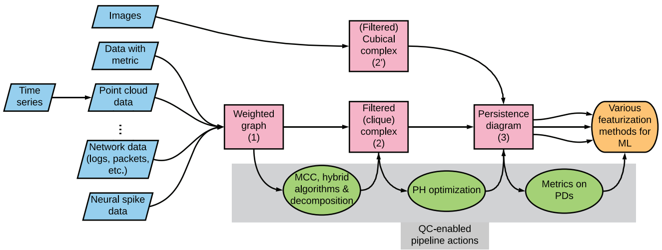


**Figure 2.** The topological data analysis pipeline for persistent homology (PH). Transformations containing portions amenable to quantum computation are highlighted in the green ovals.

$G$ can be embedded as a minor contained in $K$ is known to be NP-hard. Elegant, polynomial-time algorithms exist for embedding problem Hamiltonians onto the the QPU architecture [2]. The more efficient the embedding—the shorter the chains—the larger the problem that can be solved on the QPU.

**Example 3: Distributions of solutions through sampling.** Quantum computers are inherently probabilistic, with samples being drawn from an approximate Boltzmann distribution in the case of D-Wave's implementation. Hence, it is necessary to sample the energy landscape of the problem many times to obtain a distribution of solutions. This is one form of error correction. For example, Shor's algorithm [20] is designed to return the prime factors of a number with *high probability*. Repeated sampling will provide many possible factor pairs, leaving the final confirmation of a much smaller set of possibilities up to a classical computer.

On a quantum annealing computer, once the energy landscape is defined by $H$, and the QUBO is embedded on the QPU, a collection of hundreds or thousands of samples, $\{\mathbf{x}_i\}$, can be gathered quickly by repeatedly annealing the problem and reading out the answer. Each of the $\mathbf{x}_i$ could be located at or near a local minimum, using Hamming distance as a metric; or one may find solutions with similar energy at opposite corners of the hypercube $\mathbb{B}^N$. Combined with reverse annealing, a local search capability available on the D-Wave quantum computer, this feature provides a powerful avenue to explore regions of high probability (low energy) in multimodal systems, especially in the realm of neural networks and genetic algorithms as discussed in [3]. We provide a brief example of the sampling aspect of quantum computation applied to Wasserstein distance in the subsection "Sampling solutions."

## A Mathematical Application

Many users of the D-Wave quantum computer in recent years have focused on hybrid workflows. In the context of quantum computing, these are software pipelines that use classical computers for a majority of their work, inserting quantum computation at compute-intensive bottlenecks [18, 24]. This is a fruitful area to focus research efforts as there will always be vast amounts of pre- and post-processing within real-world pipelines. Much of that processing is not amenable to quantum acceleration, yet alleviating bottlenecks has the potential to yield significant computational gains.

In general, users seeking quantum speedup tend to isolate the tight "inner loop" of their problem, the bottleneck where computing this loop in one step will reduce the complexity of the problem by orders of magnitude. Recent work has looked at specific methods to speed up the inner loop of topological data analysis pipelines [12, 25]. In Fig. 2, we show a typical topological data analysis (TDA) pipeline. The blue boxes on the left represent various potential data sources, while the pink boxes in the middle, labeled 1, 2, $2'$, and 3, show computational bottlenecks in the TDA pipeline. The green ovals highlight the algorithms that could potentially run on the quantum computer to alleviate bottlenecks. Lastly, the final box on the right assumes further processing using the features extracted by the TDA pipeline. We used the scikit-tda package for the TDA portion of our analysis. In the next two subsections we provide a brief summary of persistent homology (PH) and Wasserstein distance. In the subsection "The Wasserstein Graph as a QUBO," we translate the Wasserstein distance to a QUBO to compare the topological signature of point clouds. While not a bottleneck per se—polynomial-time algorithms exist to compute Wasserstein distance—it provides an instructive case for translating a general problem to a QUBO and hence into a quantum annealing framework. Our work shows the interplay between the underlying mathematics of the Wasserstein distance, the construction of a QUBO to solve the combinatorial optimization problem in Definition 7, and provides an example of the additional configurations returned as a result of the probabilistic nature of quantum computation.

**Persistent homology.** Our interest in computing Wasserstein distances is rooted in the search for robust features in noisy, real-world data. This section provides a brief overview of the PH pipeline [6]. PH, one of the most widely used tools in the field of TDA, is based on the idea that analyzing noisy data through a sequence of resolutions enables one to robustly identify and quantify structure in such data. Suppose we have a set of data points in a metric space. PH uses a filtered topological space, such as a simplicial or cubical complex, to study these data at various resolutions. A typical assumption in PH is that the data under study represents a random sample of points taken from some distribution on a manifold embedded in a nice ambient space like $\mathbb{R}^d$. It is the job of PH to discover the homology of the underlying manifold from the data.

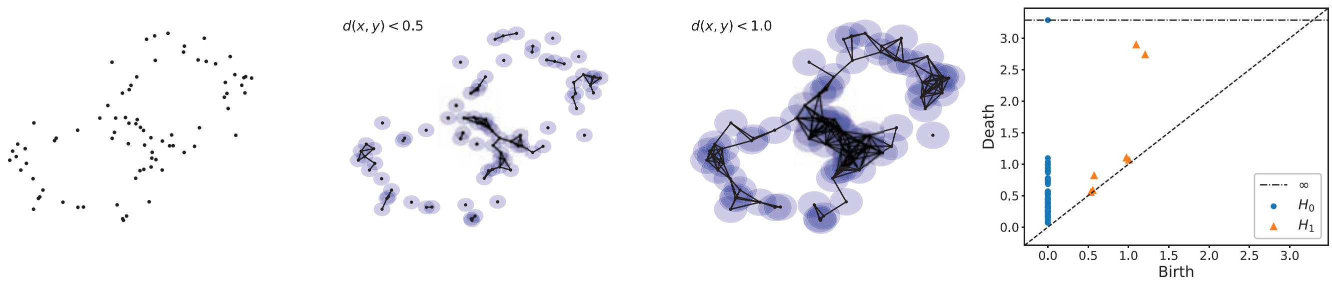


**Figure 3.** Snapshots of a filtration, with the associated persistence diagram on the right. The original data set in the left figure is generated by sampling points from two circles. A couple of steps in the filtration are shown with disks of radius $r = 0$, $r = 0.25$, and $r = 0.65$. For $H_0$, at $r \approx 0.55$ we are left with a single connected component that persists forever. In the case of $H_1$, the persistence diagram shows two long-lived generators, both born shortly after $r = 1$. When $r$ is just under 3 the central holes fill in completely, causing each to become homeomorphic to a solid disk, and killing off those generators.

As an example, consider the point cloud, $L$, on the left of Fig. 3. We take this as a collection of points in $\mathbb{R}^2$ sampled from two copies of $S^1$ and then perturbed. The concept of resolution is parameterized through the length parameter $r$. Given any $r$, we produce a *Vietoris-Rips complex*, $V_r$, from the data. A 1-simplex, or edge, is added between two points, $x, y \in L$, whenever $d(x, y) < r$. Supposing $x$ and $y$ are not part of the same connected component already, this effectively merges two connected components into one, with the corresponding disappearance of a generator in the zeroth simplicial homology group, $H_0$. Higher-dimensional simplices of dimension $k > 0$ are also included in our accounting when all $k + 1$ vertices of the simplex, $\{x_0, \dots, x_k\}$, are within distance $r$ of each other. As is clear from Fig. 3, we get that $V_r \subset V_{r'}$, for $r < r'$. This inclusion allows the PH algorithm to track the birth and death of homology generators in $H_k$ as $r$ grows (see [6]).

Given a fixed dimension $k$, we obtain a set (possibly a multiset) of intervals, $I_k = \{(b, d) \mid b, d \in \mathbb{R}\}$, defining the *birth* and *death* (appearance and disappearance) of homology classes in $H_k$ as $r$ increases. The lifetime, $b - d$, of a generator is used to infer the robustness of the corresponding topological feature. We visualize each $I_k$ by plotting the points on a *persistence diagram*. The persistence diagram on the right-hand side of Fig. 3 shows intervals for $H_0$ and $H_1$, denoted by ● and ▲, respectively. All connected components are born at $r = 0$, and merge into one component at $r \approx 1.2$. This component lives forever, as indicated by the line representing infinity. The diagram for $H_1$ indicates two long-lived homology classes at $(1.2, 2.7)$ and $(1.1, 2.9)$. These correspond to generators for the two large circles. The points just off the diagonal represent short-lived generators that correspond to small, insignificant cycles that had short lifetimes. These are often treated as noise. The intuition underlying PH is that a point in the persistence diagram far from the diagonal represents a homology class that appeared early in the filtration and died late. Such a homology class represents a robust topological feature within the noisy data.

Due to its abstract nature, PH tends towards a broad user base, with successful applications showing up in a wide variety of fields. For instance, mathematics and material engineering merge nicely in the analysis of time series obtained from rotating machines in [14]; financial crashes produce *persistence landscapes* different from stable market periods, as shown in [9]; and in [21] the authors describe a robust method for detecting holes in sensor networks.

All of these studies rely on the ability to understand trends and structures in data, which in turn requires a metric with which to compare two or more data sets. Two primary metrics used in PH are the *bottleneck distance* and the *Wasserstein distance*. For a finite dimension $k$, these metrics compute the distance between two data sets by comparing their persistence diagrams. We focus on the Wasserstein distance as we formulate an example of a quantum annealing-enabled algorithm below.

**Wasserstein distance as a graph matching problem.** In full generality, a persistence diagram is a finite multiset of points in the plane. First, define the region in the plane occupied by the persistence intervals as $\mathbb{R}^2_+ := \{(b, d) \mid d > b \text{ and } b \geq 0\}$. Second, for technical reasons, each diagram also includes an additional set of countably infinite copies of each point on the diagonal, $\Delta := \{(d, d) \mid d \geq 0\}$. The reason for this becomes clear when we define the Wasserstein distance for discrete data sets. Combining these two sets, a persistence diagram is a collection of points $\{a_1, \cdots, a_n\} \cup \Delta$, where each $a_i \in \mathbb{R}^2_+$ may occur repeatedly.

As mentioned above, in the analysis of data sets we are often interested in the distance between two persistence diagrams, $X$ and $Y$. The metric used is a discrete analog of the more general Wasserstein metric, which computes the minimal work required to transport the mass of one (continuous) probability distribution to another probability distribution. In the discrete case, we are tasked with matching points from opposing diagrams most efficiently so as to minimize the work[6] necessary to transport the configuration of points in $X$ to match the configuration of points in $Y$. In this case, we model the $p$-Wasserstein distance as

$$d_p(X,Y) = \inf_{\phi: X\to Y} \left( \sum_{a\in X} \|a-\phi(a)\|_q^p \right)^{1/p}, \tag{9}$$

where the infimum is taken over all bijections $\phi$ between points in diagrams $X$ and $Y$, $p, q \in [1,\infty)$, and $\|\cdot\|_q$ is the Euclidean $q$-norm [1]. It is convenient to use $p = q = 2$. Given a specific $\phi$, define the *cost* of the matching induced by $\phi$ as

$$C(X,Y) = \sum_{a\in X} \|a-\phi(a)\|_q^p, \tag{10}$$

where we omit reference to $\phi$, $p$, and $q$ on the left-hand side.

In practice, this is often solved by translating the bijection problem to a matching problem on a bipartite graph. Suppose $X$ and $Y$ are two persistence diagrams. We describe a method to represent the possible bijections between the diagrams $X$ and $Y$ as a weighted bipartite graph representation, that we then use to reformulate the cost function Eq. (10) as a portion of a problem Hamiltonian.

First, denote by $X_0$ and $Y_0$ the off-diagonal points in $X$ and $Y$. Define the orthogonal projections of points in $X_0$ and $Y_0$ onto the diagonal $\Delta$ by $X_0'$ and $Y_0'$, respectively. In the discrete setting, unequal numbers of points or indivisibility of mass make some matchings infeasible or impossible. In such cases the diagonal acts to absorb points in $X_0$ or $Y_0$ that cannot be matched. Then we can denote our diagrams by $X = X_0 \sqcup X_0'$ and $Y = Y_0 \sqcup Y_0'$, where we have abused notation by redefining $X$ and $Y$ to only consider the finite collection of points we will use to compute the Wasserstein distance. We now specify the graph used to construct a QUBO that we embed on the D-Wave QPU.

**Definition 11.** *Define the* **Wasserstein Graph** $W := \langle X\cup Y, E\rangle$*, where the weighted edges* $E := E_1 \cup E_2 \cup E_3$ *such that*

$$\begin{aligned} E_1 &= \{(u,v,\theta_{uv}) \mid u\in X_0, v\in Y_0\} \\ E_2 &= \{(u,u',\theta_{uu'}) \mid u\in X_0, u'\in X_0'\} \\ E_3 &= \{(v,v',\theta_{vv'}) \mid v\in Y_0, v'\in Y_0'\}, \end{aligned}$$

*and the edge weights are defined by*

$$\theta_{u,v} = \begin{cases} \|u-v\|_\infty & \text{if } v = u' \\ \|u-v\|_q & \text{otherwise.} \end{cases} \tag{12}$$

*We abbreviate edge membership, writing* $uv \in E$ *for the edge* $(u, v, \theta_{uv})$.

[6] *In the general case, the mass is variable, so transport between distributions involves the traditional* work = mass × distance *formulation. We still use this terminology, except* mass = 1*, so we neglect it.*

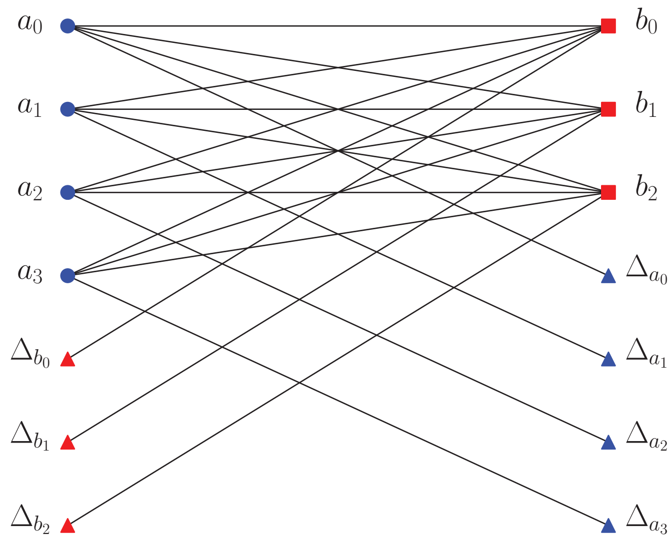


**Figure 4.** Bipartite graph with nodes on the left from $X_0$ and $Y_0'$, respectively (labeled by ●'s and ▲'s); and nodes on the right from $Y_0$ and $X_0'$, respectively (labeled by ■'s and ▲'s). The graph is complete only among off-diagonal points. (Edge weights are omitted to reduce clutter.)

Fig. 4 shows an example **Wasserstein Graph** for two persistence diagrams. In this case, $|X_0| = 4$ and $|Y_0| = 3$. Notice that off-diagonal and diagonal projection nodes are placed on opposing sides of the bipartite graph. Only the subgraph containing off-diagonal points in $X_0$ and $Y_0$ is complete. This limits the number of possible bijections between the two diagrams, a fact that helps to reduce the complexity of the QUBO as will be seen in the next section. We do not label edge weights in this example.

**The Wasserstein Graph as a QUBO.** The **Wasserstein Graph** provides a succinct example of how one might bridge mathematics and quantum computers. Given two persistence diagrams, we construct a QUBO from the associated **Wasserstein Graph**, $W$. The approach is straightforward. The QUBO must encode an objective function that minimizes the work, or cost, $C$, by "turning on" specific edges, while also enforcing certain constraints.

To make this precise, first we enumerate the edges that will map to the logical qubits. The number of edges in $W$ is $N = mn + m + n$, where $m = |X_0|$ and $n = |Y_0|$. The weighted edges in $W$ map to a set of tuples, $\{(x_{uv}, \theta_{uv}) \mid x_{uv} \in \mathbb{Z}_2\}$. An edge $uv$ is *activated* if $x_{uv} = 1$, otherwise it is *inactivated*. We now rewrite Eq. (10) in terms that include the logical qubits,

$$H_{\text{cost}}(\mathbf{x}) = \sum_{uv\in E} \theta_{uv} x_{uv},$$

where $\mathbf{x} = (x_{uv}) \in \mathbb{B}^N$. Each $\mathbf{x}$ equates to a particular matching between diagrams induced by $\phi$.

To avoid the case where setting $\mathbf{x} = \mathbf{0}$ minimizes the objective, we must add constraints. Each $u \in X_0 \cup Y_0$ must have degree exactly one in order to avoid duplication of mass and to assure that every point is transported somewhere, either between diagrams or projected to the diagonal. The diagonal nodes can have degree zero or one, depending on whether or not their off-diagonal partner connects to another off-diagonal node. From these requirements we obtain

$$H_{\text{constraint}}(\mathbf{x}) = \sum_{u \in X_0} \left(1 - \sum_{uv \in E_1 \cup E_2} x_{uv}\right)^2 + \sum_{v \in Y_0} \left(1 - \sum_{uv \in E_1 \cup E_3} x_{uv}\right)^2,$$

where the two outer summations consider only edges emanating from off-diagonal nodes. The summand, $(1 - *)$, enforces the requirement that off-diagonal nodes have degree one. If each node in $X_0 \cup Y_0$ has degree one, then $H_{\text{constraint}} = 0$; otherwise, one or both of the terms in the expression will be positive, adding a penalty to the objective function.

The restriction of edges in $H_{\text{constraint}}$ reduces the complexity of the discrete problem significantly in both the classical and quantum computing cases by limiting the number of possible bijections. It is especially beneficial in quantum situations where physical qubits are at a premium.

Combining $H_{\text{cost}}$ and $H_{\text{constraint}}$, we arrive back at the general definition of the problem in Eq. (8) with

$$H_P := H_{\text{cost}} + \gamma H_{\text{constraint}}, \tag{13}$$

where we have inserted the Lagrangian parameter $\gamma$ to balance the magnitude of the terms. Quantum computers are analog physical devices that have limited accuracy and ranges for their parameters. Thus, determining correct parameters is essential for accurate solutions.

At this point, we can study $H_P \in \mathbb{Q}[\mathbf{x}|\mathbb{B}^n]$, and also note that the linear and quadratic terms in $H_P$ define a logical graph $G$ as discussed in the section "Example 2: Compiling $H_P$ – a graph minor embedding problem." Quantum annealing requires that $H_P$ be compiled to a QMI, so at this point software converts the logical graph to the hardware graph—the grid of qubits described earlier in this section—through a minor embedding.

Eq. (13) contains an important question. While the determination of the objective function and its constituent constraints is straightforward, it is not entirely clear that minimizing Eq. (13) yields the same value as Eq. (9). In [1, Sec. 4], we prove that minimizing Eq. (13) computes the $p$-Wasserstein distance, with the caveat that the minimizer of $\mathcal{H}$ provides an equivalent solution to Eq. (9) iff $\gamma$ satisfies

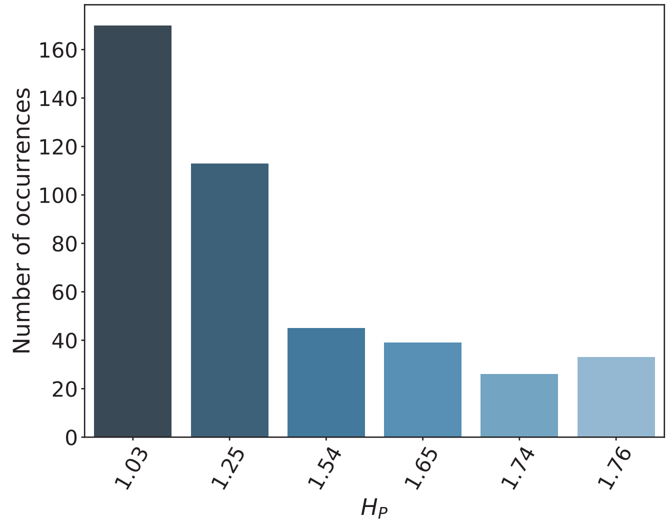


**Figure 5.** Frequency of costs, $H_P$, computed on the D-Wave quantum computer for different matchings between two persistence diagrams. The diagrams were computed from a torus and an annulus. For this example, $\gamma = 1$. Each distance on the $x$-axis corresponds to a specific $\mathbf{x} \in \mathbb{B}^N$. The occurrences on the $y$-axis denote the number of times a unique $\mathbf{x}$ was sampled. The matching corresponding to the Wasserstein distance, 1.03, is sampled most frequently.

$$\gamma > \max_{uv \in E} \theta_{uv}.$$

By keying the analysis of the quantum computational problem off of a known computable metric, we are able to determine exactly how to set hyperparameters properly. By contrast, it is often necessary in general problems to perform searches of the parameter space before a reasonable energy landscape, defined by $H_P$, can be processed accurately by the QPU.

**Sampling solutions.** As mentioned in the section "Example 3: Distributions of solutions through sampling," solutions returned by a quantum computer are probabilistic in nature. We obtain many samples by running the annealing procedure multiple times. In Fig. 5, the suite of samples we gather represents the cost of different possible matchings between the persistence diagrams of a torus and an annulus. We use $H_P(\mathbf{x})$ to compute the cost. The low-energy solutions represent valid matchings that do not violate constraints, e.g., $H_{\text{constraint}}(\mathbf{x}) = 0$. The minimum cost, 1.03, is the square of the Wasserstein distance, i.e., the infimum over all the possible valid matchings.

The different matchings and distances represent a distribution of low-energy solutions, each sample of which comes from a different $\phi$ and produces a different cost using Eq. (10). In fact, Fig. 5 represents a distribution of $\phi$'s sampled from an approximate Boltzmann distribution. In future work we plan to study the implications for statistics on persistence diagrams, along the lines outlined by Turner et al. in their work on Fréchet means in [23].

## Conclusion

In this article we covered a number of mathematical aspects of quantum computing from a high level. Nevertheless, we have hardly scratched the surface of the subject. Interesting problems can be found in many different areas, from physical applications, to theoretical improvement of embedding QUBOs on the QPU, to decomposition of large problems into QPU-sized chunks.

Mathematicians and physicists have spent many years developing algorithms designed to run faster on quantum computers. The subtlety is that many of these methods require far more qubits than are available even on the 2000-qubit D-Wave quantum computer. Luckily, even before we reach that technological state, there is still exciting and effective research that can be accomplished in the current NISQ era. We hope that in touching on the mathematics involved in programming a D-Wave quantum computer we motivate interest in the myriad problems stemming from using this novel computational tool.

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Jesse J. Berwald

**Credits**

Figures 1–5 are by the author.
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# An Invitation to Nonstandard Analysis and its Recent Applications

*Isaac Goldbring and Sean Walsh*

## Introduction

Nonstandard analysis has been used recently in major results, such as Jin's sumset theorem in additive combinatorics and Breuillard–Green–Tao's work on the structure of approximate groups. However, its roots go back to Robinson's formalization of the infinitesimal approach to calculus. After first illustrating its very basic uses in calculus in "Calculus with Infinitesimals," we go on to highlight a selection of its more serious achievements in "Selected Classical and Recent Applications," including the aforementioned work of Jin and Breuillard–Green–Tao. After presenting a simple axiomatic approach to nonstandard analysis in "Axioms for Nonstandard Extensions" we examine Jin's theorem in more detail in "The Axioms in Action: Jin's Theorem." Finally, in "The Ultraproduct Construction" we discuss how these axioms can be justified

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with a particular concrete construction (akin to the verification of the axioms for the real field using Dedekind cuts or Cauchy sequences), and in "Other Approaches" we compare our axiomatic approach to other approaches. While brief, our hope is that this survey can quickly give the reader a sense of both how nonstandard methods are being used today and how these methods can be rigorously presented and justified.

## Calculus with Infinitesimals

Every mathematician is familiar with the fact that the founders of calculus such as Newton and Leibniz made free use of *infinitesimal* and *infinite* quantities, e.g., in expressing the derivative $f'(a)$ of a differentiable function $f$ at a point $a$ as being the real number that is infinitely close to $\frac{f(a+\delta)-f(a)}{\delta}$ for every nonzero infinitesimal "quantity" $\delta$. Here, when we say that $\delta$ is infinitesimal, we mean that $|\delta|$ is less than $r$ for every positive real number $r$.

Of course, the mathematical status of such quantities was viewed as suspect and the entirety of calculus was put on firm foundations in the nineteenth century by the likes of Cauchy and Weierstrass, to name a few of the more significant figures in this well-studied part of the history of mathematics. The innovations of their "$\epsilon$-$\delta$ method" (much to the chagrin of many real analysis students today) allowed one to give rigor to the naïve arguments of their predecessors.

In the 1960s, Abraham Robinson realized that the ideas and tools present in the area of logic known as *model theory* could be used to give precise mathematical meanings to infinitesimal quantities. Indeed, one of Robinson's stated aims was to rescue the vision of Newton and Leibniz ([20]). While there are complicated historical and philosophical questions about whether Robinson succeeded entirely in this (cf. [5]), our goal is to discuss some recent applications of the methods Robinson invented.

To this end, let us turn now to the basics of Robinson's approach. On this approach, an infinitesimal $\delta$ is merely an element of an ordered field $\mathbf{R}^*$ properly containing the ordered field $\mathbf{R}$ of real numbers. While such *nonarchimedean* fields were already present in, for example, algebraic number theory, the new idea that allowed one to correctly apply the heuristics from the early days of calculus was that $\mathbf{R}^*$ is logically similar to $\mathbf{R}$, in that any elementary statement true in the one is also true in the other. (Formally, elementary statements are defined using first-order logic: see "Axioms for Nonstandard Extensions" for a self-contained presentation.) A particular feature of this approach is that one functorially associates to every function $f : \mathbf{R} \to \mathbf{R}$ an extension $f : \mathbf{R}^* \to \mathbf{R}^*$ (and similarly for relations). In this light, $\mathbf{R}$ is referred to as the *standard field of real numbers* whilst $\mathbf{R}^*$ is referred to as *a nonstandard field of real numbers* or a *hyperreal field*.

To illustrate this method, let us say that two elements $a$ and $b$ of $\mathbf{R}^*$ are *infinitely close* to one another, denoted $a \approx b$, if their difference is an infinitesimal. Then one has:

**Theorem 1.** *Suppose that $f : \mathbf{R} \to \mathbf{R}$ is a function and $a \in \mathbf{R}$. Then $f$ is continuous at $a$ if and only if: whenever $b \in \mathbf{R}^*$ and $a \approx b$, then $f(a) \approx f(b)$. Likewise, $f$ is differentiable at $a$ with derivative $L$ if and only if: whenever $b$ is a nonzero infinitesimal in $\mathbf{R}^*$, one has that $\frac{f(a+b)-f(a)}{b} \approx L$.*

*Proof.* We only discuss the continuity statement, since the differentiability statement is entirely analogous. Suppose that the assumption of the "if" direction holds and fix $\epsilon > 0$ in $\mathbf{R}$. In $\mathbf{R}^*$, by choosing an infinitesimal $\delta > 0$, the following elementary statement about $\epsilon$ is true: "there is $\delta > 0$ such that, for all $b$, if $|a - b| < \delta$, then $|f(a) - f(b)| < \epsilon$." The logical similarity mentioned above then implies that this elementary statement about $\epsilon$ is also true in $\mathbf{R}$. But this is precisely what is needed to prove that $f$ is continuous at $a$. The other direction is similar. □

Every finite element $a$ of $\mathbb{R}^*$ is infinitely close to a unique real number, called its *standard part*, denoted $\mathrm{st}(a)$. Thus, the first part of the above theorem reads: $f$ is continuous at $a$ if and only if: whenever $\mathrm{st}(b) = a$, then $\mathrm{st}(f(b)) = f(a)$. In these circumstances, one can then write $b = a + \delta x$ and $f(b) = f(a) + \delta y$ where $\delta x, \delta y$ are infinitesimals. In his elementary calculus textbook based on nonstandard methods, Keisler pictured this as a "microscope" by which one can zoom in and study the local behavior of a function at a point.

Besides extensionally characterizing familiar concepts such as continuity and differentiability, the use of infinitesimals can help to abbreviate proofs. To illustrate, let us prove the following, noting how we can avoid the usual "$\frac{\epsilon}{2}$" argument and instead appeal to the simple fact that finite sums of infinitesimals are again infinitesimal:

**Corollary 2.** *Suppose that $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ are both continuous at $a \in \mathbf{R}$. Then $f + g$ is also continuous at $a$.*

*Proof.* Consider $b \approx a$. By assumption, we have $f(b) \approx f(a)$ and $g(b) \approx g(a)$. Then since the sums of two infinitesimals is infinitesimal, we have

$$(f+g)(b) = f(b)+g(b) \approx f(a)+g(a) = (f+g)(a).$$

Since $b$ was arbitrary, we see that $f + g$ is continuous at $a$. □

## Selected Classical and Recent Applications

The primary reason for the contemporary interest in nonstandard analysis lies in its capacity for proving new results. In this section, we briefly describe a handful of the more striking results that were first proven using nonstandard methods. We remark that nonstandard methods have proven useful in nearly every area of mathematics, including algebra, measure theory, functional analysis, stochastic analysis, and mathematical economics to name a few. Here, we content ourselves with a small subset of these application areas.

**Bernstein-Robinson theorem on invariant subspaces.** The famous *Invariant Subspace Problem* asks whether every bounded operator $T$ on a separable Hilbert space $H$ has a $T$-invariant closed subspace besides $\{0\}$ and $H$. In the 1930s, von Neumann proved that *compact* operators have nontrivial closed invariant subspaces. (Recall an operator is compact if it is the norm-limit of finite-rank operators.) Little progress was made on the Invariant Subspace Problem until:

**Theorem 3** (Bernstein-Robinson [3]). *If $T$ is a* polynomially compact *operator on $H$ (meaning that there is a nonzero polynomial $p(Z) \in \mathbb{C}[Z]$ such that $p(T)$ is a compact operator), then $T$ has a nontrivial closed invariant subspace.*

One of the main ideas of the proof is to use the nonstandard version of the basic fact that operators on finite-dimensional spaces have upper-triangular representations to find many hyperfinite-dimensional subspaces of $H^*$ which are $T^*$-invariant. (Upon seeing the Berenstein-Robinson theorem, Halmos proceeded to give a proof that did not pass through nonstandard methods and the two papers were published together in the same volume.) We should point out that the Berenstein-Robinson theorem was later subsumed by Lomonosov's theorem from 1973, which says that an operator that commutes with a nonzero compact operator satisfies the conclusion of the Invariant Subspace Problem.

**Asymptotic cones.** In [11], Gromov proved the following theorem, which is one of the deepest and most beautiful theorems in geometric group theory:

**Theorem 4.** *If $G$ is a finitely generated group of polynomial growth, then $G$ is virtually solvable.*

Here, a finitely generated group has *polynomial growth* if the set of group elements that can be written as a product of at most $d$ generators and their inverses grows polynomially in $d$. The polynomial growth condition is a geometric condition describing the growth of the sizes of balls centered at the identity in the Cayley graph of the group and the amazing fact represented in this theorem is that this geometric condition has serious algebraic consequences. (The converse of the theorem is also true and much easier to prove.)

A key construction in the proof of Gromov's theorem is that of an *asymptotic cone* of a metric space. Roughly speaking, an asymptotic cone of a metric space is the result of looking at the metric space from "very far away" retaining only the large-scale geometry of the space. The prime example of this phenomenon is that an asymptotic cone of the discrete space $\mathbf{Z}$ is the continuum $\mathbf{R}$.

In [8], van den Dries and Wilkie used the nonstandard perspective to give a much cleaner account of the asymptotic cone construction. Indeed, given a metric space $(X, d)$, a fixed point $x_0 \in X$, and an infinite hyperreal number $R \in \mathbf{R}^*$, one can look at the subspace

$$X_R := \left\{ x \in X^* \ : \ \frac{d(x, x_0)}{R} \text{ is finite} \right\}.$$

One can then place the metric $d_R$ on $X_R$ given by $d_R(x, y) := \mathrm{st}(\frac{d(x,y)}{R})$. When $X$ is the Cayley graph associated to a group, then the polynomial growth condition on $\Gamma$ allows one to find $R$ so that $X_R$ is locally compact; the verification of this and other important properties of $X_R$ are very clear from the nonstandard perspective. While Gromov's original proof did not use nonstandard methods, the nonstandard perspective on asymptotic cones has now in fact become the one that is presented in courses and textbooks on the subject. See, for instance, [18].

**Jin's Sumset Theorem.** In additive combinatorics, the focus is often on densities and structural properties of subsets of $\mathbf{N}$. Given $A \subseteq \mathbf{N}$, we define the *Banach density of $A$* to be $\mathrm{BD}(A) := \lim_{n\to\infty} \max_{x\in\mathbf{N}} \frac{|A\cap[x,x+n)|}{n}$. If $\mathrm{BD}(A) > 0$, then we think of $A$ as a "large" subset of $\mathbf{N}$. An important structural property of sets of natural numbers is that of being *piecewise syndetic*, where $A$ is piecewise syndetic if there is $m \in \mathbf{N}$ such that $A + [0, m]$ contains arbitrarily long intervals. Renling Jin [14] used nonstandard analysis to prove the following:

**Theorem 5.** *If $A, B \subseteq \mathbf{N}$ both have positive Banach density, then $A + B$ is piecewise syndetic.*

Jin's theorem has paved the way for further applications of nonstandard methods in additive combinatorics, which is now a very active area; see the monograph [6]. In "The Axioms in Action: Jin's Theorem," we discuss the proof of Jin's theorem, using the formal framework for nonstandard analysis that we set out in "Axioms for Nonstandard Extensions."

**The structure of approximate groups.** Fix $K \geq 1$. A symmetric subset $A$ of a finite group $G$ is said to be a *$K$-approximate subgroup of $G$* if $A \cdot A$ is contained in $K$ (left) translates of $A$. Approximate subgroups are generalizations of subgroups (since a 1-approximate group is simply a subgroup of $G$). The *Freiman theorem for abelian groups* classifies $K$-approximate subgroups of abelian groups (they are, in some sense, built from generalized arithmetic progressions and group extensions). It was an open problem whether or not a similar result held for $K$-approximate subgroups of arbitrary (not necessarily abelian) groups.

In [13], using sophisticated methods from model theory, Hrushovski made a major breakthrough in the above so-called nonabelian Freiman problem. While he did not completely solve the problem, his breakthrough paved the way for the work of Breuillard, Green, and Tao [4], who reduced the use of sophisticated model theory in favor of mere nonstandard analysis (while using more intricate combinatorics) and, in the process, succeeded in providing a complete classification of arbitrary approximate groups.

The key insight of Hrushovski (which is also present in the Breuillard–Green–Tao work) is that a nonstandard (infinite) approximate group can be "modelled" by a locally compact group, which, by the structure theory of locally compact groups, can in turn be modeled by finite-dimensional Lie groups. (This is not unrelated to the use of nonstandard methods in the proof of Gromov's theorem described above.) Thus, in attacking this problem of finite combinatorics, one can use the infinitary tools from differential geometry and Lie theory. Blurring the distinction between the finite and the infinite is a cornerstone of many applications of nonstandard methods, allowing one to leverage tools from either side of the divide whenever it proves convenient. Indeed, this is present even in the characterization of differentiability from Theorem 1, where the infinitary process of taking a limit is replaced by the discretized finite quotient.

We mention in passing that the structure theory of locally compact groups, due to Gleason, Montgomery, and Zippin, solved the fifth problem of Hilbert. Hirschfeld [12] used nonstandard analysis to give a conceptually simpler account of this solution. The first-named author generalized Hirschfeld's account to solve the local version of Hilbert's fifth problem [10]; this was, in turn, a part of the Breuillard–Green–Tao work mentioned above.

## Axioms for Nonstandard Extensions

In this section, we describe an approach to nonstandard analysis using three axioms (NA1)-(NA3). This follows and elaborates on the treatment in the appendix of the first-named author's earlier coauthored work [7]. However, when specialized to the real numbers, axioms (NA1)-(NA2) are very similar to Goldblatt's [9] presentation of nonstandard methods.

The approach through (NA1)-(NA3) should remind the reader of the beginning of measure theory, where one sets out carefully a class of sets that form the building blocks of the subject-matter one is interested in. The most basic sets are those built into what we call the *structures* in (NA1)—and then we expand upon this class with the *definable sets* of axiom (NA2). This is similar to how, in measure theory, one might pass from basic half-opens on the real line to the $\sigma$-algebra generated by them. However, in nonstandard analysis, we build up more complicated sets from basic sets simultaneously in a structure and a nonstandard extension, similar to how one might look at varieties both over a field and over various of its field extensions. Finally, for our applications, we need a final axiom (NA3) that ensures that our class of definable sets in the nonstandard extension has certain compactness properties.

### Structures and axiom (NA1).

**Definition 6.** A *structure* $\mathbf{S} = ((S_i), (R_j))$ is a collection of sets $(S_i \ : \ i \in I)$, often called the *basic sets* of $\mathbf{S}$, together with a collection $(R_j \ : \ j \in J)$ of *distinguished relations* on the basic sets, that is, for each $j \in J$, there are $i(1), \dots, i(n) \in I$ such that $R_j \subseteq S_{i(1)} \times \cdots \times S_{i(n)}$. Distinguished relations are also called *primitives* of $\mathbf{S}$.

Later, it will be convenient to speak of the *complete structure on* $(S_i)$, which is simply the structure with basic sets $(S_i)$ and where we take *all* relations as the basic relations.

We can now state the first axiom of nonstandard extensions, which simply explains what makes them extensions:

(NA1) Each basic set $S_i$ is extended to a set $S_i^* \supseteq S_i$ and, to each distinguished relation $R_j$ as above, we associate a corresponding relation

$$R_j^* \subseteq S_{i(1)}^* \times \cdots \times S_{i(n)}^*$$

whose intersection with $S_{i(1)} \times \cdots \times S_{i(n)}$ is the original relation $R_j$.

**Definable sets and axiom (NA2).** Let $\mathbf{S}$ be a structure. Some notation: given $\bar{i} := (i(1), \dots, i(n))$, we set $S_{\bar{i}} := S_{i(1)} \times \cdots \times S_{i(n)}$. It will also be necessary to declare $S_\varnothing$ to be a one-element set. For the sake of readability, if $\bar{i}$ and $\bar{j}$ are two finite sequences, then we write $\bar{i}\,\bar{j}$ for the concatenation of the two sequences; if $\bar{i} = (i)$, then we simply write $i\,\bar{j}$ and similarly for when $\bar{j}$ is a one-element sequence.

We define the collection of $\mathbf{S}$-*definable sets* to be the Boolean algebras $\mathcal{D}_{\mathbf{S}}(\bar{i})$ (or simply $\mathcal{D}(\bar{i})$ if $\mathbf{S}$ is clear from context) of subsets of $S_{\bar{i}}$ with the following properties:

(1) $\varnothing, S_{\bar{i}} \in \mathcal{D}(\bar{i})$.
(2) For any $i$, $\{(x,y) \in S_i \times S_i \ : \ x = y\} \in \mathcal{D}(i,i)$.
(3) If $a \in S_i$, then $\{a\} \in \mathcal{D}(i)$.
(4) If $R_j \subseteq S_{\bar{i}}$ is a basic relation, then $R_j \in \mathcal{D}(\bar{i})$.
(5) If $A \in \mathcal{D}(\bar{i})$, then $A \times S_j \in \mathcal{D}(\bar{i}\,j)$ and $S_j \times A \in \mathcal{D}(j\,\bar{i})$.
(6) If $A \in \mathcal{D}(\bar{i})$ and $\vec{i} = \overline{i_1}\ j\ \overline{i_2}$ and $\pi \ : \ S_{\bar{i}} \to S_{\overline{i_1}\,\overline{i_2}}$ is the canonical projection, then $\pi(A) \in \mathcal{D}(\overline{i_1}\,\overline{i_2})$.

While the closure properties of a Boolean algebra encode propositional operations such as conjunction and disjunction, the additional postulates on the definable sets encode the operations of predicate logic: the identity relation is homophonically encoded in (2), while the last condition (6) encodes existential quantification since $\overline{a_1}\ \overline{a_2}$ is in the projection $\pi(A)$ iff there is $b$ in $S_j$ with $\overline{a_1}\ b\ \overline{a_2}$ belonging to $A$.

Now, if $D \in \mathcal{D}(\bar{i})$ is an $\mathbf{S}$-definable set, then define the $\mathbf{S}^*$-definable set $D^*$ simply by replacing all instances of basic relations $R_j$ used in the construction of $D$ by $R_j^*$. Note that not every $\mathbf{S}^*$-definable set is of the form $D^*$ for an $\mathbf{S}$-definable set, e.g., $\{a\}$ for $a \in S_i^* \backslash S_i$. And we will soon see that such elements exist.

We can now explain the second axiom of nonstandard analysis, the so-called *transfer principle*, which in the terminology of definable sets simply says that the property of basic relations from (NA1) also holds for arbitrary $\mathbf{S}$-definable sets:

(NA2) For any $\mathbf{S}$-definable set $D \in \mathcal{D}(\bar{i})$, we have $D^* \cap S_{\bar{i}} = D$.

Axiom (NA2) states precisely what it means for the nonstandard extension to be logically similar to the original structure. It is saying that any statement about elements of the original structure that can be phrased in terms of definable sets also holds within the nonstandard extension.

**Compactness and richness (NA3).** Our final axiom asks that our nonstandard extensions contain sufficiently many "ideal" elements analogous to the infinitesimal elements added to $\mathbf{R}^*$ in "Calculus with Infinitesimals." Here is the precise statement:

The structure $\mathbf{S}$ is said to be *countably rich* if for all $\vec{i}$ and every countable family $(X_m)_{m\in\mathbb{N}}$ of elements of $\mathcal{D}(\bar{i})$ with the finite intersection property, we have $\bigcap_m X_m \neq \emptyset$. Here the "finite intersection property" means that $X_{m_1} \cap \cdots \cap X_{m_k} \neq \emptyset$ for all $m_1, \ldots, m_k \in \mathbb{N}$.

Countable richness can be seen as a logical compactness property when stated in its contraposed form: if $\mathbf{S}$ is countably rich and an $\mathbf{S}$-definable set $X \in \mathcal{D}(\bar{i})$ is covered by countably many $\mathbf{S}$-definable sets $X_m \in \mathcal{D}(\bar{i})$, then $X$ is already covered by finitely many of these $X_m$.

Here is the final axiom of nonstandard extensions:

(NA3) $\mathbf{S}^*$ is countably rich.

Note that, as an ordered set, $\mathbf{R}$ is not countably rich, since $\bigcap_n (n, +\infty) = \emptyset$. Thus our initial structure $\mathbf{S}$ will usually not be rich, which is why we consider now an extension $\mathbf{S}^*$ of $\mathbf{S}$ such that (NA1), (NA2), and (NA3) hold, so $\mathbf{S}^*$ is countably rich. One consequence is that if $X$ is countable and infinite, then $X^*$ will have elements that are not in $X$.

Richness is the precise formulation of asking that our nonstandard extension have certain ideal elements analogous to the infinitesimal elements of $\mathbf{R}^*$. For example, if $G$ is a topological group, we will want there to be elements $a \in G^*$ that are infinitely close to the identity element of $G$ in the sense that $a \in U^*$ for every neighborhood $U$ of the identity in $G$. If $G$ is first countable, then countable richness will ensure the existence of such elements.

We should note that, for certain applications of nonstandard methods, countable richness is not a strong enough assumption. For example, we may want the analogous version of countable richness to hold for families of definable sets indexed by the real numbers rather than the natural numbers. (Such richness is desirable, for example, in applications to combinatorial number theory; see [6].) In this latter scenario, one would then assume that the nonstandard extension is $\mathfrak{c}$-rich, where $\mathfrak{c}$ is the cardinality of the real numbers. One can of course speak of greater assumptions of richness, but we will refrain from dwelling too much on this point.

This concludes the list of our axioms for nonstandard analysis. We remark that we could have also added distinguished functions in our definition of a structure, and then a requirement in (NA2) for extensions of distinguished functions. But working with graphs of functions, it is routine to see that such a set-up can be encoded in a purely relational structure as above.

We now stress: *all applications of nonstandard analysis can proceed from the above three axioms alone.*

**Power sets and internal sets.** The reader already familiar with nonstandard analysis will notice that we have yet to speak about a notion that is of central importance, namely the notion of internal sets. We remedy that now.

For certain basic sets $S$ of our structure $\mathbf{S}$, we often include also its power set $\mathcal{P}(S)$ as a basic set, and the membership relation $\in_S := \{(x, Y) \in S \times \mathcal{P}(S) : x \in Y\}$ as a primitive. This allows us to quantify over elements of $\mathcal{P}(S)$, and thus gives enormous expressive power. For example, with $\mathbf{R}$ and $\mathcal{P}(\mathbf{R})$ as basic sets, together with the membership relation between them and the ordering on $\mathbf{R}$ as primitives, we can express by an elementary statement the fact that every nonempty subset of $\mathbf{R}$ with an upper bound in $\mathbf{R}$ has a least upper bound in $\mathbf{R}$ (i.e., the completeness of the real line).

Given $Y \in \mathcal{P}(S)$, the formula $v \in_S Y$ (with $v$ a variable ranging over $S$) defines the subset $Y$ of $S$, so $Y$ is not only an element in our structure $\mathbf{S}$, but also an $\mathbf{S}$-definable subset of $S$. In particular, *every* subset of $S$ is now $\mathbf{S}$-definable (while not every subset of $\mathcal{P}(S)$ is $\mathbf{S}$-definable).

Next, let an extension $\mathbf{S}^*$ of $\mathbf{S}$ be given that satisfies (NA1) and (NA2). Then $S^*$ and $\mathcal{P}(S)^*$ are basic sets of $\mathbf{S}^*$, and the star extension $\in_S^*$ of $\in_S$ is among the primitives. There is no reason for the elements of $\mathcal{P}(S)^*$ to be actual subsets of $S^*$, or for $\in_S^*$ to be an actual membership relation, but we always arrange this to be the case: just replace $P \in \mathcal{P}(S)^*$ by the subset $\{a \in S^* : a \in_S^* P\}$ of $S^*$. This procedure is traditionally called *Mostowski collapse*, and it identifies $\mathcal{P}(S)^*$ with a subset of $\mathcal{P}(S^*)$.

The subsets of $S^*$ that belong to $\mathcal{P}(S)^*$ via this identification are traditionally called *internal subsets of* $S^*$. They are in fact exactly the $\mathbf{S}^*$-definable subsets of $S^*$, as is easily checked.

## The Axioms in Action: Jin's Theorem

In this section, we give some details into the proof of Jin's theorem from "Jin's Sumset Theorem."

**A few words on $\mathbb{N}^*$.** First, it behooves us to give a picture of $\mathbb{N}^*$. We begin by noting that, by transfer, every element $N$ of $\mathbb{N}^*\backslash\mathbb{N}$ is above every element $\mathbb{N}$ in the ordering. Transfer implies that any element $N$ in $\mathbb{N}^*\backslash\mathbb{N}$ has a predecessor in $\mathbb{N}^*$, which we denote by $N-1$; clearly $N-1$ is also infinite, whence we may consider its predecessor $N-2$, another infinite element, and so on. Consequently, $N$ is contained in a copy of the integers (called a $\mathbb{Z}$-chain), the entirety of which is contained in the infinite part of $\mathbb{N}^*$. Note that $2N$ is an infinite number that is contained in a strictly larger $\mathbb{Z}$-chain, and that $\frac{N}{2}$ is an infinite number contained in a strictly smaller $\mathbb{Z}$-chain. Finally, note that if $N < M$ are both infinite, then $\frac{N+M}{2}$ is contained in a $\mathbb{Z}$-chain strictly in-between the $\mathbb{Z}$-chains determined by $N$ and $M$. We summarize:

**Lemma 7.** *The set of $\mathbb{Z}$-chains determined by infinite elements of $\mathbb{N}^*$ is a dense linear order without endpoints.*

By an *interval* $I \subseteq \mathbb{N}^*$, we mean a subset of the form

$$(a,b) := \{x \in \mathbb{N}^* \ : \ a < x < b\}$$

for $a < b$ in $\mathbb{N}^*$. The interval is said to be infinite if $b-a$ is infinite.

Given a (finite) interval $I \subseteq \mathbb{N}$, every subset $D$ of $I$, being itself finite, of course has a cardinality that is an element of $[0,|I|)$. Analogously, given an infinite interval $I$ in $\mathbb{N}^*$ and a definable subset $D$ of $I$, there is a natural notion of the *definable cardinality* $|D|$ of $D$, which is an element of $[0,|I|)$. Indeed, we can view the cardinality function $|\cdot|$ as a function on the product of basic sets $\mathcal{P}(\mathbb{N}) \times \mathbb{N} \times \mathbb{N}$ given by $|\cdot|(D,a,b) := |D \cap (a,b)|$ if $a < b$ (and some other default value otherwise); the nonstandard extension of this function then assigns a cardinality to definable subsets of intervals in $\mathbb{N}^*$. By transfer, the definable cardinality of an interval is simply its length.

**Piecewise syndeticity.** Our next order of business is to give a nonstandard description of piecewise syndeticity, which we introduced in "Jin's Sumset Theorem." Given $C \subseteq \mathbb{N}^*$, a *gap* of $C$ on $I$ is a subinterval $J$ of $I$ such that $C \cap J = \emptyset$.

**Theorem 8.** *$C \subseteq \mathbb{N}$ is piecewise syndetic if and only if there is an infinite interval $I \subseteq \mathbb{N}^*$ such that $C^*$ has only finite gaps on $I$.*

*Proof.* We only prove the "if" direction. Suppose that $I$ is an infinite interval such that $C^*$ has only finite gaps on $I$. We first note that there is $m \in \mathbb{N}$ such that $C^*$ has only gaps of length at most $m$ on $I$, whence $I \subseteq C^* + [0,m]$. Indeed, if this were not the case, then the set $X_m := \{x \in I \ : \ [x,x+m) \cap C^* = \emptyset\}$ is a nonempty definable subset of $\mathbb{N}^*$ for each $m$, whence by countable saturation, there is $x \in \bigcap_m X_m$; we then have that $x, x+1, x+2, \ldots \notin C^*$, contradicting that $C^*$ has only finite gaps on $I$.

Now, given $k \in \mathbb{N}$, let $Y_k := \{x \in \mathbb{N} \ : \ [x,x+k) \subseteq C + [0,m]\}$. By the last paragraph and transfer, we have that $Y_k^* \neq \emptyset$, whence, by transfer again, we have that $Y_k \neq \emptyset$. Since $k$ was arbitrary, this proves that $C$ is piecewise syndetic. □

As an aside, we invite the reader to use the previous theorem to give a quick nonstandard proof that the class of piecewise syndetic subsets of $\mathbb{N}$ is *partition regular*, meaning that if $A$ is piecewise syndetic and $A = B \cup C$, then at least one of $B$ or $C$ is piecewise syndetic. Partition regularity is a crucial notion in Ramsey theory, and the fact that the class of piecewise syndetic sets is partition regular indicates that it is a robust structural notion of largeness.

For $x, y \in \mathbb{N}^*$, we write $x \sim_{\mathbb{N}} y$ if and only if $|x-y|$ is finite. This is an equivalence relation on $\mathbb{N}^*$ and we set $C_{\mathbb{N}}$ for the set of equivalence classes: per Lemma 7, this is just the equivalence class of $\mathbb{N}$ together with the $\mathbb{Z}$-chains. We let $\pi_{\mathbb{N}} : \mathbb{N}^* \to C_{\mathbb{N}}$ denote the canonical projection. Again by Lemma 7, the linear order on $\mathbb{N}^*$ descends to a dense linear order on $C_{\mathbb{N}}$. We can thus restate the previous theorem as: $C$ is piecewise syndetic if and only if $\pi_{\mathbb{N}}(C)$ contains an interval.

**Cuts.** We now consider a different equivalence relation on $\mathbb{N}^*$. Fix an infinite $N \in \mathbb{N}^*$ and consider instead the relation $\sim_N$ on $\mathbb{N}^*$ given by $x \sim_N y$ if and only if $\frac{|x-y|}{N}$ is infinitesimal. We let $C_N$ denote the set of equivalence classes and $\pi_N : \mathbb{N}^* \to C_N$ the canonical projection. This time, something interesting happens: as a linear order, an initial segment of $C_N$ is isomorphic to the set of positive reals (in particular, the initial segment consists of those $x$ such that $\frac{x}{N}$ is finite, and the isomorphism is given by sending $x$ to $\operatorname{st}(\frac{x}{N})$).

The equivalence relations $\sim_{\mathbb{N}}$ and $\sim_N$ are instances of a more general notion. We call an initial segment $U$ of $\mathbb{N}^*$ a *cut* if $U$ is closed under addition. Note that $\mathbb{N}$ is the smallest cut. Another example of a cut is the cut $U_N := \{K \in \mathbb{N}^* \ : \ \frac{K}{N} \text{ is infinitesimal}\}$, where $N$ is infinite. As above, for a cut $U$, defining $x \sim_U y$ if $|x-y| \in U$ yields an equivalence relation on $\mathbb{N}^*$ with set of equivalence classes $C_U$ and projection map $\pi_U$. (We chose $\sim_N$ and $\pi_N$ as opposed to $\sim_{U_N}$ and $\pi_{U_N}$ for notational cleanliness.) As before, the usual order on $\mathbb{N}^*$ descends to a linear order on $C_U$.

We now recall a classical theorem of Steinhaus: if $E$ and $F$ are subsets of $\mathbb{R}$ of positive Lebesgue measure, then $E+F$

contains an interval. Given that reals are simply equivalence classes modulo the cut $U_N$ and in proving Jin's theorem we are looking for sums of cuts modulo $U_{\mathbb{N}}$ to contain an interval, it raises the question as to whether or not there is a natural measure on any cut space $C_U$ for which the analogue of Steinhaus' theorem is true and which yields Lebesgue measure in the case of the cut space $C_N$.

Earlier, we saw that every definable subset of an interval $I$ in $\mathbb{N}^*$ has a definable cardinality. This procedure leads to a natural measure $\mu_I$ on the algebra of definable subsets of $I$ given by $\mu_I(D) := \mathrm{st}(\frac{|D|}{|I|})$. The usual Carathéodory extension procedure shows that $\mu_I$ can be extended to a $\sigma$-additive probability measure on the $\sigma$-algebra generated by the definable subsets of $I$. Indeed, countable richness of the nonstandard extension ensures that the hypotheses of the Carathéodory's extension theorem apply. The resulting measure is called the *Loeb measure* on $I$ with corresponding $\sigma$-algebra of *Loeb measurable sets*.

If we are in the situation that $I = [0, N)$ for $N$ infinite and $U$ is a cut contained in $I$, then we can push forward the Loeb measure to a probability measure on $C_U$, which we also refer to as Loeb measure.

This procedure does in fact agree with Lebesgue measure in the case of the cuts $U_N$:

**Theorem 9.** *Given an infinite $N$, the pushforward via $\pi_N$ of the Loeb measure on $[0, N)$ is the Lebesgue measure on $[0, 1]$.*

Motivated by the preceding discussion, Keisler and Leth asked whether the analog of Steinhaus' result holds for arbitrary cut spaces. Renling Jin answered this question affirmatively:

**Theorem 10** ([14]). *Suppose that $I = [0, N)$ is an infinite interval and $U$ is a cut contained in $I$. If $A$ and $B$ are definable subsets of $I$ with positive Loeb measure, then $\pi_U(A + B)$ contains an interval.*

Technically speaking, in order for this to literally be true, one needs to assume that $A$ and $B$ are contained in the first half of $I$; otherwise, one needs to view $I$ as a nonstandard cyclic group and then look at $A + B$ in the group sense. Without going into too much detail, the theorem is proven by contradiction, taking a "maximal" counterexample, and performing some nontrivial nonstandard counting.

**Finishing the proof of Jin's theorem.** We now have all of the pieces necessary to prove the sumset theorem. Suppose that $A$ and $B$ are subsets of $\mathbb{N}$ with positive Banach density. By the nonstandard characterization of limit, there is some infinite interval $I$ such that $\mathrm{BD}(A)$ is approximately equal to $\frac{|A^* \cap I|}{|I|}$. In other words, $\mathrm{BD}(A) = \mu_I(A^* \cap I)$. Likewise, there is an infinite interval $J$ such that $\mathrm{BD}(B) = \mu_J(B^* \cap J)$. Without loss of generality, we may assume that $|I| = |J| = N$ for some infinite $N$. Let $a$ and $b$ denote the left endpoints of $I$ and $J$ respectively and set $C := (A^* \cap I) - a$ and $D := (B^* \cap J) - b$. Note then that $C$ and $D$ are definable subsets of $[0, N)$ of positive Loeb measure. Without loss of generality, we may assume that $C$ and $D$ belong to the first half of $[0, N)$. Then by Jin's theorem mentioned above, $\pi_{\mathbb{N}}(C + D)$ contains an interval in $C_{\mathbb{N}}$. Translating back by $a + b$, one finds an infinite hyperfinite interval on which $A^* + B^*$ has only finite gaps, whence, by the nonstandard characterization described earlier, we see that $A + B$ is piecewise syndetic.

## The Ultraproduct Construction

Of course, the lingering question remains: given a structure $\mathbf{S}$, is there a structure $\mathbf{S}^*$ satisfying (NA1)-(NA3)? Model theorists know these axioms to be consistent by basic model-theoretic facts. However, in this section, we present a construction that is much more "mainstream" and easy to describe to nonlogicians.

**Obtaining $\mathbf{R}^*$ as an ultrapower.** As in the passage from any number system to an extension where we are trying to add desired elements (e.g., the passage from $\mathbf{N}$ to $\mathbf{Z}$ to $\mathbf{Q}$ to $\mathbf{R}$ to $\mathbf{C}$), we simply formally add the new desired elements and then see what technicalities we need to introduce to make this formal passage precise. In this case, in passing from $\mathbf{R}$ to $\mathbf{R}^*$, we are trying to add infinite elements. Following the Cauchy sequence construction of $\mathbf{R}$ from $\mathbf{Q}$, we can simply add the sequence $1, 2, 3, \ldots$ to $\mathbf{R}$ and view this sequence as an infinite element of $\mathbf{R}^*$. Of course, just as in the case of the passage from $\mathbf{Q}$ to $\mathbf{R}$, many sequences should represent the same element of $\mathbf{R}^*$, e.g., the sequence $-32, \pi, 46, 4, 5, 6 \ldots$ should represent the same sequence as $1, 2, 3, \ldots$. In general, we should identify two sequences if they agree on a *big* number of indices, where two sequences $(x_n)$ and $(y_n)$ agree on a big number of terms if the set of $n$ for which $x_n = y_n$ is a *large* subset of $\mathbf{N}$. Admittedly, the words "big" and "large" are rather vague here, so we need to isolate some properties that large subsets of $\mathbf{N}$ should have; the resulting notion is that of a *filter* on $\mathbf{N}$. Since the definition makes perfect sense for an arbitrary index set $\mathcal{I}$, we do so:

**Definition 11.** Suppose that $\mathcal{I}$ is a set. A (proper) *filter* on $\mathcal{I}$ is a collection $\mathcal{F}$ of subsets of $\mathcal{I}$ satisfying:

- $\emptyset \notin \mathcal{F}, \mathcal{I} \in \mathcal{F}$;
- if $A \in \mathcal{F}$ and $A \subseteq B \subseteq \mathcal{I}$, then $B \in \mathcal{F}$;
- if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

Denoting the set of $\mathcal{I}$-sequences by $\mathbf{R}^{\mathcal{I}}$, the above definition of a filter was engineered so that the relation $\sim_{\mathcal{F}}$ on $\mathbf{R}^{\mathcal{I}}$ defined by $x \sim_{\mathcal{F}} y$ if and only if $\{i \in \mathcal{I} : x_i = y_i\} \in \mathcal{F}$ is an equivalence relation. We denote the equivalence class of an $\mathcal{I}$-sequence $x$ from $\mathbf{R}^I$ by $[x]$, and the set of all such equivalence classes by $\mathbf{R}^{\mathcal{F}}$. By identifying an element $r \in \mathbf{R}$ with the constant sequence $c_r = (r, r, r, \ldots)$, we get a natural inclusion of $\mathbf{R}$ into $\mathbf{R}^{\mathcal{I}}$, and by passing to the equivalence class we obtain the natural inclusion of $\mathbf{R}$ into $\mathbf{R}^{\mathcal{F}}$. In a diagram:

$$\mathbf{R} \xrightarrow{r \mapsto c_r} \mathbf{R}^{\mathcal{I}} \xrightarrow{x \mapsto [x]} \mathbf{R}^{\mathcal{F}}$$

Note also that $\mathbf{R}^{\mathcal{F}}$ has a natural field structure on it extending the field structure on $\mathbf{R}$.

There are now two problems with leaving things at this level of generality. The first can be motivated by the desire to turn $\mathbf{R}^{\mathcal{F}}$ into an *ordered* field. The natural thing to do would be to declare $[x] < [y]$ if and only if $\{i \in \mathcal{I} : x_i < y_i\} \in \mathcal{F}$. However, it is entirely possible that, under the above definition, we may have $\sim_{\mathcal{F}}$-inequivalent sequences $x$ and $y$ for which $[x] \not< [y]$ and $[y] \not< [x]$. We can remedy this by adding one further requirement:

**Definition 12.** A filter $\mathcal{U}$ on $\mathcal{I}$ is called an *ultrafilter* if, for every $A \subseteq \mathcal{I}$, we have that either $A \in \mathcal{U}$ or $\mathcal{I} \backslash A \in \mathcal{U}$ (but not both).

Here is the other problem: suppose that $\mathcal{U}_{37}$ is the collection of subsets of $\mathbf{N}$ defined by declaring $A \in \mathcal{U}_{37}$ if and only if $37 \in A$. Although this hardly matches the intuition of gathering *large* subsets of $\mathbf{N}$, it is easy to see that $\mathcal{U}_{37}$ is in fact an ultrafilter on $\mathbf{N}$, called the *principal ultrafilter on* $\mathbf{N}$ *generated by* 37. Now notice that in $\mathbf{R}^{\mathcal{U}_{37}}$, every sequence is equivalent to the real number given by its 37th entry. In other words, the infinite element $1, 2, 3, \dots$ that we tried to add is not infinite at all, but rather is identified with the very finite number 37. To avoid this triviality, we add the following requirement:

**Definition 13.** If $\mathcal{I}$ is an index set and $i \in I$, we call $\mathcal{U}_i := \{A \subseteq I : i \in A\}$ the *principal ultrafilter on $I$ generated by $i$*. An ultrafilter $\mathcal{U}$ on $I$ is called *principal* if it is of the form $\mathcal{U}_i$ for some $i \in I$ and is otherwise called *nonprincipal*.

We may now summarize: for any nonprincipal ultrafilter $\mathcal{U}$ on any index set $\mathcal{I}$, $\mathbf{R}^{\mathcal{U}}$ is a proper ordered field extension of $\mathbf{R}$. Now, the ultrafilter on a set $I$ may also be viewed as a finitely additive $\{0, 1\}$-valued measure on $\mathcal{I}$, which, moreover, gives points measure 0 precisely when the ultrafilter is nonprincipal. In this way, the ultrapower construction of $\mathbf{R}$ bears much resemblance to the practice common in measure theory of identifying measurable functions if they agree almost everywhere.

**Ultrapowers of structures.** The approach of the previous subsection works to obtain, given an initial structure $\mathbf{S}$, structures $\mathbf{S}^*$ satisfying (NA1) and (NA2). Indeed, given any basic set $S_i$, we can set $S_i^* := S_i^{\mathcal{U}}$ and given any basic relation $R_j$, we can set $R_j^* := R_j^{\mathcal{U}}$ and $\mathbf{S}^* = \mathbf{S}^{\mathcal{U}}$. In this case, $\mathbf{S}^*$ is called the *ultrapower* rather than an ultraproduct. As before, we have the natural inclusions, and it is clear that $\mathbf{S}^*$ satisfies (NA1). The fact that (NA2) holds in this context is an instance of a result in model theory known as *Łos' theorem*, which is easily proven by induction on the "complexity" of definable sets. Further, that this construction works for any initial structure $\mathbf{S}$ is the fact that one invokes to show that certain axiomatic theories of nonstandard methods are conservative over certain axiomatic theories of real numbers: i.e., anything that the former theory proves the latter theory also proves (cf. [5]).

**Getting richer.** It turns out that obtaining countable richness via ultrapowers is quite straightforward:

**Theorem 14** (Keisler [15]). *Suppose that $\mathcal{U}$ is a nonprincipal ultrafilter on a countably infinite index set $\mathcal{I}$. Then for any structure $\mathbf{S}$, $\mathbf{S}^{\mathcal{U}}$ is countably rich.*

What about higher richness? If one insists on only using ultraproducts as a means of producing nonstandard extensions, then one can indeed obtain nonstandard universes with higher richness properties at the expense of dealing with some messy infinite combinatorics. Indeed, Keisler isolated a combinatorial property of ultrafilters, called *goodness*, and proved the following theorem:

**Theorem 15** (Keisler [16], [17]). *Let $\mathcal{U}$ be an ultrafilter on a set $\mathcal{I}$. Then $\mathcal{U}$ is good if and only if: for every structure $\mathbf{S}$, $\mathbf{S}^{\mathcal{U}}$ is* maximally rich.

Here, we are using the admittedly vague term *maximally rich* to mean that the structure is as rich as the cardinality of its underlying domain allows. In order for this theorem to be useful, one needs to know that good ultrafilters exist. This is indeed the case: Keisler first proved, under the assumption of the Generalized Continuum Hypothesis, that any infinite set possesses a good ultrafilter; later, Kunen proved this fact without any extra set-theoretic assumption.

## Other Approaches

Ever since its inception, a slew of different frameworks for approaching nonstandard analysis have been presented. Many of these approaches can be viewed as attempts to axiomatize different aspects of the ultraproduct construction. In this section, we briefly describe the differences between these approaches and our preferred approach using (NA1)-(NA3).

One alternative approach is just to axiomatize the behavior of the embedding of the original structure in its nonstandard extension. Depending on how formal one sought to be, one might then replace informal descriptions of the original structure by some axiomatic characterization of it. However, as the ultraproduct construction itself shows, first-order axioms cannot describe infinite structures up to isomorphism. Hence, if one sought first-order axiomatic characterizations, one might seek to add on axioms suggestive of the way in which the original structure was more canonical than its nonstandard extension. For instance, in the case of reals, one might add on first-order axioms to the effect that any bounded subset of $\mathbf{R}$ first-order definable by recourse to $\mathbf{R}^*$ had a least-upper bound (e.g., Nelson's [19, 1166] principle of standardization). Similar expressive difficulties emerge when one tries to find first-order axiomatic renditions of the richness conditions (e.g., Nelson's [19, 1166] principle of idealization, and Hrbacek's principle of bounded idealization). Since (NA1)-(NA3) makes no pretense to be a first-order axiomatization, it can avoid these niceties and just be content with our informal understanding of the real numbers and the richness conditions. In this, (NA1)-(NA3) is similar to the superstructure approach of Robinson–Zakon (cf. the Chang–Keisler model theory book), which additionally builds in the Mostowski collapse mentioned in "Power Sets and Internal Sets" and adopts a formulation of richness which does not require the notion of definable set.

A particularly influential axiomatic approach was that of Nelson's internal set theory ([19]). Nelson's aim was to create an overall set theory for nonstandard methods. In addition to the usual set-theoretic axioms, it also contained the aforementioned principles of standardization and idealization. Nelson's approach seems natural given that much of mathematics can be replicated in set theory. However, as we have sought to stress in the applications in "Selected Classical and Recent Applications," many of the most exciting applications of nonstandard analysis are to local areas of mathematics. Hence, as far as applications go, little seems to be gained by adopting the very global perspective of set theory.

Another recent approach pioneered by Benci and di Nasso [1] is to axiomatize the inclusion of the set of $\mathbf{R}$-valued sequences into the ultraproduct $\mathbf{R}^*$. Their theory is called *$\alpha$-theory* because they axiomatize the map $x \mapsto [x]$ sending a *sequence* to its equivalence class (cf. the diagram after Definition 11). They use the notation $x \mapsto x[\alpha]$, since this notation reminds one of field extensions. For instance, it follows from the ultraproduct construction that the operation sending $x$ to $x[\alpha]$ commutes with addition and multiplication:

$$x[\alpha]+y[\alpha] = (x+y)[\alpha], \qquad x[\alpha]\cdot y[\alpha] = (x\cdot y)[\alpha]$$

The idea of $\alpha$-theory is to take these identities—and others pertaining to sets of reals—as axioms, and to derive transfer from these. The choice between $\alpha$-theory and (NA1)-(NA3) is similar then to the choice between Dedekind cuts and Cauchy sequences: they are both equally good descriptions of their subject matter, but they just differ in which basic properties are derived and which are taken as primitive. One reason traditionally given for preferring Cauchy sequences over Dedekind cuts is that it more readily generalizes to other situations, e.g., complete metric spaces. Much the same is true of (NA1)-(NA3): since we can take extensions of any structure, the approach works just the same for, e.g., nonstandard $p$-adic analysis as for nonstandard real-analysis. By contrast, with $\alpha$-theory, in each case one has to isolate some basic axioms that suffice for the derivation of the full transfer principle, and in each case one has to reprove the full transfer principle.

Another recent approach due to Benci and di Nasso [2] seeks to characterize the nonstandard extensions of the reals as images of certain rings (they also have similar results for certain classes of spaces). While they state their ring-theoretic results for the reals, their proof generalizes as follows (recall the notion of complete structure given immediately after Definition 6):

**Theorem 16.** *Suppose that $\mathbf{K}$ is the complete structure of an uncountable ordered field. Then $\mathbf{K}^*$ satisfies (NA1)-(NA2) if and only if $\mathbf{K}^*$ is the image under a ring homomorphism of a composable ring over $\mathbf{K}$.*

In this, the ring is said to be *composable over* $\mathbf{K}$ if it is a subring of the ring $\mathbf{K}^{\mathcal{I}}$ of all functions from some index set $\mathcal{I}$ to the original field $\mathbf{K}$ which is closed under taking compositions with functions from $\mathbf{K}$ to $\mathbf{K}$ (and which contains all the constant functions). However, this of course is not to say that studying composable rings is always the best way to study nonstandard methods. After all, any group is the image of a free group, but it would be a mistake to approach all problems in group theory through the lens of free groups.

A unifying motivation behind these recent approaches is the desire for a more mathematically natural framework for nonstandard analysis. By contrast, Robinson himself was an instrumentalist about the methods he developed (cf. [5]): He thought that their worth was tied less to any notions of naturalness and more to their proven track record in obtaining results about standard structures, such as we have surveyed in "Selected Classical and Recent Applications." As we have sought to emphasize throughout, nonstandard methods such as (NA1)-(NA3) are easy to state and use, and their consistency is easily verifiable via the ultraproduct construction. In describing the ultraproduct construction we mentioned the analogy with the Lebesgue integral. Here is another respect in which they are similar: to use the Lebesgue integral correctly, one does not need to keep its measure-theoretic construction constantly in view, but rather one can just work with characteristic properties of it, like the Dominated Convergence Theorem. Likewise, to use nonstandard methods correctly, one does not need to keep the ultraproduct construction constantly in mind. Rather, as we sought to illustrate with Jin's Theorem in "The Axioms in Action: Jin's Theorem" one can do nonstandard analysis just using the three simple axioms (NA1)-(NA3).

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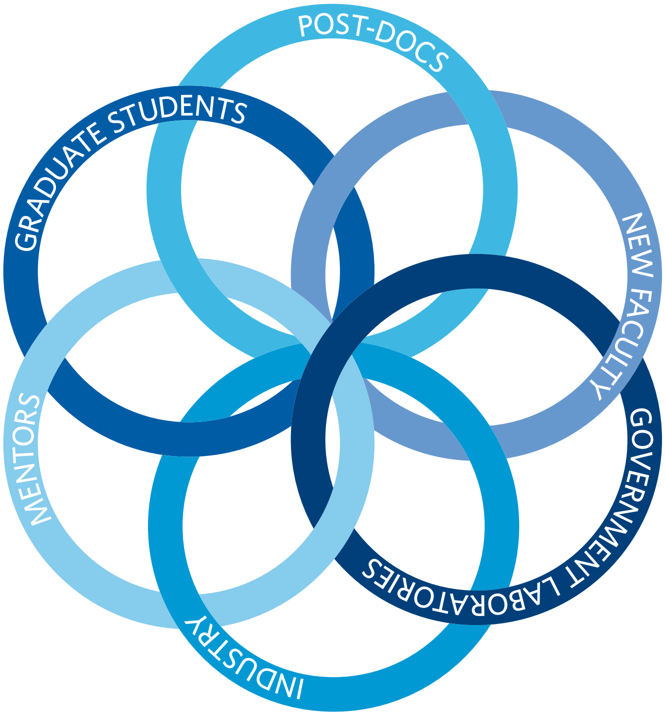
**The Early Career Section** is a new community project, featured here in the *Notices*. This compilation of articles will provide information and suggestions for graduate students, job seekers, early career academics of all types, and those who mentor them. Angela Gibney serves as the editor of this section. This month's theme is Getting Ready for the Academic Job Market. Next month's theme will be Teaching.




# Cultivating an Online Presence for the Academic Job Market

*Holly Krieger*

When you're on the market, time is your most valuable commodity. Maybe you're rushing to finish and write a nice result before your applications go out. Perhaps you're busy expanding your knowledge of the literature and thinking deeply about where you might take your work in the next five years to make your research statement as compelling as possible. There are so many time-consuming ways to invest your valuable time in improving your application that before going any further, I should explain why cultivating an online presence is worth the effort. Some mathematicians deride the notion that such a presence is necessary when on the market; however, these are often the same people who strongly encourage their students and postdocs to build reputation by "getting themselves out there" at conferences and via email.

An easily discoverable and informative online presence is the modern supplement to conference introductions and mathematical correspondence. Hiring committees are not passive entities that sit back and wait for the best candidates to approach them. Well before any deadlines arrive, some departments with open positions are seeking out and contacting qualified candidates who are most likely to suit their subject-area needs. The names that arise this way are often people unfamiliar to most or all members of a hiring committee, and the first step taken is to gather more information about these potential applicants by searching online. By curating an online presence, you lead them directly to your research and answer basic questions about your field of study, whether you're on the market, your inclination toward teaching or a research focus, and so on. Making this information easily available can lead to your application being solicited by universities you never

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dreamed might have interest in you, and by departments you didn't realize would be an excellent fit.

Solicited application or not, online content can give you an edge in presenting the strongest aspects of your file. In application files, a candidate's features and accomplishments are all presented equally dryly, and in no particular order. On the other hand, your website, blog, or social media can (and should) draw attention to your most outstanding features. Are you particularly proud of a recent paper? Include a summary for non-experts on your webpage that communicates the importance of the work. Are you an excellent speaker or teacher? Embed a video of your talks or lectures on your page, start a YouTube channel, or promote your videos or course development on social media. By creating content, you control and direct the searcher's attention to your best attributes.

**The bare minimum.** One effort you *must* make is to ensure that your online presence doesn't work against you. Not having a professional website is unusual (and frustrating) enough that it counts as a negative against you! Even if it is bare-bones, have a departmental website. It's also critical that your pages are up-to-date. Both a nonexistent and a very old site not only fail to provide useful information to searchers but also communicate that you are probably not on the job market (and possibly no longer in academia).

It is in your best interest to communicate only professional information to potential employers. If you have any social media, blogs, or websites that contain offensive, embarrassing, or controversial opinions or images, or are purely personal, you are better off making them only privately accessible. This is a matter of directing your audience's attention beneficially rather than censoring yourself. If making such content private offends your sensibilities, or it is externally hosted and cannot be made private, at least be sure that you do not link to it on any professional pages.

**Your professional webpage.** Once you've mitigated any negative online presence, the most fundamental tool to develop is your professional webpage. Your webpage should at a minimum provide your contact details and field of study, your CV, and information on or links to your research and teaching experience. If you come to a department's attention via word of mouth, there may not be anyone on the faculty who knows whether you're on the market or not. A seemingly uncommon but vastly useful line to add to the top of your webpage is: "I am on the job market in [year]." A photo on your site can make it, and your application, more memorable. A photo related to professional activities is a better choice than one from a night out drinking ten years ago—and yes, that latter happens more frequently than you might think.

Once the basics are in place, give some thought to how your webpage might emphasize your strengths; if you don't know what the strengths of your application are, ask someone! This information will tell you how to order and organize your website to your advantage, drawing attention to your successes, and including content (such as videos or paper summaries) that highlight your abilities.

**Actively engaging.** A basic professional webpage is a good starting point, but a more active online engagement can be worth the effort. Math communication and other online outreach are increasingly valued by universities, and can connect not just the general public but also other mathematicians (who might be involved in hiring) to your work. Maintaining a mathematical blog, Twitter, or YouTube account increases both your professional network and your online professional content. This content showcases your communication, outreach, and teaching abilities as no paper application can.

**Now go and do it!** In the academic job search, your goal is not only to be an excellent candidate, but also to communicate that fact as widely as possible. By cultivating an online presence, you easily and effectively disseminate the most impressive features of your application, and open lines of communication that can result in unexpected opportunities. You've worked for years building the professional experience to prepare you for this job search—now go and make sure everyone knows it.

Holly Krieger

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Photo of Holly Krieger is courtesy of the author.

# Interested in Applying to a Liberal Arts Institution?: Perspectives from Reva Kasman, Julie Rana, and Chad Topaz

*Linda Chen*

What follows was extracted from conversations with Reva Kasman, Julie Rana, and Chad Topaz about applying to faculty jobs at liberal arts institutions.

## General Comments:

**RK:** There is a wide spectrum of schools in terms of academic standards, student demographics, class size, expectations for faculty teaching load and research/scholarship, even within the broad category of "liberal arts colleges" or liberal arts-focused comprehensive universities. Applicants should have an open mind when looking at the different options for jobs and consider what will genuinely be a good fit, both professionally and personally.

**JR:** The most important thing you can do is to educate yourself about the types of jobs available: public/private, religiously-affiliated/secular, liberal arts/technical/R1, undergraduate-only vs. masters granting vs. PhD granting. Talk to people at these different types of institutions. Make sure you understand what liberal arts colleges actually are and why you want to go in that direction.

**RK:** Some liberal arts schools can have heavy teaching loads (3–4 courses in a semester), but some will also have very high research standards. So if your goal is a tenure-track job at a school with significant expectations for research output, consider whether you will be better positioned for such a job by doing a research postdoc first. (On the flip side, a research postdoc is not essential for every liberal arts tenure-track position, and having done one will not automatically rank a candidate higher than a well-qualified applicant coming right out of a PhD.)

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**CT:** At liberal arts institutions, teaching is nearly always the top institutional priority, and this priority may create opportunities to work closely with students and make a significant impact on their educational experiences. Additionally, some liberal arts school careers might involve a good deal of formal and/or informal interaction with faculty from other departments. If working closely with students appeals to you, and if you are intrigued by thinking about how mathematics complements and connects to other disciplines, a liberal arts career could be a rewarding one for you.

**Q:** *What is the most common mistake you see in applications?*

**CT:** Generic cover letters. Cover letters addressed to the wrong college. Egregious typos or, even worse, misspelled names of people at my institution. Most of all, no attempt to address the liberal arts.

**JR:** Anyone who doesn't tailor their application has little chance of making it into our top twenty.

**RK:** I don't expect everything to be customized, but if someone mischaracterizes my school or our students, that is going to reduce the chance that I perceive them to be a good fit.

**Q:** *What makes a strong cover letter?*

**RK:** The cover letter is where you make a first impression, establish connections with a particular school, and convince a committee that this is a conversation they want to continue. If there are specific criteria mentioned in a job ad, be sure to address them, and if you don't fit the requirements in an obvious way, use your letter to make a compelling case for why you still believe that you're a good fit. Be clear that you value the kind of school that you are addressing—the category of "liberal arts institution" covers a wide range of colleges and student demographics, and you should demonstrate an awareness and appreciation for the one mentioned in the specific letter. If there is something unique in your experiences, highlight it here so that it won't get overlooked in a quick skim of your CV.

**JR:** I like to see that you've taken the time to get to know our school. The best cover letters I've seen also show evidence that you're someone who takes initiative. Tell us something you've done with your teaching/research/outreach/service that goes above and beyond, and talk about how that fits into your future with us.

**CT:** The most effective letters I've read are the ones that followed directions, sounded genuinely enthusiastic, and were specific about my liberal arts institution and the reasons that the applicant wanted to come be a faculty member there.

**Q:** *What can I do one to two years before going on the job market to be better positioned at application time?*

**CT:** Search committees will evaluate you based on your teaching effectiveness, your interest in and/or commitment to the liberal arts, and, possibly, your research contributions and your ability to advise student research. Make sure you have a portfolio of evidence along these axes. For teaching, the strongest applicants will have experience as instructor of record for at least one or two courses. If you are a graduate student and the opportunity to serve as instructor of record isn't available to you, make sure you serve as a TA frequently so that you build the group of courses with which you are familiar. Search committees are often impressed by candidates with well-rounded course portfolios that demonstrate experience with courses for non-majors, introductory courses for the major, upper-division core courses, and courses on specialized topics. Educate yourself about how to be a successful teacher and document your success by saving your teaching evaluations and by asking long-term faculty in the department and/or the director of your campus's teaching center to observe you so that they can write specific things about you at application time. To demonstrate interest in the liberal arts, take every opportunity to network with faculty at liberal arts colleges and volunteer to come visit them to give a talk and/or do outreach. To amass a convincing research portfolio, try to get a paper or two into print before you apply. Give as many conference talks as you can. Network with researchers outside of your institution who then might be able to write detailed and convincing letters about you at application time. Finally, seek the opportunity to supervise student research as part of a summer program, independent study, or thesis experience.

**RK:** While every liberal arts school is different, all of them will be looking to hire someone who demonstrates a commitment to high quality undergraduate teaching. Consider whether your teaching experience might seem sparse or lack diversity. If you are a graduate student who has had primarily research-based funding, it may be worth requesting a semester of teaching to add to your CV. If your teaching has been fairly limited in scope (e.g., only running discussion sections if you are a TA or only having taught calculus), ask your department if there are opportunities to teach a course or work with a student population that will broaden your experiences. Expand your knowledge of the range of schools that are out there so that you have a better sense of what jobs might be a good fit, and then tailor your application materials appropriately. You might talk to people at smaller conferences about their jobs or read journals like *MAA FOCUS* or *PRIMUS* to increase your understanding of what is innovative in undergraduate teaching, and the kinds of scholarship and service that liberal arts faculty often do beyond traditional math research. Finally, start your job application materials early in the summer before you go on the market and get more than one person's feedback, ideally from someone at the type of school where you think you might like to work.

**JR:** Other ways to show interest in working with undergraduates: help a professor at your institution advise a summer undergraduate student (whether as part of an REU or not), get involved with your university's Putnam exam group or math club, organize outreach/enrichment activities, or help mentor an undergraduate's senior project. I would also encourage you to experiment with your teaching, and get feedback from your university's center for teaching and learning. You might also attend some sessions at MAA MathFest or the JMM specifically about teaching, and think critically about how you can incorporate some of these ideas into your own classes.

**Q:** *What do you look for in a teaching statement?*

**JR:** Above all, I'd rather see an anecdote than a philosophical description. More specifically, I look for evidence that you're truly interested in experimenting with your teaching. It's also important to me and to my institution that you've thought about and made progress toward managing diversity in the classroom. Be honest and thoughtful about what you do well, moments of failure and what you learned from them, and what you'd like to explore in the future.

**CT:** I look for authenticity, enthusiasm, organization, and pedagogy. I have hundreds of statements to read, and the best ones make clear for me what the main messages are by providing organization and by highlighting key points. The candidates who impress me the most are the ones who educate themselves about effective practices and put these practices into use.

**RK:** Avoid using trendy pedagogical "buzzwords" for their own sake, but if you genuinely subscribe to a particular philosophy or technique, then make sure to show how it tangibly appears in your classroom teaching. Your teaching statement should emphasize student learning of mathematics (which may include content knowledge acquisition, problem solving, critical thinking, applications, etc.) and how you strive to foster these skills effectively. Ideally, a teaching statement demonstrates an awareness that classrooms include students with a diversity of mathematical knowledge, interest levels, and career goals, and addresses the ways that you endeavor to create successful students across this spectrum. Remember that a faculty position at a liberal arts college can involve teaching up to four courses at a time, so consider whether the highlighted characteristics of your teaching style are going to be replicable—for instance, talking about how you spend hours with individually struggling students may seem admirable, but can convey a lack of realism about what your job will entail and what students need inside the classroom. Finally, while I expect that a teaching statement will focus on a candidate's strengths, I look for evidence that someone is reflective about what is challenging in teaching mathematics, and where they perceive themselves to have room for future growth.

**Q:** *What do you look for in a research statement?*

**CT:** The weakest research statements are ones where applicants say "this is the problem I work on" but give me no sense of why or of how it fits in to a bigger picture. The very best ones provide context for how the research fits into mathematics (or other fields) more broadly and also leave me with a fairly concrete sense of the candidate's plan. No matter how technical your research is, I will look for at least a part of the research statement that explains the work in a way that I (and even better, someone outside my department) can understand. I love when a candidate discusses research with students in the research statement, and I love when a candidate mentions potential connections to other areas of the liberal arts.

**RK:** It is fine if the main body of the statement is technical and beyond the scope of my knowledge, but I want to see that a candidate has made an effort, at least in the early paragraphs, to value me as a reader and introduce their research area in an accessible, big picture way. This introductory section should avoid notation whenever possible, and if there are necessary technical terms, then they should be defined casually at this point. Be transparent about your own contribution to the field, and include your plans for continued research. Finally, I expect that research statements for a liberal arts school be about two to three pages. If a longer, more detailed one is required for research postdocs or similar jobs, then a candidate should write two versions and submit each to the appropriate places.

**JR:** I look primarily for evidence that you can continue your research in a teaching-intensive, undergraduate-only environment. I also look for evidence that you're starting to think beyond your thesis. If you can include well-formulated ideas for projects with undergraduates, that's a huge bonus.

**Q:** *What are some characteristics of a good teaching recommendation?*

**RK:** The best recommendations are likely to come from someone with whom you have an established and ongoing relationship about your teaching. This can be a formal supervisor or simply a faculty member who has both seen you teach and had conversations with you about your course planning, assessment, and things that challenged you. Such a person has the material to write a well-rounded letter about who you are as a teacher, rather than just giving a quick summary of a single class observation or student evaluation data. If you don't have a long-standing relationship with a teaching mentor or colleague, it is still worth being proactive about obtaining an informative letter: invite a faculty member to come to your current class at least once and have follow-up discussions with that person. Remember that outside of your institution, no one will know if there is a person who typically writes everyone's teaching letters (e.g. a director of undergraduate studies), so it is more important to get an insightful and personalized letter from an educator you trust than to get a formulaic letter from someone designated to this role.

**JR:** In a nutshell, I look for any evidence that you're a productive colleague who's thoughtful about your teaching. The most helpful teaching recommendations make it clear that the letter writer has observed many different instructors and has observed you multiple times over one or two courses. This makes any comparison with your peers more convincing. I also appreciate reading recommendations from letter writers who are aware of modern teaching practices.

**CT:** They are specific and enthusiastic. For graduate students, letters from a professor or administrator who has supervised a course that say little more than "so-and-so was a good teaching assistant" convey little information and might even be read as lukewarm. As with most types of writing, specificity really matters. I get the most information from letter writers who have observed a candidate teaching and spoken to them about their pedagogical goals. From these letters, I learn how the candidate structures their course or discussion section, what they do in and out of the classroom, and how students respond to them. You, as a candidate, can encourage your letter writer to include this type of information. "Here's my assessment scheme for this course, and here's how it aligns with effective practice," you could say. "It would be so helpful if you could mention this in the letter you write for me."

**Q:** *What advice do you have for candidates when discussing diversity, either in a separate diversity statement or in other materials?*

**CT:** A strong diversity statement might include one or more of the following components: a discussion of why equity, diversity, and inclusion (EDI) issues are important; disclosure of your own identities along various axes of diversity; presentation of any formal knowledge you have about EDI; examples of EDI issues at play in teaching you've done; descriptions of professional activities related to EDI; and other relevant personal or professional thoughts and experiences. Whether you choose from among these components or include others, a committee will want to see some thoughtful discussion of and genuine interest in EDI. An excellent diversity statement can really make a candidate stand out.

**RK:** Issues related to diversity are going to be present in your job. Your application should demonstrate an awareness of equity issues that may be present in your classroom (or broader school environment) and indicate that you take seriously your responsibility for creating an inclusive and supportive learning environment for students coming from vastly different circumstances. If you have personal experience that impacts your professional perspective, enhances your ability to mentor and support students, or that you hope to use in outreach efforts, then you are most welcome to include it, but you are under no obligation to disclose characteristics of your identity for the sake of making an impact in a diversity statement. Moreover, it is important not to stereotype students by their racial, socioeconomic, or other characteristics, to present yourself as "saving" certain groups of students from their circumstances, or to draw what may be uninformed parallels between your own experiences and those of others. As with the teaching statement, be genuine and concrete in the views and examples you share in a diversity statement, avoid throwing in keywords that you think a committee wants to hear, and be reflective about your own potential for growth in this area.

**JR:** I like to see that you've both read and thought about issues of equity and inclusion, especially in the context of teaching. For example, check out the fabulous AMS inclusion/exclusion blog, `https://blogs.ams.org/inclusionexclusion/`. If you are considering including information about your personal background, think carefully about how your statement puts your experience into the context of these larger issues. If you can't address this, you may not want to include this information.

**Q:** *Besides a generic cover letter, what are other common mistakes you see?*

**RK:** I am rarely impressed by quotes from student evaluations or dramatically positive student evaluation data—hearing that someone was called "the best teacher ever" is not helpful in assessing potential as a full-time faculty member, and can sometimes be off-putting. If student evaluation quotes are included, then they should highlight something specific from a candidate's experience that enhanced student learning.

Applicants can inadvertently minimize a school's students and the courses they will teach. There is a tendency for candidates to sound like they are most excited about teaching upper-level major courses and that they put the greatest value on students who love mathematics and/or are heading to graduate school. They may talk about "lower-level classes" and mean Calculus II, or imply that struggling students are "problems" who can be fixed individually in office hours. At many schools, "lower-level" means a variety of pre-calculus, service, or remedial courses, and most of your teaching audience may be non-majors. Many mathematics majors will not be graduate school bound. Don't accidentally shoot yourself in the foot by diminishing the majority of students at a school or those that won't follow the same educational path as you did.

**CT:** In the teaching statement, do not vehemently express ideas that are incorrect. For instance, I have seen candidates write sentences such as, "Students learn best when material is explained clearly and multiple example problems are presented." Says who!? There exists a wide variety of effective pedagogical models (though they all share some common general principles grounded in learning science). The candidate's statement makes me fear that they are not open to examining and refining their teaching based on theory and evidence.

**JR:** We've had a few applicants from the same school apply for the same job. This is bound to happen, but it can mean that two applications look nearly identical, especially if everyone is helping out with the same enrichment events, attending the same conferences, teaching the same courses, etc. To make yourself stand out, take initiative to organize something on your own, and point this out in your cover letter.

**Q:** *The focus of this article is not the interview process, but do you have recommendations for resources for how to prepare for interviews, or for job advice more generally?*

**JR:** I used the interview prep materials on `theprofessorisin.com`. But whichever resources you use, make sure you practice, practice, practice! Be sure you read up a lot on the institution you're visiting, and ask a lot of questions. Be honest while you're visiting. Ask the questions you really want to know the answers to!

**CT:** One resource is people who have served on search committees at liberal arts colleges. If you have access to any such person, ask them if they'd be willing to do a 20-minute mock interview with you and give you feedback on it.

**RK:** Do practice interviews, ideally with someone who is at the type of school where you hope to work. Ask someone at a conference if they will do a mock interview with you on Skype, for example. Look at the kinds of questions you might get asked (there are good resources online) and practice giving your answers out loud, even when you're alone. Have good examples on hand to answer questions like "what was something you did well in a class" or "what was something you'd like to do differently" so that you won't go blank in an interview, or worse, come up with a spontaneous anecdote that reflects poorly on you or your teaching. Have questions ready to ask of the school—something simple like "Tell me what you like most about working in your department" can provide deep information about a potential job.

For example, some practice questions can be found at `https://blogs.ams.org/onthemarket/2013/02/04/preparing-for-an-interview-questions-by-sarah-ann-stewart-fleming-belmont-university/`.

**LC:** The November issue of the *Notices* will include advice on preparing for the Employment Center and Joint Meetings interviews for tenure track jobs.

## Other Comments:

**RK:** As you put together your application materials, keep the perspective that every item in some way showcases who you are as a teacher, communicator, and future professional colleague. Let that influence the tone and style you use to present information and engage with your audience in each document, from your cover letter through to your CV.

Remember that your job search and your goals—both professional and personal—are your own. Don't let anyone else try to define them for you. Ask many people the same questions as you go through this process. Know that you are going to get well-meaning advice from some people that is not always about you or reflective of your needs and wants, and you are free to ignore anything that doesn't feel right.

**CT:** If your goal is to have a fulfilling career, you should think deeply about what matters to you in life, and hold those values up against every single job you apply for—academic or otherwise—to see if your priorities align with those of the job.

**JR:** I'll reiterate Chad and Reva's comments: Educate yourself about the types of jobs out there, and be honest with yourself about what you want in a career. If you do that and it shows through in your application, you're on the right track!

Linda Chen

Reva Kasman

Julie Rana

Chad Topaz

### Credits

Photos of Linda Chen, Reva Kasman, Julie Rana, and Chad Topaz are courtesy of the interviewer and the interviewees.

# Applying for Grants: Why and How?

*John Etnyre*

Applying for grants is an important part of an academic research mathematician's career. There are myriad reasons why, from the obvious—directly supporting your research and pleasing your employer (or future employer) if you get the grant—to the less obvious—refocusing yourself on where your research fits into the greater mathematical world and why you are doing it, even if you do not get the grant.

Before moving on to the actual application process, let's flesh out a bit more "why you should apply." Grants will typically support some or all of the following: your travel to conferences and workshops; your ability to bring collaborators to you; a summer salary that allows you to focus on research; the undergraduate and graduate students working with you; computer and other supplies; and, in some cases, the postdocs working with you. All of these things are of course highly beneficial to your research program. In addition, university administrators are interested in faculty obtaining grant support, so demonstrating that you can do this can be advantageous to a job search or to building a case for promotion. This alone is ample reason to apply for grants. But even if you do not get a grant when you first try, the application process can be very favorable to your research. Specifically, research problems can frequently be highly specialized, and while working on them it can be easy to lose sight of why one is working on the problem. Without some reflection on what you are working on, it can be easy to head down a rabbit hole that no one really cares about. Writing a grant makes you step back and think about the big picture. Why are the problems I am working on interesting to the mathematical, or broader, community? What should I be trying to work on over the next few years? While trying to answer these questions, you can frequently come up with whole new interesting lines of research, in addition to having a better and deeper appreciation for problems on which you are already working.

There are numerous organizations that fund various types of research mathematics. Consider applying for several grants; even small grants, like grants to support travel, can have a big impact on your career, and they can be a stepping stone to larger grants. [The AMS website has a convenient list of many of these opportunities at `https://www.ams.org/opportunities`, and the *Notices* prints a Mathematical Opportunities section in every issue, where Early Career Opportunities feature prominently.] The most common funding for most mathematicians comes from either the National Science Foundation (NSF) or, more recently, the Simons Foundation. The Simons Foundation offers various types of grants, details of which can be found at `https://www.simonsfoundation.org/funding-opportunities`. The NSF also has various opportunities from research and CAREER grants, to workforce and FRG grants; see `https://www.nsf.gov/div/index.jsp?div=DMS`.

The rest of this article will primarily focus on NSF research grants, though much of the discussion will apply to other grants as well, especially other NSF grants.

First and foremost, every grant will have a grant solicitation that describes all the particular issues about the grant, such as who may apply, what funding may be requested, criteria for reviewing the grant, and other essential information. Read and carefully follow the grant solicitation when preparing your grant proposal. Secondly, for whatever grant you are applying for, ask colleagues if they can share examples of successful grants of that type. Reading as many such examples as possible is a great way to get a sense of the structure of a successful grant proposal.

As with all writing, one needs to keep the audience in mind. Typically, your NSF grant proposal will be evaluated with forty or more other grants in a similar area by a panel of ten or more experts. The panel will put them into some rough order, and then the NSF program directors will take these orderings and evaluations and combine them with other panels' evaluations for nearby areas of mathematics to determine which grants will be funded. Your grant will typically be read by three people on the panel, then their evaluations will be read to the rest of the panel, who will then discuss, give your proposal a collective evaluation, and rank it with the other proposals. It is important to keep in mind that of the three people who carefully read your proposal, most, and maybe all, are probably not experts in your specific research area. They will be in some nearby research field, and probably know something about your field, but that is all you can assume. In particular, they will probably not know the intricacies of some of the important questions of your field or why they are important. You need to explain that to them. That is, the target audience for your proposal is someone broadly working in an area similar to your own, but not necessarily an expert in the area. As such, your proposal needs to appeal to this audience, so carefully explain your terms, why your problems are interesting, and how your work fits into the broad field as a whole.

Also, keep in mind that each panelist is reading a large number of proposals, so make it easy for them to read yours and see the merits without any difficulty. It is good to make sure some non-technical indication of the main goals of your proposal and its impact on the field are included in the first page or two of the proposal. This should help keep all of your reviewers carefully reading and give them ideas to use while arguing for a better ranking for your proposal.

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If your grant is not funded—typically less than one third are, so you are in good company—then you will receive feedback on your proposal. Carefully take this feedback into account and resubmit your grant the next year. Your proposal will hopefully be stronger for this and there will be a new panel evaluating the proposal, both of which can affect how your proposal is ranked.

NSF grant proposals must discuss "intellectual merit" and "broader impacts." Most people are fairly clear on the intellectual merit of their proposal; this is usually the research that will be done if the grant is funded. Broader impacts are not as well understood by many. The NSF has a good discussion of broader impacts at `https://www.nsf.gov/pubs/2007/nsf07046/nsf07046.jsp`.

In mathematics, broader impacts usually involve your impact on other people. For example, your educational efforts beyond your standard teaching duties—such as preparing notes to introduce young mathematicians to your area of study and mentoring undergraduates, graduate students, or people at any level; your efforts to build STEM infrastructure—such as organizing seminars and conferences, developing new curricula, and partnering with researchers in industry; and your outreach activities—such as participating in math circles, giving public lectures, broadening participation in math of people from underrepresented groups, and writing general audience articles. While younger mathematicians might not have done all that much on the broader impacts front, don't worry. The panelist reading your proposal knows that it takes some time to develop a good track record; but it is not hard to start doing several things to improve the mathematical community around you, and doing so will give you some good broader impact; so, get out there and get started now!

Preparing a grant proposal can be a very rewarding endeavor, helping you to broaden and deepen your research program. And, of course, being awarded a grant can be a real boon to your research program. Therefore, now is the time to start looking into your grant options and start thinking about writing a grant proposal.


John Etnyre

### Credits

Photo of John Etnyre is by Renay San Miguel of Georgia Tech College of Sciences.

AMERICAN MATHEMATICAL SOCIETY

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The American Mathematical Society has a bicameral governance structure consisting of the Council (created when the Society's constitution was ratified in December 1889) and the Board of Trustees (created when the Society was incorporated in May 1923). These bodies have the ultimate responsibility and authority for representing the AMS membership and the broader mathematical community, determining how the AMS can best serve their collective needs, and formulating and approving policies to address these needs. The governing bodies determine what the Society does and the general framework for how it utilizes its volunteer, staff, and financial resources.

The Governance Leadership consists of the Officers (President, three Vice Presidents, Secretary, four Associate Secretaries, Treasurer, and Associate Treasurer), the Council, Executive Committee of the Council, and Board of Trustees.

The Council formulates and administers the scientific policies of the Society and acts in an advisory capacity to the Board of Trustees. Council Meetings are held twice a year (January and the spring).

The Board of Trustees receives and administers the funds of the Society, has full legal control of its investments and properties, and conducts all business affairs of the Society. The Trustees meet jointly with the Executive Committee of the Council twice a year (May and November) at ECBT Meetings.

The Council and Board of Trustees are advised by nearly 100 Committees, including five Policy Committees (Education, Meetings and Conferences, Profession, Publications, and Science Policy) and over 20 Editorial Committees for the various journals and books it publishes.

The Council and Board of Trustees are also advised by the Executive Director and the Executive Staff, who are responsible for seeing that governance decisions are implemented by the Society's 210 staff members.

Learn more at `www.ams.org/about-us/governance`.

# Memorial Tribute: L. Gaunce Lewis Jr. (1949–2006) and Mark Steinberger (1950–2018)

*J. Peter May*

We have lost Gaunce Lewis and Mark Steinberger, two excellent algebraic topologists, to early deaths. Both were students, collaborators, and friends of mine, and Mark was also my nephew. Both were struck down by brain cancer, Gaunce dying on May 17, 2006, and Mark on September 15, 2018.

Gaunce and Mark, along with other students of mine from the early 1970s, especially Bob Bruner and Jim McClure, were in at the beginning of two major current directions in algebraic topology, equivariant stable homotopy theory and structured ring spectra. I will try to give something of the flavor of the work of Gaunce and Mark, focusing in part on the two books *Equivariant Stable Homotopy Theory* [11] and *$H_\infty$ Ring Spectra and their Applications* [3] before going on to separate accounts of their later work. The first book, [11], was written by Gaunce, Mark, Jim, and me, and the second, [3], was written by Bob, Jim, Mark, and me. Both were published in 1986, based on a decade's worth of prior collaborative work. The first includes the results of Gaunce's 1978 thesis, and the two together include the results of Mark's 1977 thesis. Although Gaunce was older, Mark arrived at Chicago earlier, in 1972, so I will start with him back then.

I knew Mark as a child, although not well. His father's mother was my father's sister. That side of our family escaped from Nazi Germany in the 1930s. As teenagers, Mark's father Herbert and Herbert's brother Jack were sent to the United States on the first *Kindertransport* out of Germany in 1934. Their parents and younger brother Rudi followed in 1937. My father got out in 1936. They all started off in Chicago, strangely enough. Herbert died in 1994, Rudi in 2017, but Jack is still alive, age ninety-seven. Jack is a Nobel laureate in physics who befriended me and was my childhood role model, very much responsible for where I went to college and how I've spent my life, but that is another story.

Mark had already made up his mind to work with me before he entered graduate school, and I never knew whether that was more because of his mathematical interests or because of the family connection. He helped convince others to work in algebraic topology, spearheading a wonderful group of eight people who obtained their PhDs in just the three years 1977–79, including Gaunce, Bruner (PhD, 1977), and McClure (PhD, 1978).

Bruner wrote to me about Mark that "the first thing that comes to mind is his laugh and his ability to see things in a humorous light." Jeff Caruso (PhD, 1979) wrote, "I didn't know him very well but in our conversations he was

**Figure 1.** Mark Steinberger at the piano, 1982.

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**Figure 2.** Gaunce Lewis in his college years.

always helpful. He enjoyed explaining things, and helped me to learn about moduli spaces and other topics. I still remember vividly his witty portrayal of Prof. Rothenberg in the 1976 Beer Skit." Mark was charismatic and had a bubbling but caustic sense of humor. He was then still a teenager at heart. He had to be bailed out after one escapade, when he was caught for driving too slowly, but the details are hazy in my memory. Mathematically, he was quick, sharp, and incisive.

Gaunce, in contrast, was incredibly careful, precise, and methodical. McClure wrote to me, "Mark was a character, of course, and Gaunce was a man of great integrity." At a time when many young men were trying desperately to escape the draft, Gaunce volunteered for and served in the navy in the years 1972 to 1975 before entering graduate school. He made up for lost time by finishing his PhD in three years. In later life, Gaunce was long a teacher at the First United Methodist Church of Oswego, where he served as liturgy coordinator. He too had a great sense of humor.

To discuss their work, it seems best to start with [11] and Gaunce's contributions. It is no exaggeration to say that this book first consolidated the study of equivariant stable homotopy theory as a major branch of algebraic topology. Interest in it spiked early with Gunnar Carlsson's use of equivariant stable homotopy theory to prove the Segal conjecture [5] (also published in 1984), and it was resurrected by the extraordinary and unexpected role of equivariant stable homotopy theory in the remarkable solution by Mike Hill, Mike Hopkins, and Doug Ravenel of the Kervaire invariant problem [8] (published in 2016). To quote from Paul Goerss's Mathematics Reviews of [8], "This paper marks the renaissance and reinvigoration of equivariant stable homotopy theory. While this has been an important subfield since at least the 1970s, the unexpected application of equivariant techniques to such an important problem has brought the study of group actions in stable homotopy theory to the front of the stage. ... The foundation text remains [11]." His praise of [11] in his review of an earlier expository paper [9] was still more effusive.[1]

Gaunce's expertise, especially his remarkable application of Freyd's adjoint functor theorem to construct an adjunction between the prespectra and spectra of [11], is what made the original construction and analysis of the equivariant stable homotopy category possible. It is paradoxical that this abstract idea played this crucial role in building the approach to the stable homotopy category with the most precise point-set level description of the homotopically meaningful objects (technically, these are the strict $\Omega$ $G$-CW-spectra). I never had to take point-set topology questions seriously in our joint work, although there were serious issues to be sorted out, since I could just go to him for the answers. Several of his papers answer point-set questions of interest to algebraic topologists.

Gaunce's thesis was largely devoted to the development of highly structured Thom spectra, which are now very widely used; equivariant examples play a major role in [8]. That work and an early paper axiomatizing transfer operations were expanded and incorporated in [11]. With McClure and me, Gaunce proved that the equivariant and nonequivariant versions of the Segal conjecture are equivalent, which turned out to be a necessary step in Carlsson's proof. We also generalized that result to a result about classifying $G$-spaces that has since been used to study maps between classifying spaces. A short early paper by Gaunce has been particularly influential. It showed that a very natural set of axioms for a "convenient" category of spectra is inconsistent, meaning that to construct the best possible concrete category of spectra, one has to sacrifice having the best possible relationship to the category of spaces.

Gaunce's later work remained focused mostly on equivariant homotopy theory, both stable and unstable. He brought to it a powerful and unusual blend of categorical and computational thinking. Many fundamental features of nonequivariant algebraic topology require rethinking equivariantly. Gaunce wrote the definitive equivariant treatment of the Hurewicz theorem, the construction of Eilenberg–Mac Lane $G$-spaces associated to representations

[1] *It is unfortunate that, as the senior author in many collaborations, my name is often cited alone, giving me disproportionate credit for joint work.*

of $G$, the van Kampen theorem, and the Freudenthal suspension theorem in the papers [14, 13]. A comment by Bruner is relevant: "Gaunce found complications where people might not have expected them (or at least hoped there wouldn't be any), then found ways (often again surprising) to cope with them."

**Figure 3.** Gaunce Lewis in his mature years.

Gaunce pioneered the study of equivariant cohomology. People unfamiliar with modern algebraic topology think of equivariant cohomology with coefficients in an abelian group $A$ as $H^*(EG\times_G X;A)$. That is Borel cohomology. While it is powerful and useful, it is only a very special case of Bredon cohomology, the equivariant cohomology theory that satisfies the dimension axiom. With McClure and myself, Gaunce introduced $RO(G)$-graded Bredon cohomology. That requires Mackey functor coefficients, and it is now understood to be central to equivariant algebraic topology. For example, for the obvious reason that one cannot embed a $G$-manifold equivariantly in any $\mathbb{R}^q$ with trivial $G$-action, one cannot even make sense of Poincaré duality without $RO(G)$-grading. However, $RO(G)$-graded cohomology is extraordinarily difficult to compute. It is a stroke of luck that the only actual equivariant calculation in the solution to the Kervaire invariant problems is flukishly easy; the genius is in the reduction to that calculation.

In the papers [12, 6], Gaunce and his student Kevin Ferland carried out what to this day are some of the most difficult and interesting calculations in equivariant algebraic topology. To the best of my memory, Gaunce was the first to have the idea that equivariant cohomology should not only be $RO(G)$-graded but should also be Mackey functor *valued*. That is, instead of an abelian group for each integer, as in classical algebraic topology, one has a Mackey functor for each element of $RO(G)$; these Mackey functors are interrelated by multiplicative structure. That rich structure raised foundational questions about the homological behavior of Mackey functors, which Gaunce addressed in the full generality of compact Lie groups rather than just finite groups and, still more generally, in an illuminating categorical framework for the relevant homological algebra. He shows that standard results, like projective implies flat for modules over a ring, can actually fail in such more general contexts.[2]

In two large-scale papers [15, 16], Gaunce made great progress in understanding equivariant stable homotopy theory for incomplete universes, which involves using parts, but not all of, $RO(G)$; that is closely, but mysteriously, related to recent work by Andrew Blumberg and Mike Hill [1] that grew out of the solution to the Kervaire invariant problem. With Halvard Fausk and me, he computed the Picard group, that is the group of invertible objects, of the equivariant stable homotopy category. With Mike Mandell, he made a systematic study of the equivariant universal coefficient and Künneth theorems, and they went on to give a valuable study of modules over a monoid in a general monoidal category.

Turning to Mark's work, we return to [11]. In preparing this tribute, I was startled to find that I had forgotten the contributions by Mark that are direct precursors to current work on equivariant infinite loop space theory. Operads of $G$-spaces are no more difficult to understand than operads of spaces, but they were first taken seriously in [11], where Mark was the first to consider actions of such $G$-operads on $G$-spectra. Even today some of the results obtained there seem surprising. Such operad actions are now understood to be fundamental to the study of equivariant stable homotopy theory. A plethora of examples were predicted to exist in [1] and were shown to exist in three 2017 preprints by different authors. Nonequivariantly, Mark studies operad actions on spectra homologically via chain complexes associated to extended powers of spectra.

This work established the foundations for Mark's study of Dyer Lashof operations on highly structured ring spectra, called $E_\infty$-ring spectra or, in their weaker up-to-homotopy version, $H_\infty$-ring spectra. This work appears in [3]. These operations are homology analogues of the classical Steenrod operations on the cohomology of spaces. In particular, he computed these operations on the homology of Eilenberg–Mac Lane spectra. I foolishly had expected such operations to be trivial, but Mark proved how very wrong I was. One highlight gives very general criteria for when a $p$-local, $H_2$-ring spectrum splits as a wedge of Eilenberg–Mac Lane spectra $H\mathbf{Z}_p$, or more generally EM spectra $H\mathbf{Z}_{(p^r)}$, or Brown-Peterson spectra $BP$.

Mark's calculations were quickly used in Bökstedt's celebrated calculation of the topological Hochshild homology of $\mathbb{Z}/p\mathbf{Z}$ [2]. To quote a referee who objected that this work deserves more credit than I gave it, these calculations "underlie all the research on THH and its cousins." They have also gained interest from their use in Tyler Lawson's remarkable proof [10] (published in 2018) that $BP$ at $p$=2 is not an $E_{12}$-ring spectrum, answering a long-studied question that I asked in 1975. The book [3] as a whole (especially McClure's part) is the starting point of the study of power operations in stable homotopy theory, which now pervade that subject.

[2] *Unfortunately, much influential work of his in this direction remains unpublished.*

In contrast to Gaunce, Mark went in a different direction after his early work in algebraic topology. While his later work was still largely equivariant, it was now in geometric topology. It was done mostly in collaboration with James West and partly also with Sylvain Cappell, Julius Shaneson, and Shmuel Weinberger. I'll let Jim tell the story.

*From Jim West:*

Mark's most important body of work with me was, broadly speaking, in applications of topological simple homotopy theory. Simple homotopy theory is a fundamental ingredient in the study of the structure of manifolds. Equivariant versions of simple homotopy theory are essential to the study of the structure of group actions on manifolds. The classical formulations of simple homotopy theory are dependent on the combinatorial (or cellular) structure of simplicial (or CW) complexes. Homeomorphisms and group actions that are not differentiable or piecewise linear need not respect this combinatorial or cellular structure. It was important to find a formulation of simple homotopy theory that was invariant under topological homeomorphisms. Tom Chapman did this, using earlier work of mine, when he showed that the entire theory could be interpreted as the homeomorphism theory of compact Hilbert cube manifolds.

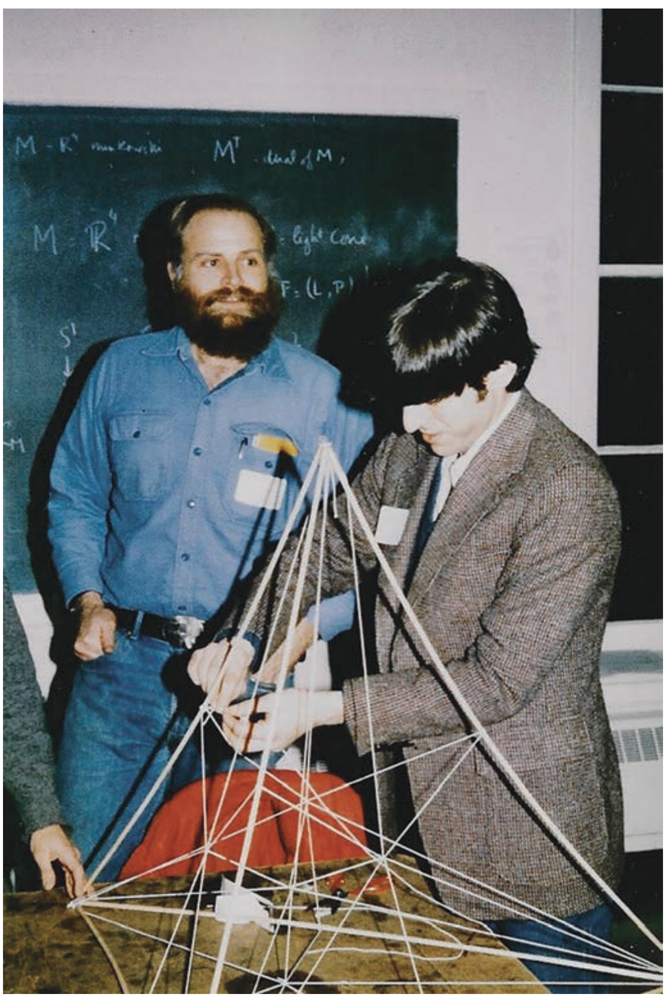

**Figure 4.** Mark Steinberger and Jim West, math and art; late 1970s.

Mark arrived at Cornell excited by the idea of working with me to apply this extension in contexts where the classical formulation was difficult or impossible to apply. We discovered that the equivariant homeomorphism theory of locally linearizable actions of finite groups on (finite dimensional) manifolds was exactly such a situation. We had a very fruitful collaboration which involved, among other things, developing an equivariant surgery theory for locally linearizable actions. I think it would be fair to say that our work opened this subject for further research. The high points of our research were Mark's *Inventiones* paper [20] and the joint paper with Cappell and Shaneson in the *American Journal* titled "Non-linear similarity begins in dimension 6" [4]. I was congratulated very warmly in person on the latter result by Georges De Rham and by Ed Floyd.

In this collaboration, we were definitely equal partners. Mark was the strategist. He was usually the one who came up with the applications that might be accessible using our techniques. Technically, he made all the algebraic calculations, while I concentrated on the controlled infinite processes.

To give a better idea of what this is all about, nonlinear similarity asks when linearly inequivalent representations of $G$ can be $G$-homeomorphic. Analysis of the Picard group of the equivariant stable homotopy category is somewhat analogous, since in part it concerns the classification of representation spheres up to $G$-homotopy type.

*Shmuel Weinberger wrote to me about the work of Mark and Jim:*

My main mathematical interactions with Mark were about the work that he had done, partly with West, on topological simple homotopy theory for $G$-manifolds. Unlike the beautiful work of Chapman and Kirby–Siebenmann, which shows that topological manifolds behave just like smooth manifolds as far as their handlebody theory was concerned, this is not true with a group action. There were many examples and it looked like a complete mess before the work of Steinberger and West, and

> around the same time, Quinn came out and explained everything. Their points of view were very different, and it took a while for the community to understand why everything worked out consistently. I also took part in a collaboration with Cappell–Shaneson–Steinberger–West on nonlinear similarity that sought to develop constructive methods simultaneously with analyzing the obstructions coming out of topological equivariant simple homotopy theory. Ultimately, this led to the development of a theory of surgery on stratified spaces.

Jim does not allude to his first paper with Mark, in which they observe that a Serre fibration between CW complexes is a Hurewicz fibration. From the point of view of model categorical foundations for homotopy theory, in which these two kinds of fibrations play vastly different roles, this geometric result is quite curious. Another pair of papers by Mark alone clarifies the nature of PL fibrations.

**Figure 5.** Mark Steinberger speaking, 2015.

Turning to another kind of contribution, Mark deserves sole credit for the creation of an impressive new journal of mathematics, namely the *New York Journal of Mathematics* (*NYJM*). Mark founded that in 1994 as the first electronic general mathematics journal, and he was its editor-in-chief. Mark wrote an article about it that was published in the January 1996 issue of the *Notices* and is available online [19]. Mark was interested in both the economic and the technical benefits of electronic journals. For the former, he, along with many others at the time, was especially concerned about the sometimes exorbitant cost of many mathematics journals, and he saw it as a major advantage of electronic journals that they could be free.[3]

He also hoped that new journals like the *NYJM* might dent the hold of the top-ranked journals. He hated snobbishness in general and the rankings of journals in particular. He would have nodded in agreement with the article [7] about the tyranny of the top five journals in economics, which could just as well have been written about mathematics. In fact, as he knew, both [3] and [11] are in large part shotgun marriages of articles unpublishable in top journals at the time, hence their late appearance.

Mark was very much focused on the technical potential of electronic journals, and he wrote two informative (if perhaps technically dated) articles that focus on the creation of the *NYJM* [19] and on the existing and potential relevant technology [18]. One interesting technical innovation in the *NYJM* can be found at `nyjm.albany.edu/search/j/ghindex.html` where one can search all past publications of the *NYJM* at once for key strings of symbols or words. However, the focus of the journal is on quality and expertise, as its very strong editorial board attests.[4]

I wish I had words to do justice to the personalities of so many friends and colleagues now gone. I can hear both Mark and Gaunce laughing at me as I try.

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[3] *I can imagine his snort of laughter if he were told that pdf files of individual chapters of [11] are on sale by the publisher for $29.95 per chapter. (In fairness, the publisher sells the entire ebook version for "only" $59.99.)*

[4] *A blog by a satisfied author gives a readable description [17].*

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J. Peter May

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COMMUNICATION

# The First Twenty-Five Winners of the AWM Alice T. Schafer Prize

*Joseph A. Gallian*

> *It is a wonderful honor to be awarded the Alice T. Schafer Prize from the Association for Women in Mathematics. I would like to thank those who established the award for their vision to recognize and encourage young women mathematicians. Mathematics, though extremely rewarding, is a difficult career to pursue, and thus it is so important for young mathematicians to feel support from the community as they pursue their careers. I want to thank the Association for Women in Mathematics for showing me such support and recognizing me among such outstanding young women mathematicians.*
>
> —Melanie Matchett Wood, 2002 Co-winner

## The Origins

The Alice T. Schafer Mathematics Prize For Excellence in Mathematics by an Undergraduate Woman was established in 1990 by the executive committee of the Association for Women in Mathematics (AWM) and is named for its second president and one of its founding members, Alice T. Schafer, who oversaw the incorporation of the AWM and championed opportunities for women in mathematics throughout her career. She retired as the Helen Day Gould Professor of Mathematics at Wellesley College in 1980. Schafer's honors include being elected a Fellow of the American Association for the Advancement of Science in 1985 and receiving the MAA Yueh-Gin Gung and Dr. Charles Y. Hu Award for Distinguished Service to Mathematics in 1998. Schafer died in 2009.

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Schafer Prize nominees must be either US citizens or have a school address in the United States and be an undergraduate when nominated. The award is presented at the AWM Reception at the Joint Mathematics Meetings (JMM) each January and at the JMM Awards Presentation. Recipients receive a $1000 prize, an honorary plaque, and are featured in an article in the AWM newsletter. The charge to the three-person AWM selection committee is, "To recognize talented young women to be evaluated on the ability for independent work in mathematics, demonstration of real interest in mathematics, quality of performance in advanced mathematics courses and special programs, and (when relevant) performance in mathematical competitions at the local or national level." To encourage multiple worthy nominees, each year there are one or two winners, up to two named runners-up, and up to three named honorable mentions.

In this article we provide a brief overview of the career paths of the first twenty-five Schafer Prize winners.

## The Winners

**1990: Linda Green** (co-winner) is a teaching assistant professor of mathematics at University of North Carolina at Chapel Hill. She obtained a bachelor's degree from the University of Chicago and a PhD at Princeton in 1996 under Cynthia Louise Curtis. Her research interests include math and statistics education, mathematical modeling of disease, and topology and geometry of three-dimensional manifolds. She has published seven papers in medical journals. In 2018 she received the UNC Goodman–Petersen Award for Excellence in Teaching.

**1990: Elizabeth Wilmer** (co-winner) is a professor of mathematics and former department head at Oberlin College.

She received a bachelor's degree from Harvard and a PhD from Harvard in 1999 under Persi Diaconis. Wilmer is a coauthor of the 2008 AMS book *Markov Chains and Mixing Times* and has published five papers in combinatorics and probability.

**1991: Jeanne Nielsen Clelland** is a professor of mathematics at the University of Colorado at Boulder. She received her bachelor's degree from Duke and PhD from Duke in 1996 under Robert Bryant. Her research area is differential geometry and its applications to differential equations. In 2018 she received the Burton W. Jones Distinguished Teaching Award from the Rocky Mountain Section of the MAA. Clelland has published more than twenty papers.

**1992: Zvezdelina E. Stankova** was a professor of mathematics at Mills College for sixteen years and is currently teaching mathematics at Berkeley. She received a bachelor's degree from Bryn Mawr and a PhD from Harvard in 1997 under Joe Harris. She has published seven papers in enumerative combinatorics that have been cited more than 250 times and is coeditor of two books on problem solving. Stankova is founder of the Berkeley Math Circle, an inaugural winner of the MAA Henry L. Alder Award for Distinguished Teaching by a Beginning College or University Mathematics Faculty Member in 2004, and the recipient of the MAA Deborah and Franklin Tepper Haimo Award for Distinguished College or University Teaching of Mathematics in 2011.

**Figure 1.** Cathy O'Neil.

**1993: Cathy O'Neil** is an author and data science consultant. She received her bachelor's degree from Berkeley and a PhD from Harvard in 1999 under Barry Mazur. Following five years as a math postdoc at MIT, she took a position at Barnard College. From 2007–2011 she worked in the finance industry. A PBS *Frontline* episode about Wall Street featured a 38-minute interview with her. She is the author of the blog mathbabe.org and was a TED talk speaker in 2017 (see [1]). O'Neil's book *Weapons of Math Destruction* was long-listed for the 2016 National Book Award for Nonfiction. At the 2019 Joint Math Meetings she received the MAA's Euler Book Prize and gave the MAA-AMS-SIAM Porter Public Lecture.

**1993: Dana Pascovici** is a biostatistician at the Australian Proteome Analysis Facility at Macquarie University, where she focuses on generating reliable methods of interpreting and analyzing data on plasma proteomics and plant proteomics. She received a bachelor's degree from Dartmouth and a PhD from MIT in 2000 under David Vogan. Pascovici was the first recipient of the Elizabeth Lowell Putnam Prize for a high score in the Putnam Competition, finishing sixteenth out of 2,356 participants.

**1994: Jing Rebecca Li** is a research scientist at Institut National de Recherche en Informatique et en Automatique in France. She received her bachelor's degree from Michigan and PhD degree from MIT in 2000 under Jacob White. Li has published more than twenty-five papers in applied math and physics journals.

**1995: Ruth Britto-Pacumio** (now Britto) is an associate professor in theoretical physics at Trinity College Dublin. She earned a bachelor's degree in mathematics from MIT and a PhD in physics from Harvard in 2002. She has held research positions at the Institute for Advanced Study, the University of Amsterdam, the Fermi National Accelerator Laboratory, and the Commissariat à l'énergie atomique. Britto is best known for her work on scattering amplitudes in high-energy collider experiments designed for discovering and analyzing new particles and new physical behaviors. Her 2005 paper with Cachazo, Feng, and Witten, which provided a recursion method for calculating scattering amplitudes, has been cited more than 1,100 times. She coauthored two other papers in 2005 which have been cited more than 1,600 times. Britto has also published six papers on black holes. She was the second winner of the Elizabeth Lowell Putnam Prize.

**1996: Ioana Dumitriu** is a professor of mathematics at the University of Washington at Seattle. In September 2019 she will join the math department at UC San Diego as a professor. Her research interests include the theory of random matrices, numerical analysis, and scientific computing. She received her bachelor's degree from NYU and a PhD from MIT in 2003 under Alan Edelman. She was the first woman Putnam Fellow (top five) and is a Fellow of the American Mathematical Society. Dumitriu has received the Leslie Fox Prize for Numerical Analysis, an NSF CAREER Award, and won the Elizabeth Lowell Putnam Prize three times. She is the author of twenty-five published papers.

**1997:** No award given due to calendar change.

**1998: Sharon Ann Lozano Gretencord** (co-winner) received her bachelor's degree from the University of Texas in 1998 and a master of science in computational and applied mathematics from Texas in 2000. She then spent two years as a lecturer in the mathematics department at UT while developing math and science curricula for a non-profit. Since then Gretencord has been home-schooling her six children.

**1998: Jessica Shepherd Purcell** (co-winner) is an associate professor of mathematics at Monash University in Australia. She received her bachelor's degree from the University of Utah and a PhD in 2004 from Stanford under Steven Kerckhoff. She has published thirty-nine papers on low-dimensional topology and has given more than seventy-five invited talks. Purcell has received an NSF CAREER Award and a Sloan Research Fellowship.

**1999: Caroline J. Klivans** is an associate professor of applied mathematics at Brown University. She received a bachelor's degree from Cornell and a PhD from MIT in 2003 under Richard Stanley. Klivans has published more than twenty papers in combinatorics.

**2000: Mariana E. Campbell Levin** is an assistant professor of mathematics, specializing in mathematics education, at Western Michigan University. Her research concerns how people think about and learn mathematics with the goal of fostering meaningful learning experiences and broad participation in mathematics. She received a bachelor's degree from the UC San Diego, a PhD in math education from Berkeley in 2011 under Alan Schoenfeld, and had a postdoctoral research position in the Program in Mathematics Education (PRIME) at Michigan State University. Levin has a book in press titled *Conceptual and Procedural Knowledge During Strategy Construction: A Complex Knowledge Systems Perspective*.

**2001: Jaclyn Kohles Anderson** received a bachelor's degree from the University of Nebraska and a PhD from Wisconsin in 2006 under Ken Ono. After finishing her PhD, she raised two children while working on mathematics as time permitted. Recently, Anderson has returned to school to study operations research and data science. She has published four papers in number theory and one in discrete dynamical systems.

**2002: Kay Kirkpatrick** (co-winner) is Blackwell Scholar in Mathematics and an associate professor of mathematics and physics at the University of Illinois at Urbana–Champaign. She received a bachelor's degree from Montana State and a PhD from Berkeley in 2007 under Fraydoun Rezakhanlou. Her research interests include statistical mechanics, PDEs, condensed matter physics, and biological computation. Kirkpatrick was an NSF Postdoctoral Fellow at MIT from 2007–2009. She has received an NSF CAREER Award, given more than thirty-five invited talks, and published eight articles in math and physics journals.

**Figure 2.** Melanie Matchett Wood.

**2002: Melanie Matchett Wood** (co-winner) is the Vilas Distinguished Achievement Professor of Mathematics at the University of Wisconsin at Madison. In September 2019 she will join the Berkeley math department as a Chancellor's Professor. She received a bachelor's degree from Duke and a PhD from Princeton in 2009 under Manjul Bhargava. Wood was the first American woman to be a Putnam Fellow and the first woman to win the AMS-MAA-SIAM Frank and Brennie Morgan Prize for Outstanding Research by an Undergraduate Student. Her many awards include AMS Fellow, NSF CAREER Award, AWM-Microsoft Research Prize, an American Institute of Mathematics Five-Year Fellowship, a Sloan Research Fellowship, and a Packard Fellowship. The website "The Best Schools" has her on the list "The Top 50 Women in STEM." She has published more than thirty-five papers in number theory and given more than 100 invited talks.

**2003: Kate Gruher Mattison** is vice president of curriculum at IXL Learning, an American educational technology company whose website offers educational practice for K–12 students. Mattison leads the content design team that creates interactive, engaging, challenging practice skills for math, English language arts, science, social studies, and Spanish. She received a bachelor's degree from Chicago and a PhD from Stanford in 2007 under Ralph Cohen.

**2004: Kimberly Spears Hopkins** is the owner of a real estate investment company specializing in industrial multitenant buildings. She received a bachelor's degree from UC Santa Barbara and a PhD in 2010 from Texas under Fernando Rodriguez-Villegas.

**2005: Melody Chan** is an assistant professor at Brown University. She received a bachelor's degree in computer science and mathematics from Yale and a PhD from Berkeley in 2012 under Bernd Sturmfels. From 2012 to 2015 she was an NSF Postdoctoral Fellow and Lecturer in the mathematics department at Harvard. Her research interests are combinatorial algebraic geometry, graph theory, and tropical geometry. From 2000–2001 Chan studied the violin at the Juilliard School with Itzhak Perlman and Dorothy DeLay. She has more than twenty publications, more than 500 citations, and has given more than ninety invited talks. She is a Sloan Research Fellow.

**2006: Alexandra Ovetsky Fradkin** is the dean of mathematics, science, and technology at the Main Line Classical Academy, an elementary school in Bryn Mawr, where she develops their math curriculum and teaches children in grades K–5. After receiving her 2006 bachelor's degree and a 2011 PhD in mathematics from Princeton under Maria Chudnovsky, Fradkin worked for several years as a professional mathematician publishing ten papers in combinatorics. Before her present position she taught enrichment math at the Golden Key Russian School to children ages 4–10. In 2017 she published *Funville Adventures*, a math-inspired children's fantasy adventure that introduces kids to the concept of mathematical functions.

**Figure 3.** Ana Caraiani.

**2007: Ana Caraiani** is a Royal Society University Research Fellow and senior lecturer in mathematics at Imperial College London. She received a bachelor's degree at Princeton where her senior thesis advisor was Andrew Wiles. She was a two-time Putnam Fellow and a member of the first place 2006 Putnam Competition Team, the only year Princeton has ever won the team competition. She won the William Lowell Putnam Fellowship for Graduate Study at Harvard, where she received a PhD in 2012 under Richard Taylor. Her research interests include the Langlands program, algebraic number theory, arithmetic geometry, and representation theory. Caraiani was an L. E. Dickson Instructor and NSF Postdoctoral Fellow at Chicago (2012–2013), a Veblen Research Instructor and NSF Postdoctoral Fellow at Princeton University and the Institute for Advanced Study (2013–2016), and a Bonn Junior Fellow at the Hausdorff Center for Mathematics (2016–2017). She has received the Whitehead Prize given by the London Mathematical Society. Caraiani has nine published papers with three running more than one hundred pages and has given more than one hundred invited talks. A paper she coauthored with nine other authors posted on arXiv in December 2018 ran 193 pages. The website "The Best Schools" has her on a list of "The Top 50 Women in STEM."

**2008: Galyna Dobrovolska** (co-winner) is an NSF Postdoctoral Fellow in mathematics at Columbia University. She obtained her bachelor's degree at MIT and a PhD from Chicago in 2014 under Roman Bezrukavnikov and Victor Ginzburg. In 2015–2016 she was a postdoc at the Max Planck Institute for Mathematics in Bonn. Dobrovolska's research interests lie in geometric representation theory and related areas of algebra, geometry, and combinatorics. She has published six papers.

**2008: Alison Miller** (co-winner) is a Benjamin Peirce Fellow and NSF Postdoctoral Fellow in mathematics at Harvard with research interests in algebraic number theory, arithmetic invariant theory, and their connections with classical knot invariants. She received a bachelor's degree from Harvard and a PhD degree from Princeton in 2014 under Manjul Bhargava. She is a three-time winner of the Elizabeth Lowell Putnam Prize finishing in the top fifteen each time. In 2018 Miller received a Harvard Excellence in Teaching Award. She has four published papers. The website "The Best Schools" has her on a list of "The Top 50 Women in STEM."

**2009: Maria Monks Gillespie** is an NSF Postdoctoral Fellow and a Krener Assistant Professor of Mathematics at UC Davis. In September 2019 she will be an assistant professor at Colorado State. She received a bachelor's degree from MIT and a PhD in 2016 from UC Berkeley under Mark Haiman. She is a winner of the Frank and Brennie Morgan Prize for Outstanding Research in Mathematics by an Undergraduate Student, a Churchill Scholar, a Hertz Fellow, and an NSF Graduate Research Fellow. Gillespie's research interests lie in algebraic combinatorics. She has eight published papers and has given more than thirty invited lectures.

**2010: Hannah Alpert** (co-winner) is a Zassenhaus Assistant Professor of Mathematics at Ohio State University and an NSF Postdoctoral Fellow. She received a bachelor's degree from Chicago and PhD from MIT in 2016 under Larry Guth.

In 2016–2017 she was a postdoctoral fellow at the Institute for Computational and Experimental Research in Mathematics (ICERM) at Brown University. Alpert has published eleven papers in geometric topology and combinatorics.

**2010: Charmaine Sia** (co-winner) is clinical assistant professor of mathematics at NYU. She received her bachelor's in mathematics and physics from MIT and a PhD from Harvard in 2015 under Michael Hopkins. Sia's research interests include algebraic topology, homotopy theory, the theory of topological modular forms, structured ring spectra, and forms of *K*-theory. Prior to joining NYU, Sia was the Zorn Postdoctoral Fellow in the department of mathematics at Indiana University Bloomington. She has published five papers.

## Comments

Eighteen of the first twenty-five Schafer Prize winners received the award as a senior. Remarkably, Pascovici and Dumitriu won as sophomores. All winners profiled here participated in an REU-like summer program and did original research. All but one winner earned a PhD. Five schools have had multiple Schafer Prize winners: MIT (4), Chicago (3), Duke (2), Harvard (2), and Princeton (2). Five schools have had more than one Schafer Prize winner who obtained a PhD degree at their institution: Harvard (6), MIT (5), Berkeley (4), Princeton (4), and Stanford (2). The criterion "quality of performance in advanced mathematics courses" gives a decided advantage to students from PhD granting institutions. All of the winners profiled here were from such schools. Five women from non-PhD granting institutions received runner-up designation. The highly positive reaction in the math community to the Schafer Prize motivated the MAA to establish the Morgan Prize in 1995 with the AMS and SIAM joining as cosponsors.

It is important to note that the Schafer Prize honors more than just the women selected. It recognizes the mentors, the departments, and the research programs that provide support, nurturing, guidance, and inspiration. The following response from 2002 co-winner Kay Kirkpatrick at the AWM reception for award winners typifies the appreciation they have for those who provide support.

> *I feel extremely honored to be numbered among today's rising women in math. The Association for Women in Mathematics is doing a wonderful thing to encourage and support aspiring mathematicians. I'll spend the rest of my life repaying this debt to AWM and to all of my professors and mentors. You all have not only supported me, but also have been true inspirations.*

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1. http://www.ted.com/talks/cathy_o_neil_the_era_of_blind_faith_in_big_data_must_end?language=en

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# The Origins of Spectra, an Organization for LGBT Mathematicians

*Robert Bryant, Ron Buckmire, Lily Khadjavi, and Douglas Lind*

On November 3, 1992, the citizens of Colorado passed an amendment to the state constitution that invalidated local ordinances in Denver, Boulder, and Aspen banning discrimination on the basis of sexual orientation. More importantly, it prohibited the passage of any further laws of this sort at the state or local level. When Colorado's Amendment 2 passed, the Joint Mathematics Meetings (JMM) were scheduled to be held in Denver in January 1995. Two mathematicians, acting independently, felt strongly that this meeting should be moved and wrote individual letters to the leadership of the American Mathematical Society (AMS) and the Mathematical Association of America (MAA) urging them to take this unprecedented action.

This article tells the story of what happened after Colorado's Amendment 2 passed and how our professional societies responded. The national consequences were profound, leading to a landmark decision in 1996 by the United States Supreme Court. Despite the prospect of serious financial consequences and possible opposition by members, the brave decision to move the 1995 JMM from Denver[1] to San Francisco affirmed to lesbian, gay, bisexual, and transgender (LGBT[2]) mathematicians that they mattered.

An informal get-together at that meeting led to annual events, and ultimately to the creation of Spectra, an organization for LGBT mathematicians and their allies. It is a story worth knowing, even a quarter-century later.

## Prologue and Colorado's Amendment 2

Today it may be difficult for some to imagine the plight of sexual minorities in the 1970s and 1980s. LGBT people faced the reality of being fired, denied housing, forcibly outed, abandoned by their families, or even imprisoned should their sexual orientation become known—or suspected—by others [16].

Many adopted secrecy and self-censorship to cope, often with very destructive outcomes. One measure of the level of anti-LGBT stigma in our society at the time is that for years the law treated gay men and lesbians as criminals, although this was only selectively enforced. In 1986 the United States Supreme Court ruled in *Bowers v. Hardwick* that Georgia's anti-sodomy statute was constitutional [3]. Georgia's law criminalized sexual behavior between consenting adults of the same sex in the privacy of their home, with penalties of up to twenty years in jail.

Overlaying this damning judicial decision was the enormous tragedy of AIDS, a time when finding a small sore or spot on one's skin could well mean a relentless descent to a painful death from a disease with no effective treatment [29]. In 1992, the year that Colorado's Amendment

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[1] *As we explain later, the climate for LGBT people in Colorado is now very different from what it was in the 1990s.*

[2] *We have adopted the acronym LGBT throughout this article to reflect the terminology and labels that were often chosen in the 1990s to describe the people who were targeted and impacted by Amendment 2. Today, more inclusive terms can be used to describe the full spectrum of members of this community.*



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2 passed, 23,411 people in the United States died from AIDS [31].

In response to the oppression faced by many LGBT people at the time, gay rights groups in the US started advocating for legal protections from the sort of blatant discrimination many had experienced [13]. In Colorado, the cities of Denver, Boulder, and Aspen all passed ordinances in the early 1980s protecting gays and lesbians from discrimination in housing, employment, and public accommodations.

A number of groups opposed to civil rights for LGBT people were either begun in or moved to Colorado. These included the Family Research Institute, Christian Civil Rights Watch, and Mass Resistance. One such organization, a nonprofit founded in Colorado Springs in 1991, was Colorado for Family Values (CFV). According to its mission statement, its role is to "pro-actively lead and assist those opposing the militant homosexual attack on traditional family values" [26].

CFV sponsored an initiative to amend the state constitution and gathered enough signatures to include it on the November 1992 ballot. The initiative, known as Amendment 2, read:

> Neither the State of Colorado, through any of its branches or departments, nor any of its agencies, political subdivisions, municipalities or school districts, shall enact, adopt or enforce any statute, regulation, ordinance or policy whereby homosexual, lesbian or bisexual orientation, conduct, practices or relationships shall constitute or otherwise be the basis of or entitle any person or class of persons to have or claim any minority status, quota preferences, protected status or claim of discrimination. This Section of the Constitution shall be in all respects self-executing.

The sweeping language of this initiative was extraordinary. As US Supreme Court Justice Sandra Day O'Connor later observed [25] during oral arguments about its constitutionality, "The literal language would seem to indicate, for example, a public library could refuse to allow books to be borrowed by homosexuals and there would be no relief from that." The proponents of this measure often summarized their argument as No Special Rights, claiming that they were only interested in prohibiting "special rights" for sexual minorities [13, 26]. Although surveys showed that the majority of Colorado residents opposed discrimination based on sexual orientation, the "no special rights" argument convinced enough people, and Amendment 2 passed by 53% to 47% [8].

## Letter Writing

When David Pengelley, a mathematician then at New Mexico State University, learned of the passage of Amendment 2, he immediately connected this with the Joint Mathematics Meetings scheduled for Denver in January 1995. Although these meetings are planned years ahead, he decided to write to the members of the AMS Council and the MAA Board of Governors, urging them to move the meeting out of Colorado. His letter [21] reads in part:

- It would be both unfair and insulting to the many homosexual members of the AMS and MAA to be asked to attend an annual meeting in an openly hostile and potentially more dangerous place.
- It is important that this dangerous and intolerant action in Colorado not become a national trend, and the AMS and MAA, along with many other organizations, can help ensure this by not being accomplices. Already many organizations like ours are making such decisions by cancelling convention bookings. One might hope that this will also influence the people of Colorado to change their actions, if not their prejudices.
- Finally, many heterosexual members, like myself, would also be unwilling to be accomplices to this trend by attending an annual meeting in Colorado, and thus attendance and program quality in Denver would suffer, and many members would be alienated.

At the same time, James Humphreys at the University of Massachusetts, Amherst, had similar misgivings. Although he felt that Denver was more progressive than most parts of the state, he thought that the symbolism of having thousands of mathematicians spend lots of money in Colorado was important to avoid. Unaware of Pengelley's efforts, he also decided to write individual letters to all AMS officers and members of the AMS Council urging them to consider moving the JMM. This was a time when a newfangled method of communication called electronic

**Figure 1.** Attendees at the 2019 JMM Spectra reception in Baltimore, including David Pengelley (second from right), his first time attending a Spectra reception.

mail was just starting to become widespread. However, it was quite difficult to find email addresses for a group of people as large as the AMS Council and MAA Board of Governors. So both Pengelley and Humphreys put their letters into individually addressed and stamped envelopes and mailed them off (more than a hundred altogether) in the first week of December 1992. They understood that the governing boards of the professional societies would be meeting shortly at the January 1993 JMM in San Antonio, and wanted this issue on their radar. Neither was optimistic his letter would result in concrete action. They were wrong.

## Societies React

This was not the first time that mathematicians had urged the professional societies to become more inclusive. There is, for example, a rich history of activism on the part of African-American mathematicians and allies against segregation [14].

The letters from Pengelley and Humphreys created a flurry of responses and activity. Time was short. The holidays were fast approaching, and the JMM was convening in early January. Nevertheless, a series of email exchanges between Pengelley and some of those contacted showed there was strong support for the idea of moving the JMM away from Colorado, and that this would be put on the agenda at the governance meetings of both the AMS and MAA.

The schedule of governance meetings was crucial. First up was the MAA Executive and Finance Committee, then the full MAA Board of Governors, and finally the AMS Council. Deborah Tepper Haimo, then president of the MAA, made sure this was on the agenda of the first meeting. She thought a move would have strong support and, indeed, said that no one she had talked with thought there would be any question about moving the JMM meeting site, despite added costs and difficulties with the relocation. She was right. The MAA Executive and Finance Committee recommended the move to their Board of Governors, which was meeting the next day. After that, the AMS Council convened and also was in general agreement to move the meeting.

At an unprecedented joint meeting of the governing boards arranged by AMS President Michael Artin and MAA President Haimo, there was strong sentiment for moving the 1995 JMM out of Colorado [9]. However some participants were opposed, citing both the unknown financial consequences and whether professional organizations should take political stands on this issue. After an hour and a half of discussion, parallel motions were prepared and voted on by the AMS Council and MAA Board. The AMS resolution [1] read:

> The Council of the AMS believes that the actions taken by the majority of those voting in Colorado in November 1992 with respect to discrimination against homosexuals were wrong. The Council of the AMS recommends that the Joint Meetings not take place in Colorado while language similar to that in Amendment 2 of the November 1992 General Election passed by the voters of Colorado remains in the Colorado constitution. One of the reasons for this resolution is that the AMS has the duty to protect all participants at their meetings from possible discrimination.
>
> The Council of the AMS delegates the responsibility for final action to the AMS Board of Trustees and the MAA Executive and Finance Committee, who will instruct the Joint Meetings Committee to make every effort to find a site for the January 1995 meeting in a state other than Colorado.
>
> The Council of the AMS requests that the sentiments of this resolution be communicated to the Governor of Colorado.

The AMS Council passed their version unanimously, and the MAA Board approved theirs by a vote of thirty-six in favor to seven against with two abstentions. The executive bodies reconvened and the votes were announced. According to Devlin [9], President Haimo's update at an MAA meeting two days later received a "large and spontaneous round of applause."

Meanwhile the AMS Meetings staff had been working with their MAA counterparts to find an alternative venue that could be part of these discussions. They recommended San Francisco—always a popular choice, and it had clear symbolic value as well. Four days after these resolutions passed, the Joint Meetings Committee met and agreed to move the meeting from Denver to San Francisco. They also resolved to obtain convention cancellation insurance for all future JMM meetings, and to alert the mayors, chambers of commerce, and convention bureaus in the future sites of JMMs about their intentions and history regarding anti-civil rights legislation. All future hotel contracts for the JMM now include a "Change of Legislation" clause.

A Denver Negotiating Team handled the terms of the cancellation in Denver. Two Denver hotels made claims on the AMS and MAA for damages. All parties settled for a total of $35,000 in damages, half paid by AMS and half paid by MAA [22]. Although not certain, it seems likely that increased attendance due to the change in location from Denver to San Francisco made up most, if not all, of this amount.

*MAA FOCUS* received six letters opposing the move. *FOCUS* Editor Keith Devlin decided to publish three [20], explaining in a preceding editorial that although he thought that these represented a minority view, they deserved to be heard.

## Colorado Boycott

The passage of Amendment 2 was the first major success of a series of similar anti-LGBT rights activities at the time in many other states, including Arizona, California, Florida, Georgia, Idaho, Iowa, Maine, Michigan, Minnesota, Missouri, Montana, Ohio, Oregon, and Washington [12]. The strategy and tactics of Colorado for Family Values, especially their No Special Rights slogan, provided a template for similar groups nationwide.


**Figure 2.** Boycott Colorado was formed as a way to fight back against attacks on anti-discrimination laws.

Alarmed by these developments, a number of individuals and groups considered ways to fight back against this wave of attacks on anti-discrimination laws. The idea of an economic boycott of Colorado gained steam, and by early 1993 the group Boycott Colorado formed as a clearinghouse to publicize and organize these efforts. The boycott sought to deter similar anti-LGBT efforts elsewhere, even encouraging business and political leaders to actively oppose copycat initiatives [2, 26]. The boycott idea proved controversial—boycotts are blunt instruments that can harm those sympathetic to its goals—but it also proved effective.

Three months into the boycott, about three dozen conventions scheduled for Colorado had been cancelled, including the 1995 JMM. By June 1993, Boycott Colorado had enlisted more than one hundred organizations and individuals to endorse this effort, including municipalities such as Chicago, Los Angeles, and New York, resulting in cancelled contracts for Colorado businesses [4]. NBC even changed the locale of its new television series *Frasier* from Denver to Seattle [26].

Estimates of the economic impact on Colorado from the boycott range from $40 million to $120 million, but, even assuming the highest estimate, this represented only 2% of the state's tourism budget [26].

More importantly, the boycott took a serious toll on Colorado's reputation. From innumerable newspaper articles and other publicity, Colorado acquired the epithet "The Hate State." Within the state, many companies and individuals did what they could to counteract this. They adopted and publicized nondiscrimination policies covering sexual orientation, and some required any vendors they did business with to adopt similar policies. Among these efforts was the Colorado Alliance for Restoring Equality, a Denver-based group of businesses and community groups devoted to overturning Amendment 2 [26].

In December 1994 the Colorado Supreme Court struck down Amendment 2 as unconstitutional [7]. The activities of Boycott Colorado were suspended, as they awaited further legal developments [26].

## Initial LGBT Reception

In the fall months of 1994, Don Goldberg of Occidental College in Los Angeles contacted others interested in organizing a social event at the January 1995 meetings, rescheduled for San Francisco. The organizers shared the belief that, in the wake of the decision by the AMS and MAA governing bodies to relocate the 1995 meetings, this was an appropriate time for mathematicians belonging to sexual minorities to establish a visible presence within the profession. The steering committee that organized the event consisted of Robert Bryant (Duke University), Don Goldberg (Occidental College), Concha Gomez (University of California, Berkeley), Steven Hillion (University of California, Berkeley), James Humphreys (University of Massachusetts at Amherst), Nadine Kowalsky (Institute for Advanced Study), Janet Ray (Seattle Central Community College), and Sandra Rhoades (now Gokey) (Smith College) [10].

The AMS staff was helpful in arranging for an announcement of an LGBT reception to be listed with other informal events in the *Meetings Daily Newsletter*. It was held at the Iron Horse, a nearby restaurant and bar, with nearly one hundred attendees. As at most social occasions at the meetings, the discussion ranged over research problems, teaching methods, mutual friends and colleagues, job-hunt networking, the forging of new friendships, and the renewal of old ones. One man, in his sixties, remarked that at meetings years ago he thought he was the only gay mathematician in attendance and was gratified by the size of the gathering. One graduate student was pleasantly surprised to see the author of a favorite book at the reception. Frank Farris [11] has recently written a personal account describing the significance of this event to him.

Many people expressed the desire to have such a gathering at every national meeting. It also became apparent that discussion of sexual orientation issues related to the mathematics profession should be continued beyond the debate over the location of a single meeting. Two email lists were set up to continue communications. These receptions became annual events at the JMM, initially organized by George Bradley of Duquesne University, who scheduled them at the conference hotel, supported them with his own funds, and gathered donations until 2009, when others agreed to take over these duties.

## Supreme Court Decision

Amendment 2 was challenged in the courts nine days after passage by a group consisting of individuals and municipalities. The lead plaintiff was Richard Evans, a gay man who worked for the mayor of Denver. Jean Dubofsky, well known in Colorado legal circles as the youngest person and first woman appointed to the Colorado Supreme Court where she served until 1987, led the legal team. A permanent injunction prevented the measure from taking effect. On October 11, 1994, almost two years after its passage, the Colorado Supreme Court ruled 2–1 that Amendment 2 was unconstitutional [7].

Supporters of the amendment then appealed to the US Supreme Court, which accepted the case in February 1995. Although the Colorado Governor Roy Romer had opposed the initiative, he was obligated to defend it in court. And so the case became known as *Romer v. Evans* [23, 24].

Dubofsky again led the team challenging the amendment, this time in federal court. The stakes were enormous, especially since a definitive ruling would have serious impacts on similar anti-LGBT initiatives that were at various stages of legal challenge around the country. As the team prepared, they were helped by John Roberts, then an appellate attorney and now Chief Justice of the US Supreme Court, as part of his *pro bono* work. Dubofsky later said that Roberts was “terrifically helpful in meeting with me and spending some time on the issue. He seemed to be very fair-minded and very astute” [28].

Oral arguments were heard on October 10, 1995. For a vivid account of the chaotic scene outside the Supreme Court building (with long lines of people trying to secure one of the few seats to witness the historic case) and the tense, dramatic legal exchanges that occurred inside, see Casey [5, 6].

The audio recording of the hour-long hearing (together with the transcript) is available at [25] and is fascinating to listen to. Justice Ruth Bader Ginsberg asked the lead lawyer for the state, “I would like to know whether in all of US history there has been any legislation like this that earmarks a group and says, you will not be able to appeal to your State legislature to improve your status.” Justice Antonin Scalia hammered away on special-rights arguments. He also asked Dubofsky point-blank, “Are you asking us to overrule *Bowers v. Hardwick*?,” referring to the earlier decision that justices were loath to revisit. She deftly showed the justices how they could find Amendment 2 unconstitutional without overturning *Bowers*.

On May 10, 1996, the US Supreme Court announced its decision in Romer v. Evans [23]. By a 6–3 majority, it ruled Amendment 2 unconstitutional, although for different reasons than those given by the Colorado Supreme Court.


THE DENVER POST

Voice of the Rocky Mountain Empire

Court rejects Amend. 2

1992 law ‘born of animosity’ toward gays

By Adriel Bettelheim and Michael Booth

WASHINGTON — The U.S. Supreme Court yesterday struck down Colorado's Amendment 2, dealing a final blow to an anti-gay-rights law that triggered an intense clash of politics and culture when Colorado voters approved it 3½ years ago.

In a 6-3 decision, the high court said the 1992 law was “born of animosity” toward gays and denied their right to equal protection under the Constitution.

Amendment 2 banned any law protecting gays from discrimination in jobs, housing or public services.



**Figure 3.** Headline announcing the US Supreme Court decision striking down Amendment 2.

Justice Kennedy, writing for the majority (with Justices Stevens, O’Connor, Souter, Ginsburg, and Breyer concurring), said that the law “is at once too narrow and too broad. It identifies persons by a single trait and then denies them protection across the board. The resulting disqualification of a class of persons from the right to seek specific protection from the law is unprecedented in our jurisprudence” and “Its sheer breadth is so discontinuous with the reasons offered for it that the amendment seems inexplicable by anything but animus toward the class that it affects; it lacks a rational relationship to legitimate state interests.” He also addressed the No Special Rights argument head-on, saying, “We find nothing special in the protections Amendment 2 withholds. These are protections taken for granted by most people either because they already have them or do not need them” [23].

A dissenting opinion authored by Justice Scalia (joined by Justice Thomas and Chief Justice Rehnquist) began, “The Court has mistaken a *Kulturkampf* for a fit of spite.” It continued, “In holding that homosexuality cannot be singled out for disfavorable treatment, the Court contradicts a decision, unchallenged here, pronounced only 10 years ago, see *Bowers v. Hardwick* … , and places the prestige of this institution behind the proposition that opposition to homosexuality is as reprehensible as racial or religious bias.” It also said that “Amendment 2 is designed to prevent piecemeal deterioration of the sexual morality favored by a majority of Coloradans, and is not only an appropriate means to that legitimate end, but a means that Americans have employed before” [23].

The *Romer* decision to strike down Amendment 2 marked a turning point in the legal battles to secure the civil rights of LGBT people. At least temporarily, it also turned back the tide of similar efforts to challenge and restrict LGBT civil rights, although later these have resurfaced using subtler tactics.

The reverberations from this decision continue to be felt. In its 2003 decision *Lawrence v. Texas*, the US Supreme Court ruled 6–3 that *Bowers v. Hardwick* had been wrongly decided, effectively decriminalizing same-sex relationships nationwide by invalidating the sodomy laws that still remained on the books in sixteen states at the time [15]. In the 2015 decision *Obergefell v. Hodges*, the Court ruled 5–4 that the fundamental right to marry is guaranteed to same-sex couples by both the Due Process Clause and the Equal Protection Clause of the Constitution [18, 19]. As in *Romer* and *Lawrence*, the majority opinion here was authored by Justice Kennedy, while Chief Justice Roberts dissented.

On November 6, 2018, the citizens of Colorado elected Jared Polis as governor, the first time in US history that an openly gay person was elected a state governor [27].

## Creation of Spectra

For many years George Bradley continued to organize and support receptions for LGBT mathematicians at both the Joint Meetings in the winter and MAA's MathFests in the summer. In 2007 Bradley organized an LGBT Math Caucus within the National Organization of Gay and Lesbian Scientists and Technical Professionals (NOGLSTP [17]), a nonprofit organization led by Rochelle Diamond and Barbara Belmont to support LGBT STEM professionals. Since then, NOGLSTP has provided financial and administrative support for the annual JMM reception, as well as serving as the place individuals can send tax-deductible donations for the reception.

At an informal meeting of LGBT mathematicians at the 2010 JMM in San Francisco, faced with Bradley's understandable desire for others to take over this role, several participants pledged some on-going financial support. Christopher Goff (University of the Pacific) stepped up to be the primary organizer of the annual JMM receptions, while Mark MacLean (Seattle University) continued as organizer of the informal off-site receptions. The sense also grew over the next few years that a more formal organization could not only sustain this activity but also provide other valuable ways to support LGBT mathematicians.

David Crombecque had already taken over for Bradley in the LGBT Math Caucus. A small group, serving as a steering committee, started brainstorming ideas of what other things could be done to support mathematicians regardless of sexual orientation, gender expression, or gender identity. Searching for a good name for the nascent organization, Robert Bryant (Duke University) and Mike Hill (University of California, Los Angeles) suggested the colorful term "Spectra," with both mathematical and cultural associations. The Spectra website at `www.lgbtmath.org` has information about the people involved and sponsored events, as well as a way to subscribe to the Spectra email list. It also has links to resources, including primary source materials used to prepare this article.

The first official Spectra event was a panel discussion at the 2015 JMM in San Antonio called "Out in Mathematics: LGBTQ Mathematicians in the Workplace." David Crombecque (University of Southern California) moderated a lively and well-attended discussion featuring Andrew Bernoff (Harvey Mudd College), Julie Blackwood (Williams College), Kristina Garrett (St. Olaf College), Mike Hill (UCLA), and Marie Vitulli (University of Oregon).

A similar panel discussion took place at the 2018 JMM in San Diego, moderated by Lily Khadjavi (Loyola Marymount University), and with panelists Shelly Bouchat (Indiana University of Pennsylvania), Juliette Bruce (University of Wisconsin–Madison), Ron Buckmire (National Science Foundation), Frank Farris (Santa Clara University), and Emily Riehl (Johns Hopkins University). Participants shared their experiences and perspectives. Gathered in a large and supportive audience, attendees raised a wide range of concerns: Should a graduate student on the job market avoid even applying for work in states where adoption would be a legal struggle for him and his husband? How does a graduate student or faculty member get an institution and colleagues to respectfully recognize their gender identity, from day-to-day interactions to official documents? How does one navigate working with an advisor who may not understand or be mindful of these issues? Reactions and responses illustrated that the environment still varies tremendously from institution to institution, as does the legal landscape from state to state. For example, a majority of states in the US do not have prohibitions against employment discrimination based on sexual orientation and gender identity [30].

**Figure 4.** Discussion leaders at the Spectra Town Hall meeting at the 2019 JMM, l to r: Christopher Goff, Douglas Lind, Alexander Hoover, Ron Buckmire, and David Crombecque.

Spectra held a Town Hall meeting at the 2019 JMM in Baltimore, where participants divided into small groups focused on topics that included teaching and job search issues, together with how Spectra can help raise the visibility of the LGBT community within their departments.

## Looking Ahead

With generous contributions from several donors, Spectra has been able to continue the tradition of annual JMM receptions, organized in recent years by Christopher Goff (University of the Pacific) and currently Douglas Lind (University of Washington). Everyone is warmly welcome to attend these events and to contribute any ideas or suggestions they may have for future Spectra activities, as well as to donate funds to support these events.

Throughout, the leadership and staff at our professional societies have been extremely receptive and supportive. Both societies have strong anti-discrimination policies. In 2015 Christopher Goff was appointed the inaugural At-Large Member for Inclusion in the MAA's Council on the Profession, where he still serves. In 2016 Helen G. Grundman was named the inaugural Director of Education and Diversity at the AMS, and she has given Spectra generous encouragement and support. As a recent example, at Spectra's urging the Joint Mathematics Meetings will now provide some well-labeled "All Gender" bathrooms. We are very grateful to all those individuals who have helped Spectra over the years.

The 2020 Joint Math Meetings will be held in Denver, the first time the JMM will be in Colorado since the events recounted here. We encourage participants to celebrate the progress already made and the role our professional organizations play in creating an inclusive environment for all attendees.

The visible presence of LGBT, nonbinary, and gender nonconforming people among mathematicians is an important sign of the diversity of the mathematics community. As educators, *all* mathematicians should be aware of the challenges facing our students and colleagues. Despite advances in recent years, the societal and professional environment is not as welcoming as it could be to those who are underrepresented mathematicians. Often they feel like they cannot participate fully in the mathematics community while simultaneously expressing all aspects of their identity. The ability to do so should be a goal of all our professional societies.


**Figure 5.** Pins that Spectra supporters could wear during the 2019 JMM to increase their visibility.

There will be many challenges ahead. But we hope that the publication of this story serves as an inspirational example of how individuals, working together with their professional societies, can advance the inclusiveness of the mathematical community in important, concrete, and visible ways.

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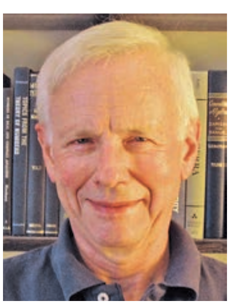
Douglas Lind


To promote community among LGBTQ+ mathematicians, who are often scattered and isolated, Harrison Bray and Autumn Kent established LG&TBQ, a conference at the University of Michigan this summer to foster collaboration and mentoring in geometry, topology, and dynamical systems. They hope this will spur similar efforts in other scientific areas.

### Credits

Figure 1 is by David Crombecque.
Figure 2 is from the personal papers of Jean Dubofsky.
Figure 3 is courtesy of the *Denver Post*.
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# Extensive Cooperation with Rugged Individualism

## George Mackey's guide for practitioners of mathematics

*Della Dumbaugh*

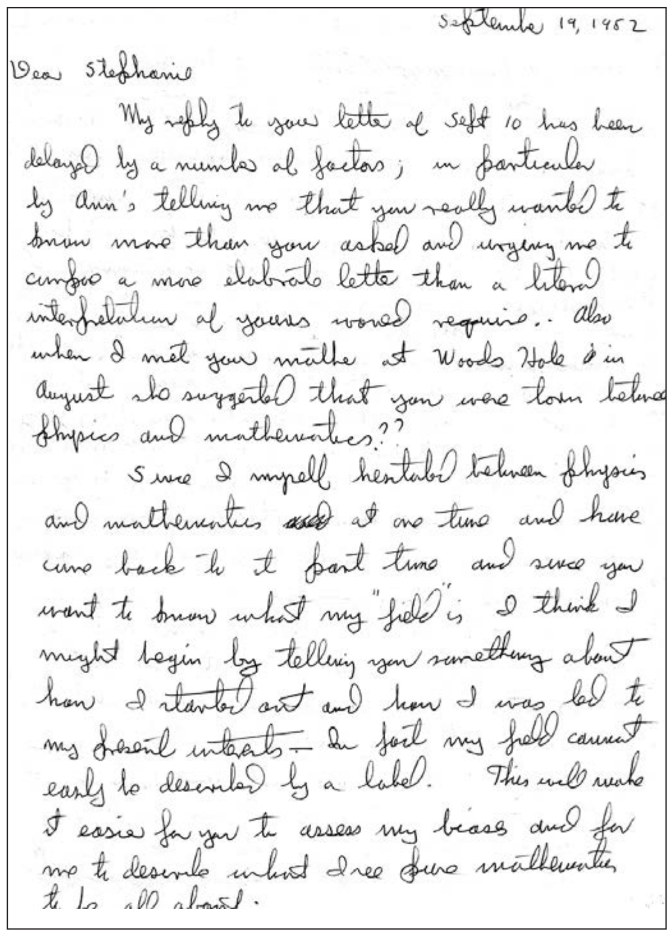
"In a curious way, the advancement of pure mathematics very effectively combines extensive cooperation with rugged individualism."

–George Mackey, "What do mathematicians do?"

### Introduction

The mathematician George Mackey (1916–2006) is often remembered both for his scholarly contributions and his methodical, solitary work habits, tempered by an eager affinity for discussing mathematics with all who took an interest. His broad view of the subject inspired his contributions in infinite-dimensional group representations, ergodic theory, and mathematical physics.

In 1982, Mackey's daughter Ann was a student at Yale University. Her friend, Stephanie Frank Singer, was a sophomore in college trying to decide whether to major in math or physics. Mackey had faced a similar dilemma as an undergraduate, and throughout his career the two disciplines competed for his attention. To help Singer with her decision-making process, Mackey wrote two letters[1] to her in September and October of 1982. He also sent her the text of a talk he had delivered on "What do mathematicians do?" in Paris in March, 1978.[2]


September 19, 1982

Dear Stephanie

My reply to your letter of Sept 10 has been delayed by a number of factors; in particular by Ann's telling me that you really wanted to know more than you asked and urging me to compose a more elaborate letter than a literal interpretation of yours would require. Also when I met your mother at Woods Hole in August she suggested that you were torn between physics and mathematics??

Since I myself hesitated between physics and mathematics ~~and~~ at one time and have come back to it part time and since you want to know what my "field" is I think I might begin by telling you something about how I started out and how I was led to my present interests. In fact my field cannot easily be described by a label. This will make it easier for you to assess my biases and for me to describe what I see pure mathematics to be all about.

First page of Mackey's letter to Stephanie Frank Singer.

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[1]*To read the letters in full see* https://www.ams.org/notices.

[2]*Mackey delivered a talk at the Harvard Club of France, located in Paris during his sabbatical there in March, 1978. It seems likely that "What do mathematicians do?" (see p. 890 for full speech) is the talk he delivered on that occasion and sent to Singer later [1, 2].*



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Stephanie Frank Singer and George Mackey in Berkeley, CA, circa 1984.

Mackey in Beijing, China, circa 1990.

The current article provides an introduction to George Mackey, including excerpts from his letters to Singer, and the complete text of his "What do mathematicians do?"[3] With an undergraduate query forming the inspiration for Mackey's letters to Singer, this work aims to shed new light on the life of this celebrated American mathematician by considering his contributions to undergraduate education.[4]

## Mackey at Work

Mackey adhered to a disciplined lifestyle that began with focus on his mathematical research each morning.[5] In the afternoons, he would normally walk the mile or so to Harvard (to his office or the "long table" at the faculty club for lunch). He ended his days with an early bedtime. He carried a clipboard at all times. He wore a seersucker jacket in warm months and a tweed jacket in cooler ones. He wrote letters about his "latest discoveries" [7, p. 847]. For Mackey, the advancement of mathematics hinged on what he described as an "extensive cooperation with rugged individualism" [4, p. 2]. He seemed to protect time for the "rugged individualism" in the morning and foster "extensive cooperation" in the form of teaching and mathematical discussions later in the day.

This combination helped Mackey make a "lasting impact" on students and colleagues [7, p. 824]. "I learned how to be a mathematician from him," Mackey's student Calvin Moore claimed in his NAS biography. Richard Palais described Mackey as a "pivotal influence" on his life: "my contacts with him, early and late, determined who I was, what I would become and how my life and career would play out" [7, p. 841]. Roger Howe took a novel approach in his measurement of Mackey's influence. He observed that of Euler's 40,000 mathematical descendants, about 300 of them come from Mackey [7, p. 832]. Mackey "made an indelible impression" on his last PhD student, Judith Packer, who reported that he improved her life as both thesis advisor and friend [7, p. 837]. These testimonies suggest the far-reaching influence of Mackey's practice of both the private and public aspects of the profession.

## Mackey at Home

The early 1960s formed an especially exciting time in Mackey's life. In December 1960, just weeks before he turned forty-five, he set aside his bachelor lifestyle of a sparse apartment with a single chair and stereo (presumably his clipboard served as his desk?) and "surprised" his colleagues by marrying Alice Willard [14, p. 11]. A Wellesley graduate, Alice had worked as a buyer for the Jordan Marsh department store in Boston. Together they welcomed many members of the mathematical community to their home on Coolidge Hill Road for elegant dinners and vibrant conversation. In 1961, Mackey delivered the prestigious Colloquium Lectures at the Annual Summer Meeting of the AMS. On this occasion, he summarized his theory of unitary representations and his ergodic theory.[6] In 1962, Mackey was elected to the National Academy of Sciences. George and Alice's daughter Ann was born in 1963. "George persisted in many of his bachelor habits," Moore wrote of Mackey's transition to marriage and family life, "while also adapting them in order to become a dutiful husband and

[3]*The* Notices *would like to thank Ann Mackey and Stephanie Frank Singer for bringing these letters to our attention and granting permission to publish them.*

[4]*Stephanie Frank Singer graduated from Yale and earned her PhD in mathematics from the Courant Institute in 1991 under the supervision of Nicholas Ercolani. She has published two books in mathematics:* Symmetry in Mechanics: A Gentle, Modern Introduction *and* Linearity, Symmetry, and Prediction in the Hydrogen Atom *[17, 18]. She currently protects democracy as a data scientist with the Verified Voting Foundation. She and Mackey's daughter Ann were roommates at Yale, and remain close friends to this day.*

[5]*"[O]ne cannot imagine," his student Caroline Series would later observe, how Mackey managed to "avoid teaching in the mornings" [7, p. 844].*

[6]*See [14, pp. 12–13] for the details of these lectures and the subsequent manuscripts.*

father" [14, p. 11]. For example, Mackey sat on a park bench with his clipboard while Ann and Alice explored local attractions.[7] His family seemed to understand his disciplined adherence to his work schedule. Ann and Alice served as "his wonderful support system…his lifeline" [7, p. 837].

## Mackey: Early Life

Born on February 1, 1916, in St. Louis, Missouri, George Mackey moved to Houston with his parents, brother, and sister in 1926 after a one-year stint in Florida whose last days included surviving the infamous Great Miami Hurricane of 1926, a dramatic tale that all three siblings retold for the rest of their lives. Although only ten years old at the time, this move would have significant consequences for Mackey. After attending public schools, Mackey enrolled in what was then Rice Institute, now Rice University, in the fall of 1934. That Rice did not charge tuition at the time made it an especially advantageous opportunity since Mackey's family did not have money to spare for college [2]. Initially, he planned to study chemical engineering in an effort to align his father's business aims for his life with his own interest in chemistry. It did not take long for his professors to identify and encourage his talent in mathematics. He found a compromise with a degree in physics—officially that is. Mackey described his undergraduate experience as a triple major in mathematics, physics, and chemistry, although this sort of official recognition of multiple majors did not exist at the time.

Mackey and daughter Ann in Cambridge, MA, circa 1964.

Mackey with wife Alice in Cambridge, MA, mid-1960s.

While at Rice, however, Mackey had the good fortune to learn from Professor Walter Leighton, who had only just earned his Harvard PhD in 1935 under the direction of Marston Morse. Leighton offered Mackey two suggestions that would markedly influence his life. He encouraged Mackey to consider studying at Harvard with John Van Vleck, a theoretical physicist with a joint appointment in the mathematics and physics departments. He also bolstered Mackey's confidence to believe he could find success at Harvard. As Mackey later put it, Leighton "assured me that I was 'good enough' for Harvard and urged me to apply" [13, p. 16]. Take a moment to consider this thought. The young George Whitelaw Mackey, who would ultimately become the distinguished American mathematician by the same name, benefitted from a faculty member's belief in him as an undergraduate.

Leighton also informed Mackey about the inaugural "William Lowell Putnam Competition" in his senior year in 1937–1938. Concurrently, Leighton promoted Mackey as the "most promising" of the mathematics students at Rice and convinced the mathematics department to nominate Mackey as their Putnam entry that year. Leighton may have understood the relationship between these two suggestions. The grand prize of the Putnam Competition included a full scholarship to Harvard graduate school. Mackey earned one of the top five scores on the Putnam that year out of 163 participants.[8] He did not win the Putnam grand prize, however. That award went to Irving Kaplansky. Although Harvard had accepted Mackey, they had not initially offered him any funding. Once they learned of his top-five performance on the Putnam, Harvard offered Mackey financial aid, including full tuition. The chairman at the University of California at Berkeley mathematics department, Griffith C. Evans, who had previously served as chair of the mathematics department at Rice, allowed Mackey to rescind his acceptance there and pursue his graduate work at Harvard.

In 2004, when Mackey was eighty-eight and not in good health, he revised his 1989 *Notices* obituary of Marshall Stone to serve as part of the introduction to *Operator Algebras, Quantization, and Noncommutative Geometry: A Centen-*

[7] *Here, Mackey's lifestyle calls to mind Leonard Dickson's equally legendary insight about his honeymoon. When a colleague asked, "how was your honeymoon?" Dickson replied, "it was great, except I only got 2 papers written" [9, p. 398].*

[8] *Gallian includes the entire 1938 Putnam exam in his [10] and the online updated version "The Putnam Competition from 1938–2013,"* `www.d.umn.edu/~jgallian/putnam.pdf`. *Mackey also wrote on the Putnam in [11].*

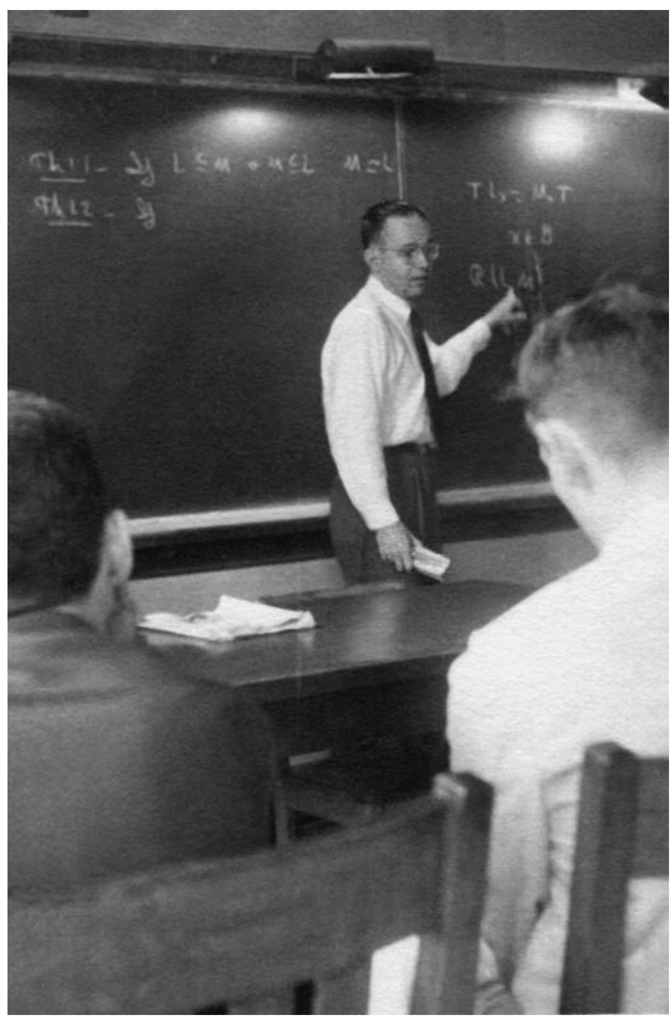
*nial celebration honoring John von Neumann and Marshall H. Stone*.[9] In this tribute ostensibly dedicated to Stone, Mackey writes about his own undergraduate experience at Rice. This choice tells us something about the value of his time at Rice in his life. He had nearly the entire arc of his life in view at that moment. Perhaps Mackey paused then, as we can do now, to reflect on the people he came into contact with as an undergraduate. There was Leighton, a student of Marston Morse at Harvard, who would ultimately lead the mathematics department at Washington University in St. Louis. Leighton recommended John Van Vleck, son of Edward Burr Van Vleck. The senior Van Vleck had studied with Felix Klein at Göttingen and later served as president of the American Mathematical Society. John Van Vleck would win the Nobel Prize in 1977. There was Evans, who would go on to become President of the American Mathematical Society in 1939–1940. After earning a PhD from Harvard in 1910 with a dissertation on Volterra's Integral Equation written under the direction of Maxime Bôcher, Evans was awarded a Sheldon Fellowship from Harvard to study in Rome with Vito Volterra. Evans joined the Rice faculty in 1912 and remained there until 1934, bringing remarkably talented mathematicians, including Benoit Mandelbrot, Tibor Rado, and Carl Menger as visiting professors. Although Evans left Rice shortly before Mackey arrived, he had helped establish a strong research tradition there. He was subsequently hired by Berkeley to do the same with their mathematics department [15, p. 127]. Mackey became acquainted with the name of Irving Kaplansky through the Putnam competition. Thus before he ever left Rice, whether he realized it or not, Mackey had come into contact with seminal figures and/or their ideas in American mathematics.

## Mackey at Harvard: Graduate Student

Mackey arrived at Harvard in the fall of 1938. "I meant to go with physics but applied to the mathematics department for admission," Mackey later described it to Singer.

> My intention was to learn some more mathematics and then come back and do physics "right." I had found physics extremely interesting—especially because of the rather advanced mathematical tools that it used. On the other hand I was quite disturbed by the loose way theory was defined in physics and by the sloppy "hand waving" proofs. I wanted somehow to combine the logical precision of mathematics with the (apparently) richer content of physics. However as my mathematical studies progressed at Harvard I gradually came to realize that pure mathematics had just as rich a content as physics and quite happily dropped physics and became a full fledged pure mathematician [4, p. 2].

A mathematical treatise helped reorient Mackey's academic interests. At the end of his first year at Harvard, Mackey "encountered a thick book in the mathematics library entitled 'Linear transformations in Hilbert space and their applications to analysis' by M. H. Stone…I found the material quite fascinating and ended up reading 60 or 70% of it during the summer and asking Stone to be my thesis advisor" [4, p. 2]. Later, Mackey would clarify that this decision to choose Stone rather than Van Vleck as his thesis advisor did not mean he had decided to abandon his desire to become a physicist. It did mean that he wanted to "learn the deeper parts of pure mathematics under the supervision of the writer of this masterly book" [13, p. 19]. Mackey described Stone's influence on his thesis as "indirect" in that when he met with Stone, he told him about what he was doing, and listened to Stone's "encouraging comments." Harvard courses by Hassler Whitney, Garrett



Mackey teaching, 1940s.

[9] *Alice typed some of his handwritten notes and ensured the article made it to print form [7, p. 826].*

Birkhoff, and the recent Polish immigrant Stanislaw Ulam had a much stronger influence on Mackey [13, p. 20]. Mackey attributed the ideas of his thesis, "The subspaces of the conjugate of an abstract Linear Space," to mathematics he learned and developed from Garrett Birkhoff and a stronger understanding of the linear algebra in Stone's influential text [13, p. 21].

Stone had a much more direct influence on what Mackey termed his "development" when he "arranged" for Mackey to have a Sheldon Traveling Fellowship for his final year at Harvard in 1941–1942. On Stone's advice, Mackey divided the time between Caltech and the Institute for Advanced Study (IAS) in Princeton. At the latter, Mackey met "such legendary figures as Albert Einstein, Oswald Veblen, and John von Neumann" along with younger PhDs including Paul Halmos, Paul Erdős, and Shizuo Kakutani [13, pp. 20–21]. Since Mackey did not yet have his PhD he could not technically join the IAS as a member. Stone, however, "took advantage of his close relationship with von Neumann to talk the Institute into making an exception in my case" [13, p. 21].[10] Kakutani became a close friend and, in particular, he and Mackey "often dined together." As David Mumford later described it, meeting for lunch was "Mackey's favorite way of keeping in touch" [7, p. 837].

While making his way from Caltech to Princeton, Mackey stopped off at an AMS meeting where he met his former Rice professor, Lester Ford. Ford had just assumed the chairmanship at the newly founded Illinois Institute of Technology, and he invited Mackey to join the department as an instructor in mathematics in 1942–1943 once he graduated from Harvard.[11] Although he did not enjoy his time at Illinois Tech, it did allow him to teach mathematics to engineers rather than serve in the military [13, p. 22]. For the next three years he contributed to war-related research at Columbia University and in High Wycombe, England [14, p. 6].

## Mackey at Harvard: Faculty Member

Mackey joined the Harvard faculty as an assistant professor in 1946, was promoted to full professor in 1956, and became the inaugural Landon T. Clay Professor of Mathematics and Theoretical Science in 1969. He retired in 1985. While at Harvard, Mackey counted himself among the "relatively small number of people in the world (perhaps a few thousand) who spend a large part of their time thinking about and trying to contribute to an esoteric subject called pure mathematics" [6, p. 1]. In his "What do mathematicians do?" Mackey set out to answer, "Whatever are these mathematicians doing? Why do they find it so interesting and what does it have to do with the rest of the world?" [6, p. 1]. The latter question surely arose from Mackey's broad view of mathematics and his longstanding interest in describing physical phenomena with mathematics. "In a word," Mackey began with a succinct answer to his questions, "pure mathematicians are refining, developing, improving and (rather rarely) discovering the intellectual tools that have proved useful in analyzing and understanding the measurable aspects of the world in which we live" [6, p. 1]. Early in his exposition, he (not surprisingly?) reduced biology to chemistry and chemistry to mathematics and claimed that this understanding allowed "the measurable aspects of the world [to] become quite pervasive" [6, p. 2]. His letters to Singer also emphasized this link between mathematics and the physical world. He often connected those developments with the people who made them. Writing to Singer about Newton, Mackey asserted that "[d]ifferential equations are what made modern physics possible and the most important thing about calculus is

Mackey working on a lecture, Paris, 1978.

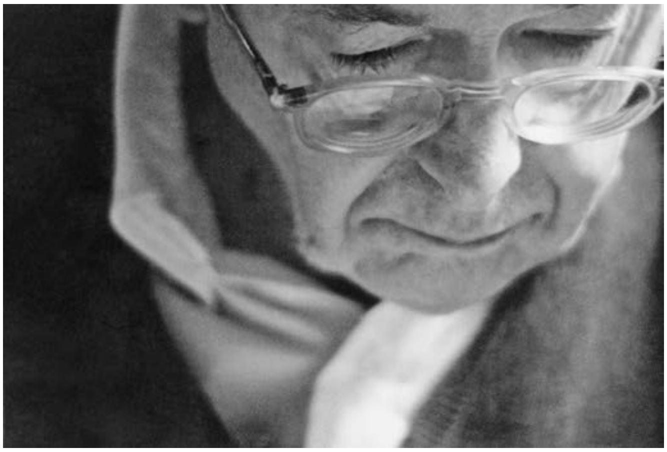
[10] *Indeed, the Institute for Advanced Study lists Mackey as a Member in Mathematics from January to June, 1942 and a Harvard PhD in the same year. See* https://www.ias.edu/scholars/george-w-mackey.

[11] *Margaret Matchett, Emil Artin's first and only woman American PhD student, would just miss Mackey at Illinois Tech. She earned her PhD in 1946 and held an instructorship at Illinois Institute of Technology from 1946–1950.*

that it makes differential equations possible" [4, p. 5]. He also included his thoughts on the pedagogy associated with these ideas when he continued with his view that "Newton's work is epoch making in the strongest sense of the word and I personally find it deplorable that these facts are so little emphasized in modern teaching" [4, p. 5]. Given Mackey's regular discussions with his Harvard colleague Andrew Gleason [7, p. 846; 16], who later became a major proponent of the teaching of calculus, one has to wonder if they also took up these concerns in their conversations.[12]

The topic of teaching was never far from Mackey in his letters to Singer and in his "What do mathematicians do?" In fact, in the latter, Mackey *linked* the life of a pure mathematician with teaching. As he described it at the very beginning of his talk, the vast majority of mathematicians "make their living by teaching in universities, their investigations being subsidized by their being given less than full time teaching loads" [6, p. 1]. He circled back around to this idea near the end when he brought up the "certain tension" that exists for mathematicians who become immersed in their research problems and long for further time to devote to them [6, p. 4]. He identified mathematical institutes as a perfect remedy for this situation. In particular, he cited the Institute for Advanced Study in Princeton and the L'Institut des Hautes Études Scientifiques just outside of Paris as examples of places where mathematicians could "find more time for their work" (rugged individualism) and "exchange ideas" (extensive cooperation) [6, p. 5].

For Mackey, that exchange of ideas included discussions of *undergraduate* teaching, which ultimately attracted extraordinary scholars to the field. Arlan Ramsey had his first class with Mackey in 1958–1959 on projective geometry, for example. "Already," Ramsey later recalled, "I found his attitude and style appealing" [7, p. 842]. Ramsey's "golden opportunity" with Mackey came the following year when Mackey taught from the notes that would become his *Mathematical Foundations of Quantum Mechanics* [12]. "This course answered many questions, and then the answers raised further questions. It was just what was needed and gave me a start on a long-term interest in quantum physics" [7, p. 842]. Richard Palais met Mackey as a sophomore in 1949 when he took his "famous Math 212 course." This course started with the foundations of mathematics and "ended up with some highly advanced and esoteric topics, such as the Peter–Weyl Theorem" [7, p. 841]. The course was just the beginning of a transformative experience for Palais. To Palais's good fortune, Mackey was a resident tutor in his dorm, Kirkland House. Mackey encouraged Palais to share meals with him and discuss his queries about the course material. These meals became increasingly frequent and the conversation stretched to other areas of mathematics and life in general. By the end of his sophomore year, Palais changed his major from physics to mathematics [7, p. 841].



Mackey in his study at Coolidge Hill Road, 1980s.

Five years later, David Mumford initially met Mackey as his Kirkland House nonresident tutor at Harvard in 1954. Through their weekly lunches, Mackey revealed to Mumford "the internal logic and coherence of mathematics… it was the lucid sequence of definitions and theorems that was so enticing—a yellow-brick road to more and more amazing places" [7, p. 836]. Although Mackey would "sometimes disclaim any interest in fostering undergraduate education," according to his daughter Ann, "he would engage passionately with anyone at any level of knowledge who expressed an interest in mathematics" [3]. With Mackey's investment in undergraduates, often at the intersection of mathematics and physics, it seems only natural that he would share his thoughts and expertise with his daughter's undergraduate friend considering similar types of questions. One might even style his letters and text on "What do mathematicians do?" as something of a two-dimensional version of his Kirkland House conversations.

## Concluding Thoughts

But there is something more. The arc of Mackey's life celebrates his own undergraduate experience and his opportunity to work with talented undergraduates at Harvard. The geography of his youth led to his transformative

[12]*In [16], the chapter on "The War and its Aftermath: Andrew Gleason, George Mackey and an Assignation in Hilbert Space" focuses more on Gleason and Mackey than on the war and its aftermath. Mackey and Gleason forged a memorable friendship at Harvard. In particular, Gleason regarded Mackey as his PhD supervisor even though he never earned a PhD. The two colleagues talked about mathematics almost daily but never collaborated on any papers. Mackey was slow and methodical, and Gleason worked with "dazzling" speed [p. 160]. The beautiful reflection on the power of their collaboration will inspire readers. Every reader will benefit from the very human insight Mackey provided when he described the "peace of mind" he gained when he stopped viewing Andrew Gleason "as a dangerous younger rival" whom he had to outdo and instead "concentrated on his own strength, which was his ability to think deeply about a subject—for years or days at a stretch—with monk-like devotion" [p. 160].*

experiences at Rice Institute. In the 1930s, in the somewhat unlikely location of Houston, Texas, talented mathematicians like Leighton, Ford, and Griffin served on the Rice faculty. Mackey benefitted from their extraordinary training and exposure to some of the most celebrated American mathematicians at the time. They not only taught Mackey mathematics but they also helped point him towards what would become his own distinguished mathematical career. Mackey carried this training forward at Harvard with his own students. Whether in his classes, in his office, or at Kirkland House, he shared his verve for mathematics with them. Singer's queries offered another venue for Mackey to do what he did best (in the afternoon), namely, cooperate extensively on mathematics. Although Mackey begins his "What do mathematicians do?" by offering an analysis of the "mathematical strength of the world" focused in five geographical areas, this consideration of his life and work actually suggests a much broader view of the strength of mathematics, beginning with the undergraduate experience.

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Della Dumbaugh

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## "What do mathematicians do?"

George W. Mackey

There are a relatively small number of people in the world (perhaps a few thousand) who spend a large part of their time thinking about and trying to contribute to an esoteric subject called pure mathematics. The more active and successful number only in the hundreds and form a world community in which every one knows or knows of everyone else. The overwhelming majority make their living by teaching in universities, their investigations being subsidized by their being given less than full time teaching loads. For complicated historical and cultural reasons the great majority live in Europe, North America, and Japan and are far from being uniformly distributed over these areas. Some European countries are almost completely unrepresented, and some, like France, are especially strong. Moreover, if the pure mathematicians of Paris, Moscow, greater Boston, Princeton, and New York City were to be eliminated, the mathematical strength of the world would probably be reduced by at least two thirds.

If a non-mathematician listens to these people talk or attempts to read their journals, he confronts an incomprehensible jargon filled with words like differential equation, group, ring, manifold, homotopy, etc. If he asks for an explanation, he is overwhelmed by a concatenation of difficult to grasp abstract concepts held together by long chains of intricate argument. Whatever are these mathematicians doing? Why do they find it so interesting and what does it have to do with the rest of the world?

In the time at my disposal I can do little to answer these questions. Nevertheless, I am going to make an attempt. In a word, pure mathematicians are refining, developing, improving, and (rather rarely) discovering the intellectual tools that have proved useful in analyzing and understanding the measurable aspects of the world in which we live. These measurable aspects are not so limited as they might seem. At the beginning there was just counting and later the measuring of distances, areas and volumes. However, the last three centuries or so have witnessed a steadily accelerating growth in the extent to which all natural phenomena can be understood in terms of relationships between measurable entities. In the 1920s, for example, the discovery of quantum mechanics went a very long way toward reducing chemistry to the solution of well-defined mathematical problems. Indeed, only the extreme difficulty of many of these problems prevents the present day theoretical chemist from being able to predict the outcome of every laboratory experiment by making suitable calculations. More recently the molecular biologists have made startling progress in reducing the study of life back to the study of chemistry. The living cell is a miniature but extremely active and elaborate chemical factory and many, if not most, biologists today are confident that there is no mysterious "vital principle," but that life is just very complicated chemistry. With biology reduced to chemistry and chemistry to mathematics, the measurable aspects of the world become quite pervasive.

At this point I must make it emphatically clear that, in spite of what I have just said, pure mathematicians concern themselves very little with the external world—even in its measurable aspects. Their concern with the intellectual tools used in analyzing the external world is not so much in using these tools as in polishing them, improving them, and very occasionally inventing brand new ones. Indeed it is their concern with the tools themselves, rather than with using the tools, that distinguishes them from applied mathematicians and the more mathematically minded scientists and engineers.

While it is natural to suppose that one cannot do anything very useful in tool making and tool improvement, without keeping a close eye on what the tool is to be used for, this supposition turns out to be largely wrong. Mathematics has sort of inevitable structure which unfolds as one studies it perceptively. It is as though it were already there and one had only to uncover it. Pure mathematicians are people who have a sensitivity to this structure and such a love for the beauties it presents that they will devote themselves voluntarily and with enthusiasm to uncovering more and more of it, whenever the opportunity presents itself.

Perhaps, because of the lack of arbitrariness in its structure, research in pure mathematics is a very cooperative activity in which everyone builds on the work of someone else and in turn has his own work built upon. On the other hand, mathematicians tend to work alone (and occasionally in pairs) and to be intensely individualistic. Thus, in a curious way, the advancement of pure mathematics very effectively combines extensive cooperation with rugged individualism. No one has enough of an overview to be at all effective in directing the development of mathematics. Indeed if anyone tried he would probably do more harm than good. Just as the social insects build marvelously designed intricate structures by apparently carrying materials around at random so have the mathematicians built a marvelously articulated body of abstract concepts by following their individual instincts with an eye to what their colleagues are doing. An interesting example occurred during the first two decades of the twentieth century. While the physicists were struggling with contradictions and anomalies in the so-called "old quantum theory," two quite distinct branches of pure mathematics were being developed by two different sets of mathematicians with no thought for one another or for physics. Then the discoveries of Schrödinger and Heisenberg in 1924-25 provided the key to the mystery, and physics found its way to that subtle refinement of Newtonian mechanics known as quantum mechanics. Almost immediately it was found that these two separate new branches of pure mathematics were not only what quantum mechanics needed for its precise formulation and further development, but they could be regarded moreover as two facets of a bigger and better unified new branch which was even more adapted to the needs of quantum physics. Several decades later this unified new branch began to have important applications to some of the oldest problems in the theory of numbers.

The set of natural numbers 1, 2, 3, . . . is perhaps the first mathematical tool discovered by man, but its study continues to provide pure mathematicians with an apparently inexhaustible supply of profound and challenging problems. Consider, for example, the problem of determining in how many different ways (if any) a given whole number can be written as a sum of two squared whole numbers. The answer to this question turns out to depend on the factorization of the number into primes. I remind you that a number is said to be a prime if it cannot be written as the product of two other positive numbers, neither of which is one. For example 2, 3, 5 and 7 are primes while 4 and 15 are not since $4 = 2\times2$, and $15 = 3\times5$. One can find an answer for the problem expressed in terms of the answer when the given number is a prime. This much is fairly easy. Much more difficult to establish is the beautiful result that solutions exist for the prime 2 and for precisely those odd primes which leave a remainder of 1 when divided by 4. This theorem was announced without proof by Fermat in the middle of the seventeenth century. One hundred years later Euler, the great eighteenth century mathematician, worked for seven years before finding a proof. Nowadays quite simple proofs exist, but they use sophisticated new tools such as group theory and field theory. Similar but slightly more complicated problems remained unsolved until quite recently. Others are still beyond our reach but may become accessible when the new tool mentioned above and which arose in physics becomes further developed.

Such problems may seem trifling to the outsider, but a major lesson taught by the development of Science in the last three and a half centuries is that the way to progress lies in fine analysis—in looking very closely at the simplest aspects of things and then building from there. Galileo began modern mathematical physics by deciding that it would be worthwhile to time a falling body and discover just how much it accelerated as it fell.

Now let me return to my statement that the great majority of pure mathematicians make their livings by teaching in universities and have their work subsidized by reduced teaching loads. Nowadays many people criticize this arrangement on the grounds that it tempts faculty members to neglect their teaching. I think that this criticism is without serious foundation. In my opinion a very high proportion enjoy the teaching they do and regard doing it well as a serious responsibility which is part of what they owe the University for supporting their research. It is, I think, a rather happy arrangement in that it makes it possible for at least some teaching to be done by genuine authorities in the field and at the same time supports an activity whose measurable economic benefits are so uncertain and so far into the future. On the other hand, there is a certain tension. One becomes extremely absorbed in one's research problems and longs for extra time in which to work on them. The summer vacation helps but is not enough. It is fortunate that other possibilities exist, such as sabbatical leaves and various institutes where one can go from time to time and concentrate exclusively on research. Actually there are all too few of the latter, and I would like to close by saying a few words about the two which I myself have visited—one of which is just a few miles outside of Paris in Bures sur Yvette.

The Institute for Advanced Study in Princeton, New Jersey is the older and became famous very quickly by having Einstein on its faculty. It was founded in 1933 and has played a very useful role in the mathematical world ever since. Its school of mathematics has an extremely distinguished permanent faculty of half a dozen or so and every year a group of 50 or 100 visitors. Most of the visitors are young—only a few years beyond the PhD. However, there is always of sprinkling of older mathematicians including a few distinguished foreigners. I have just come from a very pleasant and productive term there.

The institute at Bures sur Yvette (L'Institut des Hautes Etudes Scientifiques) is younger and has a smaller permanent faculty—but one which is probably no less distinguished. I spent an agreeable and profitable term there seven years ago. Like its older counterpart in Princeton, it plays a very important role in the mathematical world—not only by helping mathematicians find more time for their work, but by bringing those with similar interests together so they may exchange ideas.

On this visit to Paris I am not at Bures but am rather teaching a course at the University (Paris VI). However, my Harvard colleague, Professor Barry Mazur, is there and in fact is a frequent visitor. He is in the audience today and has agreed to try to answer questions any of you may have either about the nature of mathematical research or about the IHES.

# Progressions in Reasoning in K–12 Mathematics

*Dev Sinha*

"I think that it is impossible for some of our students to learn to do proofs," explained a colleague of mine. My belated response is that all students can indeed learn to do proofs. College faculty have been asking aspiring math majors to make a huge jump, from not being responsible for providing reasoning in entry-level college courses and most of their K–12 experience to making formal proofs. To ease this transition, our department at the University of Oregon has recently created "lab" courses for first-year students in addition to our "bridge" requirement, to help students by degrees gain experience with proof-based mathematics.

More broadly, we should afford all students, at all levels, practice with gradually more demanding reasoning, something which educators would call engaging in progressions in reasoning [CCSS18]. Teachers, mathematics educators, and mathematicians have been working together to develop and study such learning progressions for reasoning[1] in K–12 mathematics (see for example [SBK09, Knu02, KCJS02]). In this article I share perspective as a research mathematician who has developed and implemented reasoning-focused tasks for K–12 students of all grades, for both aspiring and current teachers, and for many types of undergraduates.

Historically, explicit calls for student reasoning had been all but absent in school curricula until Euclidean geometry. But mathematical argument does not need to wait until the onset of puberty. Indeed, children naturally ask "why questions," and based on research both the National Council of Teachers of Mathematics Principles and Standards for School Mathematics [NCTM00] and Common Core State Standards in Mathematics, in particular its third Practice Standard [CCSS10], call for more reasoning and proof throughout school mathematics.

Such calls are being answered. For example using pictures such as those given in Figure 1, many current curricula provide opportunities to reason about the sum of two odd numbers in the second or third grade.

While such arguments through pictures lack the formal trappings of proof, engaging with such arguments is valuable experience. Moreover, visual learning of arithmetic has a clear basis in the research literature [PB13, Ans16]. One could place this activity in a coherent learning progression across K–12 by revisiting arithmetic of even and odd numbers through other arguments, either based on place value and case analysis or grounded in algebra. Such a learning sequence could advance in high school with activities such as showing that the sum of two squares cannot be one less than a multiple of four.

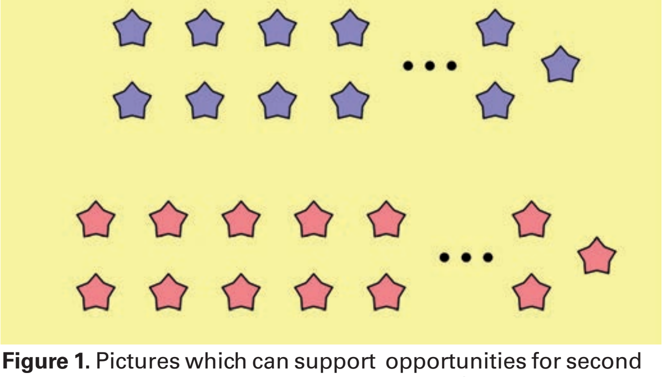

**Figure 1.** Pictures which can support opportunities for second and third graders to reason about the sum of two odd numbers.

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*Communicated by* Notices *Associate Editor William McCallum.*

[1]*We use the terms "reasoning" and "proof" interchangeably. There is a continuum of formalism in reasoning and proof, and it is pedagogically useful for students to engage at different levels, even within the same classroom activity. We prefer to emphasize this range rather than making only a binary distinction based on some hard-to-define cutoff for rigor and formalism.*



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Reasoning about even and odd numbers has been used fruitfully in a number of settings. Colleagues at the University of Oregon and I use this material as an introduction to different levels of formalism for undergraduates who aspire to be elementary school teachers [BHS]. Patrick Callahan, a mathematician who led the California Math Project, often has schools evaluate student reasoning by asking students at different grade levels to explain why the sum of odd numbers is even. Callahan reports that high schoolers generally fare no better than grade schoolers, and when presented with the argument through variables, they (including advanced students) commonly report that they did not realize it was "allowed" for variables to be used in that way. Deborah Ball and her colleagues at the University of Michigan have used student-driven discussion about the definition of even numbers as a strong component of teacher training [Bas05, BB03].

The Common Core State Standards for Mathematics were designed through progressions [CCSS18]. While the Common Core can be read as policy or as informing pedagogy, to a knowledgeable reader they also suggest proofs for all of K–12 mathematics, short of concepts that require limits, in particular working rigorously with functions over all real numbers. The commutative property of multiplication, for example, should be established through noticing that a rectangular array and its transpose are in bijective correspondence. Strong curricula engage students in such proofs in age-appropriate ways.

The Common Core also asks that the canon of elementary mathematics be taught consistently with how mathematicians practice mathematics. The multiplication table is not just a set of facts but also a rich locale for conjecture and proof. Students who type $1/7$ or $\sqrt{5}$ into a calculator and get an "answer" can be asked what that answer means in terms of inequalities, for practice at using definitions as well as reinforcement of estimation and number sense. And the story of the law of exponents, which goes from having a simple verification for positive whole exponents to being the driver of the definition for all other exponents, is a great example of the art of mathematical definition. (I enjoy the parallel between the law of exponents and the homotopy lifting property, which went from being a property of fiber bundles to being Serre's definition of a fibration.)

Reasoning at the high-school level can reach, for example, the circle of ideas centered on the fact that the sum of the first $n$ odd numbers is $n^2$. The statement itself is ripe for conjecture through seeing cases, and it is substantial work for students to make their conjectures precise. A graphical argument, as in Figure 2, is readily accessible, and then provides an opportunity to press for details—for example, why the number of blocks added in each step increases by two. Algebraically, there are multiple arguments: an inductive argument, the standard trick for summing arithmetic sequences, and one through calculating successive differences (a case in which "simplifying" has a purpose) and employing reasoning central to the fundamental theorem of calculus. Indeed, such analysis provides a first explanation, in the discrete setting, for why quadratic functions should model total displacement in the presence of constant acceleration.

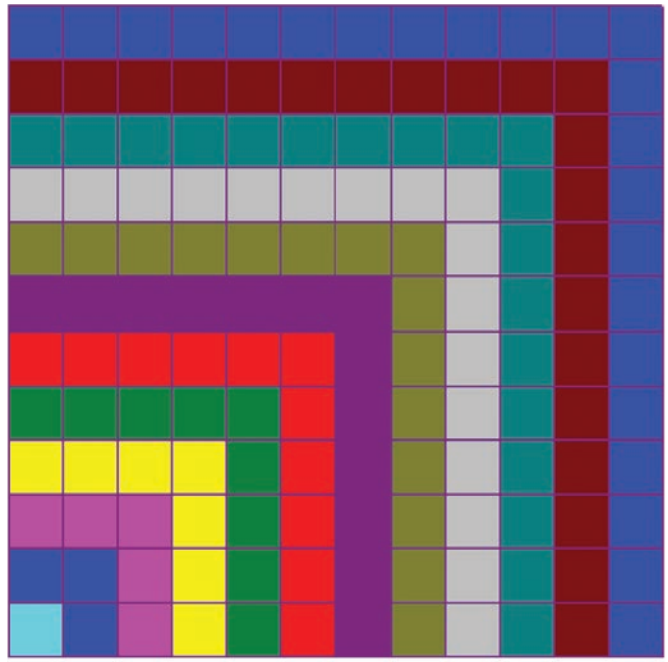

**Figure 2.** A graphical argument that the sum of the first $n$ odd numbers is $n^2$.

Asking students to provide and critique reasoning is more time-intensive than only demonstrating reasoning to them or omitting reasoning altogether. But the development of communities of reasoning is sorely needed, especially at this moment. Students benefit immensely from establishing truth through logic that is accessible to all, with contributions from themselves and peers, rather than having mathematics exclusively "handed down" through authority of teacher and textbook. Such communities foster responsibility for acknowledging errors, understanding them as a way to progress to correct understanding.

In addition to presenting mathematics consistently with the practices and values of our community, increased reasoning will pay dividends through deeper retention as well as greater transfer of reasoning ability (see for example Chapter 3 of [NRC00]). Facility with application is also supported, as students who have access to reasoning can more flexibly use mathematics to understand the world. For example, concrete visual models are now regularly used to reason about dividing fractions and in particular "remainders," as needed for applications. If it takes $1/4$ of a ton of steel to make a car, and a factory has $8\ 1/3$ tons of steel, dividing, we see that there could be 33 cars made, and there will be $1/12$ of a ton of steel left over. This is $1/3$

of the amount needed to make another car, as one sees in the quotient 33 ⅓, but in some contexts 1⁄12 would be the correct "remainder," a quantity which is not accessible if one only learns to "flip and multiply."

Progressions occur not only in the mathematics itself, as definitions and theorems build on previous such, but across many domains. A familiar domain to mathematicians is the amount of abstraction. Related, but less familiar, is the sophistication of representation, including progression from tactile models to pictures to diagrams to variables. On the more cognitive side, there are progressions in student autonomy, including whether tools are called for explicitly or demanded through problem-solving. There are progressions in how much of the process of doing mathematics is engaged in, for example in formulation of conjectures and counterexamples. Demands in language will also progress. Strong curricula attend to all of these types of progression, and address them through research-based pedagogy, in particular active learning [LBHea14, Bea16, MAA18].

Teachers need to be prepared to implement curricula that demand that students supply their reasoning. As the mathematical community plays substantial roles in their preparation—especially that of future high-school teachers—we make some recommendations for teacher training and for other related matters below. For comprehensive recommendations for teacher preparation, see [BLS+12, AMT17].

1. Restructure entry-level college courses so students are asked to autonomously provide reasoning, through proofs or in the context of applications.[2] The rest of the world, including future teachers, takes our choices for these classes as a signal about the nature of our subject. Currently, they deduce that math is only about accurately reproducing procedures!

2. Develop profound understanding of the K–12 mathematics progressions in mathematics courses for future elementary and secondary teachers.

3. Provide math majors interested in teaching with opportunities to connect formal college-level mathematics with school mathematics, as being developed in multiple NSF-funded projects: the Mathematical Association of America's META Math project; the Association of Public and Land-Grant Universities' MODULES project; and the ULTRA project run by Rutgers, Columbia, and Temple Universities. At some point—perhaps masters-level or other professional preparation—such students should learn the arguments underlying the rules for arithmetic and thus algebra, as preparation for teaching algebra as reasoning and being able to appeal to or even fill in background. Knowledge of such arguments is also helpful for entry-level college teaching.

4. Encourage development of opportunities for reasoning in high-school curricula, in particular the use of algebra as a tool of reasoning, for example to establish divisibility rules or in counting problems. Reasoning about authentic applications should also be developed, and play an especially prominent role in high-school math.

5. Support K–12 educators as they move away from harmful practices such as acceleration without understanding and tracking. Students should be challenged through depth of understanding, which is achievable in mixed-ability classrooms. Tracking has negative consequences for all students, especially those from disadvantaged backgrounds. The benefits of such system shifts, as recommended by the National Council of Teachers of Mathematics [NCTM18], have been borne out by data, for example in the work of San Francisco Unified School District [BF14, SFU18].

ACKNOWLEDGMENT. I would like to thank Yvonne Lai and Jenny Ruef for their help in writing this piece.

[2] *For example, in my department we've replaced our Intermediate Algebra offerings with classes which apply algebra to mathematical modeling.*

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Dev Sinha

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# Talent Nurturing in Hungary: The Pósa Weekend Camps

*Péter Juhász*

It is a Friday afternoon in an old mansion house in the Buda Hills. We hear the chatter of high school students and parents arriving in the courtyard. It is obvious that this is not their first time here. The next weekend math camp led by Lajos Pósa is starting soon, and the air is filled with anticipation. Pósa knows the parents quite well—many of them came to the camp when they were little. They might revive old memories in a few words or possibly talk about more recent topics.

A bit later, the parents say goodbye to their children, but this is not a hard goodbye. The kids quickly go find their usual rooms and catch up with their friends, with whom they share their rooms. They know they will see their parents in about two days, and the time in between is going to fly by. The two days will include about fourteen hours of intensive thinking about math problems—which might sound frightening to many—but for them this is something they have been waiting for for the past few months.

In fact, they were not simply waiting for the camp, they were also preparing. The camps usually start with a plenary session, in which the students discuss their progress on the homework problems from the previous time. Some of these problems are easier, and they are intended for the students to practice the new ideas they learned in a different setting, but some problems are very difficult and require substantial effort, even from the most talented students. When a problem turns out to be particularly difficult, it is not uncommon that the problem will be worked on over several camps before the solution is discussed. The students are not expected to solve all the problems, but instead to think about them persistently.

After the homework discussions are finished, about five or six new problems are assigned. The students start working on these problems, but not all by themselves. They work in teams of two, three, or sometimes four, which they form themselves. Each team occupies a different room of the mansion so they do not disturb one another.

This peaceful, stress-free, and collaborative environment is one of the most important traits of the camps. In the rooms the teams and the mathematical puzzles are locked in together; they do not know what the other teams are doing; they work at their own pace. If they want, the students can work at desks, but many prefer to think in their beds, while walking around, or simply just sitting on the ground. Some groups talk quite often and some—especially in the camps for older students—communicate only in a few words. This freedom and calmness makes the camps quite special.

The teamwork itself is also far from ordinary. One would expect that, in a team, if someone solves one of the problems, they would immediately share it with the others. In the Pósa camps this is different. The central concept of these camps is the joy of thinking, so everyone should have a chance to tackle and solve the problems themselves. If this is the case, one may ask, why are there teams in the first place? In these camps, initially everyone thinks about the problems individually and in any order they want. If they solve a problem, they are supposed to keep the solution secret; only those who have thought about it but do not know the solution yet are allowed to continue discussing that problem. If anyone solves the problem during the discussion, they also stop conversing. Solving problems is a little bit like finding a way out of a dark forest. It is exciting and we all want to participate. While the way is uncertain, it is better to have company, but once we are out and have

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experienced the joy of finding the solution, it is better to let the others find the right path themselves. Initially, this version of teamwork might be strange for the students, and in seventh or eighth grade they often break the rules. But by tenth grade no one allows the others to reveal the solutions; they have already experienced how the joy comes from finding the path themselves and not simply knowing the answer. The journey is more important than the destination.

The camps are structured by alternating plenary and teamwork sessions. The plenary sessions usually include solution discussions but sometimes also tales about the history of mathematics, quizzes, and sessions where the students are supposed to ask follow-up questions. Learning to ask good questions is a central objective of the camps. The authors of good questions receive chocolate rewards, and if a question becomes part of the camp curriculum, the authors will be mentioned in the preceding story each time the problem is assigned, and thus become part of camp history. This way students learn that raising good questions is an important part of science, often more important than finding smart answers.

Every Saturday afternoon is spent in team competition. The teams have three hours to solve five problems related to the camp curriculum. Here is an example problem that often appears in eighth-grade camps:

> *A precious piece of treasure is locked up in a safe. The door of the safe is circular and there are four indentations on it. The indentations are positioned on the vertices of a square centered at the midpoint of the circular door. Each indentation hides a binary switch, which cannot be seen from the outside but its position can be identified if we put our hands in. We are standing in front of the door and we can each put our hands into one of the indentations, and we can change their setting. However, once we pull our hands out, the door senses that it has been tampered with, and it starts rotating extremely fast*

**Lajos Pósa (1947– )** stood out as a child with his special mathematical abilities. At the age of fifteen, he wrote a joint article with Pál Erdős. The Pósa condition, which guarantees the existence of a Hamiltonian cycle in a graph, is taught in many schools around the world even today. In high school he went to the same class as László Lovász and Miklós Laczkovich. His career in mathematics started wonderfully, but after acquiring his PhD degree he gradually renounced research for mathematics education.

He taught mathematics in several normal (not special math) high schools. His focus shifted toward talented students in the 1980s. Pósa organized the first weekend camp in 1988. At that time pretty much every circumstance was different. Hungary was still part of the Soviet regime; the parents helped find the location, which was different each time; Pósa usually led the camps without helpers; and there was only a single group of students. Since then the camps have become much more organized. The location—the old mansion house—is fixed. There are two groups of students per grade, which means a total of ten groups run in parallel (from seventh to eleventh grade); each gets to camp about two or three times a year.

Today we are past our 350th camp, and more than a thousand students have had a chance to take part. Some of them are mathematicians or mathematics teachers, but many have become software engineers, economists and financial analysts, and even dramaturges and archaeologists.

Nurturing mathematical talent is a great tradition in Hungary. Experts on education, such as Tamás Varga, György Pólya, or Zoltán Dienes, have left behind long-lasting legacies. We have had countrywide competitions since the beginning of the twentieth century, and since 1894 (except for the years during the World Wars) the *KöMaL* journal has appeared every month. One of the main features of *KöMaL* is a high-quality year-long mail-in competition in mathematics for high school students. Clearly, Hungary was doing quite well in terms of nurturing talent even before Pósa.

However, what Pósa has brought to the table is something completely new and revolutionary. It is still talent nurturing, but the goal is not for students to become efficient problem solvers. It is true that each year almost all the members of the Hungarian IMO team also go to the Pósa camps, but such competition results are merely a byproduct. The goal is for the students to experience the joy of thinking, which is a delicate task, and it only happens in the right kind of environment. It requires both the structure provided by Pósa's carefully constructed curriculum and the freedom given by the supportive and stress-free atmosphere characteristic of the camps. And once the students have experienced this joy, it gives them the confidence to think, to be creative, to dare to fail. All of this is essential for solving hard problems, but more importantly, for asking good questions. Asking questions is deeply embedded in the culture of the camps; in fact, it is quite likely that the most common sentence of these camps is, "What would you ask right now?" Of course the effectiveness of these principles depends highly on how they are put into practice. This is where Pósa's vision and experience matters the most, which is now further strengthened by the alumni who strive to carry on and extend his ideas so that future generations can benefit from the same introduction to the beauty of mathematics and joyful thinking.

*until it stops at possibly a different angle than before. Unless of course all four switches are in the same position, in which case the door opens immediately and we win the treasure. Can we always win the treasure in finite time?*

The goal of the competition is again not to overstress the students or to establish a ranking among them; it is only an exciting task that helps them stay motivated. This is why at the end, the only information they receive about their results is whether they did very well, in which case the team wins two bars of chocolate, or even better, which is rewarded by three bars of chocolate.

As has already been mentioned several times, a key feature of the camps is the curriculum, which was carefully worked out by Pósa himself. The problems assigned in the camps are all building blocks that depend on one another. If someone cannot come to one of the camps, they must make an effort to make up the work; otherwise they would be left behind and could not take part in the future. Missing one or two building blocks usually does not cause serious problems, but a whole camp's worth of building blocks can be a structural hazard. Since so much happens in one weekend, making up for the material is much more difficult than attending—another reason why students rarely miss the camps.

The building blocks are organized into threads; each problem builds on the previous one. The specific elements needed from each thread might differ between teams: stronger teams go faster, weaker ones need smaller steps. These threads are tied together by the key ideas in their solution (e.g., impossibility proofs, recursive approach, starting from the extreme case, the idea of motion, etc.), instead of the traditional classification (algebra, geometry, combinatorics). Finally, the "building" is stabilized by the frequent intersection of the threads. One or two particularly important problems shed light on the rarely observed phenomenon that sometimes even distant parts of mathematics have a strong connection between them.

During breaks between the mathematical sections of the camp, students play sports or choose one of the many board games available. Bughouse chess is a returning favorite. The evenings are usually spent with camp-wide games, but the students are free to spend this time as they prefer.

The students coming to the camps are the "eccentric children" in most schools, as strong mathematical interest is atypical. In these camps they can meet their intellectual peers, with whom they can build lasting friendships, and everyone feels perfectly normal in their skin. This experience is very inspiring for them, and it affects their life outside of the camp as well.

It is now a Sunday afternoon. Parents are arriving at the old mansion house. The last session is finished, and the children quickly tidy up before going home to rest. They have worked hard for three days, which they have surely enjoyed, but it must also have tired them out considerably. On the way home they sleep in the car or pose some of the problems they solved in the camp to their parents. It is not uncommon that the parents cannot solve these problems.

It will be about three or four months before they come back and the adventure will continue. The adventure introduces them to the beauty of mathematics by letting them discover it themselves.

In case the above paragraphs did not do justice to how special these camps are, we also include a couple of testimonials by former campers who later became mathematicians or mathematics teachers.

- *The long building process, the unexpected connections filled me with such pleasure that made mathematics for me similar to a form of art.*
- *This is where I learned that "real mathematics" is discovery.*
- *In the Pósa camps we learned to walk on the path of thinking. At times we stumbled and fell, but we got up and tried again. Lajos and his well-prepared helpers paved our way with fatherly care, but they never made a step for us. This was the key. They let us discover mathematics ourselves. This way we have not just received solutions to some interesting problems, but something much more important; the hidden power of thinking.*


Photo: Dávid Ferenczy / Abcug.hu

Péter Juhász

# The Banff International Research Station for Mathematical Innovation and Discovery

*Nassif Ghoussoub*

The Banff International Research Station (BIRS) was established in 2003 to address the imperatives of collaborative research and cross-disciplinary synergy. It provides an environment for creative, intense, and prolonged interactions between mathematical scientists and researchers in related areas of science and engineering. BIRS is also about multiplying opportunities, providing intellectual access, facilitating collaborative problem solving, incubating new research projects, settling intellectual controversies, and disseminating new discoveries, while also providing a forum for accelerated training, networking, and job prospecting for new generations of mathematical scientists.

The Banff Centre Pavilion where BIRS seminars and lectures are held.

While many other research-enabling institutes exist around the globe, the BIRS model offers several distinctive features: The Station provides a setting that is conducive to uninterrupted scientific interaction by way of focused, easily organized research workshops. Each year, its programs cover a broad range of the mathematical sciences and respond quickly to new and exciting developments. Even within the category of institutions that deliver in the same format, such as the MFO (Germany), CIRM (France), and AIM (USA), BIRS distinguishes itself by being distinctly multi-disciplinary, particularly receptive to emerging—possibly risky—areas of research, and by its commitment to provide equal access to all scientists worldwide through an open and highly competitive peer-reviewed process. BIRS is also a multinational collaborative effort that illustrates

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how scientific leadership can transcend not only provincial and national, but also disciplinary boundaries in the global quest to advance scientific discovery and innovation.

BIRS embraces all aspects of quantitative and analytic research. Its programs span pure, applied, computational, and industrial mathematics; statistics; and computer science. Its workshops often involve physicists, biologists, engineers, economists, and financial mathematicians. In addition, BIRS hosts leadership retreats, student modeling camps, First Nations workshops, training sessions for math Olympiad teams, and "ateliers" in scientific writing.

The retreat-like atmosphere at BIRS is ideal for ensuring a creative environment for the exchange of ideas, knowledge, and methods within the mathematical sciences and their vast array of applications in science and engineering. An added intellectual feature of the Station is its location within the site of the world-renowned Banff Centre for Arts and Creativity in Alberta, which is already internationally recognized as a place of high culture with programs in music and sound, and the written, visual, and performing arts that draw in many hundreds of artists, scholars, and intellectual leaders from around the world.

BIRS is committed to providing intellectual access and opportunities on a large scale and in an open competitive process by making a place of cutting edge research accessible to a large number of scientists. Every year, a call for proposals is sent to the international mathematical science community as BIRS provides equal access to all researchers regardless of geographic location or scientific expertise as long as it is anchored on solid mathematical, statistical, or computational grounds. Applications are selected on a competitive basis, using the criteria of excellence and relevance, by an international scientific panel of thirty experts drawn from across the entire breadth of the mathematical sciences and related areas.

BIRS's main mode of operation is to competitively select and run weekly workshops, each devoted to a specific area of high significance while involving forty-two researchers from around the world. Each workshop has its own personality, yet they all share the common objective of encouraging an atmosphere that fosters innovative ideas and a collaborative spirit. The extraordinary response to the opportunities at BIRS leads to extremely high-quality competitions: more than 250 applications for workshops were received for the 2019 program alone. This guarantees high standards of excellence, a diverse scientific program, and a multinational collaborative effort.

BIRS also hosts teams of two to four researchers for periods of one or two weeks to allow collaborative, distraction-free research and the completion of major scientific projects. The Station's setting has also been ideal for summer schools and focused collaborative research groups. The weekends are also used for two-day workshops and university-industry interactions.

BIRS is a key resource in the global effort for promoting diversity in the sciences. In particular, it addresses in its own way the challenges of increasing participation of underrepresented groups in STEM. Every year the station hosts several exclusively "Women in X" workshops. In 2019, the X will stand for Geometry, Numerical Methods for PDEs & Applications, Analysis, and Commutative Algebra. These four workshops can count on the participation of about 150 women mathematicians from all over the world. Accommodations suitable for families are available at BIRS, with special provisions for women with infants and nursing mothers. Since 2015 BIRS has provided financial support for workshop participants who are travelling with children requiring day-care services. BIRS has led the way in addressing science and education issues for aboriginal people.

In 2012, BIRS made its programs accessible to the world's scientific community in virtual space, via live video streaming, recordings, and broadcasting produced by a state-of-the-art automated production system. About 125 new videos are recorded every month and more than 10,000 of them have already been posted on the BIRS website.

BIRS "underscores how international cooperation adds up to more than what any nation could accomplish alone." These words of the former director of the NSF, Rita Colwell, go to the heart of this project, since BIRS represents a new level of scientific cooperation in North America. Indeed, a rewarding unique aspect of BIRS is that it is a joint Canada–US–Mexico initiative, which is funded by Mexico's Consejo Nacional de Ciencia y Tecnología (CONACYT), Alberta Innovation, the US National Science Foundation (NSF), and Canada's Natural Science and Engineering Research Council (NSERC). A partnership of this scale is providing new and exciting opportunities for North American faculty and graduate students, giving them access to their international colleagues at the highest levels and across all mathematical disciplines.

This collaboration reached a new milestone in 2014, when the Government of Mexico awarded an infrastructure

The current (temporary) CMO facilities in Oaxaca.

grant for the construction of an affiliated research facility in Oaxaca so that BIRS could run an additional twenty-five workshops per year. Casa Mathematica Oaxaca (CMO) was born in 2015, and the construction of a new and permanent facility there will be starting soon thanks to generous funding from CONACYT and the Universidad Nacional Autónoma de México (UNAM). Most recently, the Simons Foundation approved funding for the operating costs of several BIRS workshops at CMO. Furthermore, in a new and promising pilot program, BIRS will run ten additional workshops in 2020 jointly with the nascent Institute of Advanced Studies in Hangzhou, China. As such, the 2020 BIRS program will consist of eighty-five weekly workshops giving access to 3,500 researchers from hundreds of institutions in more than eighty countries. This groundbreaking development opens a much-needed new era for international collaborative research in the mathematical sciences.

Nassif Ghoussoub

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A survey of math in the news:

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WHAT IS...?

# a Thin Group?

*Alex Kontorovich, D. Darren Long, Alexander Lubotzky, and Alan W. Reid*

The group $\mathrm{SL}_2(\mathbb{Z})$ of $2 \times 2$ integer matrices with unit determinant is a quintessential **arithmetic group**. By this we mean that there is an **algebraic group**, that is, a variety defined by polynomial equations, namely,

$$\mathrm{SL}_2 \;:\; \{(a, b, c, d) : ad - bc - 1 = 0\},$$

whose points over a ring happen to also form a group (under standard matrix multiplication, which is a polynomial map in the entries); then $\mathrm{SL}_2(\mathbb{Z})$ is the set of integer points in this algebraic group. More generally, an arithmetic group $\Gamma$ is a finite-index subgroup of the integer points $G(\mathbb{Z})$ of an algebraic group $G$. Roughly speaking, a "thin" group is an *infinite*-index subgroup of an arithmetic group which "lives" in the same algebraic group, as explained below.

While the term "thin group"[1] was coined in the last 10–15 years by Peter Sarnak, such groups had been studied as long as 100–150 years ago; indeed, they appear naturally in the theory of Fuchsian and Kleinian groups. For a long while, they were largely discarded as "irrelevant" to arithmetic, in part because there was not much one could do with them. More recently, thin groups have become a "hot topic" thanks to the explosion of activity in "Super Approximation" (see below). Armed with this new and massive hammer, lots of previously unrelated problems in number theory, geometry, and group theory started looking like nails. Our goal here is to describe some of these nails at a basic level; for a more advanced treatment of similar topics, the reader would do well to consult [Sar14].

Let's get to the general definition from first seeing some (non-)examples. Take your favorite pair, $A, B$, say, of $2 \times 2$ matrices in $\mathrm{SL}_2(\mathbb{Z})$ and let $\Gamma = \langle A, B\rangle$ be the group generated by them; should $\Gamma$ be called thin?

**Example 1.** Suppose you choose $A = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right)$. Then, as is well known, $\Gamma$ is all of $\mathrm{SL}_2(\mathbb{Z})$. This cannot be called "thin"; it's the whole group.

**Example 2.** If you choose $A = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1\end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1\end{smallmatrix}\right)$, then the resulting $\Gamma$ is also well-known to be a **congruence group**, meaning roughly that the group is defined by congruence relations. More concretely, $\Gamma$ turns out to be the subset of $\mathrm{SL}_2(\mathbb{Z})$ of all matrices with diagonal entries congruent to $1 \pmod 4$ and evens off the diagonal; it is a good exercise to check that these congruence restrictions do indeed form a group. It is not hard to prove that the index[2] of $\Gamma$ in $\mathrm{SL}_2(\mathbb{Z})$ is 12, so just 12 cosets of $\Gamma$ will be enough to cover all of $\mathrm{SL}_2(\mathbb{Z})$; that also doesn't qualify as thin.

**Example 3.** Say you chose $A = \left(\begin{smallmatrix} 1 & 4 \\ 0 & 1\end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 1 & 6 \\ 0 & 1\end{smallmatrix}\right)$; that will generate $\Gamma = \left(\begin{smallmatrix} 1 & 2\mathbb{Z} \\ 0 & 1\end{smallmatrix}\right)$, the group of upper triangular matrices with an even upper-right entry. This group is

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[1] *Sometimes "thin matrix group," not to be confused with other notions of thin groups in the literature.*

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[2] *If the reader was expecting this index to be 6, that would be correct in* $\mathrm{PSL}_2(\mathbb{Z})$*, or alternatively, if we added* $-I$ *to* $\Gamma$.

certainly of infinite index in $\mathrm{SL}_2(\mathbb{Z})$, so now is it thin? Still no. The reason is that $\Gamma$ fails to "fill out" the algebraic variety $\mathrm{SL}_2$. That is, there are "extra" polynomial equations satisfied by $\Gamma$ besides $\det = 1$; namely, $\Gamma$ lives in the strictly smaller **unipotent** (all eigenvalues are 1) algebraic group

$$U : \{(a,b,c,d) : ad-bc-1=a-1=d-1=c=0\}.$$

The fancy way of saying this is that $U$ is the **Zariski-closure** of $\Gamma$, written

$$U = \mathrm{Zcl}(\Gamma).$$

That is, $\mathrm{Zcl}(\Gamma)$ is the algebraic group given by all polynomial equations satisfied by all elements of $\Gamma$. And if we look at the integer points of $U$, we get $U(\mathbb{Z}) = \left(\begin{smallmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{smallmatrix}\right)$, in which $\Gamma$ has finite index (namely, two). So again $\Gamma$ is not thin.

**Example 4.** Take $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 5 & 3 \\ 3 & 2 \end{smallmatrix}\right)$. This example is a little more subtle. The astute observer will notice that $B = A^2$, so $\Gamma = \langle A \rangle$, and moreover that

$$A^n = \left(\begin{smallmatrix} f_{2n+1} & f_{2n} \\ f_{2n} & f_{2n-1} \end{smallmatrix}\right),$$

where $f_n$ is the $n$th Fibonacci number, determined by $f_{n+1} = f_n + f_{n-1}$ and initialized by $f_1 = f_2 = 1$. Again it is easy to see that $\Gamma$ is an infinite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$, and unlike Example 3, all the entries are changing. But it is still not thin! Let $\phi = \frac{1+\sqrt{5}}{2}$ be the golden mean and $K$ the golden field, $K = \mathbb{Q}(\phi)$; there is a matrix $g \in \mathrm{SL}_2(K)$ which conjugates $\Gamma$ to

$$\Gamma_1 = g\Gamma g^{-1} = \{\left(\begin{smallmatrix} \phi^{2n} & 0 \\ 0 & \phi^{-2n} \end{smallmatrix}\right) : n \in \mathbb{Z}\}.$$

The latter group lives inside the "diagonal" algebraic group

$$D : \{(a,b,c,d) : b = c = ad-1 = 0\} = \{\left(\begin{smallmatrix} a & 0 \\ 0 & 1/a \end{smallmatrix}\right)\}.$$

This group is an example of what's called an **algebraic torus**; the group of complex points $D(\mathbb{C})$ is isomorphic to the multiplicative "torus" $\mathbb{C}^\times$. When we conjugate back, the torus $D$ goes to

$$D_1 = g^{-1}Dg = \mathrm{Zcl}(\Gamma);$$

the variety $D_1$ is now defined by equations with coefficients in $K$, not $\mathbb{Q}$. The rational integer points of $D_1$ are exactly $\Gamma = D_1(\mathbb{Z})$, so $\Gamma$ is not a thin group.

**Example 5.** This time, let $A = \left(\begin{smallmatrix} 1 & 4 \\ 0 & 1 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, with $\Gamma = \langle A, B \rangle$. If we replace the upper-right entry 4 in $A$ by 1, we're back to Example 1. So at first glance, perhaps this $\Gamma$ has index 4 or maybe 8 in $\mathrm{SL}_2(\mathbb{Z})$? It turns out that $\Gamma$ actually has infinite index (see, e.g., [Kon13, §4] for a gentle discussion). What is its Zariski closure? Basically the only subvarieties of $\mathrm{SL}_2$ that are also groups look, up to conjugation, like $U$ and $D$ (and $UD$), and it is easy to show that $\Gamma$ lives in no such group. More generally, any subgroup of infinite index in $\mathrm{SL}_2(\mathbb{Z})$ that is not virtually (that is, up to finite index) abelian is necessarily thin. Indeed, being non-virtually abelian rules out all possible proper sub-algebraic groups of $\mathrm{SL}_2$, implying that $\mathrm{Zcl}(\Gamma) = \mathrm{SL}_2$.

It is now a relatively simple matter to give an almost-general definition.

**Definition 6.** Let $\Gamma < \mathrm{GL}_n(\mathbb{Z})$ be a subgroup and let $G = \mathrm{Zcl}(\Gamma)$ be its Zariski closure. We say $\Gamma$ is a **thin group** if the index of $\Gamma$ in the integer points $G(\mathbb{Z})$ is infinite. (Most people add that $\Gamma$ should be finitely generated.)

For more context, we return to the classical setting of a congruence group $\Gamma < \mathrm{SL}_2(\mathbb{Z})$. Such a group acts on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$ by fractional linear transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az+b}{cz+d},$$

and much twentieth- and twenty-first-century mathematics has been devoted to the study of:

- "Automorphic forms," meaning eigenfunctions $\varphi : \mathbb{H} \to \mathbb{C}$ of the hyperbolic Laplacian $\Delta = y^2(\partial_{xx} + \partial_{yy})$ that are $\Gamma$-automorphic, that is,

$$\varphi(\gamma z) = \varphi(z),$$

  for all $\gamma \in \Gamma$ and $z \in \mathbb{H}$, and square-integrable (with respect to a certain invariant measure) on the quotient $\Gamma\backslash\mathbb{H}$. These are called "Maass forms" for Hans Maass's foundational papers in the 1940s. Their existence and abundance in the case of congruence groups is a consequence of the celebrated Selberg trace formula, developed in the 1950s.
- "$L$-functions" attached to such $\varphi$. These are certain "Dirichlet series," meaning functions of the form

$$L_\varphi(s) = \sum_{n \geq 1} \frac{a_\varphi(n)}{n^s},$$

  where $a_\varphi(n)$ is a sequence of complex numbers called the "Fourier coefficients" of $\varphi$. When $\varphi$ is also an eigenfunction of so-called "Hecke operators" and normalizing $a_\varphi(1) = 1$, these $L$-functions are also multiplicative, enjoying Euler products of the form

$$L_\varphi(s) = \prod_p \left(1 + \frac{a_\varphi(p)}{p^s} + \frac{a_\varphi(p^2)}{p^{2s}} + \cdots\right),$$

  where the product runs over primes. Needless to say, such $L$-functions are essential in modern analytic number theory, with lots of fascinating applications to primes and beyond.
- More generally, one can define related objects (called "automorphic representations") on other arithmetic groups $G(\mathbb{Z})$, and study their

$L$-functions. The transformative insight of Langlands is the conjectured interrelation of these on different groups, seen most efficiently through the study of operations on their $L$-functions. Consequences of these hypothesized interrelations include the Generalized Ramanujan and Sato-Tate Conjectures, among many, many others. We obviously have insufficient capacity to do more than graze the surface here.

Hecke, in studying Example 5, found that his theory of Hecke operators fails for thin groups, so such $L$-functions would not have Euler products,[3] and hence no direct applications to questions about primes. Worse yet, the Selberg trace formula breaks down, and there are basically no Maass forms to speak of (never mind the $L$-functions!). So for a long while, it seemed like thin groups, although abundant, did not appear particularly relevant to arithmetic problems.

About fifteen years ago, a series of stunning breakthroughs led to the theory of "Super Approximation," as described below, and for the first time allowed a certain Diophantine analysis on thin groups, from which many striking applications soon followed. To discuss these, we first describe the more classical theory of *Strong Approximation*. In very rough terms, this theory says that from a certain algebraic perspective, "thin groups are indistinguishable from their arithmetic cousins," by which we mean the following.

It is not hard[4] to show that reducing $\mathrm{SL}_2(\mathbb{Z})$ modulo a prime $p$ gives all of $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$. What happens if we reduce the group $\Gamma$ in Example 5 mod $p$? Well, for $p = 2$, we clearly have a problem, since the generator $A = \left(\begin{smallmatrix}1&4\\0&1\end{smallmatrix}\right)$ collapses to the identity. But for any other prime $p \neq 2$, the integer 4 is a unit (that is, invertible mod $p$), so some power of $A$ is congruent to $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ mod $p$. Hence on reduction mod (almost) *any* prime, we cannot distinguish Example 5 from Example 1! That is, even though $\Gamma$ in Example 5 is thin, the reduction map $\Gamma \to \mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ is onto. The Strong Approximation theorem [MVW84] says that if $\Gamma < \mathrm{SL}_n(\mathbb{Z})$ has, say, full Zariski closure $\mathrm{Zcl}(\Gamma) = \mathrm{SL}_n$, then $\Gamma \to \mathrm{SL}_n(\mathbb{Z}/p\mathbb{Z})$ is onto for all but finitely many primes $p$. In fact, this reasoning can be reversed, giving a very easy check of Zariski density: if for a single prime $p \geq 5$, the reduction of $\Gamma$ mod $p$ is all of $\mathrm{SL}_n(\mathbb{Z}/p\mathbb{Z})$, then the Zariski closure of $\Gamma$ is automatically all of $\mathrm{SL}_n$; see [Lub99] for details.

One immediate caveat is that, if one is not careful, Strong Approximation can fail. For a simple example, try finding a $\gamma \in \mathrm{GL}_2(\mathbb{Z})$ that mod 5 gives $\left(\begin{smallmatrix}1&2\\3&4\end{smallmatrix}\right)$. The problem is that $\mathrm{GL}_n(\mathbb{Z})$ does not map onto $\mathrm{GL}_n(\mathbb{Z}/p\mathbb{Z})$, since the only determinants of the former are $\pm 1$, while the latter has determinants in all of $(\mathbb{Z}/p\mathbb{Z})^\times$. But such obstructions are well understood and classical. (In fancy language, $\mathrm{GL}_n$ is reductive, while $\mathrm{SL}_n$ is semisimple.)

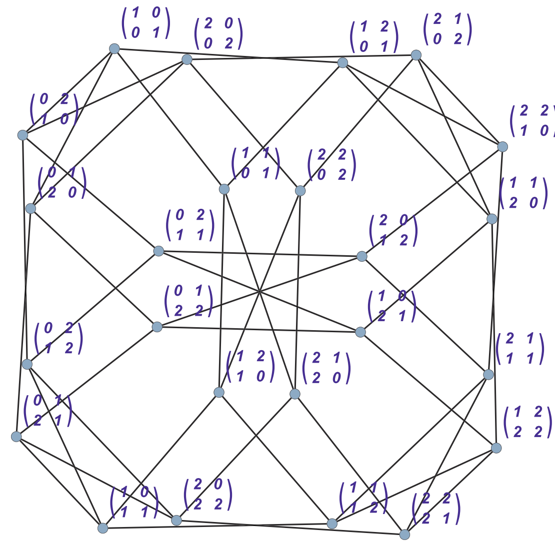

**Figure 1.** The Cayley graph with vertices $\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$ and generators $A^\pm$ and $B^\pm$ from Example 5.

For Super Approximation, we study not only *whether* these generators $A$ and $B$ in Example 5 fill out $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$, but the more refined question of *how rapidly* they do so. To quantify this question, construct for each (sufficiently large) prime $p$ the *Cayley graph*, $\mathcal{G}_p$, whose vertices are the elements of $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ and two vertices (i.e., matrices) are connected if one is sent to the other under one of the four generators $A^{\pm 1}, B^{\pm 1}$. When $p = 3$, the graph[5] is as shown in Figure 1. This is a $k$-regular graph with $k = 4$, that is, every vertex $\gamma \in \mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ has four neighbors. The "graph Laplacian" of $\mathcal{G}_p$ is the matrix $\Delta := I - \frac{1}{k}\mathcal{A}$, where $\mathcal{A}$ is the adjacency matrix of the graph. By the *spectrum* of $\mathcal{G}_p$, we mean the eigenvalues

$$\lambda_0^{(p)} \leq \lambda_1^{(p)} \leq \cdots$$

of $\Delta$. In the case of the graph above, the spectrum is:

$$\left\{0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{3}{4}, \frac{5}{4}, \cdots, \frac{1}{8}\left(7 + \sqrt{17}\right) \approx 1.39\right\}.$$

Notice that the bottom eigenvalue $\lambda_0$ is 0 (corresponding to the constant function), and has multiplicity 1; this is an instance of Strong Approximation—the graph is connected! (In general, the multiplicity of the bottom eigenvalue is the number of connected components.) Hence the first eigenvalue above the bottom, $\lambda_1^{(p)}$, is strictly positive, which by standard techniques implies that a random walk on the graph is "rapidly mixing" (see, e.g., [DSV03]).

[3] *Without Euler products, L-functions can have zeros in the region of absolute convergence; that is, the corresponding Riemann Hypothesis can fail dramatically!*

[4] *Though if you think it's completely trivial, try finding a matrix $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ whose reduction mod 5 is, say, $\left(\begin{smallmatrix}2&0\\1&3\end{smallmatrix}\right)$. The latter is indeed an element of $\mathrm{SL}_2(\mathbb{Z}/5\mathbb{Z})$, since it has determinant $6 \equiv 1 (\mathrm{mod}\, 5)$.*

[5] *This graph is begging us to identify each node $\gamma$ with $-\gamma (\mathrm{mod}\, p)$, that is, work in* $\mathrm{PSL}_2$.

But we have infinitely many graphs $\mathcal{G}_p$, one for each prime, and *a priori*, it might be the case that the mixing rate deteriorates as $p$ increases. Indeed, $\lambda_1^{(p)}$ goes from $\frac{1}{2}$ when $p = 3$ down to $\lambda_1^{(p)} \approx 0.038$ when $p = 23$, for which the graph has about $12,000$ vertices. Super Approximation is precisely the statement that this deterioration does not continue indefinitely: there exists some $\varepsilon > 0$ so that, for all sufficiently large $p$,

$$\lambda_1^{(p)} \geq \varepsilon.$$

That is, the rate of mixing is *uniform* over the entire *family* of Cayley graphs $\mathcal{G}_p$. (This is what's called an *expander* family, see [Sar04, Lub12].)

For congruence groups, Super Approximation is now a classical fact: it is a consequence of "Kazhdan's Property T" in higher rank (e.g., for groups like $\mathrm{SL}_n(\mathbb{Z})$ $n \geq 3$), and of non-trivial bounds towards the "Generalized Ramanujan Conjectures" in rank one (for example, isometry groups of hyperbolic spaces); see, e.g., [Lub10, Sar05] for an exposition. A version of Super Approximation for some more general (still arithmetic but not necessarily congruence) groups was established by Sarnak–Xue [SX91].

For thin subgroups $\Gamma < \mathrm{SL}_n(\mathbb{Z})$, major progress was made by Bourgain-Gamburd [BG08], who established Super Approximation (as formulated above) for $\mathrm{SL}_2$. This built on a sequence of striking results in Additive Combinatorics, namely the Sum-Product Theorem [BKT04] and Helfgott's Triple Product Theorem [Hel08], and prompted a slew of activity by many people (e.g., [Var12, PS16, BGT11]), culminating in an (almost) general Super Approximation theorem of Salehi–Golsefidy and Varju [SGV12].

Simultaneously, it was realized that many natural problems in number theory, groups, and geometry *require* one to treat these aspects of thin (as opposed to arithmetic) groups. Two quintessential such, discussed at length in [Kon13], are the Local-Global Problem for integral Apollonian packings [BK14b] and Zaremba's conjecture on "badly approximable" rational numbers [BK14a]. Other related problems subsequently connected to thin groups (see the exposition in [Kon16]) include McMullen's Arithmetic Chaos Conjecture and a question of Einsiedler–Lindenstrauss–Michel–Venkatesh on low-lying fundamental geodesics on the modular surface. The latter, eventually resolved in [BK17], was the catalyst for the development of the Affine Sieve [BGS10, SGS13]; see more discussion in [Kon14]. Yet a further direction was opened by the realization that the Affine Sieve can be extended to what may be called the "Group Sieve," used to great effect on problems in group theory and geometry in, e.g., [Riv08, Kow08, LLR08, LM12]. We will not rehash these topics, choosing instead to end by highlighting the difficulty of answering the slight rewording of the title:

*Can you tell... whether a given group is thin?*

**Example 7.** To ease us into a higher rank example, consider the group $\Gamma < \mathrm{SL}_3(\mathbb{Z})$ generated by

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

A moment's inspection reveals that $\Gamma$ is just a copy of $\mathrm{SL}_2(\mathbb{Z})$ (see Example 1) in the upper left $2 \times 2$ block of $\mathrm{SL}_3$. This $\Gamma$ has Zariski closure isomorphic to $\mathrm{SL}_2$, and is hence not thin.

**Example 8.** Here's a much more subtle example. Set

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & 2 & 4 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

It is not hard to show that the group $\Gamma = \langle A, B \rangle$ has full Zariski closure, $\mathrm{Zcl}(\Gamma) = \mathrm{SL}_3$. Much more striking (see [LRT11]) is that $\Gamma$ is a *faithful* representation of the "$(3, 3, 4)$ hyperbolic triangle" group

$$T = \left\langle A, B : A^3 = B^3 = (AB)^4 = 1 \right\rangle$$

into $\mathrm{SL}_3(\mathbb{Z})$; that is, the generators have these relations and no others. It then follows that $\Gamma$ is necessarily of infinite index in $\mathrm{SL}_3(\mathbb{Z})$, that is, thin.

**Example 9.** The matrices

$$A = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

generate a group $\Gamma < \mathrm{SL}_4(\mathbb{Z})$ whose Zariski closure turns out to be the symplectic group $\mathrm{Sp}(4)$. The interest in these particular matrices is that they generate the "monodromy group" of a certain (Dwork) hypergeometric equation. It was shown in [BT14] that this group is thin. For general monodromy groups, determining who is thin or not is wide open; see related work in [Ven14] and [FMS14], as well as the discussion in [Sar14, §3.5].

**Example 10.** The four matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ -2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 2 & 0 & 1 \\ 2 & 3 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -12 & -12 & 0 & -5 \end{pmatrix}$$

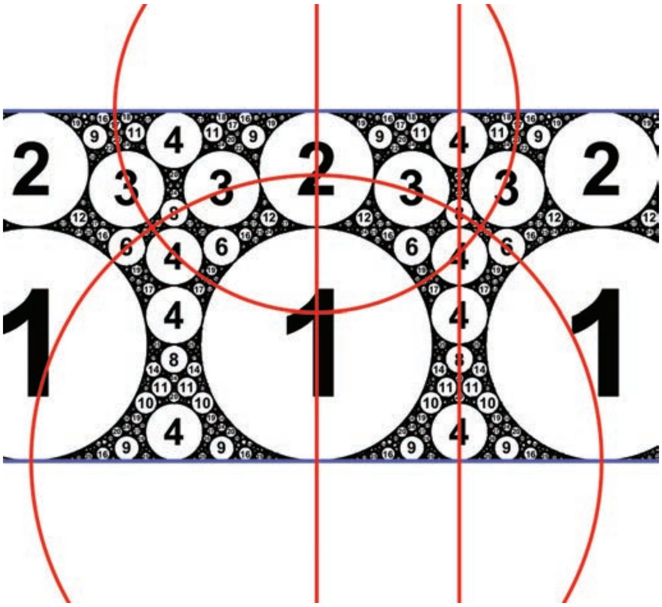

**Figure 2.** A crystallographic sphere packing.

generate a group $\Gamma < GL_4(\mathbb{Z})$. Its Zariski closure turns out to be the "automorphism group" of a certain quadratic form of signature $(3,1)$. By a standard process (see, e.g., [Kon13, p. 210]), such a $\Gamma$ acts on hyperbolic 3-space

$$\mathbb{H}^3 = \{(x_1, x_2, y) : x_j \in \mathbb{R}, y > 0\},$$

and in this action, each matrix represents inversion in a hemisphere. These inversions are shown in red in Figure 2, as is the set of limit points of a $\Gamma$ orbit (viewed in the boundary plane $\mathbb{R}^2 = \{(x_1, x_2, 0)\}$); the latter turns out to be a fractal circle packing, see Figure 2. Here circles are labeled with the reciprocal of their radii (notice these are all integers!). This limit set is an example of a "crystallographic packing," introduced (and partially classified) in [KN18] as a vast generalization of integral Apollonian circle packings. It follows from the fractal nature of this limit set that $\Gamma$ is indeed a thin group.

In a sense that can be made precise (see [LM12, Aou11, Riv10, FR17]), random subgroups of arithmetic groups are thin. But lest we leave the reader with the false impression that the theory is truly well developed and on solid ground, we demonstrate our ignorance with the following basic challenge.

**Example 11.** The following group arises naturally through certain geometric considerations in [LRT11]: let $\Gamma = \langle A, B \rangle < SL_3(\mathbb{Z})$ with

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -3 & -2 \end{pmatrix}, B = \begin{pmatrix} -2 & 0 & -1 \\ -5 & 1 & -1 \\ 3 & 0 & 1 \end{pmatrix}.$$

Reduced mod 7, this $\Gamma$ is all of $SL_3(\mathbb{Z}/7\mathbb{Z})$, so its Zariski closure is $SL_3$. Is it thin? As of this writing, nobody knows!

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## Credits

Figures 1 and 2 are by Alex Kontorovich.

# Mathematics Societies and AAAS: Natural Partners?

*Sophia D. Merow*

It is a truth universally acknowledged (by those who pause to think about it, anyway) that mathematics is "under the radar" in the American Association for the Advancement of Science (AAAS), its profile in the multidisciplinary scientific society lower than perhaps befits what Gauss called the "queen of the sciences." AAAS, likewise, goes about its science-advancing business largely unheeded by much of the mathematics community.

Which is not to say that mathematics doesn't have advocates within AAAS, and vice versa. Indeed, the would-be bridge-builders who showed up for the business meeting of the AAAS Section on Mathematics (Section A) at the Association's annual meeting in DC in February packed the assigned space so snugly that one of the morning's presenters forewent the easel paper procured for him. "There's no way I'm going to get up there without falling on somebody," he explained. During its three hours together, the capacity crowd discussed how to increase the involvement of mathematicians—and thus the visibility of mathematics—in AAAS.

## Everybody's $n^{\text{th}}$ Society (where $n$>1)

Jennifer Pearl, who directs AAAS's Science & Technology Policy Fellowships program (which she wrote about in the April 2019 *Notices*, see `https://bit.ly/2TVm0QC`), knows that AAAS can be a hard sell, and not just to mathematicians. "One of the challenges is that it's everybody's second society," she said. A specialist in ant behavior is likely to join the Entomological Society of America before AAAS; most mathematicians probably prioritize membership in at least one of MAA, SIAM, and AMS (not to mention the other professional societies under the CBMS umbrella) above AAAS membership.

Given its areas of emphasis, though, Section A Secretary Reinhard Laubenbacher believes that AAAS complements mathematical sciences professional societies nicely. "It's all about outreach, and all about policy—all important issues for the math community," he told business-meeting attendees.[1] "So AAAS is a natural partner." Former AMS president—and former Section A chair—Eric Friedlander likewise cites AAAS's advocacy activities as a reason for mathematicians to consider membership (see statement, p. 913).

## Fellows

Friedlander and the presenter who didn't risk a trip to the easel, C. T. Kelley, were elected AAAS Fellows in 2018. Of the 416 fellows in the 2018 class, six (see box below) belong to Section A. For comparison, Section U (Statistics) boasted seven fellows, Section R (Dentistry and Oral Health Sciences) five, and Section G (Biological Sciences) 112. The number of fellows allotted to each of AAAS's twenty-four

### AAAS Section on Mathematics 2018 Fellows

Eric M. Friedlander, *University of Southern California*

Ilse C. F. Ipsen, *North Carolina State University*

George Em Karniadakis, *Brown University*

C. T. Kelley, *North Carolina State University*

David E. Keyes, *King Abdullah University of Science and Technology (Saudi Arabia)*

Yi Li, *John Jay College of Criminal Justice, CUNY*

*Sophia D. Merow is Special Projects Editor and* Notices *assistant. Her email address is* merow.notices@gmail.com.



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[1] *In post-meeting correspondence with the author, Laubenbacher cautioned against losing sight of science itself as a reason for the mathematics community to ally with AAAS: "I believe strongly that an important, or maybe the most important, role this connection can play is to connect to and anchor mathematics research in the sciences, an important contributor for our continued success."*

Why belong to AAAS? One reason is to read about scientific developments in *Science* magazine and to attend broad-based scientific meetings, thereby broadening a member's intellectual interests. Another is that AAAS is an important advocate for science. With the backing of its broad membership, AAAS speaks loudly in favor of government funding of science, supports rationality in public discourse, and promotes issues of common interest to scientists such as diversity and human rights.

—**Eric Friedlander**, *Fellow of the AMS (2012), Fellow of AAAS (2018), President of AMS (2011–2012), Chair of AAAS Math Section (2015–2016)*

sections depends on section membership: Only those who have been AAAS members without lapse for the four-year period prior to nomination are eligible to be fellows, and no section's nominees in any given year can number more than 0.4 percent of its primary membership.

The 2019 Section A business meeting included presentations from fellows for the first time. Kelley told tales of applying optimization and nonlinear solvers across disciplines. Yi Li credited color blindness with getting him into mathematics. ("To be an engineer," he said, "you had to be able to tell green from red.") David Keyes capped off an "ode-like" homily on the beauty, utility, and community of mathematics with an actual (self-authored) ode titled "Anthematica." The final couplet: "At AAAS do mathematics and her partners meet | The fellowship of pilgrims here is infinitely sweet."

## Agencies Fight over These Folks

Another kind of AAAS fellow also addressed the gathering. Having received a PhD in mathematics (with an emphasis in algebra and combinatorial topology) from the University of Oregon in 2014, Tyler Kloefkorn is spending 2017–2019 as a AAAS Science and Technology Policy Fellow in the NSF's Division of Information and Intelligent Systems. Kloefkorn isn't always sure how much math to dish out when. "The other day I was trying to explain what a simplicial complex was to somebody who was trying to understand topological data science," he said. "I don't know that I was all that helpful." But he does appreciate that the fellowship allows him to explore alternatives to "very very very hard to get" assistant professor jobs.

Jennifer Pearl, herself an alumna of the program she now directs, agrees that seeing how one's mathematical knowledge and sensibility play out in a Congressional office or one of the federal agencies can open up a "completely different world" of possibilities. "There are some individuals that want to do the symplectic geometry research and stay very focused on the technical part," said Pearl, whose own dissertation work was in symplectic geometry. But the fellowship offers "opportunities for folks who want to broaden what they're looking at." Pearl's comments jibe with 2012–2013 AMS/AAAS Congressional Fellow Carla Cotwright-Williams's reasons for pursuing the fellowship (see statement below).

And the agencies and offices in DC are, to hear outgoing Section A chair Deborah Lockhart of the National Science Foundation tell it, eager to host any symplectic geometers or combinatorial topologists or the like who are keen to bring scientific expertise to the federal sector to inform better policy making. "Agencies fight over these folks," Lockhart said of fellowship applicants.

Pearl reported that in 2018–2019 there were many more offices that wanted fellows than there were qualified final-

For over 40 years, AAAS Science and Technology Policy Fellows have worked alongside congressional staffers and federal personnel helping to inform a vast number of S&T policy issues ranging from education to energy, from cybersecurity to water quality. Mathematicians apply their PhD-level analytical skills to help find solutions to the toughest problems faced by the US Congress and federal agencies.

I've had fellow mathematicians ask, "Why would you want to do that fellowship?" My response is a simple one. It is important to me to use my knowledge and skills in a different way…in a way that I might have a broader impact on the world around me. The fellowship is a unique opportunity to learn and contribute outside of some paths taken by PhD mathematicians.

Some fellows choose to remain in the federal space or private industry after the fellowship and continue in policymaking. There are fellows who return to academia with a diverse view of the needs of the federal government and the US which informs their research. They are better equipped with knowledge of the policies which make our government function and the impact these policies may have when implemented.

—**Carla Cotwright-Williams**, *2012–2013 AMS/AAAS Congressional Fellow*

**For more information about…**

AAAS membership, see `aaas.org/membership`

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AAAS annual meeting, see `meetings.aaas.org`

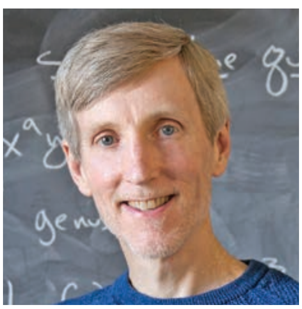
AAAS Section A, see `aaas.org/governance/section-a`

ists available. "I very much want to see more applications from the math community," she said.

## Mathematics Symposia

The AAAS Annual Meeting bills itself as "the world's largest general scientific gathering," and Section A secretary Reinhard Laubenbacher's informal polling of members of the math community indicates widespread belief that, to put it mildly hyperbolically, "there are never any symposia in mathematics at these meetings." In fact the 2019 meeting included four scientific symposia officially classified as mathematics. "If you put them together cumulatively," Laubenbacher noted, "there's a whole day's worth of mathematics you can go to." And since three of the four 2019 symposia were discussed at the 2018 Section A business meeting, generating ideas for future sessions was a priority for the 2019 convening.

Individuals submit session proposals, but each section endorses one proposal, and section endorsement vastly increases a proposal's chances of acceptance. Possible topics bandied about included the Riemann hypothesis, mathematics and gerrymandering, the mathematics of artificial intelligence, and math for social justice. Meeting attendees committed to pitching potential presenters. (See MIT mathematician Andrew Sutherland's statement, left, for his thoughts on presenting at AAAS.) Laubenbacher offered to help craft proposals,[2] expressing his aspiration for incremental growth in the visibility of mathematics at the meeting: "Hopefully next year we will have five symposia rather than four."



As we move forward into the twenty-first century, mathematicians are facing new research challenges that scientists have been dealing with for years. For example, large-scale collaborations are becoming more common in mathematics, and computational tools are playing an increasingly important role, both as generators of large data sets and tools for analyzing them. Communication with the public is another area that mathematicians have really only started to think seriously about quite recently, whereas scientists have been grappling with this issue for years.

As mathematicians, we often approach these problems from a different point of view than scientists, and in many cases we are facing them for the first time. Sharing a fresh perspective is useful for both scientists and mathematicians; we can learn from each other's experiences.

I would encourage any mathematician who is given the opportunity to consider presenting at the AAAS. We have some interesting stories to tell, and it is a great opportunity to make connections with people that you might not meet at any other venue.

—**Andrew Sutherland**, *Massachusetts Institute of Technology, presented symposium "Closing the Gap: The Polymath Project on Bounded Gaps Between Primes" at the AAAS 2016 Annual Meeting*

## Tremendous Impact

Mathematicians who would like to organize a symposium—or, to choose but one of AAAS's numerous initiatives, participate in a science communication workshop—may not be aware that the opportunity exists. The primary takeaway from the business meeting of the Section on Mathematics was that each attendee should spread the word about AAAS activities to his or her corner of the mathematics community.

A Section A web presence is in the offing, but until it goes live, information will be disseminated via other channels, such as blog posts (see, for instance, `https://bit.ly/2TToM9c`), slide presentations at regional meetings, announcements in society publications.

"Everybody should do as much as they can," Laubenbacher urged. "Collectively we could make a tremendous impact."

[2]*The deadline for proposals for the 2020 meeting has passed. No harm, however, in getting a head start on crafting a winning proposal for 2021.*

## Feed Their Curiosity

Laubenbacher (`laubenbacher@uchc.edu`) welcomes correspondence from *Notices* readers interested in any sort of involvement with AAAS and Section A. He is particularly keen for mathematicians to consider contributing to *Science* (which is published by AAAS). Many *Science* articles are heavily mathematical, but the journal seldom publishes mathematics research *per se*. Pieces about math and mathematicians do make an occasional appearance, however. "To me this indicates that the science community is curious about us," says Laubenbacher.

And why not feed that curiosity? Mathematical scientists could contribute to the journal in any number of ways, from a research article to a perspective piece to a policy forum. "Probably the most effective way at this time," says Laubenbacher, "would be to write an appropriate review of a math topic or open problem."

Sophia D. Merow

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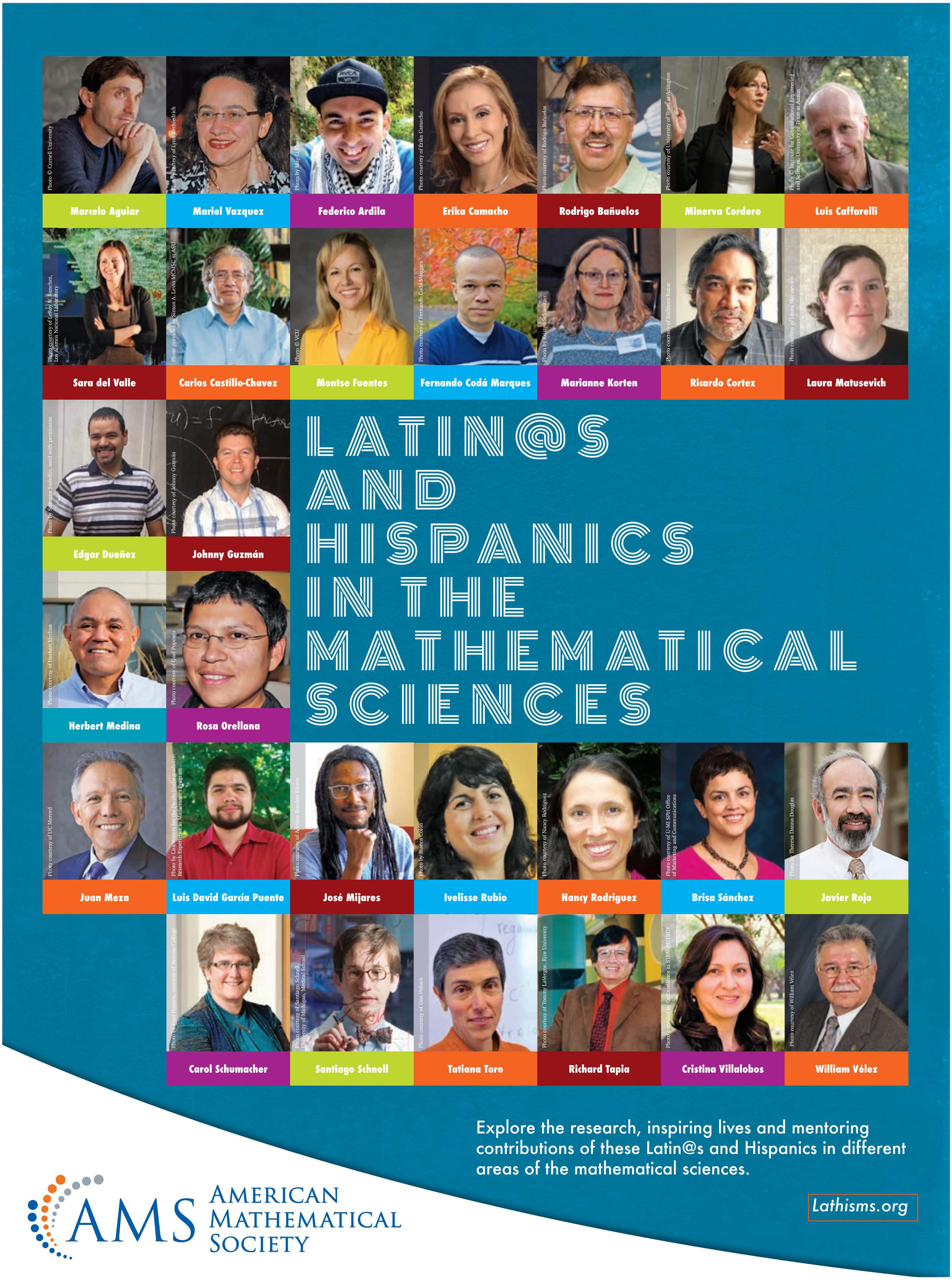
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# Ten Great Ideas about Chance

## *A Review by Mark Huber*


Princeton University Press, 2018
Hardcover, 272 pages, ISBN: 9780691174167

***Ten Great Ideas About Chance***
By Persi Diaconis and Brian Skyrms

Most people are familiar with the basic rules and formulas of probability. For instance, the chance that event *A* occurs plus the chance that *A* does not occur must add to 1. But the question of why these rules exist and what exactly probabilities are, well, that is a question often left unaddressed in probability courses ranging from the elementary to graduate level.

Persi Diaconis and Brian Skyrms have taken up these questions in *Ten Great Ideas About Chance*, a whirlwind tour through the history and philosophy of probability. Diaconis is the Mary V. Sunseri Professor of Statistics and Mathematics at Stanford University, and has long been known for his seminal work on randomization techniques such as card shuffling. Skyrms is a professor of philosophy at Stanford and Distinguished Professor of Logic and Philosophy of Science and Economics at the University of California at Irvine. His recent work has been in evolutionary game theory. The book they created is based upon a course they have taught at Stanford for the past ten years, and is aimed at a reader who has already gone through a typical undergraduate course in elementary probability.

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*Communicated by* Notices *Book Review Editor Stephan Ramon Garcia.*

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Having had Diaconis as my professor and postdoctoral advisor some two decades ago, I found the cadences and style of much of the text familiar. Throughout, the book is written in an engaging and readable way. History and philosophy are woven together throughout the chapters, which, as the title implies, are organized thematically rather than chronologically.

The story begins with the first great idea: Chance can be measured. The word *probability* itself derives from the Latin *probabilis*, used by Cicero to denote that "...which for the most part usually comes to pass" (*De inventione*, I.29.46, [2]). Even today, modern courtrooms in the United States shy away from assigning numerical values to probabilities, preferring statements such as "preponderance of the evidence" or "beyond a reasonable doubt." Those dealing with chance and the unknown are reluctant to assign an actual number to the chance that an event occurs.

The idea that chance could be measured quantitatively by a number took until the sixteenth century to arise. For instance, consider the problem of finding the probability

**Figure 1.** A six-sided die.

of rolling a 1 or a 2 on a fair six-sided die (see Figure 1). This classic problem is usually solved by assuming that each outcome of the die is equally likely. Therefore, since there are six sides and probability must sum to one, each has a one in six chance of occurring, and the probability of a 1 or 2 is two out of six, or one third.

Even in the Middle Ages, though, mathematicians such as Cardano were knowledgeable about the cheating ways of gamblers, and considered what might happen with shaved or otherwise altered dice. Gambling drove progress in probability: A historical tidbit offered up in this chapter concerns Galileo being asked by his patron to analyze a particular gambling problem to discover who had the advantage.

**Figure 2.** Blaise Pascal.

A leap forward in understanding how to measure probability came about during a correspondence between Pascal (Figure 2) and Fermat in the 1600s concerning the *problem of points* [4]. In this problem, two equally skilled opponents begin a series of games to decide who obtains a certain amount of prize money. Each winner is assigned a point, and the first to reach a fixed number (set in advance) of points wins the entire prize. Suppose they play the first game and the first player wins. At this point their match is interrupted, and so the question is: How should they split the prize money given this information?

The discussion of Pascal and Fermat is the first time the concept of expected value appears. The *expected value* is the average amount of money that each player would walk away with in their situation, and is how they should split the prize to achieve a fair division. Use of expected value enables the consistent development of probabilities where the outcomes are not all equally likely.

This was the beginning of turning chance into something numerical, something that could be dealt with mathematically rather than as a vague notion of plausibility. This path reaches its culmination in the fifth great idea: the formal definition of a probability measure within a set-theoretic framework. The definition introduced by Kolmogorov is now a standard feature of introductory texts.

These definitions allow for proving theorems and provide a set-theoretic foundation of the field, but they give little insight into why this is the proper way to calculate probabilities, or what probabilities actually are. They treat probability as a measure, but they leave unanswered the question of what exactly probability measures.

Now both of the authors are unabashed Bayesians, and upfront about this throughout the work. It is therefore unsurprising that their answers to this question lie with Bayesian philosophy. In the Bayesian approach to chance, what probabilities measure is the degree of belief by an individual that the event will occur. Given an initial set of beliefs, one can update their beliefs as new information about the events comes in. This is the center of the Bayesian philosophy.

The first Bayesian idea, and the second great idea about chance in the book, is judgment, the idea that probabilities arise from individual beliefs about how likely an event is to occur. But how can we ascertain what an individual's belief really is? And how do we derive the logical rules of probability from this point of view?

One way is to use an idea in mathematical finance. Suppose I have a security that pays \$1.00 if event $A$ occurs, and nothing if $A$ does not occur. We use $A^C$ to mean the event that $A$ does not occur, also called the complement of $A$. How much should I be willing to pay to purchase this security? Well, the expected payoff from owning this security is just the probability that event $A$ occurs. If I judge $A$ to have probability 0.47, then I should be willing to pay \$0.47 to own one share of the security (see Figure 3).

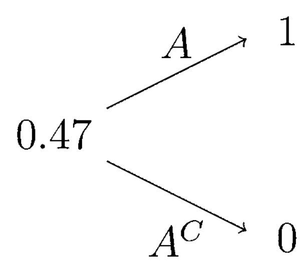


**Figure 3.** A simple security.

Now suppose I do the same thing for several events. A *Dutch book* is a portfolio of multiple securities that guarantees that the owner gains money. A *book* is the list of odds that a casino or racetrack would post before races. The person making the book is still known as a bookmaker or bookie. The origins of the Dutch part of the term are lost to history.

As an example of a Dutch book, suppose the security for $A$ costs \$0.30 and the security for $A^C$ costs \$0.60. With these prices I could buy one security that pays \$1.00 when $A$

occurs and nothing when $A^C$ occurs, plus I can buy a second security that pays \$1.00 when $A^C$ occurs and nothing when $A$ occurs. Together, these two securities form a Dutch book that costs \$0.30 + \$0.60 = \$0.90. If $A$ occurs, the first security returns \$1.00 and the second security returns nothing. If $A^C$ occurs, then the second security returns \$1.00 and the first security returns nothing. Either way, the Dutch book makes \$1.00, but at a cost of only \$0.90. Therefore I am guaranteed to receive \$0.10 regardless of the outcome. Bruno de Finetti [3] called probabilities associated with such a Dutch book *incoherent* because prices (and probability beliefs) would naturally change if this type of profit taking were possible. Either the price of $A$ or $A^C$ would rise until the possibility of a Dutch book disappeared. This kind of argument leads directly to the notion that the probability of $A$ and $A^C$ should add to 1.

The idea here is not to derive the commonly used definitions of a probability measure, but instead to explain why probabilities behave in the way that they do.

Today, these types of securities actually exist and you can buy and sell them. They are called *prediction markets*, and are a way of gauging the probabilities of future events. One such market is the PredictIt market, created by the Victoria University of Wellington to buy and sell securities based on future events. For instance, as I write this in September of 2018, the PredictIt market is giving a 36% chance that President Trump will be re-elected to his position in 2020. Those who feel the chance is higher should buy this security, while those who feel it is lower should sell.

The sixth great idea is Bayes' Theorem for updating probabilities given new information. Bayes was an English statistician and minister who created the first argument for using prior information about a parameter together with a special case of what is now known as Bayes' Theorem. There are several ways to write this theorem; one such way is by using odds. Suppose that I believe that either theory X or theory Y is true, but not both. Each has equal probability of being true. This means that I give theory X versus theory Y 1:1 odds of being true. Because this is my current belief, this is known as the *prior* in Bayesian thought.

Now suppose that I run an experiment that gives me evidence E. If theory X is true, I can calculate that there was only an 80% chance of seeing evidence E. On the other hand, if theory Y is true, then I only had a 40% chance of seeing evidence E. At this point I know the chance of E given X and E given Y. Can I use that to determine the probability of X given E and Y given E?

Note that 80% divided by 40% is 2, and so that value is called the Bayes' Factor in favor of theory X. Here is where Bayes' Theorem comes in. To find the new odds of X versus Y, take the original odds (the prior) and multiply by the Bayes' Factor. The original odds for X versus Y were 1:1 (also known as 1/1), so I multiply by the Bayes' Factor of 2 to get the new odds of 2:1 in favor of theory X. In this way, I can use experiments that I run to update the odds and also the probabilities. This updated probability is called the *posterior*.

If I then ran a second experiment, the posterior from the first experiment would become the prior for the second experiment, and so on. In this way, I could keep multiplying Bayes' Factors and updating my beliefs about the veracity of theory X versus theory Y for as long as I keep getting more evidence.

What about the practitioners who do not hold that probabilities are degrees of belief? Is it a coincidence that they use the same rules and definitions of probability? The seventh great idea provides one answer to this question. This idea is Bruno de Finetti's concept of *exchangeability*.

Exchangeability is a type of symmetry applied to sequences of random variables. For instance, consider a coin that lands heads or tails, H or T, on each flip. Then applying Bayes' Theorem, no matter what your prior is, if the coin is flipped three times, the sequences HHT, HTH, and THH should be exchangeable in the sense that each sequence is equally likely. Whatever probability each of these three sequences has (and that will depend on the prior you choose for the probability of heads), the probability of each of these three sequences will be the same.

Based on this formulation, probabilities can be derived without requiring that chance exist. Whatever probability you assign to these three sequences, you can work backward to figure out what your prior on the chance of heads must have been. In other words, if you can assign probabilities to sequences, then you are behaving as though you had a prior on the probability of heads, even if you did not derive your probabilities in that fashion. This powerful tool can be extended to other processes (such as Markov chains) that have some but not all of the total symmetry found in the independent coin flip example. See [3] for more details.

Frequentist Philosophy takes a different approach, and gets a turn as the fourth great idea of chance. In the frequentist interpretation, probability does not represent the degree of belief that something is true. Instead, there is always exactly one true probability of an event, and that can be found by repeating the identical experiment over and over again and taking the limit of the number of times the event occurs and dividing by the number of times the experiment was tried. This will converge to the probability of the event (for properly defined convergence) and is a special case of the law of large numbers proved by Jacob Bernoulli in *Ars Conjectandi* (1713) [1].

John Venn in *The Logic of Chance* (published 1866) went further, saying that the projected limit of the frequency of an event was actually the probability of the event. They are not just equal numbers: that sequence together with the limit *defines* what chance is. In this approach, I cannot speak of the probability that a single coin flip is heads; instead I must consider an infinite sequence of coin flips, all done identically and independent of one another. Only then can

I take the limit of the number of heads to the number of trials to obtain the true probability.

Of course, the deep question that leaves unaddressed is: What does that infinite sequence of independent coin flips mean? If I already have a probability for my single coin flip, I can calculate for the probability of any finite subset of the coin flips, but Venn is moving in the opposite direction, requiring that the infinite sequence exist before moving to the probability of a single coin.

Later, in 1919, Richard von Mises put forth an answer to this question, not as a direct response to Venn, but rather to David Hilbert. In 1900, Hilbert had proposed ten great problems in mathematics (he later extended these to twenty-three problems), the sixth of which was *"To treat in the same manner, by means of axioms, those physical sciences in which already today mathematics plays an important part; in the first rank are the theory of probabilities and mechanics."*

Von Mises (Figure 4) took up this challenge, and came up with something that he called a *Kollektiv*. This is a sequence that has the properties that are thought of as belonging to a random sequence of independent fair coin flips. For instance, the limiting frequency of H values should be ½, which is a first requirement that the sequence be random. However, a sequence such as HTHTHTH... has this property without being random.

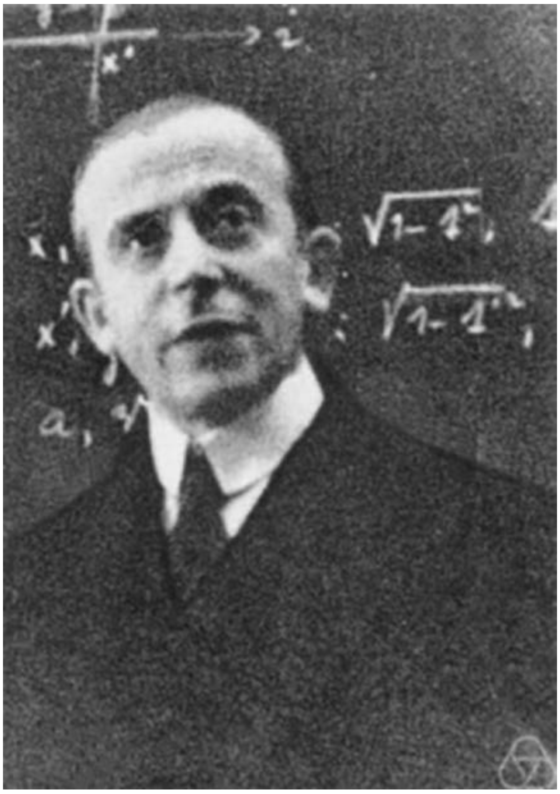

**Figure 4.** Richard von Mises.

Here the problem is if we only consider the places 1, 3, 5, 7, …, these are all H values, and 2, 4, 6, 8, … are all T values. We would like it if any subsequence of the sequence had the same ½ limiting frequency, but unfortunately any sequence with limiting frequency ½ will contain a subsequence of all H values.

The eighth great idea, algorithmic randomness, now enters the picture. In the 1930s, logicians such as Church and Turing were developing the idea of what it means to be *computable*. For instance, we could say that a sequence of integers $a_1, a_2, a_3, \ldots$. is computable if given the first *n* terms in the sequence, a Turing machine can calculate the next term in the sequence. There are only a countable number of Turing machines, and so only a countable number of such sequences exist.

It is possible to create a Kollektiv such that for any computable sequence, the limiting sequence of H and T values is always ½. However, Jean Ville found a complication in 1939. Ville found a Kollektiv such that for all computable sequences, the limit is ½. However, in the Ville Kollektiv, after a while the frequency of H values rises to be at least ½ and stays there. In a true random sequence, the frequency will rise above ½ and drop below ½ infinitely often. In particular, a gambler who was aware of Ville's property could use this fact to create a system with guaranteed returns, something that should not be possible when betting on a truly random sequence.

The remaining great ideas cover different aspects of probability. For instance, despite the laws of probability being well known and commonly understood, people often behave in defiance of these rules. The psychology of chance is the third great idea. This chapter covers the experiments of psychologists that showed that how a question is framed can lead to different decisions.

For instance, Kahneman and Tversky asked the following two questions (mixed among others) in a survey. Both questions had the same background. Suppose that the United States faces an outbreak where without intervention 600 people will die. The first question is to choose between the following two medical options:

1. A treatment where 200 people will be saved.
2. A treatment where there is a 1/3 chance that all 600 will be saved and a 2/3 chance that no one will be saved.

The second question also asked participants to choose between two different options.

1. A treatment where 400 people will die.
2. A treatment where there is a 1/3 chance that no one will die, and a 2/3 chance that 600 people will die.

They found that almost three-quarters of participants preferred the first treatment in the first question, but in the second question almost three-quarters of participants preferred the second treatment. But it only takes a short while to reflect on the questions and come to the realization that *both treatments are exactly the same*. How a question is framed can affect how we make decisions and deal with chance events.

The ninth great idea is the concept of physical randomness. This type of randomness is often the first type encountered by people as they grow up, and so often it is mistakenly thought to be the only type of randomness. Rolling dice and spinning a cage of lottery balls lead to this style of randomness. Here the final result of a die roll or a coin flip is very sensitive to initial conditions. A very slight uncertainty in the initial speed and rotation of a coin flip leads to each side being equally likely to come up in practice.

This effect appears even in simple systems like the flip of a coin, but also applies to systems such as gases where a huge number of molecules are continually changing directions and bouncing off of one another and any container. Returning to the Hilbert problems, his sixth question was driven by the need of physicists to understand what randomness is in order to create a well-defined theory of statistical mechanics.

This need grew further with the development of quantum mechanics. The Copenhagen interpretation of quantum mechanics uses probabilities as a way to connect classical thought to the many experiments that show quantum behavior. The authors argue that this does not give rise to a new theory of probability, but rather leads to the same sorts of information that can be incorporated into priors through Bayesian conditioning.

The last great idea presented is induction. The idea of induction is simply that we learn from past experience. Centuries ago, Hume laid out the basic recursive question raised by the use of induction in *A Treatise of Human Nature* [5]. How do we know that inductive reasoning works if the only way we can prove it is to use inductive reasoning? Meeting that challenge is a lens for revisiting the ideas of chance raised by Laplace, Bayes, and de Finetti earlier in the text.

All in all, this is an excellent book for a reader who already understands how to calculate probabilities. The purpose of this book is to consider the questions of what probability is, and what exactly it means. Whether or not the reader has considered these questions before, the book provides a fun and engaging introduction to some of the fascinating ways that probability has been thought about over the centuries, and would make excellent supplemental reading for a probability course.

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Mark Huber

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# Activities of the Office of Government Relations

*Karen Saxe*

This quarterly column provides information on different facets of the American Mathematical Society Office of Government Relations (OGR) portfolio and activities. This offering focuses on our activities at the Joint Mathematics Meetings and gives an overview about the new members of the 116th Congress and newly configured Congressional committees that have jurisdiction to support research and education in the mathematical sciences.

## Office of Government Relations Activities at the Joint Mathematics Meetings

The Joint Mathematics Meetings (JMM) are busy for the Office of Government Relations (OGR), which annually organizes four events.

On the day preceding the official start of the JMM, we host a workshop for department chairs.[1] This year's workshop featured four interactive sessions, encouraging networking and sharing of ideas amongst participants. Almost seventy chairs and other department leaders attended, and it was a lively and productive day. The last session of the day was a "What's on your mind?" discussion, and this proved quite interesting. Many participants wanted to focus on the changing demographics of our national undergraduate student body—what will this look like in ten or fifteen years? What about the shrinking number of eighteen-year olds? How do we address the mental health of our students? Each year, at the end of the day's activities, we hold a reception for the group and invite attendees from previous years—this is always a good time. Please join us next year in Denver on Tuesday, January 14.

The OGR works with two of the AMS policy committees—the Committee on Education and the Committee on Science Policy—helping each to organize a panel discussion.[2] This year the Committee on Education opted to hold an active session, adopting the mantra that "the audience is the panel." Organizers David Pengelley (Oregon State University), Dev Sinha (University of Oregon), and Ravi Vakil (Stanford University) redesigned the format to model this pedagogy at the center of the discussion. They recruited fifteen "discussion leaders" (rather than panelists) and officially renamed the event as a "guided discussion" on "Evidence-based teaching: How do we all get there?" as it was listed in the JMM program. The room was full the entire time—the participants were broken into small groups, and they engaged in creative problem solving on the spot. The organizers provided a bit of background and questions, which small groups then discussed, facilitated by the discussion leaders, followed by reporting out and

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[1] www.ams.org/profession/leaders/workshops/chairsworkshop



DOI: https://dx.doi.org/10.1090/noti1889

[2] www.ams.org/about-us/governance/committees/gov-committees

whole-room discussion after each small-group period. The session probed the interests of the audience in scaling up active learning in their departments, the basic steps and challenges they face in pursuing those interests, the existing resources to support those interests, and resources the audience would like to see developed to support their efforts. Some especially interesting observations and suggestions came from the early career "leading from below" group. The challenges this group feels they face are not surprising—lack of experience, lack of influence, low expectations in terms of their contributions in their departments. Their requests to overcome these challenges include offering more "on-ramps" in their department settings. One specific request asks departments to "normalize" conversations about education by including seminars and colloquia on education issues (some by distinguished visitors, just as mathematical talks are).[3]

The Committee on Science Policy decided to take advantage of the proximity to Washington, DC, by "using" their panel time to have a conversation with the National Science Foundation's (NSF) new heads of the Directorates for Mathematical & Physical Sciences (MPS) and Education & Human Resources (EHR). Dr. Anne Kinney is a PhD astrophysicist who came to NSF/MPS in January 2018. MPS supports fundamental research in astronomy, chemistry, physics, materials science, and mathematics. Dr. Karen Marrongelle holds a PhD in mathematics education and joined NSF/EHR in October 2018. EHR supports STEM education at all levels. I was to facilitate a conversation about Dr. Kinney's vision for the Division of Mathematical Sciences, Dr. Marrongelle's vision for mathematics work in EHR, and their joint views on how the mathematical sciences fit with larger programs at the NSF. Unfortunately, the NSF was closed as part of the partial government shutdown, and the session was cancelled. The NSF was shuttered because its funding is tied up in the border wall fight.[4] The shutdown has affected science in many ways: the NSF reported more than one hundred postponed panels, and there were declines in participation at many scientific conferences including at the JMM. The lingering effects of a government shutdown are felt for months, if not years.

AMS Committee on Education session at JMM.

Lastly, at each JMM we host the Congressional Fellowship Session in order to spread the word about this great opportunity to spend a year in Washington. This one-year fellowship provides a public policy learning experience, demonstrates the value of science-government interaction, and brings a technical background and external perspective to the decision-making process in Congress. Panelists this year were the current AMS Congressional Fellow James Ricci (office of Senator Amy Klobuchar, MN) and Jennifer Pearl (PhD mathematician and Director of the Science & Technology Policy Fellowships Program at the American Association for the Advancement of Science). Other alumni fellows were on hand to answer questions and share their experiences.[5] The annual application deadline for the AMS Congressional Fellowship is in mid-February.[6]

In addition to the four programs that we run, I played a role in a few other programs.

Since 2018 the AMS has hosted a Graduate Student Chapter Luncheon. In addition to the graduate students who attend, several AMS leaders (staff and elected officers) participate. This year, I was the "speaker," and talked about the importance of being involved, and how to become involved in advocacy efforts that support the mathematical community. I told them about our newly launched CASE fellowship[7] opportunity, and we had a lively debate about redistricting, including whether or not mathematicians and statisticians should play a formal role in the process.

A JMM 2019 highlight for me was the JPBM Communications Award ceremony honoring Margot Lee Shetterly, author of *Hidden Figures*. As part of *Mathemati-con*, she received her award and was interviewed by Talithia Williams of Harvey Mudd College. Each year I invite certain Congressional members to participate at the JMM; most often these are members with particular ties to mathematics and the local representatives. US Senator Chris Van Hollen joined us and gave remarks at the beginning of this event. Senator Van Hollen represents Maryland and is a cosponsor of a bill to award Congressional Gold Medals to the *Hidden Figures*: Christine Darden, Mary Jackson, Katherine Johnson, and

[3]*Slides and notes can be found at* `https://web.nmsu.edu/~davidp`.

[4]*The NSF and the potential border wall funding are both part of one of the twelve appropriations bills that package Commerce, Science, and Justice together (NSF is in science; the wall in justice). For more on how appropriations work see* `https://www.ams.org/government/dc-budgetprimer`.

[5]*I was there, and so were David Weinrich, Dan Ullman, Catherine Paolucci, and Margaret Callahan. Also, on hand were alumni Mass Media Fellows Ben Thompson and Yen Duong, and other mathematicians who have served as AAAS Fellows in the Executive Branch.*

[6]*Learn more at* `https://www.ams.org/programs/ams-fellowships/ams-aaas/ams-aaas-congressional-fellowship`.

[7]*This is a three-day workshop for students; learn more at* `https://www.ams.org/government/dc-case-fellowship`.

AMS Executive Director Catherine Roberts, *Hidden Figures* author Margot Lee Shetterly, and US Senator Chris Van Hollen (MD).

Dorothy Vaughan. The Congressional Gold Medal is the highest civilian award in the US. (An interesting piece of trivia is that the first medal went to George Washington in 1776 for his "wise and spirited conduct" in bringing about the British evacuation of Boston.[8]) The conversation between Talithia Williams and Margot Lee Shetterly was a real treat. Shetterly described what it was like working with stars Janelle Monae and Kevin Costner on the movie project. I found it interesting to learn about her next book project, which will be about early twentieth-century African American entrepreneurs in Baltimore. Williams told me that she loved hearing the author "talk so beautifully about the life and legacy of Rudy Horne," the math consultant on the movie,[9] and that Shetterly's remarks made clear "that she knew him and was intricately a part of the production of the movie, and didn't just hand her book over and step away."

Representative Jerry McNerney (CA 9),[10] a PhD mathematician and—perhaps needless to say—the only one in the US Congress, always enjoys the JMM and attends as often as his congressional schedule allows. This year he spent much of Friday with us. He went to the MAA Undergraduate Student Poster Session, spending time with students from his district in California; to talks on the mathematics behind elections; and to the Current Events Bulletin Session.

## The 2018 Election and Implications for the Mathematics Community

According to the *New York Times*, the 2018 midterm election resulted in ten new senators and 101 new members of the House of Representatives.[11] All House seats and one-third of the Senate seats are up for election every two years. All ten of the new senators graduated from college, and seven of them received graduate degrees. Almost all new House members also graduated from college, and about ninety percent of them hold graduate degrees.[12]

Seven of the newcomers in the House hold a graduate degree in a STEM field, or a medical degree:

- Sean Casten (IL 6) has his undergraduate degree in molecular biology and biochemistry, a master of engineering management, and a master of science in biochemical engineering. His career has been in

[8] https://fas.org/sgp/crs/misc/R45101.pdf

[9] *The mathematics community lost Dr. Horne in 2017:* https://www.ams.org/journals/notices/201902/rnoti-p202.pdf.

[10] *Parenthetic reference indicates that he represents California's congressional district 9, in the US House of Representatives. Senators' states are indicated in a similar manner.*

[11] https://www.nytimes.com/interactive/2018/11/28/us/politics/congress-freshman-class.html

[12] *For more on the education of members of the 116th Congress see* https://www.forbes.com/sites/michaeltnietzel/2018/12/10/the-college-profile-of-the-116th-congresss-first-year-class.

the private sector and is focused on clean energy technologies.

- Joe Cunningham (SC 1) has a bachelor's degree in ocean engineering. He worked in this field, in industry, until the 2008 recession when he returned to law school.
- Chrissy Houlahan (PA 6) has an undergraduate engineering degree from Stanford and a master of science in technology and policy from MIT. She has taught high school science and most recently worked for a nonprofit focusing on early childhood literacy in underserved populations.
- Elaine Luria (VA 2) has an undergraduate degree in physics and a master of engineering management. As an engineer, she operated nuclear reactors in the Navy.

The others are pediatrician Kim Schrier (WA 8); Lauren Underwood (IL 14), a registered nurse; and dentist Jeff van Drew (NJ 2).

Re-elected to Congress are PhD mathematician Jerry McNerney (CA 9) and PhD physicist Bill Foster (IL 11). The House lost Jacky Rosen—a computer programmer, software developer, and proponent of programs to support women and girls in science—but she won her bid for the Senate seat, so she will remain in Congress, representing Nevada.

All this is not to say that legislators with science backgrounds are our *only* allies. Congressional members can introduce legislation on any topic they want, but are most easily able to introduce legislation and promote policies through the committees on which they serve.

There are four congressional committees with power over the NSF—the "appropriating" and "authorizing" committees for each of the House and Senate. The Appropriations Committees—specifically, the Subcommittees on Commerce, Science and Justice in each chamber—decide how much money the NSF receives each year. This money, in turn, is awarded to scientists to support their research through grants. Appropriations committees are powerful, and membership on these committees is sought by many members of Congress.[13] While the Appropriations Committees in each of the House and Senate are the same in their names and subcommittee structure, this is not the case for the authorizing committees. The NSF authorizing committees are the House Committee on Science, Space, and Technology (SST) and the Senate Committee on Commerce, Science and Transportation (CST). In the House, the SST's Subcommittee on Research and Technology holds jurisdiction over the NSF, as well as university research policy and all matters relating to STEM education. In the Senate, it is the CST's Subcommittee on Science, Oceans, Fisheries, and Weather.[14]

The authorization committees provide guidance about how the NSF spends and manages its appropriated funds. In January 2017, just as he was leaving office, President Obama signed into law the most recent NSF authorizing law—the *American Innovation and Competitiveness Act*. Just to give you an idea of what sort of things are in such an authorization law, the Republicans had been pushing for Congress to have the power to determine how much money goes to each research area funded, that is, to fund the NSF "by directorate." In part due to efforts by scientists reaching out to their members of Congress and the concerted efforts of scientific society government relations staff, this particular provision did not succeed in making it to the final law, and leaves (at least for the time being) the NSF to determine for itself how to distribute funds among research areas. This effort was pushed by Lamar Smith, who retired at the end of last year but was the powerful chair of the House Committee on Science, Space, and Technology at the time the *American Innovation and Competitiveness Act* was signed into law.

Here are the leaders of the four NSF appropriating and authorizing committees (the Chair is always of the majority party, and the Ranking member is from the minority party):

**Subcommittee on Commerce, Science and Justice (of the Senate Appropriations Committee)**
Chair Jerry Moran (KS)
Ranking member Jeanne Shaheen (NH)

**Subcommittee on Commerce, Science and Justice (of the House Appropriations Committee)**
Chair José Serrano (NY 15)
Ranking member Robert Alderholt (AL 4)

**Subcommittee on Science, Oceans, Fisheries, and Weather (of the Senate Committee on Commerce, Science and Transportation)**
Chair Cory Gardner (CO)
Ranking member Tammy Baldwin (WI)

**Subcommittee on Research and Technology (of the House Committee on Science, Space, and Technology)**
Chair Haley Stevens (MI 11)
Ranking member Jim Baird (IN 4)

It is important for mathematicians to know which congressional members hold decision-making power about how much money the NSF spends and how it is spent. Lawmakers are especially interested in the concerns of their own constituents. If you have an NSF grant and are doing research that you can convince them is important, or if you

[13] *Each of the House and Senate has rules for how committee assignments are made, the structure of each committee, how many committees each member can sit on, etc. These rules are updated and evolve with each new Congress; each chamber has a website with an explanation:*

*Senate:* `https://www.senate.gov/artandhistory/history/common/briefing/Committees.htm`

*House:* `clerk.house.gov/committee_info/index.aspx`

*This site is also informative:* `https://www.congress.gov/committees`.

[14] *For more on these committees in the 116th Congress, see:* `https://blogs.ams.org/capitalcurrents/2019/02/16/which-members-of-congress-have-a-say-over-the-nsf`.

are working to engage students in STEM fields in their home district or state, they will be interested and may perhaps use your story as ammunition in arguing for Congress to fund the NSF at a robust and sustainable level in years to come. The leaders of the committees, in large part, set the priorities and agendas for the committee work, and they can push legislation that they favor.

There are, of course, other congressional committees with jurisdiction over matters of concern to many mathematicians. For example, the Senate Homeland Security and Governmental Affairs Committee has considered whether or not papers that report on federally funded research should be made available to the public for free. I wrote about "open access" in my March *Notices* column, and it is a topic to watch, perhaps especially as Plan S is implemented in Europe.[15] As another example, the House Education and Labor Committee covers topics ranging from workforce training to student financial aid in higher education to improving employment conditions for contingent faculty—those employed outside of the tenure track. The shift from a Republican-controlled House to a Democratic-controlled House means that the committees' agendas have shifted. The new Democratic leadership's ideas about how to ensure access to higher education, reduce cost, and increase graduation rates are markedly different from those of their Republican predecessors.[16] For example, the Democrats propose far more substantial investments in the Pell Grant program than do the Republicans—the Democrats support a larger maximum Pell Grant and propose to index the award amount to inflation in subsequent years. The Democrats would create a new grant program to help community colleges boost graduation and transfer rates.

I write a biweekly blog, and my posts often touch on congressional priorities and actions that might affect mathematicians and mathematics research.

Karen Saxe

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Photo of Catherine Roberts, Margot Lee Shetterly, and Chris Van Hollen is by Kate Awtrey.

Photo of Karen Saxe is courtesy of Macalester College/David Turner.

[15] *There has been much written on Plan S; National Academies of Sciences President Marcia McNutt has written an article about how it might interact with society publishers:* `https://www.pnas.org/content/early/2019/01/24/1900359116`.

[16] *See* `www.ihep.org/sites/default/files/uploads/prosper-aha_ihep_analysis_final.pdf`.

AMERICAN MATHEMATICAL SOCIETY

# 2018 Contributors


**Catherine A. Roberts**
AMS Executive Director

Dear AMS Members and Friends,

I am filled with gratitude for your remarkable generosity in supporting mathematics; the year 2018 was very special. We began to build The Next Generation Fund (NextGen), a permanent endowment to support early career mathematicians. A generous benefactor sparked this initiative by offering to match up to $1.5 million in gifts and pledges. Many of you are responding with enthusiasm to this fundraising challenge. In doing so, you are creating an enduring resource that will impact hundreds of emerging mathematicians every year at formative moments in their professional lives. Thank you!

As part of this endeavor, we introduced the Maryam Mirzakhani Fund for The Next Generation. This is a special named fund within the NextGen endowment supporting the same goals and activities as NextGen. Your donations to this fund honor Professor Mirzakhani's memory by helping the rising mathematicians of today and tomorrow.

Our donors continue to advance our strategic priorities in creative ways. Philippe Tondeur gave a significant boost to the BIG Math Network, helping to connect aspiring mathematical scientists with career opportunities in business, industry, and government. The Mary P. Dolciani Halloran Foundation funded a new AMS prize to recognize faculty members from non-doctoral mathematics departments for their active research programs and distinguished records of scholarship.

I want to acknowledge everyone who is making a legacy gift to the AMS through their estate. Your thoughtful vision will help shape mathematics well into the future. The AMS received bequests last year from the estates of Eugene and Katherine Toll, and James and Bettie Hannan, providing vital unrestricted support to AMS programs and services.

The following list includes several first time donors. We welcome you with deep appreciation for your kindness. We also celebrate our many long-time supporters who have donated to the AMS for ten years, twenty years, and longer! Your dedication fuels our mission to advance research and create connections.

You may notice our 2018 Contributors report now features an expanded Thomas S. Fiske Society list, and a special section for the Campaign for The Next Generation identifying people who made gifts and pledges to the campaign as of December 31, 2018.

Every person on the 2018 Contributors list, including those who prefer to be anonymous, helps make wonderful things happen in our mathematics community, both locally and globally. On behalf of all the people and programs benefiting from your selfless generosity, I offer you my deepest thanks.

Catherine A. Roberts
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**"The time in my working group was unlike any other experience I've had** (since second year of grad school) **and I'm extremely thankful for it!** In just a few days we were able to get enough together for at least one paper; it was time extremely well spent!"

*—Early-Career Participant in Mathematics Research Communities*

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This report reflects contributions received January 1, 2018, through December 31, 2018. Accuracy is important to us and we apologize for any errors. Please bring discrepancies to our attention by calling AMS Development at 401.455.4111 or emailing `development@ams.org`. Thank you.

# Karen Uhlenbeck Awarded Abel Prize

The Norwegian Academy of Science and Letters has awarded the Abel Prize for 2019 to **Karen Keskulla Uhlenbeck** of the University of Texas at Austin, "for her pioneering achievements in geometric partial differential equations, gauge theory and integrable systems, and for the fundamental impact of her work on analysis, geometry and mathematical physics." The Abel Prize recognizes contributions of extraordinary depth and influence in the mathematical sciences and has been awarded annually since 2003. It carries a cash award of six million Norwegian krone (approximately US$700,000).

Karen Keskulla Uhlenbeck

## Citation

Karen Keskulla Uhlenbeck is a founder of modern geometric analysis. Her perspective has permeated the field and led to some of the most dramatic advances in mathematics in the last forty years.

Geometric analysis is a field of mathematics where techniques of analysis and differential equations are interwoven with the study of geometrical and topological problems. Specifically, one studies objects such as curves, surfaces, connections, and fields, which are critical points of functionals representing geometric quantities such as energy and volume. For example, minimal surfaces are critical points of the area and harmonic maps are critical points of the Dirichlet energy. Uhlenbeck's major contributions include foundational results on minimal surfaces and harmonic maps, Yang–Mills theory, and integrable systems.

An important tool in global analysis, preceding the work of Uhlenbeck, is the Palais–Smale compactness condition. This condition, inspired by earlier work of Morse, guarantees existence of minimizers of geometric functionals and is successful in the case of 1-dimensional domains, such as closed geodesics.

Uhlenbeck realized that the condition of Palais–Smale fails in the case of surfaces due to topological reasons. The papers of Uhlenbeck, coauthored with Sacks, on the energy functional for maps of surfaces into a Riemannian manifold, have been extremely influential and describe in detail what happens when the Palais–Smale condition is violated. A minimizing sequence of mappings converges outside a finite set of singular points, and, by using rescaling arguments, they describe the behavior near the singularities as bubbles or instantons, which are the standard solutions of the minimizing map from the 2-sphere to the target manifold.

In higher dimensions, Uhlenbeck in collaboration with Schoen wrote two foundational papers on minimizing harmonic maps. They gave a profound understanding of singularities of solutions of nonlinear elliptic partial differential equations. The singular set, which in the case of surfaces consists only of isolated points, is in higher dimensions replaced by a set of codimension 3.

The methods used in these revolutionary papers are now in the standard toolbox of every geometer and analyst. They have been applied with great success in many other partial differential equations and geometric contexts. In particular, the bubbling phenomenon appears in many works in partial differential equations, in the study of the Yamabe problem, in Gromov's work on pseudoholomorphic curves, and also in physical applications of instantons, especially in string theory.

After hearing a talk by Atiyah in Chicago, Uhlenbeck became interested in gauge theory. She pioneered the study of Yang–Mills equations from a rigorous analytical point of view. Her work formed a base of all subsequent research in the area of gauge theory.

Gauge theory involves an auxiliary vector bundle over a Riemannian manifold.

The basic objects of study are connections on this vector bundle. After a choice of a trivialization (gauge), a connection can be described by a matrix valued 1-form. Yang–Mills connections are critical points of gauge-invariant functionals. Uhlenbeck addressed and solved the fundamental question of expressing Yang–Mills equations as an elliptic system, using the so-called Coulomb gauge. This was the starting point for both Uhlenbeck's celebrated compactness theorem for connections with curvature bounded in $L_p$ and for her later results on removable singularities for Yang–Mills equations defined on punctured 4-dimensional balls. The removable singularity theory for Yang–Mills equations in higher dimensions was carried out much later by Gang Tian and Terence Tao. Uhlenbeck's compactness theorem was crucial in non-Abelian Hodge theory and, in particular, in the proof of the properness of Hitchin's map and Corlette's important result on the existence of equivariant harmonic mappings.

Another major result of Uhlenbeck is her joint work with Yau on the existence of Hermitian Yang–Mills connections on stable holomorphic vector bundles over complex $n$-manifolds, generalizing an earlier result of Donaldson on complex surfaces. This result of Donaldson–Uhlenbeck–Yau links developments in differential geometry and algebraic geometry, and is a foundational result for applications of heterotic strings to particle physics.

Uhlenbeck's ideas laid the analytic foundations for the application of gauge theory to geometry and topology, to the important work of Taubes on the gluing of self-dual 4-manifolds, to the groundbreaking work of Donaldson on gauge theory and 4-dimensional topology, and many other works in this area. The book written by Uhlenbeck and Dan Freed on instantons and 4-manifold topology instructed and inspired a generation of differential geometers. She continued to work in this area, and in particular had an important result with Lesley Sibner and Robert Sibner on non-self-dual solutions to the Yang–Mills equations.

The study of integrable systems has its roots in nineteenth-century classical mechanics. Using the language of gauge theory, Uhlenbeck and Hitchin realized that harmonic mappings from surfaces to homogeneous spaces come in 1-dimensional parametrized families. Based on this observation, Uhlenbeck described algebraically harmonic mappings from spheres into Grassmannians relating them to an infinite-dimensional integrable system and Virasoro actions. This seminal work led to a series of further foundational papers by Uhlenbeck and Chuu-Lian Terng on the subject and the creation of an active and fruitful school.

The impact of Uhlenbeck's pivotal work goes beyond geometric analysis. A highly influential early article was devoted to the study of regularity theory of a system of nonlinear elliptic equations, relevant to the study of the critical map of higher order energy functionals between Riemannian manifolds. This work extends previous results by Nash, De Giorgi, and Moser on regularity of solutions of single nonlinear equations to solutions of systems.

Karen Uhlenbeck's pioneering results have had fundamental impact on contemporary analysis, geometry, and mathematical physics, and her ideas and leadership have transformed the mathematical landscape as a whole.

## Biographical Sketch

The following is taken from a biography written by Jim Al-Khalili and published on the website `www.abelprize.no/c73996/binfil/download.php?tid=74122`.

"Karen Keskulla Uhlenbeck, the eldest of four children, was born in Cleveland, Ohio, in 1942. Her father, Arnold Keskulla, was an engineer, and her mother, Carolyn Windeler Keskulla, an artist and school teacher. The family moved to New Jersey when Karen was in third grade. As a young girl, she was curious about everything. Her parents instilled in her a love of art and music, and she developed a lifelong love of the outdoors, regularly roaming the local countryside near her home.

"Most of all, she loved reading, shutting herself away whenever she could to devour advanced science books, staying up late at night and even reading secretly in class. She dreamed of becoming a research scientist, particularly if it meant avoiding too much interaction with other people; not that she was a shy child, but rather because she enjoyed the peace and solitude of her own company. The last thing she wanted to do was to follow in her mother's footsteps and end up teaching—an attitude that would change dramatically later in life.

"Uhlenbeck's love affair with mathematics developed only after she had started at university. Having been inspired in high school by the writings of great physicists such as Fred Hoyle and George Gamow, she enrolled at the University of Michigan, initially planning to major in physics. However, she soon discovered that the intellectual challenge of pure mathematics was what really excited her. It also meant she didn't have to do any lab work, which she disliked.

"Graduating in 1964, she married her biophysicist boyfriend Olke Uhlenbeck a year later and decided to embark on postgraduate study. Already well aware of the predominantly male and often misogynistic culture in academia, she avoided applying to prestigious schools such as Harvard, where Olke was heading for his PhD and where competition to succeed was likely to be fierce. Instead, she enrolled at Brandeis University where she received a generous graduate fellowship from the National Science Foundation. There, she completed her PhD in mathematics [under Richard Palais], working on the calculus of variations; a technique that involves the study of how small changes in one quantity can help us find the maximum or minimum value of another quantity—like finding the shortest distance between two points. You might think this would be a straight line, but it is not always so straightforward. For example, if you have to drive through a busy city, the quickest route is not necessarily the shortest. Needless to say, Uhlenbeck's contribution to the field was somewhat more complicated than this!

"After a brief teaching period at MIT, she moved to Berkeley, California, where she studied general relativity and the geometry of space-time—topics that would shape her future research work. Although a pure mathematician, Uhlenbeck has drawn inspiration for her work from theoretical physics and, in return, she has had a major influence in shaping it by developing ideas with a wide range of different applications.

"For example, physicists had predicted the existence of mathematical objects called instantons, which describe the behavior of surfaces in four-dimensional space-time. Uhlenbeck became one of the world's leading experts in this field. The classic textbook *Instantons and 4-Manifolds*, which she cowrote in 1984 with Dan Freed, inspired a whole generation of mathematicians.

"In 1971, she became an assistant professor at the University of Illinois at Urbana-Champaign, where she felt isolated and undervalued. So, five years later she left for the University of Illinois at Chicago. Here, there were other female professors, who offered advice and support, as well as other mathematicians who took her work more seriously. In 1983, she took up a full professorship at the University of Chicago, establishing herself as one of the preeminent mathematicians of her generation. Her interests included nonlinear partial differential equations, differential geometry, gauge theory, topological quantum field theory, and integrable systems. In 1987, she moved to the University of Texas at Austin to take up the Sid W. Richardson Foundation Regents' Chair in mathematics. There, she broadened her understanding of physics by studying with Nobel Prize–winning physicist Steven Weinberg. She would remain at the University of Texas until the end of her working career.

"Uhlenbeck's most noted work focused on gauge theories. Her papers analyzed the Yang–Mills equations in four dimensions, laying some of the analytical groundwork for many of the most exciting ideas in modern physics, from the standard model of particle physics to the search for a theory of quantum gravity. Her papers also inspired mathematicians Cliff Taubes and Simon Donaldson, paving the way for the work that won Donaldson the Fields Medal in 1986.

"Uhlenbeck, now back in New Jersey, remains a staunch advocate for greater gender diversity in mathematics and in science. She has come a long way from the young girl who wished to be alone. For a while, she struggled to come to terms with her own success, but now says she appreciates it as a privilege. She has stated that she is aware of being a role model, for young female mathematicians in particular, but that 'it's hard, because what you really need to do is show students how imperfect people can be and still succeed. Everyone knows that if people are smart, funny, pretty, or well-dressed they will succeed. But it's also possible to succeed with all of your imperfections. I may be a wonderful mathematician and famous because of it, but I'm also very human.' Karen Uhlenbeck is certainly a remarkable human."

Karen Uhlenbeck received the National Medal of Science in 2000 and the AMS Steele Prize for Seminal Contribution to Research in 2007. She is a former MacArthur and Guggenheim Fellow and is a Fellow of the American Academy of Arts and Sciences and a member of the inaugural class of AMS Fellows. She gave the AWM Noether Lecture in 1988 and became the second woman after Emmy Noether to give a plenary lecture at the International Congress of Mathematicians in 1990. She is the first woman mathematician to be elected to the National Academy of Sciences (1986). She is currently a visiting senior research scholar at Princeton University and a visiting associate at the Institute for Advanced Study.

AMS President Jill C. Pipher said: "On behalf of the American Mathematical Society, it is my great pleasure to congratulate Professor Karen Uhlenbeck, recipient of the 2019 Abel Prize. Professor Uhlenbeck has made legendary advances in several fields of mathematics. Her early groundbreaking work on harmonic maps gave rise to a new field, geometric analysis. Her analysis via gauge theory of solutions of Yang–Mills equations had and will continue to have a profound influence on all future work in this field. She transformed the fields of geometry and analysis, crossing boundaries and making deep discoveries at the interfaces."

Read more about Uhlenbeck's life and work, including "A Glimpse of the Laureate's Work" by Arne B. Sletsjøe, as well as a list of previous recipients of the Abel Prize, at `www.abelprize.no/c73996/binfil/download.php?tid=74122`. See also the article "Karen Uhlenbeck and the Calculus of Variations," *Notices of the American Mathematical Society*, March 2019.

## About the Prize

The Niels Henrik Abel Memorial Fund was established in 2002 to award the Abel Prize for outstanding scientific work in the field of mathematics. The prize is awarded by the Norwegian Academy of Science and Letters, and the choice of Abel Laureate is based on the recommendation of the Abel Committee, which consists of five internationally recognized research scientists in the field of mathematics. The Committee is appointed for a period of two years.

*—From announcements of the Norwegian Academy of Science and Letters*

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Photo of Karen Keskulla Uhlenbeck is courtesy of the Institute for Advanced Study.

NEWS

# Mathematics People

## Braverman Receives NSF Waterman Award


Mark Braverman

**Mark Braverman** of Princeton University has been selected as a cowinner of the 2019 Alan T. Waterman Award of the National Science Foundation (NSF) for his work in complexity theory, algorithms, and the limits of what is possible computationally. According to the prize citation, his work "focuses on complexity, including looking at algorithms for optimization, which, when applied, might mean planning a route—how to get from point A to point B in the most efficient way possible.

"Algorithms are everywhere. Most people know that every time someone uses a computer, algorithms are at work. But they also occur in nature. Braverman examines randomness in the motion of objects, down to the erratic movement of particles in a fluid.

"His work is also tied to algorithms required for learning, which serve as building blocks to artificial intelligence, and has even had implications for the foundations of quantum computing.

"Braverman's work includes mechanism design with applications in health care. His multidisciplinary approach is developing algorithms to address issues such as a new way to match medical residents to US hospitals and ways to implement new incentive structures in health insurance.

"Braverman has solved two puzzles that eluded researchers for decades: the Grothendieck constant and the Linial-Nisan conjecture."

Braverman received his PhD from the University of Toronto in 2008. He served on the faculty of the University of Toronto until joining Princeton in 2011. His honors include an NSF CAREER Award (2012), a Packard Fellowship (2013), the Stephen Smale Prize (2014), and the Presburger Award of the European Association for Theoretical Computer Science (2016).

The Waterman Award annually recognizes an outstanding young researcher in any field of science or engineering supported by NSF. Researchers forty years of age or younger, or up to ten years post-PhD, are eligible. Awardees receive US$1 million distributed over five years.

*—From an NSF announcement*

## Prizes of the Association for Women in Mathematics

The Association for Women in Mathematics (AWM) has awarded a number of prizes in 2019.


Catherine Sulem

**Catherine Sulem** of the University of Toronto has been named the Sonia Kovalevsky Lecturer for 2019 by the Association for Women in Mathematics (AWM) and the Society for Industrial and Applied Mathematics (SIAM). The citation states: "Sulem is a prominent applied mathematician working in the area of nonlinear analysis and partial differential equations. She has specialized on the topic of singularity development in solutions of the nonlinear Schrödinger equation (NLS), on the problem of free surface water waves, and on Hamiltonian partial differential equations. Her work on the subtle $1/\sqrt{2t\log(\log t)}$ behavior of $H^1$-solutions of the NLS at the singularity time resolved a major outstanding scientific question. Her book on the NLS is the central and most highly cited reference monograph in the field. Her continuing work on the problem of water waves, their time evolution, and their approximation by model dispersive equations is opening new territory, both in studies of wave propagation and in the analysis of the Euler equations." Sulem received her PhD from the University of Paris-Nord under the direction of Claude Bardos and held positions with CNRS and Ben Gurion University before joining the faculty at Toronto. She is a recipient of the Krieger-Nelson Prize of the Canadian Mathematical Society and of a Simons Foundation Fellowship and is a Fellow of the AMS and of the Royal Society of Canada. She is also an accomplished violinist who performs regularly in small ensembles and local orchestras. She will deliver the Kovalevsky Lecture at the 2019 ICIAM meeting in Valencia, Spain.

Anna Skripka

**Anna Skripka** of the University of New Mexico was awarded the Ruth I. Michler Memorial Prize of the AWM for the years 2019–2021. She proposes to connect some of her recent work in noncommutative analysis with the research of Michael Nussbaum of Cornell University on statistical problems of estimation, regression, and asymptotic analysis. She also hopes that interactions with other Cornell faculty members will enhance her research program. Skripka received her PhD from the University of Missouri in 2007 under the direction of Konstantin A. Makarov. She worked at Texas A&M University and the University of Central Florida before joining the faculty at New Mexico. She held invited positions at the University of California at Berkeley, Université de Franche-Comté, Besançon, and the University of New South Wales. Skripka has been awarded four single-investigator NSF awards, including a CAREER award. The Michler Prize grants a mid-career woman in academia a residential fellowship in the Cornell University mathematics department without teaching obligations.

Michelle Snider

**Michelle Snider** of the Institute for Defense Analyses (IDA) Center for Computing Sciences has been selected to receive the 2019 Service Award "for her concern, care and tireless efforts to bring AWM's message to members through aiding in building the new AWM website. Snider launched herself into learning a new piece of software, problem solving, as well as teaching and working with the rest of the website team, so as to create a communication channel for women to feel connected and to find their place in the mathematics community. Snider's goals for a renovated AWM website were for each page to communicate values, goals, and a vision for this important organization." Snider received her PhD from Cornell University in 2011 and is presently a research staff member at IDA. The prize is given for helping to promote and support women in mathematics through exceptional voluntary service to the Association for Women in Mathematics.

—*From AWM announcements*

# Prizes of the Canadian Mathematical Society

Jeremy Quastel

**Jeremy Quastel** of the University of Toronto has been awarded the 2019 Jeffery–Williams Prize for Research Excellence of the Canadian Mathematical Society (CMS) for his exceptional contributions to mathematics research. The citation reads: "Dr. Quastel is awarded the 2019 Jeffery-Williams prize for his ground-breaking results in probability and non-equilibrium statistical mechanics, in particular, his recent discovery with Matetski and Remenik of the complete integrability of TASEP, and through a scaling limit, the strong coupling fixed point of the KPZ universality class. The class contains random interface growth models and directed polymer free energies. An example is the famous Kardar–Parisi–Zhang non-linear stochastic partial differential equation, which gives the class its name; TASEP is its most popular discretization. The KPZ fixed point is expected to describe the universal long time large scale fluctuations for all such systems." Quastel received his undergraduate degree from McGill University and his PhD from the Courant Institute in 1990 under the direction of S. R. S. Varadhan. He is professor and chair of the Department of Mathematics at the University of Toronto, where he has taught since 1998. He received the CRM–Fields–PIMS prize in 2018 and is a Fellow of the Royal Society of Canada. The prize recognizes mathematicians who have made outstanding contributions to mathematical research.

Jacob Tsimerman

**Jacob Tsimerman** of the University of Toronto has been awarded the 2019 Coxeter–James Prize for his exceptional contributions to mathematics research. The prize citation reads in part: "His work is a mixture of transcendence theory, analytic number theory, and arithmetic geometry. Early in his career, Dr. Tsimerman obtained remarkable results related to the Andre-Oort conjecture. This conjecture is concerned with the behavior of collections of special points inside Shimura varieties. Dr. Tsimerman made several breakthrough advancements towards a proof of the conjecture and removing unnecessary (Riemann) hypothesis conditions. He built a reputation for his creativity and insight in this area." Tsimerman was born in Kazan, Russia, and received his PhD in pure mathematics from Princeton University in 2011 under the supervision of

Peter Sarnak, after which he held a postdoctoral position at Harvard University. He won gold medals in the International Mathematical Olympiad (IMO) in both 2003 and 2004 (perfect score), and his honors include the SASTRA Ramanujan Prize (2015) and the André Aisenstadt Prize (2017). He was an invited speaker at the 2018 International Congress of Mathematicians. The prize recognizes young mathematicians who have made outstanding contributions to mathematical research.


Julia Gordon

**Julia Gordon** of the University of British Columbia has been awarded the 2019 Krieger–Nelson Prize "for her exceptional contributions to mathematics research." The prize citation reads: "Julia Gordon works in representation theory of $p$-adic groups related to the Langlands Program, and motivic integration. In many of her results, she applies model theory (specifically, motivic integration) to arithmetic questions. In rough terms, motivic integration makes it possible to do integration on $p$-adic fields uniformly in $p$. With Raf Cluckers and Immanuel Halupczok, Gordon used this technique to prove uniform estimates on orbital integrals that have an application in the study of $L$-functions." Gordon received her PhD at the University of Michigan in 2003 under the supervision of Thomas Hales. She was a Fields Institute Postdoctoral Fellow in 2003 and a University of Toronto Postdoctoral Fellow from 2004 to 2006. She received the Michler Prize of AWM and Cornell University in 2017. The Krieger–Nelson Prize recognizes outstanding contributions in the area of mathematical research by a woman mathematician.


Andrea Fraser

**Andrea Fraser** of Dalhousie University has been awarded the 2019 Excellence in Teaching Award. The prize citation states in part: "There is an overwhelming amount of positive student feedback that speaks to Dr. Fraser's dedication and commitment to student success, and to the originality and exceptional clarity of her presentation. Students praise her ability to make difficult concepts easy and intuitive, and her lecturing style, which makes students feel they are 'discovering' the material.... Dr. Fraser's innovation in designing courses extends to the development of textbooks that contain stimulating visuals and clear explanations, and sets her apart as an outstanding instructor of mathematics." Fraser received her PhD from Princeton University in 1997 under Elias M. Stein. After a four-year lecturer position at the University of New South Wales in Australia, she returned to Canada, where she has been a faculty member at Dalhousie University since 2001. Her research interests include multiplier operators and analysis on the Heisenberg group. She tells the *Notices*: "I am an avid hiker, and enjoy kayaking and windsurfing. I also paint landscapes en plein air, a pursuit I started while living on the spectacular coastline in Sydney, Australia, during the time I was a research associate at UNSW."

*—From CMS announcements*

# Bertsekas and Tsitsiklis Awarded 2018 von Neumann Theory Prize


Dimitri P. Bertsekas


John N.Tsitsiklis

**Dimitri P. Bertsekas** and **John N. Tsitsiklis**, both of the Massachusetts Institute of Technology, have been awarded the 2018 Institute for Operations Research and the Management Sciences (INFORMS) John von Neumann Theory Prize "for contributions to parallel and distributed computation as well as neurodynamic programming."

The prize citation reads: "Working together and independently, Bertsekas and Tsitsiklis have made seminal contributions to both these fields. They unified ideas and built solid theoretical foundations while these fields were still relatively nascent, thus greatly enhancing subsequent development of rigorous theory.

"Their monograph *Parallel and Distributed Computation: Numerical Methods* represents a significant achievement in the field. The work builds on and extends the authors' extensive previous work in this area, identifying the tolerance of algorithms to asynchronous implementations and a number of positive convergence results. An antecedent work of particular significance to the operations research community is the paper by Tsitsiklis, Bertsekas, and Athans, which provides seminal analysis of asynchronous implementations of deterministic and stochastic gradient algorithms. This line of inquiry has recently found application in the analysis of descent algorithms for neural network training and other machine learning problems. Their work in distributed computation has also had significant impact on the areas of distributed network control and distributed detection.

"Their monograph *Neuro-Dynamic Programming* helped provide a unified theoretical treatment of the wide variety of reinforcement learning algorithms by building connections to the dynamic programming and distributed computation literature. This has proven extremely valuable in bringing theoretical rigor to a field of rapid, empirical innovation. The authors' contributions in this area go beyond providing a theoretical foundation that others could build on. The authors have made significant original contributions to value function learning, temporal difference methods and actor-critic algorithms.

"The work of Bertsekas and Tsitsiklis is characterized by its innovation, depth, and clarity, and it has had tremendous impact, as evident from the large number of citations. Their two joint monographs are among their individual five most cited works, making the award of a joint prize particularly appropriate. Bertsekas and Tsitsiklis have brought the fields of computer science and operations research closer together through unifying theory."

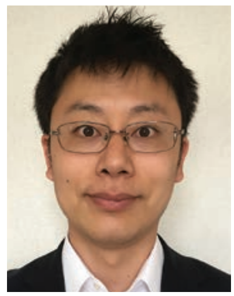
Dimitri Bertsekas was born in Athens, Greece, and received his PhD in system science from the Massachusetts Institute of Technology in 1971. Before joining the MIT faculty he taught at Stanford University and the University of Illinois at Urbana–Champaign. He is the author or coauthor of sixteen textbooks and monographs. Among his honors are the 2014 INFORMS Khachiyan Prize and the 2015 Dantzig Prize of the Society for Industrial and Applied Mathematics (SIAM) and the Mathematical Optimization Society (MOS). He was elected to the US National Academy of Engineering in 2001. He tells the *Notices*: "I remember the periods I spent researching and writing *Parallel and Distributed Computation* and *Neurodynamic Programming* as among the most exciting of my career. Both books share the characteristic that they were the first to focus on speculative fields of marginal interest at the time they were written, only to emerge as major research areas twenty years later. Asynchronous distributed algorithms became a major subject of continuing interest in machine learning in the late 2000s, while neurodynamic programming, essentially a synonym for reinforcement learning, is currently of great interest in artificial intelligence. Sharing the journey with my longtime research collaborator and friend John Tsitsiklis added greatly to this memorable experience."

John Tsitsiklis was born in Thessaloniki, Greece, and received his PhD from the Massachusetts Institute of Technology in 1984. After a year at Stanford University, he joined the MIT faculty in 1984. He currently serves as the director of the Laboratory for Information and Decision Systems and is affiliated with the Institute for Data, Systems, and Society (IDSS), the Statistics and Data Science Center, and the MIT Operations Research Center. His honors include the 1997 ICS Prize, the ACM SIGMETRICS Achievement Award (2016), and the IEEE Control Systems Award (2018). He is a member of the National Academy of Engineering and a Fellow of the IEEE and INFORMS. He tells the *Notices* that, while growing up in Greece, his hobbies were Euclidean geometry and skiing in the Greek mountains. These days, skiing has been replaced by rock climbing, his favorite outdoor activity.

*—From an INFORMS announcement*

# Prizes of the Mathematical Society of Japan

The Mathematical Society of Japan (MSJ) has awarded several prizes for 2019.



Yasunori Maekawa

**Yasunori Maekawa** of Kyoto University was awarded the MSJ Spring Prize for "outstanding contributions to new developments for mathematical analysis of fluid mechanics." The Spring Prize and the Autumn Prize are the most prestigious prizes awarded by the MSJ to its members. The Spring Prize is awarded to those under the age of forty who have obtained outstanding mathematical results.

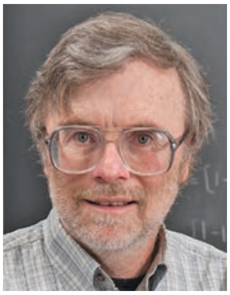
The Algebra Prizes were awarded to **Shinichi Kobayashi** of Kyushu University for contributions to the Iwasawa theory of elliptic curves and to **Shunsuke Takagi** of the University of Tokyo for work on singularities in characteristic zero and $F$-singularities.

The Outstanding Paper Prizes, given for papers published in the *Journal of the Mathematical Society of Japan*, were awarded to the following: **Masato Tsujii**, Kyushu University, for "Exponential Mixing for Generic Volume-Preserving Anosov Flows in Dimension Three," **70** (2018), no. 2; **Xun Yu**, Tianjin University, for "Elliptic Fibrations on $K3$ Surfaces and Salem Numbers of Maximal Degree" **70**, no. 3; and **Akito Futaki**, University of Tokyo and Tsinghua University, and **Hajime Ono** for "Volume Minimization and Conformally Kähler, Einstein–Maxwell Geometry," **70**, no. 4.

*—From MSJ announcements*

NEWS

# Lawler and Le Gall Awarded 2019 Wolf Prize



Gregory F. Lawler

Jean-François Le Gall

**Gregory F. Lawler** of the University of Chicago and **Jean-François Le Gall** of Université Paris-Sud Orsay have been awarded the Wolf Foundation Prize for Mathematics for 2019 by the Wolf Foundation. Lawler was honored "for his comprehensive and pioneering research on erased loops and random walks," and Le Gall was selected "for his profound and elegant works on stochastic processes." According to the prize citation, "the work undertaken by these two mathematicians on random processes and probability, which [has] been recognized by multiple prizes, became the stepping stone for many consequent breakthroughs."

## Citation

The prize citation for Lawler reads as follows: "Gregory Lawler has made trailblazing contributions to the development of probability theory. He obtained outstanding results regarding a number of properties of Brownian motion, such as cover times, intersection exponents, and dimensions of various subsets. Studying random curves, Lawler introduced a now-classical model, the Loop-Erased Random Walk (LERW), and established many of its properties. While simple to define, it turned out to be of a fundamental nature, and was shown to be related to uniform spanning trees and dimer tilings. This work formed much of the foundation for a great number of spectacular breakthroughs, which followed Oded Schramm's introduction of the SLE curves. Lawler, Schramm, and Werner calculated Brownian intersection exponents, proved Mandelbrot's conjecture that the Brownian frontier has Hausdorff dimension 4/3, and established that the LERW has a conformally invariant scaling limit. These results, in turn, paved the way for further exciting progress by Lawler and others."

The prize citation for Le Gall states that he "has been at the forefront of probability since 1983, when he established what still are the best results on pathwise uniqueness for one-dimensional stochastic differential equations. His current groundbreaking discoveries on the Brownian map ensure he remains at the cutting edge of the field today.

"Jean-François Le Gall has made several deep and elegant contributions to the theory of stochastic processes. His work on the fine properties of Brownian motions solved many difficult problems, such as the characterization of sets visited multiple times and the behavior of the volume of its neighborhood—the Brownian sausage. Le Gall made groundbreaking advances in the theory of branching processes, which arise in many applications. In particular, his introduction of the Brownian snake and his studies of its properties revolutionized the theory of super-processes—generalizations of Markov processes to an evolving cloud of dying and splitting particles. He then used some of these tools for achieving a spectacular breakthrough in the mathematical understanding of 2D quantum gravity. Le Gall established the convergence of uniform planar maps to a canonical random metric object, the Brownian map, and showed that it almost surely has Hausdorff dimension 4 and is homeomorphic to the 2-sphere."

## Biographical Notes

Gregory Lawler was born in Alexandria, Virginia, in 1955 and received his PhD from Princeton University in 1979 under the direction of Edward Nelson. He was a faculty member at Duke University from 1979 to 2001 and at Cornell University from 2001 to 2006 before joining the University of Chicago in 2006. With Oded Schramm and Wendelin Werner, he was a corecipient of the George Pólya Prize of the Society for Industrial and Applied Mathematics (SIAM) in 2006. He was a member of the Inaugural Class of AMS Fellows in 2012 and is also a Fellow of the American Academy of Arts and Sciences, the Alfred P. Sloan Foundation, and the Institute of Mathematical Statistics. He was elected to the National Academy of Sciences in 2013. He has authored or coauthored six books. He served as editor-in-chief of the *Annals of Probability* from 2006 to 2008 and was an editor of the *Journal of the American Mathematical Society* from 2009 to 2013. He cofounded the *Electronic Journal of Probability* in 1995 and served as its coeditor until 1999.

Professor Lawler studies random walks, especially strongly interacting walks "with memory" that arise in critical phenomena in statistical physics. He introduced the loop-erased random walk, which is one of the important models in the field. With Oded Schramm and Wendelin Werner, he developed the theory of the Schramm-Loewner Evolution (SLE) as a continuum limit of two-dimensional random curves. This machinery, along with Lawler's earlier work relating intersection exponents for Brownian motion with fractal properties of curve, proved a conjecture of Benoit Mandelbrot that the Hausdorff dimension of the Brownian coastline is 4/3.

Besides his research, Lawler has been involved in the tournament bridge world in investigations of cheating among top competitors, especially the use of statistics to verify allegations. When not doing math, he plays guitar and is in charge of music at the Beverly Unitarian Church in Chicago. In summers he plays on the mathematics department softball team at University of Chicago.

Jean-François Le Gall was born in 1959 in Morlaix, France, and received his PhD from the Université Pierre et Marie Curie (Paris VI) in 1982 under the direction of Marc Yor. After twenty years on the faculty of Université Pierre et Marie Curie, he joined the Université Paris-Sud Orsay in 2007. He received the Rollo Davidson Prize in 1986, the Loève Prize in 1997, and the Sophie Germain and Fermat Prizes in 2005. He was elected to the French Academy of Sciences in 2013. He is the author of the books *Spatial Branching Processes, Random Snakes and Partial Differential Equations* (Birkhäuser Verlag, 1999) and *Brownian Motion, Martingales, and Stochastic Calculus* (Springer, 2016).

### About the Prize

The Wolf Prize carries a cash award of US$100,000. The science prizes are given annually in the areas of agriculture, chemistry, mathematics, medicine, and physics. Laureates receive their awards from the President of the State of Israel in a special ceremony at the Knesset Building (Israel's Parliament) in Jerusalem. The list of previous recipients of the Wolf Prize in Mathematics is available on the website of the Wolf Foundation, `www.wolffund.org.il`.

—*Elaine Kehoe*

## 2019 Rollo Davidson Prize

Tom Hutchcroft

**Tom Hutchcroft** of the University of Cambridge and **Vincent Tassion** of ETH Zurich have been named recipients of the Rollo Davidson Prize for 2019. Hutchcroft was recognized for his many beautiful results in probability theory, including on random and self-avoiding walks, random forests, and percolation theory. Tassion was recognized for his extensive achievements in disordered systems and percolation. Hutchcroft received his PhD from the University of British Columbia under the supervision of Asaf Nachmias and Omer Angel. He received the CMS Doctoral Prize in 2018. He tells the *Notices*: "I grew up in rural England in the small town of Glastonbury, which is known for its music festival, magic trees, and monastic ruins, and according to legend is the final resting place of the Holy Grail." Tassion studied at Ecole Normale Supérieure de Lyon and did postdoctoral work with Hugo Duminil-Copin at the University of Geneva. His research involves phase transitions in statistical physics using discrete models such as percolation and the Ising model. The prize is awarded annually to early career probabilists by the Rollo Davidson Trust.

—*From a Davidson Trust announcement*

## Harrington and Veraart Awarded Adams Prize

Heather Harrington

Luitgard Veraart

**Heather Harrington** of the University of Oxford and **Luitgard Veraart** of the London School of Economics and Political Science (LSE) have been awarded the 2019 Adams Prize in this year's chosen field, the Mathematics of Networks. According to Mihalis Dafermos, chair of the prize adjudicators, "Dr. Harrington has adapted ideas from areas such as algebraic geometry and algebraic topology and applied them in a novel way to real-world problems, with particular emphasis on those arising in biology. Her broad work ranges from the mathematics of biological networks to detailed empirical studies. Dr. Veraart has developed new tools and concepts relevant for the representation and analysis of financial stability and systemic risk in banking networks. Her work has had considerable visibility and impact, both within academia and outside."

Harrington received her PhD from Imperial College London in 2010 under the supervision of Jaroslav Stark and Dorothy Buck. She was awarded a Whitehead Prize of the London Mathematical Society in 2018. She is head of the Algebraic Systems Biology group at the Mathematical Institute at Oxford. She is a member of the AMS, the London Mathematical Society, and the Society for Industrial and Applied Mathematics (SIAM). Veraart received her PhD in 2007 from the University of Cambridge. She was postdoctoral research associate at the Bendheim Center for Finance at Princeton University and an assistant professor of financial mathematics at Karlsruhe Institute of Technology before joining the Department of Mathematics at LSE. She received a George Fellowship from the Bank of England in 2016 for research on systemic risk in financial networks. She is associate editor of *Applied Mathematical Finance* and the *SIAM Journal on Financial Mathematics*.

—*From a University of Cambridge announcement*

# Wigderson Awarded Knuth Prize

Avi Wigderson

**Avi Wigderson** of the Institute for Advanced Study has been awarded the 2019 Donald E. Knuth Prize "for fundamental and lasting contributions in areas including randomized computation, cryptography, circuit complexity, proof complexity, parallel computation, and our understanding of fundamental graph properties" and for his contributions to education and as a mentor. He is the author of the book *Mathematics and Computation* (Princeton University Press). The prize is awarded by the Association for Computing Machinery (ACM) Special Interest Group on Algorithms and Computation Theory and the IEEE Technical Committee on the Mathematical Foundations of Computing to recognize major research accomplishments and contributions to the foundations of computer science over an extended period of time.

*—From an ACM/IEEE announcement*

# Hansen Awarded 2018 IMA Prize

Anders Hansen

**Anders Hansen** of the Cambridge Centre for Analysis at the University of Cambridge and the University of Oslo has been awarded the 2018 IMA Prize of the Institute for Mathematics and Its Applications (IMA). He was honored for his "work in computational mathematics, and in particular for his development of the solvability complexity index and its corresponding classification hierarchy." His work involves foundations of computational mathematics and applied functional and harmonic analysis. He is currently working on enhancing resolution in medical imaging.

Hansen received his PhD from the University of Cambridge in 2008. He received a Leverhulme Prize in Mathematics and Statistics in 2017 and is an editor of the *Proceedings of the Royal Society Series A*. Hansen tells the *Notices*: "I have a slightly unorthodox background as my original plan was to become a jazz guitarist, and I even spent some time at Berklee College of Music pursuing an education in music. And, although I have always had a strong interest in mathematics, I was a bit torn regarding the final career path. However, after spending a year sailing across the Indian Ocean, I decided to focus on math, move to California and study at UC Berkeley."

The IMA Prize is awarded annually to a mathematical scientist who received his or her PhD degree within ten years of the nomination year. The award recognizes an individual who has made a transformative impact on the mathematical sciences and their applications.

*—From an IMA announcement*

# ANZIAM Prizes Awarded

Australian and New Zealand Industrial and Applied Mathematics (ANZIAM), a division of the Australian Mathematical Society, has awarded medals for 2019 to three mathematical scientists. **Peter Taylor**, director of the Australian Research Council Centre of Excellence for Mathematical and Statistical Frontiers, has been awarded the 2019 ANZIAM Medal "for his contributions to the theory and applications of mathematics, particularly in the area of applied probability." The medal is awarded for outstanding merit in research achievements, activities enhancing applied or industrial mathematics, or both, and contributions to ANZIAM. **Scott McCue** of Queensland University of Technology has been awarded the 2019 E. O. Tuck Medal for research focusing on developing and applying theoretical and computational techniques to problems in interfacial dynamics and mathematical biology, as well as his broader contributions to industrial mathematics. The medal is a midcareer award given for outstanding research and distinguished service to the field of applied mathematics. **Ryan Loxton** of Curtin University was selected to receive the J. H. Michell Medal for research in areas such as nonlinear optimization, operations research, and system identification. The medal recognizes an outstanding young researcher in applied/industrial mathematics.

*—From an ANZIAM announcement*

# Putnam Prizes Awarded

The winners of the seventy-ninth William Lowell Putnam Mathematical Competition have been announced. The Putnam Competition is administered by the Mathematical Association of America (MAA) and consists of an examination containing mathematical problems that are designed to test both originality and technical competence. Prizes are awarded both to individuals and to teams.

The six highest ranking individuals each received a cash award of US$2,500. Listed in alphabetical order, they are:

- **Dongryul Kim**, Harvard University
- **Shyam Narayanan**, Harvard University
- **David Stoner**, Harvard University
- **Yuan Yao**, Massachusetts Institute of Technology
- **Shengtong Zhang**, Massachusetts Institute of Technology

Institutions with at least three registered participants obtain a team ranking in the competition based on the rankings of three designated individual participants. The five top-ranked teams (with members listed in alphabetical order) were:

- Harvard University, **Dongryul Kim**, **Shyam Narayanan**, **David Stoner**
- Massachusetts Institute of Technology, **Junyao Peng**, **Ashwin Sah**, **Yunkun Zhou**
- University of California, Los Angeles, **Ciprian Mircea Bonciocat**, **Xiaoyu Huang**, **Konstantin Miagkov**
- Columbia University, **Quang Dao**, **Myeonhu Kim**, **Matthew Lerner-Brecher**
- Stanford University, **David Kewei Lin**, **Hanzhi Zheng**, **Yifan Zhu**

The first-place team receives an award of US$25,000, and each member of the team receives US$1,000. The awards for second place are US$20,000 and US$800; for third place, US$15,000 and US$600; for fourth place, US$10,000 and US$400; and for fifth place, US$5,000 and US$200.

**Danielle Wang** of the Massachusetts Institute of Technology was awarded the Elizabeth Lowell Putnam Prize for outstanding performance by a woman in the competition. She received an award of US$1,000.

*—From an MAA announcement*

# AWM Essay Contest Winners

The Association for Women in Mathematics (AWM) has announced the winners of its 2019 essay contest, "Biographies of Contemporary Women in Mathematics." The grand prize was awarded to **Dominique Alexander** of Douglas High School, Minden, Nevada, for the essay "How Bees Sting," about Christine Ensign of Douglas High School. The essay also won first place in the high school level category and will be published in the *AWM Newsletter*. First place in the college undergraduate category was awarded to **Liyaan Maskati** of Brown University for the essay "Nothing Ventured, Nothing Gained" about Ellie Pavlick of Brown University. First place in the middle school category was awarded to **Farren Stainton** of the Sharon Academy, Sharon, Vermont, for the essay "My Teacher Makes the Irrational Perfectly Rational," about Sandy Thorne of the Sharon Academy.

*—From an AWM announcement*

# Simons Fellows in Mathematics

The Simons Foundation Mathematics and Physical Sciences (MPS) division supports research in mathematics, theoretical physics, and theoretical computer science. The MPS division provides funding for individuals, institutions, and science infrastructure. The Fellows Program provides funds to faculty for up to a semester-long research leave from classroom teaching and administrative obligations. The mathematical scientists who have been awarded 2019 Simons Fellowships are:

- **Federico Ardila**, San Francisco State University
- **Nir Avni**, Northwestern University
- **Yuri Berest**, Cornell University
- **Christopher Bishop**, Stony Brook University
- **Sergey Bobkov**, University of Minnesota–Twin Cities
- **Vyjayanthi Chari**, University of California, Riverside
- **Ivan Cherednik**, University of North Carolina at Chapel Hill
- **Gheorghe Craciun**, University of Wisconsin–Madison
- **Philippe Di Francesco**, University of Illinois at Urbana–Champaign
- **William Duke**, University of California, Los Angeles
- **Sergey Fomin**, University of Michigan
- **Joshua Greene**, Boston College
- **Changfeng Gui**, University of Texas at San Antonio
- **Robert Guralnick**, University of Southern California
- **Juhi Jang**, University of Southern California
- **Victor Kac**, Massachusetts Institute of Technology
- **Matthew Kahle**, Ohio State University
- **Nets Katz**, California Institute of Technology
- **Rinat Kedem**, University of Illinois at Urbana–Champaign
- **Autumn Kent**, University of Wisconsin–Madison

- **Inwon Christina Kim**, University of California, Los Angeles
- **Sergiu Klainerman**, Princeton University
- **Slava Krushkal**, University of Virginia
- **Kai-Wen Lan**, University of Minnesota–Twin Cities
- **Xiaochun Li**, University of Illinois at Urbana–Champaign
- **Irina Mitrea**, Temple University
- **Toan Nguyen**, Pennsylvania State University
- **Denis Osin**, Vanderbilt University
- **Dmitry Panchenko**, University of Toronto
- **Irena Peeva**, Cornell University
- **Malabika Pramanik**, University of British Columbia
- **Eric Rowell**, Texas A&M University
- **Andreas Seeger**, University of Wisconsin–Madison
- **Evgueni Tevelev**, University of Massachusetts, Amherst
- **Tatiana Toro**, University of Washington
- **Jared Weinstein**, Boston University
- **Michael Wolf**, William Marsh Rice University
- **Paul Yang**, Princeton University
- **Guoliang Yu**, Texas A&M University

*—From a Simons Foundation announcement*

# Guggenheim Fellowship Awards to Mathematical Scientists

The John Simon Guggenheim Memorial Foundation has announced the names of the scholars, artists, and scientists who were selected as Guggenheim Fellows for 2019. Selected as fellows in the mathematical sciences were:

- **Mohammad T. Hajiaghayi**, University of Maryland, applied mathematics
- **David Jerison**, Massachusetts Institute of Technology, mathematics
- **Per A. Mykland**, University of Chicago, applied mathematics

Selected as a Fellow in computer sciences was **Georg Essl**, University of Wisconsin–Milwaukee.

Guggenheim Fellows are appointed on the basis of impressive achievement in the past and exceptional promise for future accomplishments.

*—From a Guggenheim Foundation announcement*

# Regeneron Science Talent Search

Two young scientists whose work involves the mathematical sciences are among the top winners in the 2019 Regeneron Science Talent Search.

**Ana Humphrey**, eighteen, of Alexandria, Virginia, received the first-place award of US$250,000 for her mathematical model to determine the possible locations of exoplanets—planets outside our solar system—that may have been missed by NASA's Kepler Space Telescope. She used her model to find "unpacked" spaces where as many as 560 new planets might fit and identified ninety-six locations as primary search targets. Her research could aid our understanding of the formation of planets and inform our search for life in outer space. **Adam Areishar**, seventeen, of Alexandria, Virginia, was awarded third place and US$150,000 for his project combining a classic previously unsolved math problem called the "coupon collector problem" with extreme value theory. The theory is used to determine the likelihood of a maximal event, such as a 1,000-year flood. He developed a way to calculate the average maximum values of distributional datasets, which could be applied to predicting the expected amount of time for a given number of different randomly-timed events to occur.

The Regeneron Science Talent Search is the United States' oldest and most prestigious science and mathematics competition for high school seniors. It is administered by the Society for Science and the Public.

*—From a Society for Science and the Public announcement*

# 2019 SIAM Fellows Elected

The Society for Industrial and Applied Mathematics (SIAM) has elected its class of fellows for 2019. Their names and institutions follow.

- **Mihai Anitescu**, Argonne National Laboratory and University of Chicago
- **David A. Bader**, Georgia Institute of Technology
- **Francesco Bullo**, University of California, Santa Barbara
- **José Antonio Carrillo de la Plata**, Imperial College London
- **Stephen Jonathan Chapman**, University of Oxford
- **Pierre Comon**, CNRS
- **Wolfgang A. Dahmen**, University of South Carolina, Columbia
- **Jesus Antonio De Loera**, University of California, Davis
- **Froilán Dopico**, Universidad Carlos III de Madrid

- **Ernesto Estrada**, University of Zaragoza and ARAID Foundation
- **Fariba Fahroo**, Air Force Research Laboratory
- **Andreas Frommer**, Universität Wuppertal
- **Roger G. Ghanem**, University of Southern California
- **Sigal Gottlieb**, University of Massachusetts, Dartmouth
- **Michael Allen Heroux**, Sandia National Laboratories and St. John's University
- **Misha Kilmer**, Tufts University
- **Ron Kimmel**, Technion-Israel Institute of Technology and Intel Corporation
- **Gitta Kutyniok**, Technische Universität Berlin
- **Irena Lasiecka**, University of Memphis
- **Juan C. Meza**, University of California, Merced
- **Jill C. Pipher**, Brown University
- **Mason A. Porter**, University of California, Los Angeles
- **Sebastian Reich**, Universität Potsdam
- **Carla D. Savage**, North Carolina State University
- **Zuowei Shen**, National University of Singapore
- **Joel A. Tropp**, California Institute of Technology
- **Yin Zhang**, Rice University and Chinese University of Hong Kong, Shenzhen
- **Jun Zou**, Chinese University of Hong Kong

*—From a SIAM announcement*

# AAAS Fellows Elected

The American Academy of Arts and Sciences (AAAS) has elected its new Fellows and foreign honorary members for 2019. The new members in the section on Mathematical and Physical Sciences are:

- **Michael H. Harris**, Columbia University
- **Mikhail Lyubich**, Stony Brook University
- **Sylvia Serfaty**, New York University
- **Michael J. Shelley**, New York University
- **András Vasy**, Stanford University
- **Ofer Zeitouni**, Weizmann Institute of Science

Elected in the section on Computer Sciences were:

- **Francine D. Berman**, Rensselaer Polytechnic Institute
- **David R. Karger**, Massachusetts Institute of Technology

*—From an AAAS announcement*

# Fellows of the Royal Society

The Royal Society has announced the names of fifty-one new fellows, ten foreign members, and one honorary fellow for 2019. The new fellows whose work involves the mathematical sciences are:

- **Manjul Bhargava**, Princeton University
- **Caucher Birkar**, University of Cambridge
- **Sarah C. Darby**, University of Oxford
- **Christoper Hacon**, University of Utah
- **Peter Haynes**, University of Cambridge
- **Richard Jozsa**, University of Cambridge
- **Roy Kerr**, University of Cambridge, New Zealand
- **Marta Kwiatkowska**, University of Oxford
- **Robert Tibshirani**, Stanford University
- **Ashkay Venkatesh**, Princeton University

Elected as a foreign member was **Jack Dongarra**, University of Tennessee, Oak Ridge National Laboratory, and the University of Manchester.

*—From a Royal Society announcement*

# NAE Elections

The National Academy of Engineering (NAE) has elected eighty-six new members and eighteen foreign members. Below are the mathematical scientists who were elected for 2019:

- **Joseph Halpern**, Cornell University
- **William Jordan**, Jordan Analytics
- **Mahta Moghaddam**, University of Southern California

Elected as foreign members were:

- **Michael Cates**, University of Cambridge
- **Gilbert Laporte**, HEC Montreal

*—From an NAE announcement*

# Hertz Foundation Fellowships

The Fannie and John Hertz Foundation has announced its Graduate Fellowship awards for 2019. The new Fellows whose work involves the mathematical sciences are **Noah Golowich**, Harvard University; **Melissa Mai**, Johns Hopkins University; **Nitya Mani**, Stanford University; and **Nina Zubrilina**, Stanford University. The new Fellows will receive up to five years of academic support valued at up to US$250,000 to pursue innovative research without constraints.

*—From a Hertz Foundation announcement*

NEWS

# NSF Graduate Research Fellowships

The National Science Foundation (NSF) has awarded a number of Graduate Research Fellowships for fiscal year 2019. Further awards may be announced later in the year. This program supports students pursuing doctoral study in all areas of science and engineering and provides a stipend of US$30,000 per year for a maximum of three years of full-time graduate study. Information about the solicitation for the 2020 competition will be published in the "Mathematics Opportunities" section of an upcoming issue of the *Notices*.

Following are the names of the awardees in the mathematical sciences selected so far in 2019, followed by their undergraduate institutions (in parentheses) and the institutions at which they plan to pursue graduate work.

- **David J. Altizio** (Carnegie-Mellon University), Carnegie-Mellon University
- **Allen Alvarez Loya** (California State University, Fullerton), University of Colorado at Boulder
- **Montie S. Avery** (University of New Mexico), University of Minnesota–Twin Cities
- **Julius Baldauf-Lenschen** (Massachusetts Institute of Technology), Massachusetts Institute of Technology
- **William J. Barham** (University of Colorado at Boulder), University of Colorado at Boulder
- **Savannah V. Bates** (Jacksonville University), North Carolina State University
- **Olivia M. Bernstein** (Biola University), University of California, Irvine
- **Adam B. Block** (Columbia University), Columbia University
- **Sarah Brauner** (Reed College), University of Minnesota–Twin Cities
- **Thomas Brazelton** (Johns Hopkins University), University of Pennsylvania
- **Madelyne M. Brown** (Bucknell University), University of North Carolina at Chapel Hill
- **Katherine Brumberg** (Yale University), Yale University
- **Amanda Burcroff** (University of Michigan–Ann Arbor), University of Michigan–Ann Arbor
- **Alois Cerbu** (Yale University), University of California, Berkeley
- **Ryan C. Chen** (Princeton University), Princeton University
- **Tyler Chen** (Tufts University), University of Washington
- **Karina Cho** (Harvey Mudd College), Harvey Mudd College
- **Sarah (Sally) Collins** (Boston College), Georgia Institute of Technology
- **Destiny Diaz** (State University of New York at Buffalo), State University of New York at Buffalo
- **Michael Dotzel** (University of Missouri–Columbia), University of Missouri–Columbia
- **Hindy Drillick** (Stony Brook University), Stony Brook University
- **Rebecca F. Durst** (Williams College), Brown University
- **Patrick J. Dynes** (Clemson University), University of Oklahoma–Norman
- **Yael Eisenberg** (Yeshiva University), Yeshiva University
- **Sabrina E. Enriquez** (University of Southern California), University of California, Davis
- **Katherine Gallagher** (University of Notre Dame), University of Notre Dame
- **Aaron S. George** (University of Maryland), University of Maryland
- **Shay Gilpin** (University of California, Santa Cruz), University of Colorado at Boulder
- **Nikolay D. Grantcharov** (University of California, Berkeley), University of California, Berkeley
- **Jiajing Guan** (George Mason University), George Mason University
- **Mitchell S. Harris** (Yale University)
- **Brice Huang** (Massachusetts Institute of Technology), Massachusetts Institute of Technology
- **DeVon M. Ingram** (Georgia Institute of Technology), Georgia Institute of Technology
- **Adam Q. Jaffe** (Stanford University), Stanford University
- **Jennifer N. Jones** (Universidad de Guanajuato/CIMAT), Universidad de Guanajuato/CIMAT
- **Sydney N. Kahmann** (University of California, Los Angeles), University of California, Los Angeles
- **Caleb Ki** (Amherst College), University of Michigan–Ann Arbor
- **Yujin Hong Kim** (Columbia University), Columbia University
- **Nathaniel J. Kroeger** (Baylor University), William Marsh Rice University
- **Cameron Krulewski** (Harvard University), Harvard University
- **Andrew Kwon** (Carnegie-Mellon University), Carnegie-Mellon University
- **Jaylen M. Lee** (James Madison University), University of California, Irvine
- **Ishan Levy** (Princeton University), Princeton University
- **Jared D. Lichtman** (Dartmouth College), University of Cambridge
- **Hyun Jae Lim** (Harvard College), University of California, Berkeley

- **Jessica Maghakian** (Massachusetts Institute of Technology), Stony Brook University
- **Scott A. Mahan** (Arizona State University), University of California, San Diego
- **Yelena Mandelshtam** (Stanford University), Stanford University
- **Nitya Mani** (Stanford University), Stanford University
- **Sofia R. Martinez Alberga** (University of California, Riverside), University of California, Riverside
- **Amanda S. Mason** (Wofford College), University of Colorado at Boulder
- **Bijan H. Mazaheri** (Williams College), California Institute of Technology
- **Sean McGrath** (McGill University)
- **Theo McKenzie** (Harvard University), University of California, Berkeley
- **Andres N. Mejia** (Bard College), Yale University
- **Anya Michaelsen** (Williams College), Williams College
- **Samantha C. Moore** (University of Northern Colorado), University of North Carolina at Chapel Hill
- **Keshav B. Patel** (University of North Carolina at Chapel Hill), University of North Carolina at Chapel Hill
- **Luke K. Peilen** (Yale University), New York University
- **Thomas Reeves** (Princeton University), Cornell University
- **Tristan Reynoso** (University of Central Florida), University of Central Florida
- **Daniel D. Richman** (Massachusetts Institute of Technology)
- **Ryan A. Robinett** (Massachusetts Institute of Technology), Massachusetts Institute of Technology
- **Sarah Robinson** (University of Georgia), William Marsh Rice University
- **Samuel P. Rosin** (Harvard College), University of North Carolina at Chapel Hill
- **Bryce T. Rowland** (Centre College of Kentucky), University of North Carolina at Chapel Hill
- **Gabriel Ruiz** (University of California, Riverside), University of California, Los Angeles
- **Andrew I. Sack** (University of Florida), University of Florida
- **Sandeep B. Silwal** (Massachusetts Institute of Technology), Massachusetts Institute of Technology
- **Yousuf Soliman** (Carnegie-Mellon University), California Institute of Technology
- **Melissa I. Spence** (University of California, Davis), University of California, Merced
- **Samuel A. Spiro** (University of Miami), University of California, San Diego
- **Chandler Squires** (Massachusetts Institute of Technology), Massachusetts Institute of Technology
- **Stephanie E.-T. Stacy** (Williams College), University of California, Los Angeles
- **Zofia Stanley** (Brown University), University of Colorado at Boulder
- **George Stepaniants** (University of Washington), University of Washington
- **David W. Stoner** (Harvard University), Harvard University
- **Austin J. Stromme** (University of Washington), Massachusetts Institute of Technology
- **Naomi Sweeting** (University of Chicago), University of Chicago
- **Tiffany Tang** (William Marsh Rice University), University of California, Berkeley
- **Evan H. Toler** (William Marsh Rice University), New York University
- **Matthew Tyler** (Princeton University), Princeton University
- **Dario Verta** (Johns Hopkins University), George Washington University
- **Alexander M. Vetter** (Villanova University), Villanova University
- **Nathan A. Wagner** (Bucknell University), Washington University
- **Danielle Y. Wang** (Massachusetts Institute of Technology), Massachusetts Institute of Technology
- **Joshua X. Wang** (Princeton University), Harvard University
- **Luya Wang** (Princeton University), University of California, Berkeley
- **Scott Weady** (Yale University), New York University
- **Michael C. Wigal** (West Virginia University), Georgia Institute of Technology
- **Emily T. Winn** (College of the Holy Cross), Brown University
- **David H. Yang** (Massachusetts Institute of Technology), Harvard University
- **Andrew Yarger** (Saint Olaf College), University of Michigan–Ann Arbor
- **Michelle Yu** (College of the Holy Cross), University of California, Berkeley
- **David K. Zhang** (Vanderbilt University), Vanderbilt University
- **Edith Zhang** (University of Virginia), University of Virginia
- **John C. Zito** (Kenyon College)
- **Nina Zubrilina** (Stanford University), Stanford University

*—NSF announcement*

## NEWS

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# Community Updates

## AMS–AAAS Mass Media Fellowship Awarded

Leila Sloman

**Leila Sloman**, a doctoral student in mathematics at Stanford University, has been chosen as the 2019 AMS–AAAS Mass Media Fellow. Leila is in the third year of her PhD studies and works in the areas of applied mathematics, partial differential equations, and probability theory. She will work at *Scientific American* this summer.

The Mass Media Science and Engineering Fellows program is organized by the American Association for the Advancement of Science (AAAS). This program is designed to improve public understanding of science and technology by placing advanced undergraduate, graduate, and postgraduate science, mathematics, and engineering students in media outlets nationwide. The Fellows work for ten weeks over the summer as reporters, researchers, and production assistants alongside media professionals to sharpen their communication skills and increase their understanding of the editorial process by which events and ideas become news.

Now in its forty-fifth year, this program has placed more than 700 Fellows in media organizations nationwide as they research, write, and report today's headlines. The program is designed to report science-related issues in the media in easy-to-understand ways so as to improve public understanding and appreciation for science and technology.

For more information on the AAAS Mass Media Science and Engineering Fellows program, visit the website `www.aaas.org/programs/mass-media-fellowship`. Follow on Twitter @AAASMassMedia for program highlights and news.

*—Anita Benjamin*
*AMS Office of Government Relations*

## Epsilon Awards Announced

The AMS has chosen nineteen summer mathematics programs to receive Epsilon grants for 2019. These summer programs give students a chance to see aspects of mathematics that they may not see in school and allow them to share their enthusiasm for mathematics with like-minded students.

The programs that received Epsilon grants for 2019 are:

- **AIMC Math Camp at Navajo Prep**, Navajo Preparatory School, Tatiana Shubin, Director
- **All Girls/All Math Summer Camp**, University of Nebraska, Lincoln, Amanda Laubmeier, Director
- **Baa Hózhó Math Camp**, Navajo Technical University, Crownpoint, New Mexico, David Auckly, Director
- **Bridge to Enter Advanced Mathematics (BEAM)**, Bard College and Union College, Daniel Zaharopol, Director
- **Broward Young People's Project**, Broward County Public Schools, Beverly Kerner, Director
- **Canada/USA Mathcamp**, Lewis and Clark College, Marisa Debowsky, Director
- **Center for Mathematical Talent**, Courant Institute of Mathematical Sciences at New York University, Selin Kalaycioglu, Director
- **Euclid Lab**, online, David Gay, Director
- **GirlsGetMath@ICERM**, Brown University, Brendan Edward Hassett, Director
- **GirlsGetMath@Rochester**, University of Rochester, Amanda Tucker, Director
- **MathILy**, Bryn Mawr College, sarah-marie belcastro, Director
- **MathILy-Er**, Bowdoin College, Alice Mark, Director
- **Mathworks Honors Summer Math Camp**, Texas State University, Max Warshauer, Director
- **PROMYS (Programs in Mathematics for Young Scientists)**, Boston University, Glenn Stevens, Director
- **PROTaSM (Puerto Rico Opportunities for Talented Students in Math)**, University of Puerto Rico, Mayaguez, Luis F. Caceres, Director
- **QTM Math Circle**, Emory University's Institute for Quantitative Theory, Steven Olsen, Director
- **Research Science Institute (RSI)**, Massachusetts Institute of Technology, Charles Farmer, Director

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- **Ross Mathematics Program**, Ross Mathematics Foundation, Jim Fowler, Director
- **TexPREP-Lubbock**, Texas Tech University, Jim Brown, Director

*—AMS announcement*

# From the AMS Public Awareness Office

The Mathematical Imagery web page now presents images in a larger format and employs MathJax to display mathematics in the descriptions. View nearly 700 art works with descriptions in forty-nine galleries, and find links to museums, galleries, articles and resources, at `https://www.ams.org/math-imagery`.

*—Annette Emerson and Mike Breen*
*AMS Public Awareness Officers*
*paoffice@ams.org*

# From the AMS Committee on Education

The AMS Committee on Education is sponsoring a one-day mini-conference on October 25, 2019, in DC. This one-day mini-conference will focus on "Mathematics Departments and the Explosive Growth of Computational and Quantitative Offerings in Higher Education." If you are interested in attending the meeting, please register at: `www.ams.org/minireg`.

Registration opens June 19, 2019. Please register no later than **September 27, 2019**.

For additional information, contact `amsdc@ams.org`.

*—AMS announcement*

# Mathematics Opportunities

*Listings for upcoming mathematics opportunities to appear in* Notices *may be submitted to* `notices@ams.org`.

**Early Career Opportunity**

## Call for Nominations for Adams Prize

The University of Cambridge will award the 2020 Adams Prize in the field of algebra. The prize will be awarded to a mathematician who is under forty years of age and in a university or other institution in the United Kingdom for outstanding achievement in algebra. The deadline for applications is **October 31, 2019**. See `https://www.maths.cam.ac.uk/adams-prize`.

*—From a University of Cambridge announcement*

## Call for Nominations for 2020 W. K. Clifford Prize

The W. K. Clifford Prize is awarded to young researchers for excellence in theoretical and applied Clifford algebras, their analysis, and geometry. The prize will be awarded at the 12th Conference on Clifford Algebras and Their Applications in Mathematical Physics in 2020. The deadline for nominations is **September 30, 2019**. Nominations should be sent to `CliffordPrize2020@gmail.com`. See `https://wkcliffordprize.wordpress.com`.

*—G. Stacey Staples*
*Southern Illinois University*

## IMA Prize in Mathematics and Its Applications

The Institute for Mathematics and Its Applications (IMA) awards the annual Prize in Mathematics and its Applications to an individual who has made a transformative impact on the mathematical sciences and their applications. The deadline for nominations is **July 19, 2019**. See `www.ima.umn.edu/prize`.

*—From an IMA announcement*

**Early Career Opportunity**

## NSF Mathematical Sciences Postdoctoral Research Fellowships

The National Science Foundation (NSF) solicits proposals for the Mathematical Sciences Postdoctoral Research Fellowships. The deadline for full proposals is **October 16, 2019**. See `https://www.nsf.gov/funding/pgm_summ.jsp?pims_id=5301`.

*—From an NSF announcement*

## NSF CAREER Awards

The National Science Foundation (NSF) solicits proposals for the Faculty Early Career Development Awards. The deadline for full proposals is **July 17, 2019**. See `www.nsf.gov/funding/pgm_summ.jsp?pims_id=503214`.

*—From an NSF announcement*

*The most up-to-date listing of NSF funding opportunities from the Division of Mathematical Sciences can be found online at:* `www.nsf.gov/dms` *and for the Directorate of Education and Human Resources at* `www.nsf.gov/dir/index.jsp?org=ehr`. *To receive periodic updates, subscribe to the DMSNEWS listserv by following the directions at* `www.nsf.gov/mps/dms/about.jsp`.

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## NSF Mathematical Sciences Innovation Incubator

The Mathematical Sciences Innovation Incubator (MSII) activity of the National Science Foundation (NSF) provides funding to support the involvement of mathematical scientists in research areas in which the mathematical sciences are not yet playing large roles—for example, security and resilience of critical infrastructure, emerging technologies, innovative energy technology, and foundational biological and health research. For details, see `www.nsf.gov/funding/pgm_summ.jsp?pims_id=505044&org=DMS`.

*—From an NSF announcement*

## Research Experiences for Undergraduates

The Research Experiences for Undergraduates (REU) program supports student research in areas funded by the National Science Foundation (NSF) through REU sites and REU supplements. See `www.nsf.gov/funding/pgm_summ.jsp?pims_id=5517`. The deadline date for proposals from institutions wishing to host REU sites is **August 28, 2019**. Dates for REU supplements vary with the research program (contact the program director for more information). Students apply directly to REU sites. See `www.nsf.gov/crssprgm/reu/list_result.jsp?unitid=5044` for active REU sites.

*—From an NSF announcement*

### Early Career Opportunity

## Fulbright Israel Postdoctoral Fellowships

The United States–Israel Educational Foundation (USIEF) plans to award eight grants to US postdoctoral scholars who seek to pursue research at Israeli institutions of higher education. The deadline for applications is **September 16, 2019**. See `https://awards.cies.org/content/clone-postdoctoral-fellowship`.

*—From a USIEF announcement*

## 2019 Clay Research Conference and Workshops

The Clay Mathematics Institute (CMI) will hold the 2019 Clay Research Conference on October 2, 2019, at the Mathematical Institute of the University of Oxford. The plenary speakers are:

- **Benedict Gross**, University of California San Diego
- **Alex Lubotzky**, Hebrew University of Jerusalem
- **Oscar Randal-Williams**, University of Cambridge
- **Geordie Williamson**, University of Sydney

Associated workshops will be held throughout the week of the conference, September 29–October 4, 2019:

- Beyond Spectral Gaps (Emmanuel Breuillard, Assaf Naor)
- Modular Representation Theory (Geordie Williamson, Ivan Losev, Matthew Emerton)
- Patterns in Cohomology of Moduli Spaces (Søren Galatius, Oscar Randal-Williams)
- Periods, Representations, and Arithmetic: Recent Advances on the Gan-Gross-Prasad Conjectures and Their Applications (Christopher Skinner, Wei Zhang)

Registration for the conference is free but required. Participation in the workshops is by invitation; a limited number of additional places are available. Limited accommodation is available for PhD students and early career researchers. To register for the conference and to register interest in a workshop, email Naomi Kraker at `admin@claymath.org`. For full details, including the schedule, titles, and abstracts when they become available, see `www.claymath.org`.

*—From a CMI announcement*

**Early Career Opportunity**

## News from MSRI

The Mathematical Sciences Research Institute (MSRI) will hold the following workshops during the fall of 2019. Established researchers, postdoctoral fellows, and graduate students are invited to apply for funding. It is the policy of MSRI to actively seek to achieve diversity in its workshops. Thus a strong effort is made to remove barriers that hinder equal opportunity, particularly for those groups that have been historically underrepresented in the mathematical sciences. MSRI has a resource to assist visitors with finding child care in Berkeley. For more information, please contact Sanjani Varkey at `sanjani@msri.org`.

The workshops are as follows:

- **August 15–16, 2019**: Connections for Women: Holomorphic Differentials in Mathematics and Physics. See `www.msri.org/workshops/894`
- **August 19–23, 2019**: Introductory Workshop: Holomorphic Differentials in Mathematics and Physics. See `www.msri.org/workshops/895`
- **August 29–30, 2019**: Connections for Women: Microlocal Analysis. See `www.msri.org/workshops/896`
- **September 3–6, 2019**: Introductory Workshop: Microlocal Analysis. See `www.msri.org/workshops/897`
- **October 14–18, 2019**: Recent Developments in Microlocal Analysis. See `www.msri.org/workshops/899`
- **November 18–22, 2019**: Holomorphic Differentials in Mathematics and Physics. See `www.msri.org/workshops/898`

MSRI has been supported from its origins by the National Science Foundation, now joined by the National Security Agency, more than 100 academic sponsor departments, a range of private foundations, and generous and farsighted individuals.

*—MSRI announcement*

# Classified Advertising

*Employment Opportunities*

## CHINA

**Tianjin University, China**
**Tenured/Tenure-Track/Postdoctoral Positions at the Center for Applied Mathematics**

Dozens of positions at all levels are available at the recently founded Center for Applied Mathematics, Tianjin University, China. We welcome applicants with backgrounds in pure mathematics, applied mathematics, statistics, computer science, bioinformatics, and other related fields. We also welcome applicants who are interested in practical projects with industries. Despite its name attached with an accent of applied mathematics, we also aim to create a strong presence of pure mathematics. Chinese citizenship is not required.

Light or no teaching load, adequate facilities, spacious office environment and strong research support. We are prepared to make quick and competitive offers to self-motivated hard workers, and to potential stars, rising stars, as well as shining stars.

The Center for Applied Mathematics, also known as the Tianjin Center for Applied Mathematics (TCAM), located by a lake in the central campus in a building protected as historical architecture, is jointly sponsored by the Tianjin municipal government and the university. The initiative to establish this center was taken by Professor S. S. Chern. Professor Molin Ge is the Honorary Director, Professor Zhiming Ma is the Director of the Advisory Board. Professor William Y. C. Chen serves as the Director.

TCAM plans to fill in fifty or more permanent faculty positions in the next few years. In addition, there are a number of temporary and visiting positions. We look forward to receiving your application or inquiry at any time. There are no deadlines.

Please send your resume to `mathjobs@tju.edu.cn`.

For more information, please visit `cam.tju.edu.cn` or contact Ms. Erica Liu at `mathjobs@tju.edu.cn`, telephone: 86-22-2740-6039.

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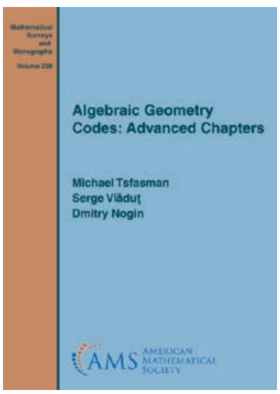
# New Books Offered by the AMS

## Algebra and Algebraic Geometry




### Algebraic Geometry Codes: Advanced Chapters

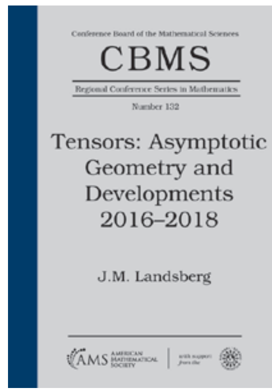
**Michael Tsfasman**, *CNRS, Laboratoire de Mathématiques de Versailles, France, Institute for Information Transmission Problems, Moscow, Russia, and Independent University of Moscow, Russia*, **Serge Vlăduţ**, *Aix Marseille Université, France, and Institute for Information Transmission Problems, Moscow, Russia*, and **Dmitry Nogin**, *Institute for Information Transmission Problems, Moscow, Russia*

*Algebraic Geometry Codes: Advanced Chapters* is devoted to the theory of algebraic geometry codes, a subject related to several domains of mathematics. Whereas most books on coding theory start with elementary concepts and then develop them in the framework of coding theory itself within, this book systematically presents meaningful and important connections of coding theory with algebraic geometry and number theory.

*This item will also be of interest to those working in applications.*

**Mathematical Surveys and Monographs**, Volume 238
July 2019, 453 pages, Hardcover, ISBN: 978-1-4704-4865-3, LC 2019003782, 2010 *Mathematics Subject Classification:* 14Hxx, 94Bxx, 14G15, 11R58; 11R04, 11T71, 11M38, 11R37, 11R42, 14D22, 14J20, List US$129, **AMS members US$103.20**, **MAA members US$116.10**, Order code SURV/238

`bookstore.ams.org/surv-238`




### Tensors: Asymptotic Geometry and Developments 2016–2018

**J.M. Landsberg**, *Texas A&M University, College Station, TX*

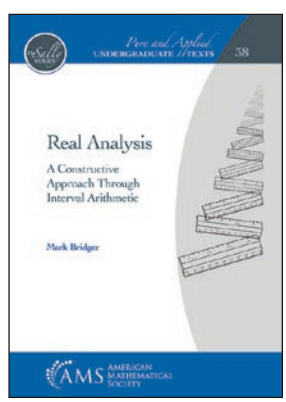
Tensors are used throughout the sciences, especially in solid state physics and quantum information theory. This book brings a geometric perspective to the use of tensors in these areas. Numerous open problems appropriate for graduate students and post-docs are included throughout.

*This item will also be of interest to those working in applications and mathematical physics.*

**CBMS Regional Conference Series in Mathematics**, Number 132
June 2019, 152 pages, Softcover, ISBN: 978-1-4704-5136-3, 2010 *Mathematics Subject Classification:* 68Q17, 14L30, 15A69, 81P45, List US$55, **AMS members US$44**, **MAA members US$49.50**, Order code CBMS/132

`bookstore.ams.org/cbms-132`

## Analysis




### Real Analysis

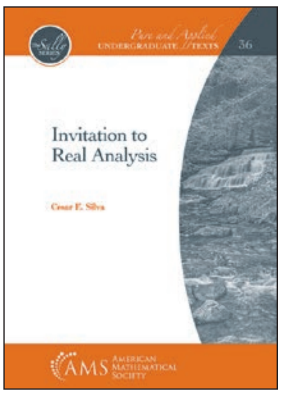
#### A Constructive Approach Through Interval Arithmetic

**Mark Bridger**, *Northeastern University, Boston, MA*

*Real Analysis: A Constructive Approach Through Interval Arithmetic* presents a careful treatment of calculus and its theoretical underpinnings from the constructivist point of view. Throughout the book the emphasis on rigorous and direct proofs is supported by an abundance of examples, exercises, and projects at the end of every section. The exposition is informal but exceptionally clear and well motivated throughout.

**Pure and Applied Undergraduate Texts**, Volume 38
June 2019, 302 pages, Hardcover, ISBN: 978-1-4704-5144-8, LC 2019006280, 2010 *Mathematics Subject Classification:* 03F60, 03F55, 97–01, 97Ixx, 26–01, 26Axx, List US$82, **AMS members US$65.60**, **MAA members US$73.80**, Order code AMSTEXT/38

bookstore.ams.org/amstext-38




### Invitation to Real Analysis

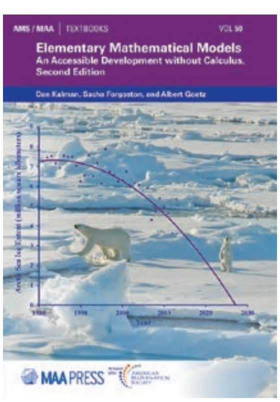
**Cesar E. Silva**, *Williams College, Williamstown, MA*

This book is an introduction to real analysis for a one-semester course aimed at students who have completed the calculus sequence and preferably one other course, such as linear algebra. It does not assume any specific knowledge and starts with all that is needed from sets, logic, and induction.

**Pure and Applied Undergraduate Texts**, Volume 36
June 2019, 303 pages, Hardcover, ISBN: 978-1-4704-4928-5, LC 2018058742, 2010 *Mathematics Subject Classification:* 26–01, 26A03, 26A06; 26A24, 26A42, List US$99, **AMS members US$79.20**, **MAA members US$89.10**, Order code AMSTEXT/36

bookstore.ams.org/amstext-36

# Applications




### Elementary Mathematical Models

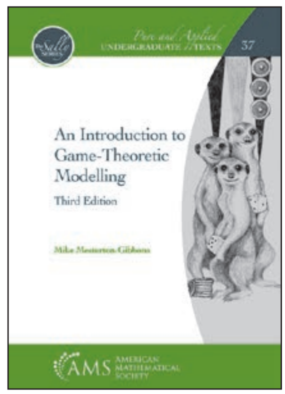
#### An Accessible Development without Calculus, Second Edition

**Dan Kalman**, *American University, Washington, DC*, **Sacha Forgoston**, and **Albert Goetz**

*Elementary Mathematical Models* offers instructors an alternative to standard college algebra, quantitative literacy, and liberal arts mathematics courses. Presuming only a background of exposure to high school algebra, the text introduces students to the methodology of mathematical modeling, which plays a role in nearly all real applications of mathematics.

**AMS/MAA Textbooks**, Volume 50
July 2019, approximately 522 pages, Hardcover, ISBN: 978-1-4704-5001-4, LC 2018053676, 2010 *Mathematics Subject Classification:* 00–01, 39–01, List US$75, **AMS Individual member US$56.25**, AMS Institutional member US$60, **MAA members US$56.25**, Order code TEXT/50

bookstore.ams.org/text-50




### An Introduction to Game-Theoretic Modelling

#### Third Edition

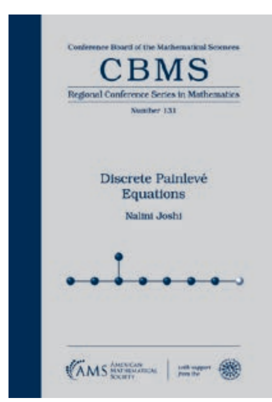
**Mike Mesterton-Gibbons**, *Florida State University, Tallahassee, FL*

This book introduces game theory and its applications from an applied mathematician's perspective, systematically developing tools and concepts for game-theoretic modelling in the life and social sciences. In the present third edition, the author has added substantial new material on evolutionarily stable strategies and their use in behavioral ecology.

**Pure and Applied Undergraduate Texts**, Volume 37
July 2019, approximately 398 pages, Hardcover, ISBN: 978-1-4704-5029-8, LC 2019000060, 2010 *Mathematics Subject Classification:* 91–01, 91A10, 91A12, 91A22, 91A40, 91A80; 92–02, 92D50, List US$82, **AMS members US$65.60**, **MAA members US$73.80**, Order code AMSTEXT/37

bookstore.ams.org/amstext-37

# Differential Equations




### Discrete Painlevé Equations

**Nalini Joshi**, *University of Sydney, Australia*

Discrete Painlevé equations are nonlinear difference equations, which arise from translations on crystallographic lattices. The deceptive simplicity of this statement hides immensely rich mathematical properties, connecting dynamical systems, algebraic geometry, Coxeter groups, topology, special functions theory, and mathematical physics. Instead of technical theorems or complete proofs, this book relies on providing essential points of many arguments about this topic through explicit examples, with the hope that they will be useful for applied mathematicians and physicists.

*This item will also be of interest to those working in algebra and algebraic geometry, analysis, and applications.*

**CBMS Regional Conference Series in Mathematics**, Number 131
June 2019, 146 pages, Softcover, ISBN: 978-1-4704-5038-0, LC 2018058757, 2010 *Mathematics Subject Classification:* 33E17, 39A45, 14E07, 14E15, 32S45, List US$55, **AMS members US$44**, **MAA members US$49.50**, Order code CBMS/131

bookstore.ams.org/cbms-131

# General Interest


## The Math Behind the Magic

### Fascinating Card and Number Tricks and How They Work

**Ehrhard Behrends**, *Freie Universität, Berlin, Germany*
Translated by David Kramer

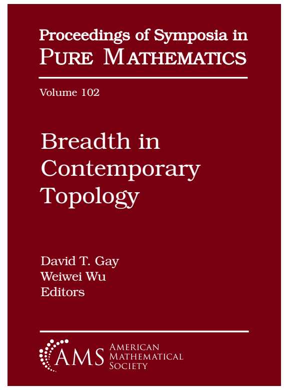
A magician appears able to banish chaos at will: a deck of cards arranged in order is shuffled— apparently randomly—by a member of the audience. Then, hey presto! The deck is suddenly put back in its original order! In this rich, colorfully illustrated volume, Ehrhard Behrends presents around 30 card tricks and number games that are easy to learn and have an interesting mathematical foundation, with no prior knowledge required.

*This item will also be of interest to those working in discrete mathematics and combinatorics and number theory.*

June 2019, 208 pages, Softcover, ISBN: 978-1-4704-4866-0, LC 2019004263, 2010 *Mathematics Subject Classification:* 00A08, 97A20, List US$30, **AMS members US$24**, **MAA members US$27**, Order code MBK/122

bookstore.ams.org/mbk-122

# Geometry and Topology




## Breadth in Contemporary Topology

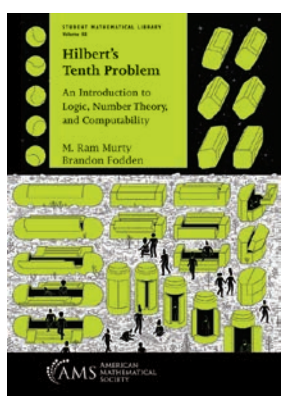
**David T. Gay**, *University of Georgia, Athens, GA*, and **Weiwei Wu**, *University of Georgia, Athens, GA*, Editors

This volume contains the proceedings of the 2017 Georgia International Topology Conference, held from May 22–June 2, 2017, at the University of Georgia, Athens, Georgia. The papers contained in this volume cover topics ranging from symplectic topology to classical knot theory to topology of 3- and 4-dimensional manifolds to geometric group theory. Several papers focus on open problems, while other papers present new and insightful proofs of classical results.

**Proceedings of Symposia in Pure Mathematics**, Volume 102
June 2019, 282 pages, Hardcover, ISBN: 978-1-4704-4249-1, LC 2019001949, 2010 *Mathematics Subject Classification:* 57–06, 20F36, 53D35, 53D12, 55P48, 57M25, 57M27, 57R17, 57R91, List US$133, **AMS members US$106.40**, **MAA members US$119.70**, Order code PSPUM/102

bookstore.ams.org/pspum-102

# Number Theory




## Hilbert's Tenth Problem

### An Introduction to Logic, Number Theory, and Computability

**M. Ram Murty**, *Queen's University, Kingston, ON, Canada*, and **Brandon Fodden**, *Carleton University, Ottawa, ON, Canada*

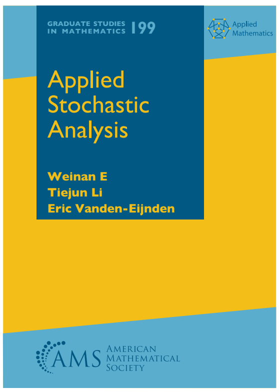
This book gives a complete solution to Hilbert's tenth problem using elementary number theory and rudimentary logic. It is self-contained.

*This item will also be of interest to those working in logic and foundations.*

**Student Mathematical Library**, Volume 88
June 2019, 239 pages, Softcover, ISBN: 978-1-4704-4399-3, LC 2018061472, 2010 *Mathematics Subject Classification:* 11U05, 12L05, List US$55, **All Individuals US$44**, AMS Institutional member US$44, **MAA members US$44**, Order code STML/88

bookstore.ams.org/stml-88

# Probability and Statistics




### Applied Stochastic Analysis

**Weinan E**, *Princeton University, NJ*, **Tiejun Li**, *Peking University, Beijing, China*, and **Eric Vanden-Eijnden**, *Courant Institute of Mathematical Sciences, New York, NY*

Presenting the basic mathematical foundations of stochastic analysis as well as some important practical tools and applications, this textbook is for advanced undergraduate students and beginning graduate students in applied mathematics. The book strikes a nice balance between mathematical formalism and intuitive arguments, a style that is most suited for applied mathematicians.

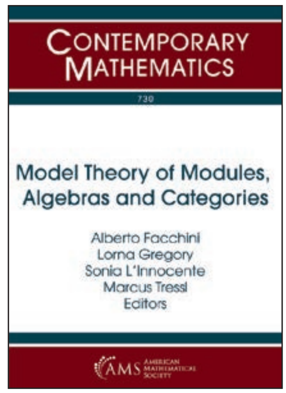
*This item will also be of interest to those working in analysis.*

**Graduate Studies in Mathematics**, Volume 199
June 2019, 305 pages, Hardcover, ISBN: 978-1-4704-4933-9, LC 2019000898, 2010 *Mathematics Subject Classification:* 60–01, 60J22, 60H10, 60H35, 62P10, 62P35, 65C05, 65C10, 82–01, List US$85, **AMS members US$68**, **MAA members US$76.50**, Order code GSM/199

bookstore.ams.org/gsm-199

# New in Contemporary Mathematics

## Algebra and Algebraic Geometry




### ModelTheory of Modules, Algebras and Categories

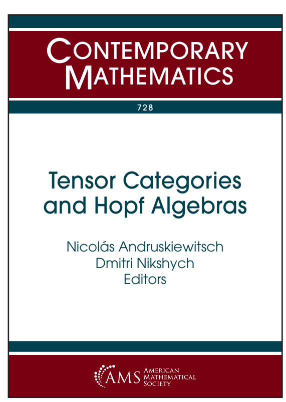
**Alberto Facchini**, *Università Degli Studi Di Padova, Italy*, **Lorna Gregory**, *Università Degli Studi Della Campania "Luigi Vanvitelli", Caserta, Italy*, **Sonia L'Innocente**, *Università Di Camerino, Italy*, and **Marcus Tressl**, *University of Manchester, United Kingdom*, Editors

This volume contains the proceedings of the international conference Model Theory of Modules, Algebras and Categories, held from July 28–August 2, 2017, at the Ettore Majorana Foundation and Centre for Scientific Culture in Erice, Italy. Papers contained in this volume cover recent developments in model theory, module theory and category theory, and their intersection.

*This item will also be of interest to those working in logic and foundations.*

**Contemporary Mathematics**, Volume 730
June 2019, 237 pages, Softcover, ISBN: 978-1-4704-4367-2, LC 2018047324, 2010 *Mathematics Subject Classification:* 03C60, 13D09, 13F30, 16D40, 16D70, 16G20; 18D10, 18E05, 37A35, List US$117, **AMS members US$93.60**, **MAA members US$105.30**, Order code CONM/730

bookstore.ams.org/conm-730



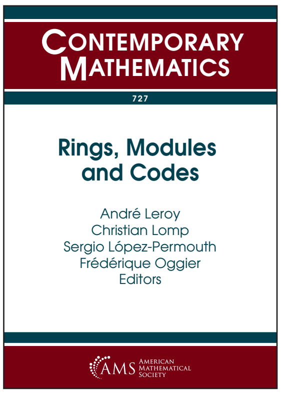

### Tensor Categories and HopfAlgebras

**Nicolás Andruskiewitsch**, *Universidad Nacional de Córdoba, Argentina*, and **Dmitri Nikshych**, *University of New Hampshire, Durham, NH*, Editors

This volume contains the proceedings of the scientific session "Hopf Algebras and Tensor Categories", held from July 27–28, 2017, at the Mathematical Congress of the Americas in Montreal, Canada. Papers highlight the latest advances and research directions in the theory of tensor categories and Hopf algebras.

**Contemporary Mathematics**, Volume 728
May 2019, 194 pages, Softcover, ISBN: 978-1-4704-4321-4, LC 2018042384, 2010 *Mathematics Subject Classification:* 16D90, 16E65, 16T05, 16T20, 16W50, 17B37, 17D05, 18D10, 20G05, List US$117, **AMS members US$93.60**, **MAA members US$105.30**, Order code CONM/728

bookstore.ams.org/conm-728




### Rings, Modules and Codes

**André Leroy**, *Université d'Artois, Arras, France*, **Christian Lomp**, *Universidade do Porto, Portugal*, **Sergio López-Permouth**, *Ohio University, Athens, OH*, and **Frédérique Oggier**, *Nanyang Technological University, Singapore*, Editors

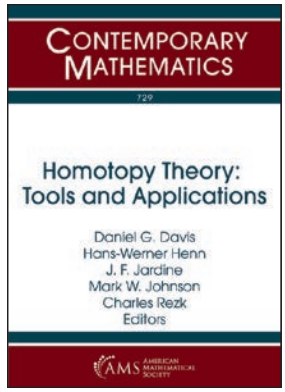
This book contains the proceedings of the Fifth International Conference on Noncommutative Rings and their Applications, held from June 12–15, 2017 at the University of Artois, Lens, France. The papers are related to noncommutative rings, covering topics such as: ring theory, with both the elementwize and more structural approaches developed; module theory with popular topics such as automorphism invariance, almost injectivity, ADS, and extending modules; and coding theory, both the theoretical aspects such as the extension theorem and the more applied ones such as Construction A or Reed–Muller codes.

*This item will also be of interest to those working in applications.*

**Contemporary Mathematics**, Volume 727
May 2019, 355 pages, Softcover, ISBN: 978-1-4704-4104-3, LC 2018041947, 2010 *Mathematics Subject Classification:* 16D40, 16D70, 16W20, 16S36, 11T71, 13P05, 08A05, List US$117, **AMS members US$93.60**, **MAA members US$105.30**, Order code CONM/727

bookstore.ams.org/conm-727

## Geometry and Topology




### Homotopy Theory: Tools and Applications

**Daniel G. Davis**, *University of Louisiana at Lafayette, LA*, **Hans-Werner Henn**, *Université de Strasbourg, France*, **J. F. Jardine**, *Western University, London, Ontario, Canada*, **Mark W. Johnson**, *Pennsylvania State University, Altoona, PA*, and **Charles Rezk**, *University of Illinois at Urbana-Champaign, IL*, Editors

This volume contains the proceedings of the conference Homotopy Theory: Tools and Applications, in honor of Paul Goerss's 60th birthday, held from July 17–21, 2017, at the University of Illinois at Urbana-Champaign, Urbana, IL. The articles cover a variety of topics spanning the current research frontier of homotopy theory.

**Contemporary Mathematics**, Volume 729
June 2019, 268 pages, Softcover, ISBN: 978-1-4704-4244-6, LC 2018047360, 2010 *Mathematics Subject Classification:* 55P43, 55N22, 55N91, 18D50, 55Q45, 55Q51, 55T15, 54B40, 55U10, 55S35, List US$117, **AMS members US$93.60**, **MAA members US$105.30**, Order code CONM/729

bookstore.ams.org/conm-729

NEW BOOKS

# New in Memoirs of the AMS

## Algebra and Algebraic Geometry

### Automorphisms of Two-Generator Free Groups and Spaces of Isometric Actions on the Hyperbolic Plane

**William Goldman**, *University of Maryland, College Park, Maryland*, **Greg McShane**, *Institut Fourier, Grenoble, France*, **George Stantchev**, *University of Maryland, College Park, Maryland*, and **Ser Peow Tan**, *University of Singapore, Singapore*

*This item will also be of interest to those working in geometry and topology.*

**Memoirs of the American Mathematical Society**, Volume 259, Number 1249
June 2019, 78 pages, Softcover, ISBN: 978-1-4704-3614-8, 2010 *Mathematics Subject Classification:* 57M05, 22D40, List US$81, **AMS Individual member US$48.60**, AMS Institutional member US$64.80, **MAA members US$72.90**, Order code MEMO/259/1249

bookstore.ams.org/memo-259-1249

### Flat Rank Two Vector Bundles on Genus Two Curves

**Viktoria Heu**, *Institut de Recherche Mathématique Avancée (IRMA), Strasbourg, France*, and **Frank Loray**, *Institut de Recherche Mathématique de Rennes (IRMAR), France*

**Memoirs of the American Mathematical Society**, Volume 259, Number 1247
June 2019, 107 pages, Softcover, ISBN: 978-1-4704-3566-0, 2010 *Mathematics Subject Classification:* 14H60; 34Mxx, 32G34, 14Q10, List US$81, **AMS Individual member US$48.60**, AMS Institutional member US$64.80, **MAA members US$72.90**, Order code MEMO/259/1247

bookstore.ams.org/memo-259-1247

### Moufang Sets and Structurable Division Algebras

**Lien Boelaert**, *Ghent University, Belgium*, **Tom De Medts**, *Ghent University, Belgium*, and **Anastasia Stavrova**, *St. Petersburg State University, Saint Petersburg, Russia*

**Memoirs of the American Mathematical Society**, Volume 259, Number 1245
June 2019, 88 pages, Softcover, ISBN: 978-1-4704-3554-7, 2010 *Mathematics Subject Classification:* 16W10, 20E42, 17A35, 17B60, 17B45, 17Cxx, 20G15, 20G41, List US$81, **AMS Individual member US$48.60**, AMS Institutional member US$64.80, **MAA members US$72.90**, Order code MEMO/259/1245

bookstore.ams.org/memo-259-1245

## Analysis

### Geometric Pressure for Multimodal Maps of the Interval

**Feliks Przytycki**, *Polish Academy of Sciences, Warszawa, Poland*, and **Juan Rivera-Letelier**, *University of Rochester, NY*

**Memoirs of the American Mathematical Society**, Volume 259, Number 1246
June 2019, 81 pages, Softcover, ISBN: 978-1-4704-3567-7, 2010 *Mathematics Subject Classification:* 37E05, 37D25, 37D35, List US$81, **AMS Individual member US$48.60**, AMS Institutional member US$64.80, **MAA members US$72.90**, Order code MEMO/259/1246

bookstore.ams.org/memo-259-1246

## Differential Equations

### On Space-Time Quasiconcave Solutions of the Heat Equation

**Chuanqiang Chen**, *Zhejiang University of Technology, Hangzhou, China*, **Xinan Ma**, *University of Science and Technology of China, Hefei, China*, and **Paolo Salani**, *Università di Firenze, Italy*

**Memoirs of the American Mathematical Society**, Volume 259, Number 1244
June 2019, 83 pages, Softcover, ISBN: 978-1-4704-3524-0, 2010 *Mathematics Subject Classification:* 35K20; 35B30, List US$81, **AMS Individual member US$48.60**, AMS Institutional member US$64.80, **MAA members US$72.90**, Order code MEMO/259/1244

bookstore.ams.org/memo-259-1244

## Mathematical Physics

### Spinors on Singular Spaces and the Topology of Causal Fermion Systems

**Felix Finster**, *Universität Regensburg, Germany*, and **Niky Kamran**, *McGill University, Montreal, Canada*

**Memoirs of the American Mathematical Society**, Volume 259, Number 1251
June 2019, 82 pages, Softcover, ISBN: 978-1-4704-3621-6, 2010 *Mathematics Subject Classification:* 53–02; 53Z05, 53C80, 53C27, 57R22, List US$81, **AMS Individual member US$48.60**, AMS Institutional member US$64.80, **MAA members US$72.90**, Order code MEMO/259/1251

bookstore.ams.org/memo-259-1251

### Distribution of Resonances in Scattering by Thin Barriers

**Jeffrey Galkowski**, *McGill University, Montreal, Canada*

**Memoirs of the American Mathematical Society**, Volume 259, Number 1248
June 2019, 153 pages, Softcover, ISBN: 978-1-4704-3572-1, 2010 *Mathematics Subject Classification:* 35P20, 35P25, 31B10, 31B20, 31B35; 45C05, 47F05, List US$81, **AMS Individual member US$48.60**, AMS Institutional member US$64.80, **MAA members US$72.90**, Order code MEMO/259/1248

bookstore.ams.org/memo-259-1248

## Probability and Statistics

### Time Changes of the Brownian Motion: Poincaré Inequality, Heat Kernel Estimate and Protodistance

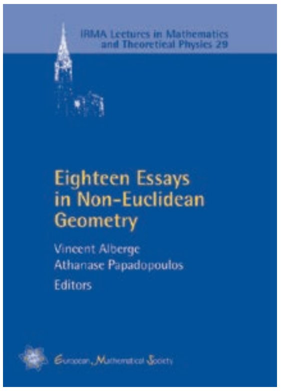
**Jun Kigami**, *Kyoto University, Japan*

**Memoirs of the American Mathematical Society**, Volume 259, Number 1250
June 2019, 118 pages, Softcover, ISBN: 978-1-4704-3620-9, 2010 *Mathematics Subject Classification:* 31E05, 60J35, 60J60; 28A80, 30L10, 43A99, 60J65, 80A20, List US$81, **AMS Individual member US$48.60**, AMS Institutional member US$64.80, **MAA members US$72.90**, Order code MEMO/259/1250

bookstore.ams.org/memo-259-1250

# New AMS-Distributed Publications

## Algebra and Algebraic Geometry




### Eighteen Essays in Non-Euclidean Geometry

**Vincent Alberge**, *Fordham University, Bronx, NY*, and **Athanase Papadopoulos**, *Université de Strasbourg, France*, Editors

This book consists of a series of self-contained essays in non-Euclidean geometry in a broad sense, including the classical geometries of constant curvature (spherical and hyperbolic), de Sitter, anti-de Sitter, co-Euclidean, co-Minkowski, Hermitian geometries, and some axiomatically defined geometries. Some of these essays deal with very classical questions and others address problems that are at the heart of present-day research, but all of them are concerned with fundamental topics.

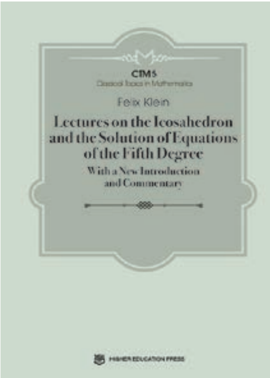
All the essays are self-contained, and most of them can be understood by the general educated mathematician. They should be useful to researchers and to students of non-Euclidean geometry and are intended to be references for the various topics they present.

*This item will also be of interest to those working in geometry and topology.*

A publication of the European Mathematical Society. Distributed within the Americas by the American Mathematical Society.

**IRMA Lectures in Mathematics and Theoretical Physics**, Volume 29
March 2019, 475 pages, Hardcover, ISBN: 978-3-03719-196-5, 2010 *Mathematics Subject Classification:* 03B30, 51A05, 51F15, 51F20, 51M04, 51M05, 51M09, 51M10, 51M16, 51M20, 51M25, 51N15, 52A15, 52A55, 52B15, 52C25, 05C62, 53–01, 53A05, 53A30, 53A35, 53C35, 53C45, List US$88, **AMS members US$70.40**, Order code EMSILMTP/29

bookstore.ams.org/emsilmtp-29




### Lectures on the Icosahedron and the Solution of Equations of the Fifth Degree

With a New Introduction and Commentary

**Felix Klein**

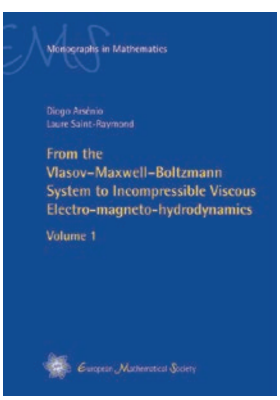
The highest achievement of Greek mathematics is the classification of regular solids, the five so-called Platonic solids. The most complicated solid is the icosahedron. Up to and through the 19th century, the most important problem in mathematics was solving algebraic equations. In this classic book, Klein showed how to relate these two seemingly unrelated topics and also tied them together with another new theory of mathematics: hypergeometric functions and monodromy groups.

A publication of Higher Education Press (Beijing). Exclusive rights in North America; non-exclusive outside of North America. No distribution to mainland China unless order is received through the AMS bookstore. Online bookstore rights worldwide. All standard discounts apply.

**Classical Topics in Mathematics**, Volume 5
May 2019, 306 pages, Hardcover, ISBN: 978-7-04-051022-5, 2010 *Mathematics Subject Classification:* 12D10, 30C15, 13A50, 51M20, 33C05, List US$99, **AMS members US$79.20**, Order code CTM/5

bookstore.ams.org/ctm-5

# Analysis




### From the Vlasov–Maxwell–Boltzmann System to Incompressible Viscous Electro-magneto-hydrodynamics

Volume 1

**Diogo Arsénio**, *Université Paris Diderort, France*, and **Laure Saint-Raymond**, *École Normale Supérieure, Lyon, France*

The Vlasov–Maxwell–Boltzmann system is a microscopic model to describe the dynamics of charged particles subject to self-induced electromagnetic forces. At the macroscopic scale, in the incompressible viscous fluid limit, the evolution of the plasma is governed by equations of Navier–Stokes–Fourier type, with some electromagnetic forcing that may take on various forms depending on the number of species and on the strength of the interactions. From the mathematical point of view, these models have very different behaviors. Their analysis, therefore, requires various mathematical methods which this book aims to present in a systematic, painstaking, and exhaustive way.

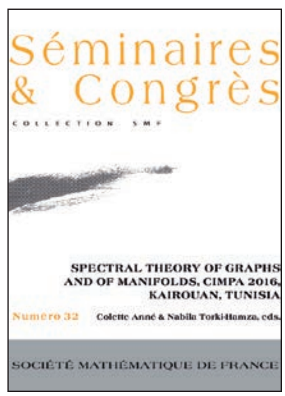
*This item will also be of interest to those working in mathematical physics.*

A publication of the European Mathematical Society (EMS). Distributed within the Americas by the American Mathematical Society.

**EMS Monographs in Mathematics**, Volume 9
April 2019, 418 pages, Hardcover, ISBN: 978-3-03719-193-4, 2010 *Mathematics Subject Classification:* 76P05, 76W05, 82C40, 35B25, List US$88, **AMS members US$70.40**, Order code EMSMONO/9

bookstore.ams.org/emsmono-9

# Applications




### Spectral Theory of Graphs and of Manifolds

CIMPA 2016, Kairouan, Tunisia

**Colette Anné**, *Laboratoire de Mathématiques Jean Leray, Nantes, France*, and **Nabila Torki-Hamza**, *Universitaire de Mahdia, Tunisie*, Editors

This volume is devoted to the Spectral Theory on Graphs and Manifolds, the CIMPA Research School which took place at Kairouan (Tunisia) in November 2016. The school offered six courses and two conferences.

This volume contains the redaction of five of the presentations: Spectral Theory on Combinatorial and Quantum Graphs, by E. M. Harrell; Introduction to Spectral Theory of Unbounded Operators, by H. Najar; On the Absolute Continuous Spectrum of Discrete Operators, by S. Golénia; Random Schrödinger Operators on Discrete Structures, by C. Rojas-Molina; Critical Points at Infinity Theory on CR Manifolds, by N. Gamara. Geometric Bounds on the Eigenvalues of Graphs, by N. Anantaraman, is just summarized, as it was podcasted and is still available on the Internet. The volume concludes with the text of L. Hillairet's conference on Two Applications of the Dirichlet-Neumann Bracketing.

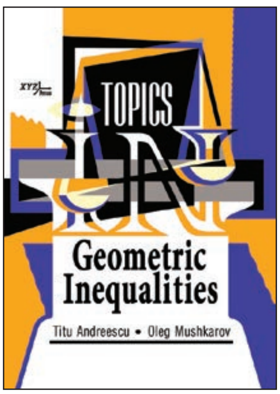
*This item will also be of interest to those working in geometry and topology.*

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

**Séminaires et Congrès**, Number 32
December 2018, 261 pages, Softcover, ISBN: 978-2-85629-895-4, 2010 *Mathematics Subject Classification:* 35P05, 39A05, 58J05, 05C05, 47A05, 81Q35, 32V30, List US$42, **AMS members US$33.60**, Order code SECO/32

bookstore.ams.org/seco-32

# Math Education




## Topics in Geometric Inequalities

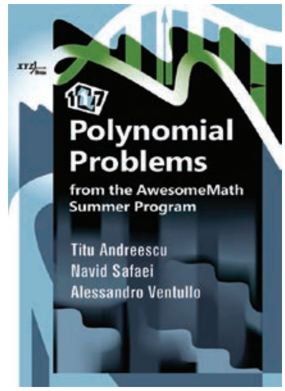
**Titu Andreescu**, *University of Texas at Dallas*, and **Oleg Mushkarov**, *Bulgaria Academy of Science*

Cross discipline discoveries with multiple solutions provided makes *Topics in Geometric Inequalities* a must have for dedicated problem solvers that will strengthen readers' abilities to analyze, dissect, and invent creative methods—all skills that are necessary to succeed in mathematics competitions. Readers will also learn innovative and powerful techniques in geometric arguments.

A publication of XYZ Press. Distributed in North America by the American Mathematical Society.

**XYZ Series**, Volume 34
February 2019, 420 pages, Hardcover, ISBN: 978-0-9993428-3-1, 2010 *Mathematics Subject Classification:* 00A05, 00A07, 97U40, 97D50, List US$59.95, **AMS members US$47.96**, Order code XYZ/34

bookstore.ams.org/xyz-34




## 117 Polynomial Problems from the AwesomeMath Summer Program

**Titu Andreescu**, *University of Texas at Dallas*, **Navid Safaei**, *Sharif University of Technology, Tehran, Iran*, and **Alessandro Ventullo**, *University of Milan, Italy*

Polynomials form the cornerstone of modern mathematics and other discrete fields, and this book will showcase the true beauty of polynomials through a discerning collection of problems from mathematics competitions and intuitive lectures. Through the problems, lecture, and theory, readers will gain the knowledge, strategies, and tricks to fully appreciate, solve, and enjoy solving polynomials.

A publication of XYZ Press. Distributed in North America by the American Mathematical Society.

**XYZ Series**, Volume 33
February 2019, 234 pages, Hardcover, ISBN: 978-0-9993428-4-8, 2010 *Mathematics Subject Classification:* 00A07, 97U40, 97D50, List US$59.95, **AMS members US$47.96**, Order code XYZ/33

bookstore.ams.org/xyz-33

# CELEBRATING WOMEN MATHEMATICIANS

### RESEARCHERS & ROLE MODELS

## Rosemary Guzman

Topology and its applications in other areas of mathematics

"Form your networks with intent, understanding that the formation of different kinds of networks—research, mentoring, and teaching—serve distinct purposes."

## Fern Y. Hunt

Discrete approximation of dynamical systems and applications of Markov chains

"Attend as many research conferences as time and money allow, make use of online resources such as the arXiv and MathSciNet, and organize a special session at an annual conference."

## Emily Riehl

Category theory, particularly as related to homotopy theory

"Ultimately, how you want to spend your time engaging with mathematical ideas is up to you. Give talks and take on extracurricular writing or teaching projects if this makes you happier, even if this means less time for research and other things."

## Éva Tardos

Algorithms and algorithmic game theory

"Be open to exciting new opportunities in research."

## Amie Wilkinson

Dynamical systems

"Enjoy the process of discovery, there's an infinite zoo of possibilities to explore and that's the joy of mathematics."

## Melody Chan

Tropical geometry, combinatorialalgebraic geometry and combinatorics

"Do examples! Try to do as much mathematics as you can standing at the board, writing things down, and explaining them to people."

## Tara S. Holm

Symplectic geometry and its applications in other areas of mathematics

"Find collaborators, especially ones who are close enough to your field so that you can talk but far enough that they will teach you some new mathematics."

## Andrea Nahmod

Nonlinear Fourier and harmonic analysis and partial differential equations

"Have a broad mathematical culture, follow your intuition, keep a long view about research, and love what you do."

## Gigliola Staffilani

Partial differential equations that model nonlinear wave phenomena

"Work with other mathematicians. The model of the lonely researcher in an ivory tower does not match with most of the mathematicians I know. This is a myth that definitely needs to be busted: it is dangerous and not encouraging."

## Chelsea Walton

Noncommutative algebra and noncommutative algebraic geometry

"Find, value, and support your network of people, in math or not, who can selflessly give you words of encouragement, because the happier you are, the more math you will do!"

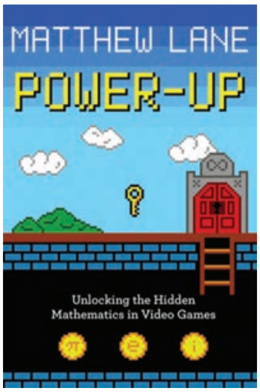
Read about the mathematical research, inspiring stories, and advice of these AMS members and more women researchers and role models at

**www.ams.org/women-18**

AMS American Mathematical Society

# BOOKSHELF

## *New and Noteworthy Titles on our Bookshelf*
## June/July 2019




Princeton, 2017, 264 pages

***Power-Up:***
*Unlocking the Hidden Mathematics in Video Games*
by Matthew Lane

This book consists of nine chapters, the first eight of which cover aspects of video games that prompt lively mathematical digressions. A good calculus student with some patience should be able to follow most of the mathematics. The final chapter ("The Value of Games") discusses the complex relationship between mathematics, education, and society, often quoting from enlightening sources.

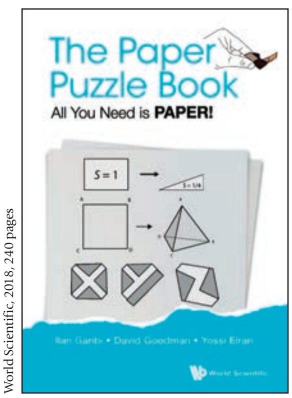
The games discussed vary from old favorites such as *Pac-Man* and *Minesweeper*, to newer games such as *Assassin's Creed* and *The Sims*. Although some familiarity with the games might make the reading experience more enjoyable, it is not a prerequisite. The relevant game mechanics are explained with plenty of illustrations, many of which are in color.

The mathematics involved is mostly of the Cartesian geometry, calculus, and probability sort, although deeper topics such as the uncountability of real numbers occasionally arise. These topics are not explored in great detail, but rather sprinkled throughout the book to suggest added riches just below the surface.

How exactly does math enter a discussion about video games? Here are a few examples. What is the best way to gather power-up items that are spread throughout an open "sandbox"-style game? This leads to a discussion of the traveling salesman problem, complexity theory, and the *P* versus *NP* problem. Why do quiz-type games seem to repeat the same questions so often? This prompts a discussion of probability and the birthday paradox. In a similar manner, software developers' attempts to rank game features by user satisfaction ratings lead to an account of voting theory and Arrow's Impossibility Theorem.

Although the mathematics surveyed is mostly elementary, the algebraic details are often messy. Fortunately, addendum sections attached to some of the chapters contain the more onerous calculations, a few of which stretch on for several pages. The gory details are worked out in full detail and with plenty of explanations. This approach permits the main text to proceed in a conversational and elegant fashion.

This book is suitable for any mathematically inclined reader who enjoys video games. For example, this might be a wonderful book to give to a high-school calculus student with a passion for video games.




World Scientific, 2018, 240 pages

***The Paper Puzzle Book:***
*All You Need is Paper!*
by Ilan Garibi, David Goodman, Yossi Elran

This book contains almost one hundred clever paper-based puzzles, tricks, and gags in the spirit of Martin Gardner, whom the authors cite as an inspiration. Those who delight in puzzles will find this book fascinating and enjoyable. Illustrations appear on almost every page, and the book is easy to follow.

The puzzles are loosely organized according to themes such as "just folding," "strips of paper," and "flexagons." Hints are occasionally provided, and complete solutions appear at the end of each chapter. The puzzles often require paper of specific dimensions; each puzzle is accompanied by an icon that denotes those requirements. The puzzles are assigned a difficulty level from one to four stars. Some are relatively easy, although even the simplest require a bit of thought, visualization, and experimentation. For example, one instructs the reader to "fold an equilateral triangle from a square sheet of paper." Another wants one to take a paper strip in proportion 1:7 and fold it into a cube of size 1:1:1. The harder problems are true brain benders of the most pleasant sort.

Although the problems are implicitly infused with mathematics, the book is accessible to almost anyone who has taken trigonometry and knows about square roots. It could even be enjoyed by a motivated middle-school student.

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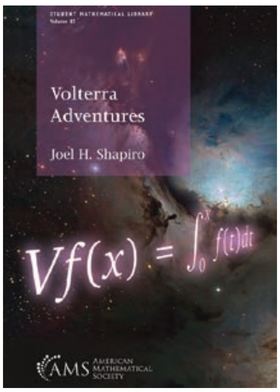
*The Bookshelf is prepared monthly by* Notices *Associate Editor Stephan Ramon Garcia.*

*Appearance of a book in the* Notices *Bookshelf does not represent an endorsement by the* Notices *or by the AMS.*

*Suggestions for the Bookshelf can be sent to* `notices-booklist@ams.org`.

# AMS BOOKSHELF

*The AMS Book Program serves the mathematical community by publishing books that further mathematical research, awareness, education, and the profession while generating resources that support other Society programs and activities. As a professional society of mathematicians and one of the world's leading publishers of mathematical literature, we publish books that meet the highest standards for their content and production.* Visit **bookstore.ams.org** to explore the entire collection of AMS titles.




AMS Student Mathematical Library, 219 pages

## *Volterra Adventures*

Joel H. Shapiro

In his preface, Shapiro quotes Louis Auslander, "Mathematics is like a river. You just jump in...the current will take you where you need to go." Shapiro chooses to jump in at the Volterra operator, $V : C([0, a]) \to C^1([0, a])$:

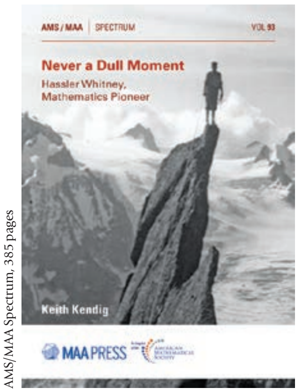
$$V(f)(x) = \int_0^x f(t)\, dt.$$

He then follows a meandering, but natural, current through regions of functional analysis. First, we observe that the operator acts linearly on $C(0, a)$ and the natural linear algebra questions all make perfect sense in this infinite-dimensional setting. The essential idea communicated early on is that we can conceive of functions as points in these spaces and do analysis and algebra there. He converts the initial-value problem for a mass-spring system into an integral equation and notes that existence of a unique solution to the IVP is equivalent to invertibility of the derived linear operator.

So now we're asking questions about kernels of operators and it's just a short step to considering the Banach space of bounded operators between Banach spaces. Then a simple geometric series argument establishes the invertibility of the operator associated with a Volterra kernel. We're led to convolution operators and, eventually, a proof of the Titchmarsh Convolution Theorem.

The goal from the beginning has been the Volterra Invariant Subspace Theorem. The only invariant subspaces of $V$ are exactly the spaces of functions identically zero on $[0, b]$, as Shapiro says, just the "obvious" ones.

Shapiro is a terrific guide carefully pointing out where we're going, why the questions are natural, and how they relate to what the reader already knows. This book would make a strong independent study course for a motivated undergraduate, or a lovely preview for a graduate student.




AMS/MAA Spectrum, 385 pages

## *Never a Dull Moment*

*Hassler Whitney, Mathematics Pioneer*

Keith Kendig

As an undergraduate at Yale in the 1920s, Hassler Whitney took only one mathematics class, an advanced class in complex variables. He was admitted to Harvard for graduate study in physics in 1930. Whereupon he realized with dismay that physicists had to remember stuff, which he was bad at. He switched to mathematics and began a lifelong pattern. He looked around for an easily stated problem upon which he could use his instinct for experimentation. He wanted a problem that would yield to devoted study of geometric examples. He found one in the four-color theorem of graph theory. He discovered a new characterization of planarity; a simple condition that guaranteed the existence of a Hamiltonian cycle; and some tricks for computing chromatic polynomials. It was enough for a PhD under G.D. Birkhoff.

Whitney's extension theorems, his embedding theorems, his work on singularities, and his (co-)invention of the cup product all followed a similar pattern. He would manipulate geometric examples until he found those that typified the behavior he was trying to understand. Then he would express the results for publication, carefully hiding all traces of the geometric inspiration.

Keith Kendig first met Whitney in the 1950s. With shared passions for music and geometry, the two became lifelong friends. As a result, Kendig became very familiar with Whitney's mathematical thinking; that familiarity is on display here. He also learned Whitney family lore (Whitney's grandfather Simon Newcomb was the fourth president of the AMS) and Whitney's habits and hobbies: mountain climbing, mechanical tinkering, chamber music. This closeness makes for a very intimate—personally and mathematically—biography of a great mathematician.

*The AMS Bookshelf is prepared bimonthly by AMS Acquisitions Specialist for MAA Press titles Stephen Kennedy. His email address is* kennedy.maapress@ams.org.

# Meetings & Conferences of the AMS

## June/July Table of Contents

The Meetings and Conferences section of the *Notices* gives information on all AMS meetings and conferences approved by press time for this issue. Please refer to the page numbers cited on this page for more detailed information on each event.

Invited Speakers and Special Sessions are listed as soon as they are approved by the cognizant program committee; the codes listed are needed for electronic abstract submission. For some meetings the list may be incomplete. Information in this issue may be dated.

The most up-to-date meeting and conference information can be found online at: `www.ams.org/meetings`.

**Important Information About AMS Meetings:** Potential organizers, speakers, and hosts should refer to page 127 in the January 2019 issue of the *Notices* for general information regarding participation in AMS meetings and conferences.

**Abstracts:** Speakers should submit abstracts on the easy-to-use interactive Web form. No knowledge of LaTeX is necessary to submit an electronic form, although those who use LaTeX may submit abstracts with such coding, and all math displays and similarly coded material (such as accent marks in text) must be typeset in LaTeX. Visit `www.ams.org/cgi-bin/abstracts/abstract.pl`. Questions about abstracts may be sent to `abs-info@ams.org`. Close attention should be paid to specified deadlines in this issue. Unfortunately, late abstracts cannot be accommodated.

### Associate Secretaries of the AMS

**Central Section:** Georgia Benkart, University of Wisconsin–Madison, Department of Mathematics, 480 Lincoln Drive, Madison, WI 53706-1388; email: `benkart@math.wisc.edu`; telephone: 608-263-4283.

**Eastern Section:** Steven H. Weintraub, Department of Mathematics, Lehigh University, Bethlehem, PA 18015-3174; email: `steve.weintraub@lehigh.edu`; telephone: 610-758-3717.

**Southeastern Section:** Brian D. Boe, Department of Mathematics, University of Georgia, 220 D W Brooks Drive, Athens, GA 30602-7403; email: `brian@math.uga.edu`; telephone: 706-542-2547.

**Western Section:** Michel L. Lapidus, Department of Mathematics, University of California, Surge Bldg., Riverside, CA 92521-0135; email: `lapidus@math.ucr.edu`; telephone: 951-827-5910.

## Meetings in this Issue



*See* `www.ams.org/meetings` *for the most up-to-date information on the meetings and conferences that we offer.*

# Meetings & Conferences of the AMS

IMPORTANT information regarding meetings programs: AMS Sectional Meeting programs do not appear in the print version of the *Notices*. However, comprehensive and continually updated meeting and program information with links to the abstract for each talk can be found on the AMS website. See `www.ams.org/meetings`.

Final programs for Sectional Meetings will be archived on the AMS website accessible from the stated URL.

## Madison, Wisconsin

*University of Wisconsin–Madison*

### September 14–15, 2019

*Saturday – Sunday*

### Meeting #1150

Central Section

Associate secretary: Georgia Benkart

Announcement issue of *Notices*: June 2019
Program first available on AMS website: July 23, 2019
Issue of *Abstracts*: Volume 40, Issue 3

### Deadlines

For organizers: Expired
For abstracts: July 16, 2019

*The scientific information listed below may be dated. For the latest information, see* `www.ams.org/amsmtgs/sectional.html`.

### Invited Addresses

**Nathan Dunfield**, University of Illinois, Urbana–Champaign, *Fun with finite covers of 3-manifolds: connections between topology, geometry, and arithmetic*.

**Teena Gerhardt**, Michigan State University, *Invariants of rings via equivariant homotopy*.

**Lauren Williams**, University of California, Berkeley, *Title to be announced* (Erdős Memorial Lecture).

### Special Sessions

*If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at* `www.ams.org/cgi-bin/abstracts/abstract.pl`.

*Algebraic and Geometric Combinatorics* (Code: SS 12A), **Benjamin Braun**, University of Kentucky, **Marie Meyer**, Lewis University, and **McCabe Olsen**, Ohio State University.

*Analysis and Probability on Metric Spaces and Fractals* (Code: SS 10A), **Guy C. David**, Ball State University, and **John Dever**, Bowling Green State University.

*Applications of Algebra and Geometry* (Code: SS 38A), **Shamgar Gurevich** and **Jose Israel Rodriguez**, University of Wisconsin–Madison.

*Arithmetic of Shimura Varieties* (Code: SS 26A), **Chao Li**, Columbia University, and **Solly Parenti** and **Tonghai Yang**, University of Wisconsin–Madison.

*Association Schemes and Related Topics—in Celebration of J.D.H. Smith's 70th Birthday* (Code: SS 8A), **Kenneth W. Johnson**, Penn State University Abington, and **Sung Y. Song**, Iowa State University.

*Automorphic Forms and L-Functions* (Code: SS 16A), **Simon Marshall** and **Ruixiang Zhang**, University of Wisconsin–Madison.

*Categorical Gromov-Witten Invariants and Mirror Symmetry* (Code: SS 42A), **Andrei Caldararu**, University of Wisconsin–Madison, and **Junwu Tu**, University of Missouri–Columbia and Shanghai Tech University.

*Classical and Geophysical Fluid Dynamics: Modeling, Reduction and Simulation* (Code: SS 17A), **Nan Chen**, University of Wisconsin–Madison, and **Honghu Liu**, Virginia Tech University.

*Combinatorial Algebraic Geometry* (Code: SS 21A), **Juliette Bruce** and **Daniel Erman**, University of Wisconsin–Madison, **Chris Eur**, University of California Berkeley, and **Lily Silverstein**, University of California Davis.

*Commutative Algebra: in Celebration of the 150th Birthday of Roger and Sylvia Wiegand* (Code: SS 22A), **Nicholas Baeth**, Franklin & Marshall College, and **Graham Leuschke**, Syracuse University.

*Computability Theory in honor of Steffen Lempp's 60th birthday* (Code: SS 6A), **Joseph S. Miller**, **Noah D. Schweber**, and **Mariya I. Soskova**, University of Wisconsin–Madison.

*Connecting Network Structure and Behavior of Biological Interaction Systems* (Code: SS 31A), **David Anderson**, **Gheorghe Craciun**, and **Abhishek Deshpande**, University of Wisconsin–Madison.

*Connections between Noncommutative Algebra and Algebraic Geometry* (Code: SS 15A), **Jason Gaddis** and **Dennis Keeler**, Miami University.

*Extremal Graph Theory* (Code: SS 14A), **Józef Balogh**, University of Illinois, and **Bernard Lidický**, Iowa State University.

*Floer Homology in Dimensions 3 and 4* (Code: SS 29A), **Jianfeng Lin**, UC San Diego, and **Christopher Scaduto**, University of Miami.

*Fully Nonlinear Elliptic and Parabolic Partial Differential Equations, Local and Nonlocal* (Code: SS 25A), **Fernando Charro**, Wayne State University, **Stefania Patrizi**, The University of Texas at Austin, and **Peiyong Wang**, Wayne State University.

*Functional Analysis and Its Applications* (Code: SS 30A), **Clement Boateng Ampadu**, Boston, MA, and **Waleed Al-Rawashdeh**, Montana Tech University.

*Geometry and Topology in Arithmetic* (Code: SS 41A), **Rachel Davis**, University of Wisconsin–Madison.

*Geometry and Topology of Singularities* (Code: SS 13A), **Laurentiu Maxim**, University of Wisconsin–Madison.

*Hall Algebras, Cluster Algebras and Representation Theory* (Code: SS 27A), **Xueqing Chen**, UW-Whitewater, and **Yiqiang Li**, SUNY at Buffalo.

*Hodge Theory in Honor of Donu Arapura's 60th Birthday* (Code: SS 11A), **Ajneet Dhillon**, University of Western Ontario, **Kenji Matsuki** and **Deepam Patel**, Purdue University, and **Botong Wang**, University of Wisconsin–Madison.

*Homological and Characteristic $p > 0$ Methods in Commutative Algebra* (Code: SS 1A), **Michael Brown**, University of Wisconsin–Madison, and **Eric Canton**, University of Michigan.

*Homotopy Theory* (Code: SS 34A), **Gabe Angelini-Knoll** and **Teena Gerhardt**, Michigan State University, and **Bertrand Guillou**, University of Kentucky.

*Large Scale Properties of Interacting Stochastic Systems* (Code: SS 33A), **Timo Seppalainen**, **Hao Shen**, and **Benedek Valko**, University of Wisconsin–Madison.

*Lie Representation Theory* (Code: SS 19A), **Mark Colarusso**, University of South Alabama, **Michael Lau**, Université Laval, and **Matt Ondrus**, Weber State University.

*Model Theory* (Code: SS 5A), **Uri Andrews** and **Omer Mermelstein**, University of Wisconsin–Madison.

*Nonlinear Dispersive Equations and Water Waves* (Code: SS 37A), **Mihaela Ifrim**, University of Wisconsin–Madison, and **Daniel Tataru**, University of California, Berkeley.

*Number Theory and Cryptography* (Code: SS 40A), **Eric Bach**, University of Wisconsin–Madison, and **Jon Sorenson**, Butler University.

*Quasigroups and Loops—in honor of J.D.H. Smith's 70th birthday* (Code: SS 35A), **J.D. Phillips**, Northern Michigan University, and **Petr Vojtechovsky**, University of Denver.

*Recent Developments in Harmonic Analysis* (Code: SS 3A), **Theresa Anderson**, Purdue University, and **Joris Roos**, University of Wisconsin–Madison.

*Recent Trends in the Mathematics of Data* (Code: SS 39A), **Sebastien Roch**, University of Wisconsin–Madison, **David Sivakoff**, Ohio State University, and **Joseph Watkins**, University of Arizona.

*Recent Work in the Philosophy of Mathematics* (Code: SS 4A), **Thomas Drucker**, University of Wisconsin–Whitewater, and **Dan Sloughter**, Furman University.

*Relations Between the History and Pedagogy of Mathematics* (Code: SS 32A), **Emily Redman**, University of Massachusetts, Amherst, **Brit Shields**, University of Pennsylvania, and **Rebecca Vinsonhaler**, University of Texas, Austin.

*Several Complex Variables* (Code: SS 7A), **Hanlong Fang** and **Xianghong Gong**, University of Wisconsin–Madison.

*Special Functions and Orthogonal Polynomials* (Code: SS 2A), **Sarah Post**, University of Hawaiʻi at Manoa, and **Paul Terwilliger**, University of Wisconsin–Madison.

*Stochastic Partial Differential Equations and Related Fields* (Code: SS 28A), **Igor Cialenco**, Illinois Institute of Technology, **Yu Gu**, Carnegie Mellon University, and **Hyun-Jung Kim**, Illinois Institute of Technology.

*Supergeometry, Poisson Brackets, and Homotopy Structures* (Code: SS 36A), **Ekaterina Shemyakova**, University of Toledo, and **Theodore Voronov**, University of Manchester.

*Topics in Graph Theory and Combinatorics* (Code: SS 20A), **Songling Shan** and **Papa Sissokho**, Illinois State University.

*Topology and Descriptive Set Theory* (Code: SS 18A), **Tetsuya Ishiu** and **Paul B. Larson**, Miami University.

*Uncertainty Quantification Strategies for Physics Applications* (Code: SS 9A), **Qin Li**, University of Wisconsin–Madison, and **Tulin Kaman**, University of Arkansas.

*Wave Phenomena in Fluids and Relativity* (Code: SS 24A), **Sohrab Shahshahani**, University of Massachusetts, and **Willie W.Y. Wong**, Michigan State University.

*Zero Forcing, Propagation, and Throttling* (Code: SS 23A), **Josh Carlson**, Iowa State University, and **Nathan Warnberg**, University of Wisconsin–La Crosse.

## Accommodations

Participants should make their own arrangements directly with the hotel of their choice. Special discounted rates were negotiated with the hotels listed below. Rates quoted do not include the Wisconsin state hotel tax (5.5%), city tax (10%), and other hotel fees. Participants must state that they are with the **American Mathematical Society's (AMS) Fall Central Sectional Meeting** to receive the discounted rate. The AMS is not responsible for rate changes or for the quality of the accommodations. **Hotels have varying cancellation and early checkout penalties; be sure to ask for details.**

**AC Hotel Madison Downtown,** 1 North Webster Street, Madison, WI 53703; (608) 286-1337; `www.achotelmadison.com`. Rates are **US$145** per night for a standard room. Amenities include AC Kitchen serving a fresh European-influenced breakfast, Eno Vino Wine Bar and Bistro located on the ninth and tenth floors with panoramic views of Madison, and AC Lounge serving espresso, tapas, and cocktails. Valet parking is available for **US$20** per car a night. The parking rate is subject to change and the parking fee in effect during the dates of your stay will be charged. This property is located about a five-minute drive from campus. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. The deadline for reservations at this rate is **noon local time on August 14, 2019.**

**Best Western Plus Inn Towner,** 2424 University Avenue, Madison, WI 53726; (608) 233-8778. Rates are **US$119** per night for a room with one king or two queen beds. Guests should visit `www.inntowner.com/reservations` to make a reservation online. In the 'Group Code' field on the left-hand side of the page enter AMS2019 and hit 'enter.' Amenities include complimentary hot breakfast buffet, complimentary airport shuttle, swimming pool/whirlpool, business center, fitness center, and free self-parking. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. This property is located about a six-minute drive from campus. The deadline for reservations at this rate is by **August 14, 2019.**

**Best Western Premier Park Hotel,** 22 S. Carroll Street, Madison, WI 53703; (608) 285-8000. Rates are **US$189** per night for a traditional room with one or two beds. Guests should visit `https://www.bestwestern.com/en_US/book/hotel-rooms.50061.html?groupId=W11NQ1A1` to make a reservation online. Amenities include free shuttle service to and from the airport, free Wi-Fi in all guest rooms and hotel common areas, refrigerators and microwaves in all guest rooms, on-site restaurant and lounge, and modern fitness center. The valet parking rate is **US$15** a day for one vehicle per room. The parking rate is subject to change and the parking fee in effect during the dates of your stay will be charged. This property is located about an eight-minute drive from the campus. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. The deadline for reservations at this rate is **August 13, 2019.**

**Doubletree by Hilton Hotel Madison,** 525 W. Johnson Street, Madison, WI 53703; (608) 251-5511. Rates are **US$159** per night for a single or double bed room. Guests should visit `https://doubletree.hilton.com/en/dt/groups/personalized/M/MSNDTDT-AMS-20190913/index.jhtml?WT.mc_id=POG` to make a reservation online or call the hotel at (608) 251-5511 and reference code 'AMS.' Amenities include complimentary parking, complimentary airport and downtown shuttle service, complimentary Wi-Fi, twenty-four-hour fitness center, business center, on-site restaurant, Starbucks Café, and microwaves and refrigerators in every room. This property is located about a five-minute drive from campus. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. The deadline for reservations at this rate is **August 14, 2019.**

**Graduate Madison,** 601 Langdon Street, Madison, WI 53703; (608) 257-4391. Rates are **US$169** per night for a standard room. Guests should visit `https://gc.synxis.com/rez.aspx?Hotel=77206&Chain=21643&arrive=9/13/2019&depart=9/16/2019&adult=1&child=0&group=190913AMER` to book a reservation online. Amenities include complimentary bike rentals, complimentary shuttle service, fitness center, lobby-level café, and rooftop hangout. Current valet parking rates include unlimited in and out privileges for **US$25** per day. The parking rate is subject to change and the parking fee in effect during the dates of your stay will be charged. This property is located about a block from campus. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. The deadline for reservations at this rate is **August 13, 2019.**

**Hilton Madison Monona Terrace,** 9 E. Wilson Street, Madison, WI 53703; (608) 255-5100; `https://www3.hilton.com/en/hotels/wisconsin/hilton-madison-monona-terrace-MSNMHHF/index.html`. Rates are **US$179** per night for a standard room. Amenities include Pavilion Pantry Market, bar area, gift shop, guest activity/recreation desk, laundry/valet service, local area transportation, lounge, on-site convenience store, and room service. The current on-site self-parking rate is **US$17** per day and the current valet parking rate is **US$20** per day. The parking rates are subject to change and the parking fee in effect during the dates of your stay will be charged. This property is located about an eight-minute drive from campus. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. The deadline for reservations at this rate is **August 14, 2019.**

**Lowell Center,** 610 Langdon Street, Madison, WI 53703; (608) 256-2621; `https://pyle.wisc.edu/hotel-accommodations`. Rates are **US$149** based on double occupancy per night in a deluxe room. More than two guests will be **US$12** per person per night. To book online visit `bit.ly/mathassociation`. Amenities include complimentary Wi-Fi, indoor swimming pool, sauna and fitness rooms, and complimentary breakfast buffet served daily in the dining room. On-site parking is **US$10** a day for overnight guests and complimentary parking is available nearby. The parking rate is subject to change and the parking fee in effect during the dates of your stay will be charged. This property is located about a fifteen-minute walk from meeting space on campus. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. The deadline for reservations at this rate is **August 13, 2019.**

**Sheraton Madison,** 706 John Nolen Drive, Madison, WI 53713; (608) 251-2300. Rates are **US$139** for a standard room. Guests should visit `https://www.marriott.com/event-reservations/reservation-link.mi?id=1553525302604&key=GRP&app=resvlink` to make a reservation online. Amenities include complimentary shuttle service to and from the Dane County Regional Airport and downtown Madison area, free Wi-Fi available in the lobby business center and throughout the hotel, twenty-four-hour fitness center, free self-parking, indoor pool, and hot tub. This property is located about an eleven-minute drive from campus. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. The deadline for reservations at this rate is **5:00pm local time on August 14, 2019.**

**The Edgewater,** 1001 Wisconsin Avenue, Madison, WI 53703; (608) 535-8200; `https://www.theedgewater.com`. Rates are **US$199** for a king room. A one-night deposit of room, tax, and resort fee is required to secure a reservation. All guest rooms will be assessed a nightly resort fee at the current rate of **US$18** per night. The resort fee includes: wireless and wired Internet, local calls, transportation to and from Dane County Airport (based upon availability), transportation within a two-mile radius of the hotel (based upon availability), two bottles of water in your guest room per day, business center, The Edgewater Fitness Club, and access to The Edgewater Spa relaxation pool and steam room. This property is located about a five-minute drive from campus. Overnight parking rates are **US$25** for valet and **US$18** for self-parking. Both overnight options include unlimited in and out privileges. The parking rate is subject to change and the parking fee in effect during the dates of your stay will be charged. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. The deadline for reservations at this rate is **August 1, 2019.**

**The Madison Concourse Hotel and Governor's Club,** 1 W. Dayton Street, Madison, WI 53703; (608) 257-6000; `www.concoursehotel.com`. Rates are **US$189** for a concourse premier level guest room. To book online visit `https://reservations.travelclick.com/6388?groupID=2515736.` Amenities include complimentary wireless Internet access, compact refrigerator, weekday newspaper, Starbucks, and in-room dining. The current parking rate for overnight guests is **US$15.** Additional parking facilities are located within one block of the hotel. The parking rate is subject to change and the parking fee in effect during the dates of your stay will be charged. This property is located about a five-minute drive from campus. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. The deadline for reservations at this rate is **August 18, 2019.**

## Housing Warning

Please beware of aggressive housing bureaus that target potential attendees of a meeting. They are sometimes called "room poachers" or "room-block pirates" and these companies generally position themselves as a meeting's housing bureau, convincing attendees to unknowingly book outside the official room block. They call people who they think will more likely than not attend a meeting and lure them with room rates that are significantly less than the published group rate—for a limited time only. People who find this offer tempting may hand over their credit card data, believing they have scored a great rate and their housing is a done deal. Unfortunately, this often turns out to be the start of a long, costly nightmare.

Note that some of these room poachers create fake websites on which they represent themselves as the organizers of the meeting and include links to book rooms, etc. The only official website for this meeting is `https://www.ams.org` and one that has the official AMS logo.

These housing bureaus are not affiliated with the American Mathematical Society or any of its meetings, in any way. The AMS would never call anyone to solicit reservations for a meeting. The only way to book a room at a rate negotiated for an AMS Sectional Meeting is via a listing on AMS Sectional Meetings pages or *Notices of the AMS*. The AMS cannot be responsible for any damages incurred as a result of hotel bookings made with unofficial housing bureaus.

## Food Services

**On Campus:** Please visit `https://www.housing.wisc.edu/dining` for further details on dining options. The most up-to-date list of university restaurants hours is located at `https://www.housing.wisc.edu/dining/locations.`

- **Carson's Market**, main level of Carson Gulley Center; open 12:00 pm–8:00 pm on Saturdays and Sundays; offers pizza, deli sandwiches, ethnic entrées, and baked goods.
- **Gordon Avenue Market**, Gordon Dining & Event Center; open 9:00 am–9:00 pm on Saturdays and Sundays; serves home-style comfort foods, coffee, ice cream, fresh baked goods, daily pasta selections, pizza, burgers, and ethnic entrées.

**Off Campus:** During fall you'll find farm-to-table comfort foods on many menus. For an interactive directory of downtown restaurants and attractions visit `https://www.visitmadison.com/restaurants`.

Some options for coffee include:

- **Aldo's Cafe**, 330 N. Orchard Street, Madison; (608) 204-3943; `https://aldoscafemadison.com`; café and deli featuring locally roasted coffee from Just Coffee Cooperative.
- **Ancora Coffee & Tea**, 107 King Street, Madison; (608) 255-0285; `https://ancoracoffee.com`; artisan roasted coffees, delicious soups and sandwiches.
- **Poindexter Coffee**, 601 Langdon Street, Madison; (608) 257-4391; `https://www.graduatehotels.com/madison/restaurant/poindexter-coffee`; located in the lobby of the Graduate Madison, serving coffee, fresh juices, and snacks on the fly.

Some options for downtown dining include:

- **Avenue Club and the Bubble Up Bar**, 1128 E. Washington Avenue, Madison; (608) 257-6877; `https://avenueclubmadison.com`; midcentury modern ambiance, serving cocktails and casual dining in an upbeat atmosphere.
- **Badgerland Bar & Grill**, 525 W. Johnson Street, Madison; (608) 251-5511; `badgerlandbarandgrill.com`; located inside the DoubleTree by Hilton Madison Hotel with many large TV screens.
- **Bassett Street Brunch Club**, 440 W. Johnson Street, Madison; (608) 467-5051; `https://brunchclubmadison.com`; serving playful versions of comfort food, breakfast classics, and unique cocktails in a modern atmosphere.
- **BelAir Cantina Capitol Square**, 111 Martin Luther King Jr. Boulevard, Madison; (608) 620-6040; `https://belaircantina.com`; serving fresh tacos, burritos, and salsas.
- **The Brass Ring**, 701 E. Washington Avenue, Madison; (608) 256-9359; `https://thebrassringmadison.com`; serving craft beers and upscale pub food within a historic venue including billiards tables, shuffleboard, or live trivia.
- **Camp Trippalindee**, 601 Langdon Street, Fl. 7, Madison; (608) 257-4391; `https://www.graduatehotels.com/madison/restaurant/camp-trippalindee`; located in the Graduate Madison and inspired by 1980s camp movies, this fun and easy restaurant serves tacos, double stacked burgers, and craft beers.

## Registration and Meeting Information

**Advance Registration:** Advance registration for this meeting opens on **July 22, 2019.** Advance registration fees will be **US$75** for AMS members, **US$110** for nonmembers, and **US$15** for students, unemployed mathematicians, and emeritus members. Participants may cancel registrations made in advance by emailing `mmsb@ams.org`. The deadline to cancel is the first day of the meeting.

**On-site Information and Registration:** The registration desk, AMS book exhibit, and coffee service will be located in Van Hise Hall. The Invited Address lectures, including the Erdős Lecture, will be in Ingraham Hall, room 10. Special

Sessions and Contributed Paper Sessions will take place in the nearby classrooms in Van Hise Hall and Ingraham Hall. Please look for additional information about specific session room locations on the web and in the printed program. For further information on building locations, a campus map is available at `map.wisc.edu/s/dna0uzcn`.

The registration desk will be open on Saturday, September 14, 7:30 am–4:00 pm and Sunday, September 15, 8:00 am–12:00 pm. The same fees listed above apply for on-site registration and are payable with cash, check, or credit card.

### Other Activities

**Book Sales:** Stop by the on-site AMS bookstore to review the newest publications and take advantage of exhibit discounts and free shipping on all on-site orders! AMS members receive 40% off list price. Nonmembers receive a 25% discount. AMS Members receive additional discounts on books purchased at meetings, subscriptions to Notices and Bulletin, discounted registration for world-class meetings and conferences, and more!

**Complimentary coffee** will be served courtesy of the AMS Membership Department.

**AMS Editorial Activity:** An acquisitions editor from the AMS book program will be present to speak with prospective authors. If you have a book project that you wish to discuss with the AMS, please stop by the book exhibit.

### Special Needs

It is the goal of the AMS to ensure that its conferences are accessible to all, regardless of disability. The AMS will strive, unless it is not practicable, to choose venues that are fully accessible to the physically handicapped.

If special needs accommodations are necessary in order for you to participate in an AMS Sectional Meeting, please communicate your needs in advance to the AMS Meetings Department by:

- Registering early for the meeting
- Checking the appropriate box on the registration form, and
- Sending an email request to the AMS Meetings Department at `mmsb@ams.org` or `meet@ams.org`.

The closest gender inclusive restroom is in the Gordon Dining and Event Center, located on the first floor on the right in between other restrooms. This restroom is wheelchair accessible. It is labeled as "Restroom."

### AMS Policy on a Welcoming Environment

The AMS strives to ensure that participants in its activities enjoy a welcoming environment. In all its activities, the AMS seeks to foster an atmosphere that encourages the free expression and exchange of ideas. The AMS supports equality of opportunity and treatment for all participants, regardless of gender, gender identity or expression, race, color, national or ethnic origin, religion or religious belief, age, marital status, sexual orientation, disabilities, or veteran status.

### Local Information and Maps

This meeting will take place on the campus of The University of Wisconsin–Madison. A campus map can be found at `map.wisc.edu/s/dna0uzcn`. Information about The University of Wisconsin–Madison Mathematics Department can be found at `www.math.wisc.edu`. Please visit the university website at `https://www.wisc.edu` for additional information on the campus.

Please watch the AMS website at `www.ams.org/meetings/sectional/sectional.html` for additional information on this meeting.

### Parking

There are over twenty parking lots on the University of Wisconsin–Madison campus that allow free parking all day Saturday and Sunday. A complete list of lot hours of control can be found on the lot locations and hours page at `https://transportation.wisc.edu/parking-lots/locations-and-hours/#oncampus`.

Suggested free parking lots are Lot 62 located at 525 Easterday Lane and Lot 34 located at 1480 Tripp Circle. These lots include a reasonable walk to the meeting space. Visit `http://map.wisc.edu/` for an interactive campus map including parking lots.

Paid visitor parking is available in the Nancy Nicholas Garage (Lot 27) located at 1330 Linden Drive and Observatory Drive Ramp (Lot 36) located at 1645 Observatory Drive. These lots are controlled at all times, 24/7. This is a "pay on exit" system where visitors pull a ticket upon entry and then pay by credit card upon exit. The daytime hourly visitor parking rate at time of printing is **US$1** per thirty minutes for the first three hours, **US$1** per hour thereafter, and **US$15** daily maximum. Please note campus parking rates are subject to change. Inquiries about visitor parking can be sent to `visitorparking@fpm.wisc.edu`.

## Travel

This meeting will take place on the main campus of University of Wisconsin–Madison located in Madison, Wisconsin.

### By Air:

**Dane County Regional Airport** (airport code MSN) is located minutes from downtown Madison and is the most convenient air travel choice. Please visit the airport website for a list of airlines and lists of cities with daily direct flights; www.msnairport.com.

There are several options available for transportation to and from the airport.

On-airport transportation network companies (ride share) are located between doors #3 and #4 on roadway median. Taxis are located at the north end of baggage claim at door #7. For a list of taxi companies visit www.msnairport.com/parking_transportation/ground_transportation.

Many area hotels provide courtesy vehicles to handle your transportation to or from the Dane County Regional Airport. You may use the courtesy phone located at the hotel board between Bag Claims 1 and 2, or call your hotel directly. For a list of properties visit www.msnairport.com/parking_transportation/ground_transportation.

On-airport rental car booths are located in the baggage claim area and outside door #6. For phone numbers and further information visit www.msnairport.com/parking_transportation/car_rental.

**Madison Metro Transit System** Route 20 runs between the North Transfer Point and East Towne Mall via the airport every thirty minutes during weekdays, and hourly weeknights, weekends, and holidays. For service from the airport to downtown Madison or the University of Wisconsin–Madison campus area, passengers should board buses reading "Route 20—North Transfer Point." Buses reading "Route 20—East Towne Mall" would carry passengers to points east of the airport, including the MATC campus area, and eventually, East Towne Mall. For more information call the Madison Metro Transit System at (608) 266-4466 or visit www.mymetrobus.com.

### By Bus:

Long-distance intercity bus services providing scheduled service to Madison include **Badger Coach**, **Greyhound**, **Jefferson Lines**, **Megabus**, and **Van Galder** bus lines. There are daily connections to Milwaukee and Chicago airports, and **Amtrak** in Chicago. Many buses stop at Memorial Union. Visit https://wisconsindot.gov/Pages/travel/pub-transit/bus-service.aspx for more details.

### By Car:

**From Milwaukee** (*1½ hours*)**:** From the East via Interstate 94: As you approach the city, follow Highway 30 into Madison. Take the "State Capitol" exit off of Highway 30 and you will then be on East Washington Avenue, which leads directly to the Capitol Square.

**From Dubuque** (*1¾ hour*)**:** From the Southwest via Highways 18 and 151: Take the appropriate exit to Highway 151 going north toward the State Capitol. In Madison, Highway 151 will become South Park Street. Follow Park Street to its conclusion to reach University of Wisconsin–Madison. Or, from Park, take a right onto Johnson Street to reach the State Street/Capitol Square area.

**From Chicago** (*2½ hours*)**:** From the Southeast via Interstate 90: Take the Hwy 12/18 (Beltline) exit to Madison. If you are going to the Alliant Energy Center or Monona Terrace, exit at John Nolen Drive. If you are going to University of Wisconsin–Madison, exit at Park Street and follow north.

**From Green Bay** (*2½ hours*)**:** From the Northeast via Highway 151: Follow Highway 151 toward the State Capitol. You will then be on East Washington Avenue, which leads directly to the Capitol Square.

**From Minneapolis** (*4½ hours*)**:** From the North via Interstate 90/94: Take I-90/94 to the Highway 151 exit (Southwest) going toward the State Capitol. You will then be on East Washington Avenue, which leads directly to the Capitol Square.

**Car Rental:** Hertz is the official car rental company for the meeting. To make a reservation accessing our special meeting rates online at www.hertz.com, click on the box "I have a discount," and type in our convention number (CV): CV#04N30009. You can also call Hertz directly at 800-654-2240 (US and Canada) or 1-405-749-4434 (other countries). At the time of reservation, the meeting rates will be automatically compared to other Hertz rates and you will be quoted the best comparable rate available.

For directions to campus, inquire at your rental car counter.

## Local Transportation

**Campus Bus Routes:** The University of Wisconsin–Madison campus bus routes are fare-free for all riders. All campus bus stops with real-time pick-up information can be found on the interactive campus map at www.map.wisc.edu. Routes 80 and 84 provide daytime service. Routes 80, 81, and 82 provide nighttime service. The service calendar for Routes 80 and 84 is posted at www.cityofmadison.com/metro/routes-schedules/uw-service-calendar.

The closest westbound bus stop to the meeting is on the 1200 block of Linden Drive. The closest eastbound bus stop to the meeting is across the street from Van Hise Hall. This stop serves routes 11, 28, 38, 44, 80, 84. Visit the interactive campus map at map.wisc.edu for bus stop locations on campus.

When riding the campus bus, be sure to remain behind the yellow standee line at the front of the bus for safety. Standing beyond this line obscures the driver's view and can lead to crashes. All campus buses are kneeling buses, capable of transporting wheelchairs and other mobility equipment.

**Madison Metro City Routes:** Madison Metro routes run within campus and throughout the city of Madison. Select routes also run to nearby areas, such as Fitchburg, Middleton, and Monona. For more information, visit the Madison Metro website at www.cityofmadison.com/metro or call (608) 266-4466.

For routes and schedules use the online Ride Guide at www.cityofmadison.com/metro/routes-schedules or pick up a hard copy on any Madison Metro bus or at any Transportation Services office.

**Other options:** Other local transportation options include **Uber**; www.uber.com and **Lyft**; www.lyft.com.

## Weather

Madison averages a daily maximum temperature for September of 66°F, rarely falling below 54°F or exceeding 86°F. The daily low temperature is 58°F to 47°F, rarely falling below 36°F or exceeding 68°F. The month of September in Madison experiences gradually increasing cloud cover, with overcast or mostly cloudy weather. Attendees are advised to dress in layers.

## Social Networking

Attendees and speakers are encouraged to use the hashtag #AMSmtg to tweet about the meeting.

## Information for International Participants

Visa regulations are continually changing for travel to the United States. Visa applications may take from three to four months to process and require a personal interview, as well as specific personal information. International participants should view the important information about traveling to the US found at https://travel.state.gov/content/travel/en.html. If you need a preliminary conference invitation in order to secure a visa, please send your request to cro@ams.org.

If you discover you do need a visa, the National Academies website (see above) provides these tips for successful visa applications:

* Visa applicants are expected to provide evidence that they are intending to return to their country of residence. Therefore, applicants should provide proof of "binding" or sufficient ties to their home country or permanent residence abroad. This may include documentation of the following:

- Family ties in home country or country of legal permanent residence
- Property ownership
- Bank accounts
- Employment contract or statement from employer stating that the position will continue when the employee returns;

* Visa applications are more likely to be successful if done in a visitor's home country than in a third country;

* Applicants should present their entire trip itinerary, including travel to any countries other than the United States, at the time of their visa application;

* Include a letter of invitation from the meeting organizer or the US host, specifying the subject, location and dates of the activity, and how travel and local expenses will be covered;

* If travel plans will depend on early approval of the visa application, specify this at the time of the application;

* Provide proof of professional scientific and/or educational status (students should provide a university transcript).

This list is not to be considered complete. Please visit the websites above for the most up-to-date information.

# Binghamton, New York

*Binghamton University*

## October 12–13, 2019

*Saturday – Sunday*

## Meeting #1151

Eastern Section
Associate secretary: Steven H. Weintraub

Announcement issue of *Notices*: August 2019
Program first available on AMS website: August 29, 2019
Issue of *Abstracts*: Volume 40, Issue 3

## Deadlines

For organizers: Expired
For abstracts: August 20, 2019

*The scientific information listed below may be dated. For the latest information, see* `www.ams.org/amsmtgs/sectional.html`.

## Invited Addresses

**Richard Kenyon**, Brown University, *What polygons can be tiled with squares?*

**Tony Pantev**, University of Pennsylvania, *Geometry and topology of wild character varieties*.

**Lai-Sang Young**, New York University, *A dynamical model for controlling of infectious diseases via isolation*.

## Special Sessions

*If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at* `www.ams.org/cgi-bin/abstracts/abstract.pl`.

*Algebraic Combinatorics on the Occasion of the 75th Birthday of Thomas Zaslavsky* (Code: SS 13A), **Nathan Reff**, State University of New York, The College at Brockport, and **Lucas Rusnak**, Texas State University.

*Analysis and Applications of Deterministic and Stochastic Evolution Equations* (Code: SS 10A), **Vincent Martinez**, City University of New York, Hunter College, and **Kazuo Yamazaki**, University of Rochester.

*Commutative Algebra* (Code: SS 9A), **Bethany Kubik**, University of Minnesota, Duluth, and **Denise Rangel Tracy**, Central Connecticut State University.

*Effective and Quantitative Advances in Low Dimensional Topology and Geometric Group Theory* (Code: SS 3A), **Jenya Sapir**, Binghamton University, and **Edgar Bering**, Temple University.

*Group Actions on Manifolds and Related Spaces* (Code: SS 12A), **Thomas Koberda**, University of Virginia, **Yash Lodha**, École Polytechnique Féderalé de Lausanne, Switzerland, and **Matt Zaremsky**, University at Albany, State University of New York.

*Groups and Their Representations* (Code: SS 1A), **Jamison Barsotti** and **Rob Carman**, College of William and Mary, and **Daniel Rossi** and **Hung P. Tong-Viet**, Binghamton University.

*Homotopy Theory and Algebraic K-theory* (Code: SS 5A), **Cary Malkiewich**, Binghamton University, **Marco Varisco**, University at Albany, and **Inna Zakharevich**, Cornell University.

*Invariants of Knots, Links, and Low-dimensional Manifolds* (Code: SS 15A), **Moshe Cohen**, Vassar College, **Adam Giambrone**, Elmira College, **Adam Lowrance**, Vassar College, and **Jonathan Williams**, Binghamton University.

*Operator Theory and Complex Analysis* (Code: SS 17A), **Gabriel T. Prajitura** and **Ruhan Zhao**, College at Brockport, SUNY.

*Oriented Matroids and Related Topics* (Code: SS 7A), **Laura Anderson**, **Michael Dobbins**, and **Benjamin Schroeter**, Binghamton University.

*Percolation, Random Graphs, and Random Geometry* (Code: SS 11A), **Shishendu Chatterjee** and **Jack Hanson**, City University of New York, City College.

*Recent Trends in Geometrical PDEs and Mathematical Physics* (Code: SS 6A), **Xiangjin Xu** and **Gang Zhou**, Binghamton University.

*Representations of Lie Algebras, Vertex Operators, and Related Topics* (Code: SS 2A), **Alex Feingold**, Binghamton University, and **Christopher Sadowski**, Ursinus College.

*Statistics* (Code: SS 14A), **Sanjeena Dang**, **Aleksey Polunchenko**, **Xingye Qiao**, and **Anton Schick**, Binghamton University.

*Stochastic Evolution of Discrete Structures* (Code: SS 8A), **Vladislav Kargin**, Binghamton University.

*What's New in Group Theory?* (Code: SS 4A), **Luise-Charlotte Kappe**, Binghamton University, and **Justin Lynd** and **Arturo Magidin**, University of Louisiana at Lafayette.

*p-adic Analysis in Number Theory* (Code: SS 16A), **C. Douglas Haessig**, University of Rochester, and **Rufei Ren**.

# Gainesville, Florida

*University of Florida*

### November 2–3, 2019

*Saturday – Sunday*

### Meeting #1152

Southeastern Section

Associate secretary: Brian D. Boe

Announcement issue of *Notices*: September 2019

Program first available on AMS website: September 19, 2019

Issue of *Abstracts*: Volume 40, Issue 4

### Deadlines

For organizers: Expired

For abstracts: September 10, 2019

*The scientific information listed below may be dated. For the latest information, see* www.ams.org/amsmtgs/sectional.html.

### Invited Addresses

**Jonathan Mattingly**, Duke University, *Title to be announced*.

**Isabella Novik**, University of Washington, *Title to be announced*.

**Eduardo Teixeira**, University of Central Florida, *Geometric regularity theory for diffusive processes and their intrinsic free boundaries*.

### Special Sessions

*If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at* www.ams.org/cgi-bin/abstracts/abstract.pl.

*Čech-Stone Compactification of Semigroups: Algebra, Topology, Dynamics, and Combinatorics* (Code: SS 17A), **Dana Bartošová** and **Jindřich Zapletal**, University of Florida.

*Algebras, Analysis and Physics* (Code: SS 6A), **Craig A. Nolder**, Florida State University, **Carmen Judith Vanegas Espinoza**, Technical University of Manabi (Ecuador), and **Soren Krausshar**, Universitat Erfurt (Germany).

*Analysis of Geometric and Evolutionary PDEs* (Code: SS 16A), **Yi Hu**, **Yongki Lee**, **Yuanzhen Shao**, and **Shijun Zheng**, Georgia Southern University.

*Applications of Differential Equations in Mathematical Biology* (Code: SS 25A), **Nehal Shukla**, Columbus State University.

*Applied Topology: Theory and Applications* (Code: SS 12A), **Peter Bubenik**, University of Florida, and **Nataša Jonoska**, University of South Florida.

*Combinatorial Lie Theory* (Code: SS 3A), **Erik Insko**, Florida Gulf Coast University, **Martha Precup**, Washington University in St. Louis, and **Edward Richmond**, Oklahoma State University.

*Crystallographic and Highly Symmetric Structures* (Code: SS 15A), **Milé Krajčevski** and **Gregory McColm**, University of South Florida.

*Effective Equations of Quantum Physics* (Code: SS 18A), **Israel Michael Sigal**, University of Toronto, and **Avy Soffer**, Rutgers University.

*Experimental Mathematics in Number Theory and Combinatorics* (Code: SS 20A), **Hannah Burson**, University of Illinois at Urbana–Champaign, **Tim Huber**, University of Texas, Rio Grande Valley, and **Armin Straub**, University of South Alabama.

*Extremal and Probabilistic Combinatorics* (Code: SS 19A), **Linyuan Lu**, University of South Carolina, and **Yi Zhao**, Georgia Southern University.

*Fractal Geometry and Dynamical Systems* (Code: SS 2A), **Mrinal Kanti Roychowdhury**, University of Texas Rio Grande Valley.

*Geometric Structures on Manifolds* (Code: SS 14A), **Sam Ballas**, Florida State University, **Luca Di Cerbo**, University of Florida, and **Kate Petersen**, Florida State University.

*Geometric and Topological Combinatorics* (Code: SS 1A), **Bruno Benedetti**, University of Miami, **Steve Klee**, Seattle University, and **Isabella Novik**, University of Washington.

*Geometry of Gauge Theoretic Moduli Spaces* (Code: SS 23A), **Chris Kottke**, New College of Florida, and **Ákos Nagy**, Duke University.

*Homological Methods in Algebra* (Code: SS 11A), **Luigi Ferraro** and **W. Frank Moore**, Wake Forest University.

*New Developments in Mathematical Biology* (Code: SS 21A), **Maia Martcheva**, University of Florida, **Necibe Tuncer**, Florida Atlantic University, and **Libin Rong**, University of Florida.

*Nonlinear Elliptic Partial Differential Equations* (Code: SS 13A), **Mark Allen**, Brigham Young University, and **Eduardo V. Teixeira**, University of Central Florida.

*Nonlinear PDEs in Fluid Dynamics* (Code: SS 10A), **Ming Chen**, University of Pittsburgh, **Aseel Farhat**, Florida State University, and **Cheng Yu**, University of Florida.

*Nonlinear Solvers and Acceleration Methods* (Code: SS 9A), **Sara Pollock**, University of Florida, and **Leo Rebholz**, Clemson University.

*Partition Theory and Related Topics* (Code: SS 22A), **Dennis Eichhorn**, University of California, Irvine, **Frank Garvan**, University of Florida, and **Brandt Kronholm**, University of Texas, Rio Grande Valley.

*Patterns in Permutations* (Code: SS 7A), **Miklós Bóna** and **Vince Vatter**, University of Florida.

*Probabilistic and Geometric Tools in High-Dimension* (Code: SS 4A), **Arnaud Marsiglietti**, University of Florida, and **Artem Zvavitch**, Kent State University.

*Recent Progress in Operator Theory* (Code: SS 8A), **Mike Jury**, **Scott McCullough**, and **James Pascoe**, University of Florida.

*Recent Trends in Extremal Graph Theory* (Code: SS 24A), **Theodore Molla** and **Brendan Nagle**, University of South Florida.

*Topological Complexity and Related Topics* (Code: SS 5A), **Daniel C. Cohen**, Louisiana State University, and **Alexander Dranishnikov** and **Yuli B. Rudyak**, University of Florida.

# Riverside, California

*University of California, Riverside*

### November 9–10, 2019

*Saturday – Sunday*

### Meeting #1153

Western Section

Associate secretary: Michel L. Lapidus

Announcement issue of *Notices*: September 2019

Program first available on AMS website: September 12, 2019

Issue of *Abstracts*: Volume 40, Issue 4

### Deadlines

For organizers: Expired

For abstracts: September 3, 2019

*The scientific information listed below may be dated. For the latest information, see* `www.ams.org/amsmtgs/sectional.html`.

### Invited Addresses

**Robert Boltje**, University of California, Santa Cruz, *Title to be announced*.

**Jonathan Novak**, University of California, San Diego, *Title to be announced*.

**Anna Skripka**, University of New Mexico, Albuquerque, *Title to be announced*.

### Special Sessions

*If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at* `www.ams.org/cgi-bin/abstracts/abstract.pl`.

*AWM, with Emphasis on Geometry and Dynamics* (Code: SS 13A), **Weitao Chen**, **Savanna Gee**, **Paige Helms**, and **Qixuan Wang**, University of California, Riverside.

*Advances in Functional Analysis* (Code: SS 17A), **Marat Markin**, California State University, Fresno, and **Yunied Puig De Dios**, University of California, Riverside.

*Advances in Operator Algebras* (Code: SS 31A), **Scott Atkinson**, Vanderbilt University, **Rolando de Santiago**, UCLA, and **Feng Xu**, UC Riverside.

*Algebraic and Combinatorial Structures in Knot Theory* (Code: SS 6A), **Jieon Kim**, Pusan National University, and **Sam Nelson**, Claremont McKenna College.

*Analysis of Nonlinear Partial Differential Equations and Applications* (Code: SS 9A), **Nam Q. Le**, Indiana University, Bloomington, and **Connor Mooney**, University of California, Irvine.

*Applied Category Theory* (Code: SS 12A), **John Baez** and **Joe Moeller**, University of California, Riverside.

*Applied Partial Differential Equations and Optimization* (Code: SS 24A), **Yat Tin Chow** and **Amir Moradifam**, University of California, Riverside.

*Arithmetic Geometry in Finite Characteristic* (Code: SS 22A), **Nathan Kaplan** and **Vlad Matei**, University of California Irvine.

*Canonical Bases, Cluster Structures and Non-commutative Birational Geometry* (Code: SS 11A), **Arkady Berenstein**, University of Oregon, Eugene, **Jacob Greenstein**, University of California, Riverside, and **Vladimir Retakh**, Rutgers University.

*Celebrating MM Rao's Many Mathematical Contributions as he Turns 90 Years Old* (Code: SS 27A), **Jerome Goldstein**, University of Memphis, and **Michael Green**, **Alan Krinik**, **Randall J. Swift**, and **Jennifer Switkes**, California State Polytechnic University, Pomona.

*Computational Methods in Hyperbolic Geometry* (Code: SS 35A), **Brian Benson**, University of California, Riverside, and **Jeffrey S. Meyer**, California State University, San Bernardino.

*Data Science* (Code: SS 16A), **Shuheng Zhou**, University of California, Riverside.

*Differential Equation, Differential Geometry and Mathematical General Relativity* (Code: SS 26A), **Po-Ning Chen** and **Michael McNulty**, University of California, Riverside.

*Dynamical Systems and Ergodic Theory* (Code: SS 10A), **Nicolai Haydn**, University of Southern California, **Huyi Hu**, Michigan State University, and **Zhenghe Zhang**, University of California, Riverside.

*Fluid Dynamics: from Theory to Numerics* (Code: SS 18A), **James P Kelliher** and **Ali Pakzad**, University of California, Riverside.

*Fractal Geometry, Dynamical Systems, and Related Topics* (Code: SS 30A), **Tim Cobler**, Fullerton College, **Therese Landry**, University of California, Riverside, **Erin Pearse**, California Polytechnic State University, San Luis Obispo, and **Goran Radunovic**, University of Zagreb.

*Geometric Methods in Representation Theory* (Code: SS 25A), **Mee Seong Im**, United States Military Academy, West Point, **Neal Livesay**, University of California, Riverside, and **Daniel Sage**, Louisiana State University.

*Geometric Partial Differential Equations and Variational Methods* (Code: SS 4A), **Longzhai Lin**, University of California, Santa Cruz, **Xiangwen Zhang**, University of California, Irvine, and **Xin Zhou**, University of California, Santa Barbara.

*Geometry and Representation Theory of Quantum Algebras and Related Topics* (Code: SS 19A), **Mee Seong Im**, United States Military Academy, West Point, **Bach Nguyen**, Temple University, **Hans Nordstrom**, University of Portland, and **Karl Schmidt**, University of California, Riverside.

*Graph Theory* (Code: SS 29A), **Zhanar Berikkyzy** and **Mei-Chu Chang**, University of California, Riverside.

*Integrating Forward and Inverse Modeling: Machine Learning and Multiscale, Multiphysics Challenges* (Code: SS 21A), **Mark Alber**, University of California, Riverside, and **William Cannon**, Pacific Northwest National Laboratory.

*Invariants of Knots and Spatial Graphs* (Code: SS 5A), **Alissa Crans**, **Blake Mellor**, and **Patrick Shanahan**, Loyola Marymount University.

*Inverse Problems* (Code: SS 3A), **Hanna Makaruk**, Los Alamos National Laboratory, and **Robert Owczarek**, University of New Mexico, Albuquerque and University of New Mexico, Los Alamos.

*Mathematical Biology: Multi-Scale Modeling of Complex Biological Systems* (Code: SS 15A), **Suzanne Sindi** and **Mikahl Banwarth-Kuhn**, University of California, Merced.

*Mathematical Modeling in Developmental Biology* (Code: SS 14A), **Weitao Chen** and **Qixuan Wang**, University of California, Riverside.

*Random Matrices and Related Structures* (Code: SS 2A), **Jonathan Novak**, University of California, San Diego, and **Karl Liechty**, De Paul University.

*Representations of Finite Groups and Related Topics* (Code: SS 7A), **Robert Boltje**, University of California at Santa Cruz, **Klaus Lux**, University of Arizona at Tucson, and **Amanda Schaeffer Fry**, Metropolitan State University of Denver.

*Research in Mathematics by Early Career Graduate Students* (Code: SS 20A), **Marat Markin** and **Khang Tran**, California State University, Fresno.

*Several Complex Variables and Complex Dynamics* (Code: SS 34A), **Xin Dong**, University of California, Irvine, and **Sara Lapan** and **Bun Wong**, University of California, Riverside.

*Topics in Algebraic Geometry* (Code: SS 23A), **Jose Gonzalez**, **Ziv Ran**, and **Zhixian Zhu**, University of California, Riverside.

*Topics in Extremal and Structural Graph Theory* (Code: SS 32A), **Andre Kundgen**, California State University San Marcos, and **Craig Timmons**, California State University Sacramento.

*Topics in Global Geometric Analysis* (Code: SS 8A), **Fred Whilhelm** and **Qi Zhang**, University of California, Riverside.

*Topics in Operator Theory* (Code: SS 1A), **Anna Skripka** and **Maxim Zinchenko**, University of New Mexico.

*Undergraduate Research in Mathematics: Presentations on Research and Mentorship* (Code: SS 28A), **David Weisbart**, University of California, Riverside.

# Denver, Colorado

*Colorado Convention Center*

### January 15–18, 2020

*Wednesday – Saturday*

### Meeting #1154

*Joint Mathematics Meetings, including the 126th Annual Meeting of the AMS, 103rd Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM)*

Associate secretary: Michel L. Lapidus

Announcement issue of *Notices*: October 2019

Program first available on AMS website: November 1, 2019

Issue of *Abstracts*: Volume 41, Issue 1

### Deadlines

For organizers: Expired

For abstracts: September 17, 2019

*The scientific information listed below may be dated. For the latest information, see* www.ams.org/amsmtgs/national.html.

### AMS Invited Addresses

**Maleafisha J. P. S. Tladi**, University of Limpopo, *Well-Posedness and Long-Time Dynamics of the Rotating Boussinesq and Quasigeostrophic Equations* (AMS Josiah Willard Gibbs Lecture).

# Charlottesville, Virginia

*University of Virginia*

### March 13–15, 2020

*Friday – Sunday*

### Meeting #1155

Southeastern Section

Associate secretary: Brian D. Boe

Announcement issue of *Notices*: January 2020

Program first available on AMS website: February 4, 2020

Issue of *Abstracts*: Volume 41, Issue 2

### Deadlines

For organizers: August 15, 2019

For abstracts: January 21, 2020

*The scientific information listed below may be dated. For the latest information, see* www.ams.org/amsmtgs/sectional.html.

### Invited Addresses

**Moon Duchin**, Tufts University, *Title to be announced* (Einstein Public Lecture in Mathematics).

### Special Sessions

*If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at* www.ams.org/cgi-bin/abstracts/abstract.pl.

*Advances in Difference, Differential, Fractional Differential and Dynamic Equations with Applications* (Code: SS 2A), **Muhammad Islam** and **Youssef Raffoul**, University of Dayton.

*Advances in Infectious Disease Modeling: From Cells to Populations* (Code: SS 4A), **Lauren Childs**, **Stanca Ciupe**, and **Omar Saucedo**, Virginia Tech.

*Curves, Jacobians, and Abelian Varieties* (Code: SS 1A), **Andrew Obus**, Baruch College (CUNY), **Tony Shaska**, Oakland University, and **Padmavathi Srinivasan**, Georgia Institute of Technology.

*Numerical Methods for Partial Differential Equations: A Session in Honor of Slimane Adjerid's 65th Birthday* (Code: SS 3A), **Mahboub Baccouch**, University of Nebraska at Omaha.

# Medford, Massachusetts

*Tufts University*

### March 21–22, 2020

*Saturday – Sunday*

### Meeting #1156

Eastern Section
Associate secretary: Steven H. Weintraub

Announcement issue of *Notices*: January 2020
Program first available on AMS website: February 11, 2020
Issue of *Abstracts*: Volume 41, Issue 2

### Deadlines

For organizers: August 22, 2019
For abstracts: January 28, 2020

*The scientific information listed below may be dated. For the latest information, see* www.ams.org/amsmtgs/sectional.html.

### Invited Addresses

**Daniela De Silva**, Columbia University, *Title to be announced*.
**Enrique Pujals**, City University of New York, *Title to be announced*.
**Chris Woodward**, Rutgers, the State University of New Jersey, *Title to be announced*.

### Special Sessions

*If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at* www.ams.org/cgi-bin/abstracts/abstract.pl.

*Anomalous Diffusion Processes* (Code: SS 3A), **Christoph Borgers**, Tufts University, and **Claude Greengard**, New York University and Foss Hill Partners.

*Modeling and Analysis of Partial Differential Equations in Fluid Dynamics and Related Fields: Geometric and Probabilistic Methods* (Code: SS 1A), **Geng Chen**, University of Kansas, **Siran Li**, Rice University and Centre de Recherches Mathématiques, Université de Montréal, and **Kun Zhao**, Tulane University.

*Recent Advances in Schubert Calculus and Related Topics* (Code: SS 2A), **Christian Lenart** and **Changlong Zhong**, State University of New York at Albany.

# West Lafayette, Indiana

*Purdue University*

### April 4–5, 2020

*Saturday – Sunday*

### Meeting #1157

Central Section
Associate secretary: Georgia Benkart

Announcement issue of *Notices*: February 2020
Program first available on AMS website: February 18, 2020
Issue of *Abstracts*: Volume 41, Issue 2

### Deadlines

For organizers: September 5, 2019
For abstracts: February 4, 2020

*The scientific information listed below may be dated. For the latest information, see* www.ams.org/amsmtgs/sectional.html.

### Special Sessions

*If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at* www.ams.org/cgi-bin/abstracts/abstract.pl.

*Harmonic Analysis* (Code: SS 2A), **Brian Street** and **Shaoming Guo**, University of Wisconsin–Madison.

*Low-dimensional Topology* (Code: SS 4A), **Matthew Hedden**, **Katherine Raoux**, and **Lev Tovstopyat-Nelip**, Michigan State University.

*Mathematical Methods for Inverse Problems* (Code: SS 3A), **Isaac Harris** and **Peijun Li**, Purdue University.

*The Interface of Harmonic Analysis and Analytic Number Theory* (Code: SS 1A), **Theresa Anderson**, Purdue University, **Robert Lemke Oliver**, Tufts University, and **Eyvindur Palsson**, Virginia Tech University.

# Fresno, California

*California State University, Fresno*

### May 2–3, 2020

*Saturday – Sunday*

### Meeting #1158

Western Section
Associate secretary: Michel L. Lapidus

Announcement issue of *Notices*: March 2020
Program first available on AMS website: March 19, 2020
Issue of *Abstracts*: Volume 41, Issue 2

### Deadlines

For organizers: October 3, 2019
For abstracts: March 3, 2020

# El Paso, Texas

*University of Texas at El Paso*

### September 12–13, 2020

*Saturday – Sunday*

### Meeting #1159

Central Section
Associate secretary: Georgia Benkart

Announcement issue of *Notices*: June 2020
Program first available on AMS website: July 28, 2020
Issue of *Abstracts*: Volume 41, Issue 3

### Deadlines

For organizers: February 20, 2020
For abstracts: July 14, 2020

*The scientific information listed below may be dated. For the latest information, see* `www.ams.org/amsmtgs/sectional.html`.

### Special Sessions

*If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at* `www.ams.org/cgi-bin/abstracts/abstract.pl`.

*Algebraic Structures in Topology, Logic, and Arithmetic*, **John Harding**, New Mexico State University, and **Emil Schwab**, University of Texas at El Paso.

*High-Frequency Data Analysis and Applications*, **Maria Christina Mariani**, University of Texas at El Paso, **Michael Pokojovy**, University of Texas at El Paso, and **Ambar Sengupta**, University of Connecticut.

*Nonlinear Analysis and Optimization*, **Behzad Djafari-Rouhani**, University of Texas at El Paso, and **Akhtar A. Khan**, Rochester Institute of Technology.

# State College, Pennsylvania

*Pennsylvania State University, University Park Campus*

### October 3–4, 2020

*Saturday – Sunday*

### Meeting #1160

Eastern Section
Associate secretary: Steven H. Weintraub

Announcement issue of *Notices*: August 2020
Program first available on AMS website: August 25, 2020
Issue of *Abstracts*: Volume 41, Issue 3

### Deadlines

For organizers: March 3, 2020
For abstracts: August 11, 2020

## MEETINGS & CONFERENCES

# Chattanooga, Tennessee

*University of Tennessee at Chattanooga*

### October 10–11, 2020

*Saturday – Sunday*

### Meeting #1161

Southeastern Section
Associate secretary: Brian D. Boe

Announcement issue of *Notices*: August 2020
Program first available on AMS website: September 1, 2020
Issue of *Abstracts*: Volume 41, Issue 4

### Deadlines

For organizers: March 10, 2020
For abstracts: August 18, 2020

# Salt Lake City, Utah

*University of Utah*

### October 24–25, 2020

*Saturday – Sunday*

### Meeting #1162

Western Section
Associate secretary: Michel L. Lapidus

Announcement issue of *Notices*: August 2020
Program first available on AMS website: September 17, 2020
Issue of *Abstracts*: Volume 41, Issue 4

### Deadlines

For organizers: March 24, 2020
For abstracts: September 1, 2020

### Invited Addresses

Saturday, October 24, 2020, 6:00 pm – 7:10 pm, Erdős Memorial Lecture.

# Washington, District of Columbia

*Walter E. Washington Convention Center*

### January 6–9, 2021

*Wednesday – Saturday*
*Joint Mathematics Meetings, including the 127th Annual Meeting of the AMS, 104th Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).*
Associate secretary: Brian D. Boe
Announcement issue of *Notices*: October 2020
Program first available on AMS website: November 1, 2020
Issue of *Abstracts*: To be announced

### Deadlines

For organizers: April 1, 2020
For abstracts: To be announced

# Grenoble, France

*Université de Grenoble-Alpes*

### July 5–9, 2021

*Monday – Friday*
Associate secretary: Michel L. Lapidus
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

### Deadlines

For organizers: To be announced
For abstracts: To be announced

# Buenos Aires, Argentina

*The University of Buenos Aires*

### July 19–23, 2021

*Monday – Friday*
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

### Deadlines

For organizers: To be announced
For abstracts: To be announced

# Omaha, Nebraska

*Creighton University*

### October 9–10, 2021

*Saturday – Sunday*
Central Section
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

### Deadlines

For organizers: To be announced
For abstracts: To be announced

# Seattle, Washington

*Washington State Convention Center and the Sheraton Seattle Hotel*

### January 5–8, 2022

*Wednesday – Saturday*
Associate secretary: Georgia Benkart
Announcement issue of *Notices*: October 2021
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

### Deadlines

For organizers: To be announced
For abstracts: To be announced

# Boston, Massachusetts

*John B. Hynes Veterans Memorial Convention Center, Boston Marriott Hotel, and Boston Sheraton Hotel*

### January 4–7, 2023

*Wednesday – Saturday*
Associate secretary: Steven H. Weintraub
Announcement issue of *Notices*: October 2022
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

### Deadlines

For organizers: To be announced
For abstracts: To be announced

# *Call for Nominations*

2020

## Frank and Brennie Morgan AMS-MAA-SIAM Prize

### for Outstanding Research in Mathematics by an Undergraduate Student

The prize is awarded each year to an undergraduate student (or students for joint work) for outstanding research in mathematics. Any student who is an undergraduate in a college or university in the United States or its possessions, Canada, or Mexico is eligible to be considered for this prize.

The prize recipient's research need not be confined to a single paper; it may be contained in several papers. However, the paper (or papers) to be considered for the prize must be completed while the student is an undergraduate; they cannot be written after the student's graduation. The research paper (or papers) may be submitted for the committee's consideration by the student or a nominator. Each submission for the prize must include at least one letter of support from a person, usually a faculty member, familiar with the student's research. Publication of research is not required.

The recipients of the prize are to be selected by a standing joint committee of the AMS, MAA, and SIAM. The decisions of this committee are final. Nominations for the 2020 Morgan Prize are due no later than June 30, 2019. Those eligible for the 2020 prize must have been undergraduates in December 2018.

Questions may be directed to:

James Sellers, Secretary
Mathematical Association of America

Penn State University
University Park, PA 16802

Telephone: 814-865-7528
Email: sellersj@psu.edu

Nominations and submissions should be sent to:

Carla Savage, Secretary
American Mathematical Society

Computer Science Department
North Carolina State University
Raleigh, NC 27695-8206

or uploaded via the form available at:
www.ams.org/nominations

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