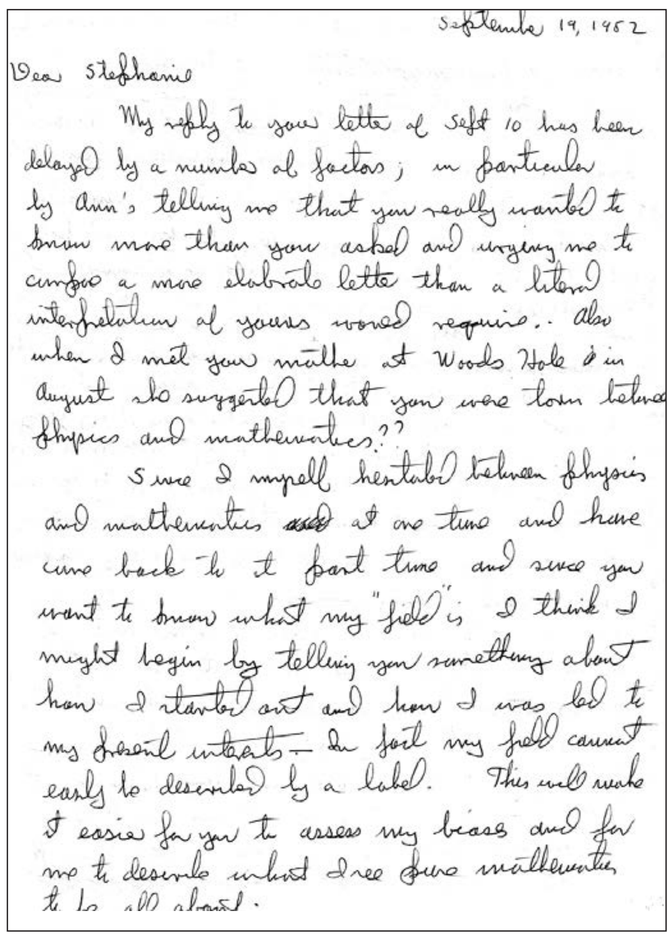
Extensive Cooperation with Rugged Individualism

George Mackey's guide for practitioners of mathematics

Della Dumbaugh

"In a curious way, the advancement of pure mathematics very effectively combines extensive cooperation with rugged individualism."

–George Mackey, "What do mathematicians do?"



September 19, 1982

Dear Stephanie

My reply to your letter of Sept 10 has been delayed by a number of factors; in particular by Ann's telling me that you really wanted to know more than you asked and urging me to compose a more elaborate letter than a literal interpretation of your words required. Also when I met your mother at Woods Hole in August she suggested that you were torn between physics and mathematics??

Since I myself hesitated between physics and mathematics ~~and~~ at one time and have come back to it part time and since you want to know what my "field" is I think I might begin by telling you something about how I started out and how I was led to my present interests. In fact my field cannot easily be described by a label. This will make it easier for you to assess my biases and for me to describe what I see pure mathematics to be all about.

First page of Mackey's letter to Stephanie Frank Singer.

Introduction

The mathematician George Mackey (1916–2006) is often remembered both for his scholarly contributions and his methodical, solitary work habits, tempered by an eager affinity for discussing mathematics with all who took an interest. His broad view of the subject inspired his contributions in infinite-dimensional group representations, ergodic theory, and mathematical physics.

In 1982, Mackey's daughter Ann was a student at Yale University. Her friend, Stephanie Frank Singer, was a sophomore in college trying to decide whether to major in math or physics. Mackey had faced a similar dilemma as an undergraduate, and throughout his career the two disciplines competed for his attention. To help Singer with her decision-making process, Mackey wrote two letters[1] to her in September and October of 1982. He also sent her the text of a talk he had delivered on "What do mathematicians do?" in Paris in March, 1978.[2]

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[1]*To read the letters in full see* https://www.ams.org/notices.

[2]*Mackey delivered a talk at the Harvard Club of France, located in Paris during his sabbatical there in March, 1978. It seems likely that "What do mathematicians do?" (see p. 890 for full speech) is the talk he delivered on that occasion and sent to Singer later [1, 2].*

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Stephanie Frank Singer and George Mackey in Berkeley, CA, circa 1984.

Mackey in Beijing, China, circa 1990.

The current article provides an introduction to George Mackey, including excerpts from his letters to Singer, and the complete text of his "What do mathematicians do?"[3] With an undergraduate query forming the inspiration for Mackey's letters to Singer, this work aims to shed new light on the life of this celebrated American mathematician by considering his contributions to undergraduate education.[4]

Mackey at Work

Mackey adhered to a disciplined lifestyle that began with focus on his mathematical research each morning.[5] In the afternoons, he would normally walk the mile or so to Harvard (to his office or the "long table" at the faculty club for lunch). He ended his days with an early bedtime. He carried a clipboard at all times. He wore a seersucker jacket in warm months and a tweed jacket in cooler ones. He wrote letters about his "latest discoveries" [7, p. 847]. For Mackey, the advancement of mathematics hinged on what he described as an "extensive cooperation with rugged individualism" [4, p. 2]. He seemed to protect time for the "rugged individualism" in the morning and foster "extensive cooperation" in the form of teaching and mathematical discussions later in the day.

This combination helped Mackey make a "lasting impact" on students and colleagues [7, p. 824]. "I learned how to be a mathematician from him," Mackey's student Calvin Moore claimed in his NAS biography. Richard Palais described Mackey as a "pivotal influence" on his life: "my contacts with him, early and late, determined who I was, what I would become and how my life and career would play out" [7, p. 841]. Roger Howe took a novel approach in his measurement of Mackey's influence. He observed that of Euler's 40,000 mathematical descendants, about 300 of them come from Mackey [7, p. 832]. Mackey "made an indelible impression" on his last PhD student, Judith Packer, who reported that he improved her life as both thesis advisor and friend [7, p. 837]. These testimonies suggest the far-reaching influence of Mackey's practice of both the private and public aspects of the profession.

Mackey at Home

The early 1960s formed an especially exciting time in Mackey's life. In December 1960, just weeks before he turned forty-five, he set aside his bachelor lifestyle of a sparse apartment with a single chair and stereo (presumably his clipboard served as his desk?) and "surprised" his colleagues by marrying Alice Willard [14, p. 11]. A Wellesley graduate, Alice had worked as a buyer for the Jordan Marsh department store in Boston. Together they welcomed many members of the mathematical community to their home on Coolidge Hill Road for elegant dinners and vibrant conversation. In 1961, Mackey delivered the prestigious Colloquium Lectures at the Annual Summer Meeting of the AMS. On this occasion, he summarized his theory of unitary representations and his ergodic theory.[6] In 1962, Mackey was elected to the National Academy of Sciences. George and Alice's daughter Ann was born in 1963. "George persisted in many of his bachelor habits," Moore wrote of Mackey's transition to marriage and family life, "while also adapting them in order to become a dutiful husband and

[3] *The* Notices *would like to thank Ann Mackey and Stephanie Frank Singer for bringing these letters to our attention and granting permission to publish them.*

[4] *Stephanie Frank Singer graduated from Yale and earned her PhD in mathematics from the Courant Institute in 1991 under the supervision of Nicholas Ercolani. She has published two books in mathematics:* Symmetry in Mechanics: A Gentle, Modern Introduction *and* Linearity, Symmetry, and Prediction in the Hydrogen Atom *[17, 18]. She currently protects democracy as a data scientist with the Verified Voting Foundation. She and Mackey's daughter Ann were roommates at Yale, and remain close friends to this day.*

[5] *"[O]ne cannot imagine," his student Caroline Series would later observe, how Mackey managed to "avoid teaching in the mornings" [7, p. 844].*

[6] *See [14, pp. 12–13] for the details of these lectures and the subsequent manuscripts.*

father" [14, p. 11]. For example, Mackey sat on a park bench with his clipboard while Ann and Alice explored local attractions.[7] His family seemed to understand his disciplined adherence to his work schedule. Ann and Alice served as "his wonderful support system…his lifeline" [7, p. 837].

Mackey: Early Life

Born on February 1, 1916, in St. Louis, Missouri, George Mackey moved to Houston with his parents, brother, and sister in 1926 after a one-year stint in Florida whose last days included surviving the infamous Great Miami Hurricane of 1926, a dramatic tale that all three siblings retold for the rest of their lives. Although only ten years old at the time, this move would have significant consequences for Mackey. After attending public schools, Mackey enrolled in what was then Rice Institute, now Rice University, in the fall of 1934. That Rice did not charge tuition at the time made it an especially advantageous opportunity since Mackey's family did not have money to spare for college [2]. Initially, he planned to study chemical engineering in an effort to align his father's business aims for his life with his own interest in chemistry. It did not take long for his professors to identify and encourage his talent in mathematics. He found a compromise with a degree in physics—officially that is. Mackey described his undergraduate experience as a triple major in mathematics, physics, and chemistry, although this sort of official recognition of multiple majors did not exist at the time.

Mackey and daughter Ann in Cambridge, MA, circa 1964.

Mackey with wife Alice in Cambridge, MA, mid-1960s.

While at Rice, however, Mackey had the good fortune to learn from Professor Walter Leighton, who had only just earned his Harvard PhD in 1935 under the direction of Marston Morse. Leighton offered Mackey two suggestions that would markedly influence his life. He encouraged Mackey to consider studying at Harvard with John Van Vleck, a theoretical physicist with a joint appointment in the mathematics and physics departments. He also bolstered Mackey's confidence to believe he could find success at Harvard. As Mackey later put it, Leighton "assured me that I was 'good enough' for Harvard and urged me to apply" [13, p. 16]. Take a moment to consider this thought. The young George Whitelaw Mackey, who would ultimately become the distinguished American mathematician by the same name, benefitted from a faculty member's belief in him as an undergraduate.

Leighton also informed Mackey about the inaugural "William Lowell Putnam Competition" in his senior year in 1937–1938. Concurrently, Leighton promoted Mackey as the "most promising" of the mathematics students at Rice and convinced the mathematics department to nominate Mackey as their Putnam entry that year. Leighton may have understood the relationship between these two suggestions. The grand prize of the Putnam Competition included a full scholarship to Harvard graduate school. Mackey earned one of the top five scores on the Putnam that year out of 163 participants.[8] He did not win the Putnam grand prize, however. That award went to Irving Kaplansky. Although Harvard had accepted Mackey, they had not initially offered him any funding. Once they learned of his top-five performance on the Putnam, Harvard offered Mackey financial aid, including full tuition. The chairman at the University of California at Berkeley mathematics department, Griffith C. Evans, who had previously served as chair of the mathematics department at Rice, allowed Mackey to rescind his acceptance there and pursue his graduate work at Harvard.

In 2004, when Mackey was eighty-eight and not in good health, he revised his 1989 *Notices* obituary of Marshall Stone to serve as part of the introduction to *Operator Algebras, Quantization, and Noncommutative Geometry: A Centen-*

[7] *Here, Mackey's lifestyle calls to mind Leonard Dickson's equally legendary insight about his honeymoon. When a colleague asked, "how was your honeymoon?" Dickson replied, "it was great, except I only got 2 papers written" [9, p. 398].*

[8] *Gallian includes the entire 1938 Putnam exam in his [10] and the online updated version "The Putnam Competition from 1938–2013,"* www.d.umn.edu/~jgallian/putnam.pdf. *Mackey also wrote on the Putnam in [11].*

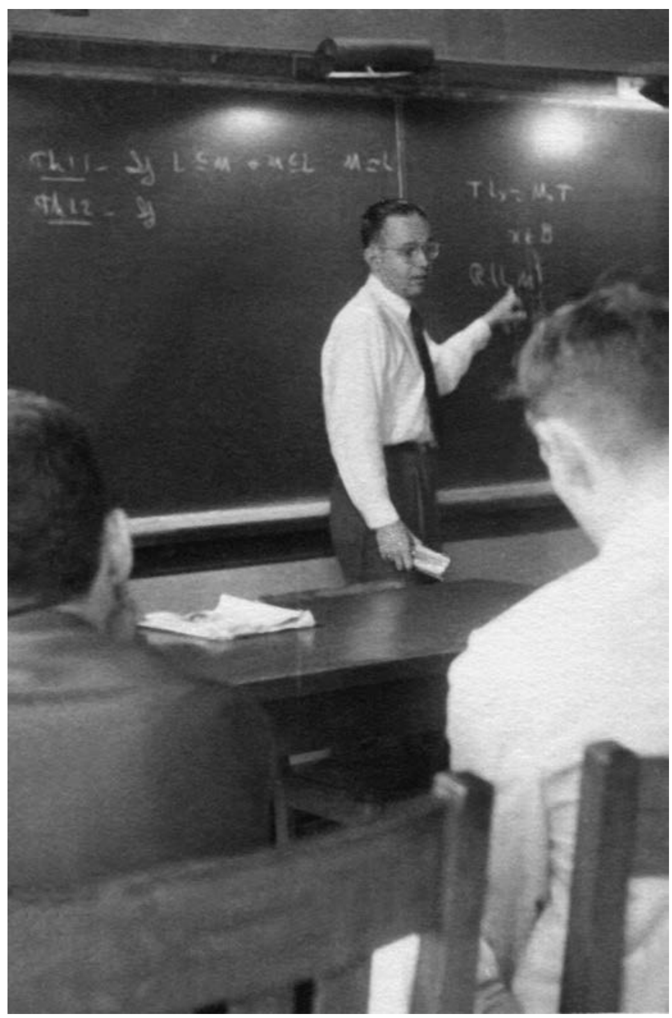
nial celebration honoring John von Neumann and Marshall H. Stone.[9] In this tribute ostensibly dedicated to Stone, Mackey writes about his own undergraduate experience at Rice. This choice tells us something about the value of his time at Rice in his life. He had nearly the entire arc of his life in view at that moment. Perhaps Mackey paused then, as we can do now, to reflect on the people he came into contact with as an undergraduate. There was Leighton, a student of Marston Morse at Harvard, who would ultimately lead the mathematics department at Washington University in St. Louis. Leighton recommended John Van Vleck, son of Edward Burr Van Vleck. The senior Van Vleck had studied with Felix Klein at Göttingen and later served as president of the American Mathematical Society. John Van Vleck would win the Nobel Prize in 1977. There was Evans, who would go on to become President of the American Mathematical Society in 1939–1940. After earning a PhD from Harvard in 1910 with a dissertation on Volterra's Integral Equation written under the direction of Maxime Bôcher, Evans was awarded a Sheldon Fellowship from Harvard to study in Rome with Vito Volterra. Evans joined the Rice faculty in 1912 and remained there until 1934, bringing remarkably talented mathematicians, including Benoit Mandelbrot, Tibor Rado, and Carl Menger as visiting professors. Although Evans left Rice shortly before Mackey arrived, he had helped establish a strong research tradition there. He was subsequently hired by Berkeley to do the same with their mathematics department [15, p. 127]. Mackey became acquainted with the name of Irving Kaplansky through the Putnam competition. Thus before he ever left Rice, whether he realized it or not, Mackey had come into contact with seminal figures and/or their ideas in American mathematics.

Mackey at Harvard: Graduate Student

Mackey arrived at Harvard in the fall of 1938. "I meant to go with physics but applied to the mathematics department for admission," Mackey later described it to Singer.

> My intention was to learn some more mathematics and then come back and do physics "right." I had found physics extremely interesting—especially because of the rather advanced mathematical tools that it used. On the other hand I was quite disturbed by the loose way theory was defined in physics and by the sloppy "hand waving" proofs. I wanted somehow to combine the logical precision of mathematics with the (apparently) richer content of physics. However as my mathematical studies progressed at Harvard I gradually came to realize that pure mathematics had just as rich a content as physics and quite happily dropped physics and became a full fledged pure mathematician [4, p. 2].

A mathematical treatise helped reorient Mackey's academic interests. At the end of his first year at Harvard, Mackey "encountered a thick book in the mathematics library entitled 'Linear transformations in Hilbert space and their applications to analysis' by M. H. Stone…I found the material quite fascinating and ended up reading 60 or 70% of it during the summer and asking Stone to be my thesis advisor" [4, p. 2]. Later, Mackey would clarify that this decision to choose Stone rather than Van Vleck as his thesis advisor did not mean he had decided to abandon his desire to become a physicist. It did mean that he wanted to "learn the deeper parts of pure mathematics under the supervision of the writer of this masterly book" [13, p. 19]. Mackey described Stone's influence on his thesis as "indirect" in that when he met with Stone, he told him about what he was doing, and listened to Stone's "encouraging comments." Harvard courses by Hassler Whitney, Garrett



Mackey teaching, 1940s.

[9] *Alice typed some of his handwritten notes and ensured the article made it to print form [7, p. 826].*

Birkhoff, and the recent Polish immigrant Stanislaw Ulam had a much stronger influence on Mackey [13, p. 20]. Mackey attributed the ideas of his thesis, "The subspaces of the conjugate of an abstract Linear Space," to mathematics he learned and developed from Garrett Birkhoff and a stronger understanding of the linear algebra in Stone's influential text [13, p. 21].

Stone had a much more direct influence on what Mackey termed his "development" when he "arranged" for Mackey to have a Sheldon Traveling Fellowship for his final year at Harvard in 1941–1942. On Stone's advice, Mackey divided the time between Caltech and the Institute for Advanced Study (IAS) in Princeton. At the latter, Mackey met "such legendary figures as Albert Einstein, Oswald Veblen, and John von Neumann" along with younger PhDs including Paul Halmos, Paul Erdős, and Shizuo Kakutani [13, pp. 20–21]. Since Mackey did not yet have his PhD he could not technically join the IAS as a member. Stone, however, "took advantage of his close relationship with von Neumann to talk the Institute into making an exception in my case" [13, p. 21].[10] Kakutani became a close friend and, in particular, he and Mackey "often dined together." As David Mumford later described it, meeting for lunch was "Mackey's favorite way of keeping in touch" [7, p. 837].

While making his way from Caltech to Princeton, Mackey stopped off at an AMS meeting where he met his former Rice professor, Lester Ford. Ford had just assumed the chairmanship at the newly founded Illinois Institute of Technology, and he invited Mackey to join the department as an instructor in mathematics in 1942–1943 once he graduated from Harvard.[11] Although he did not enjoy his time at Illinois Tech, it did allow him to teach mathematics to engineers rather than serve in the military [13, p. 22]. For the next three years he contributed to war-related research at Columbia University and in High Wycombe, England [14, p. 6].

Mackey at Harvard: Faculty Member

Mackey joined the Harvard faculty as an assistant professor in 1946, was promoted to full professor in 1956, and became the inaugural Landon T. Clay Professor of Mathematics and Theoretical Science in 1969. He retired in 1985. While at Harvard, Mackey counted himself among the "relatively small number of people in the world (perhaps a few thousand) who spend a large part of their time thinking about and trying to contribute to an esoteric subject called pure mathematics" [6, p. 1]. In his "What do mathematicians do?" Mackey set out to answer, "Whatever are these mathematicians doing? Why do they find it so interesting and what does it have to do with the rest of the world?" [6, p. 1]. The latter question surely arose from Mackey's broad view of mathematics and his longstanding interest in describing physical phenomena with mathematics. "In a word," Mackey began with a succinct answer to his questions, "pure mathematicians are refining, developing, improving and (rather rarely) discovering the intellectual tools that have proved useful in analyzing and understanding the measurable aspects of the world in which we live" [6, p. 1]. Early in his exposition, he (not surprisingly?) reduced biology to chemistry and chemistry to mathematics and claimed that this understanding allowed "the measurable aspects of the world [to] become quite pervasive" [6, p. 2]. His letters to Singer also emphasized this link between mathematics and the physical world. He often connected those developments with the people who made them. Writing to Singer about Newton, Mackey asserted that "[d]ifferential equations are what made modern physics possible and the most important thing about calculus is

Mackey working on a lecture, Paris, 1978.

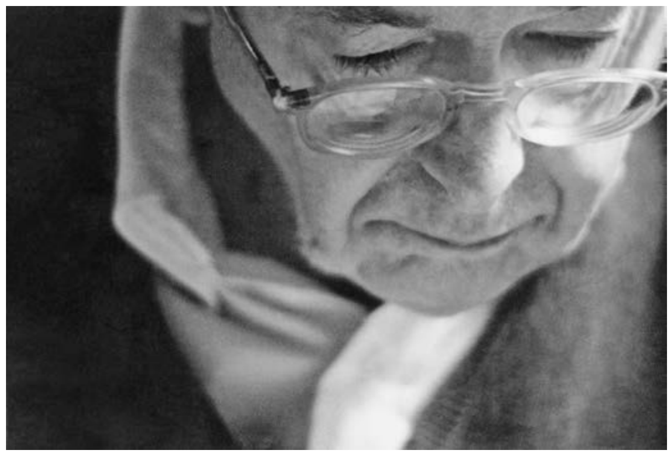
[10] *Indeed, the Institute for Advanced Study lists Mackey as a Member in Mathematics from January to June, 1942 and a Harvard PhD in the same year. See* `https://www.ias.edu/scholars/george-w-mackey`.

[11] *Margaret Matchett, Emil Artin's first and only woman American PhD student, would just miss Mackey at Illinois Tech. She earned her PhD in 1946 and held an instructorship at Illinois Institute of Technology from 1946–1950.*

that it makes differential equations possible" [4, p. 5]. He also included his thoughts on the pedagogy associated with these ideas when he continued with his view that "Newton's work is epoch making in the strongest sense of the word and I personally find it deplorable that these facts are so little emphasized in modern teaching" [4, p. 5]. Given Mackey's regular discussions with his Harvard colleague Andrew Gleason [7, p. 846; 16], who later became a major proponent of the teaching of calculus, one has to wonder if they also took up these concerns in their conversations.[12]

The topic of teaching was never far from Mackey in his letters to Singer and in his "What do mathematicians do?" In fact, in the latter, Mackey *linked* the life of a pure mathematician with teaching. As he described it at the very beginning of his talk, the vast majority of mathematicians "make their living by teaching in universities, their investigations being subsidized by their being given less than full time teaching loads" [6, p. 1]. He circled back around to this idea near the end when he brought up the "certain tension" that exists for mathematicians who become immersed in their research problems and long for further time to devote to them [6, p. 4]. He identified mathematical institutes as a perfect remedy for this situation. In particular, he cited the Institute for Advanced Study in Princeton and the L'Institut des Hautes Études Scientifiques just outside of Paris as examples of places where mathematicians could "find more time for their work" (rugged individualism) and "exchange ideas" (extensive cooperation) [6, p. 5].

For Mackey, that exchange of ideas included discussions of *undergraduate* teaching, which ultimately attracted extraordinary scholars to the field. Arlan Ramsey had his first class with Mackey in 1958–1959 on projective geometry, for example. "Already," Ramsey later recalled, "I found his attitude and style appealing" [7, p. 842]. Ramsey's "golden opportunity" with Mackey came the following year when Mackey taught from the notes that would become his *Mathematical Foundations of Quantum Mechanics* [12]. "This course answered many questions, and then the answers raised further questions. It was just what was needed and gave me a start on a long-term interest in quantum physics" [7, p. 842]. Richard Palais met Mackey as a sophomore in 1949 when he took his "famous Math 212 course." This course started with the foundations of mathematics and "ended up with some highly advanced and esoteric topics, such as the Peter–Weyl Theorem" [7, p. 841]. The course was just the beginning of a transformative experience for Palais. To Palais's good fortune, Mackey was a resident tutor in his dorm, Kirkland House. Mackey encouraged Palais to share meals with him and discuss his queries about the course material. These meals became increasingly frequent and the conversation stretched to other areas of mathematics and life in general. By the end of his sophomore year, Palais changed his major from physics to mathematics [7, p. 841].



Mackey in his study at Coolidge Hill Road, 1980s.

Five years later, David Mumford initially met Mackey as his Kirkland House nonresident tutor at Harvard in 1954. Through their weekly lunches, Mackey revealed to Mumford "the internal logic and coherence of mathematics… it was the lucid sequence of definitions and theorems that was so enticing—a yellow-brick road to more and more amazing places" [7, p. 836]. Although Mackey would "sometimes disclaim any interest in fostering undergraduate education," according to his daughter Ann, "he would engage passionately with anyone at any level of knowledge who expressed an interest in mathematics" [3]. With Mackey's investment in undergraduates, often at the intersection of mathematics and physics, it seems only natural that he would share his thoughts and expertise with his daughter's undergraduate friend considering similar types of questions. One might even style his letters and text on "What do mathematicians do?" as something of a two-dimensional version of his Kirkland House conversations.

Concluding Thoughts

But there is something more. The arc of Mackey's life celebrates his own undergraduate experience and his opportunity to work with talented undergraduates at Harvard. The geography of his youth led to his transformative

[12] *In [16], the chapter on "The War and its Aftermath: Andrew Gleason, George Mackey and an Assignation in Hilbert Space" focuses more on Gleason and Mackey than on the war and its aftermath. Mackey and Gleason forged a memorable friendship at Harvard. In particular, Gleason regarded Mackey as his PhD supervisor even though he never earned a PhD. The two colleagues talked about mathematics almost daily but never collaborated on any papers. Mackey was slow and methodical, and Gleason worked with "dazzling" speed [p. 160]. The beautiful reflection on the power of their collaboration will inspire readers. Every reader will benefit from the very human insight Mackey provided when he described the "peace of mind" he gained when he stopped viewing Andrew Gleason "as a dangerous younger rival" whom he had to outdo and instead "concentrated on his own strength, which was his ability to think deeply about a subject—for years or days at a stretch—with monk-like devotion" [p. 160].*

experiences at Rice Institute. In the 1930s, in the somewhat unlikely location of Houston, Texas, talented mathematicians like Leighton, Ford, and Griffin served on the Rice faculty. Mackey benefitted from their extraordinary training and exposure to some of the most celebrated American mathematicians at the time. They not only taught Mackey mathematics but they also helped point him towards what would become his own distinguished mathematical career. Mackey carried this training forward at Harvard with his own students. Whether in his classes, in his office, or at Kirkland House, he shared his verve for mathematics with them. Singer's queries offered another venue for Mackey to do what he did best (in the afternoon), namely, cooperate extensively on mathematics. Although Mackey begins his "What do mathematicians do?" by offering an analysis of the "mathematical strength of the world" focused in five geographical areas, this consideration of his life and work actually suggests a much broader view of the strength of mathematics, beginning with the undergraduate experience.

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Credits

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"What do mathematicians do?"

George W. Mackey

There are a relatively small number of people in the world (perhaps a few thousand) who spend a large part of their time thinking about and trying to contribute to an esoteric subject called pure mathematics. The more active and successful number only in the hundreds and form a world community in which every one knows or knows of everyone else. The overwhelming majority make their living by teaching in universities, their investigations being subsidized by their being given less than full time teaching loads. For complicated historical and cultural reasons the great majority live in Europe, North America, and Japan and are far from being uniformly distributed over these areas. Some European countries are almost completely unrepresented, and some, like France, are especially strong. Moreover, if the pure mathematicians of Paris, Moscow, greater Boston, Princeton, and New York City were to be eliminated, the mathematical strength of the world would probably be reduced by at least two thirds.

If a non-mathematician listens to these people talk or attempts to read their journals, he confronts an incomprehensible jargon filled with words like differential equation, group, ring, manifold, homotopy, etc. If he asks for an explanation, he is overwhelmed by a concatenation of difficult to grasp abstract concepts held together by long chains of intricate argument. Whatever are these mathematicians doing? Why do they find it so interesting and what does it have to do with the rest of the world?

In the time at my disposal I can do little to answer these questions. Nevertheless, I am going to make an attempt. In a word, pure mathematicians are refining, developing, improving, and (rather rarely) discovering the intellectual tools that have proved useful in analyzing and understanding the measurable aspects of the world in which we live. These measurable aspects are not so limited as they might seem. At the beginning there was just counting and later the measuring of distances, areas and volumes. However, the last three centuries or so have witnessed a steadily accelerating growth in the extent to which all natural phenomena can be understood in terms of relationships between measurable entities. In the 1920s, for example, the discovery of quantum mechanics went a very long way toward reducing chemistry to the solution of well-defined mathematical problems. Indeed, only the extreme difficulty of many of these problems prevents the present day theoretical chemist from being able to predict the outcome of every laboratory experiment by making suitable calculations. More recently the molecular biologists have made startling progress in reducing the study of life back to the study of chemistry. The living cell is a miniature but extremely active and elaborate chemical factory and many, if not most, biologists today are confident that there is no mysterious "vital principle," but that life is just very complicated chemistry. With biology reduced to chemistry and chemistry to mathematics, the measurable aspects of the world become quite pervasive.

At this point I must make it emphatically clear that, in spite of what I have just said, pure mathematicians concern themselves very little with the external world—even in its measurable aspects. Their concern with the intellectual tools used in analyzing the external world is not so much in using these tools as in polishing them, improving them, and very occasionally inventing brand new ones. Indeed it is their concern with the tools themselves, rather than with using the tools, that distinguishes them from applied mathematicians and the more mathematically minded scientists and engineers.

While it is natural to suppose that one cannot do anything very useful in tool making and tool improvement, without keeping a close eye on what the tool is to be used for, this supposition turns out to be largely wrong. Mathematics has sort of inevitable structure which unfolds as one studies it perceptively. It is as though it were already there and one had only to uncover it. Pure mathematicians are people who have a sensitivity to this structure and such a love for the beauties it presents that they will devote themselves voluntarily and with enthusiasm to uncovering more and more of it, whenever the opportunity presents itself.

Perhaps, because of the lack of arbitrariness in its structure, research in pure mathematics is a very cooperative activity in which everyone builds on the work of someone else and in turn has his own work built upon. On the other hand, mathematicians tend to work alone (and occasionally in pairs) and to be intensely individualistic. Thus, in a curious way, the advancement of pure mathematics very effectively combines extensive cooperation with rugged individualism. No one has enough of an overview to be at all effective in directing the development of mathematics. Indeed if anyone tried he would probably do more harm than good. Just as the social insects build marvelously designed intricate structures by apparently carrying materials around at random so have the mathematicians built a marvelously articulated body of abstract concepts by following their individual instincts with an eye to what their colleagues are doing. An interesting example occurred during the first two decades of the twentieth century. While the physicists were struggling with contradictions and anomalies in the so-called "old quantum theory," two quite distinct branches of pure mathematics were being developed by two different sets of mathematicians with no thought for one another or for physics. Then the discoveries of Schrödinger and Heisenberg in 1924-25 provided the key to the mystery, and physics found its way to that subtle refinement of Newtonian mechanics known as quantum mechanics. Almost immediately it was found that these two separate new branches of pure mathematics were not only what quantum mechanics needed for its precise formulation and further development, but they could be regarded moreover as two facets of a bigger and better unified new branch which was even more adapted to the needs of quantum physics. Several decades later this unified new branch began to have important applications to some of the oldest problems in the theory of numbers.

The set of natural numbers 1, 2, 3, . . . is perhaps the first mathematical tool discovered by man, but its study continues to provide pure mathematicians with an apparently inexhaustible supply of profound and challenging problems. Consider, for example, the problem of determining in how many different ways (if any) a given whole number can be written as a sum of two squared whole numbers. The answer to this question turns out to depend on the factorization of the number into primes. I remind you that a number is said to be a prime if it cannot be written as the product of two other positive numbers, neither of which is one. For example 2, 3, 5 and 7 are primes while 4 and 15 are not since $4 = 2\times2$, and $15 = 3\times5$. One can find an answer for the problem expressed in terms of the answer when the given number is a prime. This much is fairly easy. Much more difficult to establish is the beautiful result that solutions exist for the prime 2 and for precisely those odd primes which leave a remainder of 1 when divided by 4. This theorem was announced without proof by Fermat in the middle of the seventeenth century. One hundred years later Euler, the great eighteenth century mathematician, worked for seven years before finding a proof. Nowadays quite simple proofs exist, but they use sophisticated new tools such as group theory and field theory. Similar but slightly more complicated problems remained unsolved until quite recently. Others are still beyond our reach but may become accessible when the new tool mentioned above and which arose in physics becomes further developed.

Such problems may seem trifling to the outsider, but a major lesson taught by the development of Science in the last three and a half centuries is that the way to progress lies in fine analysis—in looking very closely at the simplest aspects of things and then building from there. Galileo began modern mathematical physics by deciding that it would be worthwhile to time a falling body and discover just how much it accelerated as it fell.

Now let me return to my statement that the great majority of pure mathematicians make their livings by teaching in universities and have their work subsidized by reduced teaching loads. Nowadays many people criticize this arrangement on the grounds that it tempts faculty members to neglect their teaching. I think that this criticism is without serious foundation. In my opinion a very high proportion enjoy the teaching they do and regard doing it well as a serious responsibility which is part of what they owe the University for supporting their research. It is, I think, a rather happy arrangement in that it makes it possible for at least some teaching to be done by genuine authorities in the field and at the same time supports an activity whose measurable economic benefits are so uncertain and so far into the future. On the other hand, there is a certain tension. One becomes extremely absorbed in one's research problems and longs for extra time in which to work on them. The summer vacation helps but is not enough. It is fortunate that other possibilities exist, such as sabbatical leaves and various institutes where one can go from time to time and concentrate exclusively on research. Actually there are all too few of the latter, and I would like to close by saying a few words about the two which I myself have visited—one of which is just a few miles outside of Paris in Bures sur Yvette.

The Institute for Advanced Study in Princeton, New Jersey is the older and became famous very quickly by having Einstein on its faculty. It was founded in 1933 and has played a very useful role in the mathematical world ever since. Its school of mathematics has an extremely distinguished permanent faculty of half a dozen or so and every year a group of 50 or 100 visitors. Most of the visitors are young—only a few years beyond the PhD. However, there is always of sprinkling of older mathematicians including a few distinguished foreigners. I have just come from a very pleasant and productive term there.

The institute at Bures sur Yvette (L'Institut des Hautes Etudes Scientifiques) is younger and has a smaller permanent faculty—but one which is probably no less distinguished. I spent an agreeable and profitable term there seven years ago. Like its older counterpart in Princeton, it plays a very important role in the mathematical world—not only by helping mathematicians find more time for their work, but by bringing those with similar interests together so they may exchange ideas.

On this visit to Paris I am not at Bures but am rather teaching a course at the University (Paris VI). However, my Harvard colleague, Professor Barry Mazur, is there and in fact is a frequent visitor. He is in the audience today and has agreed to try to answer questions any of you may have either about the nature of mathematical research or about the IHES.