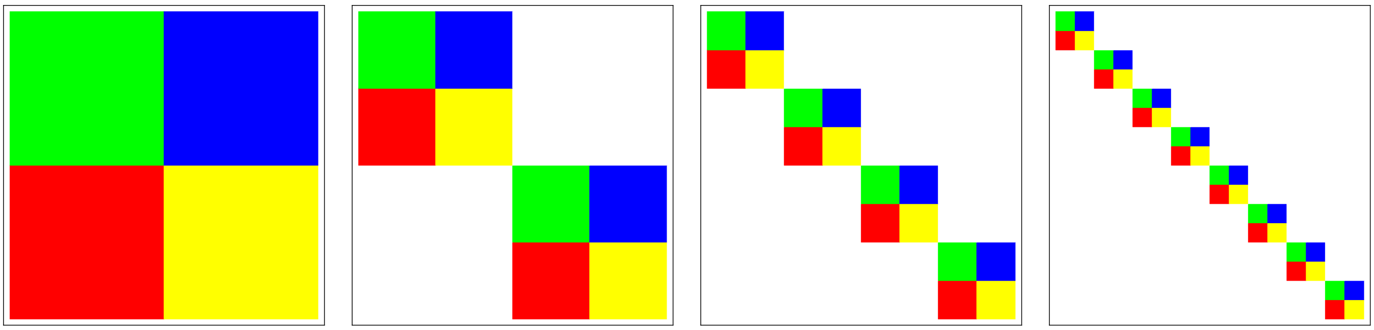
$\mathcal{R}$ We Living in the Matrix?



Roy Araiza and Rolando de Santiago

Introduction

In the early years of the twentieth century, the foundations for quantum mechanics were laid out by Dirac, Heisenberg, Bohr, Schrödinger, and others. In his work on the foundations of quantum mechanics, John von Neumann postulated that physical phenomena should be modeled in terms of Hilbert spaces and operators, with observables corresponding to self-adjoint operators and states corresponding to unit vectors. Motivated by his interest in the theory of single operators, he would introduce the notion of what is now termed a von Neumann algebra. Von Neumann and Francis Murray subsequently published a series of fundamental papers, beginning with "On rings of operators" [13], that develop the basic properties of these algebras and establish operator algebras as an independent field of study.

In the years after Murray and von Neumann's initial work, the field of operator algebras developed rapidly and split into subfields including C^*-algebras and von Neumann algebras. Moreover, operator algebraists began to examine generalizations of these objects, such as operator spaces and operator systems. The importance of operator algebras can be witnessed by its applications in Voiculescu's free probability theory, Popa's deformation/rigidity theory, and Jones' theory of subfactors. These areas give us insight into numerous fields, including random matrix theory, quantum field theory, ergodic theory, and knot theory.

In a landmark paper unraveling the isomorphism classes of injective von Neumann algebras, Connes proves that it is possible to construct a sequence of approximate embeddings for a large class of von Neumann algebras into finite-dimensional matrix algebras; Connes somewhat casually remarks that this property should hold for all separable von Neumann algebras. Formally, Connes' embedding problem, as this assertion is now called, asks if every type II_1 factor acting on a separable Hilbert space is embeddable into an ultrapower of the hyperfinite II_1 factor via a nonprinciple ultrafilter.

Our goal is to unravel the meaning behind Connes' embedding problem and to highlight its significance by providing equivalent formulations that have driven research in the field.

Background

Let X be a compact Hausdorff space. Then the set $C(X)$ of continuous complex-valued functions on X, endowed with pointwise addition and multiplication, is an algebra over $\mathbb{C}$. This algebra admits an anti-linear involution $f^* := \overline{f}$ and a norm

$$\|f\|_\infty := \sup_{x\in X} |f(x)|$$

that are related by the identity

$$\|f^*f\|_\infty = \|f\|_\infty^2. \tag{1}$$

Moving towards the noncommutative setting, we consider the algebra $M_n(\mathbb{C})$ of $n \times n$ complex matrices. The *operator norm* of $m \in M_n(\mathbb{C})$ is

$$\|m\|_\infty := \sup\{\|m\xi\|_2 : \xi \in \mathbb{C}^n,\ \|\xi\|_2 \leq 1\},$$

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in which $\|\xi\|_2$ denotes the Euclidean norm of $\xi \in \mathbb{C}^n$. The conjugate transpose $*$ is an anti-linear involution on $M_n(\mathbb{C})$ that satisfies

$$\|m^*m\|_\infty = \|m\|_\infty^2. \tag{2}$$

In both situations, the operation $*$ satisfies $(ab)^* = b^*a^*$ for all elements of their domain. To generalize $M_n(\mathbb{C})$, we replace $\mathbb{C}^n$ with an appropriate Hilbert space.

Definition 1. A *Hilbert space* is a complex inner-product space $\mathcal{H}$ that is complete in the norm $\|\xi\|_{\mathcal{H}} = \sqrt{(\xi|\,\xi)}$ induced by the inner product.

A standard example of an infinite-dimensional Hilbert space is the space of square-summable sequences of complex numbers

$$\ell_2(\mathbb{N}) := \left\{ (x_j)_{j=1}^\infty : \sum_{j=1}^\infty |x_j|^2 < \infty \right\},$$

with inner product

$$(x|\,y) := \sum_{j=1}^\infty x_j \overline{y_j}.$$

It turns out that every separable, infinite-dimensional Hilbert space is isometrically isomorphic to $\ell_2(\mathbb{N})$. For example, this applies to

$$L_2(X,\mu) := \left\{ f : X \to \mathbb{C} : \int_X |f(x)|^2 \, d\mu(x) < \infty \right\},$$

in which (X,μ) is a σ-finite measure space and

$$(f|\,g) := \int_X f(x)\overline{g(x)}\, d\mu(x).$$

Operators on Hilbert spaces: Formalizing infinite matrices. Operators on Hilbert spaces provide a framework with sufficient versatility to be applied in physics, representation theory, partial differential equations, and other fields. We focus here on the class of *bounded operators*.

Definition 2. Given a Hilbert space $\mathcal{H}$ and a linear operator $x : \mathcal{H} \to \mathcal{H}$, the *operator norm* of x is defined by

$$\|x\| := \sup\{\, \|x\xi\|_{\mathcal{H}} : \xi \in \mathcal{H},\ \|\xi\|_{\mathcal{H}} \le 1\}.$$

A linear operator x is *bounded* if $\|x\|$ is finite. $B(\mathcal{H})$ will denote the collection of all bounded linear operators on $\mathcal{H}$.

It can be shown that an operator is bounded if and only if it is continuous. Since the composition of bounded linear operators is again bounded, it follows that $B(\mathcal{H})$ is an algebra over $\mathbb{C}$. The algebra $B(\mathcal{H})$ shares several properties of the algebra $M_n(\mathbb{C})$ outlined above. It is a complete, normed algebra that admits an anti-linear involution $*$ (the adjoint). The *adjoint* of $x \in B(\mathcal{H})$ is the unique element $x^* \in B(\mathcal{H})$ satisfying

$$(x\xi|\,\eta) = (\xi|\,x^*\eta)$$

for all $\xi, \eta \in \mathcal{H}$. The norm and the involution satisfy an identity analogous to (1) and (2); for $x \in B(\mathcal{H})$, we have

$$\|x^*x\| = \|x\|^2.$$

C^*- and von Neumann algebras. There are multiple infinite-dimensional generalizations of $M_n(\mathbb{C})$, an important one being $B(\mathcal{H})$. We examine two such generalizations: one that arises from elevating the properties of $B(\mathcal{H})$ to axioms and another that comes from considering a weaker notion of convergence than the one arising from the operator norm. This yields two branches of operator algebras: C^*-algebras and von Neumann algebras.

Definition 3. Let $(\mathcal{A}, \|\cdot\|)$ be a complex algebra endowed with an anti-linear involution $*$ that is complete with respect to $\|\cdot\|$. If

$$\|ab\| \le \|a\|\,\|b\|$$

for all $a, b \in \mathcal{A}$, then $\mathcal{A}$ is a *Banach-$*$ algebra*. If the *C^*-identity*

$$\|a^*a\| = \|a\|^2 \tag{3}$$

holds for all $a \in \mathcal{A}$, then $\mathcal{A}$ is a *C^*-algebra.* The algebra $\mathcal{A}$ is *unital* if there is a $1_{\mathcal{A}} \in \mathcal{A}$ such that $1_{\mathcal{A}}a = a1_{\mathcal{A}} = a$ for all $a \in \mathcal{A}$.

The classic example of a unital, abelian C^*-algebra is $C(X)$, with the unit being the constant function 1. By a theorem of Gelfand, every unital, abelian C^*-algebra can be identified with $C(X)$ for some compact Hausdorff space X. Examples of nonabelian unital C^*-algebras are $M_n(\mathbb{C})$ and $B(\mathcal{H})$. The Gelfand–Naimark–Segal (GNS) construction produces a representation of a C^*-algebra given any state (unital positive linear functional) on that algebra.

Theorem 4 (Gelfand–Naimark). *Given a C^*-algebra $\mathcal{A}$, there exists a C^*-subalgebra $\tilde{\mathcal{A}} \subset B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ such that $\mathcal{A}$ and $\tilde{\mathcal{A}}$ are isometrically $*$-isomorphic.*

Therefore, any "abstract" C^*-algebra can be realized as a "concrete" C^*-algebra via the GNS construction and the Gelfand–Naimark theorem above.

The *weak operator topology* (*WOT*) on $B(\mathcal{H})$ is defined by the family of seminorms

$$\rho_{\xi,\eta}(x) = (x\xi|\,\eta)$$

where $\xi, \eta \in \mathcal{H}$. In practice, a sequence x_n of operators converges to x in the weak operator topology if and only if for every $\xi, \eta \in \mathcal{H}$,

$$((x_n - x)(\xi)|\,\eta) \to 0.$$

This loosely means that the "matrix coefficients" of x_n converge to those of x.

Definition 5. A unital $*$-subalgebra $M \subset B(\mathcal{H})$ is a *von Neumann algebra* if M is closed in the WOT.

The WOT is weaker than the norm topology, and thus every von Neumann algebra is in fact a C^*-algebra. Letting

$$M' := \{y \in B(\mathcal{H}) : xy = yx \ \forall x \in M\}$$

denote the *commutant* of M, von Neumann proved that a unital, self-adjoint subset $M \subset B(\mathcal{H})$ is a von Neumann algebra if and only if $M = (M')'$, where $(M')'$ denotes the *double commutant* of M. This characterization is known as the double commutant theorem. In contrast, no purely algebraic characterization of C^*-algebras is known.

To contrast von Neumann algebras against general C^*-algebras, note that if we consider a finite Borel measure space (X, μ), functions in $L_\infty(X, \mu)$ act on $L_2(X, \mu)$ by way of multiplication as bounded operators. Naturally, we have the following chain of inclusions:

$$C(X) \subset L_\infty(X, \mu) \subset B(L_2(X, \mu)).$$

While $C(X)$ is a C^*-algebra, it is not a von Neumann algebra, since $(C(X)')'$ contains $L_\infty(X, \mu)$. It can be shown that $L_\infty(X, \mu)$ is a maximal abelian subalgebra in $B(L_2(X, \mu))$ and is thus a von Neumann algebra by the double commutant theorem. In fact, all abelian von Neumann algebras on a separable Hilbert space can be identified with $L_\infty(X, \mu)$ for some measure space (X, μ).

Classification of factors. A von Neumann algebra M is a *factor* if its center is trivial: $M \cap M' = \mathbb{C}1_M$. Every von Neumann algebra admits a direct integral decomposition over its center into factors. Thus, the classification of von Neumann algebras reduces to the study of factors. Furthermore, the classification of factors involves the *cone of positive* elements

$$M_+ = \{t^*t : t \in M\}$$

in a factor M and the *lattice of projections*

$$\mathcal{P}(M) = \left\{p \in M : p = p^2, p = p^*\right\}$$

in M.

A linear map $w : M_+ \to [0, \infty]$ is a *tracial weight* on M if $w(t^*t) = w(tt^*)$ for all $t \in M$. A factor M is

- type I_n if $w(\mathcal{P}(M)) = \{0, 1/n, 2/n, \ldots, 1\}$
- type I_∞ if $w(\mathcal{P}(M)) = \{0, 1, 2, \ldots, \}$
- type II_1 if $w(\mathcal{P}(M)) = [0, 1]$
- type II_∞ if $w(\mathcal{P}(M)) = [0, \infty)$
- type III if $w(\mathcal{P}(M)) = \{0, \infty\}$

for some tracial weight w. Whenever M is a factor, any tracial weight is unique up to a positive scalar, making type decomposition well-defined. Factors of type I_n are isomorphic to $B(\mathcal{H})$ for some Hilbert space of dimension n. Type III factors can be further decomposed into so-called type III_λ factors for $\lambda \in [0, 1]$, but we will not elaborate the distinguishing features here.

Given a type I_n or II_1 factor M, the unique weight $w : M_+ \to [0, \infty)$ satisfying $w(1_M) = 1$ extends to a linear functional on M. Observing that every element $x \in M$ decomposes as a linear combination

$$x = x_+ - x_- + iy_+ - iy_-$$

of four elements in M_+, the *trace* $\tau : M \to \mathbb{C}$ is the linear functional defined by

$$\tau(x) := w(x_+) - w(x_-) + iw(y_+) - iw(y_-).$$

The functional τ satisfies the *trace property* $\tau(xy) = \tau(yx)$ for all $x, y \in M$ and is uniquely determined if we insist that $\tau(1_M) = 1$. If M is a type I_n factor, then $\tau : M \to \mathbb{C}$ is the normalized trace on $M_n(\mathbb{C})$.

Group algebras. Von Neumann observed that group representation theory provides us with a potentially rich collection of von Neumann algebras. We direct our attention to the *left regular representation* of a countable group Γ endowed with the discrete topology. Consider the Hilbert space $\ell_2(\Gamma)$ with basis $\{\delta_\gamma\}_{\gamma \in \Gamma}$. The *left regular representation* is the unitary representation $\lambda : \Gamma \to B(\ell_2(\Gamma))$ defined by linearly extending the map $\lambda_\sigma(\delta_\gamma) = \delta_{\sigma\gamma}$ for all $\sigma, \gamma \in \Gamma$. This embeds a copy of the group ring $\mathbb{C}[\Gamma]$ into $B(\ell_2(\Gamma))$. Taking either the norm closure or the WOT closure of $\mathbb{C}[\Gamma]$ gives either $C_\lambda(\Gamma)$ or $L(\Gamma)$, the *reduced group C^*-algebra* of Γ or the *group von Neumann algebra* of Γ, respectively.

In both of these cases, there is a canonical trace given by $\tau(x) = (x\delta_e | \delta_e)$, where $e \in \Gamma$ is the identity. A group von Neumann algebra is a II_1 factor precisely when Γ is an *i.c.c. group*, that is, a group for which the conjugacy class of every nontrivial element has infinite order. Examples of i.c.c. groups are $S_\infty = \bigcup_{n \geq 2} S_n$, the group of finitely supported permutations of $\mathbb{N}$, and $\mathbb{F}_n$, the free group on $n \geq 2$ generators. Murray and von Neumann used this construction to give examples of two nonisomorphic von Neumann algebras, namely $L(S_\infty)$ and $L(\mathbb{F}_2)$. The free groups factor problem, which in one form asks whether $L(\mathbb{F}_2) \not\cong L(\mathbb{F}_3)$, is a question whose answer is currently unknown. Efforts to understand this led to the development of Voiculescu's free probability theory, a topic we visit in the subsection "Free probability theory."

The hyperfinite II_1 factor. An algebra (C^*- or von Neumann) is *hyperfinite* if there exists an increasing sequence of finite-dimensional subalgebras whose union is dense (with respect to the relevant topology) in the algebra. For example, $L(S_\infty) = \overline{\bigcup_{n \geq 2} L(S_n)}^{WOT}$ is a hyperfinite type II_1 factor. Murray and von Neumann were able to prove that any two hyperfinite II_1 factors are necessarily isomorphic. We denote by

$$\mathcal{R} := L(S_\infty)$$

the unique hyperfinite II_1 factor, up to isomorphism. One of the consequences of Connes' work shows that $\mathcal{R}$ is the "smallest" II_1 factor in the sense that every II_1 factor admits an embedding of $\mathcal{R}$.

A canonical construction of a hyperfinite algebra arises by considering the embeddings

$$M_2(\mathbb{C}) \hookrightarrow M_4(\mathbb{C}) \hookrightarrow M_8(\mathbb{C}) \hookrightarrow$$

mapping $x \in M_{2^n}(\mathbb{C})$ to

$$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \in M_{2^{n+1}}(\mathbb{C}).$$

Provided we use normalized traces, this forms a sequence of trace-preserving embeddings. The inductive limit A of this procedure is an infinite-dimensional $*$-algebra with a tracial state $\tau_0 : A \to \mathbb{C}$ and trace-preserving embeddings of $M_{2^n}(\mathbb{C}) \hookrightarrow A$. The GNS construction with respect to τ_0 gives us a Hilbert space $\mathcal{H}$ and a faithful representation $\pi : A \to B(\mathcal{H})$. Closing these objects in either norm (resp., WOT) produces a uniformly hyperfinite (UHF) (resp., hyperfinite) C^*- or von Neumann algebra, respectively.

We may attempt to produce nonisomorphic examples by emulating this procedure and taking trace-preserving embeddings of the form

$$M_{k_1}(\mathbb{C}) \hookrightarrow M_{k_2}(\mathbb{C}) \hookrightarrow M_{k_3}(\mathbb{C}) \hookrightarrow$$

for some sequence $\{k_n\}$ in $\mathbb{N} \setminus \{1\}$. A theorem of Glimm shows that there exist uncountably many nonisomorphic UHF C^*-algebras arising in this way, and they are distinguished by the generating sequence $\{k_n\}$. The von Neumann algebra case is vastly different. Connes' classification of injective factors proves that all von Neumann algebras produced in this manner are isomorphic to $\mathcal{R}$. Indeed $\mathcal{R}$ is an example of a type II_1 factor with the trace playing the role of the weight in its type decomposition.

We would be remiss if we did not expand on the classification of injective factors acting on a separable Hilbert space. This endeavor nearly came to a close with Connes' landmark result, which earned him the Fields Medal in 1982. In [2], Connes establishes the equivalence between the notions of injectivity and hyperfiniteness for von Neumann algebras. This settled numerous open problems, such as whether any subfactor of the hyperfinite II_1 factor is hyperfinite. Building on his previous works, Connes goes on to show that for each of the types II_1, II_∞, or III_λ with $\lambda \in (0, 1)$, there is a unique injective von Neumann algebra and it is hyperfinite. While it is well known that type III_0 are not classifiable through "simple" means, Haagerup was able to prove that there is exactly one hyperfinite type III_1 factor, essentially closing the book on the classification of injective factors.

Amenability. Why it is so difficult to differentiate one group II_1 factor from another? The heart of this problem lies in understanding representation-theoretic aspects of the group itself.

A discrete group Γ is *amenable* if there is a sequence of finite subsets $\{F_n\}$ of Γ so that for all $\gamma \in \Gamma$,

$$\lim_{n\to\infty} \frac{|\gamma F_n \Delta F_n|}{|F_n|} = 0,$$

where $F \Delta G$ denotes the symmetric difference of sets. Equivalently, we ask if the left regular representation contains a sequence of unit vectors $\{\xi_n\} \in \ell_2(\Gamma)$ so that

$$\lim_{n\to\infty} \|\lambda_\gamma(\xi_n) - \xi_n\| = 0$$

for every $\gamma \in \Gamma$. In essence, this means that the group and its group operation can be approximated by finite structures. The prototypical example is the integers $\mathbb{Z}$, though this class contains others such as $(\oplus_{\mathbb{Z}} \mathbb{Z}/2) \rtimes \mathbb{Z}$, $(\oplus_{\mathbb{Z}} \mathbb{Z}) \rtimes \mathbb{Z}$, and S_∞. Connes' classification of injective factors demonstrates that in the case of group von Neumann algebras, an i.c.c. group Γ is amenable if and only if $L(\Gamma) \simeq \mathcal{R}$.

To provide nonexamples, we use an equivalent formulation of amenability: a group Γ is *amenable* if there does not exist a finitely additive mean $\mu : \mathcal{P}(\Gamma) \to [0, 1]$ that satisfies $\mu(\gamma \cdot E) = \mu(E)$. It is a standard exercise to show that $\mathbb{F}_n$ is a nonamenable group whenever $n \geq 2$. An argument establishing this fact is similar to that which demonstrates the Banach–Tarski paradox and is in fact a necessary first step in the proof of the paradox. For this reason, nonamenable groups are often considered to be groups that admit "paradoxical decompositions."

Tensor products. We will take various objects, such as Hilbert spaces, C^*-algebras, and von Neumann algebras, and add more structure to their algebraic tensor products, recalling that the algebraic tensor product is just the tensor product of vector spaces. For example, given two Hilbert spaces $\mathcal{H}, \mathcal{K}$, we start with the algebraic tensor product $\mathcal{H} \otimes \mathcal{K}$ and produce the *Hilbertian tensor product* $\mathcal{H} \otimes_2 \mathcal{K}$, the completion of the pre-Hilbert space $\mathcal{H} \otimes \mathcal{K}$ with respect to the inner product on finite sums given by

$$\left(\sum_i \xi_i \otimes \eta_i \middle| \sum_j \xi_j \otimes \eta_j \right)_{\mathcal{H} \otimes_2 \mathcal{K}} := \sum_{i,j} (\xi_i | \xi_j)_{\mathcal{H}} \, (\eta_i | \eta_j)_{\mathcal{K}}.$$

Let $\mathcal{A}$ and $\mathcal{B}$ be either C^*- or von Neumann algebras. Their algebraic tensor product $\mathcal{A} \otimes \mathcal{B}$ is a $*$-algebra with the product and involution given on finite sums by

$$\left(\sum_i a_i \otimes b_i \right) \left(\sum_j c_j \otimes d_j \right) = \sum_{i,j} a_i c_j \otimes b_i d_j$$

and

$$\left(\sum_i a_i \otimes b_i \right)^* = \sum_i a_i^* \otimes b_i^*.$$

The tensor product of two von Neumann algebras $M \subseteq B(\mathcal{H}), N \subseteq B(\mathcal{K})$ is formed by first taking the Hilbertian tensor product $\mathcal{H} \otimes_2 \mathcal{K}$ and the algebraic tensor product $M \otimes N$. This implements a representation of operators acting on $\mathcal{H} \otimes_2 \mathcal{K}$ via the formula

$$(m \otimes n)(\xi \otimes \eta) = (m\xi) \otimes (n\eta)$$

for every $m \in M$, $n \in N$, $\xi \in \mathcal{H}$, and $\eta \in \mathcal{K}$. We then take the WOT closure in $B(\mathcal{H} \otimes_2 \mathcal{K})$ of these operators to form the *von Neumann algebra tensor product* $M\bar{\otimes}N$. For group von Neumann algebras, this construction gives an identification between $L(\Gamma \times \Lambda)$ and $L(\Gamma)\bar{\otimes}L(\Lambda)$.

Constructing C^*-algebras from tensor products is far more interesting and is explored in the subsection "The minimal and maximal C^*-algebra tensor products."

Ultraproduct constructions. Let us take a II_1 factor $M \subset B(\mathcal{H})$. Connes' work tells us that $\mathcal{R}$ is the only II_1 factor that may be approximated by finite-dimensional structures. But what if instead we attempt to approximate a II_1 factor M by $\mathcal{R}$? In what sense are we approximating M by $\mathcal{R}$? To answer the second question, we first need to extend the notion of a limit.

A (proper) *filter* $\mathcal{F}$ on $\mathbb{N}$ is a collection $\mathcal{F}$ of subsets of $\mathbb{N}$ such that

(1) $\mathbb{N} \in \mathcal{F}$,
(2) $\varnothing \notin \mathcal{F}$,
(3) if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$, and
(4) if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

Informally, a filter tells us which subsets of $\mathbb{N}$ are considered to be large. An important filter is the *cofinite filter*, denoted by $\mathcal{F}_0$, which consists of all subsets of $\mathbb{N}$ whose complement in $\mathbb{N}$ is a finite set.

A filter $\mathcal{F}$ is an *ultrafilter* if for every $A \subset \mathbb{N}$, either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$. A standard maximality argument shows that every filter can be extended to an ultrafilter, though this completion is not unique. Ultrafilters containing $\mathcal{F}_0$ are called *nonprincipal ultrafilters*.

Throughout this discussion, we fix a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. The collection $\mathcal{U}$ may be used to form limits of bounded sequences $\{s_n\}_{n=1}^{\infty} \in X$ in a proper metric space (X, d), even when a classical limit does not exist. For example, if $x_n = (-1)^n$ is a sequence in $\mathbb{R}$, then the ultralimit of the sequence, which we denote by $\lim_{n \to \mathcal{U}} x_n$, will either be -1 or 1 depending on whether $\mathcal{U}$ contains the set of even or odd numbers. In essence, the ultralimit along an ultrafilter $\mathcal{U}$ is a preferred choice of a convergent subsequence. Moreover, if X admits an algebra structure, the operation $\lim_{n \to \mathcal{U}}$ obeys traditional limit laws such as linearity and multiplicativity.

Let $(M_n, \tau_n)_{n=1}^{\infty}$ be a countably infinite family of II_1 factors with traces $\tau_n : M_n \to \mathbb{C}$. We define the space

$$\ell_\infty(M_n) := \{(x_n) : x_n \in M_n, \sup_{n \in \mathbb{N}} \|x_n\| < \infty\}$$

and the ideal $\mathcal{I}_\mathcal{U}$ of $\ell_\infty(M_n)$,

$$\mathcal{I}_\mathcal{U} := \{(x_n) \in \ell_\infty(M_n) : \lim_{n \to \mathcal{U}} \tau_n(x_n^* x_n) = 0\}.$$

The *ultraproduct* of $(M_n, \tau_n)_{n=1}^{\infty}$ along $\mathcal{U}$ is the II_1 factor $\prod M_n/\mathcal{U} = \ell_\infty(M_n)/\mathcal{I}_\mathcal{U}$ with trace $\tau(x_n) = \lim_{n \to \mathcal{U}} \tau_n(x_n)$. It is a nontrivial exercise to show that the ultraproduct of II_1 factors is again a II_1 factor (see [1,4]). If $M_n = M$ for every $n \in \mathbb{N}$, then we write $\prod M/\mathcal{U} := M^\mathcal{U}$, and in this case we say that $M^\mathcal{U}$ is the *ultrapower* of M. It should be noted that a II_1 factor arising from an ultraproduct of II_1 factors is necessarily represented on a nonseparable Hilbert space.

Connes' Embedding

Before stating Connes' embedding problem we wish to make a few remarks. As was mentioned, Connes nearly completed the classification of injective factors in his seminal paper [2], where (among other things) he shows that each of the factors of types II_1, II_∞, and III_λ for $\lambda \in (0, 1)$ is isomorphic to a unique hyperfinite von Neumann algebra of the corresponding type. Connes proves that the free group factors $L(\mathbb{F}_n), n \geq 2$, admit a sequence of approximate algebraic embeddings into matrix algebras. Furthermore, these embeddings can be chosen in such a way that they approximately preserve the trace. He further remarks that such a construction ought to hold for any separable II_1 factor. The precise statement is as follows:

Conjecture 6 (Connes' embedding problem). *If M is a separable II_1 factor with trace τ then for any $\varepsilon > 0$ and finite subset $F \subset M$, there exists $n \in \mathbb{N}$ and a function $\varphi : M \to M_n(\mathbb{C})$ so that $\varphi(1_M) = 1_{M_n}$ and so that for all $x, y \in F$:*

(1) $\|\varphi(x + y) - \varphi(x) - \varphi(y)\|_{2,n} < \varepsilon$,
(2) $\|\varphi(xy) - \varphi(x)\varphi(y)\|_{2,n} < \varepsilon$,
(3) $|\tau(x) - \tau_n(\varphi(x))| < \varepsilon$,

where $\|m\|_{2,n} := \sqrt{\tau_n(m^*m)}$ for all $m \in M_n(\mathbb{C})$.

Properties (1) and (2) state that $\varphi : M \to M_n(\mathbb{C})$ is approximately an algebraic homomorphism with respect to the Hilbert–Schmidt norm on $M_n(\mathbb{C})$, and condition (3) shows that φ respects the trace. Taking a sequence of $\varepsilon_n > 0$ that tend to 0, we may restate Connes' embedding problem in the following manner:

Conjecture 7. *Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$. Then every separable II_1 factor M admits a trace-preserving embedding into $\mathcal{R}^\mathcal{U}$.*

In the words of S. Popa, the elements of a von Neumann algebra that embed into $\mathcal{R}^\mathcal{U}$ may be "simulated" in an appropriate sense by sequences valued in the unique hyperfinite II_1 factor $\mathcal{R}$.

One may inquire whether there exists some II_1 factor N into which every II_1 factor embeds in a trace-preserving manner. In [14], Ozawa shows that any such II_1 factor must necessarily be nonseparable. It is a well-known fact that $\mathcal{R}^{\mathcal{U}}$ is not faithfully represented on a separable Hilbert space and thus is a viable natural candidate.

Hyperlinear groups. For each $n \in \mathbb{N}$, let $U(n) \subseteq M_n(\mathbb{C})$ denote the group of $n \times n$ unitary matrices, equipped with the Hilbert–Schmidt norm

$$d_{HS}(u, v) = \sqrt{\tau_n((u - v)^*(u - v))}.$$

The ultraproduct of $(U(n_i) \subseteq M_{n_i}(\mathbb{C}))_{i=1}^{\infty}$ is the group

$$\prod U(n_i)/\mathcal{U} := \frac{\prod_{i=1}^{\infty} U(n_i)}{\{(g_i)_{i=1}^{\infty} : \lim_{i \to \mathcal{U}} d_{HS}(g_i, 1) = 0\}}.$$

A group Γ is *hyperlinear* if there exist a sequence of unitary groups $\{U(n_i)\}_{i=1}^{\infty}$, an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, and an injective group homomorphism $\Gamma \hookrightarrow \prod U(n_i)/\mathcal{U}$. A deeper treatment of ultraproducts of groups and their properties may be found in [1, 17].

It is conceivable that every group is hyperlinear, and in fact the existence of a nonhyperlinear group is sufficient to disprove Connes' embedding conjecture:

Theorem 8 (Radulescu). *The following are equivalent:*

(1) $L(\Gamma) \hookrightarrow \mathcal{R}^{\mathcal{U}}$ *for all countable i.c.c. groups.*
(2) *Every i.c.c. group is hyperlinear.*

Free probability theory. Viewing the trace $\tau : M \to \mathbb{C}$ on a II_1 factor M as a noncommutative analogue of the integral on a probability space, Voiculescu defined notions of independence and entropy for tuples of self-adjoint elements in M. This abstract formalism allows one to generalize the classical theory of probability to the more general theory of noncommutative probability theory by replacing random variables and their expectations with self-adjoint operators and their traces. Surprisingly, Connes' embedding conjecture has an equivalent formulation in this realm.

We fix a finite collection of n self-adjoint elements in a II_1 factor $x_1, \dots, x_n \in M$. Given an element of the form $(y = x_{i_1} \cdots x_{i_j}) : 1 \leq j \leq n,\ i_1, \dots, i_j \in \{1, \dots, n\}$, the *mixed moment* of y is $\tau(y)$. The set

$$\{\tau(x_{i_1} \cdots x_{i_j}) : 1 \leq j \leq n,\ i_1, \dots, i_j \in \{1, \dots, n\}\}$$

is the collection of all possible mixed moments of the n-tuple $(x_1, \dots, x_n)$, that is, the collection of values of the trace over every possible product of the x_is.

Mimicking the classical notion of Shannon entropy, Voiculescu defined the following analogue of microstates for the noncommutative setting. We let $\tau_k : M_k(\mathbb{C}) \to \mathbb{C}$ denote the normalized trace on the $k \times k$ matrices. Given $R > 0$, $m, k \in \mathbb{N}$, and $\varepsilon > 0$, the set of *approximating matricial microstates*, denoted by $\Gamma_R((x_1, \dots, x_n), m, k, \varepsilon)$, is the collection of n-tuples of $k \times k$ matrices $(r_1, \dots, r_n)$ such that

$$|\tau(x_{i_1} \cdots x_{i_j}) - \tau_k(r_{i_1} \cdots r_{i_j})| < \varepsilon$$

for every $j \in \{1, \dots, m\}$ and $i_1, \dots, i_j \in \{1, \dots, n\}$ with $\|r_i\| \leq R$. We note that the set $\Gamma_R((x_1, \dots, x_n), m, k, \varepsilon)$ may be empty for certain choices of parameters.

An n-tuple of self-adjoint elements in M has *microstates* if for every $\varepsilon > 0$, there exist parameters $R > 0$, $m, k \in \mathbb{N}$, so that $\Gamma_R((x_1, \dots, x_n), m, k, \varepsilon) \neq \emptyset$. Heuristically, the existence of microstates is equivalent to having the ability to model the noncommutative probability distribution of n-tuples of operators in M by sequences of matrices. This hints at Voiculescu's observation connecting microstates and Connes' embedding conjecture [19].

Theorem 9. *Let M be a II_1 factor. The following are equivalent.*

(1) *Every set of self-adjoint elements $x_1, \dots, x_n \in M$ has microstates.*
(2) $M \hookrightarrow \mathcal{R}^{\mathcal{U}}$.

Continuous model theory. In light of a classical theorem of Łoś, ultrapower constructions play a foundational role in model theory. The underlying idea in continuous model theory is to transform classical predicate logic into a continuous one. Here, the standard truth values $\{T, F\}$ are replaced with the interval $[0, 1]$, quantifiers $\forall$ and $\exists$ are replaced by sup and inf, and continuous functions from $[0, 1]^n \to [0, 1]$ will be our connectives. The model theory of von Neumann algebras takes this one step further by viewing a II_1 factor M with trace τ as a logical structure with a metric arising from the 2-norm $\|x\|_2 := \sqrt{\tau(x^*x)}$ for all $x \in M$. The interplay between the operator norm and the 2-norm on M introduces complexity when considering M as a logical structure. We remark that the understanding of continuous model theory of II_1 factors is currently in its infancy (see [4, 5]).

Recall that one of the outcomes of Connes' work shows that $\mathcal{R}$ embeds into any II_1 factor M. On the other hand, Connes' embedding conjecture asserts that $M \hookrightarrow \mathcal{R}^{\mathcal{U}}$. Using the language of model theory, a positive solution to Connes' embedding states that $\mathcal{R}^{\mathcal{U}}$ is a locally universal object in the category of II_1 factors. But another question may be whether or not a locally universal II_1 factor exists at all. Using model theoretic techniques, there does in fact exist such a II_1 factor; a positive solution to Connes' embedding will show that $\mathcal{R}$ is locally universal [7].

Kirchberg's Conjecture

In the 1990s Eberhard Kirchberg discovered a highly nontrivial equivalent form of Connes' embedding conjecture [11]. Explaining the equivalence would take us outside the scope of this article, but we point out that in this landmark paper Kirchberg presented many equivalences to Connes' original problem and what we will refer to as Kirchberg's conjecture is only one such equivalence. In order to explain Kirchberg's conjecture we must start at its foundations.

The minimal and maximal C^*-algebra tensor products. Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras and let

$$\pi : \mathcal{A} \to B(\mathcal{H}) \quad \text{and} \quad \sigma : \mathcal{B} \to B(\mathcal{K})$$

denote the faithful representations afforded by the GNS construction. For a finite sum $x = \sum_i a_i \otimes b_i \in \mathcal{A} \otimes \mathcal{B}$, the *minimal C^*-tensor product norm* is

$$\|x\|_{C^*\text{-min}} := \left\| \sum_i \pi(a_i) \otimes \sigma(b_i) \right\|_{B(\mathcal{H}\otimes_2\mathcal{K})} . \tag{4}$$

As before, we let $\otimes_2$ denote the Hilbertian tensor product. The completion of $\mathcal{A} \otimes \mathcal{B}$ with respect to $\|\cdot\|_{C^*\text{-min}}$ is a C^*-algebra denoted $\mathcal{A} \otimes_{C^*\text{-min}} \mathcal{B}$ and called the *minimal C^*-algebra tensor product* of $\mathcal{A}$ and $\mathcal{B}$. It can be shown that the minimal C^*-algebra tensor product is independent of choice of representations.

Given $x \in \mathcal{A} \otimes \mathcal{B}$, its *maximal C^*-tensor product norm* is

$$\|x\|_{C^*\text{-max}} := \sup\{\, \|\pi(x)\| \}, \tag{5}$$

where the supremum runs over all Hilbert spaces $\mathcal{H}$ and all representations $\pi : \mathcal{A} \otimes \mathcal{B} \longrightarrow B(\mathcal{H})$. We denote the completion, which is again a C^*-algebra, of the algebraic tensor product with respect to $\|\cdot\|_{C^*\text{-max}}$ by $\mathcal{A} \otimes_{C^*\text{-max}} \mathcal{B}$ and call it the *maximal C^*-algebra tensor product*.

As we might expect from the terminology, it follows that given any norm $\|\cdot\|_\alpha$ on $\mathcal{A} \otimes \mathcal{B}$ that is submultiplicative, preserved under the involution, satisfies the C^*-identity (3), and yields a C^*-algebra after completing, then

$$\|\cdot\|_{C^*\text{-min}} \leq \|\cdot\|_\alpha \leq \|\cdot\|_{C^*\text{-max}} .$$

When does equality hold in the preceding?

Definition 10. A C^*-algebra $\mathcal{A}$ is *nuclear* if for all C^*-algebras $\mathcal{B}$,

$$\mathcal{A} \otimes_{C^*\text{-min}} \mathcal{B} = \mathcal{A} \otimes_{C^*\text{-max}} \mathcal{B}.$$

Examples of nuclear C^*-algebras are the matrix algebras $M_n(\mathbb{C})$ for all $n \geq 1$. In particular, every finite-dimensional C^*-algebra is nuclear. If X is a compact Hausdorff space, then $C(X)$ is nuclear, and thus every abelian C^*-algebra is nuclear.

Lance's weak expectation property. It can be checked that given an inclusion of algebras $\mathcal{A} \subset \mathcal{A}_o$, then $\mathcal{A} \otimes \mathcal{B} \subset \mathcal{A}_o \otimes \mathcal{B}$ for all algebras $\mathcal{B}$. This *inclusion principle* quickly fails once we put more structure on the algebraic tensor product. If $\mathcal{A} \subset \mathcal{A}_o$ and $\mathcal{B}$ are C^*-algebras, then since every representation of $\mathcal{A}_o \otimes \mathcal{B}$ is a representation of $\mathcal{A} \otimes \mathcal{B}$ by restricting, the maximal norm on $\mathcal{A}_o \otimes \mathcal{B}$ will in general be smaller than the maximal norm on $\mathcal{A} \otimes \mathcal{B}$. Thus, in general, $\otimes_{C^*\text{-max}}$ does not satisfy the inclusion principle.

In the 1970s Christopher Lance introduced a notion for C^*-algebras that he called the *weak expectation property* (WEP), and he proved that this notion was directly related to $\otimes_{C^*\text{-max}}$ satisfying the inclusion principle [12].

Definition 11. A unital C^*-algebra $\mathcal{A}$ has *Lance's WEP* if the maximal C^*-algebra tensor product satisfies the inclusion principle with respect to $\mathcal{A}$.

Thus, $\mathcal{A}$ has the WEP if for all C^*-algebras $\mathcal{A}_o \supset \mathcal{A}$ containing $\mathcal{A}$ as a C^*-subalgebra, we have the inclusion of C^*-algebras

$$\mathcal{A} \otimes_{C^*\text{-max}} \mathcal{B} \subset \mathcal{A}_o \otimes_{C^*\text{-max}} \mathcal{B}$$

for all C^*-algebras $\mathcal{B}$. The nth *amplification* of a linear map $\varphi : \mathcal{A} \to \mathcal{B}$ is

$$\varphi^{(n)} : M_n(\mathcal{A}) \to M_n(\mathcal{B}), \quad [a_{ij}]_{i,j} \mapsto [\varphi(a_{ij})]_{i,j} .$$

We say that φ is a *contractive completely positive map* (c.c.p.) if $\varphi^{(n)}$ is positive for all $n \in \mathbb{N}$ and contractive for $n = 1$. Here is an equivalent formulation of Lance's WEP. Realizing $\mathcal{A}$ as a C^*-subalgebra of $B(\mathcal{H})$, then given the completely isometric inclusion $\iota : \mathcal{A} \hookrightarrow \mathcal{A}^{**}$, where $\mathcal{A}^{**}$ denotes the double dual of $\mathcal{A}$, ι extends to a unital, completely positive map $\tilde{\iota} : B(\mathcal{H}) \to \mathcal{A}^{**}$ such that $\tilde{\iota}(a) = a$ for all $a \in \mathcal{A}$.

Kirchberg's conjecture. If Γ is a discrete group, then the *full group C^*-algebra*, $C^*(\Gamma)$, is the completion of the group ring $\mathbb{C}[\Gamma]$ with respect to the norm $\|\cdot\|$ given for $x \in \mathbb{C}[\Gamma]$ by

$$\|x\| = \sup\{\|\pi(x)\|\},$$

where the supremum is taken over all representations $\pi : \mathbb{C}[\Gamma] \longrightarrow B(\mathcal{H})$ and all Hilbert spaces $\mathcal{H}$. At this point we are ready to state the equivalent formulation to Connes' embedding problem known as Kirchberg's conjecture [11]. Consider the free group $\mathbb{F}_\infty$ on a countably infinite number of generators and its full group C^*-algebra $C^*(\mathbb{F}_\infty)$.

Conjecture 12 (Kirchberg's conjecture). *$C^*(\mathbb{F}_\infty)$ has Lance's WEP.*

This was shown to be equivalent to a weaker nuclearity property of $C^*(\mathbb{F}_\infty)$, whether there was only one C^*-structure on $C^*(\mathbb{F}_\infty) \otimes C^*(\mathbb{F}_\infty)$. In other words, does it follow that

$$C^*(\mathbb{F}_\infty) \otimes_{C^*\text{-min}} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{C^*\text{-max}} C^*(\mathbb{F}_\infty)?$$

It can be shown that every unital C^*-algebra $\mathcal{A}$ can be written as a quotient of $C^*(\mathbb{F})$ for some free group $\mathbb{F}$. We say that a unital C^*-algebra is *QWEP* if it is the quotient of a C^*-algebra with Lance's WEP. Kirchberg conjectured that every unital C^*-algebra had QWEP, which if true, would give an affirmative answer to Connes' original conjecture.

Conjecture 13 (Kirchberg's QWEP conjecture). *Every C^*-algebra is QWEP.*

Tsirelson's problem. In our discussion on Connes' original problem the recurring theme was the approximation of infinite dimensions by finite dimensions. An experiment arising in quantum information theory concerns two independent observers making measurements on a quantum system. In 1980 Boris Tsirelson noticed that Bell's inequality had intimate connections with certain famous inequalities arising in analysis [18]. Here we present the operator algebraic reformulation of this experiment and Tsirelson's observations.

Consider an m-tuple $(P_i)_{i=1}^m$ of pairwise orthogonal projections on some Hilbert space $\mathcal{H}$ such that $\sum_i P_i = I$. For specific reasons this object is also called a *projection-valued measure* (*PVM*) with m-outputs. Let $(Q_j)_{j=1}^m$ be another such tuple of pairwise orthogonal projections on $\mathcal{H}$ such that $P_iQ_j = Q_jP_i$ for all i and j. If $\xi \in \mathcal{H}$ is a unit vector, then by looking at matrices of the form $\left[(\xi|\, P_iQ_j\xi)\right]_{i,j}$ we ask whether these matrices can be approximated by matrices of the same form but where we only consider finite-dimensional Hilbert spaces. The short answer to this question is yes. A much more complicated scenario is where we consider a d-tuple of PVMs each with m-outputs. Thus, we consider the scenario where $(P^1, \dots, P^d)$ and $(Q^1, \dots, Q^d)$ are tuples of PVMs where for each $1 \le a \le d$, $(P_i^a)_{i=1}^m$ is itself an m-tuple of pairwise orthogonal projections on the Hilbert space $\mathcal{H}$, and similarly for each $(Q_j^b)_{j=1}^m$, $1 \le b \le d, 1 \le j \le m$. The condition of commutativity implies that $P_i^aQ_j^b = Q_j^bP_i^a$ for all $1 \le i, j \le m$, and $1 \le a, b \le d$.

Definition 14. In the above scenario, a *covariance matrix* will be a matrix of the form $\left[(\xi|\, P_i^aQ_j^b\xi)\right]_{a,i;b,j}$, and the set of all such matrices will be denoted $Q_c(m,d)$.

The space of covariance matrices with PVMs restricted to finite-dimensional Hilbert spaces will be denoted $Q_s(m,d)$. One can show that $Q_c(m,d)$ is closed (limits of covariance matrices are covariance matrices), and since we have that $Q_s(m,d) \subset Q_c(m,d)$ then $\overline{Q_s(m,d)} \subset Q_c(m,d)$.

Problem 15 (Tsirelson's problem). *Does it follow that* $\overline{Q_s(m,d)} = Q_c(m,d)$ *for all* m *and* d*?*

An affirmative answer to Tsirelson's problem would yield an affirmative answer to Kirchberg's conjecture and therefore to Connes' original problem. We refer the interested reader to [3, 8, 16] for excellent treatments of this equivalence.

Does local lifting imply Lance's WEP? We wish to finish our discussion on Kirchberg's conjecture with one final equivalence that merits further analysis. Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. A self-adjoint (closed under involution) unital subspace of $\mathcal{A}$ will be called an *operator subsystem* of $\mathcal{A}$. Given an ideal (closed, two-sided) $\mathcal{J} \subset \mathcal{B}$, consider a c.c.p. map $\varphi : \mathcal{A} \to \mathcal{B}/\mathcal{J}$. If $q : \mathcal{B} \to \mathcal{B}/\mathcal{J}$ denotes the canonical quotient map of $\mathcal{B}$ onto the quotient C^*-algebra $\mathcal{B}/\mathcal{J}$, then φ *lifts locally* if given any finite-dimensional operator subsystem $\mathcal{E} \subset \mathcal{A}$, there exists a c.c.p. map $\psi : \mathcal{E} \to \mathcal{B}$ such that $q \circ \psi = \varphi|_{\mathcal{E}}$.

Definition 16. A C^*-algebra $\mathcal{A}$ has the *local lifting property* (*LLP*) if for all C^*-algebras $\mathcal{B}$ and ideals $\mathcal{J} \subset \mathcal{B}$, every c.c.p. map $\varphi : \mathcal{A} \to \mathcal{B}/\mathcal{J}$ lifts locally.

Kirchberg showed that $C^*(\mathbb{F})$ has the LLP for any free group $\mathbb{F}$. Thus, we are led into our next and final equivalence of Kirchberg's conjecture.

Conjecture 17. *LLP implies Lance's WEP.*

It is worth pointing out that in [11] Kirchberg asked if Lance's WEP implied LLP. This was proven to be false by Junge and Pisier in [9], where they showed that

$$B(\mathcal{H}) \otimes_{C^*\text{-min}} B(\mathcal{H}) \neq B(\mathcal{H}) \otimes_{C^*\text{-max}} B(\mathcal{H}),$$

where $\mathcal{H}$ is an infinite-dimensional Hilbert space. This proves that Lance's WEP does not imply the LLP, since $B(\mathcal{H})$ can be seen to have Lance's WEP, recalling the equivalent formulation of Definition 11, and it can be shown that a unital C^*-algebra $\mathcal{A}$ has the LLP if and only if

$$\mathcal{A} \otimes_{C^*\text{-min}} B(\mathcal{H}) = \mathcal{A} \otimes_{C^*\text{-max}} B(\mathcal{H})$$

for all Hilbert spaces $\mathcal{H}$.

Closing

Our brief treatment of Connes' embedding problem and Kirchberg's conjecture is only part of the story that makes up this exciting area of operator algebras. Other equivalences to Connes' embedding problem with connections to quantum information theory and operator system theory can be found in [6] and [10], respectively. For more detailed surveys of Connes' embedding problem, we refer the interested reader to [1, 15].

As was already mentioned in the beginning of the article, Murray and von Neumann developed the foundations of operator algebras after von Neumann had begun to formalize the mathematics of quantum mechanics. It is known that during the famous debates between Niels Bohr and Albert Einstein with regard to the probabilistic tendencies of the universe, Einstein said, "God does not play dice." Thus, in this same spirit we leave you with a similar farewell. In the words of Vaughan Jones, "God may or may not play dice, but he sure does love a von Neumann algebra."

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