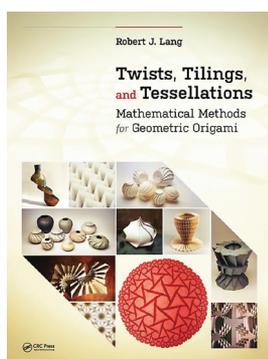




Twists, Tilings, and Tessellations

A Review by Yossi Elran



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Twists, Tilings, and Tessellations
Mathematical Methods
for Geometric Origami
by Robert J. Lang

Introduction

“Everyone and no-one knows what origami, the Japanese art of folding, is” is a fine and fitting first sentence for Robert Lang’s excellent new book *Twists, Tilings, and Tessellations: Mathematical*

Methods for Geometric Origami. Lang’s motivation for writing the book is given a few sentences later: “...in the latter part of the 20th century, it [origami] exploded in variety, complexity, and artistry, with numerous genres and specializations.” He could not have spelled out the contemporary origamist’s quandary better and could not have provided us with a better solution: a well-written, exceptionally structured book that makes a perfect compendium of geometric origami.

The Book

Twists, Tilings, and Tessellations has 736 pages, so it is a very long read from cover to cover. However, as Lang writes in the introduction, the reader can “jump among the various chapters, trying out things that look interesting and skipping what makes your eyes glaze over.” Indeed, the book is perfectly structured to allow the reader to do just that. Every chapter is self-contained, meticulously divided

into subsections, with many examples, illustrations, and beautiful photos.

The book focuses on geometric and mathematical origami as opposed to representational or figurate origami, folding a subject that looks like something—a crane, or a hat, or whatever. Geometric origami is the paper folding of geometric structures, using math both for the design and for the description of the beautiful outcomes. One of its main topics is origami tessellations, the division of the plane into decorative patterns (usually) using just a single sheet of paper. The ten chapters proceed in increasing order of structural complexity, from the very basics of modeling origami with crease patterns, through twists, tilings, and tessellations to 3D analysis and rotational solids.

The book is truly as exhaustive as possible within reasonable limits. There are some origami structures that are not included in the book, notably, elementary shapes such as polygons and polyhedra and, on the other hand, some of the more modern structures, like Ron Resch tessellations and curved folds. This is already noted by Lang, and, I would say, quite acceptable, given that the book is meant to be a comprehensive work on contemporary geometric origami, itself very much focused on tilings and tessellations. The exceptionally detailed instructions and analyses of those structures that do appear give the reader all the knowledge needed to explore other structures that are not in the book.

From a didactic point of view, *Twists, Tilings, and Tessellations* is a masterpiece. Lang really bends over backwards (so to speak) to make sure that his readers wholly understand the text, and he does a great job of it. First, he uses the introduction to explain the purpose of the book, the motivation for writing it, and what it does and does not include. He then explains what to expect and what you need, going on to define the book’s intended audience: anyone who wants to learn about geometric origami and the math behind it, experts and novices alike, whose mathematical skills are anywhere from high school level upwards. Throughout the book, a system of asterisk symbols accurately indicates the

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level of math needed for each section. Having said that, Lang makes sure with his eloquent writing skills that even those with elementary math skills enjoy and understand the book. Terminology is very important for origami folding, and Lang makes sure that the definitions used throughout the book are understood, using illustrations and photographs that give the reader multiple viewpoints of what is being defined.

Each chapter is structured from the basics to the more complicated material, with elaborate explanations, illustrations, and folding instructions. The math regarding the designs is also presented using the same meticulous method and thorough explanations, going from elementary definitions all the way through to theorems. Every chapter includes bibliographic references and ends with a list of all the terms that are used within. Lang does justice to many fine origami artists and mathematicians throughout the book, generously and liberally referencing their work and presenting much of their work in beautiful color photos. One might expect the artistic value of geometric origami to go unnoted given that the book's main theme is mathematical methods. However Lang refers to the artistic side of geometric origami throughout the book many times. Here is one such (beautiful) excerpt from a discussion on Archimedean tiling:

Mathematics provides a description of pattern and regularity and it can sometimes precisely enumerate the small numbers of elegant regularities within a field. One of those elegant regularities is that there are precisely 12 Archimedean tilings. This is a universal truth: anyone who strives to use regular polygons and have every vertex the same is bound to arrive at one of those 12 tilings, whether designing now, in the past, or in the future. But we still have aesthetic choices that let us express individuality: in choices of pleat width, tile size, tile type, crease assignment, materials, overall artwork size, and even context and execution. We can create work that expresses our own individual vision, but if we choose to work within a strongly mathematical genre like tessellations, we should not be put off to find that others have (or will) find the same resonances within the artistic universe. [p. 381]

The Chapters

The first chapter of the book is, in my mind, the most important. It begins with the very basics of the mathematical description of origami. In this chapter, and indeed, throughout the book, the author emphasizes the difference between mathematical origami and the actual physical process of building an origami structure. There are many reasons for this, the most basic being the fact that a math-

ematical model of origami usually assumes zero-thickness of the paper. The chapter introduces crease patterns, which is the map or guide to folding a shape, the different kinds of folds (valley and mountain folds), crease angles, and vertices, the points where the crease lines come together. Even the most basic of concepts are explained using a variety of didactic tools such as detailed explanations and illustrations. Once the initial definitions have been set, the mathematical description of these concepts (folds, crease maps, and vertices) is given, along with its implications for the actual folding process (for example, which folds are mathematically possible). It is here where you will find thorough explanations of some of the most important math needed to really enjoy the book, most of which are at high school math level, such as the Kawasaki-Justin theorem [1, 2]. For the advanced reader, a full linear algebraic description of origami is presented. This is great for those who want to delve deeper or those interested in developing or applying computer algorithms in origami design.

Most of *Twists, Tilings, and Tessellations* is devoted to origami design of tessellations. It is therefore a wise choice to devote the second chapter to periodicity. Once again, the basics are laid out at the beginning of the chapter, and the concept of tiles is introduced. Contrary to common presumption, origami tiles, as defined by the author, are not individual paper structures that are somehow joined together. They are a repeating folding pattern, made within a single sheet of paper. You can see this, for example, in Figure 1 (Figure 2.41 on page 116 in the book). On the left, you can see the crease pattern for a 16-tile Miura-ori pattern, where the red dashed lines are valley creases and the solid black lines mountain creases. The partially folded form is shown in the middle (it is turned over, so it looks like the inverse of the crease pattern), and on the right is the flat-folded form. The different lengths of the four folds that meet at each vertex in the figure and the parallelogram angles (60° and 120°) are responsible for the beautiful outcome. Throughout the chapter, periodic and near-periodic tessellations of many prominent origami artists are presented and analyzed. These include works by Ilan Garibi, Paul Jackson, Shuzo Fujimoto, Yuri and Katrin Shumakov, and Yuko Nishimura, to name but a few.

If Chapters 1 and 2 lay down the foundations for the rest of the book, Chapters 3, 4, and 5 really give readers almost

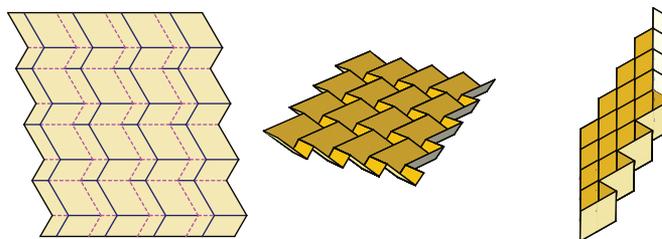


Figure 1. (This is Figure 2.41 from page 116 in the book.) A Miura-ori pattern with four fold lengths around each vertex. Left: crease pattern. Middle: partially folded form (turned over). Right: flat-folded form.

all they need to know about (arguably) the core business of modern geometric origami: twists and twist tilings. A twist is an origami pattern where a localized region of the paper is rotated with regard to the other parts of the paper. A *regular* polygonal twist is one where a regular polygon is rotated with respect to the rest of the paper. It turns out that for each n -gon there is a certain number of crease assignments, which grows exponentially as n increases. The author presents the exact formula for the (theoretical) number of crease assignments for each n , but then goes on to explain why some of these cannot lead to a flat-folded origami state [p. 222], based on the math principles from the previous chapters. This is an excellent example of how math comes into play in origami design and how the author walks the reader through the mathematical thought process to reach the correct conclusion.

Many designs are used to illustrate the different types of twist tilings and tessellations, one of which can be seen in Figure 2. Momotani's wall by Yoshihide Momotani, explained in depth and complete with folding instructions and illustrations, is another good example of a tessellation [p. 203]. It is made from an array of twists and is quite easy to fold.

I really liked the author's comprehensive explanation of vertex figures [p. 291], an elegant notation sequentially listing regular polygons that fit around a single vertex without leaving any gaps. Again, this is not just thrust upon the reader. It arises naturally from the solution of two simple Diophantine equations, and then the mathematical consequences are discussed at length. One such consequence is that there are only 21 possible vertex figures, composed of regular polygons, that tile the plane. The twist tiling in Figure 2 is made out of two regular pentagons that meet up with a dodecahedron at a single vertex. Sharp readers might be able to see why this tiling cannot be continued further in any direction.

From Chapter 6 until the end of the book, more advanced structures are discussed, such as primal-dual tessellations (Chapter 6), issues of foldability (Chapter 7), spherical vertices (Chapter 8), 3D origami (Chapter 9), and rotational solids (Chapter 10). Each and every one of these chapters is a feast for origami artists and mathematicians alike. For example, origami tilings in the hyperbolic plane that resemble the artistic work of Dutch artist M. C. Escher and Canadian geometer H. S. M. Coxeter [p. 437] are discussed in Chapter 6. These are fine examples of shrink-rotate tessellations and lead to a very enlightening explanation of Voronoi diagrams and Delaunay triangulations [pp. 439–442]. Many more gems can be found on almost every page. By the time readers get to the last page of the book, they will know almost everything about origami tilings and tessellations in the plane and in 3D. They will be able to understand and even reconstruct some of the most beautiful origami work, including the work of the author,

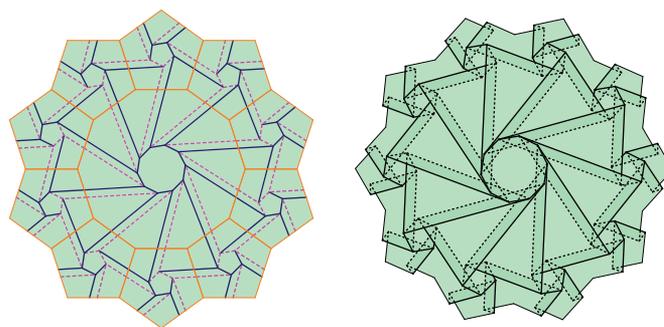


Figure 2. (This is Figure 4.30 from page 296 in the book.) A tiling of centered twist tiles made up of two pentagons that meet a dodecahedron at each vertex. Left: crease pattern tiling. Right: folded form.

who so generously makes more room in his book for other artists than for his own magnificent designs (Figure 3).

Conclusion

Twists, Tilings, and Tessellations: Mathematical Methods for Geometric Origami is a real gem. Design algorithms, folding instructions, mathematical principles, photographs—this book has it all. It is a “must have” book for anyone who wants to learn about geometric origami and, in particular, those interested in origami tilings and tessellations. There are really no special prerequisites to enjoy the book, although you really do need to have at least a high school math level to go beyond the first chapters. The painstaking effort Robert Lang takes to make sure that the reader gains a holistic yet detailed view of each and every origami structure in the book is truly admirable, especially given the difficulty many origami artists have conveying paper folding instructions in written word. Lang's efforts pay off, as *Twists, Tilings, and Tessellations: Mathematical Methods for Geometric Origami* is a remarkably well-written, well-structured, comprehensive, and absorbing read. Robert Lang is not only one of the world's greatest origami artists; this book establishes him as a wonderful educator, origami communicator, and author.



Figure 3. (This is Figure 10.28 from page 671 in the book.) An assortment of pots using the thick-flange algorithm with various cross sections, folded from Canson and Elephant Hide paper (2004–2013), by Robert J. Lang.

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Yossi Elran

Credits

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