A Primer on Generated Jacobian Equations: Geometry, Optics, Economics

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Jacobians and Dualities

A problem that appears in different guises all across mathematics is that of finding, given two spaces X and Y, a **measure-preserving map** from X to Y within a given family of admissible mappings (defined via some **monotonicity condition**). Here are some of the situations leading to this problem: rearranging one mass distribution onto another in an optimal way, analysis of semigeostrophic flows, coupling two probability measures to maximize their covariance, and designing the shape of a lens so that it refracts a given light source onto a desired pattern.

More concretely, we are given two open domains Ω, $\overline{\Omega} \subset \mathbb{R}^d$ and respective mass densities $f : \Omega \to \mathbb{R}$, $g : \overline{\Omega} \to \mathbb{R}$. Then, we are first of all looking for a mapping $\tau : \Omega \to \overline{\Omega}$ that is smooth and has a smooth inverse that *maps f into g* (see Figure 1).

That is, for every Borel set $E \subset \Omega$ we have (Figure 1)

$$\int_{\tau(E)} g(y)\, dy = \int_E f(x)\, dx.$$

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Using the change of variables formula we obtain the following equation for the Jacobian of any such τ:

$$\det(D\tau(x)) = \frac{f(x)}{g(\tau(x))} \text{ in } \Omega. \tag{1}$$

Conversely, if (1) holds for an invertible smooth map τ, then τ will map f into g. The Jacobian of the map τ is prescribed in terms of x and its image under τ. If τ is treated as an unknown, then (4) is a highly nonlinear first-order partial differential equation.

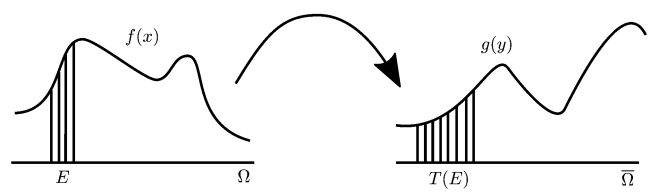


Figure 1. Mapping one mass density onto another.

Furthermore, we are not looking for any such τ, but one that satisfies a certain monotonicity property. For example, we could ask that τ be cyclically monotone, which requires that for any finite sequence $x_0, x_1, \dots, x_N$,

$$(\tau(x_1), x_2 - x_1) + \cdots + (\tau(x_N), x_1 - x_N) \le 0. \tag{2}$$

This condition is much stronger than mere monotonicity, which consists of requiring the above just for $N = 2$:

$$(\tau(x_2) - \tau(x_1), x_2 - x_1) \ge 0.$$

Also commonplace in mathematics is the appearance of a "duality," which in vague terms amounts to a correspondence between sets of real-valued functions over two spaces. One of the most important examples is the Legendre transform. This transform assigns to a function v another function u via

$$u(x) = \sup_x \{\langle x, y\rangle - v(y)\}.$$

Such a function u will be, of course, convex. A map is said to admit a *convex potential* if

$$\tau(x) = \nabla u(x),$$

where u is as above. A famous theorem of Rockafellar says (essentially) that a map $\tau : \mathbb{R}^d \to \mathbb{R}^d$ satisfies (2) if and only if it admits a convex potential.

If τ admits a convex potential, then it maps f into g if u solves the Monge-Ampère equation in Ω:

$$\det(D^2 u(x)) = \frac{f(x)}{g(\nabla u(x))}. \tag{3}$$

Therefore, finding solutions to (1) reduces to finding solutions of (3) (a scalar PDE) if we are interested only in cyclically monotone maps.

If we replace the expression $\langle x, y\rangle - v$ with an arbitrary function $G(x, y, v)$ we arrive at the realm of **generated Jacobian equations (GJE)**, which has its own notion of transform, a notion of convexity, and a respective Monge-Ampère-type equation. Although such equations have appeared in various forms in the past, the framework of generated Jacobian equations (and the term itself) was introduced by Trudinger in [18]. Trudinger was motivated by problems in geometric optics that led to Monge-Ampère-type equations that were similar to but not among those covered by optimal transport methods. Generating functions also appear naturally in economics, where they have been independently studied for some time and in particular by Nöldeke and Samuelson [15].

This is an introduction to generated Jacobian equations and generating functions and in particular to what is known as the second boundary value problem for GJE (see Problem 2). We will discuss the basic definitions and some of the basic questions, along with specific examples from optimal transportation, geometry, optics, and economics. We will end with a survey of recent results as well as a discussion of open problems.

The basic problem, roughly speaking. The basic problem we are concerned with requires a number of preliminary definitions. Therefore let us first give an incomplete description of this problem, leaving the exact formulation for after we have introduced all the necessary concepts (see the section "The Basic Problem, Recast"). The missing ingredients below will be introduced and described in the section "Generating Functions: The Basics."

The Data

We are given the following:

(1) domains $\Omega, \overline{\Omega}$ of dimension d,
(2) a *generating function* $G : \mathfrak{g} \subset \Omega \times \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$,
(3) probability densities f and g in Ω and $\overline{\Omega}$.

For such data, we pose the following problem.

Problem 1. Find a map $\tau : \Omega \to \overline{\Omega}$ such that:

(a) The Jacobian of τ solves

$$\det(D\tau(x)) = \frac{f(x)}{g(\tau(x))} \quad \text{in } \Omega. \tag{4}$$

(b) There is a scalar function $u : \Omega \to \mathbb{R}$ such that

$$\tau(x) = E(x, u(x), Du(x))$$

for some $E : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \overline{\Omega}$.

(c) The map τ satisfies a "monotonicity" condition.

The exact relation between G and E, and G and the "monotonicity" condition will be explained in the following section. Observe that the mapping τ involves first derivatives of u, so the Jacobian should be a second-order expression in u. Assuming u and E are sufficiently differentiable, the chain rule says that (we are denoting the variables of E as $E(x, z, p)$)

$$D\tau(x) = D_x E + D_z E \otimes Du + D_p E\, D^2 u.$$

If $D_p E$ is invertible, then this can be rewritten as

$$D\tau(x) = D_p E(D^2 u(x) + A(x, u, Du)),$$
$$A(x, u, Du) := (D_p E)^{-1}(D_x E + D_z E \otimes Du).$$

Lastly, we write

$$\psi(x, u, p) := \det(D_p E(x, u, p)^{-1}) \frac{f(x)}{g(E(x, u, p))}.$$

The equation (4) seen in terms of u takes the form

$$\det(D^2 u + A(x, u, Du)) = \psi(x, u, Du), \tag{5}$$

and this is what is known as a generated Jacobian equation. By far the most studied example is the real Monge-Ampère equation (3). As is the case with (3), GJEs in (4) are degenerate elliptic equations at least when considering the right "convexity" class.

Problem 1 consists of solving the PDE (5), with the condition $\tau(\Omega) = \overline{\Omega}$ playing the role of a boundary condition. In the PDE literature a problem with such a boundary condition is known as the *second boundary value problem*. One may approach Problem 1 from the perspective of classical solutions (a priori estimates, continuity method) or weak solutions (construction of weak solutions and a respective regularity theory). The existence of classical solutions requires certain structural hypotheses on the data, and in general weak solutions may have singularities. Both perspectives have their strengths and weaknesses and are

more convenient in different situations (see the section "An Overview of the Literature" for a review of the relevant literature).

Generating functions and the rich world behind them. The three ingredients in Problem 1 are all aspects of the same structure, and this structure is determined by a generating function. Associated to every generating function are a notion of gradient/subdifferential, a transform between functions, a notion of convex functions, and even a notion of segments and convex sets (this latter one will not be discussed here; see [8, section 4]). In the case of the linear generating function these notions reduce to the standard ones in convex analysis.

These notions have a different meaning in each example. In geometric optics, the graph of the function u is the reflective/refractive surface, and the respective map τ (which plays the role of the gradient map) is the "ray-tracing map" (see the section "Generating Functions in Geometric Optics"). Meanwhile, in the matching problem in economics $u(x)$ represents the maximal utility for a buyer x, and the map $\tau(x)$ assigns to every buyer x the commodity $\tau(x)$ that they should buy to maximize utility (see the section "Stable Matchings: Generating Functions in Economics").

One may think of the generating function G as determining the type of GJE we are considering in the same way a Riemannian metric determines a linear elliptic equation; that is, $\Delta_g u = 0$ where Δ_g is the Laplace-Beltrami operator for the metric.

Outline. The next two sections introduce the basic notions associated to a given generating function and describe the basic problem in more careful terms. All of this will be illustrated with concrete (or not so concrete) examples from standard convex analysis/geometry and optimal transport, followed in the fourth and fifth sections by a more extended discussion of generating functions in geometric optics and economics.

Generating Functions: The Basics

Consider two domains $\Omega, \overline{\Omega}$ (open, bounded) of Riemannian manifolds[1] M and $\overline{M}$ (both of the same dimension d). A *generating function* G is a function

$$G : \mathfrak{g} \subset \Omega \times \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$$

that has the following properties:

(1) G is C^1 in $\mathfrak{g}$, which is an open set.
(2) For all $(x, y, z) \in \mathfrak{g}$, we have $\frac{\partial}{\partial z} G(x, y, z) < 0$.
(3) For fixed $x \in \Omega$, the map

$$(y, z) \mapsto (D_x G(x, y, z), G(x, y, z))$$

is differentiable and invertible on its image, and this inverse is also differentiable.

The *dual generating function* H is defined by

$$G(x, y, H(x, y, u)) = u,$$

for all (x, y, u), where the above expression is well defined (the domain of H is usually denoted by $\mathfrak{h}$).

Remark 1. The simplest example of a generating function is the linear generating function

$$G(x, y, z) = x \cdot y - z,$$

where $\Omega = \overline{\Omega} = \mathbb{R}^d$ and $\mathfrak{g} = \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$. In this case the dual function is G itself, so $H(x, y, u) = x \cdot y - u$.

Duality and convexity. The right way to think about G is to see it as a "duality structure" between functions in Ω and functions in $\overline{\Omega}$, an operation that transforms a function in Ω into a function in $\overline{\Omega}$ and vice versa. In the case where $G(x, y, z) = x \cdot y - z$ this duality amounts to the Legendre transform (see Example 1).

Definition 1. The *transform* of a function $v : \overline{\Omega} \to \mathbb{R}$ is a new function, $v^* : \Omega \to \mathbb{R}$, defined by

$$v^*(x) = \sup_{\bar{x} \in \overline{\Omega}} G(x, \bar{x}, v(\bar{x})).$$

Likewise, the transform of a function $u : \Omega \to \mathbb{R}$ is a function $u^* : \overline{\Omega} \to \mathbb{R}$ given by the formula

$$u^*(\bar{x}) = \sup_{x \in \Omega} H(x, \bar{x}, u(x)).$$

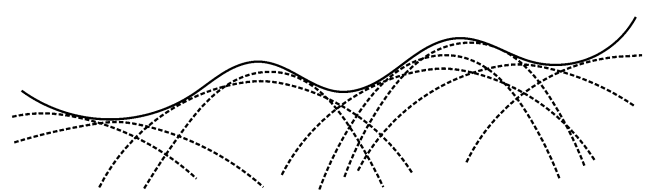


Figure 2. The solid line represents the function u; the dotted lines are the graphs of the functions $G(\cdot, y, v(y))$ for various values of y.

Remark 2. A pair of functions $u, v : \Omega, \overline{\Omega} \to \mathbb{R}$ will be called a *conjugate pair* if $u = v^*$ and $v = u^*$. Note that if (u, v) are conjugate, then

$$G(x, y, v(y)) \le u(x) \quad \text{and}$$
$$H(x, y, u(x)) \le v(y) \quad \text{for all } x \in \Omega,\ y \in \overline{\Omega}.$$

Moreover, for each x there is at least one y for which the first inequality becomes an equality, and for each y there is at least one x for which the second inequality becomes an equality.

That $u = v^*$ for some $v : \overline{\Omega} \to \mathbb{R}$ means the graph of u is "enveloped" by graphs of functions $G(\cdot, y, v(y))$; see

[1] *For most of our examples the manifolds are $\mathbb{R}^d$ and $\mathbb{S}^d$.*

Figure 2. In particular, for any $x_0 \in \Omega$ we expect[2] there to be $y_0 \in \overline{\Omega}$ such that

$$u(x_0) = G(x_0, y_0, v(y_0)),$$
$$u(x) \geq G(x, y_0, v(y_0)) \quad \text{for any } x \in \Omega.$$

Definition 2. A function $u : \Omega \to \mathbb{R}$ will be called *G-convex* or simply *convex* if for every $x_0 \in \Omega$ there is a $y_0 \in \overline{\Omega}$ and $z_0 \in \mathbb{R}$ such that

$$u(x) \geq G(x, y_0, z_0) \quad \forall\, x \in \Omega,$$
$$u(x_0) = G(x_0, y_0, z_0).$$

In a similar manner we shall talk of a function $v : \overline{\Omega} \to \mathbb{R}$ being *H-convex* or simply *convex*. Observe that if u and v form a conjugate pair, then u is *G-convex* and v is *H-convex*.

Example 1 (Legendre transform). The most popular instance of this is the Legendre transform between functions in Euclidean space,

$$v^*(x) := \sup_{y \in \mathbb{R}^d} \{x \cdot y - v(y)\},$$

which corresponds in the definition above to the special case where $G(x, y, z) = x \cdot y - z$ (note that in this case $H(x, y, z) = G(x, y, z)$). Furthermore, G-convex functions here are just the standard convex functions.

Example 2 (The shipper's dilemma). This problem is perhaps better known as the *dual problem* in optimal transportation; however, it is of interest in its own right [2]. In this problem we are given:

(1) two domains $\Omega, \overline{\Omega}$ (assume bounded, for simplicity),
(2) a probability measure μ in Ω and a probability measure ν in $\overline{\Omega}$,
(3) a function $c : \Omega \times \overline{\Omega} \to \mathbb{R}$ known as the *cost function* (assumed uniformly continuous, for simplicity).

Then the problem asks us to find a pair of functions $u : \Omega \to \mathbb{R}$, $v : \overline{\Omega} \to \mathbb{R}$ maximizing

$$\int_\Omega u(x)\, d\mu(x) + \int_{\overline{\Omega}} v(y)\, d\nu(y)$$

among all pairs of functions (u, v) satisfying the pointwise constraint

$$u(x) + v(y) \leq c(x, y).$$

A basic fact in optimal transportation is that this problem admits at least one solution, and the solution is given by a pair of functions u, v such that

$$u(x) = \inf_{y \in \overline{\Omega}} \{c(x, y) - v(x)\},$$
$$v(y) = \inf_{x \in \Omega} \{c(x, y) - u(x)\}.$$

In other words, the functions $-u$ and $-v$ form a conjugate pair with respect to $G(x, y, z) = -c(x, y) - z$.

[2] *This holds under mild assumptions on ν and $\overline{\Omega}$.*

The subdifferential and the exponential.

Definition 3. Consider a G-convex function u and $x_0 \in \Omega$. The subdifferential of u at x_0, denoted $\partial u(x_0)$, is defined as the set

$$\{y \in \overline{\Omega} \mid u(x) \geq G(x, y, H(x_0, y, u(x_0))) \quad \forall\, x \in \Omega\}.$$

Likewise we define the subdifferential $\partial v(y_0)$ of an H-convex function v at a point $y_0 \in \overline{\Omega}$. Often for emphasis we write ∂_G or ∂_H to clarify which type of subdifferential is being talked about, but we will not do so here.

The subdifferential of a G-convex function is always nonempty, and in general it may contain more than one point. However, at points where u is smooth the subdifferential contains exactly one element.

Proposition 1. *If u is differentiable at x_0, then $\partial u(x_0)$ has exactly one element y_0, determined by*

$$D_x G(x_0, y_0, z_0) = Du(x_0),$$
$$\text{where} \quad z_0 = H(x_0, y_0, u(x_0)).$$

Proof. We already know $\partial u(x_0)$ has at least one element. Let's show there can only be one. Take any $y_0 \in \partial u(x_0)$. Then with $z_0 = H(x_0, y_0, u(x_0))$ we have

$$u(x) \geq G(x, y_0, z_0) \quad \forall\, x \in \Omega.$$

Since u and $G(\cdot, y_0, z_0)$ are both differentiable at x_0, it follows that

$$Du(x_0) = D_x G(x_0, y_0, z_0).$$

However, as assumed in the definition, this equation can have at most one solution, so y_0 is uniquely determined. □

Of course the best known and most used instance of the G-subdifferential is the standard one in convex analysis: for convex function u and point x_0 the subdifferential $\partial u(x_0)$ gives the set of all vectors p representing the slopes of hyperplanes that support the graph of u at the point $(x_0, u(x_0))$. Another example arising in convex analysis is related to the study of convex bodies and their duals.

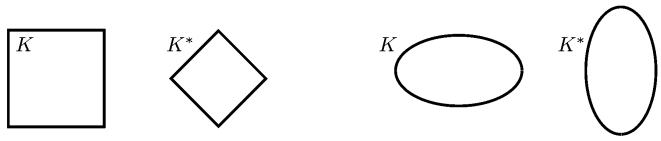
Example 3 (Polar dual). A convex body is a bounded convex set $K \subset \mathbb{R}^d$ with nonempty interior. The *polar dual* of a convex body is the convex set (see Figure 3)

$$K^* := \{\xi \in \mathbb{R}^d \mid \eta \cdot \xi \leq 1 \quad \text{for all } \eta \in K\}.$$

It is well known that if K is convex and $0 \in K$, then $(K^*)^* = K$, so this defines a duality between convex bodies. This duality can be captured in terms of a generating function when expressed in the right variables.

First, we must describe convex bodies in terms of a function. If 0 lies in the interior of K, then K can be described by the radial function

$$\rho_K : \mathbb{S}^{d-1} \to \mathbb{R}, \qquad \rho_K(x) := \sup\{t \mid tx \in K\}.$$



$K = \{\max\{|x|, |y|\} \leq 1\}$ $\quad$ $K = \{(x/a)^2 + (y/b)^2 \leq 1\}$

$K^* = \{|x| + |y| \leq 1\}$ $\quad$ $K^* = \{(ax)^2 + (by)^2 \leq 1\}$

Figure 3. Two examples of a convex body together with its respective polar dual.

The function ρ_K is such that its radial subgraph is equal to K. A straightforward calculation leads to the following formula expressing ρ_{K^*} in terms of ρ_K, namely,

$$\rho_{K^*}(y) = \frac{1}{\sup\limits_{x \in \mathbb{S}^{d-1}} (x \cdot y)\rho_K(x)}.$$

Let $u(x) = \frac{1}{\rho_K(x)}$ and $v(y) = \frac{1}{\rho_{K^*}(y)}$. The above formula implies that u and v form a conjugate pair with respect to the generating function

$$G(x, y, z) := \frac{x \cdot y}{z},$$

defined for all $z > 0$ and for $x, y \in \mathbb{S}^d$ such that $x \cdot y > 0$. In this case the subdifferential $\partial u(x_0)$ represents the set of (outward pointing) normals to hyperplanes that support K at the point $\rho_K(x_0)x_0 \in \partial K$.

Up to this point we have not used condition (3) in the definition of a generating function. This condition is needed for the G-exponential map.

Definition 4. Let G be a generating function. Recall that for every fixed x_0 the map

$$(\bar{x}, z) \to (DG(x_0, \bar{x}, z), G(x_0, \bar{x}, z)) \tag{6}$$

is invertible with a (differentiable) inverse. In particular this means that for each pair x, u, there are an open set $D_{x,u}$ inside the tangent space to Ω at x and a differentiable map (Figure 4)

$$\exp_{x,u} : (\mathrm{T}\Omega)_x \to \overline{\Omega},$$

such that for $Z_{x,u}(p) := H(x, \exp_{x,u}(p), u)$ we have

$$(DG)(x, \exp_{x,u}(p), Z_{x,u}(p)) = p.$$

The map $\exp_{x,u}$ is called the G-exponential map, or simply exponential map, at (x, u).

In other words, the point $y = \exp_{x,u}(p)$ is the one point in $\overline{\Omega}$ with the property that the graph of $x \mapsto G(x, y, z)$ has a slope given by p at x. Since $\exp_{x,u}$ maps a set of a vector space into $\overline{\Omega}$, it gives a chart for this domain. One can likewise talk of an H-exponential map $\exp_{(y,z)}$ (see [8, section 4] for a more complete discussion).

Definition 5. Given a map $\tau : \Omega \to \overline{\Omega}$ and a scalar u we will say that u is a G-potential (or simply a potential) of τ if $\tau(x) = \exp_{x,u(x)}(Du(x))$.

The condition $\tau(x) = E(x, u(x), Du(x))$ hinted at in Problem 1 is nothing but the condition that τ admits a G-potential u and the map E is simply the G-exponential map.

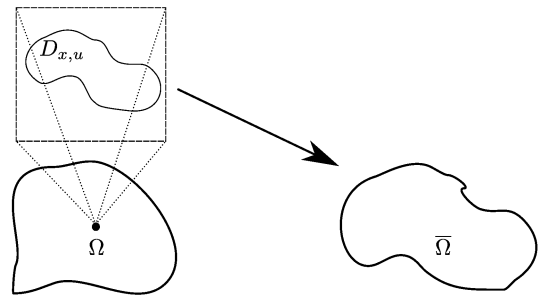


Figure 4. The G-exponential.

The G-exponential is a generalization of the exponential map in Riemannian geometry, from which it takes the name. Let $\Omega = \overline{\Omega}$, and let g be a Riemannian metric on Ω. Then consider the generating function given by

$$G(x, y, z) = \frac{1}{2}d(x, y)^2 - z,$$

where $d(x, y)$ denotes the geodesic distance from x to y with respect to the metric g. Using exponential coordinates at a point x_0, it is not difficult to show, provided y is sufficiently close to x_0, that

$$D_x G(x_0, y, z) = d(x_0, y) D_x d(x_0, y) = p,$$

where p is such that $\exp_{x_0}(p) = y$ (where $\exp_{x_0}$ is the exponential map of the metric g at x_0). Then $\exp_{x,u}$, which is independent of u, is nothing but the exponential map of the metric based at the point x.

The Basic Problem, Recast

We are now ready to describe Problem 1 in full detail.

The Data

(1) Domains $\Omega \subset M$, $\overline{\Omega} \subset \overline{M}$ for d-dimensional Riemannian manifolds M and $\overline{M}$.

(2) A generating function $G : \mathfrak{g} \subset \Omega \times \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, where $\mathfrak{g}$ is some open nonempty subset.

(3) Probability measures $\mu \in \mathcal{P}(\Omega)$ and $\nu \in \mathcal{P}(\overline{\Omega})$.

Problem 2. Find a map $\tau : \Omega \to \overline{\Omega}$, smooth and invertible, given by

$$\tau(x) = \exp_{x,u}(Du(x))$$

for a G-convex function u which sends μ into ν.

In general, Problem 2 does not have any smooth solutions, and even when it does, the best way to get a hold

on them is through weak solutions. The following two notions of weak solutions are the most important.

Definition 6. A (G-)convex function u is said to be a solution in the sense of Aleksandrov if for every Borel set $E \subset \Omega$ we have

$$\nu(\partial u(E)) = \mu(E).$$

To avoid some technical issues, we only consider the other notions of solutions for μ and ν that are absolutely continuous with respect to Lebesgue measure (denoted by $dVol(x)$ and $dVol(y)$).

Definition 7. Let $\mu = f dVol(x)$ and $\nu = g dVol(y)$. A (G-)convex function u is said to be a solution in the sense of Brenier if for every $\eta \in C(\Omega)$,

$$\int_{\Omega} \eta(\tau(x))\, f(x) dx = \int_{\overline{\Omega}} \eta(y) g(y)\, dVol(y),$$

where a.e. x we have $\tau(x) = \exp_{x,u(x)}(Du(x))$.

Remark 3. Each notion has its advantages and disadvantages. The existence of Brenier solutions comes naturally in optimal transport, whereas regularity results work directly with Aleksandrov solutions. It was Caffarelli who first determined when a Brenier solution is also an Aleksandrov solution and provided an example of a Brenier solution that is not an Aleksandrov solution.

Suppose that the function G is in fact twice differentiable, and suppose that $\mu = f(y)dx$ and $\nu = g(y)dy$. Then, if there is a solution u that is twice differentiable, the map

$$\tau_u(x) = \exp_{x,u}(Du(x))$$

is such that

$$\det(D\tau_u(x)) = \frac{f(x)}{g(\tau(x))}.$$

The following example deals with an important model in geometric optics that is a simpler (and limiting) version of those discussed in the next section. We present it first here as a natural instance of Problem 2.

Example 4 (The far field reflector problem). The (far field) reflector problem in geometric optics involves surfaces given in polar form

$$\Sigma_\rho = \{\rho(x)x \mid x \in \Omega \subset \mathbb{S}^d\}$$

for some positive function $\rho : \Omega \to \mathbb{R}$. In this problem we look for a ρ such that rays emanating from the origin in directions $x \in \Omega$ reflect off Σ_ρ, in accordance with Snell's law, into a new set of directions $\overline{\Omega}$, and ρ must be such that Σ_ρ reflects a source power density on directions Ω onto a target power density on directions $\overline{\Omega}$. Let us elaborate on this. Denote by $\tau : \Omega \to \overline{\Omega}$ the map that associates to a ray emanating in the direction x the new direction $y = \tau(x)$ it has after reflection. This is known as the ray-tracing map. Conservation of energy means that if f is the power density of the rays emanating at Ω and g is the power density of the rays in the direction $\overline{\Omega}$, then τ must map f into g (see the next section for a more complete discussion and references).

Lastly, the surface Σ_ρ is required to be enveloped by conics, specifically by paraboloids of revolution whose focus is placed at the origin. This means ρ must be such that

$$\rho(x) = \inf_{y \in \overline{\Omega}} \frac{1}{\rho^*(y)(1 - \langle x, y\rangle)}$$

for some function $\rho^* : \overline{\Omega} \to \mathbb{R}$. Thus $u = 1/\rho$ and $\nu = 1/\rho^*$ form a conjugate pair with respect to the generating function defined for $x, y \in \mathbb{S}^d$ and $z > 0$ by

$$G(x, y, z) = \frac{1 - \langle x, y\rangle}{z}.$$

Furthermore, if ρ is smooth (and thus u) the ray-tracing map is given by $\tau(x) = \exp_{x,u(x)}(Du(x))$, the exponential being with respect to G. This u solves Problem 2, where $\mu = f d\sigma(x)$ and $\nu = g d\sigma(y)$ (σ is the standard measure on the sphere).

Generating Functions in Geometric Optics

Now we shall describe a family of problems arising when designing the geometry of optical surfaces. The goal is to design an optical surface (i.e., reflector or refractor) so that a given power density emanating from a source surface is reflected/refracted into a given power density on a target surface. As explained in Westcott and Norris [16], the law of reflection/refraction together with conservation of energy determines the shape of the optical surface according to the source and target power densities.[3] The determination happens through a differential equation, and it was in [16] where the connection to the Monge-Ampère equation was made explicit in the setting of the far field problem (Example 4). For various near field problems the respective equations are GJE.

Throughout this section we will discuss several variations on this problem leading to different GJEs depending on the configuration of the source and the target and whether we are dealing with a reflection or refraction problem. We will use d to denote the unspecified dimension of the domains Ω and $\overline{\Omega}$ (so the light beams are traveling in $\mathbb{R}^{d+1}$), but obviously all of the examples discussed are of physical interest only in the case $d = 2$.

Reflection and refraction. Let us first review the laws of reflection and refraction. Take an interface Σ separating two homogeneous media. If light traveling in medium I hits Σ, then it will reflect back into medium I (if Σ is a

[3] *See also* [16] *for further discussion of the underlying physics.*

reflective surface) or else it will refract into medium II (say if Σ is a lens).

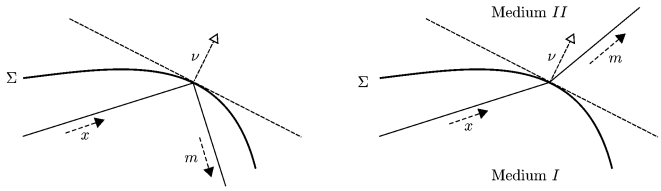


Figure 5. Law of Reflection (7) and Refraction (8).

Let x and m be the unit vectors denoting the direction of light before and after it hits Σ. If Σ is a reflective surface, then the Law of Reflection says that

$$m = x - 2\langle x, \nu\rangle \nu, \tag{7}$$

where ν denotes the normal to Σ at the reflection point (Figure 5). On the other hand, if Σ is a refractive lens, then the new direction m is related to x via Snell's law. The law takes into account the speed of light in each medium: let n_1 denote the refractive index of medium I and n_2 the index for medium II, and let $\kappa = n_1/n_2$. Snell's law says that

$$m = \frac{1}{\kappa}x - \lambda\nu, \quad \text{for some } \lambda \in \mathbb{R}, \tag{8}$$

and such m exists if and only if

$$1 - \langle x, \nu\rangle^2 \leq \kappa^2.$$

Here we will discuss only the case where $\kappa \geq 1$, so this condition will always be satisfied.

Design of optical surfaces. Let us describe the general configuration common to every problem before we go into particular examples. We are always considering an apparatus where, out of every point x on a domain Ω of a d-dimensional oriented surface $S \subset \mathbb{R}^{d+1}$ we shoot out a ray in the normal direction. Such rays eventually hit an optical surface Σ and reflect or refract until they hit a domain $\overline{\Omega}$ of another d-dimensional oriented surface $T \subset \mathbb{R}^{d+1}$. The surface S is referred to as the source, and the surface T is referred to as the target. The two most commonly studied sources are what are known as "point sources," corresponding to $S = \mathbb{S}^d$, and parallel sources, where S is planar $S = \mathbb{R}^d \times \{0\}$. The problem consists of designing Σ so that a given power density (radiation emanating) at Ω is reflected/refracted into a given power density in $\overline{\Omega}$.

The power density at $x \in \Omega$ is denoted by $f(x)$, while the intensity hitting at $y \in \overline{\Omega}$ is denoted by $g(y)$. These two functions are nonnegative, and by conservation of energy we must have

$$\int_\Omega f(x)\, d\sigma(x) = \int_{\overline{\Omega}} g(y)\, d\sigma(y),$$

so without loss of generality we assume this common value to be 1. Lastly, the optical surfaces we consider are given as the graph (normal or radial, depending on the nature of S) of a positive function $u : \Omega \to \mathbb{R}$, so that

$$\Sigma_u := \begin{cases} \{(x, u(x)) \mid x \in \Omega\}, & \text{when } S \subset \mathbb{R}^d, \\ \{u(x)x \mid x \in \Omega\}, & \text{when } S \subset \mathbb{S}^d. \end{cases}$$

For a given Σ_u we will denote by $\tau_u(x)$ the point on $\overline{\Omega}$ hit by the ray originally emanating from x. This map, which goes from Ω to $\overline{\Omega}$, is known as the *ray-tracing map* of the surface Σ_u. Assuming this map is smooth and with a smooth inverse conservation of mass ultimately requires that (see [16])

$$\det(D\tau_u(x)) = \frac{f(x)}{g(\tau_u(x))} \quad \text{for all } x \in \Omega.$$

Last but not least, there are surfaces (often portions of conic sections) with the property that all rays emanating from S reflect (or refract) so that they all end up passing through a given point $y \in \overline{\Omega}$. One may think of this family of surfaces as solving the above problem when ν is a Dirac delta placed at $y \in \overline{\Omega}$. Moreover, the ray-tracing map of an optical surface Σ_u at a point x is determined by which surface in this special family touches Σ_u at the point corresponding to x.

This means we have a function $u : \Omega \to \mathbb{R}$ whose graph or radial graph yields the reflecting/refracting surface and which is enveloped by a family of special surfaces. In particular,

$$u(x) = \inf_{y \in R} G(x, y, z(y)), \tag{9}$$

and the ray-tracing map is nothing but the G-gradient map of u

$$\tau_u(x) = \exp_{x,u(x)}(Du(x)).$$

Thus the study of these optical surfaces amounts to the study of a GJE for the right generating function.

Remark 4. Observe in (9) that there is an inf and not a sup, and thus it makes sense to talk about G-concave functions and not G-convex functions. This is largely a matter of convention: one can consider instead $-u$ and $-G$ and consider G-convex functions if one insists on doing so. The definitions and theory in the second and third sections extend, with trivial modifications, to consider G-concave functions, so we will refer to GJE with G-concave functions throughout this section without further ado.

The parallel near field reflector. Consider the case where light is shot vertically upward, with Ω denoting the horizontal cross section of the whole beam, and the target domain $\overline{\Omega}$ is also flat and lies on the same plane as Ω (Figure 6). While more complicated targets are possible, in

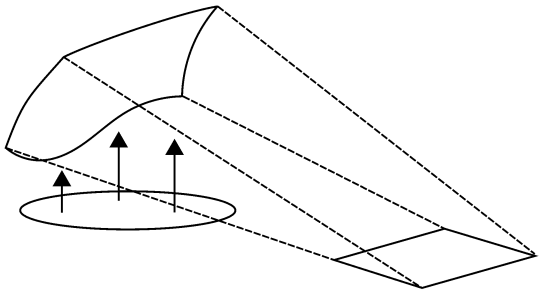


Figure 6. The parallel near field reflector.

this case the form of G and the ray-tracing map are particularly simple. Firstly, G takes the form

$$G(x,y,z) = \frac{1}{4z} - z|x-y|^2.$$

The ray-tracing map is given by

$$\tau_u(x) = x + \frac{2u(x)}{(1-|Du(x)|^2)}Du(x).$$

Observe that $x \mapsto G(x,y,z)$ yields a concave paraboloid (see Figure 7). This was to be expected, as these surfaces reflect vertical rays into the focus y.

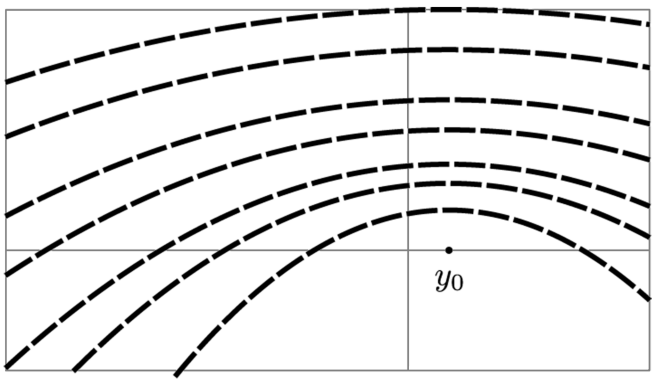


Figure 7. Cross sections of a family of paraboloids with a common focus y_0 for various values of z.

The parallel near field refractor. We will only describe the parallel near field refractor in the case $n_1 \le n_2$, so light beams are going into a denser medium. In this setting the natural optical surfaces are portions of ellipsoids of revolution.

We fix domains $\Omega, \overline{\Omega} \subset \mathbb{R}^d$. The horizontal cross section of the totality of the source beam is given by $\Omega \times \{0\}$. The target is a portion of a different horizontal plane that lies a distance L above the plane of Ω, so it is given by $\overline{\Omega} \times \{L\}$ (see Figure 8).

We will write $\kappa = n_1/n_2$, so $0 \le \kappa \le 1$. Let

$$\mathfrak{g} = \{(x,y,z) \mid z > 0 \quad \text{and} \quad \alpha|x-y|^2 \le z^2\}$$

where $\alpha := 1-\kappa^2$. With this setup, the corresponding generating function (which is defined only on $\mathfrak{g}$) is given by the formula

$$G(x,y,z) = L - \frac{1}{\alpha}\left(\kappa z - \sqrt{z^2 - \alpha|x-y|^2}\right).$$

The graph of this section parametrizes the lower half of an ellipsoid of revolution of eccentricity κ and semimajor axis z/α whose foci are (y,L) and $(y, L - 2\frac{\kappa}{\alpha}z)$. The ray-tracing map τ_u has a more complicated expression in comparison to the parallel reflector. The system of equations that determines τ_u takes the form

$$Du(x) = \frac{x-\tau(x)}{\sqrt{z^2 - \alpha|x-\tau(x)|^2}}, \; u(x) = G(x,\tau(x),z).$$

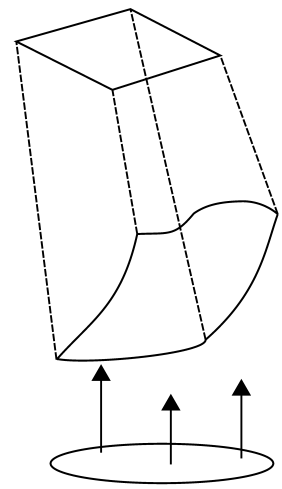


Figure 8. Near field parallel refractor.

The point source near field reflector. This is one of the models that has received the most attention in recent years. Since the rays are coming out in radial directions we have a function $u : \Omega \to \mathbb{R}$ (Ω is a set of unit directions so $\Omega \subset \mathbb{S}^d$), and the radial graph of u describes the reflecting surface.

Let $\overline{\Omega}$ denote an open domain of some d-dimensional surface $\overline{M}$ in $\mathbb{R}^{d+1}$. For any $y \in \overline{\Omega}$ consider an ellipsoid of revolution (of any eccentricity) with one focus placed at the origin and another placed at y. Then any ray emanating from the origin and reflecting off the ellipsoid will eventually pass through y. Therefore the optical surfaces in this case are ellipsoids of revolution where one focus lies at the origin and the other lies at y. The polar representation of an ellipsoid (where the origin is placed at one of its foci) yields the generating function for this problem,

$$G(x,y,z) = \frac{z^{-2} - \frac{1}{4}|y|^2}{z^{-1} - \frac{1}{2}\langle x,y\rangle}.$$

Karakhanyan and Wang [12] derived the Monge-Ampère equation for this problem. For fixed data, the corresponding weak solutions form a one-parameter family (which

foliates a portion of space). They showed that some elements in this family are $C^{2,\alpha}$ under natural conditions on the target $\overline{\Omega}$ and assuming the source and target densities are smooth. They also showed that if some of the natural conditions fail, then there are smooth densities for which some of the weak solutions in the family are smooth while some are not even differentiable. This phenomenon does not occur in optimal transport and illustrates how the issue for regularity is relatively more complicated for GJE.

Revisiting the far field problem. Consider the point source near the field reflector. If the target $\overline{\Omega}$ is very far (i.e., we push the target away and let the displacement tend to infinity), then the point where the reflected rays hit the target becomes irrelevant, and it suffices to care about in which direction they were reflected.

In the limit, the ellipsoids of revolution converge to paraboloids of revolution whose focus is located at the origin. The polar representation for these surfaces yields the generating function

$$G(x, y, z) = \frac{1}{z}\frac{1}{1 - \langle x, y\rangle}.$$

Unlike near field problems, the dependence on the variable z is multiplicative, so making the substitution $z' = -\log(z)$ and introducing the cost function $c(x, y) = -\log(1 - (x, y))$ we have

$$\log(G(x, y, z)) = c(x, y) - z.$$

This means we can bring this problem to the framework of optimal transport by working with $\log(u(x))$. This observation is behind existence and uniqueness results for weak solutions of the far field problem. Such weak solutions have been shown to be differentiable (and even classical) under appropriate conditions on the energy densities and the shape of the source Ω and set of target directions $\overline{\Omega}$.

Stable Matchings: Generating Functions in Economics

Consider a situation where we have a set of types of buyers (represented by Ω) and a set of types of sellers (represented by $\overline{\Omega}$). We are given a probability measure μ representing the distribution of buyers and a probability measure ν representing the distribution of sellers.

A type of buyer and a type of seller can decide to engage in a transaction resulting in a utility for each of them. The generating function $G(x, y, v)$ will represent this utility, interpreted as follows:

If x engages in a transaction with y yielding a utility of v for y, then x obtains a utility of $G(x, y, v)$.

The dual function $H(x, y, u)$ then has the obvious corresponding interpretation as the utility y obtains when it proceeds to a transaction with x resulting in a utility of u to x.

We are interested in studying *outcomes*. This refers to three things: a function $u(x)$, a function $v(y)$, and a measure λ over $\Omega \times \overline{\Omega}$ with marginals μ and ν, respectively, such that for λ-a.e (x, y),

$$u(x) = G(x, y, v(y)) \quad \text{and} \quad v(y) = H(x, y, u(x)).$$

All of this simply says that pairs (x, y) given by λ describe buyers and sellers engaging in a transaction resulting for each in a utility of $u(x)$ and $v(y)$, respectively.

A *stable outcome* is an outcome where each buyer x is doing the best they can do given the utility profile of sellers, and conversely each seller is doing the best they can do given the utility profile of buyers. This means that in addition to the condition above, we have

$$u(x) \geq G(x, y, v(y)) \quad \forall\ x \in \Omega, y \in \overline{\Omega},$$
$$v(y) \geq H(x, y, u(x)) \quad \forall\ y \in \Omega, x \in \overline{\Omega}.$$

In other words, if (u, v, λ) is a stable outcome, then in particular (u, v) must be a conjugate pair.

Note that if one has a stable outcome (u, v, λ), then for each buyer x_0 there is at least one y_0 achieving the maximum of

$$G(x_0, y, v(y)).$$

The set of such sellers represents those whom x could buy from to realize their maximum utility. Therefore finding such a y_0 is the same as finding $y_0 \in \partial_G u(x_0)$. If there is a map $x \to y(x)$ with the property that $y(x)$ maximizes $G(x, y, v(y))$, then we say that $y(x)$ is a map **implemented** by $v(y)$ (note that there could be more than one map). We also say y is **implementable**.

Perfectly transferable utilities. For utility functions of the form

$$G(x, y, z) = b(x, y) - z$$

the problem falls within the framework of optimal transportation, a fruitful fact that has led to many contributions by Carlier, Ekeland, McCann, and many others. Such utility functions are known in the economics literature as *quasi-linear utility functions* and correspond to what is known as *perfectly transferable utility*.

Problems with *imperfectly transferable utility* that are used to model e.g. high income and taxation effects require utility functions that are not quasilinear; see Galichon et al. [6]. Nöldeke and Samuelson [15] in particular show the existence of stable matchings for nonquasilinear utilities using the formalism of generating functions (which, they note, corresponds to what is known as a Galois connection) while also studying a principal-agent model with nonquasilinear utility.

An Overview of the Literature

The framework of generated Jacobian equations was set forth by Trudinger in [18], where the main ideas were introduced. A complete review of the elements of generating functions (including G-convexity of domains, G-segments) can also be found in my work with Kitagawa [8, section 4].

Among several works, we mention that of Jiang and Trudinger [10], where they derived (a priori) estimates for the second derivative of solutions provided the G satisfies the generating function analogue of the celebrated A3w condition of Ma, Trudinger, and Wang in optimal transport. My latter work with Kitagawa [8] deals with the question of differentiability of weak (Aleksandrov-type) solutions to Problem 2 under minimal assumptions on G and the rest of the data. A key assumption on G is that it satisfies a version of the A3w condition introduced in optimal transport by Ma, Trudinger, and Wang (and later described by Trudinger in the context of GJE).

There have been various numerical results. We first mention work of De Leon, Gutiérrez, and Mawi [4] that dealt specifically with the far field refractor. Subsequently, Abedin and Gutiérrez [1] obtained an estimate on the number of steps needed to complete an iterative method to find solutions of GJE with a discrete target measure.

We briefly mention a few references dealing with geometric optics (the field is too vast to do it justice in this space). The work by Galindo [7] in the 1960s is among the first to do a systematic treatment of antenna design. Subsequently, more complicated surfaces were treated in works of Westcott and Norris [16], where the role of the Monge-Ampère equation was first recognized. Decades later, works of Oliker [17] brought more sophisticated mathematical tools from nonlinear PDE and differential geometry to the problem. Another essential reference is the work of Kochengin and Oliker [13], which covered both theory and numerics.

Regarding the existence of smooth solutions for near field problems, we already mentioned [18]. In [11], Karakhanyan and Sabra show $C^{2,\alpha}$ regularity for weak solutions (Aleksandrov and Brenier) of the near field refractor problem assuming $f, g \in C^2$ (along with other natural assumptions) in the case $n_1 \geq n_2$. As for $C^{1,\alpha}$ regularity results in geometric optics with densities that might not be necessarily smooth, we mention work of Wang [19] for the far field problem (this problem was an instrumental special case for the development of optimal transport).

A few more words regarding the economics literature. Nöldeke and Samuelson also noted in [15] that the duality structure given by a generating function yields what is known in the computer science and economics literature as a *Galois connection*. Another relevant work concerning nonquasilinear utility functions is that of McCann and Zhang [14]. In [14] the authors study a monopolist problem with an underlying generating function that is not necessarily quasilinear. They identify necessary and sufficient conditions for the concavity of this problem. The work [14] shows that the monopolist problem (or principal-agent problem) is equivalent to a concave maximization problem on a convexity constraint, the necessary and sufficient conditions on G being a Ma–Trudinger–Wang-type condition. Previous related work by Figalli, Kim, and McCann proved a similar result for quasilinear utilities. Furthemore, Zhang [20] has obtained an existence result for a screening problem with nonquasilinear utilities.

Open Problems

Several of the problems discussed below deal with regularity estimates for generated Jacobian equations. Many of these estimates are well understood in the case of optimal transport, but it is unclear to what extent they hold in general. As has been shown for the point source near field reflector [12], the regularity of potential functions in GJE is more complicated than in the optimal transport case. Another set of questions deals with structural issues such as uniqueness of weak solutions or variational characterization of solutions, which are also more complicated than in the optimal transport setting.

(1) A feature of many geometric optics models is that G satisfies the strong A3 condition of Ma, Trudinger, and Wang. This condition implies, for instance, pointwise bounds that play an important role in the regularity theory for weak solutions, as seen for example in [9]. While these results have a stronger assumption on G, they do not require a localization procedure as in [8] and may cover more general weak solutions. It is of interest in proving a regularity result for an abstract GJE under the strong A3 condition.

(2) An important result by Caffarelli [3] yields a $W^{2,p}$ estimate for weak solutions of the real Monge-Ampère equation. Subsequently, $W^{2,p}$ estimates were obtained for the optimal transportation problem. There is no result of its type for any GJE beyond those covered by optimal transport, including special problems like near field problems in geometric optics.

(3) In a different direction, it is well known that weak solutions of the real Monge-Ampère equation are much better behaved in two dimensions than their higher dimensional counterparts. Figalli and Loeper showed this is still the case for two-dimensional optimal transportation problems [5]. Does a similar result hold for generated Jacobian equations? This question is especially pertinent to GJEs arising in geometric optics, as they are naturally two-dimensional problems.

(4) To date, all regularity results for weak solutions of GJE are interior estimates. Based on what is known in the

optimal transport case, boundary regularity for GJE should require at least G-convexity of the source and the target. However, for G's that satisfy only the A3 weak condition, further conditions on u may be necessary, such as its being *nice* in the sense of [8].

(5) We mentioned Rockafellar's theorem. A generalization for c-convex functions is well known. If the generating function is not quasilinear (i.e., is not of the form $c(x, y) - z$), is there a simple characterization of mappings τ which admits a G-convex potential? Or, in the terminology of economics [15], when is $y(x)$ an implementable map with respect to a given utility function?

(6) In optimal transport proofs uniqueness of weak solutions relies on the structure of the Kantorovich problem and its dual. No uniqueness result for weak solutions is known in general for GJE except in the case where the target measure consists of a finite number of point masses, as was shown by Gutiérrez and Huang using an abstract framework to deal with the near field refractor.

(7) Is there a variational characterization of solutions for GJE equations, as in optimal transport? More concretely, given G, μ, and ν is there a functional $J(u, v)$ such that its minimizers (or critical points) are automatically a conjugate pair with respect to G and such that τ_u sends μ to ν?

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