The Number Line: Unifying the Evolving Definition of Number in K–12 Mathematics

Brigitte Lahme, Cam McLeman, Michael Nakamaye, and Kristin Umland

Introduction

Beginning in the spring of 2016, a team of more than thirty mathematics teachers, mathematics educators, and mathematicians at Illustrative Mathematics began to write a grades 6–8 mathematics curriculum that was released as an open education resource in the summer of 2017. Since then the team has expanded, and Illustrative Mathematics released a high school curriculum in the summer of 2019 and will release a grades K–5 mathematics curriculum in the summer of 2021. The four authors of this article are mathematicians who have worked together since the inception of Illustrative Mathematics in 2011 on these and other K–12 mathematics projects.

Over the time we have worked together, we have enjoyed the surprising variety of interesting mathematical issues that arise when writing about K–12 mathematics. For example, choosing a unique best definition for a given mathematical concept may be a sensible undertaking for a textbook covering a single semester or a year's worth of content. But for a comprehensive K–12 curriculum, definitions and notation must evolve alongside student understanding, and diagrams and other representations must change over time to accommodate and facilitate the expansion of these ideas. In this article, we discuss some of the mathematical and pedagogical nuances of making K–12 curriculum-wide decisions about the selection and adoption of a coherent and rigorous set of definitions, notation, and graphical conventions that both hold up to mathematical scrutiny and also serve the needs of the students to whom they are introduced.

In particular, we will focus our attention on curriculum design issues surrounding the development of the real numbers from kindergarten to grade 12, keeping in mind the work that students might do in later years. Because our work is rooted in the Common Core State Standards for Mathematics [1], we constrain our discussion to the development of a K–12 mathematics curriculum designed with these standards in mind, and all subsequent discussion of grade-level work refers to the sequential development of the mathematics in these standards.

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Numbers and the Number Line

Developing the notion of a *number* requires attention to both the relevant mathematics and how students learn that mathematics. Students need some understanding of the real number system by the time they complete high school, and yet the real numbers are notoriously more subtle to define and work with than is typically evident from their treatment in instructional materials. None of the classical constructions or definitions from a real analysis course (via Cauchy sequences, Dedekind cuts, or as the unique ordered field satisfying certain axioms) are particularly useful for most K–12 students (though neither are they far from our minds, e.g., infinite decimal expansions as Cauchy sequences of finite approximations). Of course, even a set-theoretic mathematical definition of the counting numbers is out of the question, and yet the need remains: students learn about and make use of the counting numbers starting in kindergarten and expand and refine their notion of what a number is many times in the twelve years of schooling that follow. Students need working definitions of the various types of numbers they encounter that allow them to reason deductively within the framework of their current stage of learning. Because the concept of a number expands from year to year, students' working definitions must evolve accordingly over time.

H. S. Wu advocates that we define a real number as a point on a number line [3]. However, even if this is the definition adopted in later grades, students in kindergarten and grade 1 need something more concrete to reason from. We agree that number line diagrams should be the primary representation of numbers once students are sophisticated enough to learn about and interpret these types of representations. By tying any understanding of a number to the number line, students can expand their understanding of the real number system in an intuitive but not mathematically misleading way. While we must attend to long-term goals for how we want students to think about mathematical concepts, we must also respect the intellectual work appropriate to each grade level to ensure that students have a strong foundation on which to build a more abstract understanding of number in later grades. Curriculum writers must pay careful attention to the evolution of the working definitions that students will use at various stages of learning. We assert that a good curriculum will help students build their understanding of the counting and rational numbers as telling "how many/how much" and then help them connect this understanding to the representation of a number as a point on the number line. This stance raises a number of important mathematical questions related to the use of number line diagrams in curriculum materials.

For our work *the number line* begins as a line in the usual Euclidean sense, which is used as a geometric model for the real numbers. Two distinct points are chosen and identified with the numbers 0 and 1 (which establishes a metric on the line). As new sets of numbers are introduced, a process for locating them on the number line is also introduced, so that the difference between the two numbers is reflected in the displacement on the line between the two corresponding points. The mechanism for locating further points is grade-level dependent: in early grades students locate points on the number line that correspond to the whole numbers by placing the number 2 on the opposite side of 1 from 0 so that 1 is equidistant from both, and so on. Later, students will reason through the location of the positive rational numbers on the number line by partitioning the segment from 0 to 1 into equal-length pieces to identify unit fractions (fractions of the form $\frac{1}{b}$, for b a positive integer). Then, by adding a whole number, a, of copies of $\frac{1}{b}$ they can place any fraction $\frac{a}{b}$, just as the location of the whole number a was deduced from the location of 1. Negative rational numbers are located by finding the point that is the same distance from but on the opposite side of 0 as its positive counterpart. The real numbers are located via an exploration of decimal expansions (with some hand-waving around limits). Thus, as students' understanding of *number* evolves from whole numbers to rational numbers to real numbers, we enlarge the subset of points on the number line that correspond to numbers, eventually filling out that initial Euclidean line as a visual representation of the set of all real numbers.

A *number line diagram* is a drawing that represents an interval of the number line. The question of how to draw number line diagrams is tied intimately with pedagogical decisions about the use of the number line itself, and so we are left with a slew of questions. Among others, of principal interest are the following:

- When should number lines first be introduced?
- What conventions for drawing number line diagrams should be followed at each grade level?
- Should operations be represented on a number line diagram? If so, how?

We address these and other questions below.

Working Definitions and Representations for Number Systems

We begin by describing the development of the real number system across grade levels. Students study the whole numbers in kindergarten through grade 2, the positive rational numbers in grades 3–6, the rational numbers in grades 6 and 7, and the real numbers in grade 8 and beyond (although they are introduced to π in grade 7, they do not explore the more general notion of nonrational numbers until grade 8). Table 1 shows the way in which the number system expands across K–8. Beyond the counting numbers, students first study a new set of numbers as objects in their own right before they study the four arithmetic operations ($+ - \times \div$) in the context of each system.

To support the understanding expected of students in later grades, early elementary students need working definitions for the whole numbers, addition, and subtraction that are developmentally appropriate, robust enough to support mathematical reasoning, and engineered to admit

Grade	K	1	2	3	4	5	6	7	8
$\mathbb{Z}_{\geq 0}$	$+$ $-$	$+$ $-$	$+$ $-$	$+$ $-$ $\times$ $\div$					
$\mathbb{Q}_{\geq 0}$				numbers	$+$ $-$ $\times$	$+$ $-$ $\times$ $\div$	$+$ $-$ $\times$ $\div$		
$\mathbb{Q}$							numbers	$+$ $-$ $\times$ $\div$	
$\mathbb{R}$									numbers

Table 1. Evolution of the real number system K–8.

(or at least minimally interfere with) an expanding definition of the number system. We believe that properly engineered working definitions for numbers and operations that systematically evolve over time can help bootstrap students to an increasingly more sophisticated and unified understanding of the real number system.

This is no easy task: the early development of students' understanding of the counting numbers, addition, and subtraction is surprisingly complex and one of the most well-researched areas in mathematics education (see [2]). Once they have learned to count and know that the count represents "how many," kindergartners understand addition as the operation that represents putting two amounts together or adding one amount to another and understand subtraction as splitting or removing. They use the working definition of addition as "putting together or adding to" to reason about things like the value of $2 + 3$, and the primary representations of numbers that they study include concrete objects and discrete diagrams (see Figure 1), which they use to build meaning for numerals. In grade 1, students expand their definition of addition

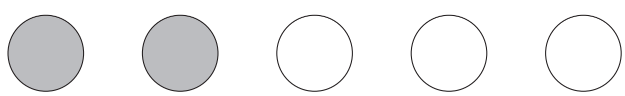

Figure 1. A discrete diagram.

to include situations that involve comparing two amounts. They continue to use discrete diagrams to understand the operations, including diagrams that show base-ten structure.

Should kindergarten and grade 1 students also study number line diagrams, which require an understanding of length? Students compare objects by length in kindergarten, "express the length of an object as a whole number of length units" in grade 1, and measure lengths in standard units in grade 2. When in this sequence of learning should students be introduced to the number line and number line diagrams? Research suggests that number lines can be difficult to understand for children below grade 2 and that related representations like the "number path," such as the one shown in Figure 2, are more accessible at this age (see [2]). Discrete representations are the primary way that students in grades K and 1 represent numbers and operations, and a number path can provide a bridge to continuous representations like number line diagrams. Figure 2 can be viewed as an intermediary between Figures 1 and 3.

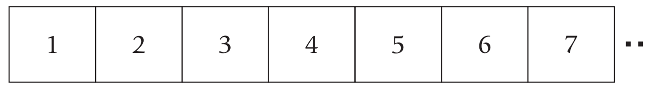

Figure 2. A number path representation of the counting numbers.

In grade 2, students extend their understanding of addition and subtraction to larger whole numbers and are ready for work with number line diagrams, which are excellent tools for, e.g., comparing numbers. In grade 3, fractions (nonnegative rational numbers) are introduced. The fraction $\frac{a}{b}$ is defined to be the sum of a copies of a number $\frac{1}{b}$, where the sum of b copies of $\frac{1}{b}$ equals 1. Fractions are understood both in terms of quantities ($\frac{1}{4}$ of a cup of juice) and as points on the number line. The number line is a powerful tool for placing $\frac{1}{4}$ in the same universe as the counting numbers, and number line diagrams can be used to compare fractions by comparing their relative positions. They can also be used to illustrate instances of fraction equivalence; for example, students can locate $\frac{12}{4}$ on a number line diagram (Figure 3) using the definition above to see that it occupies the same location as 3.

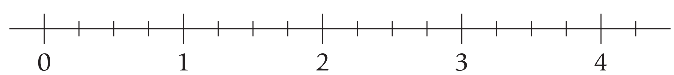

Figure 3. A number line diagram that can illustrate why $\frac{12}{4} = 3$.

While the number line is useful for understanding order and equivalence, we note that number systems are not merely sets of numbers; they are endowed with one or more operations. This raises a new question: should one represent operations with number line diagrams? If yes, which operations, with which numbers, and how should the operations be represented? In grade 2, students can use the number line to represent adding or subtracting as moving to the right or left (respectively) on the number line (Figure 4), which helps students visualize sums and differences with larger numbers. Representing addition and subtraction on the number line in grade 2 and be-

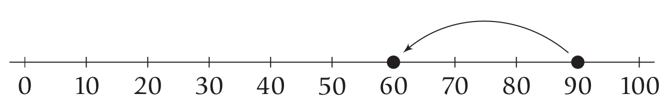

Figure 4. Representing $90 - 30$ in grade 2.

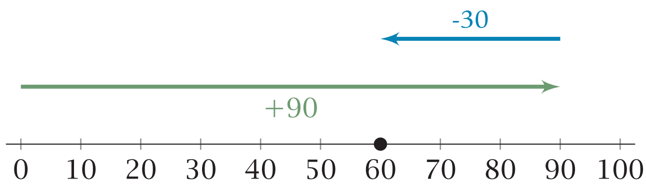

Figure 5. Representing $90 + (-30)$ in grade 7.

yond also lays the foundation for understanding adding and subtracting signed numbers in grade 7 (see Figure 5).

In grades 4–6, students use both quantities and the number line to develop their understanding of operations on fractions. For example, in grade 5, a number line diagram like the one shown in Figure 3 can also help students determine how many $\frac{1}{4}$'s there are in 3 as a way to understand $3 \div \frac{1}{4}$.

Students learn in grade 8 that the myriad rational numbers do not fill up the number line. They learn there are numbers, such as π and $\sqrt{2}$, that cannot be expressed as rational numbers and that each real number (that is, each point on a number line) can be written via a decimal expansion. Building on their work representing rational numbers on a number line diagram, students can visualize finite and then infinite decimal expansions (see Figure 6 for a number line diagram inspired by [4]).

In high school, students look at operations on real numbers and study what happens when we add or multiply an irrational number by a rational number. For example, continuing the development of ideas begun in grade 2, they may use visualizations based on the number line to reason about sums of irrational numbers. Since numbers like π or $\sqrt{2}$ are more difficult to locate on a number line diagram than rational numbers, older students will have to make informed choices; e.g., $\pi + \sqrt{2}$ is somewhere in the vicinity of 4.5, since π is slightly to the right of 3 and $\sqrt{2}$ is slightly to the left of 1.5. This work prepares them for more abstract reasoning about the real numbers in later mathematics.

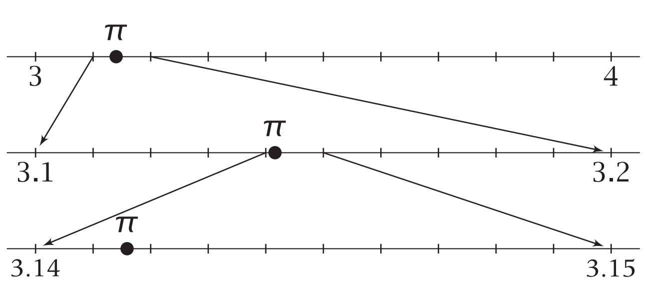

Figure 6. Placing π on a number line.

Beyond K–12

While the primary goal of the curriculum is to prepare all K–12 students for their mathematical futures regardless of their eventual destination, it is worth touching upon how this work ties to students' mathematical work in subsequent years if they do pursue more mathematics in college. Even as far as a real analysis course, before students write proofs we often ask them to draw a number line diagram, pick arbitrary points, and visualize the general argument. For example, to show that an interval (a, b) is open they might draw a picture like the one in Figure 7.

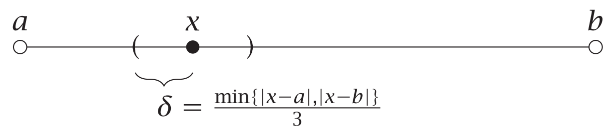

Figure 7. Showing an arbitrary open interval in an open set.

Students who have used the number line throughout K–12 to extend their understanding of the number system and who have used number line diagrams to solve problems are in a better position to continue using them as a tool to visualize and make sense of abstract ideas in later mathematical work. Ideally, they will have the inclination to draw helpful diagrams before they attempt writing proofs.

We note that in such exercises, students are asked to draw upon both their mathematical content knowledge and the metamathematical practice of using, adapting, or adopting conventions for drawing number line diagrams to convey their arguments. Figure 7 reflects several such conventions, using both open circles and open parentheses to indicate the openness of intervals, filled circles to indicate the relevant point, and various adornments for naming and indicating points and lengths. There are no "proofs by pictures" without good pictures.

Relatedly, though formal definitions of real numbers were mentioned as a bit of a bogeyman earlier in this article, students who *do* encounter a definition of the real numbers as, say, Dedekind cuts, will continue to make use of these kinds of diagrams. The aforementioned grade 8 exercise of identifying $\pi + \sqrt{2}$ as a point on the number line is almost direct preparation for exercises like proving the commutative law for real number addition in this framework. And regardless of the formal definition adopted, students must eventually run into the nested interval property (or one of its equivalents) that an infinite intersection of nested closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

is necessarily nonempty.

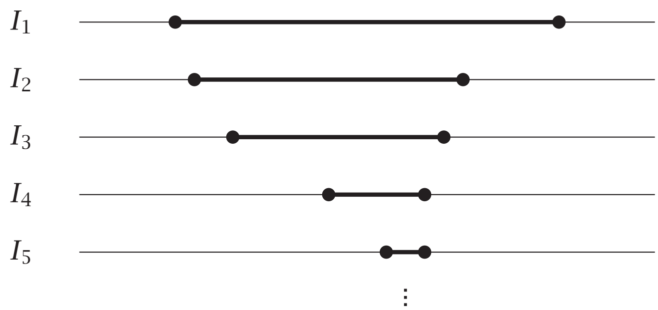

Figure 8. Representing a sequence of nested closed intervals.

It is hard to imagine inspiration for a proof of this property that does not begin with a diagram closely akin to Figure 8.

The number line serves as a common unifying representation of the real numbers that begins in the early grades and continues throughout students' mathematical careers. Though the proportion of K–12 students who will pursue mathematics to this depth is small, carefully chosen conventions starting even as early as grade 2 can help all students develop a robust understanding of numbers while also laying the foundations for a more formal understanding of the real numbers if students do decide to pursue it.

Answering Our Last Question

The curriculum writing team at IM intends that the diagrams in the curriculum support both the current mathematics that students are learning and, to the extent possible, the mathematics that they will learn in the future. Earlier we showed several visual milestones that reflect the progression of student understanding of the real numbers, and it is with that progression in mind that we ask our final question: What conventions should be followed for drawing number line diagrams that will best support students' evolving understanding of the real number system? Figure 9 shows some of the possible variations. Which should we use in elementary school? In middle or high school? In an undergraduate or graduate textbook? What are the reasons for these choices?

Should number line diagrams include arrows on each end (as in some textbooks), indicating that the numbers "go on forever in each direction," even though students are not introduced to negative numbers until grade 6? Should diagrams include an arrow just to indicate the positive end, as is often used for axes in the coordinate plane? Should they be omitted entirely, because arrows in number line diagrams can also be used to represent operations and signed numbers?

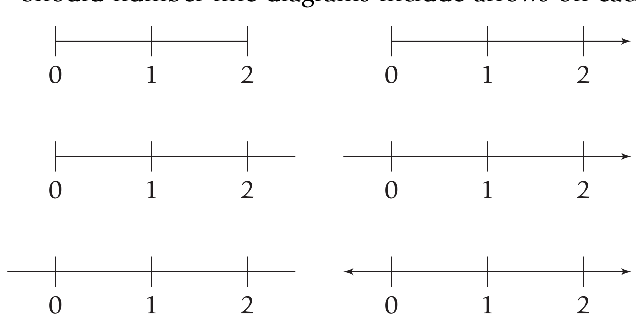

Figure 9. Different examples of number line diagrams.

Our choices were heavily influenced by how students' understanding of numbers changes over time and how number line diagrams will be used in later grades. In grade 7, we used directed line segments to represent signed numbers, with an arrow to depict the direction, and we lined the directed segments up tip to tail to represent addition, as shown in Figure 5. As one consequence of this decision, we elected to suppress the arrows that show the numbers "go on forever in each direction" in number line diagrams in grades 6 and 7 and then, to be consistent, in all other number line diagrams in the curriculum. Decisions about other conventional features of the diagram required similar reflection. For example, the grade 2 diagrams include a segment to the left of zero, as shown in Figure 4. This feature suggests that there is territory to be explored to the left of zero, in much the same way that the space between the whole number tick marks leaves room for the rationals. We also decided that the conventions chosen for representing addition and subtraction in number line diagrams in grade 2 should build on students' understanding of addition and subtraction from grades K and 1 while anticipating their use in the middle school curriculum. The conventions adopted in Figure 4 reflect these considerations.

Final Thoughts

The K–12 evolution of working definitions for numbers and operations and representations of the real number system is an illustration of a set of much larger decisions that have to be made when writing K–12 curricula. Even though abuse of notation, sloppy diagrams, and imprecise language occur routinely in dialogue between proficient users of mathematics, curriculum writers need to be especially careful not to let their own fluency interfere with more deliberate and principled usage. This need is at its greatest when standardly accepted uses of a mathematical term include instances where there are multiple implicit (or explicit) definitions of the term. Fluent practitioners of mathematics typically come to coalesce these separate definitions as facets of the same Platonic mathematical object, but for first learners, a multitude of dissonant definitions may cause more harm than good. One example that a certain subset of the mathematical community has been debating passionately for years is the term "ratio." Here are examples of two commonly accepted uses of the term:

- The ratio of bees to flies is $2 : 5$.
- The ratio of the side-lengths of the triangle is $\frac{2}{5}$.

In the first instance, the implicit definition of a ratio is an ordered pair, and in the second it is a single number. So what "should" the definition of ratio be? One can argue this point ad nauseam—and some people have! We chose to distinguish between the *ratio* $a : b$ and the *value of the ratio*, $\frac{a}{b}$, in grades 6–8 and to then follow common usage (and the implicit shift in the language of the standards) in high school, which refers to either of these as the ratio of a to b.

When writing curricula, one has to make choices and then live with the logical, pedagogical, and psychological consequences in all downstream materials (and accept the fact that everyone who would have made different choices will wish to readjudicate the issues forever). However, there is a tension between being internally consistent and being reflective of how language and notation are used by various mathematical demographics (e.g., by fluent practitioners, previous generations of learners, or other textbooks, etc.). In our work, we (attempt to) choose meanings and representations that require minimal alteration through the grades, except as needed to accommodate the expansion of concepts as students mature in their mathematical understanding. Finally, we note we are not by any means alone in these endeavors: the authors of the EngageNY curriculum [5], for example, have given a lot of thought to these issues and have written a very detailed description of the definitions they use and the grades at which those definitions are introduced in that curriculum [6]. Since our curriculum is still a work in progress, we do not yet have such detailed documentation of our own decisions, but we hope the examples we have given here help convey the complexity and nuance of this kind of work.

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