WHAT IS...

a Hereditarily Indecomposable Banach Space?

Hossein Hosseini Giv

The story began in 1993 with the paper [10] by Gowers and Maurey, in which a negative answer was given to the long-standing *unconditional basic sequence problem*. The problem, whose first formal appearance was in [6], was to determine whether every Banach space contains an unconditional basic sequence, and the space X constructed by Gowers and Maurey provided the negative answer. It was pointed out by W. B. Johnson, prior to the publication of [10], that the space had the remarkable property of being *hereditarily indecomposable*. This is the property we aim to discuss in this note.

H.I. Spaces and Their Basic Properties

Lindenstrauss asked in [13] if every infinite-dimensional Banach space is *decomposable*, that is, can be written as the topological direct sum $Y \oplus Z$ of two closed infinite-dimensional subspaces. This problem is closely related to the existence of complemented subspaces: If B is a Banach space, a closed subspace Y is said to be (topologically) *complemented* in B if there exists a closed subspace Z such that $B = Y \oplus Z$. So, a Banach space is decomposable if and only if it has a nontrivial infinite-dimensional complemented subspace. The simplest case for the study of decomposability is clearly that of a Hilbert space, because given any closed subspace Y, we can take Z to be $Y^{\perp}$, the orthogonal complement of Y. Lindenstrauss and Tzafriri proved the remarkable result that a Banach space in which every closed subspace is complemented is isomorphic to a Hilbert space. The problem is, of course, more delicate in the general context of Banach spaces.

The importance of the space X constructed in [10] is that every infinite-dimensional subspace of X (and hence X itself) is indecomposable, and this is what allows us to call it a *hereditarily indecomposable* (H.I.) Banach space. The space X, which is also reflexive, is indeed a development of Schlumprecht's space, which is in turn a Tsirelson-type arbitrarily distortable Banach space, together with the Maurey–Rosenthal space [14]. Tsirelson's space was constructed in [15].

The paper [10] contains some remarkable results about H.I. spaces. First, every bounded linear operator T from a complex H.I. space into itself can be written as $T = \lambda I + S$, where $\lambda \in \mathbb{C}$, I is the identity operator, and S is a strictly singular operator. Second, an H.I. space is not isomorphic to any proper subspace, and the space cannot be, accordingly, isomorphic to its hyperplanes.

Suggestions for Further Reading

The paper [10] was a pioneering work that established a new and interesting field of research in the theory of Banach spaces. Here we briefly mention some of the research done in this direction that we believe to be appropriate for further reading. These works can also be considered as evidence of the importance of H.I. spaces in Banach space theory.

Argyros and Felouzis proved in [2] that every Banach space either contains a subspace isomorphic to ℓ^1 or has a subspace that is a quotient of an H.I. space ([2, Theorem 1.1]). They observed that separable Hilbert spaces, L^p spaces for $1 < p < \infty$, and c_0 are quotients of H.I. spaces and proved that some bounded linear operators between Banach spaces factor through H.I. spaces ([2, Theorem 1.2]).

Argyros also proved in [1] that every separable Banach space universal for the class of reflexive H.I. spaces contains

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$C[0,1]$ isomorphically and is therefore universal for all separable spaces. This result shows that separable H.I. spaces define a large class of Banach spaces. Its proof is based on Bourgain's techniques of constructing reflexive Banach spaces connected to well-founded trees, developed in [7].

The remarkable *dichotomy* of Gowers ([9, Theorem 1.4]) states that every infinite-dimensional Banach space has a subspace that either has an unconditional basis or is H.I. The theorem was used to solve *Banach's homogeneous space problem*.

In [5] a general method for the construction of H.I. spaces is proposed which is then used to present a nonseparable H.I. space Y. This space has the interesting property that every bounded linear operator from Y into itself is of the form $\lambda I + W$, with W being a weakly compact operator. Each such operator therefore has a separable range.

In [12], Koszmider constructed $C(K)$ spaces that are indecomposable and more precisely such that every operator is a weakly compact perturbation of a multiplication operator. These spaces are indecomposable without being H.I.

Ferenczi gave an example of a real H.I. space in [8] that, up to isomorphisms, has exactly two complex structures that are totally incomparable. His example proved that two Banach spaces that are isometric as real spaces may be totally incomparable as complex ones. This extended Bourgain's result that real isomorphic complex Banach spaces need not be complex isomorphic.

Related to the aforementioned result obtained in [10] for operators on a complex H.I. space, a problem formulated by Lindenstrauss, known as the *scalar-plus-compact problem*, was recently answered in the affirmative by Argyros and Haydon in [3]. The problem was as follows. Can we find a Banach space such that every bounded operator on the space is of the form $\lambda I + K$, with K being a compact operator? Notice that a related result was obtained earlier in [5], with compact replaced with weakly compact, as we mentioned above.

The book [4] is a valuable resource that can be considered as a nice starting point for researchers interested in the study of H.I. spaces. It describes the deep techniques that are used in the theory in a well-organized and accessible style.

Finally, we encourage the reader to take a look at the open problems that are related to H.I. spaces in [11, Section 1.2]!

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References

[1] Argyros SA, A universal property of reflexive hereditarily indecomposable Banach spaces, *Proc. AMS*, Vol. 129, No. 11 (2001), 3231–3239. MR1844998

[2] Argyros SA, Felouzis V, Interpolating hereditarily indecomposable Banach spaces, *J. AMS*, Vol. 13, No. 2 (2000), 243–294. MR1750954

[3] Argyros SA, Haydon RG, A hereditarily indecomposable $\mathcal{L}_\infty$-space that solves the scalar-plus-compact problem, *Acta Mathematica*, Vol. 206, No. 1 (2011), 1–54. MR2784662

[4] Argyros SA, Todorcevic S, *Ramsey methods in analysis*, Birkhäuser Verlag, 2005. MR2145246

[5] Argyros SA, Tolias A, Methods in the theory of hereditarily indecomposable Banach spaces, *Mem. AMS*, Vol. 170, Issue 806, 2004. MR2053392

[6] Bessaga C, Pełczyński A, A generalization of results of R. C. James concerning absolute bases in Banach spaces, *Studia Math.*, Vol. 17 (1958), 165–174. MR0115071

[7] Bourgain J, On separable Banach spaces universal for all separable reflexive spaces, *Proc. AMS*, Vol. 79 (1980), 241–246. MR0565347

[8] Ferenczi V, Uniqueness of complex structure and real hereditarily indecomposable Banach spaces, *Adv. Math.*, Vol. 213, Issue 1 (August 2007), 462–488. MR2331251

[9] Gowers WT, An infinite Ramsey theorem and some Banach-space dichotomies, *Ann. Math., Second Series*, Vol. 156, No. 3 (2002), 797–833. MR1954235

[10] Gowers WT, Maurey B, The unconditional basic sequence problem, *J. AMS*, Vol. 6, No. 4 (1993), 851–874. MR1201238

[11] Guirao AJ, Montesinos V, Zizler V, *Open problems in the geometry and analysis of Banach spaces*, Springer International Publishing Switzerland, 2016. MR3524558

[12] Koszmider P, Banach spaces of continuous functions with few operators, *Math. Ann.*, Vol. 330, Issue 1 (2004), 151–183. MR2091683

[13] Lindenstrauss J, Some aspects of the theory of Banach spaces, *Adv. Math.*, Vol. 5 (1970), 159–180. MR0279566

[14] Maurey B, Rosenthal HP, Normalized weakly null sequences with no unconditional subsequence, *Studia Math.*, Vol. 61, Issue 1 (1977), 77–98. MR0438091

[15] Tsirelson BS, Not every Banach space contains an imbedding of l^p or c_0, *Funct. Anal. Appl.*, Vol. 8, Issue 2 (1974), 138–141.

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