A Tale of Two Integrals

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In this lecture I will describe my personal struggle—spanning eight years and $\log_2 8$ academic positions—to solve an open problem in mathematics. The problem itself is captivating: while of an ostensibly analytic nature, ideas from combinatorics, geometry, probability, and representation theory are needed to break through to a solution. So, I hope that my lecture will be appealing to those who enjoy seeing interactions between different parts of mathematics. But more than this, I will candidly discuss the ups and downs, the successes and defeats encountered along the path to a solution—experiences that, I believe, are universal among mathematicians and that ultimately make our discipline so rewarding.

Here is the problem. Consider the integrals

$$I_N = \int_{\mathrm{U}(N)} e^{zN\mathrm{Tr}AUBU^{-1}}\mathrm{d}U$$

and

$$J_N = \int_{\mathrm{U}(N)} e^{zN\mathrm{Tr}(AU+BU^{-1})}\mathrm{d}U.$$

These are not your garden-variety calculus integrals; rather, they are integrals over the compact group of $N \times N$ unitary matrices against unit mass Haar measure. Their integrands depend on a complex parameter, z, and a pair of $N \times N$ complex matrices, A and B. Unless you have spent long periods of time thinking about matrix integrals, you probably have no feeling for these objects and perhaps see no reason why they should be of any interest. I assure you that they are both interesting and important, sufficiently so that they have names: I_N is known as the Harish-Chandra/Itzykson–Zuber integral, while its additive counterpart J_N is often referred to as the Brézin–Gross–Witten integral.

As these names suggest, I_N and J_N have been considered by physicists. Indeed, they were first singled out for focused study in a cluster of 1980 papers on lattice gauge theory, a mathematical apparatus for modeling strong interactions in particle physics. The authors of these works—Claude Itzykson, Jean-Bernard Zuber, Édouard Brézin, David Gross, Edward Witten, and others—wanted to know about the asymptotic behavior of I_N and J_N in order to understand the large N behavior of $\mathrm{U}(N)$ lattice gauge theory. The $N \to \infty$ asymptotic behavior of I_N and J_N has remained a subject of perennial interest in various parts of theoretical physics since the 1980s; see the survey [15].

The main mathematical motivation for studying I_N and J_N comes from random matrix theory, one of the most active topics in contemporary probability theory. The basic goal of random matrix theory is to understand the statistical features of eigenvalues of random matrices and to identify the universal laws that govern them in the large dimension limit. This has been accomplished in the case of a single random matrix, but the spectral analysis of multiple coupled random matrices is more difficult and less fully developed. For a large family of multimatrix models, a successful analysis of the joint spectrum hinges on understanding I_N, which emerges naturally when attempting to diagonalize the model.

The intense interest in the asymptotics of I_N and J_N stemming from mathematical physics and random matrix theory has produced many predictions and conjectures over the years. In the early 2000s, these were codified into precise mathematical statements by Benoît Collins [1, Section 5.1] and Alice Guionnet [9, Section 4.3], leading to the following conjecture.

Conjecture 1. *There exists $\varepsilon > 0$ such that, for $|z| \leq \varepsilon$ and $\|A\|, \|B\| \leq 1$, we have the $N \to \infty$ approximations*

$$I_N = e^{N^2(F_N+o(1))} \quad \text{and} \quad J_N = e^{N^2(G_N+o(1))}$$

as $N \to \infty$, where the error term is uniform and

$$F_N = \sum_{d=1}^{\infty}\left(\sum_{\alpha,\beta\vdash d} F(\alpha,\beta)\frac{p_\alpha(a_1,\dots,a_N)}{N^{\ell(\alpha)}}\frac{p_\beta(b_1,\dots,b_N)}{N^{\ell(\beta)}}\right)\frac{z^d}{d!},$$

$$G_N = \sum_{d=1}^{\infty}\left(\sum_{\beta\vdash d} G(\beta)\frac{p_\beta(c_1,\dots,c_N)}{N^{\ell(\beta)}}\right)\frac{z^{2d}}{d!}$$

are analytic functions of z, the eigenvalues $a_1,\dots,a_N$ of A, the eigenvalues $b_1,\dots,b_N$ of B, and the eigenvalues $c_1,\dots,c_N$ of $C = AB$. Moreover, the coefficients $F(\alpha,\beta)$, $G(\beta)$ are integers.[1]

The existence of the "free energies" F_N and G_N is tentatively assumed in the physics literature, where the focus is on computing the numbers $F(\alpha,\beta)$ and $G(\beta)$, which depend solely on α and β. The belief that these universal coefficients are integers is based on a fundamental principle in quantum field theory due to Gerard 't Hooft [14], according to which $F(\alpha,\beta)$ and $G(\beta)$ ought to enumerate some unspecified class of planar combinatorial structures.

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[1] *The notation $\alpha \vdash d$ means that α is a Young diagram with d cells. The quantity $\ell(\alpha)$ is the number of rows in α, and $p_\alpha(x_1,\dots,x_N)$ is the power sum symmetric polynomial in N variables.*

A real variables version of Conjecture 1 was proved in 2009 by Benoît Collins, Alice Guionnet, and Édouard Maurel-Segala [2]. Using a mathematically rigorous version of the Schwinger–Dyson "loop" equations, these authors were able to show that the asymptotic approximation of I_N claimed by Conjecture 1 holds if z is real and A, B are self-adjoint and that the asymptotic approximation of J_N holds if z is real and C is self-adjoint. Moreover, they showed that the coefficients $F(\alpha,\beta)$ and $G(\beta)$ enumerate certain complicated classes of planar maps. Although the loop equations approach has been generalized and refined by Alice Guionnet in collaborations with Alessio Figalli [3] and myself [10], it remains unclear whether it can be implemented in the complex case, i.e., when I_N and J_N are oscillatory integrals.

In the spring of 2011, Ian Goulden, Mathieu Guay-Paquet, and I began to develop a new approach to Conjecture 1 that we hoped would be robust enough to handle complex parameters. This new approach was driven by a crucial insight: the mysterious combinatorial structure underlying Conjecture 1 is Hurwitz theory, a classical branch of enumerative algebraic geometry concerned with counting maps between Riemann surfaces. We realized that if Conjecture 1 is true, then it must be the case that

$$F(\alpha,\beta) = (-1)^{\ell(\alpha)+\ell(\beta)}\vec{H}_0(\alpha,\beta),$$
$$G(\beta) = (-1)^{d+\ell(\beta)}\vec{H}_0(1^d,\beta),$$

where $\vec{H}_g(\alpha,\beta)$ is a newfangled "monotone" version of the classical double Hurwitz number $H_g(\alpha,\beta)$. The double Hurwitz numbers $H_g(\alpha,\beta)$, so-named by Andrei Okounkov [13], count degree d maps from a compact connected Riemann surface of genus g to the Riemann sphere that have ramification profiles α and β over two given points on the sphere, with the remainder of the branching locus as simple as possible. These numbers are truly remarkable: despite being relatively simple combinatorial objects, they encode more than enough information to yield the famous Kontsevich–Witten theorem relating intersection theory in moduli spaces of curves to integrable hierarchies of partial differential equations [11].

The realization that Conjecture 1 is linked to Hurwitz theory opened up a whole new perspective on the problem and ultimately led to a solution—Conjecture 1 is now a theorem. However, the path from problem to solution was not an easy one; it took Ian, Mathieu, and me a total of six papers to make the complete trek [4–8, 12]. We got lost several times along the way and at one particularly daunting precipice almost gave up, but eventually we reached the summit. You are warmly invited to come to my lecture and hear about our journey, which may well intersect with your own.

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Credits

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