

Face Numbers: Centrally Symmetric Spheres versus Centrally Symmetric Polytopes

Isabella Novik

We start by introducing the main players. A *polytope* P is the convex hull of a set of finitely many points in \mathbb{R}^d . For instance, a (geometric) simplex is the convex hull of a set of affinely independent points. We say that P is *centrally symmetric* (cs, for short) if it is symmetric about the origin, that is, if $P = -P$. An important example of a cs polytope is the d -dimensional cross-polytope $C_d^* = \text{conv}(\pm e_1, \pm e_2, \dots, \pm e_d)$, where e_1, e_2, \dots, e_d are the endpoints of the standard basis in \mathbb{R}^d . The cross-polytope is a *simplicial polytope*, meaning that all of its (proper) faces are geometric simplices.

Our other main player is a *centrally symmetric* (or cs) *simplicial complex*. Recall first that a simplicial complex Δ on a (finite) vertex set V is a collection of subsets of V that is closed under inclusion. The elements of Δ are called faces. The dimension of a face F is one less than the size of F , and the dimension of Δ is the maximum dimension of its faces. The k -skeleton of Δ is the subcomplex of Δ consisting of all faces of dimension $\leq k$. We refer to i -dimensional faces as i -faces; their number is usually denoted by f_i and is called the i th face number.

Each simplicial complex Δ admits a geometric realization $|\Delta|$ that contains a geometric i -simplex for each i -face of Δ . This allows us to talk about *simplicial spheres*—simplicial complexes whose geometric realizations are homeomorphic to a sphere. For instance, if P is a simplicial polytope, then the empty set along with the collection of vertex sets of proper faces of P is a simplicial sphere called the boundary complex of P . However, as works of Kalai, and Pfeifle and Ziegler show, the boundary complexes of simplicial polytopes of dimensions $d \geq 4$ form only a tiny fraction of $(d - 1)$ -dimensional simplicial spheres.

Isabella Novik is Robert R. & Elaine F. Phelps Professor of Mathematics at the University of Washington. She is supported by NSF grant DMS-1664865. Her email address is novik@math.washington.edu.

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A map $\phi : V \rightarrow V$ induces a *simplicial map* on Δ if ϕ takes faces to faces. A simplicial complex Δ is centrally symmetric (cs) if it is equipped with an involution ϕ on V that induces a free simplicial involution on Δ . In other words, for every nonempty face F of Δ the following holds: $\phi(F)$ is also a face of Δ , $\phi(F) \neq F$, but $\phi(\phi(F)) = F$. For instance, the boundary complex of any cs simplicial polytope P is a cs simplicial sphere with the involution induced by the map $\phi(v) = -v$ on the vertex set of P .

The celebrated Upper Bound Theorem of McMullen (for polytopes) and Stanley (for spheres) asserts that in the class of all $(d - 1)$ -dimensional simplicial spheres with n vertices, the boundary complex of a certain polytope (called the cyclic polytope) simultaneously maximizes all the face numbers. One amazing property of the d -dimensional cyclic polytope on n vertices is that its $[d/2]$ -skeleton coincides with that of an $(n - 1)$ -dimensional simplex (for all values of $n \geq d + 1$). In particular, every two vertices of the cyclic polytope of dimension $d \geq 4$ are connected by an edge! (Here and below by k -skeleton of a simplicial polytope P we mean the k -skeleton of the boundary complex of P .)

The existence of cyclic polytopes and the statement of the Upper Bound Theorem along with recently discovered tantalizing connections (initiated by Donoho) between cs polytopes with many faces and seemingly unrelated areas of error-correcting codes and sparse signal reconstruction motivate the following questions in the centrally symmetric world. For a fixed $d \geq 4$ and $N \geq d$, what is the largest number of i -faces that a cs simplicial sphere of dimension $d - 1$ with $2N$ vertices can have? What is the largest number of i -faces that a cs simplicial polytope of dimension d with $2N$ vertices can have? Do there exist cs simplicial spheres of dimension $d - 1$ with $2N$ vertices (for arbitrarily large N) whose $[d/2]$ -skeleton coincides with that of the N -dimensional cross-polytope? What about cs simplicial polytopes with the same parameters? Are the answers for cs spheres and cs polytopes the same or (drastically) different?

We will discuss these and other questions, summarizing some of the surprising and fascinating developments in this field over the last fifty years. Our discussion will include results from a very recent joint project with Hailun Zheng of the University of Michigan.



Isabella Novik

Credits

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