Operator Integrals in Theory and Applications

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ABSTRACT. Operator integration is a powerful tool enabling analysis of functions with noncommuting arguments. Such functions arise, for example, in matrix analysis, mathematical physics, noncommutative geometry, and statistical estimation. Over some seventy years of its development, the theory underlying multilinear operator integration has accumulated many deep results and important applications. We will discuss major advancements made in recent years and their impact on differentiation and approximation of operator functions.

Introduction

"Multilinear operator integration" refers to methods and techniques designed for treating noncommutativity and obtaining properties of operator functions analogous to those of scalar functions. Given self-adjoint matrices or, more generally, infinite-dimensional operators A and B on a separable Hilbert space, a scalar function f, and the operator function $f(A)$ defined by the functional calculus, the inequalities

$$A \leq B \Rightarrow f(A) \leq f(B), \tag{1}$$

$$\|f(A)B - Bf(A)\| < \infty, \tag{2}$$

$$\begin{aligned}&\Big\|f(A) - f(B) - \sum_{k=1}^{n-1} \frac{1}{k!}\frac{d^k}{dt^k} f(B + t(A-B))\big|_{t=0}\Big\| \quad (3)\\ &\leq c_n \|f^{[n]}\| \, \|A-B\|^n,\end{aligned}$$

where $\|\cdot\|$ stands for an appropriate norm and $f^{[n]}$ for the nth divided difference of f, are products of the operator integration approach. Indeed, under suitable assumptions on A, B, and f, we have the representations

$$f(A) - f(B) = T^{A,B}_{f^{[1]}}(A-B), \tag{4}$$

$$f(A)B - Bf(A) = T^{A,A}_{f^{[1]}}(AB - BA), \tag{5}$$

$$\begin{aligned}&f(A) - f(B) - \sum_{k=1}^{n-1} \frac{1}{k!}\frac{d^k}{dt^k} f(B + t(A-B))\big|_{t=0} \quad (6)\\ &= T^{A,B,\dots,B}_{f^{[n]}}(A-B, \dots, A-B),\end{aligned}$$

where $T^{A_1,A_2,\dots,A_{n+1}}_{f^{[n]}}$ is an n-linear operator integral. Thus, a proper analysis of the latter transformation ultimately leads to (1)–(3).

The relation (4) was applied to derive (1) by K. Löwner in his work on characterization of matrix monotone functions in 1934. In that setting, $T^{A,B}_{f^{[1]}}$ is a Schur multiplier by the matrix $\{f^{[1]}(\lambda_j, \mu_k)\}^d_{j,k=1}$, where λ_j and μ_k are the eigenvalues of the self-adjoint matrices A and B, respectively. The relation (5) is useful for obtaining (2). When A is a Dirac operator, B is an operator of pointwise multiplication by a function g, and f is a sign function, the condition (2) means that the quantized derivative of g is summable in a certain sense. The relation (6) is essential for proving bounds like the one in (3), which arise in approximation problems of operator functions. For instance, to estimate the function $f(A)$ or the functional $\mathrm{Tr}\,(f(A))$ of an unknown covariance matrix A, one can replace A by the sample covariance matrix B constructed from collected data. Other examples arise in mathematical physics, where B is a discrete Laplacian or differential operator with known properties and A is a perturbed operator. In that context, one often considers the trace of the Taylor remainder (6), which encodes information about quantitative characteristics of the perturbation.

The proof of (6) requires both originality and technical proficiency, especially in the case of infinite-dimensional operators A and B, even when one does not intend to include the most general functions f and operators A, B. In particular, the proof utilizes the boundedness of the multilinear operator integral

$$T^{A_1,A_2,\dots,A_{n+1}}_{\varphi} : \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \to \mathcal{X} \tag{7}$$

defined on the product of Banach spaces $\mathcal{X}_1, \dots, \mathcal{X}_n$ with values in the Banach space $\mathcal{X}$. Proving existence of the derivative $\frac{d^k}{dt^k} f(B + t(A-B))$ in a fairly general infinite-dimensional setting is also a nontrivial task that depends on efficient bounds for (7). Moreover, as suggested by the examples (1)–(3), the boundedness of $T^{A_1,A_2,\dots,A_{n+1}}_{\varphi}$ along with the respective inequalities is an ultimate goal of the operator integration. The investigation of this question was initiated by Yu. L. Daletskii and S. G. Krein in 1956 and eventually led to a deep, comprehensive theory.

The existing norm bounds for the transformation (7) depend on the norms of the domain and target spaces, on the type of the symbol φ, and sometimes on the operators $A_1, A_2, \dots, A_{n+1}$. The best bound for (7) holds when

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DOI: https://doi.org/10.1090/noti1965

$\mathcal{X}_1 = \cdots = \mathcal{X}_n$ equals the Hilbert–Schmidt ideal S^2. More precisely, the results by B. S. Birman, M. Z. Solomyak, and B. S. Pavlov in the 1960s provide

$$\|T_\varphi^{A_1,A_2,\ldots,A_{n+1}} : S^2 \times \cdots \times S^2 \to S^2\| \leq \|\varphi\|_\infty \quad (8)$$

for every Borel function $\varphi : \mathbb{R}^{n+1} \to \mathbb{C}$. The second best bound holds when $\mathcal{X}_j$ equals the Schatten–von Neumann ideal S^{p_j} and $\mathcal{X} = S^p$, where $1 < p, p_j < \infty$, $j = 1, \ldots, n$, and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_n}$. Namely, D. Potapov, A. Skripka, and F. Sukochev established in 2013 that

$$\|T_{f^{[n]}}^{A_1,A_2,\ldots,A_{n+1}} : S^{p_1} \times \cdots \times S^{p_n} \to S^p\| \leq c_{p_1,\ldots,p_n} \|f^{(n)}\|_\infty \quad (9)$$

for every n times continuously differentiable function f. When $\mathcal{X}_1 = \cdots = \mathcal{X}_n = \mathcal{X}$ equals the space $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space or, more generally, equals a symmetrically normed ideal $\mathcal{I}$ of a semifinite von Neumann algebra with certain properties, we have the bound

$$\|T_{f^{[n]}}^{A_1,A_2,\ldots,A_{n+1}} : \mathcal{I} \times \cdots \times \mathcal{I} \to \mathcal{I}\| \leq \|f^{[n]}\|_\otimes \quad (10)$$

for f in the Besov space $B^n_{\infty 1}(\mathbb{R})$, where $\|f^{[n]}\|_\otimes \leq c_n \|f\|_{B^n_{\infty 1}(\mathbb{R})}$ is the integral projective tensor product norm, which is generally greater than the norm $\|f^{(n)}\|_\infty$. The bound (10) for $\mathcal{I} = \mathcal{B}(\mathcal{H})$ was established by V. V. Peller in 2006 and for a more general ideal $\mathcal{I}$ by N. A. Azamov, A. L. Carey, P. G. Dodds, and F. A. Sukochev independently in 2009.

The methods leading to (8) and (9) benefit from special features of the domain and target spaces. More specifically, the bound (8) is a consequence of the Hilbert space structure of S^2, and (9) is based on harmonic analysis of the UMD space S^p, $1 < p < \infty$. There are counterexamples showing that (9) does not extend to the case of the non-UMD space S^1 (corresponding to $p = 1$), and then (10) works as a possible replacement of (9) for sufficiently smooth functions f. The approach behind (10) relies on existence of a factorization for the function $f^{[n]}(\lambda_1, \ldots, \lambda_{n+1})$ separating the variables $\lambda_1, \ldots, \lambda_{n+1}$, which inevitably leads to a larger norm of the symbol $f^{[n]}$ that makes the bound ineffectual in certain situations. To compare, the separation of variables is avoided in the derivation of (9), and instead an intricate recursive procedure that essentially preserves the symbol $f^{[n]}$ is applied.

To specify the parameters for which (3) holds, we need, in particular, to summarize results on existence of the operator derivatives. Differentiability of operator functions with respect to the Schatten S^p-norms, $1 < p < \infty$, holds under minimal assumptions on the respective scalar functions. For instance, f is n times Fréchet S^p-differentiable at every bounded self-adjoint operator A if and only if $f \in C^n(\mathbb{R})$. This result in the case $n = 1$ was proved by E. Kissin, D. Potapov, V. Shulman, and F. Sukochev in 2012 and in the case $n \geq 2$ by C. Le Merdy and F. Sukochev in 2019. Its proof utilizes (9) and, when $n \geq 2$, the recent approach to the multilinear operator integration on $S^2 \times \cdots \times S^2$ by C. Coine, C. Le Merdy, and F. Sukochev. Using (10), V. V. Peller established differentiability of operator functions with respect to the operator norm for every $f \in B^1_{\infty 1}(\mathbb{R}) \cap B^n_{\infty 1}(\mathbb{R})$ in 2006. Thus, (3) with norms of the operators equal to the S^p-norm, $1 < p < \infty$, and $\|f^{(n)}\|_\infty$ standing for $\|f^{[n]}\|$ holds for $f \in C^n(\mathbb{R})$ if A, B are bounded; (3) with norms of the operators equal to the operator norm and $\|f\|_{B^n_{\infty 1}(\mathbb{R})}$ standing for $\|f^{[n]}\|$ holds for $f \in B^1_{\infty 1}(\mathbb{R}) \cap B^n_{\infty 1}(\mathbb{R})$.

The talk is based on [1] and will address the aforementioned and related results along with some open questions.

References

[1] Skripka A, Tomskova A. *Multilinear operator integrals: theory and applications*, Lecture Notes in Mathematics 2250, Springer, 2019. ISBN 978-3-030-32406-3.

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