The cover design is based on imagery from "3264 Conics in a Second," page 30.
1:00 PM | Jordan S. Ellenberg  
University of Wisconsin-Madison  

Geometry, inference, and democracy  
There is no democracy without representation. How can mathematics help to ensure it?

2:00 PM | Bjorn Poonen  
Massachusetts Institute of Technology  

A \( p \)-adic approach to rational points on curves  
A new proof of a famous theorem!

3:00 PM | Sunčica Ćanić  
University of California, Berkeley  

Recent progress on moving boundary problems  
Cardiovascular interventions with the help of math (and trying to understand how we swim).

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You may well wonder what is entailed in running a society’s publishing division. On the one hand we are operationally focused, acquiring content, enabling content submission, producing and publishing the content efficiently in online and print formats, marketing and then selling the content around the world. On the other hand, it is about strategy, ensuring that as a society serving the needs of mathematicians, we understand how to publish for all generations of mathematicians, how to innovate with caution, recommend publishing policies in context of the culture of mathematics, and grow revenues that allow the Society to provide mathematicians with programs and services. My job is to steer and advise, working with AMS Governance to position the AMS as a top-flight publisher. We are also involved in the wider publishing community (https://scholarlykitchen.sspnet.org/2019/10/03/why-scholarly-societies-are-vitally-important-to-the-academic-ecosystem/). We are active in developing AMS open access publishing policies. Working together with the AMS Associate Executive Director and Head of Government Relations, Karen Saxe, we talk directly with the Office of Science and Technology Policy (OSTP) in the White House, as well as funders and libraries, and actively engage with initiatives such as Plan S (https://www.coalition-s.org/) on behalf of the mathematics community.

Academic publishing, including mathematics publishing, is embracing the ideal of openness—a term I use to describe the concept of open research leading to open publishing through the range of open access (open access content is open to all, with no access fees) publishing models available today. A range of forces—some complementary, some contradictory—are steering research and publications that derive from research into an open setting. Funding agencies are committed to openness across all areas of research, all types of publications, and data. Just take a look at the FAIR initiative (Findable, Accessible, Interoperable, Reusable; see https://go.nature.com/2kmDGUT) with data principles embraced in the earliest stages of research leading to data, though this has not yet impacted mathematics in any meaningful way. Many institutions are following suit, embracing openness on behalf of their faculty and students, setting up institutional repositories, and developing open access policies. Yet, for many academics and academic societies, open access publishing is not their first priority.

The AMS understands and supports the ideal of openness, recognizing it is fundamental to research culture. Openness should really be considered in the context of the culture of the academic discipline. In mathematics, many assume that almost all research output appears on the preprint server arXiv and that mathematics is by nature an open field. As a mathematician you want to be free to do your research, perhaps write a paper, with versions of your paper appearing on arXiv, culminating with publication in a high-ranking journal. As a researcher you want simple and straightforward article submission and a communicative, thoughtful publisher shepherding your work to publication. The complexities of how to pay for publishing and where you may be allowed to publish, or not, are not central to your life as a mathematician. Yet those complexities exist.

At the AMS we see the drive to openness as both a threat and an opportunity. In a truly open world, how do we balance the costs of publishing when publications are essentially free to read and open to all? A quarter of AMS journal authors are funded by federal funding. But where will authors without such funding find the money to pay for article processing charges? Gold open access works for well-funded fields but not for fields where research is not grant driven. Indeed, the AMS publishes two gold open access journals (Proceedings of the American Mathematical Society B and Transactions of the American Mathematical Society B), allowing an author of an article accepted to Proceedings of the AMS or Transactions of the AMS to pay a small article processing charge and be published open...
access with an appropriate Creative Commons License (https://bit.ly/2bNifVP), but only a handful of authors choose this option. Yet the AMS is keen to embrace openness—so how do we do it and still provide our mathematical community services? It is important to keep in mind that we provide the entire mathematical community with a range of services including employment and educational initiatives, government relations representing the AMS in science policy discussions at the federal level, research prizes, student travel grants, summer courses such as the Mathematics Research Communities (MRCs), and so on. Seventy percent of the AMS operating budget comes from sales of its publications, including books, journals, and MathSciNet®. Were this revenue to evaporate, the AMS would suffer, as would the wider mathematical community we serve.

The mathematics community is asking: How can we thrive in an open publishing world? We know that mathematicians still love books and still love books in print. We know that AMS journals are among the best. We know that MathSciNet is regarded as the authoritative discovery database for mathematical research. Adding to complexity is that funders and academic institutions in Europe are taking a starkly different political view on openness in research and publishing than those in the US. While the ideal of openness is the same, the mechanisms are different, with Europe moving to mandate that their grantees publish in journals that specifically meet gold open access requirements set by funders—but there, researchers have access to funding to support the publication of their articles. This approach is exemplified by the rise of Plan S. While I will not delve into the details of Plan S here, it is worth investigating further if you have the time and energy to explore this European initiative. In the US, the approach is very different. Funders are mandated by government to provide open access, a form of open access that involves provision of the article in its final form by the publisher in repositories or at the publisher’s site after an agreed-upon embargo period, usually twelve months. The US approach as led by the OSTP is not to burden the author with complexity and cost for managing a gold route to openness. I do encourage you to delve further into these issues by reading our freely available AMS Primer on Open Access (https://bit.ly/2JAmdme).

The AMS knows that its books, journals, and MathSciNet database are needed and relied upon around the world. The AMS knows that it must preserve its ability to bring in revenue from publishing. The AMS also knows that you cannot compare the business of an independent nonprofit society to a giant commercial publisher able to strike massive deals with government consortia. A recent example of this may be seen in Springer Nature’s “Transformative Publish and Read” arrangement with Projekt DEAL (a consortium of more than 700 publicly and privately funded academic and research organizations in Germany; see https://bit.ly/2msUZow) to make more than 13,000 articles by German scholars and scientists published open access. Overall, Projekt DEAL will pay a per-article fee of €2,750 on behalf of authors for publication in Springer’s hybrid journals along with a 20 percent discount on the list price for publication in gold open access titles. Faculty, students, and staff at more than 700 institutions across Germany will receive access to the entire Springer journal portfolio, though not the Nature journals. The scale of this is huge and not something an independent society publisher can match.

For the AMS, faced with uncertainty, and a community that has not wholeheartedly embraced open access publishing, the approach needs to be balanced. We believe that we can embrace the ideal of openness in parts of the AMS publishing program while still recognizing that there are significant costs involved, to be recouped through subscriptions or other sources of revenue. The AMS will experiment but do so cautiously, recognizing that our community of mathematicians is our primary concern. In the end, we are committed to publishing the best books and journals and above all not burdening our authors and readers with complexity and cost.
Subscription prices for Volume 67 (2020) are US$720 list; US$576 institutional member; US$432 individual member; US$648 corporate member. (The subscription price for members is included in the annual dues.) A late charge of 10% of the subscription price will be imposed upon orders received from non-members after January 1 of the subscription year. Add for postage: Domestic—US$6; International—US$40; expedited delivery to destinations in North America—US$35; elsewhere—US$120. Subscriptions and orders for AMS publications should be addressed to the American Mathematical Society, PO Box 845904, Boston, MA 02284-5904 USA. All orders must be prepaid.

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A case of mistaken identity

Della Dumbaugh’s entertaining and thought-provoking piece on George Mackey in the June/July Notices contains an amusing instance of mistaken identity. On page 886, it is asserted that “[Griffith] Evans joined the Rice faculty in 1912 and remained there until 1934, bringing remarkably talented mathematicians including Benoit Mandelbrot, Tibor Rado, and Carl [sic] Menger as visiting professors.” In point of fact, it was not Benoit Mandelbrot, but his uncle Szolem Mandelbrojt (1899–1983), who spent the academic year 1926/27 (when Benoit was two years old) at Rice.

A student of Sergei Bernstein in Kharkov and of Hadamard in Paris, Mandelbrojt was among the founding members of Bourbaki. He would go on to an illustrious career as Hadamard’s successor at the Collège de France (1938–1972) and, ultimately, a member of the French Académie des Sciences. His doctoral students include, among others, Shmuel Agmon, J.-P. Kahane, Paul Malliavin, and Yitzhak Katznelson. Szolem Mandelbrojt was also on the original Conseil de Rédaction (Advisory Board) of Journal d’Analyse Mathématique, along with Einstein (!) and such other worthies as Borel, Denjoy, Hadamard, Littlewood, Montel, and Weyl.

Lawrence Zalcman,
Jerusalem, Israel

(Received October 7, 2019)

Fair Research Data, A New Development

I would like to bring the topic of FAIR research data to your attention.

In many European countries, e.g. Germany, there is a strong movement that all research data should be freely available according to the FAIR principles (findable, accessible, interoperable and reusable). See https://libereurope.eu/wp-content/uploads/2017/12/LIBER-FAIR-Data.pdf

In principle, FAIR is a laudable goal that will improve the openness of science. Of course, this is a major challenge for scientists who produce massive data, e.g. from numerical simulations, or physical experiments. In addition, the way data and standards are defined also poses some serious challenges for mathematical
research as a whole, by including non-traditional forms of “data,” such as mathematical formulas and theorems.

How and in which form can we standardize the way to find mathematical formulas or mathematical theorems, when different communities use different terminology for the same objects while the same formulas for different objects?

The German Research Foundation DFG has just started a large call for building research data infrastructures to deal with this, see e.g. https://www.dfg.de/en/service/press/press_releases/2018/press_release_no_58/index.html.

Most people in the mathematical community seem to ignore these developments, but this may lead to real threats for the community if we do not join the movement right from the beginning. Examples of such threats could be that standards will be fixed that are incompatible with our current way of producing mathematical articles (in LATEX) and PDF, or that the way formulas are stored is just graphically. Another problem may be that standards for model generation, mathematical software, or simulation data are cumbersome or impractical. It is clear that commercial code providers are heavily lobbying with governments to make standards that are good for them and that IT companies and data analytics people have their own views of how data should be addressed.

The mathematical community must unite in a common quest to be on board right away in the developments (the German math community has already decided to do this) and to make these principles realistic for mathematics and the neighbouring sciences and to preserve and improve established publishing standards like to be able to deal with the future developments. This may require also the construction of new and uniform concepts, such as semantic annotation of formulas or theorems.

Volker Mehrmann,
President of the European Mathematical Society
(Received September 26, 2019)

From the Academics for Peace
I am writing to you to give the latest news about the Academics for Peace in Turkey, and also thank the AMS for its support.

After penalizing about 200 Peace Academics, finally courts in Istanbul started to give acquittals. I was acquitted in early September, and the remaining four mathematicians (Tuna Altinel, Feza Arslan, Kıvanç Ersoy and Özgür Martin) were acquitted later in September or October. Many Peace Academics are being acquitted every day thanks to the Constitutional Court’s decision of July 26.

Those 200 academics (including mathematicians Öznur Yaşar Diner and Özlem Beyarslan) who had already been penalized by the courts will object to their sentences and demand new trials. It is expected that they will all be acquitted too.

We believe that the Constitutional Court’s decision is no accident and was planned by the top officials. We think that the administrators could not handle the international pressure and degrading reputation. So, it is thanks to international solidarity, thanks to the mathematical associations, but especially thanks to the Human Rights Committee of the AMS, that we are being let free from this legal nightmare. By publishing statements about me, Betül Tanbay, Tuna Altinel, and other mathematicians on trial, the AMS made the biggest public impact. Thank you very much.

Not only did the AMS support lead to acquittals, but it has been a great psychological help to know that we have friends and colleagues out there who are thinking about us, who are ready to support us. Thanks to my colleagues, I understand and have experienced at first hand what solidarity really means.

Thank you very much once more. I feel extremely lucky that I am a mathematician, and I am proud to be a member of the American Mathematical Society.

Best wishes in solidarity,
Aysê Berkman

Note: The AMS statements about Turkish mathematicians can be found at the following links.
https://www.ams.org/news?news_id=4893

(Received October 16, 2019)
Some Recent Trends in Motivic Homotopy Theory

Marc Levine

Introduction

Since its inception in the 1990s by Morel and Voevodsky, and Voevodsky’s application to the proofs of the Milnor conjecture and the Bloch–Kato conjecture, \( \mathbb{A}^1 \)-homotopy theory or, as it is also known, motivic homotopy theory has experienced an explosive development. I had planned in this survey to give an overview of progress and vistas in motivic homotopy theory as it has developed in the past ten years or so. I was both dismayed and heartened to find that such an undertaking in the limited space available was quite impossible: there has simply been too much going on to report on in this format. I was therefore compelled to select a few of the many areas I had planned to discuss for inclusion in this article. For this reason, I have not written anything at all about such topics as the recent applications of the various Postnikov-like towers in motivic homotopy theory; the Calmés–Fasel theory of Chow–Witt motives and its relation to the motivic sphere spectrum; the beautiful work of Ananyevskiy–Druzhinin–Garkusha–Panin–Neshitov–Sosnilo and Elmanto–Hoyois–Khan–Sosnilo–Yakerson on motivic recognition principles; the study of the tensor triangulated geometry of the motivic stable homotopy category by Andrews, Gheorghe, Heller, Hornbostel, Joachimi, Kelly, and Thornton; connections with equivariant homotopy theory by Hu, Heller, Kriz, Ormsby, and others; the computations of unstable homotopy groups with applications to splitting vector bundles by Asok and Fasel; or Ayoub’s work on motives for integrable connections. Some of these topics are covered in my memorial article for Vladimir Voevodsky in the Bulletin; others may be found in the lovely survey article [7] of Isaksen–Østvær, as well as many other articles and workshop notes. Nonetheless, I hope that this perhaps idiosyncratic selection of mine will still be of interest to those non-experts who would like to get a taste of a few of the many directions in which motivic homotopy theory is moving today. Finally, I would like to thank the AMS for giving me the opportunity to present this material to the mathematical community and the referees for their very helpful comments and suggestions.

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What Is Motivic Homotopy Theory?

In a nutshell, motivic homotopy theory is a particular homotopy theory of algebraic varieties. Motivic homotopy is distinguished from other homotopical approaches to algebraic geometry, which rely on constructions such as the homotopy theory of complex algebraic varieties or the étale homotopy theory of schemes, by being essentially internal. That is, motivic homotopy theory does not rely on giving individual algebraic varieties a topology (or Grothendieck topology) and constructing thereby a homotopy type in the usual homotopy theory of topological spaces, but builds new categories of a homotopical nature that include algebraic varieties themselves as objects and in some sense are generated by these objects. Additionally, the affine line \( \mathbb{A}^1 \) plays a privileged role as being a version of the classical interval, so one can mix classical notions of homotopy or homotopy equivalence with notions of \( \mathbb{A}^1 \)-homotopy and \( \mathbb{A}^1 \)-homotopy equivalence, which in turn may be viewed as refinements of the classical notion of rational equivalence of algebraic cycles.

We begin with a brief sketch of the main players. Classical homotopy theory begins with choosing a suitable category of spaces, \( \text{Spc} \), for example CW complexes or, for the more combinatorially minded, simplicial sets (we will stick with this latter choice here). Given a small category \( \mathcal{C} \), one can enlarge \( \text{Spc} \) to the functor category \( \text{Spc}(\mathcal{C}) := \text{Fun}(\mathcal{C}^{\text{op}}, \text{Spc}) \), that is, presheaves of spaces on \( \mathcal{C} \). Viewing the category of sets as the category of discrete spaces or constant simplicial sets, the Yoneda embedding \( \mathcal{C} \hookrightarrow \text{Fun}(\mathcal{C}^{\text{op}}, \text{Sets}) \) gives an embedding \( \mathcal{C} \hookrightarrow \text{Spc}(\mathcal{C}) \), with \( X \in \mathcal{C} \) going to the functor \( Y \mapsto \text{Hom}_\mathcal{C}(Y, X) =: X(Y) \). We also have the embedding \( \text{Spc} \hookrightarrow \text{Spc}(\mathcal{C}) \) sending a space \( K \) to the constant presheaf on \( \mathcal{C} \) with value \( K \). Thus \( \text{Spc}(\mathcal{C}) \) gives a framework for combining \( \mathcal{C} \) with \( \text{Spc} \).

As a functor category, \( \text{Spc}(\mathcal{C}) \) inherits many good properties of \( \text{Spc} \), for example, products and coproducts, limits and colimits, all computed sectionwise. As \( \text{Spc} \) is generated by the set of \( n \)-simplices \( \{ \Delta^n \mid n = 0, 1, \ldots \} \), this gives us the set of generators \( \{ X \times \Delta^n \mid X \in \mathcal{C}, n = 0, 1, \ldots \} \) for \( \text{Spc}(\mathcal{C}) \). One has a similar construction of pointed spaces on \( \mathcal{C} \), \( \text{Spc}_{\mathbb{S}}(\mathcal{C}) \), by taking presheaves of pointed spaces.

For a presheaf \( X \in \text{Spc}(\mathcal{C}) \), one has the associated presheaf of sets \( \pi_0(X) \), \( \pi_0(X)(x) = \pi_0(X(x)) \), and for a pointed presheaf \( X \in \text{Spc}_{\mathbb{S}}(\mathcal{C}) \) the presheaves of groups \( \pi_n(X) \) (for \( n \geq 1 \), abelian for \( n \geq 2 \)). This gives the notion of a weak homotopy equivalence of pointed presheaves, namely, a map \( f : X \to Y \) that induces an isomorphism on \( \pi_n \) for all \( n \) (in the unpointed case, one needs to make sense of the notion of “choice of basepoint,” but this can be done). This gives us the unstable homotopy category \( \mathcal{H}^{\mathbb{S}}(\mathcal{C}) \) and the pointed version \( \mathcal{H}^{\mathbb{S}}(\mathcal{C})_{\mathbb{S}} \) by formally inverting the respective weak equivalences. For \( \mathcal{C} \) the one-point category, we have \( \text{Spc}(\mathcal{C}) = \text{Spc} \), \( \mathcal{H}^{\mathbb{S}}(\mathcal{C}) \) is the classical unstable homotopy category \( \mathcal{H} \), and similarly for the pointed versions.

There are technical problems with this that we mention briefly: as the categories \( \text{Spc}(\mathcal{C}) \) and \( \text{Spc}_{\mathbb{S}}(\mathcal{C}) \) are not small (they contain after all \( \text{Sets} \) and \( \text{Sets}_{\mathbb{S}} \)), it is not clear that the localizations \( \text{Spc}(\mathcal{C})[\text{WE}^{-1}] \) and \( \text{Spc}_{\mathbb{S}}(\mathcal{C})[\text{WE}^{-1}] \) exist. Also, we would really like to have a “homotopy theory” rather than just a homotopy category. Without saying exactly what we mean by this, we want to be able to generalize the standard constructions of classical homotopy theory, such as homotopy fibers and cofibers, in this presheaf setting. There are a number of solutions to these problems; the first really successful one is based on Quillen’s theory of model categories. We already have the notion of a weak equivalence in \( \text{Spc}(\mathcal{C}) \); to have a model category structure one adds that of a fibration and cofibration, all satisfying suitable axioms. One has an even nicer structure, that of a simplicial model category, which enriches the category \( \text{Spc}(\mathcal{C}) \) in spaces by defining the mapping space as the simplicial set

\[
\text{Maps}_{\text{Spc}(\mathcal{C})}(X, Y) = [n \mapsto \text{Hom}_{\text{Spc}(\mathcal{C})}(X \times \Delta^n, Y)].
\]

In any case, the model category structure solves the existence problem for the homotopy categories and gives the added structures needed to define a reasonable homotopy theory of presheaves of spaces.

The motivic story starts by applying this construction in the case \( \mathcal{C} = \mathbb{S} \mathbb{M} / S \), where \( S \) is a chosen base-scheme and \( \mathbb{S} \mathbb{M} / S \) is the category of smooth finite type \( S \)-schemes. This gives us the category of spaces over \( S \), \( \text{Spc}(S) := \text{Spc}(\mathbb{S} \mathbb{M} / S) \), and the similarly defined category of pointed spaces over \( S \), \( \text{Spc}_{\mathbb{S}}(S) \), together with their homotopy categories \( \mathcal{H}^{\mathbb{S}}(S) \), \( \mathcal{H}^{\mathbb{S}}(S)_{\mathbb{S}} \).

We have decorated the homotopy categories because these are not the categories we are ultimately looking for. In order to have some reasonable descent properties, we need to incorporate a suitable Grothendieck topology \( \tau \) and we also need to make the affine line an interval-object. This is accomplished by enlarging the weak equivalences to include maps that induce isomorphisms on the associated homotopy sheaves (rather than the presheaves) and the projection maps \( \mathbb{A}^1 \to \mathbb{X} \). Again invoking the techniques of model categories, this leads to a good notion of \( \mathbb{A}^1 \tau \)-weak equivalence \( WE_{\mathbb{A}^1 \tau} \), homotopy categories \( \mathcal{H}^{\mathbb{A}^1 \tau}(S) := \text{Spc}(S)[W_{E_{\mathbb{A}^1 \tau}}^{-1}] \), \( \mathcal{H}^{\mathbb{A}^1 \tau}_{\mathbb{S}}(S) := \text{Spc}_{\mathbb{S}}(S)[W_{E_{\mathbb{A}^1 \tau}}^{-1}] \), and associated homotopy theories. This was carried out in the seminal work of Morel–Voevodsky [17]. Recent innovations give corresponding constructions on the level of infinity categories.

The standard choice for the Grothendieck topology \( \tau \) is the Nisnevich topology, where a cover of a scheme \( X \) is an étale cover \( U \to X \) that is surjective on \( L \)-valued points.
for all fields $L$. We write $\mathcal{H}(S) = \mathcal{H}^{\text{et},Nis}(S), \mathcal{H}_.(S) = \mathcal{H}^{\text{et},Nis}(S)$. The étale topology is also useful, but we will mainly discuss the Nisnevich version.

In classical topology, one is also interested in stable phenomena, that is, information that is preserved by suspension with respect to the circle $S^1$. This stability property is built into the very definition of generalized (co)homology via the suspension axiom. One can also see suspension arising via Spanier–Whitehead duality, in that the Spanier–Whitehead dual involves a negative suspension, so only defines a true duality after inverting the suspension operator with respect to $S^1$. In fact, the Spanier–Whitehead dual of a smooth compact manifold $M$ is the Thom space of its virtual normal bundle $M^\nu = \text{Th}(-T_M)$, with $T_M$ the tangent bundle. To make some sense of this, one can find a vector bundle $ν_M$ and an isomorphism $ν_M \oplus T_M \cong M \times \mathbb{R}^n$ for some large $n$. The Thom space of a vector bundle $V \to M$ can be defined as the disk bundle modulo the sphere bundle, so $\text{Th}(\mathbb{R}^n) = S^n$. The isomorphism $ν_M \oplus T_M \cong M \times \mathbb{R}^n$ translates into the "identity"

$$\text{Th}(ν_M) = \text{Th}(\mathbb{R}^n) \cong \text{Th}(\mathbb{R}^n) \cong S^n \cong \text{Th}(T_M) \cong \text{Th}(S^n).$$

In other words, if we had an inverse to the suspension operator $S^1$, we could define $\text{Th}(T_M) := S^n \text{Th}(S^n)$. One inverts suspension by passing to a category of $S^1$-suspension spectra, where an object is a sequence $(E_0, E_1, \ldots)$ of pointed spaces together with bonding maps $ε_n : E_n \wedge S^1 \to E_{n+1}$, and then introducing a suitable notion of stable weak equivalence (see below). The $S^1$-suspension is then realized as shifting the sequence to the left, which has, as a stable inverse, the shift to the right (this replaces $E_0$ with the one-point space, but this new spectrum is stably equivalent to our original one).

In the motivic setting one has an analog of Spanier–Whitehead duality, where instead of inverting suspension with respect to $S^1$, one needs to invert suspension with respect to $\mathbb{P}^1$. To explain how this $\mathbb{P}^1$ suspension arises, there is a purely algebraic definition of the Thom space of an algebraic vector bundle $V \to X$ by setting $\text{Th}(V) := P(V \oplus O_X)/P(V)$, where $P(V) \to X$ is the bundle of projective spaces associated to the vector bundle $V \to X$. Classical excision shows that one can also use this formula to define the Thom space in the topological setting. Morally speaking, one has the same problem in the motivic setting as in the classical one: one needs to invert the operator of smash product with $\text{Th}(\mathbb{A}^1_\mathbb{Z}) = P^\circ_B/P^B_{B-1}$ to be able to define the dual of a smooth projective $B$-scheme $X$ as $\text{Th}(-T_{X/B})$. For $n = 1$, instead of $\text{Th}(\mathbb{R}^1) = S^1$ we get $\text{Th}(\mathbb{A}^1_\mathbb{Z}) = P^1$ (pointed by $\infty$), and one can rather easily show that $\text{Th}(\mathbb{A}^1_\mathbb{Z}) = (P^1)^{1/n}$ in $\mathcal{H}_.(B)$. This leads to the category of $\mathbb{P}^1$-spectra, where a $\mathbb{P}^1$-spectrum is a sequence $E = (E_0, E_1, \ldots)$ with $E_n \in \mathcal{Spc}_\mathbb{Z}(S)$, together with bonding maps $ε_n : E_n \wedge \mathbb{P}^1 \to E_{n+1}$, where we consider $\mathbb{P}^1$ as pointed by $\infty$. This gives us the category of $\mathbb{P}^1$-spectra over our base-scheme $S$, $\text{Sp}_{\mathbb{P}^1}(S)$.

To define a suitable notion of stable equivalence in the topological setting, one uses the suspension map $- \wedge S^1 : π_n(X) = [S^n, X]_{S^n} \to [S^{n+1} \wedge S^1, X]_{S^n} = π_{n+1}(X \wedge S^1)$, giving the stable homotopy groups $π_n(E)$ for $E = (E_0, E_1, \ldots)$; a spectrum:

$$π_n(E) := \text{colim}_n π_{n+N}(E_N), \quad n \in \mathbb{Z},$$

using the suspension map and bonding map $ε_n : E_n \wedge S^1 \to E_{n+1}$ to define the colimit. A map $f : E \to F$ of suspension spectra is a stable weak equivalence if $f$ induces an isomorphism on $π_n$ for all $n$ and the stable homotopy category is formed from the category of suspension spectra by inverting the stable weak equivalences.

In the motivic setting, there is no nice relation of $π_*(X)$ and $π_*(X \wedge \mathbb{P}^1)$, but one does have the Tate circle $G_m := (\mathbb{A}^1 \setminus \{0\}, \{1\}) \in \mathcal{Spc}(B)$ and the isomorphism in $\mathcal{H}_.(B)$$\mathbb{P}^1 \cong S^1 \wedge G_m$. This suggests introducing the bi-graded $\mathbb{A}^1$-homotopy sheaf $π_{a,b}^1(X)$ associated to the presheaf

$$U \in \mathcal{Smp/B} \mapsto [U \wedge S^{a-b} \wedge G_m^b, X]_{\mathcal{Spc}(S)}$$

for $a \geq b \geq 0$. $\mathbb{P}^1$-suspension induces the map

$$(-) \wedge \mathbb{P}^1 : π_{a,b}^1(X) \to π_{a+2,b+1}^1(X \wedge \mathbb{P}^1),$$

and thus for a $\mathbb{P}^1$-spectrum $E = (E_0, E_1, \ldots)$, we can use $ε_n : E_n \to E_{n+1} \wedge \mathbb{P}^1$ to define

$$π_{a,b}^1(E) := \text{colim}_n π_{a+n, b+n}^1(E_N)$$

for all $a, b \in \mathbb{Z}$. A map $f : E \to F$ of $\mathbb{P}^1$-spectra is then defined to be a motivic stable weak equivalence if $f$ induces an isomorphism on $π_{a,b}^1$ for all $a, b \in \mathbb{Z}$. Inverting the motivic stable weak equivalence, one forms the motivic stable homotopy category $SH(S) := \text{Sp}_{\mathbb{P}^1}(S)[W^{-1}]$. As for spaces, this localization should be put in the setting of stable model categories or stable infinity categories to have a well-defined homotopy category and a stable homotopy theory.

**Some Useful Motivic Cohomology Theories**

The classical stable homotopy category $SH$ is the category of generalized cohomology theories, where a spectrum $E$ gives rise to the cohomology theory (on spaces $X$) $E^n(X) := [\Sigma^n X, E]_{SH}$. Here $\Sigma^n X$ is the infinite suspension spectrum $(X, ΣX, Σ^2X, \ldots)$ with identity bonding maps $ε_n$. The invertibility of $Σ_{p_1}$ on $SH(S)$ gives the two-parameter family of suspensions $Σ_{a,b}$ corresponding to $Σ^{a-b}G_m$ for $a \geq b \geq 0$. A $\mathbb{P}^1$-spectrum $E$ thus gives rise to bi-graded cohomology on $\mathcal{Smp}/S$ or even on $\mathcal{Spc}_\mathbb{Z}(S)$ by

$$E^{a,b}(X) = [\Sigma^a X, \Sigma^{b}E]_{SH(S)}.$$
Here we introduce a few of the most important examples of motivic cohomology theories with their classical counterparts.

One of the first results in motivic homotopy theory was the representability of algebraic $K$-theory. In topology, stable, virtual vector bundles are represented by $\mathbb{Z} \times BU$, $BU$ being the classifying space of the infinite unitary group. Bott periodicity extends this to a 2-periodic spectrum $KU := (\mathbb{Z} \times BU, \Sigma \mathbb{Z} \times BU, Z \times BU, ...)$ representing topological $K$-theory.

The unstable representability of algebraic $K$-theory was proven by Morel–Voevodsky (over a regular base-scheme) [17, Theorem 4.3.13]. One can take as model for the classifying scheme $BGL/S$ the doubly infinite Grassmannian

$$Gr_S = \text{colim}_{n,m} Gr(n, n + m).$$

Morel–Voevodsky first show that $\mathbb{Z} \times BGL/S$ represents the Grothendieck group of algebraic vector bundles $X \to K_0(X)$ on $\mathbf{Sm}/S$ via an isomorphism

$$[X, \mathbb{Z} \times BGL/S]_{Gr(S)} \cong K_0(X)$$

that extends to higher $K$-theory using the isomorphism

$$K_n(X) \cong K_0(X \times \Delta^n; X \times \partial \Delta^n)$$

for $X$ regular and affine. Here $\Delta^n$ is the algebraic version of the $n$-simplex

$$\Delta^n = \text{Spec } \mathbb{Z}[t_0, ..., t_n]/(\sum_{i=0}^n t_i - 1),$$

and $\partial \Delta^n$ is the union of faces $t_i = 0, i = 0, ..., n$. Using the contractibility of $\Delta^1$ in $H(S)$, they show that

$$[\mathbb{A}^n \times X, BGL/S]_{Gr(S)} \cong K_0(X \times \Delta^n; X \times \partial \Delta^n) \cong K_n(X),$$

establishing the representability of algebraic $K$-theory by $\mathbb{Z} \times BGL/S$, in case of regular $S$.

This extends to a stable representability. Replacing Bott periodicity is an algebraic version: let $V_n \to BGL_n = Gr(n, \infty)$ be the universal $n$-plane bundle and let $O(-1) \to P^1 = Gr(1, 2)$ be the tautological line bundle. This gives the tensor product bundle $p_1^* V_n \otimes p_2^* O(-1) \to BGL_n \times P^1$.

Applying the unstable representability to the virtual bundle $[p_1^* V_n \otimes p_2^* O(-1)] - [p_1^* V_n] - [p_2^* O(-1)] + [O_{BGL_n \times P^1}]$ gives maps $\gamma_n : BGL_n/S \times P^1 \to BGL/S$, compatible in $n$, which induce $\gamma : BGL/S \times P^1 \to BGL/S$, defining the $P^1$-spectrum $KGL := (\mathbb{Z} \times BGL, \mathbb{Z} \times BGL, ...).$ This is explained in Voevodsky’s ICM talk [18].

Another important cohomology theory represented in $SH(S)$ is motivic cohomology, here for $S$ smooth over a field $k$. In the case of $S = \text{Spec } k, \text{char } k = 0$, the motivic cohomology spectrum $M\mathbb{Z}$ is represented by infinite symmetrized powers, relying on a motivic analog of the Dold–Thom theorem, which for $T$ a connected pointed CW complex gives the identity $\pi_n(\text{Sym}^\infty T) = H_n(T, \mathbb{Z})$.

The Dold–Thom theorem implies that the classical Eilenberg–MacLane spectrum $HZ$ can be constructed as $HZ = (\text{Sym}^\infty S^0, \text{Sym}^\infty S^1, ..., \text{Sym}^\infty S^n, ...)$; $HZ$ represents singular cohomology on $\text{Spc}$: $HZ^n(T) = H^n(T, \mathbb{Z})$.

Replacing $S^1$ with $P^1$, one has the motivic version

$$M\mathbb{Z} := (\text{Sym}^\infty S_+, \text{Sym}^\infty P^1, ..., \text{Sym}^\infty (P^1)^n, ...).$$

For $S = \text{Spec } k, k$ a perfect field, and $X \in \mathbf{Sm}/k$, one has Voevodsky’s motivic cohomology $H^n(X, \mathbb{Z}(b))$ defined using his triangulated category of motives $DM(k)$. In characteristic $p$, this theory is represented by $M\mathbb{Z} : H^n(X, \mathbb{Z}(b)) = M\mathbb{Z}^{a, n}(X)$ (see [18]). In characteristic $p$, this still holds after inverting $p$ or by modifying the definition of $M\mathbb{Z}$ suitably. There are extensions of $DM(-)$ to more general bases, and the results for $k$ extend to $S$ smooth over a perfect field (see for example [4]).

Both of these theories represent constructions that were available before the introduction of the motivic stable homotopy category: Quillen’s higher algebraic $K$-theory goes back to the early 1970s, and motivic cohomology is represented by Bloch’s higher Chow groups, first defined in the mid-1980s, while the idea that a good theory of motivic (co)homology could be achieved by a suitable algebraic translation of the Dold–Thom theorem goes back to Suslin and his construction of Suslin homology, introduced shortly after Bloch constructed his higher Chow groups. Voevodsky [18] introduced a completely new theory, algebraic cobordism, by defining the $P^1$ spectrum $\text{MGL}$ modeled closely on the Thom spectrum $\text{MU} = (\text{MU}_0, \text{MU}_1, ...),$. Recall that $\text{MU}_{2n}$ is the Thom space $\text{Th}(V_{2n})$, where $V_{2n} \to BU(n)$ is the universal rank $n$ complex vector bundle over the classifying space of the unitary group $BU(n)$. To define $\text{MGL}$, one simply replaces the classifying space $BU(n)$ and its universal bundle $V_{2n} \to BU(n)$ with the algebraic version $E_{2n} \to BGL_n$ and the Thom space $\text{MU}_{2n} := \text{Th}(V_{2n})$ with the algebraic Thom space $\text{MGL}_{2n} := \text{Th}(E_{2n})$.

Completely analogous to the topological setting, the fact that the $\text{MGL}_n$ fit together to form a $P^1$ spectrum follows from the identity $\text{Th}(E \oplus O_X) \cong \text{Th}(E) \wedge P^1$ for $E \to X$ a vector bundle and the fact that the restriction of $E_{2n+1}$ to $BGL_n$ is $E_{2n} \boxtimes O_{BGL_n}$.

This opened the door to constructions of a whole slew of new cohomology theories for algebraic varieties, many of which are modeled on classical topological theories, others being strikingly new. One now has available motivic versions of all the Morava $K$-theories, cobordism theories $\text{MSL}$ and $\text{MSp}$, modeled on the special linear groups and the symplectic groups, and Eilenberg–MacLane-type theories $\text{EM}(M_p)$ built out of Morel’s homotopy modules (see [11, §5.2]). There are also many (in fact infinitely many) different ways of constructing families of connective covers of a given theory. We will discuss some of these in more
The theories $MZ$, KGL, and MGL are oriented theories, which mean they admit Thom isomorphisms
\[ \theta_V : \mathcal{E}^{a,b}(X) \sim \mathcal{E}^{a+2r,b+r}(\text{Th}(V)) \]
for each rank $r$ vector bundle $V \to X$. This motivic notion is an algebraic analog of the topological theory of complex oriented spectra. In the topological setting, giving a spectrum $E$ a complex orientation gives rise to pushforward maps in $E$-cohomology for proper maps endowed with a complex structure on the stable normal bundle, for instance, for any proper complex analytic map of complex manifolds. In the algebraic setting, an oriented spectrum $\mathcal{E}$ gives rise to pushforward maps in $\mathcal{E}$-cohomology for proper maps of smooth schemes.

Of course, a rank $r$ vector bundle is just a Zariski locally trivial fiber bundle with fiber $\mathbb{A}^r$ and group $GL_r$. One can look at "structured" vector bundles, where the extra structure is given by restricting the structure group. For instance, a rank $r$ SL-vector bundle is a vector bundle with group $SL_r$, a rank $2r$ symplectic bundle is a vector bundle with group $Sp_{2r} \subset GL_{2r}$, and so on. One can describe these structures more intrinsically: an SL-oriented vector bundle $V \to X$ together with an isomorphism $\det V \sim O_X$, and a symplectic vector bundle is a vector bundle together with a symplectic form, i.e., a section $\omega$ of $\Lambda^2 V^\vee$ that is non-degenerate on each fiber.

Each of these structures gives rise to a notion of orientation. An SL-oriented theory is a commutative ring spectrum $\mathcal{E} \in SH(k)$ together with Thom isomorphisms
\[ \theta_{V,\rho} : \mathcal{E}^{*,*}(X) \to \mathcal{E}^{2r+*,*}(\text{Th}(V)) \]
for each rank $r$ vector bundle $V \to X$ with trivialized determinant bundle $\rho : \det V \to O_X$. These must satisfy a few natural axioms. An Sp-oriented theory $\mathcal{E} \in SH(k)$ together with Thom isomorphisms
\[ \theta_{V,\omega} : \mathcal{E}^{*,*}(X) \to \mathcal{E}^{2r+*,*}(\text{Th}(V)) \]
for each rank $r$ symplectic vector bundle $V \to X$ with symplectic form $\omega \in H^0(X, \Lambda^2 V^\vee)$. The corresponding cobordism theories are $MSL$ and $MSp$. For the sake of uniformity of notation, we sometimes refer to an oriented theory as a GL-oriented theory.

An important example of an SL-oriented theory is hermitian $K$-theory. This may be constructed as a version of Quillen $K$-theory, where one works in a category with an exact category with duality. This gives us the presheaf of spectra on $\text{Sm}/k$, $X \mapsto K(X)$, which is represented by a $\mathbb{P}^1$-spectrum $KQ$ (see [13]). Just as $KGL_{2r,r}(X)$ is the "geometric" part $K_0(X)$ of $K$-theory, the classical Grothendieck group of vector bundles, $KQ_{2r,r}(X)$, has a description as a Grothendieck–Witt group of quadratic forms. More precisely, one has the notion of a quadric form in the derived category of perfect complexes on $X$, namely, a map $q : C^* \otimes_{O_X}^L C^* \to O_X$ in $D_{\text{perf}}(X)$, which is symmetric with respect to the symmetry isomorphism $\tau : C^* \otimes_{O_X}^L C^* \to C^* \otimes_{O_X}^L C^*$, and which is non-degenerate, in that the adjoint map to $q$, $C^* \to \text{Hom}(C^*, O_X)$, is an isomorphism in $D_{\text{perf}}(X)$. One can also introduce a shifted duality by using quadratic forms with values in $O_X[r]$ for some $r$, and this, modulo a suitable derived isometry relation, gives the group $KQ_{2r,r}(X)$.

**Fundamental Invariants**

In many ways, the most fundamental motivic theory is the one represented by the unit, the motivic sphere spectrum $S_k := \bigoplus_S \text{Spec } k$. This is the motivic analog of the classical sphere spectrum $S := \bigoplus S^0$, which is the unit in $SH$.

First a toy model, the derived category. For a commutative ring $R$, much of the basic structure of the derived category of $R$-modules, $D(R)$, follows from the structure of the (graded) endomorphism ring of the unit $R$:
\[ \text{Hom}_{D(R)}(R, R[n]) = \begin{cases} R & \text{for } n = 0, \\ 0 & \text{else.} \end{cases} \]

For the classical stable homotopy category, things are much more mysterious: $\text{Hom}_{SH}(S, S^nS)$ is exactly the $n$th stable homotopy group $\pi_n(S)$. These have been intensively studied, but only the first fifty or so have been completely computed.

At least one does know that $\pi_n(S) = 0$ for $n < 0$, a reflection of the fact that $\pi_n(S^m) = 0$ for $0 < m < n$, and that $\pi_n(S) = \mathbb{Z}$, since $\pi_n(S^n) = H_\ast(S^n, \mathbb{Z}) = \mathbb{Z}$, this following from the connectivity of $S^n$ and the Hurewicz theorem. Additionally, Serre’s finiteness theorem and the Freudenthal suspension theorem imply that $\pi_n(S)$ is a finite group for all $n > 0$.

In the motivic setting life is already made more complicated by having a two-variable family of stable homotopy groups, $\pi_{a,b}(S_k)(k) : \text{Hom}_{SH(k)}(\Sigma^a b S_k, S_k)$ and homotopy sheaves $\pi_{a,b}(S_k)$ on $\text{Sm}/k$.

Morel’s homotopy $t$-structure [11, §5.2] leads one to reindex these as $\pi_{n}(S_k)_m := \pi_{n-m}(S_k)$ and consider $\pi_n(S_k)_m := \bigoplus_m \pi_{n-m}(S_k)_m$ as the motivic analog of $\pi_n(S)$.

For instance, Morel’s stable $\mathbb{A}^1$-connectivity theorem [11, Theorem 4.2.10, Lemma 4.3.11] implies that $\pi_n(S_k)_0 = 0$ for $n < 0$. A beautiful analog of the identity $\pi_0(S) = \mathbb{Z}$ is Morel’s computation of the motivic 0-stem [11, Theorem 6.4.1]
\[ \pi_0(S_k)_0(k) = K^M_{MW}(k). \]

Here $K^M_{MW}(k)$ is the $n$th Milnor–Witt $K$-group of the field $k$, defined via explicit generators and relations by Hopkins and Morel [11, §6.3].
The unstable algebraic Hopf map is the map \( \eta : \mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^1 \) defining \( \mathbb{P}^1 \) as the quotient of \( \mathbb{A}^2 \setminus \{0\} \) modulo the \( \mathbb{G}_m \)-action, that is, \( \eta((x, y)) = [x : y] \). One has isomorphisms in \( \mathcal{H}(k) \), \( (\mathbb{A}^2 \setminus \{0\}, (1, 0)) \cong S^1 \wedge \mathbb{G}_m \wedge \mathbb{G}_m \), and \( \mathbb{P}^1 \cong S^1 \wedge \mathbb{G}_m \), so stably \( \eta \) gives the class \( \eta \in \pi_0(S_k)_{-1}(1) \) and, by Morel’s theorem, the corresponding element \( \eta \in K^{MW}_1(k) \).

The Milnor K-theory of \( k \), \( K^M(k) \), is simply the tensor algebra on the group of units \( k^\times \), modulo the ideal generated by \( \{a \otimes (1 - a) \mid a \in k \setminus \{0, 1\} \} \). There is a surjective ring homomorphism \( K^MW(k) \to K^M(k) \) with kernel the two-sided ideal generated by \( \eta \). In fact, it follows from the very definition that \( K^MW(k) \) is generated as a graded ring by \( \eta \in K^{MW}_1(k) \) and elements \( [u] \in K^{MW}_1(k) \) for \( u \in k^\times \), corresponding via Morel’s theorem to \( u \) viewed as a map \( Spec\ k_+ \to G_m \), with the following relations [11, Definition 6.3.1]:

1. \( \eta \cdot [u] = [u] \cdot \eta \) for all \( u \in k^\times \).
2. \( [u] \cdot [1 - u] = 0 \) for \( u \in k \setminus \{0, 1\} \).
3. \( [uv] = [u] + [v] + \eta([u][v]) \) for \( u, v \in k^\times \).
4. \( \eta \cdot (2 + \eta - 1) = 0 \).

The map \( K^MW(k) \to K^M(k) \) is defined by sending \( \eta \) to zero and \( [u] \) to the class \( [u] \) of \( u \).

The assignment of a field \( F \) to the Milnor–Witt ring \( K^MW(F) \) or the Milnor K-theory ring \( K^M(F) \) both extend to Nisnevich sheaves \( K^MW, K^M \) on \( Sm/k \), and Morel’s isomorphism extends to an isomorphism of sheaves

\[ \pi_0(S_k)_* \cong \pi_* K^M. \]

The Milnor–Witt ring thus combines the Milnor K-groups \( K^M(k) \) with information having to do with quadratic forms. \( K^MW \) is exactly the Grothendieck–Witt ring \( GW(k) \) of nondegenerate symmetric bilinear forms over \( k \), and for \( n < 0 \), \( K^MW_n(k) \) is the Witt ring, that is, \( GW(k) \) modulo the hyperbolic form. Explicitly, the rank one form \( q_u(x) = ux^2 \), \( u \in k^\times \), gets sent to \( (u) := 1 + \eta\bar{u} \in K^MW_1(k) \). The relation (3) translates into the multiplicativity \( (uv) = (u) \cdot (v) \), which the relation \( q_u q_v = q_{uv} \) in \( GW(k) \) requires. The hyperbolic form \( h(x, y) = x^2 - y^2 \) gets sent to \( 2 + \eta(-1) \) and is thus \( \eta^* h = 0 \) in \( K^MW_1(k) \) for all \( n > 0 \). This shows that the map \( GW(k) \to K^MW(k) \) descends to \( W(k) \to K^MW(k) \) for all \( n > 0 \). Morel shows that all these maps are isomorphisms [11, Lemmas 6.3.8 and 6.3.9].

For \( n > 0 \), there is an exact sequence

\[ 0 \to \Gamma^{n+1} \to K^MW_n(k) \to K^M(k) \to 0, \]

where \( I \subset GW(k) \) is the augmentation ideal of quadratic forms of virtual rank zero [11, Theorem 6.4.5]. All this extends to the sheaf setting without change; in particular, \( K^MW_0 Guatemala \) is the sheaf \( GW \) of Grothendieck–Witt rings on \( Sm/k \).

One can try to understand the sheaves \( \pi_n(S_k)_* \) via unit maps to various motivic ring spectra. For the oriented theories \( E = MZ, KGL, MGL \), \( \eta \) acts as zero, \( \pi_0(E)_* = K^M \), and the respective unit map induces the surjection \( K^M \to K^M \). As a first attempt, consider the unit map \( u_{KGL} : S_k \to KGL \). For \( * = 0 \), this is simply the rank map \( GW \to \mathbb{Z} \), so these theories do not carry any of the quadratic forms information in \( K^M \).

Hermitian K-theory gives a better approximation to the sphere spectrum than \( MZ, KGL \), or MGL. In fact the unit map \( S_k \to KQ \) induces an isomorphism on \( \pi_0(-)_* \) for \( * \leq 3 \) by work of Suslin and Asok–Fasel. This is perhaps not so surprising, at least for \( \pi_0(S_k) \) for \( GW = KQ \), the first identity from Morel’s theorem and the second more or less by construction. In her 2019 doctoral thesis, Maria Yakerson [20] considers \( \pi_0(MSL)_* \) and shows by a direct computation that this is also \( K^M \).

Relying heavily on properties of \( KQ \), Röndigs–Spitzweck–Östvær [15] compute the motivic stable 1-stem. Here is the result; for simplicity we give the statement only in characteristic zero, although suitably modified, it remains true in arbitrary characteristic \( \neq 2 \).

**Theorem 1** [15, Theorem 5.5]. Let \( k \) be a field characteristic 0. The unit map \( S_k \to KQ \) induces a short exact sequence of Nisnevich sheaves

\[ 0 \to \mathcal{K}^{M}_{2+n/24} \to \pi_1(S_k)_n \to \pi_1(f_0 KQ)_n. \]

The right-hand map is surjective for \( n \leq 4 \). In addition, \( \pi_1(f_0 KQ)_n = 0 \) for \( n \leq -2 \); in particular \( \pi_1(S_k)_{-2} \cong \mathbb{Z}/24 \) and \( \pi_1(S_k)_{n} = 0 \) for \( n < -2 \).

Here \( f_0 \) is the so-called effective cover of \( KQ \). This involves Voevodsky’s slice filtration, a motivic analog of the classical Moore–Postnikov tower. The effective cover \( f_0 KQ \) to \( KQ \) induces an isomorphism on \( \pi_1(-) \) for all \( n \leq 0 \), so in that range, one can replace \( f_0 \) with hermitian K-theory \( KQ \) itself. We refer the reader to the excellent survey article [7] as well as the original paper [15] for details on the theorem, the slice filtration, and many other interesting topics.

A remarkable and perhaps unexpected application of the motivic theory has been to give a new and quite effective tool for computing classical stable homotopy groups of spheres. The (topological) Adams and Adams–Novikov spectral sequences promote higher multiplicative structures in singular cohomology or complex cobordism to yield computations of the stable homotopy groups of spheres. Computations of the differentials are often very difficult.

There are motivic analogs of these spectral sequences, using the motivic cohomology spectrum or the algebraic cobordism spectrum as motivic replacements for singular cohomology and complex cobordism. The motivic versions involve an additional grading, the Tate twist, which tends to split the classical differentials into a number of summands. This additional information is often very helpful in resolving ambiguities and determining the
differentials in the classical spectral sequence. Using this approach in a number of papers, Isaksen, Xu, and Wang (see for example [6], [19]) have corrected earlier computations and have begun pushing beyond; their paper on the 61st stem has the beautiful geometric consequence: the 61-sphere has a unique smooth structure, and it is the last odd-dimensional case. Pushing up to $\pi_{126}$ could resolve the last remaining case of the Kervaire invariant one conjecture. We refer the reader to [7] for a much more complete survey of this fascinating topic, as well as many other results on the motivic homotopy groups of spheres.

**Plus and Minus**

One fascinating aspect of motivic homotopy theory is how it unifies two different homotopical worlds via complex and real embeddings. An algebraic variety $X$ over $\mathbb{R}$ determines simultaneously two topological spaces, namely, the space of $\mathbb{C}$-points $X(\mathbb{C})$ and the space of $\mathbb{R}$-points $X(\mathbb{R})$, with the topology induced by $\mathbb{C}$, $\mathbb{R}$, respectively. As a simple example, the algebraic projective line $\mathbb{P}^1$ has $\mathbb{P}^1(\mathbb{C}) \cong S^2$ and $\mathbb{P}^1(\mathbb{R}) \cong S^1$. This dual nature extends to realization functors $\text{Re}_{\mathbb{C}} : \text{SH}(\mathbb{C}) \to \text{SH}$, $\text{Re}_{\mathbb{R}} : \text{SH}(\mathbb{R}) \to \text{SH}$. Morel has introduced the heuristic principle, that a conjecture in $\text{SH}(\mathbb{R})$ can be tested by seeing if it remains true in $\text{SH}(\mathbb{C})$.

As an elementary but fundamental example, consider the symmetry isomorphism $\tau : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$, exchanging the factors; clearly $\tau^2 = \text{Id}$. Under the complex realization, this is $S^4 = S^2 \times S^2 \to S^2 \times S^2 = S^4$, which is homotopic to the identity map, while under the real realization, this is $S^2 = S^1 \times S^1 \to S^1 \times S^1 = S^2$, which is homotopic to minus the identity map. Thus, the map $\tau$ gives a way of distinguishing between $\text{Re}_{\mathbb{C}}$ and $\text{Re}_{\mathbb{R}}$, even after passing to the stable setting. We can use the additive nature of $\text{SH}(k)$ to form the two idempotent endomorphisms of the sphere spectrum $S_k$ (after inverting 2).

$$\tau_+ := \frac{1 + \tau}{2}, \quad \tau_- := \frac{1 - \tau}{2}.$$

This splits $S_k$ in $\text{SH}(k)[1/2]$ as

$$S_k = S_k^+ \oplus S_k^-,$$

with $\tau_+$ acting as the identity on $S_k^+$ and as zero on $S_k^-$, and with $\tau_-$ having the opposite behavior. Since $S_k$ is the unit for the monoidal structure on $\text{SH}(k)$, this splitting of $S_k$ into plus and minus factors splits the whole category $\text{SH}(k)[1/2]$ as the product of two tensor triangulated categories

$$\text{SH}(k)[1/2] = \text{SH}(k)_+ \times \text{SH}(k)_-.$$

The fact that $\tau$ induces $\text{Id}$ under $\text{Re}_{\mathbb{C}}$ and $-\text{Id}$ under $\text{Re}_{\mathbb{R}}$ says that $\text{Re}_{\mathbb{C}}$ factors through the projection to $\text{SH}(k)_+$ and $\text{Re}_{\mathbb{R}}$ factors through the projection to $\text{SH}(k)_-$ after inverting 2.

The two factors $\text{SH}(k)_+$ and $\text{SH}(k)_-$ behave quite differently. One aspect of this is the relation with Voevodsky’s triangulated category of motives, $\text{DM}(k)$. There is a motivic Eilenberg–MacLane functor $\text{EM}_{\text{mot}} : \text{DM}(k) \to \text{SH}(k)$ that formally mimics the classical one $\text{EM} : \text{D}(\text{Ab}) \to \text{SH}$. The fact that the higher stable homotopy groups of the sphere spectrum $S$ in $\text{SH}$ are all torsion implies that the classical Eilenberg–MacLane functor is an equivalence after $\mathbb{Q}$-localization. As the symmetry operator $\tau : \mathbb{Z}(1) \otimes \mathbb{Z}(1) \otimes \mathbb{Z}(1)$ is the identity, $\text{EM}_{\text{mot}}$ factors through $\text{SH}(k)_+$ (after inverting 2). Thus, for fields $k$ such that $\text{SH}(k)_-$ survives $\mathbb{Q}$-localization, it cannot be the case that $\text{EM}_{\text{mot}}$ is a $\mathbb{Q}$-equivalence. On the other hand, a result of Cisinski–Degrèse [4, Theorems 16.1.4 and 16.2.13] tells us that $\text{EM}_{\text{mot}} : \text{DM}(k)_{\mathbb{Q}} \to \text{SH}(k)_{+\mathbb{Q}}$ is an equivalence, so $\text{EM}_{\text{mot}} : \text{DM}(k)_{\mathbb{Q}} \to \text{SH}(k)_{\mathbb{Q}}$ is an equivalence exactly when $\text{SH}(k)_{-\mathbb{Q}} = 0$.

For which fields $k$ does $\text{SH}(k)_-$ survive $\mathbb{Q}$-localization? Again, the answer to this lives in $\text{End}(S_k) = \text{GW}(k)$. The unit in $\text{DM}(k)$ is the weight-zero Tate motive $\mathbb{Z}(0)$, and it is not hard to show that $$\text{End}_{\text{DM}(k)}(\mathbb{Z}(0)) = \mathbb{Z} \cdot \text{Id}_{\mathbb{Z}(0)}.$$ Now, $\text{SH}(k)_-$ survives $\mathbb{Q}$-localization if and only if $\text{End}_{\text{SH}(k)}(S_k)_{\mathbb{Q}} \neq \{0\}$. Also, as the map $\text{End}_{\text{SH}(k)}(S_k) \to \text{End}_{\text{DM}(k)}(\mathbb{Z}(0))$ induced by the adjoint to $\text{EM}_{\text{mot}}$ is the rank homomorphism $\text{GW}(k) \to \mathbb{Z}$, we see that $\text{SH}(k)_-$ survives $\mathbb{Q}$-localization if and only if the augmentation ideal $I := \text{ker}(\text{rnk}) \subset \text{GW}(k)$ is nontorsion. This property of $k$ has been studied classically, and the answer is: $I$ is not a torsion group if and only if $k$ admits an ordering or, equivalently, $-1$ is not a sum of squares in $k$. In particular, $\text{SH}(k)_-$ survives $\mathbb{Q}$-localization for any field $k$ that admits an embedding into $\mathbb{R}$.

To summarize: the motivic stable homotopy category $\text{SH}(k)$ splits into two pieces after inverting 2, the plus part and the minus part. Rationally, the plus part is the same as Voevodsky’s triangulated category of motives, a motivic reflection of the fact that the classical stable homotopy category is rationally the same as the derived category of abelian groups. The minus part survives $\mathbb{Q}$-localization exactly when $-1$ is not a sum of squares in $k$, for example, if $k$ admits an embedding into $\mathbb{R}$.

This suggests that the mysterious minus part could be approached via the real realization for each embedding $k \hookrightarrow \mathbb{R}$. As a simple example of Morel’s heuristic, look back at our friend the involution $\tau : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$. This induces an endomorphism of $S_k$, giving us an element $\lfloor \tau \rfloor$ of $\text{GW}(k)$. What element is this? We know under complex realization $\mathbb{P}^1$ becomes an $S^2$, so under $\text{GW}(k) \to \text{GW}(\mathbb{C}) = \mathbb{Z}$ (which is the rank map) $\lfloor \tau \rfloor$ maps to 1. One can show that the map $\text{GW}(k) \to \text{End}_{\text{SH}}(S) = \mathbb{Z}$ under real realization is the signature map $\text{GW}(k) \to \text{GW}(\mathbb{R}) \xrightarrow{\text{sig}} \mathbb{Z}$. As $\mathbb{P}^1$...
Another way to get at the mysterious minus part of verified the above guess for $GW(\mathbb{Z}) =$ additional 2-torsion element), this is not too bad, but still $GW(\mathbb{Z}[1/2])$ out of reach, but perhaps not as far as the complete computation of $GW(\mathbb{Z})$. The best we can say is that the image of $GW(\mathbb{Z}) \to GW(\mathbb{R})$ there are in fact quite fundamental problems that stand out inverting 2. This suggests that $SH(k)[\eta^{-1}, 1/2] \cong SH(k)_\eta$, which is in fact the case. This identity follows from a basic relation in $GW(k)$.

As mentioned above, Morel has shown that $[\tau] = (-1) = 1 + \eta[-1]$. Splitting $SH(k)[1/2]$ into plus and minus parts corresponds to the two idempotents $\tau_+ := (1 + [\tau])/2$, $\tau_- := (1 - [\tau])/2$, with $\tau_+$ acting by the identity on $SH(k)_\eta$ and by zero on $SH(k)_{\eta^-}$. As $\tau_- = (-1/2)\eta[-1]$, both $\eta$ and $[-1]$ are invertible on $SH(k)$. Similarly, as $\tau_+ = 1d + (1/2)\eta[-1], \eta[-1] = 0$ on $SH(k)_\eta$. In fact, $SH(k)_{\eta^{-1}} = 0$.

As we have already seen, $\eta^n \cdot K^W(k) = K^W_n(k) = W(k)$ for all $n > 0$, and thus

$$K^W_n(k)[\eta^{-1}] = \operatorname{colim}_{n \to \infty} K^W_n(k) = W(k).$$

The augmentation ideal $I \subset GW(k)$ is also an ideal in $W(k)$, and $W(k)/I = \mathbb{Z}/2$; and as the projection onto the plus part gives the rank map $GW(k)[1/2] \to \mathbb{Z}/[2]$, we see that the projection of $\operatorname{End}_{GW(k)}(\eta)[\eta^{-1}, 1/2]$ to $\operatorname{End}_{GW(k)}(\mathbb{Z})[\eta^{-1}, 1/2]$ is the zero map, and thus $SH(k)_{\eta^{-1}} = 0$.

After inverting 2, inverting $[-1]$ has exactly the same effect as inverting $\eta$. Let $\rho : S^0_k \to G_m$ be the map corresponding to $-1 \in G_m(k)$. We also use $\rho : S^0_k \to G_m \wedge S^1_k$ for the stable version, corresponding to $[-1] \in K^W_1(k)$. Then $SH(k)_{\rho^{-1}} = SH(k)_\eta$ and $SH(k)_\rho[\rho^{-1}] = 0$. We have seen the interpretation of $\eta$ as the classical Hopf map via the complex realization and as multiplication by 2 on $S^1$ via the real realization. We can do the same for the map $\rho$. The complex realization of $G_m$ is $\mathbb{C} \approx S^1$, and $\operatorname{Re}_c([-1])$ is the inclusion of $S^0$ to $S^1$ as $\{\pm 1\}$. As this map is clearly homotopic to 0, inverting $\operatorname{Re}_c(\rho)$ will kill the entire stable homotopy category. The real realization of $\rho$ is $S^0 \approx \mathbb{R}^*$, again being the inclusion of $\{\pm 1\}$ into $\mathbb{R}^*$, but this time the map is a homotopy equivalence, even without inverting 2. This suggests that $SH(k)[\rho^{-1}]$ should be closely related to the classical stable homotopy category via the real realization.

The $\rho$-inverted Motivic Stable Homotopy Category

Bachmann has proven a much more precise statement about the $\rho$-localization of $SH(k)$. We first give the simple version.

**Theorem 2** (Bachmann [3]). The real realization $\operatorname{Re}_r : \operatorname{SH}(\mathbb{R}) \to \operatorname{SH}$ induces an equivalence $\operatorname{SH}(\mathbb{R})[\rho^{-1}] \to \operatorname{SH}$.

This is just a special case. In general, Bachmann considers the motivic stable homotopy category over a base scheme $X$. One has the topological space of "real" points of $X$, $\mathcal{R}(X)$. Here, the points of $\mathcal{R}(X)$ are pairs $(x, \alpha)$ consisting of a point $x$ of $X$ and an ordering $\alpha$ on the residue field $\kappa(x)$. The topology is formed by incorporating the
orderings and the Zariski topology on $X$. As any topological space, $\mathcal{R}(X)$ has a homotopy theory of sheaves of simplicial sets, giving rise to a stable homotopy category $\text{SH}(\mathcal{R}(X))$. Bachmann’s general theorem is

**Theorem 3.** There is an equivalence of $\text{SH}(X)[\rho^{-1}]$ with $\text{SH}(\mathcal{R}(X))$.

For $X = \text{Spec} \mathbb{R}$, there is a unique ordering on the residue field $\mathbb{R}$ of the unique point of $X$, so $\text{SH}(\mathcal{R}(\text{Spec} \mathbb{R}))$ is just the usual stable homotopy category $\text{SH}(\mathbb{R})$, giving back the special case Theorem 2.

In particular, Bachmann’s theorem gives a canonical object in $\text{SH}(\mathbb{R})[\rho^{-1}]$ with real realization $\text{MU}$. Recognizing the important role that complex cobordism plays in understanding the structure of $\text{SH}$, it would be interesting to see to what extent one can lift this object to $\text{SH}(\mathbb{Q})$ and how unique such a lifting would be. For lifting to $\text{SH}(\mathbb{R})$, an obvious candidate would be the cobordism spectrum built out of the restriction of scalars $R_{\mathbb{C}/\mathbb{R}} \text{GL}_n$, $n = 0, 1, 2, ...$, but the general picture is unclear.

As we have seen, $\text{SH}(\mathbb{Q})[\rho^{-1}]$ and $\text{SH}(\mathbb{R})[\eta^{-1}]$ both agree with $\text{SH}(\mathbb{Q})_-$ after inverting 2, so Bachmann’s theorem gives a workable description of $\text{SH}(\mathbb{Q})_-$, with some quite beautiful consequences. One of these is a new proof of a result of Ananyevskiy–Levine–Panin [1], an analog of Serre’s theorem on the finiteness of residues field theorem on the finiteness of $\text{SH}(\mathbb{F})$ for all $\mathbb{F}$. Theorem 3.

**Theorem 4.** The homotopy sheaves $\pi_i(S_k)$ are torsion for all $i \neq 0$.

Combining the Cisinski–Dégidé result identifying $\text{DM}(\mathbb{Q})_{\eta}$ with $\text{SH}(\mathbb{Q})_{+\eta}$, Morel’s $\mathbb{A}^1$-connectivity theorem ($\pi_i(S_k)_\eta = 0$ for $i < 0$), and Morel’s theorem $\pi_0(S_k)_\eta = K^{\text{MW}}$, we thus have a complete description of $\pi_i(S_k)_\eta$ in terms of the Witt sheaves $\mathcal{W}$ and motivic cohomology sheaves $\mathcal{H}^{p,q}$.

One sees the complexity of motivic homotopy theory as compared to the classical case, as well as an interesting parallel, in the following tables:

<table>
<thead>
<tr>
<th>$\pi_i(S)$</th>
<th>$\pi_i(S_k)_\eta$</th>
<th>$\pi_0(S_k)_\eta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>$K^{\text{MW}}$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>$\mathcal{H}^{n-i,n}_Q$</td>
</tr>
<tr>
<td>$\mathcal{W}$</td>
<td>$\mathcal{H}^{n-i,n}_Q$</td>
<td>0</td>
</tr>
</tbody>
</table>

We think of $\mathcal{H}^{n-i,n}$ as the “cohomology of a point,” we see another way in which the motivic theory differs from the classical one: a point (that is, the spectrum of a field) can have nontrivial higher cohomology in the motivic case.

To finish off the story, we note that $\mathcal{H}^{p,q}_Q = 0$ for $q < 0$, $\mathcal{H}^{p,0} = 0$ for $p \neq 0$, and $\mathcal{H}^{0,0} = \mathcal{W}$. The Beilinson–Soulé conjectures assert that the sheaf $\mathcal{H}^{p,q}_Q$ is zero for $p \leq 0$ and $q > 0$, but this is known only for $q = 1$. Thus, in addition to Morel’s vanishing ($\pi_i(S_k)_\eta = 0$ for $i < 0$), we have $\pi_i(S)_{\eta Q} = 0$ for $i > 0$ and $n \leq 0$, and the Beilinson–Soulé conjectures imply the additional vanishing $\pi_i(S)_{\eta Q} = 0$ for $n > 0$ and $i \geq n$.

**Cohomology Theories, Orientations, and Characteristic Classes**

Let $\mathcal{E}$ be a $G$-oriented theory ($G = GL, SL, SP$) and let $V \to X$ be a $G$-vector bundle of rank $r$. Via the Thom isomorphism $\mathcal{E}^{a,b}(X) \to \mathcal{E}^{2r+a,r+b}(\text{Th}(V))$, the unit $1_X \in \mathcal{E}^{0,0}(X)$ gives the Thom class $\text{th}(V) \in \mathcal{E}^{2r,r}(\text{Th}(V))$. The Euler class $e(V) \in \mathcal{E}^{2r,r}(X)$ is the pullback of $\text{th}(V)$ by the zero-section. Starting with the Euler class, there are often ways of constructing other quite useful characteristic classes for vector bundles.

The theory of Chern classes lies at the very roots of our modern theory of motives. Grothendieck showed how to make Chern’s topological theory algebraic, yielding a theory of Chern classes of algebraic vector bundles with values in the Chow ring. His approach relies on the existence of a first Chern class for line bundles and the projective bundle formula, which for a vector bundle $E \to X$ expresses the Chow ring of a projective space bundle $P(E) \to X$ as a free module over the Chow ring of $X$, with generator powers of the first Chern class of the tautological line bundle. In fact, Grothendieck worked in the abstract setting, using a contravariant functor $A^*$ from smooth varieties to graded rings, satisfying certain axioms. A version of these axioms is satisfied (suitably modified for bi-graded theories) for the functors $X \mapsto \mathcal{E}^*(X)$, where $\mathcal{E}^*$ is an oriented commutative ring spectrum in the sense indicated in “Some Useful Motivic Cohomology Theories,” and are all compatible via the complex realization functor with the corresponding topological theory of characteristic classes for $\text{Re}_c(\mathcal{E})$. The Euler class construction is recovered as the top Chern class, in particular, for a line bundle $L$, $e(L) = c_1(L)$, so one can use the Euler class as the starting point for the whole construction.

For example, one has a motivic theory of Conner–Floyd Chern classes for $\text{MGL}$, which under $\mathcal{C}$-realization maps to the classical Conner–Floyd Chern classes for $\text{MU}$. In fact, once one has set up the machinery, the motivic setting for oriented theories is formally exactly the same as the topological setting for complex oriented theories. This is made more precise by the universality of $\text{MGL}$ as a motivic-oriented theory, proved by Panin–Pimenov–Röndigs [12], exactly parallel to the universality of $\text{MU}$ among complex oriented theories, and by the fact that the coefficient ring $\text{MGL}^{2n,*}(k)$ is equal to the coefficient ring of $\text{MU}$, namely, the Lazard ring, classifying rank one commutative formal group laws. This fundamental result was first stated by Hopkins–Morel and was given a complete proof by Hoyois [5]. In positive characteristics, one needs to invert the characteristic for the proof, although it is conjectured that the result remains true integrally.
Complex cobordism and the theory of formal group laws give rise to some of the most basic theorems on the structure of the stable homotopy category, for instance, the nilpotence theorem of Devinatz–Hopkins–Smith. One might expect that via the close connection of MGL with $MU$, one could lift these results to $SH(k)$. However, the algebraic Hopf map $\eta$ acts as the zero map on $MGL^{**}$, so $MGL^{**}$ dies after inverting $\eta$, and one cannot use the theory of oriented spectra to say anything much about $SH(k)[1/\eta]$ or $SH(k)$ itself. However, theories with $SL$ or $Sp$ orientations do not automatically die after inverting $\eta$, and these will thus be useful for understanding $SH(k)$.

Panin–Walter [14] have considered the symplectically oriented theories as motivic analogs of the classical quaternionic theories. They have constructed a motivic analog of quaternionic projective space as the symplectic Grassmanian $\mathbb{H}P^n$, this being the open subscheme of the usual Grassmann variety of 2-planes in $2n+2$-space, $Gr(2, 2n+2)$, of those 2-planes for which the standard symplectic form $\omega_{2n+2}$ on $\Lambda^{2n+2}$ is nondegenerate, and thus the restriction $E_2 \rightarrow \mathbb{H}P^n$ of the tautological 2-plane bundle on $Gr(2, 2n + 2)$ is a symplectic vector bundle.

This generalizes to the setting of a rank $2n + 2$ symplectic bundle $(V \rightarrow X, \omega_V)$: one has the Grassmann bundle $Gr(2, V) \rightarrow X$ and the open subbundle $\mathbb{H}P(V) \rightarrow X$ of 2-planes for which $\omega_V$ restricts to a nondegenerate form. Just as for $\mathbb{H}P^n$, we have restriction $E_{2V} \rightarrow \mathbb{H}P(V)$ of the tautological 2-plane bundle, which gives us the Borel class $b_1(E_{2V})$, defined as the Euler class $e(E_{2V}) \in \mathcal{E}^{4-2}(\mathbb{H}P(V))$. The main result of Panin–Walter is that $\mathcal{E}^{**}(\mathbb{H}P(V))$ is a free $\mathcal{E}^{**}(X)$-module with basis the powers $b_1(E_{2V}), \ldots, b_{2n}(E_{2V})^n$.

They can then apply a suitably modified version of the Grothendieck approach to define the higher Borel classes $b_i(V) \in \mathcal{E}^{4i}(X)$, $i = 1, \ldots, n$.

This is fine for symplectic bundles, but what about an arbitrary vector bundle $V \rightarrow X$ without a symplectic structure? Here one has an analogy with the Pontryagin classes of real topological vector bundles. The Pontryagin classes $p_i(V)$ for a real vector bundle $V \rightarrow B$ are defined by taking the even Chern classes of the complexification of $V$: $p_i(V) = (-1)^i c_{2i}(V^\mathbb{C}) \in H^{4i}(B, \mathbb{Z})$. The odd Chern classes $c_{2i+1}(V^\mathbb{C})$ are all 2-torsion, so the $p_i(V)$ have better properties, such as satisfying a Whitney formula, after inverting 2.

In the motivic world, we symplectify a vector bundle $V \rightarrow X$ by forming $V \oplus V^\vee$ with the canonical symplectic form $\omega(L, v) = L(v)$, $\omega(v, u') = 0 = \omega(L, L')$ for $v, u'$ sections of $V, L, L'$ sections of the dual bundle $V^\vee$. We then define $p_i(V) := (-1)^i b_{2i}((V \oplus V^\vee, \omega)) \in \mathcal{E}_{2i}^{4i}(X)$.

Just as the topological Pontryagin classes are better behaved after inverting 2, the motivic Pontryagin classes are better behaved after inverting $\eta$, since the odd Borel classes are $\eta$-torsion. We also note that as the symplectic form for a rank $2n$ symplectic vector bundle $(V \rightarrow X, \omega)$ defines an isomorphism $\omega^n : det V \rightarrow O_X$, every symplectic vector bundle is an $SL$-vector bundle, and thus an $SL$-oriented theory is also $Sp$-oriented. We therefore have a good theory of Pontryagin classes for arbitrary vector bundles with values in a given $SL$-oriented theory $E$ for which $\eta$ is already invertible.

We do have a wealth of theories that do not die after inverting $\eta$, ranging in complexity from the Eilenberg–MacLane spectrum $EM(X_{M^W})$ to hermitian K-theory $KQ$, special linear cobordism $MSL$, or symplectic cobordism $MSp$. In topology, all the trouble is at the prime 2, and as the real realization of $\eta$ is multiplication by 2, it seems reasonable to look first at these more complicated theories after inverting $\eta$.

The isomorphism of $X_{M^W}[1/\eta]$ with the sheaf of Witt rings $\mathcal{W}$ leads to the identification $EM(X_{M^W})([1/\eta]) \cong EM(\mathcal{W})$, where $\mathcal{W}$ is the homotopy module $W_0 = W$. We also have the simple description of $EM(\mathcal{W})$-theory and $\mathcal{W}$-cohomology: $EM(\mathcal{W}) = H^a-b(X, \mathcal{W})$. As $W(\mathcal{R}) = \mathbb{Z}$, one can view $H^a(-, \mathcal{W})$ as a good analog of singular cohomology $H^a(-, \mathbb{Z}[1/2])$ via the real realization. In fact, it follows from a theorem of Panin–Walter that in $SH(k)$, $EM(X_{M^W})([1/\eta])$ represents usual singular cohomology of the real points. Thus, as the real realization sends $\eta$ to 2, $H^a(X, \mathcal{W}[1/2]) = H^a(X(\mathcal{R}), \mathbb{Z}[1/2])$ for $X$ a smooth $\mathcal{R}$-scheme.

Also, the corresponding Pontryagin classes $p_i(V)$ for a vector bundle $V \rightarrow X$ live in $EM(\mathcal{W})^{4i}(X) = H^{4i}(X, \mathcal{W})$, another good analogy with the topological setting. I suspect that the motivic Pontryagin class $p_i(V) \in H^{4i}(X, \mathcal{W})$ goes over to the topological one $p_i(V(\mathcal{R})) \in H^{4i}(X(\mathcal{R}), \mathbb{Z}[1/2])$ under real realization, but as far as I know this has not been checked. In any case, the $\eta$-inverted theories in $SH(k)$ are useful tools in pursuing analogies between motivic homotopy theory and the classical setting via real realizations.

The theory of characteristic classes for theories has been extensively studied by Ananyevskiy [2]. Here are a few of his results. The first one says that for $SL$-vector bundles, the Pontryagin classes and (for even rank bundles) the Euler class generate all characteristic classes in an $\eta$-inverted, $SL$-oriented theory.

Theorem 5 (Ananyevskiy [2, Theorem 10]). Let $E \in SH(k)$ be an $\eta$-inverted, $SL$-oriented commutative ring spectrum. Then $E^{**}(BSL_n)$

$$
\left\{ \begin{array}{ll}
E^{**}(k)[p_1, \ldots, p_m, e]/(p_m - e^2) & \text{for } n = 2m \text{ even}, \\
E^{**}(k)[p_1, \ldots, p_m] & \text{for } n = 2m + 1 \text{ odd}.
\end{array} \right.
$$

Here $p_1 = p_1(E_n)$, where $E_n \rightarrow BSL_n$ is the universal vector bundle, and $e = e(E_n)$ is the Euler class.
As is well known, the projective bundle formula for Chern classes gives rise to the splitting principle, which says that in order to check a natural identity in Chern classes of vector bundles, it suffices to check the identity for Chern classes of direct sums of line bundles. This principle is not available for SL-bundles, as the only line bundle that is an SL-bundle is the trivial line bundle. The following theorem of Ananyevskiy is the next best thing.

**Theorem 6** (Ananyevskiy [2, Theorem 6]). Let \( E \in \text{SH}(k) \) be an \( \eta \)-inverted, SL-oriented commutative ring spectrum. Then the block diagonal embedding \((SL_2)^m \to SL_{2m}\) induces an injection

\[
E^*(SL_{2m}) \to E^*((SL_2)^m).
\]

Moreover, \( E^*((SL_2)^m) = E^*(k)[e_1, \ldots, e_m] \), where \( e_i \) is the pullback of the Euler class \( e(E) \) via the \( i \)th projection \((SL_2)^m \to SL_2\). Finally, the inclusion

\[
E^*(SL_{2m}) = E^*(k)[p_1, \ldots, p_m, e]/(p_m - e^2) \to E^*(k)[e_1, \ldots, e_m]
\]

identifies \( E^*(SL_{2m}) \) with the invariants of \( E^*(k)[e_1, \ldots, e_m] \) under the action of the semidirect product \( \{\pm 1\}^{m-1} \rtimes S_m \), which acts by sending \( (e_1, \ldots, e_m) \) to \((e_1\sigma(e_1), \ldots, e_m\sigma(e_m))\), where \( \sigma \in S_m \) is a permutation and \( e_i \in \{\pm 1\} \) with \( \prod e_i = 1 \). Explicitly,

\[
p(E_{2m}) := 1 + p_1(E_{2m}) + \cdots + p_m(E_{2m}) = \prod_{i=1}^{m} (1 + e_i^2), \quad e = \prod_{i=1}^{m} e_i.
\]

This reduction to rank 2 bundles still leaves open a basic question. Given an \( SL_n \)-vector bundle \( V \to X \) and a representation \( \rho : SL_n \to SL_N \) one has the associated \( SL_N \)-bundle \( V^\rho \to X \). What are the Pontryagin classes and Euler class of \( V^\rho \) in terms of those of \( V \)? For the Chern classes, this is answered by the classical splitting principle and the theory of symmetric functions. Ananyevskiy’s \( SL_2 \) splitting principle reduces one to understanding the characteristic classes of \( \text{Sym}^nE_2 \), with \( E_2 \to BSL_2 \) the tautological bundle.

The solution lies in a further splitting principle: reduction to the normalizer \( N \) of the maximal torus \( T \subset SL_2 \). \( T \) is isomorphic to \( G_m \) via

\[
t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix},
\]

and \( N \) is generated by \( T \) and the element \( \sigma \),

\[
\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

Based on a motivic version of the classical Becker–Gottlieb transfer, we show in [10] that for any \( E \in \text{SH}(k) \), the pullback via \( \pi : BN \to BSL_2 \) induces an injection

\[
\pi^* : E^*(BSL_2) \to E^*(BN).
\]

This is useful, as the irreducible representations of \( N \) are all one- or two-dimensional, so the bundle \( \pi^*\text{Sym}^nE_2 \) splits as a direct sum of bundles of rank at most two. One can write these bundles explicitly, and we have computed their Euler classes in Witt cohomology. This yields the following formula, inspired by an analogous formula of Okonek–Teleman for real topological bundles.

**Theorem 7** (Levine [10, Theorem 8.1]). Suppose \( k \) has characteristic 0 or characteristic \( p > n \). Then in \( H^{n+1}(BSL_2, W) \), we have

\[
e(\text{Sym}^nE_2) = \begin{cases} 0 & \text{for } n \text{ even}, \\ n! e(E_2)^{m+1} & \text{for } n = 2m + 1 \text{ odd}. \end{cases}
\]

Here \( n! = n(n - 2) \cdots 3 \cdot 1 \) for \( n \) odd. Moreover, the total Pontryagin class \( p(\text{Sym}^nE_2) \in H^*(BSL_2, W) \) is given by

\[
p(\text{Sym}^nE_2) = \prod_{i=0}^{[n/2]} (1 + (n - 2i)^2 e(E_2)^2).
\]

We note in [10, Example 8.2] that the above computation can be used to give a “count” of the lines on a general hypersurface of degree \( 2d - 1 \) in \( \mathbb{P}^{d+1} \) as

\[
C_d = n! (1 + \frac{1}{2} (N_d - n!)) h \in GW(k),
\]

where \( h \) is the hyperbolic form \((1) + (-1), N_d \) is the degree of the top Chern class \( c_2d(\text{Sym}^{2d+1}E_2^{2d+2}) \), and \( E_{2,d+2} \to \text{Gr}(2, d + 2) \) is the tautological bundle on the Grassmannian of 2-planes in \( d + 2 \)-space. The rank of \( C_d \) gives the classical count of the number of lines on a general hypersurface of degree \( 2d - 1 \) in \( \mathbb{P}^{d+1} \), while the signature of \( C_d \) has an interpretation of a signed count of real lines on a general real hypersurface of degree \( 2d - 1 \) in \( \mathbb{P}^{d+1} \), another example of how the \( \eta \)-invertible theories relate to topology over the reals.

The quadratic form \( C_d \) itself is constructed as the push-forward of the Euler class

\[
e(\text{Sym}^{2d-1}E_{2,d+2}) \in H^{2d}(\text{Gr}(2, d + 2), \mathbb{K}_0^{MW})
\]

to \( GW(k) = H^0(\text{Spec} k, \mathbb{K}_0^{MW}) \), that is, the quadratic degree of the Euler class. The case of lines on a cubic was treated earlier using different motivic methods by Kass–Wickelgren [9]. We are hopeful that pursuing these invariants in Witt cohomology and other motivic theories, one can enrich classical enumerative geometry by providing invariants that have interesting arithmetic content through the theory of quadratic forms.

In another direction, the computation of the Euler class of symmetric powers given by Theorem 7 is limited to the case of the classes in Witt cohomology. As \( H^*(-, W) \) is in some sense an analogy of classical singular cohomology, one would expect a more complicated behavior for the characteristic classes in other theories, such as KQ, MSL, or MSP, even after inverting \( \eta \). For oriented theories, the
degree of complexity is captured by the associated formal group law. Is there such an algebraic invariant of an SL-orientated, \(\eta\)-inverted theory?

Yet another question involves comparing the plus and minus parts of theories. For example, Witt cohomology is formed by inverting \(\eta\) on bi-graded Milnor–Witt cohomology \(H^*(-, \mathbb{A}^{MW})\). This latter theory (in appropriate degrees) specializes to the Chow ring by killing \(\eta\) and to Witt cohomology by inverting \(\eta\); one has similar behavior for the other theories such as \(KQ\), \(MSL\), and \(MSp\). Can one exploit this to find comparisons between behavior of characteristic classes or solutions of enumerative problems over \(\mathbb{C}\) and over \(\mathbb{R}\)? There are for instance conjectures of Itenberg–Kharlamov–Shustin comparing the growth of classical enumerative invariants with their real counterparts [8]. Can these questions be approached using motivic methods?

References


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Cryptography

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How to Keep Your Secrets in a Post-Quantum World

Kristin Lauter

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What Do We Mean by “Hard Math Problem”?

In practical applications of cryptography, we have a relatively well-agreed-upon meaning for the term “hard math problem”: if the input is represented by \( m \) bits, then the best-known attack on the system runs in

\[
\text{exponential time in } m, \text{ e.g., } O(2^m) \text{ time}
\]

or

\[
\text{subexponential time in } m, \text{ e.g., } L(\frac{1}{3}, c) = O(e^{c/(m(\log m)^3)}) \text{ time, where } c \text{ is a constant.}
\]

For example, to factor the number \( n = p \cdot q \) where \( m = \log n \), trial division takes \textit{exponential time}. This is with respect to classical algorithms, which are represented on today’s computers with circuits and with inputs and outputs given in terms of “classical” bits, i.e., sequences of 0s and 1s. \textit{Polynomial time} algorithms run in time that is a \textit{polynomial} in \( m \), which often means in practice that an attack based on a polynomial time algorithm will succeed in a realistic amount of time and render the cryptographic system insecure.

Cryptographic Standards

There is a complex process for deploying new cryptographic protocols, especially when based on new hardness assumptions in mathematics. First, the research community needs to reach consensus on the above described process of modeling and giving precise, concrete cost estimates for the best-known attacks that solve the underlying math problem. Second, detailed standards are created through community or government processes such as:

1. a government agency like NIST (National Institute of Standards and Technology) in the United States runs a multiyear, open, international competition, e.g., the block cipher competition that standardized AES or the hash function competition that standardized SHA-3;
2. a professional society such as IEEE (Institute of Electrical and Electronics Engineers) or IETF (Internet Engineering Task Force) convenes a working group or a committee to develop a draft standard, which is updated and revised over time, e.g., the IEEE P1363 that provided a foundational standard for elliptic curve cryptography (ECC);
3. a consortium consisting of researchers from a collection of interested parties in industry, government, and academia works together to publish a draft standard for reference, e.g., the PKCS standards governing the deployment of the RSA system or the new draft standard for Homomorphic Encryption HES 1.0 [HES 2018].

There can be substantial overlap in these first two stages of standardization. Once draft standards have been developed, there is a regulatory layer that is often developed requiring the deployment or adherence to various standards. Specialized standards are often developed for protocols to be used in vertical segments of the economy, such as when ANSI (American National Standards Institute) produced the X9.62 and X9.63 ECC standards for using elliptic curve key exchange and digital signature protocols in the financial services industry. Other examples of protocol-level standards include specifications for secure browser sessions (https: SSL/TLS); signed, encrypted email (S/MIME); virtual private networking (IPSec); and authentication (X.509 certificates).

Finally, there may be an ecosystem of third-party vendors that spring up to respond to the need to verify compliance with regulations. This is the current process for establishing public trust in the cryptographic systems we deploy. It is important that much of this process be public so that everyone can see that the systems were not cooked up in a back alley with some secret trapdoors or weaknesses built in.

The possibility of new, sometimes unexpected, attacks on fundamental cryptographic problems in mathematics, combined with the lengthy and complex standardization process, leaves us in a difficult and sometimes precarious position. Recent advances and substantial new investment in the development of quantum computers represent such a potential threat to our currently widely deployed public key cryptographic systems. This is due to the existence of a polynomial time quantum attack [Shor97] on practically all of our currently deployed public key cryptosystems, which will be feasible to implement once a quantum computer can be built at a large enough scale. In response, NIST has launched a new, multiyear process to standardize post-quantum cryptography (PQC): i.e., cryptographic systems based on hard math problems for which we do not currently know polynomial time quantum attacks. The NIST PQC competition was launched in November 2017, and the twenty-six submissions for key exchange and digital signatures that have advanced to the second round were announced in January 2019. Round 2 is expected to be a 12–18-month process. There may be a third round before NIST announces the post-quantum algorithms that will be recommended.

Pre-quantum (Classical) Systems

The NIST PQC selection process aims to identify candidates to supplement or replace three standards considered to be most vulnerable to a quantum attack: FIPS 186–4, which specifies how to use digital signatures, and NIST SP 800–56A and NIST SP 800–56B, which are specifications
for key exchange. These currently widely deployed systems are based on “classically” hard problems, for which we do not know any classical polynomial time algorithms: RSA, Diffie–Hellman, and ECC. The RSA cryptosystem for encryption was proposed in the 1970s and is based on the hardness of factoring large integers that are the product of two prime numbers of equal size. Diffie–Hellman key exchange is based on the hardness of solving the discrete logarithm problem in the multiplicative group of integers modulo a large prime number. Elliptic curve cryptosystems are based on the hardness of solving ECDLP, the discrete logarithm problem in the abelian group of points on an elliptic curve over a finite field. Although there is a rich and beautiful mathematical theory of elliptic curves, developed over the course of more than one hundred years by mathematicians, cryptographers often think of an elliptic curve as simply the set of solutions to an affine equation in a finite field $F_q$. In characteristic not equal to 2 or 3, this equation is given in short Weierstrass form:

$$E: y^2 = x^3 + ax + b,$$

where $a$ and $b$ are constants in the base field $F_q$. The set of affine solutions, along with a “point at infinity” that can be seen in the projective version of the equation, forms a group where the point at infinity is the identity element. The group law can be described with concrete rational functions and has been widely implemented in industry to enable cryptographic systems, starting with Windows Vista and OpenSSL in 2005.

For RSA and Diffie–Hellman systems, classical subexponential attacks are known: the number field sieve and the index calculus attack; see the Notices article by Pomerance [Po96] for the history. Current key sizes for ECC systems are much smaller than for RSA or Diffie–Hellman because there are no known subexponential classical attacks on ECDLP for generic, ordinary elliptic curves. In 2006, the NSA published the Suite B algorithms, which provided guidance recommending adoption of ECC and mandated it for systems used by government contractors. In 2016, new guidance was released, recommending larger key sizes for ECC: 384 bits minimum instead of a 256-bit minimum. The revised guidance on the bit size raises the bar on the size of a quantum computer required to mount a successful quantum attack on ECC.

**An Emerging Threat: The Quantum Computer**

Many researchers, industrial labs, and governments are actively working on developing a quantum computer that can handle large-scale computation, such as the work at Station Q, Microsoft’s quantum computing headquarters. While classical computers—phones, tablets, laptops, servers, and so on—store and process information in the form of bits (strings of zeros and ones), quantum computers will process quantum bits, which are two-state quantum mechanical systems called qubits. In contrast to a classical bit, a qubit can simultaneously hold all values between zero and one, with each value having a specified probability. Then, when measured, the state of the qubit collapses to either zero or one. Small-scale quantum computers already exist, and estimates vary as to how many years it will take before researchers and engineers succeed in building a quantum computer that can handle computations involving thousands of qubits. However, when that day arrives, the consequences for the world’s e-commerce and security infrastructure will be enormous.

Basic arithmetic on a quantum computer is different than on a classical computer. Computation on qubits is specified via quantum circuits consisting of quantum gates. Quantum logic gates are represented by unitary matrices. It remains to be seen which quantum gates and architectures will be achieved and scaled up in practice.

In 1994, Shor [Shor97] introduced a quantum algorithm that can factor large integers in polynomial time, given a quantum computer that can accurately process those computations on a large enough number of qubits. A variant of this idea also allows polynomial-time quantum attacks on all of the other currently widely deployed public key cryptosystems used in industry and government today. Shor’s algorithm for factoring on a quantum computer runs in $4m^3$ time and requires $2m$ qubits, where $m$ is the number of bits required to represent the number to be factored ([PZ03]). The current standard minimum for RSA moduli is $m = 2048$ bits. The Proos–Zalka estimates for attacking the elliptic curve discrete logarithm problem were updated in [RNS17] to $9n + 2 \log_2 n + 10$ qubits using a quantum circuit of at most $448n^3\log_2 n + 4090n^3$ Toffoli gates for an ordinary elliptic curve over $F_q$ where $n = \log_2 q$. The conclusion is that 2048-bit RSA and elliptic curve cryptography for $n = 256$ or 384 will not be resistant to quantum attacks once a quantum computer exists at scale.

**Post-Quantum Cryptography**

The NIST Post-Quantum Cryptography (PQC) competition aims to select post-quantum cryptosystems that are not currently known to be breakable in polynomial time by a full-scale quantum computer. The following are the four main types of proposals for post-quantum systems based on hard math problems, in order of when the hard problem was first proposed in cryptography. Code-based cryptography has been studied for more than four decades, for example, whereas supersingular isogeny graphs have been studied for only about fifteen years. There are trade-offs in size, performance, and security for each proposal.

1. Code-based systems are based on “classically” hard problems, for which we do not know any classical polynomial time algorithms: RSA, Diffie–Hellman, and ECC. The RSA cryptosystem for encryption was proposed in the 1970s and is based on the hardness of factoring large integers that are the product of two prime numbers of equal size. Diffie–Hellman key exchange is based on the hardness of solving the discrete logarithm problem in the multiplicative group of integers modulo a large prime number. Elliptic curve cryptosystems are based on the hardness of solving ECDLP, the discrete logarithm problem in the abelian group of points on an elliptic curve over a finite field. Although there is a rich and beautiful mathematical theory of elliptic curves, developed over the course of more than one hundred years by mathematicians, cryptographers often think of an elliptic curve as simply the set of solutions to an affine equation in a finite field $F_q$. In characteristic not equal to 2 or 3, this equation is given in short Weierstrass form:

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possible due to an algorithm of Patterson ([Pa75]). The security of the schemes also relies on disguising the Goppa code as a general linear code.

2. Multivariate cryptosystems are based on the difficulty of solving systems of many nonlinear equations in many variables over a finite field $\mathbb{F}_q$. Imai and Matsumoto introduced the C* scheme in [MI88], and variants were introduced by Patarin and others in follow-up work. Although many proposed multivariate cryptographic systems have been broken, there are still viable proposals that have been submitted to the NIST PQC competition, such as Rainbow for signature schemes.

3. Lattice-based systems are based on the hardness of finding short vectors in lattices. Lattice-based cryptography was introduced in the mathematics community in 1996, when Hoffstein, Pipher, and Silverman ([HPS98]) proposed the system called NTRU. NTRU can be interpreted as a lattice-based system that is especially efficient because of its description in a special kind of number ring.

   A lattice is a linear space generated by a choice of basis vectors. One can imagine it in Euclidean space, where a random set of linearly independent vectors is specified and the lattice consists of all points that are integer linear combinations of these vectors. Given an arbitrary basis with very long vectors in very large dimensions, it is a hard problem to find the shortest vector in the lattice. The best-known algorithms for solving the shortest vector problem run in exponential time in $n$, the dimension of the lattice. There are well-known polynomial time algorithms ([LLL82]) for finding approximate solutions, but the ratio of the length of the approximate vector to the length of the shortest vector is exponentially bad.

4. Supersingular isogeny graph (SIG) systems were introduced in [CGL06] based on the hard problem of finding paths between random vertices in large, random-looking graphs. In particular, Charles, Goren, and Lauter proposed and implemented cryptographic hash functions based on supersingular isogeny graphs and presented it at the 2005 NIST Hash Function Workshop.

   For more information on the first three proposed approaches and hard problems, see the NIST PQC website or the IEEE Security and Privacy magazine issue on post-quantum cryptography, which has short articles on each proposed candidate [BLM17].

   The rest of this article is devoted to explaining the mathematics of supersingular isogeny graphs and their applications in cryptography. Although this is the newest proposal among the four main approaches and thus requires further study to gain confidence in the security, the mathematics is interesting and compelling enough to merit exposition.

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Supersingular Isogeny Graphs

Supersingular isogeny graphs (SIG) were introduced as a hard problem into cryptography by Charles, Goren, and Lauter at the NIST Hash Function competition in 2005. The hard problem is routing in these graphs; i.e., given two nodes or vertices in the graph, find a path between them (or find a path of a certain length between them).

We will define these graphs precisely later, but first, Figure 1 is a picture of a very small SIG, which we produced to appear in Science magazine in 2008 [Ma08]. To get a feel for the hard problem underlying this proposal, pick two random points in the graph and try to find a path between them. Then try to imagine this same problem in a graph that has $10^{75}$ times as many vertices as this one.

![Figure 1. Supersingular isogeny graph for $p = 2521$.](image)

Definition of Supersingular Isogeny Graphs

Supersingular elliptic curves. Let $p$ and $\ell$ be two distinct prime numbers. For our cryptographic applications, $p$ will be the characteristic of a finite field, which is a very large prime of cryptographic size, while $\ell$ will be the degree of a map and very small, typically $\ell = 2$ or 3. Elliptic curves were described above, and since the characteristic of the finite field is not equal to 2 or 3, we can work with the short Weierstrass equation for the elliptic curve: $E : y^2 = x^3 + ax + b$.

An elliptic curve over a finite field of characteristic $p$ is supersingular if it has no $p$-torsion over its base field or any extension field. It is known that each isomorphism class of supersingular elliptic curves modulo $p$ has a representative over the finite field of $p^2$ elements. Elliptic curves that are not supersingular are called ordinary. The $j$-invariant is an isomorphism invariant of an elliptic curve, and it can be easily computed as a rational function of the coefficients of the curve equation:
A hash function maps bit strings to bit strings:

\[ h: \{0,1\}^n \rightarrow \{0,1\}^m. \]

A hash function \( h \) is said to be collision resistant if it is computationally infeasible to find two distinct inputs, \( x,y \), that hash to the same output, \( h(x) = h(y) \). It is preimage resistant if, given any output of \( h \), it is computationally

\[ j(E) = 1728 \frac{4a^3}{4a^3 + 27b^2}. \]

An isogeny between two elliptic curves is a morphism that preserves the group structure. The degree of a separable isogeny is the size of its kernel, so to construct an isogeny of degree \( \ell \) from one elliptic curve \( E \) to another, take a subgroup \( C \) of size \( \ell \), and take the quotient \( E/C \). In our setting, the prime \( \ell \) is different from \( p \), so the isogenies are all separable. Explicit formulae for isogenies of degree \( \ell \) and the equation for \( E/\ell \) were given by Velu [Velu71].

The graphs. Define the supersingular isogeny graph \( G(p,\ell) \) to have vertex set equal to the set of isomorphism classes of supersingular elliptic curves over the algebraic closure of the finite field with \( p \) elements. The number of vertices of \( G(p,\ell) \) is the Eichler class number, which is roughly \( p^{3/2} \) and depends on the congruence class of \( p \) modulo 12 ([Sil09]). Vertices are labeled with their j-invariants, which can be computed directly from the curve equation.

The edges of the graph \( G(p,\ell) \) are the isogenies of degree \( \ell \) between elliptic curves, up to an automorphism of the target curve. Since \( \ell \) is prime and not equal to \( p \), the number of distinct edges coming out of each vertex is \( \ell + 1 \), because there are \( \ell + 1 \) distinct subgroups of order \( \ell \) of the \( \ell \)-torsion of \( E \). To make the graph undirected, we can associate an isogeny in one direction with its dual isogeny in the opposite direction. If we impose the congruence condition \( p \equiv 1 \pmod{12} \), then there is no ambiguity and we can consider the graphs to be undirected [Pizer90, CGL06].

Expansion and Ramanujan Properties of the Supersingular Isogeny Graphs

We now summarize the basic properties of \( G(p,\ell) \). These are connected graphs (see [Mestre86] or a special case of [CGL09, Theorem 4.1]) with roughly \( \frac{p}{12} \) vertices [Sil09, Theorem 4.1]. If \( p \equiv 1 \pmod{12} \), then they are undirected and \( \ell + 1 \)-regular, with vertices labeled by j-invariants.

In the next section we will describe cryptographic applications of these graphs. In particular [CGL06] defined a hash function based on random walks on the graphs \( G(p,\ell) \) for which the output should be as close to uniformly distributed as possible. But first we must define the concept of an expander graph and its expansion constant, which are closely correlated to this property. An expander graph with vertex set \( V \) and \( N \) vertices has expansion constant \( c > 0 \) if for any subset \( U \) of \( V \) of size

\[ |U| \leq \frac{N}{2}, \]

the boundary (neighbors of \( U \) not in \( U \)) satisfies

\[ |\Gamma(U)| \geq c|U|. \]

The adjacency matrix of an undirected graph is symmetric, and therefore all its eigenvalues are real. For a connected \( k \)-regular graph, the largest eigenvalue is \( k \), and all others are strictly smaller:

\[ k > \mu_1 \geq \mu_2 \geq \ldots \geq \mu_{N-1}. \]

The expansion constant \( c \) can be expressed in terms of the eigenvalues as follows:

\[ c \geq 2 \frac{k - \mu_1}{3k - 2\mu_1}. \]

Therefore, the smaller the eigenvalue \( \mu_1 \), the better the expansion constant, and the distance between the first and second eigenvalues, \( k - \mu_1 \), is referred to as the spectral gap. A theorem of Alon–Boppana says that for an infinite family of connected, \( k \)-regular graphs, \( X_m \) indexed by \( m \), with the number of vertices in the graphs tending to infinity,

\[ \lim \inf_{m \to \infty} \mu_1 (X_m) \leq 2\sqrt{k-1}. \]

We define a Ramanujan graph to be a \( k \)-regular connected graph with optimal expansion properties in the sense that it satisfies

\[ \mu_1 \leq 2\sqrt{k-1}. \]

A random walk on an expander graph mixes very fast, so the output of the hash function will be roughly uniform, provided the walk is long enough. The output of a random walk on an expander graph with \( N \) vertices tends to the uniform distribution after roughly \( O(\log(N)) \) steps, where the exact distance from the uniform distribution depends in a precise way on the expansion constant.

Supersingular isogeny graphs \( G(p,\ell) \) are optimal expander graphs when \( p \equiv 1 \pmod{12} \) in the sense that they are Ramanujan graphs (see [Pizer90, Prop. 4.7] or a special case of [CGL09, Theorem 4.2]). The Ramanujan property of this graph follows from the fact that the adjacency matrix (called the Brandt matrix) gives the action of a Hecke operator on the space of weight 2 cusp forms of level \( p \). So the bound on the eigenvalues follows from the corresponding result for modular forms (the Ramanujan–Petersson conjecture).

Applications

Cryptographic hash functions. A hash function maps bit strings to bit strings:

\[ h: \{0,1\}^n \rightarrow \{0,1\}^m. \]

A hash function \( h \) is said to be collision resistant if it is computationally infeasible to find two distinct inputs, \( x,y \), that hash to the same output, \( h(x) = h(y) \). It is preimage resistant if, given any output of \( h \), it is computationally
infeasible to find an input, $x$, that hashes to that output. To be useful in cryptographic applications and protocols, hash functions should have at least the following properties: they should be easy to compute, unkeyed (do not require a secret key to compute output), collision resistant, and preimage resistant, with an approximately uniformly distributed output.

The cryptographic hash function proposed in [CGL06] based on hardness of routing in supersingular isogeny graphs was defined as follows. A fixed vertex in the graph is specified as the starting point. The input bit string is divided into blocks and used as directions for walking around the graph. At each step in the walk, the choice of the next step to follow is determined by the next block of bits of the input. No backtracking is allowed, since that would allow for trivial collisions of walks that go forward and backward along an edge at two different steps in the walk! The output of the hash function is the label for the final vertex of the walk. A family of hash functions can be defined by allowing the starting vertex to vary. For a $k$-regular expander graph with $k - 1 = 2^e$ being a power of 2, the bits are read off in chunks of length $e$. For example, if $k = 3$, then $e = 1$ and bits are processed one at a time.

![Figure 2. Walk on a 3-regular graph: 110 is the green path and 101 is the red path.](image)

In order to avoid collisions in cryptographic hash functions based on isogeny graphs, it is best if the graph has no short cycles. Charles, Goren, and Lauter show in [CGL06] how to ensure that isogeny graphs do not have short cycles by carefully choosing $p$ to satisfy various congruence conditions. For example, they compute that a 2-isogeny graph does not have double edges (i.e., cycles of length 2) when $p$ is a power of 2, the bits are read off in chunks of length $e$. For example, if $k = 3$, then $e = 1$ and bits are processed one at a time.

The security of the hash function relies on the hardness of finding paths, or routing, in this graph. If you can find a path between two given vertices of this graph, then you have found a preimage for the hash function specified by that starting point. Collisions and preimages can be found in the graph using the generic birthday attack, which involves randomly walking around the graph from two different starting points until a collision is detected. The birthday attack runs in time proportional to the square root of the size of the graph, $O(\sqrt{p})$. No better classical attacks are currently known. To achieve 128-bits of security against the birthday attack, in practice we pick $p$ so that $\log p = 256$. The best-known quantum algorithm for computing isogenies between supersingular elliptic curves runs in time $O(p^{1/4})$, ignoring log factors [BJS14].

**Key exchange.** One of the fundamental public key protocols being standardized in the NIST post-quantum cryptography competition is key exchange. Key exchange refers to a protocol for two parties to: 1. specify their public parameters; 2. each pick a secret; 3. publicly exchange information with each other; and 4. compute a common key that only the two parties know. The following key exchange protocol was proposed in [DFJP14].

Let $E$ be a supersingular elliptic curve defined over the finite field with $p^e$ elements, where

$$p = \ell_A^m \ell_B^n \pm 1$$

and $\ell_A$ and $\ell_B$ are distinct small primes and $m$ and $n$ are balanced. In practice $\ell_A = 2$ and $\ell_B = 3$. In that case, $m$ and $n$ are roughly equal to $\frac{1}{2} \log_2 p$ and $\frac{1}{2} \log_3 p$, respectively.

Suppose two parties, A (for Alice) and B (for Bob), wish to engage in a key-exchange protocol with the goal of establishing a shared secret key by communicating via a (possibly insecure) channel. Alice and Bob generate their public parameters: Alice picks two points $P_A$ and $Q_A$ that generate the $\ell_A^m$-torsion, and Bob picks two points $P_B$ and $Q_B$ that generate the $\ell_B^n$-torsion.

Alice then secretly picks two random positive integers $m_A$ and $n_A$, which will be her secret parameters. She then computes the isogeny $\Phi_A$ from $E$ to another curve $E_{A'}$ which corresponds to taking the quotient of $E$ by the subgroup generated by $m_A P_A + n_A Q_A$. Bob does the same and secretly picks two random positive integers $m_B$ and $n_B$. He then computes the secret isogeny $\Phi_B$ by taking the quotient of $E$ by the subgroup generated by $m_B P_B + n_B Q_B$.

So far, Alice and Bob have constructed the diagram shown in Figure 3.

In the next stage of the exchange protocol, Alice computes $\Phi_A(P_B)$ and $\Phi_A(Q_B)$ and sends $\{\Phi_A(P_B), \Phi_A(Q_B), E_A\}$ to Bob. Similarly, Bob computes and sends $\{\Phi_B(P_A), \Phi_B(Q_A), E_B\}$ to Alice. Both players now have enough information to construct the diagram shown in Figure 4, where $E_{AB} = E / (m_A P_A + n_A Q_A, m_B P_B + n_B Q_B)$. Alice can use the secret information $m_A$ and $n_A$ to compute the isogeny $\Phi_B$ by taking the quotient of $E_B$ by the subgroup generated by $m_A \Phi_B(P_A) + n_A \Phi_B(Q_A)$ to obtain $E_{AB}$. Bob can use the secret information $m_B$ and $n_B$ to compute the isogeny $\Phi_A$ by taking the quotient of $E_A$ by the subgroup generated by $m_B \Phi_A(P_B) + n_B \Phi_A(Q_B)$. So far, Alice and Bob have constructed the diagram shown in Figure 3.

![Figure 3. First stage of supersingular isogeny key exchange.](image)
A Security Reduction

The security of the supersingular isogeny key-exchange protocol (SIKE) is based on a hardness assumption stated in [DFJP14], called the supersingular computational Diffie–Hellman (SSCDH) problem. However, the connection with the path-finding problem introduced in [CGL06] was not published until the paper by Costache, Feigon, Lauter, Massiereer, and Puskas [CFLMP19], which showed that the SSCDH problem is no harder than the CGL-path-finding problem, and it is entirely possible that it is easier to solve, given that there is more auxiliary information available in the SSCDH problem.

Theorem [CFLMP19]. Assume as for the key exchange setup that \( p = \ell_A^m \ell_B^n \pm 1 \) is a prime of cryptographic size, i.e., \( \log p \geq 256 \), \( \ell_A \) and \( \ell_B \) are distinct small primes, and \( m \) and \( n \) are balanced so that \( \ell_A^m \) is approximately \( \ell_B^n \). In practice \( \ell_A = 2 \) and \( \ell_B = 3 \). Given an algorithm to solve the CGL path-finding problem in supersingular isogeny graphs, it can be used to break the supersingular key exchange with overwhelming probability. The failure probability is roughly \( \frac{1}{2^\phi} \).

Conclusion

While mathematicians have been researching the hard problem of factoring large integers for centuries, we are now faced with the prospect that our future security may depend on the hardness of mathematical problems that have been studied by mathematicians for only a matter of decades. This disconcerting fact is made worse by the fact that there is an urgent need to understand both the classical and the quantum security of these new proposals. So the current answer to the question in my title is “We don’t know yet!” It is clear, though, that there are very interesting mathematical problems that could serve as the basis of the next generation of post-quantum secure cryptosystems—we just need more mathematicians working on them to understand the security!

To apply for NSF funding for research projects in cryptography and cybersecurity, visit the program solicitation for Secure and Trustworthy Cyberspace (SaTC).7

References


7https://www.nsf.gov/funding/pgm_summ.jsp?pims_id=504709


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The *AMS Page a Day Calendar* is a collection of 366 mathematical morsels. Each day features a fun math fact, a tidbit of math history, a piece of art made using mathematics, a mathematical puzzle or activity, or another mathematical delight. Topics range from the serious to the silly, from the abstract to the very real. The calendar features mathematics done by people from different races, genders, geographic locations, and time periods. Anyone interested in mathematics will learn something new and have their imagination sparked by something they find in the calendar. It will be a mathematical companion for your year.


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This article and its accompanying web interface present Steiner’s conic problem and a discussion on how enumerative and numerical algebraic geometry complement each other. The intended audience is students at an advanced undergrad level. Our readers can see current computational tools in action on a geometry problem that has inspired scholars for two centuries. The take-home message is that numerical methods in algebraic geometry are fast and reliable.

We begin by recalling the statement of Steiner’s conic problem. A conic in the plane $\mathbb{R}^2$ is the set of solutions to a quadratic equation

$A(x, y) = 0$, where

$$A(x, y) = a_1 x^2 + a_2 xy + a_3 y^2 + a_4 x + a_5 y + a_6. \quad (1)$$

If there is a second conic

$$U(x, y) = u_1 x^2 + u_2 xy + u_3 y^2 + u_4 x + u_5 y + u_6, \quad (2)$$

then the two conics intersect in four points in $\mathbb{C}^2$, counting multiplicities and counting intersections at points at infinity, provided $A$ and $U$ are irreducible and not multiples of each other. This is the content of Bézout’s theorem. To take into account the points of intersection at infinity, algebraic geometers like to replace the affine plane $\mathbb{C}^2$ with the complex projective plane $\mathbb{P}^2_{\mathbb{C}}$. In the following, when we write “count,” we always mean counting solutions in projective space. Nevertheless, for our exposition we work with $\mathbb{C}^2$.

A solution $(x, y)$ of the system $A = U = 0$ has multiplicity $\geq 2$ if it is a zero of the Jacobian determinant

$$\frac{\partial A}{\partial x} \cdot \frac{\partial U}{\partial y} - \frac{\partial A}{\partial y} \cdot \frac{\partial U}{\partial x} = 2(a_1 u_2 - a_2 u_1)x^2 + 4(a_1 u_3 - a_3 u_1)xy + \cdots + (a_4 u_5 - a_5 u_4). \quad (3)$$

Geometrically, the conic $U$ is tangent to the conic $A$ if (1), (2), and (3) are zero for some $(x, y) \in \mathbb{C}^2$. For instance, Figure 1 shows a red ellipse and five other blue conics, which are tangent to the red ellipse. Steiner’s conic problem asks the following question:

How many conics in the plane are tangent to five given conics in general position?

The number is five, because each tangency condition removes one of the five degrees of freedom in a conic.

The present article concerns the following two subject areas and how they approach Steiner’s problem:

**Enumerative algebraic geometry:**

How many conics are tangent to five conics?

**Numerical algebraic geometry:**

How do we find all conics tangent to five conics?

The first question is the original conic problem, first asked in 1848 by Steiner, who suggested the answer 7776. That
number turned out to be incorrect. In the year 1864 Chasles gave the correct answer of 3264. This was further developed by Schubert, whose 1879 book led to Hilbert’s 15th problem and thus to the twentieth-century development of enumerative algebraic geometry. The number 3264 appears prominently in the title of the textbook by Eisenbud and Harris [EH16]. A delightful introduction to Steiner’s problem was presented by Bashelor, Ksir, and Traves in [BKT08].

Numerical algebraic geometry is a younger subject. It started about forty years ago, going back at least to [GZ79]. The textbook by Sommese and Wampler [SW05] is a standard reference. It focuses on numerical solutions to polynomial equations. The field is now often seen as a branch of applied mathematics. But, as we demonstrate in this article, its methodology can be used in pure mathematics too.

An instance of our problem is given by a list of $30 = 5 \times 6$ coefficients in $\mathbb{R}$ or $\mathbb{C}$:

$$

A(x, y) = a_1 x^2 + a_2 xy + a_3 y^2 + a_4 x + a_5 y + a_6, \\
B(x, y) = b_1 x^2 + b_2 xy + b_3 y^2 + b_4 x + b_5 y + b_6, \\
C(x, y) = c_1 x^2 + c_2 xy + c_3 y^2 + c_4 x + c_5 y + c_6, \\
D(x, y) = d_1 x^2 + d_2 xy + d_3 y^2 + d_4 x + d_5 y + d_6, \\
E(x, y) = e_1 x^2 + e_2 xy + e_3 y^2 + e_4 x + e_5 y + e_6.

$$

By eliminating the two unknowns $x$ and $y$ from the three equations (1), (2), and (3), we can write the tangency condition directly in terms of the $12 = 6 + 6$ coefficients $a_1, \ldots, a_6, u_1, \ldots, u_6$ of $A$ and $U$:

$$

\mathcal{I}(A, U) = 256a_1^4u_1^4u_6^4 - 128a_1^4a_2u_3u_5u_6^3 + 16a_1^4a_3u_3^2u_6^3 + \cdots + a_5^4a_6u_1^2u_2^4.

$$

The polynomial $\mathcal{I}$ is a sum of 3210 terms. It is of degree six in the variables $a_1, \ldots, a_6$ and of degree six in $u_1, \ldots, u_6$. Known classically as the tact invariant, it vanishes precisely when the two conics are tangent.

If the coefficients are general, we can assume that each conic $U$ that is tangent to $A, B, C, D, E$ has nonzero constant term $u_6$. We can then set $u_6 = 1$. Steiner’s problem for the conics $A, B, C, D, E$ now translates into a system of five polynomial equations in five unknowns, $u_1, u_2, u_3, u_4, u_5$. Each of the five tangency constraints is an equation of degree six:

$$

\mathcal{I}(A, U) = \mathcal{I}(B, U) = \cdots = \mathcal{I}(E, U) = 0.

$$

Steiner used Bézout’s theorem to argue that these equations have $6^5 = 7776$ solutions. However, this number overcounts, because there is a Veronese surface of extraneous solutions $U$, namely, the squares of linear forms. These degenerate conics have the form

$$

U(x, y) = (x, y, 1) \cdot \ell \cdot (x, y, 1)^T,

$$

where $\ell = (\ell_1, \ell_2, \ell_3)$ is a row vector in $\mathbb{C}^3$. Since $U(x, y) = (x, y, 1)^T \begin{pmatrix} 2u_1 \ u_2 \ u_4 \\ u_2 \ 2u_3 \ u_5 \\ u_4 \ u_5 \ 2u_6 \end{pmatrix} (x, y, 1)^T$, the condition for $U$ to be a square is equivalent to

$$

\text{rank} \begin{pmatrix} 2u_1 & u_2 & u_4 \\ u_2 & 2u_3 & u_5 \\ u_4 & u_5 & 2u_6 \end{pmatrix} \leq 1.

$$

This discussion leads us to the following algebraic reformulation of Steiner’s conic problem:

Find all solutions $U$ of the equations (6) such that the matrix in (7) has rank $\geq 2$.

Ronga, Tognoli, and Vust [RTV97] proved the existence of five real conics whose 3264 conics all have real coefficients. In their argument they do not give an explicit instance but rather show that in the neighborhood of some particular conic arrangement there must be an instance that has all of the 3264 conics real. Hence, this raises the following problem:

Find an explicit instance $A, B, C, D, E$ such that the 3264 solutions $U$ to (8) are all real.

Using numerical algebraic geometry we discovered the solution in Figure 2. We claim that all the 3264 conics that are tangent to those five conics are real.

**Proposition 1.** There are 3264 real conics tangent to those given by the $5 \times 6$ matrix in Figure 2.
The construction of our example originates from an ar-
duce our web browser interface.

tangent to our five conics are real. But, first, let us intro-
arrangement of double lines, which we call the
etry software

the conics in (4). After specifying five conics, you press
Here you can type in your own
computer-assisted proof that indeed all of the
construction

of the pentagon construction, and we present a rigorous
for further details.

sect a blue line, but they are actually points where the red
conic seems to inter-
closeups around two of the five points of tangency. The red
conic is tangent to one of the two branches of the blue
hyperbola.

We provide an animation showing all the 3264 real con-
ics of this instance at this URL:

The construction of our example originates from an ar-
angement of double lines, which we call the **pentagon con-
struction**. One can see the pentagon in the middle of Fig-
ure 3. There are points where the red conic seems to inter-
sect a blue line, but they are actually points where the red
conic touches one branch of a blue hyperbola. See [Sot] for further details.

Later we shall discuss the algebro-geometric meaning of the pentagon construction, and we present a rigorous computer-assisted proof that indeed all of the 3264 conics tangent to our five conics are real. But, first, let us intro-
duce our web browser interface.

**Do It Yourself**

In this section we invite you, the reader, to choose your own instance of five conics. We offer a convenient way for you to compute the 3264 complex conics that are tangent to your chosen conics. Our web interface for solving in-
stances of Steiner’s problem is found at

The five conics from Proposition 1.

![Figure 2](image)

Figure 3. The five blue conics in the central picture are those in Proposition 1. Shown in red is one of the 3264 real conics that are tangent to the blue conics. Each blue conic looks like a pair of lines, but it is a thin hyperbola whose branches are close to each other. The two pictures on the sides show closeups around two of the five points of tangency. The red conic is tangent to one of the two branches of the blue hyperbola.

open source Julia package described in [BT18]. Those playing with the web interface need not worry about the inner workings. But, if you are curious, please read our section titled “How Does This Work?”

Shortly after the user submits their instance, by entering real coefficients, the web interface reports whether the instance was generic enough to yield 3264 distinct complex solutions. These solutions are computed numerically. The browser displays the number of real solutions, along with a picture of the instance and a rotating sample of real solutions. As promised in our title, the computation of all solutions takes only around one second.

**Remark 2.** We always assume that the five given conics are real and generic. This ensures that there are 3264 complex solutions, and these conics are tangent to the given conics at $5 \times 3264$ distinct points. The number of real solutions is even, and our web interface displays them sequentially. For every real solution, the points of tangency on the given conics are also real. This fact uses the genericity assumption, since two particular real conics can be tangent at two complex conjugate points. For instance, the conics defined by $x^2 - y^2 + 1$ and $x^2 - 4y^2 + 1$ are tangent at the points $(i : 0)$ and $(-i : 0)$ where $i = \sqrt{-1}$.

Figure 4 shows what the input and the visual output of our web interface look like. The user inputs five conics, and the system shows these in blue. After the user clicks the “compute” button, it responds with the number of complex and real conics that were found. The 3264 conics, along with all points of tangency, are available to the user upon request. The real conics are shown in red, as seen on the right in Figure 4.

When seeing this output, the user might ask a number of questions. For instance, among the real conics, how many are ellipses and how many are hyperbolas? Our web interface answers this question. The distinction between ellipses and hyperbolas is characterized by the eigenvalues of the real symmetric matrix

$$
\begin{bmatrix}
2u_1 & u_2 & u_3 \\
u_2 & u_3 & 2u_3 \\
u_3 & 2u_3 & 2u_3
\end{bmatrix}
$$

If the two eigenvalues of this matrix have opposite signs, then the conic is a hyperbola. If they have the same sign, then the conic is an ellipse. Among the ellipses, we might

![Figure 3](image)
ask for the solution that looks most like a circle. Our program does this by minimizing the expression

$$\left( u_1 - u_3 \right)^2 + u_2^2.$$ 

Users with a numerical analysis background might be interested in maximizing the distance to the degenerate conics. Equivalently, we ask: Among all 3264 solutions, which 3 × 3 matrix in (7) has the smallest condition number?

You can adapt all of this for your favorite geometry problems. As pointed out above, the Julia package `HomotopyContinuation.jl` is available to everyone—follow the link at [BT18]. This may enable you to solve your own polynomial systems in record time.

**Chow Rings and Pentagons**

We next present the approach to deriving the number 3264 that would be taught in an algebraic geometry class, along the lines of the article [BKT08]. Thereafter we explain the geometric degeneration we used to construct the fully real instance in Proposition 1.

Steiner phrased his problem as that of solving five equations of degree six on the five-dimensional space $\mathbb{P}_C^5$. The incorrect count occurred because of the locus of double conics in $\mathbb{P}_C^5$. This is a surface of extraneous solutions. One fixes the problem by replacing $\mathbb{P}_C^5$ with another five-dimensional manifold, namely, the space of complete conics. This space is the blow-up of $\mathbb{P}_C^5$ at the locus of double lines. It is a compactification of the space of nonsingular conics that has desirable geometric properties. A detailed description of this construction, suitable for a first course in algebraic geometry, can be found in [BKT08, §5.1].

In order to answer enumerative geometry questions about the space of complete conics, one considers its Chow ring, as explained in [BKT08, §5.2]. Elements in the Chow ring of the space of complete conics correspond to subvarieties of this space—more precisely, to classes of subvarieties. Two subvarieties belong to the same class if and only if they are rationally equivalent. Rational equivalence is a technical concept. We refer interested readers to the textbook by Eisenbud and Harris [EH16]. The Chow ring for the space of complete conics is worked out in [EH16, §8.2.4]. Nevertheless, the idea behind studying Chow rings is crystal clear: taking intersections of varieties is translated to multiplication in the Chow ring. In the remainder of this section we will see this in action.

The Chow ring of the space of complete conics contains two special classes $P$ and $L$. The class $P$ encodes the conics passing through a fixed point, while the class $L$ encodes the conics tangent to a fixed line. The following relations hold in the Chow ring:

$$P^5 = L^5 = 1, \quad P^4L = PL^4 = 2, \quad P^3L^2 = P^2L^3 = 4.$$ 

These relations are derived in [BKT08, §§4.4–5.3]. For instance, the first equation means that if we take five general conics passing through a fixed point, then the intersection contains one point (namely, the point we fixed in the first place). See [BKT08, Table 3] for the geometric meaning of the other equations.

We write $C$ for the class of conics that are tangent to a given conic. In the Chow ring, we have

$$C = 2P + 2L.$$ 

This identity is derived in [BKT08, equation (8)]. Our preferred proof is to inspect the first three terms in the expression (5) for the tact invariant $\mathcal{I}(A, U)$:

$$\mathcal{I} = 16 \cdot u_6^2(4u_5u_6 - u_3^2) \cdot a_1^2a_2^2 \mod(a_2, a_3^2, a_4, a_5, a_6).$$ 

This has the following intuitive interpretation. We assume that the given fixed conic $A$ satisfies

$$|a_1| \gg |a_3| \gg \max\{|a_2|, |a_4|, |a_5|, |a_6|\}. \quad (11)$$ 

Thus the conic $A$ is close to $x^2 - \varepsilon y^2$, where $\varepsilon$ is a small quantity. The process of letting $\varepsilon$ tend to zero is understood as a degeneration in the sense of algebraic geometry. With this, the condition for $U$ to be tangent to $A$ degenerates to $u_6^2 \cdot (4u_5u_6 - u_3^2)^2 = 0$. 

**Figure 4.** Input and output of the web interface (10).
The first factor $u_6$ represents all conics that pass through the point $(0,0)$. The second factor $4u_3u_6 - u_3^2$ represents all conics tangent to the line $\{x = 0\}$. The Chow ring classes of these factors are $P$ and $L$. Each of these arises with multiplicity 2, as seen from the exponents. The desired intersection number is now obtained from the Binomial Theorem:

$$C^5 = 32(L + P)^5$$

$$= 32(L^5 + 5L^4P + 10L^3P^2 + 10L^2P^3 + 5LP^4 + P^5)$$

$$= 32(1 + 5 \cdot 2 + 10 \cdot 4 + 10 \cdot 4 + 5 \cdot 2 + 1)$$

$$= 32 \cdot 102 = 3264.$$ The final step in turning this into a rigorous proof of Chasles’ result is carried out in [BKT08, §7].

The degeneration idea in (11) can be used to construct real instances of Steiner’s problem whose line spanned by the edge. By the count above, there are Fulton’s ideas. Apparently, they did not know about Sottile, who then wrote down Fulton’s proof in detail [Sot95, Sot]. Ronga, Tognoli, and Vust [RTV97] independently gave a proof. Apparently, they did not know about Fulton’s ideas.

Fix a convex pentagon in $\mathbb{R}^2$ and one special point somewhere in the relative interior of each edge. Consider all conics $C$ such that, for each edge of the pentagon, $C$ either passes through the special point or is tangent to the line spanned by the edge. By the count above, there are $(L + P)^5 = 102$ such conics $C$. If the pentagon is chosen sufficiently asymmetric, then the 102 conics are all real. We now replace each pointed edge by a nearby hyperbola, satisfying (11). For instance, if the edge has equation $x = 0$ and $(0,0)$ is its special point, then we take the hyperbola $x^2 - \epsilon y^2 + \delta$, where $\epsilon > \delta > 0$ are very small. After making appropriate choices of these parameters along all edges of the pentagon, each of the 102 conics splits into 32 conics, each tangent to the five hyperbolae. Here “splits” means, if the process is reversed, then the 32 different conics collapse into one solution of multiplicity 32. By construction, all 3264 conics are real.

The argument shows that there exists an instance in the neighborhood of the pentagon whose 3264 conics are all real, but it does not say how close they should be. Serious hands-on experimentation was necessary for finding the instance in Proposition 1.

We next present an alternative formulation of Steiner’s conic problem. The idea is to remember the five points of tangency on each solution conic. The five sextics in (6) did not involve these points. They were obtained directly from the tact invariant. The next system of equations avoids the use of the tact invariant. It uses five copies of the equations (1)–(3), each with a different point of tangency $(x_i,y_i)$, for $i = 1, 2, 3, 4, 5$. The ten equations from (1) and (2) are quadrics. The five equations from (3) are cubics. Altogether, we get the following system of equations, which we display as a $5 \times 3$ matrix $F_{(A,B,C,D,E)}$:

$$\begin{align*}
A(x_1,y_1) & U(x_1,y_1) (\frac{\partial A}{\partial x} \frac{\partial U}{\partial x} - \frac{\partial A}{\partial y} \frac{\partial U}{\partial y})(x_1,y_1) \\
B(x_2,y_2) & U(x_2,y_2) (\frac{\partial B}{\partial x} \frac{\partial U}{\partial x} - \frac{\partial B}{\partial y} \frac{\partial U}{\partial y})(x_2,y_2) \\
C(x_3,y_3) & U(x_3,y_3) (\frac{\partial C}{\partial x} \frac{\partial U}{\partial x} - \frac{\partial C}{\partial y} \frac{\partial U}{\partial y})(x_3,y_3) \\
D(x_4,y_4) & U(x_4,y_4) (\frac{\partial D}{\partial x} \frac{\partial U}{\partial x} - \frac{\partial D}{\partial y} \frac{\partial U}{\partial y})(x_4,y_4) \\
E(x_5,y_5) & U(x_5,y_5) (\frac{\partial E}{\partial x} \frac{\partial U}{\partial x} - \frac{\partial E}{\partial y} \frac{\partial U}{\partial y})(x_5,y_5)
\end{align*} \quad (12)$$

Each matrix entry is a polynomial in the 15 variables $u_1, \ldots, u_6, x_1, y_1, \ldots, x_5, y_5$. The parameters of this system are the coefficients of the conics $A, B, C, D, E$. The system of five equations seen in (6) is obtained by eliminating the 10 variables $x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4, x_5, y_5$ from the new system $F_{(A,B,C,D,E)}(x)$ introduced in (12).

At first glance, it looks like the new formulation (12) is worse than the one in (6). Indeed, the number of variables has increased from 6 to 15, and the Bézout number has increased from $6^5 = 7776$ to $10^2 \cdot 3^5 = 248832$. However, the new formulation is better suited for the numerical solver that powers our website. We explain this in the last section.

Approximation and Certification

Steiner’s conic problem amounts to solving a system of polynomial equations. Two formulations were given in (6) and (12). But what does “solving” actually mean? One answer is suggested in the textbook by Cox, Little, and O’Shea [CLO13]: Solving means computing a Gröbner basis $\mathcal{G}$. Indeed, crucial invariants, such as the dimension and degree of the solution variety, are encoded in $\mathcal{G}$. The number of real solutions is found by applying techniques like deriving Sturm sequences from the polynomials in $\mathcal{G}$. Yet Gröbner bases can take a very long time to compute. We found them impractical for Steiner’s problem.

Computing 3264 conics in a second requires numerical methods. Our encodings of the solutions are not Gröbner bases but numerical approximations. How does one make this rigorous? This question can be phrased as follows. Suppose $u_1, \ldots, u_6$ are the true coordinates of a solution and $u_1 + \Delta u_1, \ldots, u_6 + \Delta u_6$ are approximations of those complex numbers. How small must the entries of $\Delta u_1, \ldots, \Delta u_6$ be before it is justified to call them approximations? This question is elegantly circumvented by using Smale’s definition of approximate zero [BCSS98, Definition 1 in §8].

In short, an approximate zero of a system $F(x)$ of $n$ polynomials in $n$ variables is any point $z \in \mathbb{C}^n$ such that Newton’s method when applied to $z$ converges quadratically fast towards a zero of $F$. Here is the precise definition.

Definition 3 (Approximate zero). Let $J_F(x)$ be the $n \times n$ Jacobian matrix of $F(x)$. A point $z \in \mathbb{C}^n$ is an approximate zero of $F$ if there exists a zero $\zeta \in \mathbb{C}^n$ of $F$ such that the...
sequence of Newton iterates

\[ z_{k+1} = N_F(z_k), \]  
where \( N_F(x) = x - J_F(x)^{-1}F(x) \),

starting at \( z_0 = z \), satisfies for all \( k = 1, 2, 3, \ldots \) that

\[ \|z_{k+1} - \zeta\| \leq \frac{1}{2}\|z_k - \zeta\|^2. \]

If this holds, then we call \( \zeta \) the associated zero of \( z \).

Here \( \|x\| := \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}} \) is the standard norm in \( \mathbb{C}^n \), and

the zero \( \zeta \) is assumed to be nonsingular; i.e., \( \det(J_F(\zeta)) \neq 0 \).

The reader should think of approximate zeros as a data structure for representing solutions to polynomial systems, just as a Gröbner basis is a data structure. Different types of representations of data provide different levels of accessibility to the desired information. For instance, approximate zeros are not well suited for computing algebraic features of an ideal. But they are a powerful tool for answering geometric questions in a fast and reliable manner.

Suppose that \( z \) is a point in \( \mathbb{C}^n \) whose real and imaginary parts are rational numbers. How can we tell whether \( z \) is an approximate zero of \( F \)? This is not clear from the definition.

It is possible to certify that \( z \) is an approximate zero without dealing with the infinitely many Newton iterates. We next explain how this works. This involves Smale’s \( \gamma \)-number and Smale’s \( \alpha \)-number:

\[ \gamma(F, z) = \sup_{k \geq 1} \left\| \frac{1}{k!} J_F(z)^{-1} D^k F(z) \right\|^{\frac{1}{k!}}, \]

\[ \alpha(F, z) = \| J_F(z)^{-1} F(z) \| \cdot \gamma(F, z). \]

Here \( D^k F(z) \) denotes the tensor of order-\( k \) derivatives at the point \( z \), the tensor \( J_F(z)^{-1} D^k F(z) \) is understood as a multilinear map \( A : (\mathbb{C}^n)^k \to \mathbb{C}^n \), and the norm of this map is \( ||A|| := \max_{\|v\| = 1} ||A(v, \ldots, v)|| \).

Shub and Smale [SS93] derived an upper bound for \( \gamma(F, z) \) that can be computed exactly. Based on the next theorem [BCSS98, Theorem 4 in Chapter 8], one can thus decide algorithmically if \( z \) is an approximate zero, using only data of the point \( z \) itself.

**Theorem 4** (Smale’s \( \alpha \)-theorem). If \( \gamma(F, z) < 0.03 \), then \( z \) is an approximate zero of \( F(x) \). Furthermore, if \( y \in \mathbb{C}^n \) is any point with \( \|y - z\| < (20 \gamma(F, z))^{-1} \), then \( y \) is also an approximate zero of \( F \) with the same associated zero \( \zeta \) as \( z \).

Actually, Smale’s \( \alpha \)-theorem is more general in the sense that \( \alpha_0 = 0.03 \) and \( t_0 = 20 \) can be replaced by any two positive numbers \( \alpha_0 \) and \( t_0 \) that satisfy a certain list of inequalities.

Hauenstein and Sottile [HS12] use Theorem 4 in an algorithm, called **alphaCertified**, that decides if a point \( z \in \mathbb{C}^n \) is an approximate zero and if two approximate zeros have distinct associated solutions. An implementation is publicly available. Furthermore, if the polynomial system \( F \) has only real coefficients, then **alphaCertified** can decide if an associated zero is real. The idea behind this is as follows: Let \( z \in \mathbb{C}^n \) be an approximate zero of \( F \) with associated zero \( \zeta \). If the coefficients of \( F \) are all real, then the Newton operator \( N_F(x) \) from Definition 3 satisfies \( N_F(\bar{x}) = N_F(x) \). Hence \( \bar{z} \) is an approximate zero of \( F \) with associated zero \( \bar{\zeta} \). If \( \|z - \bar{z}\| < (20 \gamma(F, z))^{-1} \), then, by Theorem 4, the associated zeros of \( z \) and \( \bar{z} \) are equal. This means \( \zeta = \bar{\zeta} \).

A fundamental insight is that Theorem 4 allows us to certify candidates for approximate zeros regardless of how they were obtained. Typically, candidates are found by inexact computations using floating point arithmetic. We do not need to know what happens in that computation, because we can certify the result a posteriori. Certification constitutes a rigorous proof of a mathematical result. Let us see this in action.

**Proof of Proposition 1.** Fix five nondegenerate conics with rational coefficients listed after equation (9). We apply HomotopyContinuation.jl [BT18] to compute 3264 solutions in a second in 64-bit floating point arithmetic. The output is inexact. Each coefficient \( u_i \) of each true solution \( U \) is a complex number that is algebraic of degree 3264 over \( Q \). The floating point numbers that represent these coefficients are rational numbers, and we treat them as elements of \( Q \).

Our proof starts with the resulting list of 3264 vectors \( x \in \mathbb{Q}^{15} \) corresponding to the 15 variables of (12). The computation was mentioned to make the exposition more friendly. It is not part of the proof.

We are now given 3264 candidates for approximate zeros of the polynomial system in (12). These candidates have rational coordinates. We use them as input to the software **alphaCertified** from [HS12]. That software performs exact computations in rational arithmetic. Its output shows that the 3264 vectors \( x \) are approximate zeros, that their associated zeros \( \zeta \) are distinct, and that they all have real coordinates. This is shown in Figure 5.
output data of alphaCertified is available for download through the arXiv version of this article.

This was a rigorous proof of Proposition 1, just as trustworthy as a computer-assisted proof by symbolic computation (e.g. Gröbner bases and Sturm sequences) might have been. Readers who are experts in algebra should not get distracted by the appearance of floating point arithmetic: it is not part of the proof! Floating point numbers are only a tool for obtaining the 3264 candidates. The actual proof is carried out by exact symbolic computations.

How Does This Work?

In this section we discuss the methodology and software that powers the web interface (10).

We use the software HomotopyContinuation.jl that was developed by two of us [BT18]. This is a Julia [BEKV17] implementation of a computational paradigm called homotopy continuation. The reasons we chose Julia as the programming language are threefold: the first is that Julia is open source and free for anyone to use. The second is that Julia can be as fast as well-written C. For instance, we use Julia’s JIT compiler for fast evaluation of polynomials. Finally, the third reason is that, despite its high performance, Julia still provides an easy high-level syntax. This makes our software accessible for users from many backgrounds.

Homotopy continuation works as follows: We wish to find a zero in $\mathbb{C}^n$ of a system $F(x)$ of $n$ polynomials in $n$ variables. Let $G(x)$ be another such system with a known zero $G(\xi) = 0$. We connect $F$ and $G$ in the space of polynomial systems by a path $t \mapsto H(x,t)$ with $H(x,0) = G(x)$ and $H(x,1) = F(x)$.

The aim is to approximately follow the solution path $x(t)$ defined by $H(x(t),t) = 0$. For this, the path is discretized into steps $t_0 = 0 < t_1 < \cdots < t_k = 1$. If the discretization is fine enough, then $\xi$ is also an approximate zero of $H(x,t_1)$. Hence, by Definition 3, applying the Newton operator $N_{H(x,t_1)}(x)$ to $\xi$, we get a sequence $\xi_0, \xi_1, \xi_2, \cdots$ of points that converges towards a zero $\xi$ of $H(x,t_1)$. If $s_2 - t_1$ and $\|\xi_i - \xi\|$ are small enough, for some $i \geq 0$, then the iterate $\xi_i$ is an approximate zero of $H(x,t_2)$.

We may repeat the procedure for $H(x,t_2)$ and starting with $\xi_i$. Inductively, we find an approximate zero of $H(x,t_j)$ for all $j$. In the end, we obtain an approximate zero for the system $F(x) = H(x,1)$. Most implementations of homotopy continuation, including Bertini [BHSW06] and HomotopyContinuation.jl [BT18], use heuristics for setting both the step sizes $t_{j+1} - t_j$ and the number of Newton iterations.

Our homotopy for Steiner’s conic problem computes zeros of the system $F_{A,B,C,D,E}(x)$ from (12). We prefer formulation (12) over (6), because the equations in the former formulation have lower degrees and fewer terms. It is known that high degrees and many terms introduce numerical instability in the evaluation of polynomials. We use the homotopy

$$H(x,t) = F_{t(A, B, C, D, E)} + (1-t)(A', B', C', D', E')(x).$$

(13)

The conic $ta + (1-t)A'$ is defined by the coefficients $ta_i + (1-t)a_i'$, where $a_i$ and $a_i'$ are the coefficients of $A$ and $A'$. This is called a parameter homotopy in the literature. Geometrically, (13) is a straight line in the space of quintuples of conics. An alternative would have been

$$\tilde{H}(x,t) = tF_{A,B,C,D,E} + (1-t)\bar{F}_{A', B', C', D', E'}.$$  

(14)

The advantage of (13) over (14) is that the path stays within the space of structured systems

$$\{F_{A,B,C,D,E}(x) \mid (A, B, C, D, E) \text{ are conics}\}.$$

The structure of the equations is preserved. The system in (13) has 3264 solutions for almost all $t$, whereas (14) has 7776 solutions for random $t$. In the language of algebraic geometry, we prefer the flat family (13) over (14), which is not flat.

The last missing piece in our Steiner homotopy is a start system. That is, we need five explicit conics together with all 3264 solutions. For this, we construct a generic instance $(A, B, C, D, E)$ by randomly sampling complex coefficients for the conics. Then, we compute the 3264 solutions using standard homotopy continuation techniques [SW05]. Those 3264 solutions are saved and used for further computations. This initial computation is significantly more expensive than tracking 3264 solutions along the homotopy (13), but it only has to be done once.

In closing, we emphasize the important role played by enumerative geometry for solving polynomial systems. It gives a criterion for deciding if the initial numerical computation found the correct number of solutions. This is why numbers like 3264 are so important and why a numerical analyst might care about Chow rings and the pentagon construction. We conclude that enumerative algebraic geometry and numerical algebraic geometry complement each other.

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Structures in Representation Stability

Steven V Sam

Introduction

Representation theory is applicable in many other areas of mathematics because it can be used to exploit symmetries to simplify calculations. There are many cases where the relevance is clear, such as the action of the invertible matrices on vector spaces, linear maps, tensor products, etc., via change of basis or the action of the symmetric group by permuting coordinates or points in a space. In the examples we will discuss, we start with a sequence of objects with group actions. Next, we construct a sequence of vector spaces that are associated to the sequence of objects that carry an induced linear action of the groups. Two common examples of such groups are symmetric groups and general linear groups. At least over the field of complex numbers, the representation theory of these two groups is well understood and allows us to group together vectors by considering irreducible decompositions. Sometimes this isn’t enough, though: the dimension of these vector spaces might grow fast compared to the number of isomorphism classes of irreducible representations of these groups, and hence these groupings can get unwieldy.

The phrase “representation stability,” as it will be discussed in this article, refers to two related situations and goals. The common theme is that the sequence of symmetry groups is governed by a larger algebraic structure that controls how they interact with one another. The first situation involves showing that these representations follow some predictable pattern or stabilize in an appropriate sense. In other situations, the representations may have additional structure, such as being rings, and the explicit patterns themselves are not of interest. Instead, one may like to find bounds on invariants, such as degree of generation, or at least deduce their existence. The goal of this article is to survey a few examples of both kinds.

An Example of Using Symmetry

Before launching into examples of representation stability, we start with an example that illustrates using symmetry to simplify calculations and how it naturally leads to the issues studied in representation stability.

Let $V_1, \ldots, V_n$ be complex finite-dimensional vector spaces and let $V = V_1 \otimes \cdots \otimes V_n$ denote their tensor product,
a vector space (whose elements are called tensors) spanned by symbols of the form \( v_1 \otimes \cdots \otimes v_n \) with \( v_i \in V_i \) subject to relations to make them multilinear in the factors. The elements of the form \( v_1 \otimes \cdots \otimes v_n \) are called simple tensors. We define the rank of a tensor to be the minimal \( r \) such that it can be expressed as a linear combination of \( r \) simple tensors.

When \( n = 2 \), one can identify \( V_1 \otimes V_2 \) with the space of linear maps from \( V_1^* \) to \( V_2 \), and this notion of rank coincides with the usual rank of a linear map. While all mathematicians know how to compute the rank of a linear map, say, using Gaussian elimination, a less traditional way is that a linear map has rank \( \leq r \) if and only if every \( r \times r \) square submatrix (once bases are chosen) has determinant equal to 0. This describes the locus of rank \( \leq r \) matrices as the zero locus of a collection of polynomials; a basic question is to find a similar description when \( n > 2 \).

This is a largely open problem, and we will focus on the case when \( n = 5 \), \( \dim V_i = 2 \) for all \( i \), and \( r = 5 \). In this case, it is known that the dimension of the rank \( \leq 5 \) locus is 2 less than the dimension of \( V \), and so one might hope to find two polynomials whose zero locus is this set. We set off on this task in [OS]: we easily found a polynomial of degree 6 that is identically 0 on this set, and experimental computation suggested that the other polynomial has degree 16. Finding such a polynomial amounts to a linear algebra problem: we first evaluate all monomials of degree 16 in \( 2^5 = 32 \) variables on some set \( S \) of rank 5 tensors and organize the result into a matrix. Kernel elements of this matrix give polynomials that are 0 on \( S \), and if \( S \) is sufficiently large and “general,” then it gives the desired polynomial. Practically, this is impossible: the number of such monomials is \( \binom{47}{31} = 1503232609098 \) (approximately 1.5 trillion).

However, we can take advantage of the fact that tensor rank is invariant under change of basis on each \( V_i \), i.e., the group \( G = \text{GL}_2(\mathbb{C}) \times \text{GL}_2(\mathbb{C}) \times \text{GL}_2(\mathbb{C}) \times \text{GL}_2(\mathbb{C}) \). General principles tell us that if this degree 16 polynomial exists, then we can find one that is invariant under \( G \) up to scaling. This is much better since the space of such polynomials is only 1313-dimensional, but we can do even better by noting that there is also an action of the symmetric group \( S_5 \) that permutes the tensor factors and also leaves tensor rank unaffected. Again, such a polynomial must be invariant under this extra \( S_5 \)-action up to scaling. All together this space is 49-dimensional, which was enough of a reduction for us to find the polynomial.

The obvious objections the reader might have: Why \( n = 5 \)? Why \( \dim V_i = 2 \)? And why \( r = 5 \)? What is so special about these parameters? Aside from the fact that the codimension is 2 in this case (codimension 1 is much easier to understand), nothing in particular is special. One would like to understand general \( n \), with \( \dim V_i \) general and \( r \) general. However, the information obtained in this case can be used to find equations in other cases: “inheritance” and “flattening” (which we won’t make precise here) allow one to lift these equations whenever \( n \geq 5 \) and whenever \( \dim V_i \geq 2 \) (but keeping \( r \) fixed). Hence one can think of these two equations as generating further equations. A better problem is to find all of the generators (say, when \( r = 5 \)) under these operations rather than studying the problem one set of parameters at a time.

Even more fundamental questions are: How do we axiomatize this algebraic structure given by the operations of inheritance and flattening? As we vary over all choices of \( n \) and \( \dim V_i \), is the set of (minimal) equations generated by a finite number of equations under these operations? The remainder of the article is devoted to surveying other situations where a similar setup can be found. Surprisingly, the examples come from very different parts of mathematics, but the ideas needed to address the questions overlap in fundamental ways.

Stability and Patterns in Representations

In this section, we will consider a sequence of objects \( X_0, X_1, X_2, \ldots \), usually with a group action, and how we can find or study patterns that exist in numerical and linear invariants of the \( X_i \). We will focus on situations where one can find some large algebraic structure that controls all of them at once and what can be deduced from this structure. We will stick to complex vector spaces for simplicity of exposition.

A perspective on homological stability. Let \( k \) be a field and let \( X_n \) be the group \( \text{GL}_n(k) \) of \( n \times n \) invertible matrices with coefficients in \( k \). We would like to understand the \( i \)th homology of \( X_n \) for \( i \) fixed and \( n \) varying. Put \( M_n = H_i(X_n, \mathbb{C}) \), i.e., the \( i \)th left derived functor of the functor \( V \mapsto V \otimes_{X_n} \mathbb{C} \). The inclusion \( X_n \subseteq X_{n+1} \) given by \( A \mapsto \begin{pmatrix} \mathbb{I}_n & A \\ 0 & 0 \end{pmatrix} \) induces a map \( M_n \to M_{n+1} \). Thus we obtain the following system of vector spaces:

\[ M_0 \to M_1 \to M_2 \to \ldots \]

Putting \( M = \bigoplus_{n \geq 0} M_n \), we see that \( M \) is a graded module over the polynomial ring in one variable \( \mathbb{C}[t] \), where \( t \) has degree one (multiplication by \( t \) on an element in \( M_n \) is defined by the map \( M_n \to M_{n+1} \)); this exactly captures the structure we see. The most obvious question to ask is, is \( M \) finitely generated as a \( \mathbb{C}[t] \)-module? If the answer to the question is yes, then the structure theorem for finitely

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1Standard caveat: The set of tensors of rank \( \leq r \) is generally not closed in the Zariski topology, so in what follows, we will implicitly be dealing with its Zariski closure.

2Any multiple of the degree 6 polynomial will be identically 0 on the set of rank \( \leq 5 \) tensors and provides no new information. So we are implicitly discussing minimal equations; i.e., none of them are obtained from the others by multiplication and addition.
generated $\mathbb{C}[t]$-modules tells us that $M$ decomposes as $T \oplus F$, where $T$ is a finitely generated torsion module and $F$ is a finite rank free module. This translates to a concrete statement about the original invariants: once $n$ exceeds the maximal degree in $T$, the map $H_i(X_n, \mathbb{C}) \to H_i(X_{n+1}, \mathbb{C})$ is an isomorphism; that is, the homology of $\text{GL}_n(\mathbb{k})$ stabilizes. In fact, it is known that the homology stabilizes for all fields $\mathbb{k}$ and for a large class of coefficient rings besides $\mathbb{C}$.

**FI-modules and cohomology of configuration spaces.** Let $Y$ be a fixed topological space and let $X_n$ be the configuration space of $n$ distinct labeled points in $Y$. Thus $X_n$ is the open subset of the Cartesian power $Y^n$ where the coordinates are required to be distinct. Configuration spaces appear in many places in mathematics, and it is an important problem to understand their cohomology. For example, if $Y = \mathbb{R}^2$ is the plane, then $X_n$ is an Eilenberg–Mac Lane space for the $n$th pure braid group, and so its cohomology is the cohomology of this important group.

As in the previous example, we fix a cohomological degree $i$ and let $n$ vary. Thus put $M_n = H^i(X_n, \mathbb{C})$. The $n$th symmetric group acts on $X_n$ by permuting coordinates, and this induces an action on $M_n$. In particular, we are in the situation described in the introduction: we have a sequence $M_n$ of $S_n$-representations and we want to understand patterns that occur as we vary $n$. For special cases of $Y$, this can be worked out explicitly. For example, a classical calculation of Arnol’d determines these spaces when $Y = \mathbb{R}^2$.

However, we want to find some intrinsic structure that might help us deduce things about a general class of manifolds. Permutations are bijective functions, and the key observation is that injective functions give extra symmetries if we consider all $X_n$ at once. For notation, set $[m] = \{1, 2, \ldots, m\}$ for a nonnegative integer $m$. Given an injection $f: [m] \to [n]$, we get a map $f^*: X_n \to X_m$ defined by $f^*(x_1, \ldots, x_n) = (x_{f(1)}, \ldots, x_{f(m)})$. Thus, after taking cohomology, we obtain a linear map $f_*: M_n \to M_m$. These satisfy $(f \circ g)_* = f_* \circ g_*$ for any $g: [n] \to [p]$. If $f$ is a permutation, then $f_*: M_n \to M_n$ is the action we had before, so we are extending the action of permutations on $\bigoplus_i M_n$ to an action of all injective functions ($f_*$ acts by 0 on $M_0$ if [p] is not the domain of $f$).

To formalize this, let $FI$ (= finite injections) be the category whose objects are $[n]$ for $n \geq 0$ and whose morphisms are injective functions. Then $M$ is a functor from $FI$ to the category of vector spaces or, more briefly, an $FI$-module: for every $n$ we have a vector space $M_n$, and for every morphism $[m] \to [n]$ in $FI$ we have a linear map $M_m \to M_n$ so that composition is respected. For a fixed $n$, the set of injections $[n] \to [n]$ is closed under composition and gives the symmetric group $S_n$. So, for every $FI$-module $M$, $M_n$ is a representation of $S_n$. Hence an $FI$-module is a sequence of $S_n$-representations together with transition maps between the different representations.

Alternatively, we say that an $FI$-module is a representation of the category $FI$. More generally, given a category $\mathcal{C}$, a representation of $\mathcal{C}$ (or $\mathcal{C}$-module) is a functor from $\mathcal{C}$ to the category of vector spaces.

**Finite generation.** Unfortunately, the structure of an $FI$-module by itself is not helpful: any sequence of $S_n$-representations can be upgraded to an $FI$-module by declaring that $f_*$ is the 0 map whenever $f$ is not a bijection. From the example of $\mathbb{C}[t]$-modules in the subsection “A perspective on homological stability,” we see the same phenomenon: any sequence of vector spaces can be made into a graded $\mathbb{C}[t]$-module by having $t$ act by 0 so that there is no control at all over their dimensions. But we saw that requiring finite generation avoids this problem.

So we ask this question: Given the $FI$-module $M$ coming from the space $Y$, is $M$ finitely generated; that is, are there finitely many cohomology classes that give rise to all cohomology classes by applying the $FI$-operations and taking linear combinations? Independently of any structure theorem for $FI$-modules, such a result is of interest since it is a kind of bound on the complexity of the cohomology classes of $X_n$ as $n$ grows. However, there is a structure theorem for finitely generated $FI$-modules. Before stating it, we recall that irreducible complex representations of $S_n$ are indexed by integer partitions of $n$; we denote the representation corresponding to a partition $\lambda$ by $\mathbf{M}_\lambda$. If $M$ is a finitely generated $FI$-module, then the $M_n$’s are “representation stable” in the sense of Church–Farb; this means that there are partitions $\lambda^1, \ldots, \lambda^t$ such that for $n \gg 0$ the decomposition of $M_n$ into Specht modules (the irreducible representations of $S_n$) is $\bigoplus_{i=1}^t \mathbf{M}_{\lambda^i}$, where $\lambda^i[n] = (n - |\lambda^i|, \lambda^i_2, \ldots)$. A concrete consequence is that if $M$ is a finitely generated $FI$-module, then the sequence $n \to \dim M_n$ agrees with a polynomial function for $n \gg 0$.

**Example 1.** To illustrate some of these ideas in a toy example, consider the $FI$-module $M$ where $M_n = \mathbb{C}^n$, and for $f: [m] \to [n]$, define $f_*: \mathbb{C}^m \to \mathbb{C}^n$ by $f_*(e_i) = e_{f(i)}$, where $e_1, e_2, \ldots$ are the standard basis vectors. Then $M$ is finitely generated by one element $e_1 \in M_1$. When $n \geq 2$, $M_n$ decomposes into two irreducible representations: the subspace $\{ (x_1, \ldots, x_n) | x_1 + \cdots + x_n = 0 \}$ and the line spanned by $e_1 + \cdots + e_n$. The first is $\mathbf{M}_{(n-1, 1)}$ while the latter is $\mathbf{M}_{(n)}$. In this case, the partitions are $\lambda^1 = (1)$ and $\lambda^2 = \emptyset$.

One of the first main results about $FI$-modules, due to Church, Ellenberg, and Farb [CEF], is that the $FI$-module $M$ associated to $Y$ is finitely generated when it is a manifold with some mild restrictions. There is a wealth of literature surrounding $FI$-modules and their applications; see [Fa] for some further references.
So it is desirable to establish finite generation of certain representations of categories. In most applications, this is done in two steps:

1. Show that representations of the category in question have the noetherian property: any subrepresentation of a finitely generated representation is again finitely generated. For \( \mathbf{FI} \)-modules, this was the main result of [CEF].

2. Use the noetherian property to prove finite generation of the specific representations in question. (For example, Church, Ellenberg, and Farb use that \( H^i(X_n) \) can be computed by a spectral sequence of \( \mathbf{FI} \)-modules, and each module on the initial page is easily shown to be finitely generated.)

In joint work with Snowden [SS2], we construct a general theory that establishes the noetherian property for many of the categories of interest in representation stability.

Other examples. We now give a brief survey of some of the other categories that have come up in representation stability. In the interest of saving space, we will not motivate the definitions or explain consequences of finite generation.

There are various ways to enlarge the category \( \mathbf{FI} \) in order to deduce stronger properties about examples such as cohomology of configuration spaces. One such example is the category of noncommutative finite sets: a morphism \( \eta \to m \) is an ordinary set function \( f : [n] \to [m] \) together with a choice of total ordering on each fiber of \( f \). This is used in [EWG] to study configuration spaces of manifolds with a nowhere vanishing vector field.

One can define \( q \)-analogues of \( \mathbf{FI} \)-modules: Given a finite field \( \mathbb{F}_q \), we define a category \( \mathbf{VIC}(\mathbb{F}_q) \) whose objects are nonnegative integers, and a morphism \( m \to n \) that is an injective \( \mathbb{F}_q \)-linear map between vector spaces \( f : \mathbb{F}_q^m \to \mathbb{F}_q^n \) together with a choice of complementary subspace \( C \subseteq \mathbb{F}_q^n \) to the image \( f(\mathbb{F}_q^m) \). More generally, \( \mathbb{F}_q \) can be replaced by other finite commutative rings like \( \mathbb{Z}/\ell \). There is also a symplectic variant \( \mathbf{SL}(\mathbb{F}_q) \) where the linear maps are required to be compatible with the standard symplectic form on \( \mathbb{F}_q^{2m} \) and \( \mathbb{F}_q^{2n} \) (the complement is then chosen to be the orthogonal complement with respect to this form).

These examples were studied in [PS] in connection with the homology of congruence subgroups, which are kernels of the following kinds of homomorphisms: \( \mathbf{GL}_n(\mathbb{Z}) \to \mathbf{GL}_n(\mathbb{Z}/\ell) \), \( \mathbf{Sp}_{2n}(\mathbb{Z}) \to \mathbf{Sp}_{2n}(\mathbb{Z}/\ell) \), maps from the mapping class group of a surface of genus \( g \) to \( \mathbf{Sp}_{2g}(\mathbb{Z}/\ell) \), and maps from the automorphism group of a free group on \( n \) generators to \( \mathbf{GL}_n(\mathbb{Z}/\ell) \). Each of these examples has the structure of a finitely generated representation of one of these categories.

The moduli space \( \mathcal{M}_{g,n} \) of genus \( g \) Riemann surfaces with \( n \) marked points has an important compactification \( \overline{\mathcal{M}}_{g,n} \) which was constructed by Deligne and Mumford. Its homology and cohomology both admit the structure of an \( \mathbf{FI} \)-module but fail to be finitely generated. However, a variant can be used: define \( \mathbf{FS}^{op} \) to be the category whose objects are nonnegative integers and such that a morphism \( m \to n \) is a surjection \( [n] \to [m] \) (the op refers to opposite direction). Tosteson showed in [To] that the homology of \( \overline{\mathcal{M}}_{g,n} \) carries a representation of \( \mathbf{FS}^{op} \) that is finitely generated.

Let \( \mathbf{OI} \) be the subcategory of \( \mathbf{FI} \) where injections are required to be order-preserving. Every \( \mathbf{FI} \)-module is automatically an \( \mathbf{OI} \)-module by restricting the action. However, there are also naturally occurring examples of \( \mathbf{OI} \)-modules that don’t come from \( \mathbf{FI} \). One such involves the homology of groups of upper-triangular matrices (the ordering on the basis elements becomes important) that are shown to be finitely generated in [PSS].

### Existence of Uniform Bounds

The functorial perspective from the section “Stability and Patterns in Representations” was initially driven by topological examples in the works of Church, Ellenberg, Farb, Putman, etc. In parallel, other forms of representation stability were being used in commutative algebra and algebraic geometry, as we hinted at in “An Example of Using Symmetry.” Many of these results are of a much different character, though the underlying theme of proving a noetherianity result remains the same. For a survey on this side of the story, we also refer to [Dr1].

Equivariant rings and modules. Let \( R \) be a ring on which a group \( G \) acts by ring automorphisms. An equivariant \( R \)-module is an \( R \)-module \( M \) equipped with an action of \( G \) that is compatible with its action on \( R \) in the sense that \( g(ax) = (ga)(gx) \) for all \( g \in G, a \in R, x \in M \). Equivariant modules are ubiquitous, and the novel aspect in representation stability is that the objects involved tend to be “large.” Some examples:

- Take \( R \) to be the infinite variable polynomial ring \( \mathbb{C}[x_1, x_2, ...] \) and \( G \) to be the infinite symmetric group \( S_\infty \), acting by permuting the variables.
- Take \( R = \mathbb{C}[x_1, x_2, ...] \) as above and \( G \) to be the infinite general linear group \( \mathbf{GL}_\infty \), acting by linear substitutions in the variables.
- Take \( R = \mathbb{C}[x_{i,j}]_{i,j \geq 1} \) with \( x_{i,j} = x_{j,i} \), and take \( G = \mathbf{GL}_\infty \). The variables are the entries of an infinite symmetric matrix \( A = (x_{i,j}) \), and \( gx_{i,j} \) is the \((i, j)\) entry of \( gA^T \).
- Generalizing the previous two: let \( V \) be a representation of \( G = \mathbf{GL}_\infty \) and let \( G \) act on the symmetric algebra \( \text{Sym}(V) \) (i.e., picking a basis for \( V \) identifies \( \text{Sym}(V) \) with the polynomial ring with those basis elements as variables). In the first case,
Why infinitely many variables? In applications, we would really be interested in finitely many variables, like $\mathbb{C}[x_1, ..., x_n]$ under the action of a smaller group such as the $n$th symmetric group $S_n$. Much like in the previous section, our calculation or object of interest varies with $n$, and so we get a sequence. This can sometimes be rephrased as a single object in the case when $n \to \infty$. Hence this offers a different perspective: prove properties about a large algebraic structure as opposed to a structure that governs sequences of representations.

In each case, for applications we are interested in whether an analogue of a noetherian property holds. For example, we can ask whether ideals closed under the group action are finitely generated up to this action; i.e., given such an ideal $I$, we can ask if there are $f_1, ..., f_r \in I$ such that every $f \in I$ is a linear combination of elements of the form $g \cdot f_i$ where $g \in G$. This property is known to hold for the first three examples, but not in the level of generality of the fourth one. For an application of the first example in algebraic statistics, we point to [HS].

There is a (seemingly) more general question to ask: If $M$ is a finitely generated equivariant $R$-module, are all equivariant submodules of $M$ also finitely generated? In standard commutative algebra, this property holds once we know that it holds for ideals, but we have thus far found no such formal implication that covers these cases. Again, this stronger property holds for the first three examples but is unknown for the fourth example.3

Topological noetherianity. On the other hand, there is a weaker property we can ask for. In algebraic geometry, an ideal $I$ in a polynomial ring $\mathbb{C}[x_1, ..., x_n]$ (we allow $n = \infty$) corresponds to an algebraic set: this is the set of points in $\mathbb{C}^n$ that gives 0 when substituted into any polynomial in $I$. Finite generation of the ideal implies that there is a finite list of polynomials $f_1, ..., f_r$ so that membership in the corresponding algebraic set can be tested by evaluating just these $r$ polynomials at the point. When $n$ is finite, every ideal is finitely generated by the Hilbert basis theorem. This can be rephrased as saying that $\mathbb{C}[x_1, ..., x_n]$ is a noetherian ring. This implies that $\mathbb{C}^n$ is topologically noetherian; i.e., testing membership in any algebraic set can be done with a finite list of polynomials. Note that being topologically noetherian says less than the ring being noetherian, since different ideals can give the same algebraic set. In our equivariant infinite-dimensional context, $\mathbb{C}^\infty$ carries an action of the group $G$ that acts on $\mathbb{C}[x_1, x_2, ...]$, and so we will be interested in algebraic sets closed under the $G$-action. Now we can ask if the space is topologically $G$-noetherian: for every equivariant algebraic set, we want a finite list of polynomials $f_1, ..., f_r$ so that membership of $x$ can be tested by deciding if $(g \cdot f_i)(x) = 0$ for all $g \in G$ and $i = 1, ..., r$. The fourth example above is known by work of Draisma [Dr2] to be topologically noetherian for a large class of $V$, the polynomial representations. Two particularly interesting classes of polynomial representations are:

- Points of the symmetric power $\text{Sym}^d \mathbb{C}^\infty$ are degree $d$ homogeneous polynomials in $x_1, x_2, ...$, and hence when $V = \bigoplus_{i=1}^r \text{Sym}^d \mathbb{C}^\infty$, points parametrize tuples of homogeneous polynomials $(f_1, ..., f_r)$ with $\deg(f_i) = d_i$, which can be used to parametrize ideals in any polynomial ring in finitely many variables. This perspective gives one a way to study invariants of ideals and, in particular, stabilization of such invariants in families when the degrees of the generators of our ideal are fixed in advance. Projective dimension is an important example of such an invariant, which connects this circle of ideas with the work on Stillman’s conjecture [AH,ESS1].

- Points of the exterior power $\bigwedge^d \mathbb{C}^\infty$ represent degree $d$ skew-commutative polynomials in $x_1, x_2, ...$. The BGG correspondence allows one to study cohomology of sheaves on projective space in terms of linear algebra computations with skew-commutative polynomials. This perspective allows one to prove stabilization properties of cohomology of sheaves in various kinds of families; see [ESS2].

Bounded rank tensors. In this last part, we return to the questions raised in “An Example of Using Symmetry.”

First, the notions of inheritance and flattening are axiomatized by $\Delta$-modules in the sense of [Sn]. These are functors from a category $\Delta$ (whose definition is too involved to give here) to the category of vector spaces. The objects of $\Delta$ are finite tuples of finite-dimensional vector spaces, and the assignment of a tuple to the space of degree $d$ polynomials that are 0 on the locus of rank $\leq r$ tensors is an example of a $\Delta$-module. In fact, it is finitely generated, which answers the question posed earlier, with the caveat that both $r$ and $d$ must be fixed.

If we want also to allow $d$ to vary, then we need a more sophisticated algebraic structure. It is currently unknown whether it is possible to find such a structure that acts on all vanishing polynomials in a finitely generated way. However, Draisma and Kuttler [DK] give a topological version by proving the following statement: If we fix $r$, there is a constant $C(r)$ such that for every tuple of vector spaces, there exists a finite collection of polynomials of degree...
\( \leq C(r) \) that can be used to test membership in the rank \( \leq r \) locus.\(^4\) To prove this statement, Draisma and Kuttler take an appropriate limit of the tensor product \( V_1 \otimes \cdots \otimes V_n \) (as \( n \to \infty \) and allowing \( \text{dim } V_i \) to vary) to get an infinite-dimensional topological space with an action of a group \( G \) and prove that the space is topologically \( G \)-noetherian.

Finally, there are several important variations of this problem with different answers given. First, instead of tensor products \( V_1 \otimes \cdots \otimes V_n \), we can consider exterior powers \( \wedge^n V \) and symmetric powers \( \text{Sym}^n V \). Simple tensors are, respectively, of the form \( v_1 \wedge \cdots \wedge v_n \) and \( v_1^n \), so that we have analogues of tensor rank. The analogue of Draisma and Kuttler’s result for \( \wedge^n V \) is proven by Draisma and Eggermont [DE]. An answer about the full ideal (not just a topological statement) in the spirit of the original question is proven by Laudone [La] for \( \wedge^n V \) and by me [Sa] for \( \text{Sym}^n V \).

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\(^4\) The subtle difference here is about finding set-theoretic equations versus finding all equations or, more technically, about finding all generators for an ideal versus finding enough polynomials that generate some ideal with the same radical.
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Emergence of Operator Integration

Operator integration (OI) is a collection of powerful methods and techniques that enable analysis of functions with noncommuting arguments. Such functions arise, in particular, in various problems of applied matrix analysis, mathematical physics, noncommutative geometry, and statistical estimation.

Single operator integrals are basic tools in the classical functional calculus. For instance, a self-adjoint operator $A$ densely defined in a separable Hilbert space admits the operator integral decomposition

$$A = \int_{\mathbb{R}} \lambda \, d\mathcal{E}_A(\lambda),$$

where $\mathcal{E}_A$ is the projection-valued spectral measure of $A$, and the function of this operator $f(A)$ is given by the operator integral

$$f(A) = \int_{\mathbb{R}} f(\lambda) \, d\mathcal{E}_A(\lambda).$$

When $A$ is a finite matrix, the above integral degenerates to a finite sum

$$f(A) = \sum_k f(\lambda_k) \mathcal{E}_A(\lambda_k).$$

This spectral integral (or sum) decomposition induces a straightforward estimate of an operator function in terms of the scalar function

$$\|f(A)\| \leq \|f\|_{\infty}.$$
represent this difference as the integral
\[
f(A) - f(B) = \int_{\mathbb{R}^2} \frac{f(\lambda) - f(\mu)}{\lambda - \mu} \, dE_{A+iB}(\lambda, \mu)(A - B)
\]
with respect to the spectral measure of the normal operator \(A + iB\). By submultiplicativity of the operator norm and properties of the spectral integral, we deduce the bound
\[
\|f(A) - f(B)\| \leq \|f\|_{\text{Lip}(\mathbb{R})} \|A - B\|.
\]
Similar bounds for noncommuting \(A\) and \(B\) are deep results grounded on double or iterated operator integral decompositions
\[
f(A) - f(B) = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{f(\lambda) - f(\mu)}{\lambda - \mu} \, dE_A(\lambda)(A - B) \, dE_B(\mu).
\]
The right-hand side above can be interpreted as the value of a linear transformation \(T^{A,B}_\varphi\) on \(A - B\), where \(\varphi(\lambda, \mu) = \frac{f(\lambda) - f(\mu)}{\lambda - \mu}\), and, thus,
\[
f(A) - f(B) = T^{A,B}_\varphi(A - B).
\]
More generally, the OI approach reduces analysis of different noncommutative expressions to the analysis of a multilinear transformation \(T^{A_1,\ldots,A_{n+1}}_\varphi\) acting on a Cartesian product of matrices or infinite-dimensional operators. The parameters \(\varphi, A_1, \ldots, A_{n+1}\) are determined by the model in question. The values of the transformation \(T^{A_1,\ldots,A_{n+1}}_\varphi\) are often decomposed into integrals with operator-valued integrands or measures. Properties of \(T^{A_1,\ldots,A_{n+1}}_\varphi\) depend on the space where it acts, on the type of symbol \(\varphi\), and sometimes on the spectral properties of the operators \(A_1, \ldots, A_{n+1}\).

Organization
Our first acquaintance with multiple operator integration in this note will occur in the finite-dimensional setting since \(T^{A_1,\ldots,A_{n+1}}_\varphi\) on tuples of matrices admits a finite sum representation. We will see a nonchronological summary of major results and glimpses of fundamental ideas along with questions approachable by the OI method.

Then we will become familiar with OI in the general setting of noncommutative \(L^p\)-spaces, where in addition to puzzles of noncommutativity one deals with convergence issues. We will touch upon several constructions known under the name “multiple operator integral,” each one supplying a particular type of estimate for noncommutative expressions arising in different setups.

From the beginning of its development, the theory of operator integration has been motivated and guided by applications, and this synergy will be reflected in this note. We will demonstrate applicability of OI to questions arising in the study of smoothness properties of operator functions, spectral shift, spectral flow, quantum differentiability, and smoothness of noncommutative \(L^p\)-norms.

Technical details will be omitted, but an interested reader is invited to find them along with a continued discussion in the recent book [19]. Due to the restriction on the allowed number of references many important contributions in the field will not be cited here, but can be found in [19].

Operator Integrals on Finite Matrices
Methods of operator integration have been actively used and rediscovered in matrix analysis, often with many particular cases treated separately without appeal to a general theory. In this section we give an overview of general results supplied by the OI approach along with types of problems where they can be applied.

Linear case. Let \(\ell^2_d\) denote the \(d\)-dimensional Hilbert space equipped with the Euclidean inner product and let \(B(\ell^2_d)\) denote the Banach space of linear operators (or \(d \times d\) matrices) on \(\ell^2_d\) equipped with the operator (spectral) norm. Let \(A, B \in B(\ell^2_d)\) be self-adjoint matrices, let \(\{g_j\}_{j=1}^d, \{h_k\}_{k=1}^d\) be complete systems of orthonormal eigenvectors, and let \(\{\lambda_j\}_{j=1}^d, \{\mu_k\}_{k=1}^d\) be sequences of the corresponding eigenvalues of \(A\) and \(B\), respectively. Let \(P_h\) denote the orthogonal projection onto the vector \(h \in \ell^2_d\) and let \(\varphi : \mathbb{R}^2 \rightarrow \mathbb{C}\) be a function.

The double operator integral constructed from the spectral data of \(A\), \(B\), and the symbol \(\varphi\) is the bounded linear transformation
\[
T^{A,B}_\varphi : B(\ell^2_d) \rightarrow B(\ell^2_d)
\]
given by
\[
T^{A,B}_\varphi(X) = \sum_{j=1}^d \sum_{k=1}^d \varphi(\lambda_j, \mu_k) P_{g_j} X P_{h_k}
\]
for \(X \in B(\ell^2_d)\). In other words, \(T^{A,B}_\varphi\) acts on \(X\) as the entrywise multiplier of the matrix of \(X\) in the bases \(\{g_j\}_{j=1}^d\) and \(\{h_k\}_{k=1}^d\) by the matrix \((\varphi(\lambda_j, \mu_k))_{j,k=1}^d\):
\[
(x_{jk})_{j,k=1}^d \mapsto (\varphi(\lambda_j, \mu_k) x_{jk})_{j,k=1}^d.
\]
Due to this interpretation, the transformation \(T^{A,B}_\varphi\) is also called a Schur multiplier.

Given a differentiable function \(f : \mathbb{R} \rightarrow \mathbb{C}\) and the matrix functions \(f(A)\) and \(f(B)\) defined by the functional calculus,
\[
f(A) = \sum_{j=1}^d f(\lambda_j) P_{g_j} \quad \text{and} \quad f(B) = \sum_{k=1}^d f(\mu_k) P_{h_k},
\]
we have the representation
\[ f(A) - f(B) = T^{A,B}_{f[n]}(A - B). \]  

(3)

Here \( f^{[1]} \) is the divided difference of \( f \) given by
\[ f^{[1]}(\lambda, \mu) = \begin{cases} f(\lambda) - f(\mu) & \text{if } \lambda \neq \mu, \\ f'(\lambda) & \text{if } \lambda = \mu. \end{cases} \]

(4)

The representation (3) was derived by K. Löwner in 1934 in his work on characterization of matrix monotone functions by means of basic spectral theory. From (3) and the Schur multiplier property (2) of \( T^{A,B}_{f[1]} \) we conclude that if the matrix \( (f^{[1]}(\lambda_j, \mu_k))_{j,k=1}^d \) is positive definite, then the function \( f \) is matrix monotone, that is, \( A \preceq B \) implies \( f(A) \preceq f(B) \).

The representation (3) along with (1) can be used to prove existence and the Schur multiplier property (2) of the matrix derivative
\[ \frac{d}{dt} f(A + t(A - B)) \bigg|_{t=0} = T^{B,B}_{f[1]}(A - B). \]  

(4)

The double operator integral also calculates the quasicommutator
\[ f(A)X - Xf(B) = T^{A,B}_{f[1]}(AX - XB). \]  

(5)

The representations (3) – (5) indicate that increments and derivatives of operator functions as well as quasicommutators can be studied and estimated in a unified way in the double operator integration framework. We will consider a variety of such estimates in different norms.

Multilinear case. Multilinear operator integrals arise as natural extensions of double ones in higher-order perturbation problems.

Let \( n \) be a natural number, let \( A_1, \ldots, A_{n+1} \in \mathcal{B}(\ell^2_d) \) be self-adjoint matrices, let \( \{g_i^{[j]}\}_{i=1}^d \) be an orthonormal basis of eigenvectors, and let \( \{\lambda_i^{[j]}\}_{i=1}^d \) be the corresponding sequence of eigenvalues of \( A_j \) for \( j = 1, \ldots, n + 1 \). Let \( \varphi : \mathbb{R}^{n+1} \to \mathbb{C} \) be a function.

The multilinear operator integral constructed from the spectral data of \( A_1, \ldots, A_{n+1} \) and the symbol \( \varphi \) is the bounded \( n \)-linear transformation
\[ T^{A_1,\ldots,A_{n+1}}_\varphi : \mathcal{B}(\ell^2_d) \times \cdots \times \mathcal{B}(\ell^2_d) \to \mathcal{B}(\ell^2_d) \]  

given by
\[ T^{A_1,\ldots,A_{n+1}}_\varphi(X_1, X_2, \ldots, X_n) = \sum_{r_1, r_2, \ldots, r_{n+1}=1}^{d} \varphi(\lambda_1^{(1)}, \ldots, \lambda_{n+1}^{(n)}) \cdot P_{g_1^{[1]}X_1}P_{g_2^{[2]}X_2} \cdots P_{g_{n+1}^{[n+1]}X_n}. \]  

(6)

for \( X_1, X_2, \ldots, X_n \in \mathcal{B}(\ell^2_d) \). It is also called an \( n \)-linear Schur multiplier.

The transformation \( T^{A_1,\ldots,A_{n+1}}_{\varphi} \) generalizes both the entrywise product \( (2) \) (when \( n = 1 \)) and the usual matrix product \( X_1 \cdots X_n \) (when \( \varphi \equiv 1 \)). In the bases of eigenvectors of \( A \), the transformation \( T^{A,A,A}_{\varphi} \) acts on the pair \((X,Y)\) by
\[ \left( \sum_{j,k=1}^{d} \varphi(\lambda_i^{(1)}, \lambda_j^{(2)}, \lambda_k^{(3)}) X_{ij} Y_{jk} \right)_{i,k=1}^d. \]

This transformation appears in the representation for the Taylor remainder
\[ f(A) - f(B) - \sum_{k=1}^{n-1} \frac{1}{k!} \frac{d^k}{dt^k} f(t(A + B)) \bigg|_{t=0} = T^{A,B,B}_{f[1]}(A - B - \ldots - B) \]  

(7)

where \( f \) is \( n \) times continuously differentiable whose derivative \( f^{[n]} \) is bounded on a segment that contains the spectra of \( A \) and \( B \), and \( f^{[n]} \) is the \( n \)th order divided difference of \( f \), which is defined recursively by
\[ f^{[m+1]} = (f^{[m]})^{[1]}. \]

The relation (7) reduces questions on Taylor approximation and convexity of matrix functions to the analysis of \( T^{A,B,B}_{f[1]} \).

A multilinear Schur multiplier also calculates the higher-order Fréchet matrix derivative
\[ \frac{d^k}{dt^k} f(t(A + B)) \bigg|_{t=0} = T^{A,B,B}_{f[1]}(A - B - \ldots - B) \]  

(8)

where \( f \) is \( k \) times continuously differentiable. The idea of involving OI in higher-order differentiation of operator functions was introduced by Yu. L. Daletskii and S. G. Krein in 1956, while the best up-to-date results in this area are consequences of modern approaches to OI.

Norm estimates for Schur multipliers. One of the main objectives in the study of multiple operator integrals is finding useful estimates for their norms. While dimension dependent bounds for multilinear Schur multipliers follow directly from their definitions, delicate dimension independent bounds stand on a decades-long development of a comprehensive theory.

The best dimension independent bound is
\[ \|T^{A_1,\ldots,A_{n+1}}_{\varphi} : S^2_d \times \cdots \times S^2_d \to S^2_d \| = \max_{1 \leq r_1, \ldots, r_{n+1} \leq d} |\varphi(\lambda_1^{(1)}, \ldots, \lambda_{n+1}^{(n+1)})|, \]  

(9)
where $\mathcal{S}^d_2$ is the space $\mathcal{B}(\ell^2_d)$ equipped with the Hilbert–Schmidt (Frobenius) norm. The result of (9) is also the simplest one to derive in the collection of dimension independent bounds. Indeed, the inequality $\leq$ in (9) for $n = 1$ quickly follows from the entrywise multiplier property of $T^A_B$, and for $n \geq 2$ from the repeated application of Hölder’s inequality.

The analog of (9) for $\mathcal{S}^d_p$, which is the space $\mathcal{B}(\ell^2_d)$ equipped with the $p$th Schatten–von Neumann norm $\|\cdot\|_p$, takes the form

$$\left\|T_{f^{[n]}_1}^{A_1} \cdots T_{f^{[n]}_n}^{A_n+1} : \mathcal{S}^p_1 \times \cdots \times \mathcal{S}^p_n \to \mathcal{S}^p_d\right\| \leq c_{p_1 \cdots p_n} \|f^{(n)}\|_{\infty}. \quad (10)$$

Here $1 < p, p_j < \infty$, $j = 1, \ldots, n$,$$
\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_n},
$$
$f$ is an $n$ times differentiable function with essentially bounded $f^{(n)}$, and $\|\cdot\|_{\infty}$ denotes the sup norm. A similar bound holds for the seminorm $\|\cdot\|_1$, of the multilinear Schur multiplier

$$\left\|\text{Tr}(T_{f^{[n]}_1}^{A_1} \cdots T_{f^{[n]}_n}^{A_n+1}(X_1, \ldots, X_n))\right\| \leq c_n \|f^{(n)}\|_{\infty} \|X_1\|_n \cdots \|X_n\|_n, \quad (11)$$

where $\text{Tr}$ is the canonical matrix trace and $\|X\|_n = (\text{Tr}(|X|^n))^{1/n}$ is the $n$th Schatten–von Neumann norm of $X$.

The result of (10) is based on harmonic analysis of Banach spaces with unconditional martingale differences (UMD) and an intricate recursive procedure reducing the order yet preserving the nature of the symbol $f^{[n]}$. The respective approach for $n = 1$ was introduced and implemented in [16], its higher-order equivalent in a more technical setting of $n \geq 2$ in [12]. The estimate (11) follows from a suitable generalization of (10) from $f^{[n]}$ to more general symbols given by polynomial integral momenta and from Hölder’s inequality.

There is a more universal approach to estimating $T_{f^{[n]}_1}^{A_1} \cdots T_{f^{[n]}_n}^{A_n+1}$, which results in bounds for more general norms but with a coarser dependence on $f^{[n]}$ and a smaller set of admissible functions $f$. Namely,

$$\left\|T_{f^{[n]}_1}^{A_1} \cdots T_{f^{[n]}_n}^{A_n+1} : J \times \cdots \times J \to J\right\| \leq \left\|f^{[n]}\right\|_\infty, \quad (12)$$

where $J$ is the space $\mathcal{B}(\ell^2_d)$ equipped with a unitarily invariant norm, $f$ is a function with smoothness properties stronger than $n$-times but weaker than $(n+1)$-times continuous differentiability, and $\left\|f^{[n]}\right\|_\infty$ is the integral projective tensor product norm of $f^{[n]}$. This norm is generally greater than the norm of $f$ appearing in (10):

$$\frac{1}{n!} \|f^{[n]}\|_{\infty} = \left\|f^{[n]}\right\|_{\infty} \leq \left\|f^{[n]}\right\|_\infty.$$

The inequality (12) is based on a factorization of the function $f^{[n]}(\lambda_1, \ldots, \lambda_{n+1})$ separating the variables $\lambda_1, \ldots, \lambda_{n+1}$,

$$f^{[n]}(\lambda_1, \ldots, \lambda_{n+1}) = \int_\Omega a_1(\lambda_1, \omega) \cdots a_{n+1}(\lambda_{n+1}, \omega) d\nu(\omega),$$

which was implemented in [11] and [2]. This factorization is in the spirit of the celebrated A. Grothendieck’s characterization of bounded linear Schur multipliers on $\mathcal{B}(\ell^2)$.

In the next section, we will see alternatives to the bound (12) that reveal dependence on smoothness and decay properties of the function $f$.

From the estimates (9) – (12) and representations (7) and (8) we immediately deduce analogous bounds for derivatives and Taylor remainders of matrix functions.

We note that the Lipschitz-type dimension independent bound

$$\left\|f(A) - f(B)\right\|_p \leq c_p \left\|f\right\|_{\text{Lip}(\mathbb{R})} \|A - B\|_p, \quad (13)$$

$1 < p < \infty$, which follows for every Lipschitz function $f$ by the same method as (10), does not extend to the trace class norm ($p = 1$) or operator norm. In particular, E. B. Davies showed in 1988 that given $d \in \mathbb{N}$, there exist self-adjoint $A, B \in \mathcal{B}(\ell^2_{2d})$ such that

$$\|A - B\|_1 \geq \text{const} \log(d) \|A - B\|_1.$$

There are also continuously differentiable functions $f$ with bounded derivative for which the supremum of $\|f(A) - f(B)\|_1/\|A - B\|_1$ over distinct self-adjoint $A, B \in \mathcal{B}(\ell^2_{2d})$ grows like $\sqrt{\log(d)}$. Higher-order matrix Taylor remainders possess a similar behavior with respect to the trace class norm [15]. Different lower bounds derived in [20] provide a theoretical limitation on the accuracy of matrix Taylor approximations.

Bounds supplied by OI can be used to predict sensitivity of computational problems to small perturbations or rounding errors as well as the accuracy of estimators by sample data. For instance, the norm of a matrix Fréchet derivative calculates a condition number for problems with matrix functions in numerical analysis. For another example, if $A$ is an unknown covariance matrix whose function or functional $f(A)$ is to be estimated and $B$ is a sample covariance matrix, then the bounds for matrix derivatives and Taylor remainders can help to gain insight into the quality of the estimator $f(B)$.
Operator Integrals on Noncommutative $L^p$-spaces

In the noncommutative analysis involving functions of infinite-dimensional operators with continuous spectra, one works with suitable replacements of the transformations (1) and (6) that still satisfy properties like (3) – (5) and (7) – (12). Nowadays we have a rather comprehensive theory of multilinear operator integration as a cumulative product of many groundbreaking results.

Before proceeding, we briefly recall noncommutative $L^p$-spaces where operator integrals are defined. Let $\mathcal{M}$ be a semifinite von Neumann algebra of bounded linear operators acting on a separable Hilbert space $\mathcal{H}$ and let $\tau$ be a normal faithful semifinite trace on $\mathcal{M}$. The noncommutative $L^p$-space $L^p(\mathcal{M}, \tau)$, $1 \leq p < \infty$, associated with $(\mathcal{M}, \tau)$ consists of operators affiliated with $\mathcal{M}$ and satisfying

$$\|X\|_p = (\tau(\|X\|^p))^{1/p} < \infty.$$  

The space $L^\infty(\mathcal{M}, \tau)$ is identified with the algebra $\mathcal{M}$.

When $\mathcal{M}$ equals the algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on $\mathcal{H}$, the noncommutative $L^p$-space coincides with the Schatten–von Neumann ideal $S^p$. The latter consists of compact operators on $\mathcal{H}$ whose sequences of singular values belong to $\ell^p$, and the $S^p$-norm is defined to be $\|X\|_p = (\text{Tr}(\|X\|^p))^{1/p}$. The reader unfamiliar with general noncommutative $L^p$-spaces can assume the particular case of $S^p$ throughout the note.

**Linear case.** The first transformation receiving the name “double operator integral” was introduced by Yu. L. Daletskii and S. G. Krein in 1956, following Löwner’s utilization of what nowadays is called double operator integration (which was mentioned in the previous section). They proved existence of the derivative $\frac{d}{dt}f(A + tX)|_{t=0}$ and estimated its operator norm based on the representation (4), where $T_{\varphi}^{A, B}(\cdot)$ is an iterated Riemann–Stieltjes integral with respect to the spectral family of $A$ and where the class of scalar functions $f$ in their approach is assumed to be more restrictive than turned out to be necessary for existence of the operator derivative. Later in the 1960s, M. S. Birman and M. Z. Solomyak developed several approaches to $T_{\varphi}^{A, B}$ and substantially extended the range of applicability of the double operator integration method, which we sketch below along with more recent approaches.

Let $A, B$ be self-adjoint operators densely defined in $\mathcal{H}$ and let $\mathcal{E}_A, \mathcal{E}_B$ be their spectral measures. The product $E$ of $\mathcal{E}_A$ and $\mathcal{E}_B$ defined on the rectangular sets $\sigma_1 \times \sigma_2$ of $\mathbb{R}^2$ by

$$E(\sigma_1 \times \sigma_2)(X) = \mathcal{E}_A(\sigma_1)X\mathcal{E}_B(\sigma_2)$$

for every $X$ in the Hilbert space $S^2$ of Hilbert–Schmidt operators on $\mathcal{H}$ is the spectral measure itself. The double operator integral $T_{\varphi}^{A, B}$ on $S^2$ is then defined as the spectral integral

$$T_{\varphi}^{A, B}(X) = \int_{\text{spec}(\mathcal{M})} \varphi(\lambda, \mu) dE(\lambda, \mu)(X),$$

(14)

where $\varphi : \mathbb{R}^2 \to \mathbb{C}$ is a bounded Borel function. The transformation $T_{\varphi}^{A, B}$ inherits all the nice properties of a spectral integral, including the inequality

$$\|T_{\varphi}^{A, B} : S^2 \to S^2\| \leq \|\varphi\|_{\infty},$$

(15)

which is an analog of (9). M. S. Birman and M. Z. Solomyak showed that the transformation given by (14) satisfies (3) for $A - B \in S^2$ and $f \in \text{Lip}(\mathbb{R})$. The definition (14) and property (15) extend from $S^2$ to a general noncommutative $L^2$-space $L^2(\mathcal{M}, \tau)$.

If the transformation $T_{\varphi}^{A, B}$ extends from a dense subset $L^2(\mathcal{M}, \tau) \cap L^p(\mathcal{M}, \tau)$ of the noncommutative $L^p$-space $L^p(\mathcal{M}, \tau)$, $1 \leq p < \infty$, to a bounded transformation on $L^p(\mathcal{M}, \tau)$, then it inherits properties of a double operator integral, as was realized by B. de Pagter, H. Witvliet, and F. Sukochev in [6]. By the duality $(S^1)^* = \mathcal{B}(\mathcal{H})$, the transformation $T_{\varphi}^{A, B}$ also extends from $S^1$ to $\mathcal{B}(\mathcal{H})$, as was noted by M. S. Birman and M. Z. Solomyak in the 1960s.

Sufficient conditions for existence of double operator integrals on non-Hilbert spaces and useful estimates for their norms build on analysis of their symbols. One approach relies on existence of a factorization of the symbol $\varphi(\lambda, \mu)$ that separates its variables $\lambda$ and $\mu$. It was first suggested by M. S. Birman and M. Z. Solomyak in 1966 and then generalized by V. V. Peller in 1985 to characterize bounded $T_{\varphi}^{A, B}$ on $\mathcal{B}(\mathcal{H})$. Later, the separation of variables approach to operator integrals was implemented on many symmetrically normed ideals $\mathcal{J}$ of a semifinite von Neumann algebra by N. A. Azamov, A. L. Carey, P. G. Dodds, and F. Sukochev in [2]. These ideals possess a so-called property (F) and include $L^p(\mathcal{M}, \tau) \cap L^p(\mathcal{M}, \tau)$ and $L^{p, \infty}(\mathcal{M}, \tau) \cap L^p(\mathcal{M}, \tau)$, $1 < p < \infty$. This method is explained below.

Let $\varphi : \mathbb{R}^2 \to \mathbb{C}$ admit the representation

$$\varphi(\lambda, \mu) = \int_{\Omega} a_1(\lambda, \omega) a_2(\mu, \omega) d\nu(\omega)$$

(16)

with some finite measure space $(\Omega, \nu)$ and bounded Borel functions $a_i(\cdot, \cdot) : \mathbb{R} \times \Omega \to \mathbb{C}, i = 1, 2$, satisfying

$$\int_{\Omega} \|a_1(\cdot, \omega)\|_{\infty} \|a_2(\cdot, \omega)\|_{\infty} d\nu(\omega) < \infty.$$  

The double operator integral $T_{\varphi}^{A, B}$ is then affixed to the factorization (16) via

$$T_{\varphi}^{A, B}(X)(y) = \int_{\Omega} (a_1(A, \omega)X a_2(B, \omega))(y) d\nu(\omega)$$

(17)
for every \( y \in \mathcal{H} \). For an ideal \( J \) as above, we obtain
\[
\left\| T^A,B_\varphi : J \to \mathcal{J} \right\| \leq \| \varphi \|_{\otimes}
\]
by the symmetric property of \( J \). Here the norm \( \| \cdot \|_{\otimes} \) is defined by
\[
\| \varphi \|_{\otimes} = \inf_{\Omega} \int_{\Omega} \| a_1(\cdot, \omega) \|_{\omega} \| a_2(\cdot, \omega) \|_{\omega} d\nu(\omega)
\]
with the infimum taken over all possible representations (16). We note that the transformation given by (17) satisfies (3) for \( A - B \in \mathcal{B}(\mathcal{H}) \) and \( f \in \text{Lip}(\mathbb{R}) \) with \( \varphi = f^{[1]} \) admitting the representation (16). The decomposition (17) is also convenient for passing to the limit with respect to the parameters \( A, B \), which was used to prove differentiability results for operator functions.

The existence of the representation (16) for \( \varphi = f^{[1]} \) is determined by smoothness and decay properties of the function \( f \). Namely, if \( f \) belongs to the Besov space \( B^1_{\omega_1}(\mathbb{R}) \), then \( f^{[1]} \) satisfies (16) and
\[
\| f \|_{\text{Lip}(\mathbb{R})} \leq \| f^{[1]} \|_{\otimes} \leq \text{const} \| f \|_{B^1_{\omega_1}(\mathbb{R})}
\]
as established by means of the Fourier analysis in V. V. Peller’s work in 1985. To see how this analysis works in a nontechnical case, we assume that both \( f \) and its Fourier transform \( \hat{f} \) are integrable. By the Fourier inversion,
\[
f^{[1]}(\lambda, \mu) = \frac{f(\lambda) - f(\mu)}{\lambda - \mu} = \frac{1}{\sqrt{2\pi}} \frac{1}{\lambda - \mu} \int_{\mathbb{R}} (e^{i\lambda t} - e^{i\mu t}) \hat{f}(t) dt
\]
\[
= \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} \left( \int_{0}^{t} e^{i(\lambda s + \mu(-s))} ds \right) \hat{f}(t) dt
\]
\[
= \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}^2} e^{i\lambda u} e^{i\mu v} \hat{f}(u + v) du dv.
\]
The latter integral is in the form (16), where \( \Omega = \mathbb{R}^2 \),
\[
d\nu(u, v) = \frac{i}{\sqrt{2\pi}} \hat{f}(u + v) du dv, \quad a_1(u, v, \lambda) = e^{i\lambda u}, \quad a_2(u, v, \mu) = e^{i\mu v}
\]
for \( u, v \in \mathbb{R} \). The above calculation and further Fourier analysis induce the bounds
\[
\left\| f^{[1]} \right\|_{\otimes} \leq \| \hat{f} \|_{L^2(\mathbb{R})}
\]
\[
\leq \sqrt{2}(\| f^{''} \|_{L^2(\mathbb{R})} + \| f^{'''} \|_{L^2(\mathbb{R})}).
\]
Sharpening dependence on the symbol \( f^{[1]} \) for double operator integrals on UMD spaces was crucial for resolution of several open problems. It was implemented by D. Potapov and F. Sukochev in [16] by taking a novel approach and carrying out a different type of harmonic analysis on \( f^{[1]} \). The respective transformation \( T^A,B_\varphi \) is constructed by discretizing the spectra of \( A, B \), then following the pattern of the finite-dimensional case (1), and finally taking limits to return to the initial operators. More precisely,
\[
T^A,B_\varphi(X) = \lim_{m \to \infty} \lim_{N \to \infty} \sum_{i_1, i_2 = -N}^{N} \varphi \left( \frac{l_1}{m}, \frac{l_2}{m} \right) \mathcal{E}_{A,i_1,m}X \mathcal{E}_{B,i_2,m},
\]
where \( \mathcal{E}_{A,i,m} = \mathcal{E}_A \left( \frac{1}{m}, \frac{i+1}{m} \right) \) and \( \mathcal{E}_A \) is the spectral measure of \( A \). The transformation given by (18) exists for \( \varphi = f^{[1]} \), where \( f \in \text{Lip}(\mathbb{R}) \), and satisfies
\[
\| T^A,B_\varphi : \mathcal{L}^p(M, \tau) \to \mathcal{L}^p(M, \tau) \| \leq c_p \| f \|_{\text{Lip}(\mathbb{R})}
\]
for \( 1 < p < \infty \). The constant in (19) is known to behave like
\[
c_p \sim \frac{p^2}{p-1},
\]
as established by M. Caspers, S. Montgomery-Smith, D. Potapov, and F. Sukochev in 2014. The transformations given by (17) and (18) coincide on \( \varphi = f^{[1]} \) for a large set of functions \( f \), including the space \( B^1_{\omega_1}(\mathbb{R}) \).

There are several proofs of the estimate (19), each one using deep results from harmonic analysis. To demonstrate a few ideas from the original proof, let \( \psi \) be a Schwartz function satisfying \( \psi(t) = e^t \) for \( t \in [\log \alpha, \log \beta] \), where \( [\alpha, \beta] \subset (0, \infty) \). Then, the Fourier inversion implies that
\[
x = \psi(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \psi(s) e^{ist} ds
\]
\[
= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{\psi}(s) x^{is} ds, \quad x \in [\alpha, \beta].
\]
Hence,
\[
f^{[1]}(\lambda, \mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{\psi}(s) f(\lambda) - f(\mu) |s| \mu^{-1s} ds
\]
for those \( f, \lambda, \mu \) for which \( f^{[1]}(\lambda, \mu) \in [\alpha, \beta] \). The latter decomposition of \( f^{[1]} \) induces a representation of \( T^A,B_\varphi \) as the integral of compositions of the double operator integrals \( T^A,B_\varphi \) with \( \varphi(\lambda, \mu) = |s| \mu^{-1s} \) and \( \varphi(\lambda, \mu) = f(\lambda) - f(\mu) |s| \mu^{-1s} \). Such transformations are bounded by the Marcinkiewicz–Mihlin multiplier theory. The respective bounds along with integrability of the moments of \( \psi \) ultimately imply the boundedness of \( T^A,B_\varphi \) and estimate (19).

**Multilinear case.** Attempts to extend linear double operator integrals to the multilinear case and derive analogous useful bounds were driven by applications and made concurrently with development of the double OI theory.

The first practical multiple operator integral construction was implemented on the product of Hilbert spaces in B. S. Pavlov’s work in 1969. If \( A_1, \ldots, A_{n+1} \) are self-adjoint operators densely defined in \( \mathcal{H} \) with spectral measures
\( E_{A_1, \ldots, E_{A_{n+1}}} \), respectively, and \( X_1, \ldots, X_n \in S^2 \), then the \( S^2 \)-valued set function \( m \) defined on the rectangular sets of \( \mathbb{R}^{n+1} \) by
\[
 m(\delta_1 \times \delta_2 \times \cdots \times \delta_{n+1}) \\
= E_{A_1}(\delta_1)X_1E_{A_2}(\delta_2)X_2 \cdots X_nE_{A_{n+1}}(\delta_{n+1})
\]
admits extension to a countably additive measure with semifavorable bounded by
\[
\|m\| \leq \|X_1\|_2 \cdots \|X_n\|_2.
\]
The multiple operator integral defined by
\[
 T_{\varphi}^{A_1 \cdots A_{n+1}}(X_1, \ldots, X_n) = \int_{\mathbb{R}^{n+1}} \varphi(\omega) \, d\mu(\omega) \tag{20}
\]
for a bounded function \( \varphi \) that is measurable with respect to the product of scalar-valued spectral measures of \( A_1, \ldots, A_{n+1} \) and \( X_1, \ldots, X_n \in S^2 \) enjoys the bound
\[
\|T_{\varphi}^{A_1 \cdots A_{n+1}} : S^2 \times \cdots \times S^2 \rightarrow S^2\| \leq \|\varphi\|_{\infty}.
\]
The definition (20) does not extend beyond the \( S^2 \) setting, as confirmed by K. Dykema and A. Skripka in 2009.

There is a more recent approach to the transformation (20) by C. Coine, C. Le Merdy, and F. Sukochev in [5], which adds new technical opportunities. They realized this transformation as a \( w^* \)-continuous contractive map \( \varphi \rightarrow T_{\varphi}^{A_1 \cdots A_{n+1}} \) acting on elementary tensors by
\[
 T_{f_1 \otimes \cdots \otimes f_{n+1}}^{A_1 \cdots A_{n+1}}(X_1, \ldots, X_n) = f_1(A_1)X_1 \cdots X_n f_{n+1}(A_{n+1}).
\]
This realization makes \( T_{f_1 \otimes \cdots \otimes f_{n+1}}^{A_1 \cdots A_{n+1}} \) amiable to approximation arguments with respect to the parameters \( A_1, \ldots, A_{n+1} \). It was recently used to substantially extend results on differentiability of operator functions and obtain the representation (7) for a broad set of functions.

The double operator integral (17) was extended to the multilinear transformation
\[
 T_{\varphi}^{A_1 \cdots A_{n+1}} : \mathcal{B}(\mathcal{H}) \times \cdots \times \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})
\]
for \( \varphi \) admitting a separation of variables analogous to (16) and to the transformation
\[
 T_{\varphi}^{A_1 \cdots A_{n+1}} : J \times \cdots \times J \rightarrow J,
\]
where \( J \) is a symmetrized normed ideal with property (F), by V. V. Peller in [11] and by N. A. Azamov, A. L. Carey, P. G. Dodds, and F. Sukochev in [2], respectively. This transformation satisfies the bound
\[
\|T_{\varphi}^{A_1 \cdots A_{n+1}} : J \times \cdots \times J \rightarrow J\| \leq c_\varphi \|\varphi\|_{\otimes}.
\]
If \( f \in B_{w^*}^n(\mathbb{R}) \), then
\[
\|f^{[n]}\|_{\otimes} \leq \text{const} \|f\|_{B_{w^*}^n(\mathbb{R})}.
\]
As in the linear case, this OI construction is convenient for passing to the limit with respect to the parameters \( A_1, \ldots, A_{n+1} \).

A transformation \( T_{\varphi}^{A_1 \cdots A_{n+1}} \) that captures nice properties of UMD spaces and sharpens dependence on the symbol \( \varphi \) was constructed by D. Potapov, A. Skripka, and F. Sukochev in [12] to solve open problems in higher-order noncommutative analysis. This transformation is given by a limit generalizing (18), and it satisfies the bounds (10) and (11) as well as their variants in noncommutative \( L^p \)-spaces. We will discuss the importance of (10) and (11) in the next section.

The estimate (10) is established inductively through an elaborate reduction of the order of \( T_{f^{[n]} \cdots A_{n+1}} \) that preserves major features of its symbol. For instance, a fragment of this reduction for \( n = 2 \) and \( \lambda < \mu < \nu \) involves the decomposition
\[
 f^{[2]}(\lambda, \mu, \nu) = \int_{\mathbb{R}} (\mu - \lambda)^i \varphi_\nu(\lambda, \mu)(\nu - \lambda)^{-i} \hat{\varphi}(s) \, ds + \int_{\mathbb{R}} (\nu - \mu)^i \varphi_\nu(\nu, \mu)(\nu - \lambda)^{-i} \hat{\varphi}(s) \, ds,
\]
where \( \psi \) is a Schwartz function and
\[
\varphi_\nu(\lambda, \mu) = \int_{\mathbb{R}^+} t f^{[\nu]}(\lambda + (\mu - \lambda) t) \, dt.
\]

The decomposition of the symbol \( f^{[2]} \) given above leads to the representation of the triple operator integral \( T_{f^{[2]} \cdots A_{n+1}} \), subject to the restriction \( \lambda < \mu < \nu \), as the integral of compositions of the double operator integrals \( T_{\varphi}^{A_\lambda \cdots A_\nu} \) with \( \varphi(\lambda, \mu) = (\lambda - \mu)^{\pm i s} \) (discussed in the previous subsection) and \( \varphi(\lambda, \mu) = \varphi_\nu(\lambda, \mu) \), which is similar to \( (f')^{[1]}(\lambda, \mu) \) (also discussed in the previous subsection). The sets like \( \lambda < \mu < \nu \) are carved out by triangular truncation operators.

Below we consider several important problems that were solved by methods of multilinear operator integration implemented in different technical setups, each one determined by the intrinsic nature of the problem.

**Smoothness of Operator Functions**

In the 1960s, M. G. Krein posed a question on Lipschitz continuity of operator functions that opened a new direction of research in noncommutative analysis. Although existing results of K. Löwner, Yu. L. Daletskii, and S. G. Krein suggested that OI would be a natural tool in questions on operator smoothness, this idea was successfully implemented only in this century after development of a deep and beautiful OI theory.

More precisely, M. G. Krein asked for which functions \( f : \mathbb{R} \rightarrow \mathbb{R} \) the respective operator function is Lipschitz in
for all self-adjoint \( A, B \) such that \( A - B \in S^p \). The answer to M. G. Krein’s question is twofold. The set of scalar functions that are Lipschitz in the \( S^p \)-norm, \( 1 < p < \infty \), coincides with \( \text{Lip}(\mathbb{R}) \), and the bound (13) holds, as established by D. Potapov and F. Sukochev in [16]. However, not every \( f \in \text{Lip}(\mathbb{R}) \) is Lipschitz with respect to the \( S^1 \)-norm or operator norm, as noted by Yu. B. Farforovskaya in the 1970s. Examples of \( A, B \) with \( A - B \in S^1 \) and continuously differentiable \( f \in \text{Lip}(\mathbb{R}) \) for which \( f(A) - f(B) \notin S^1 \) can be constructed by taking direct sums of matrices \( A_d, B_d \) of increasing dimension \( d \) for which the quotient \( \|f(A_d) - f(B_d)\|/\|A_d - B_d\|_1 \) grows logarithmically with \( d \), as discussed in the section on finite-dimensional OI.

The set of \( S^1 \)-Lipschitz functions is known to contain those \( f \in \text{Lip}(\mathbb{R}) \) for which \( f^{[1]} \) admits the decomposition (16). It follows from the infinite-dimensional version of (3) and the bound for the respective double operator integral that

\[
\|f(A) - f(B)\|_1 \leq \|f^{[1]}\|_\infty \|A - B\|_1.
\]

We also have the bound in the weak noncommutative \( L^1 \) quasi-norm

\[
\|f(A) - f(B)\|_{L^1,\infty(M,\tau)} \leq \text{const} \|f\|_{\text{Lip}(\mathbb{R})} \|A - B\|_{L^1(M,\tau)}
\]

for every \( f \in \text{Lip}(\mathbb{R}) \), as established by M. Caspers, D. Potapov, F. Sukochev, and D. Zanin in [4].

As distinct from Lipschitzness, the operator functions inherit their Hölder property with \( 0 < \alpha < 1 \) in the operator norm from scalar functions, according to the result of A. B. Aleksandrov and V. V. Peller [1]. That is,

\[
\|f(A) - f(B)\| \leq c (1 - \alpha)^{-1} \|A - B\|_\alpha
\]

for all self-adjoint \( A, B \) with bounded \( A - B \) and scalar functions \( f \) in the Hölder class \( \text{Lip}(\mathbb{R}) \).

Another aspect of smoothness is differentiability. A function \( f \) is \( n \) times continuously Fréchet \( S^p \)-differentiable, \( 1 < p < \infty \), at every bounded self-adjoint operator \( A \) on \( \mathcal{H} \) if and only if \( f \in C^n(\mathbb{R}) \). This result was obtained by E. Kissin, D. Potapov, V. Shulman, and F. Sukochev in [8] and by C. Le Merdy and A. Skripka in [9] for \( n = 1 \) and \( n \geq 2 \), respectively. The proof for \( n \geq 2 \) combines advantages of two approaches to OI introduced in [12] and [5]. The respective \( k \)th Fréchet differential is expressed in terms of the multilinear operator integral

\[
D^k_p f(A)(X_1, ..., X_k) = \sum_{\sigma \in \text{Sym}_k} T^{f^{(k)}}_{f^{(1)}}(A)(X_{\sigma(1)}, ..., X_{\sigma(k)}),
\]

where \( \text{Sym}_k \) is the group of all permutations of the set \( \{1, ..., k\} \).

When \( A \) is unbounded or when the direction of differentiation belongs to another ideal, we do not have a full characterization of Fréchet differentiability. Nonetheless, we have sufficient conditions for existence of the \( n \)th-order Fréchet derivatives stated in terms of smoothness and decay properties of \( f \) and also a characterization of the \( n \)th-order Gâteaux \( S^p \)-differentiability, \( 1 < p < \infty \). The respective \( k \)th-order Gâteaux derivative is given by (8).

Thanks to the representation (7) in the infinite-dimensional case, the estimates for operator integrals apply to operator Taylor remainders

\[
R_{n,f,B}(A - B) = f(A) - f(B) - \sum_{k=1}^{n-1} \frac{1}{k!} d^k f(B + t(A - B))|_{t=0}.
\]

In particular, when the derivatives \( d^k f(B + t(A - B)) \) are evaluated in the operator norm, we have

\[
\|R_{n,f,B}(A - B)\|_p \leq c_{p,n} \|f(n)\|_\infty \|A - B\|_p^n, \quad 1 < p < \infty,
\]

for a self-adjoint bounded \( B \) and \( f \in C^n(\mathbb{R}) \). When the operator derivatives are calculated in the \( S^p \)-norm, \( 1 < p < \infty \), the bound (21) extends to an unbounded \( B \) and \( n \)-times differentiable \( f \) with bounded derivatives. We also have

\[
\|R_{n,f,B}(A - B)\|_1 \leq c_n \|f\|_{B^n(\mathbb{R})} \|A - B\|_n^n \quad \text{(22)}
\]

for \( f \in B^1(\mathbb{R}) \cap B^\infty(\mathbb{R}) \). The bound (22) follows from the work of V. V. Peller [11], and it does not extend to all \( f \in C^n(\mathbb{R}) \) by the counterexamples constructed in the work of D. Potapov, A. Skripka, F. Sukochev, and A. Tomskova [15].

Trace Formulas in Mathematical Physics

Trace formulas relate spectral properties of two operators, where one operator is well understood and the other is viewed as its perturbation whose spectral properties are unknown. This line of research originates from M. G. Krein’s seminal 1953 work addressing I. M. Lifshits’s findings and questions in physics in the 1940s. Its development and several breakthroughs benefited from the method of multilinear operator integration.

Let \( H \) and \( V \) be self-adjoint operators, \( H \) possibly unbounded. If \( V \) belongs to the Schatten–von Neumann ideal \( S^n \) for some \( n \in \mathbb{N} \), then there exists a unique real-valued function \( \eta_n \in L^1(\mathbb{R}) \) depending only on \( n, H, V \) such that

\[
\|\eta_n\|_1 \leq c_n \|V\|_n^n
\]
\[ \text{and} \]
\[ \text{Tr} \left( f(H + V) - \sum_{k=0}^{n-1} \frac{1}{k!} \frac{d^k}{dt^k} f(H + tV) \big|_{t=0} \right) \]
\[ = \int_{\mathbb{R}} f^{(n)}(t) \eta_n(t) \, dt \]
\[ (23) \]
for sufficiently nice function. This result was established by M. G. Krein in 1953, L. S. Koplienko in 1984, and D. Potapov, A. Skripka, and F. Sukochev in [12] in the cases \( n = 1, n = 2, \) and \( n \geq 3 \), respectively. The function \( \eta_n \), called the spectral shift function, plays a fundamental role in perturbation theory, but little is known about properties of \( \eta_n \) when \( n \geq 3 \) due to its complexity.

The proof of (23) for \( n = 2 \) utilizes the double operator integrals introduced by M. S. Birman and M. Z. Solomyak to carry out an approximation of \( V \in S^2 \) by a sequence of finite rank \( V_k \), for which \( \eta_2 \) can be calculated explicitly in terms of \( \eta_1 \). The trace formula for \( n \geq 3 \) crucially relies on the estimate (11). Indeed, by the integral representation for the remainder and OI representation for the operator derivative (8),

\[ f(H + V) - \sum_{k=0}^{n-1} \frac{1}{k!} \frac{d^k}{dt^k} f(H + tV) \big|_{t=0} = \frac{1}{(n-1)!} \int_0^1 (1-t)^{n-1} T_{[f]}^{H_t - H_0}(V, ..., V) \, dt, \]

where \( H_t = H + tV. \) Then the estimate (11) implies existence of the measure \( \mu_n \) such that the left-hand side of (23) equals \( \int_{\mathbb{R}} f^{(n)}(t) \, d\mu_n(t) \) and the total variation of \( \mu_n \) is bounded by \( c_n \|V\|_n^2 \). Finally, an approximation argument involving OI and reduction to the lower-order case confirms absolute continuity of \( \mu_n \) or, equivalently, existence of \( \eta_n \).

While the condition \( V \in S^n \) can hold for perturbations of discrete Schrödinger operators, it does not hold for typical perturbations of differential operators. Instead, when \( H \) is a continuous Schrödinger or Dirac operator, its perturbation \( V \) can possibly satisfy

\[ (H + V - iI)^{-1} - (H - iI)^{-1} \in S^n. \]

Analogs of the trace formula (23) were obtained in this extended setting with modified right- and, in some cases, also left-hand sides and the respective \( \eta_n \) integrable with a weight. These results are based on the theory of spectral shift functions for unitary and contractive operators with perturbations in \( S^n \), a change of variables in multiple operator integrals, and creation of summable weights within the OI framework. The condition \((H + V - iI)^{-1} - (H - iI)^{-1} \in S^n\) in the cases \( n = 1 \) and \( n = 2 \) was handled via a reduction to unitaries by M. G. Krein and H. Neidhardt in 1962 and 1988, respectively. The case \( n \geq 2 \) was treated via a reduction to contractions by D. Potapov, A. Skripka, and F. Sukochev in [14] and \( n \geq 3 \) via a reduction to unitaries by A. Skripka in [18]. The respective higher-order spectral shift theory for contractions and unitaries builds on OI methods developed in [13] and [18].

The formula (23) extends to more general traces and unsummable perturbations in semifinite von Neumann algebras that arise in noncommutative geometry. In particular, the trace formula with locally integrable \( \eta_n \) holds for an arbitrary bounded self-adjoint \( V \) and unbounded self-adjoint \( H \) whose resolvent \((H - iI)^{-1} \) is \( \tau \)-compact. This property is satisfied, for instance, by differential operators on compact Riemannian manifolds. The trace formulas also hold for perturbations in symmetrically normed ideals equipped with continuous, including singular, traces. They were established for general ideals and \( n = 1, 2 \) by K. Dykema and A. Skripka in [7] and for the dual Macek ideal and \( n \geq 3 \) by D. Potapov, A. Usachev, F. Sukochev, and D. Zanin in 2015.

**Spectral Flow**

The concept of the spectral flow stems from the celebrated work of M. F. Atiyah, V. K. Patodi, and I. M. Singer in the 1970s, where it was introduced primarily in a topological sense. At the International Congress of Mathematicians in 1974, I. M. Singer communicated that the spectral flow can be expressed as an integral of 1-form. After J. Phillips introduced an analytic approach to the spectral flow in the context of von Neumann algebras in the 1990s, the implementation of I. M. Singer’s suggestion was pursued in the framework of noncommutative geometry. This representation via the integral of a 1-form was confirmed in a general setting without summability restrictions by incorporating double operator integration techniques in the work of N. A. Azamov, A. L. Carey, and F. Sukochev [3].

Let \( D_0, D_1 \) be self-adjoint operators affiliated with \( \mathcal{M} \) whose resolvents are \( \tau \)-compact so that \( V = D_0 - D_1 \) is bounded. If \( D_0 \) and \( D_1 \) are unitarily equivalent, then the spectral flow \( \text{sf}(D_0, D_1) \) from \( D_0 \) to \( D_1 \) along the path \( r \to D_0 + rV \) can be calculated by

\[ \text{sf}(D_0, D_1) = \frac{1}{\|f\|_{L^1(\mathbb{R})}} \int_0^1 \tau(V f(D_0 + rV)) \, dr \]

for every nonnegative \( f \in C_0^\infty(\mathbb{R}) \). The expression

\[ \int_0^1 \tau(V f(D_0 + rV)) \, dr \]

is the result of integration of the closed exact 1-form \( d\theta_D^f \) given by

\[ d\theta_D^f(X) = \frac{d}{ds} \left( \int_0^1 \tau((V + sX) f(D_0 + rV + sX)) \, dr \right) \bigg|_{s=0}. \]
The above conclusion builds on OI methods for functions of operators with \( \tau \)-compact resolvents developed by analogy with OI for summable perturbations discussed in the previous section.

When \( D_0 \) and \( D_1 \) are not unitarily equivalent, the representation \((24)\) is modified by the truncated \( \eta \)-invariants and \( \tau \)-dimensions of the kernels of \( D_0 \) and \( D_1 \). The spectral flow \( sf(D_0 - \lambda I, D_1 - \lambda I) \) coincides with the spectral shift function \( \eta_1(\lambda) \) discussed above up to the kernel correction terms.

**Quantum Differentiability**

A quantized differential was introduced by A. Connes in the 1980s to replace the differential calculus in noncommutative differential geometry by an operator theoretic notion involving a commutator. The asymptotic behavior of the singular values of the quantized derivative determines the dimension of an infinitesimal in the quantized calculus.

Let \( \delta \in L^\infty(\mathbb{R}^d) \), \( d \in \mathbb{N} \), and let \( M_f \) be the operator of pointwise multiplication by \( f \). If \( D_0 \) is the Dirac operator densely defined in \( C^1_{\tilde{\alpha}}(\mathbb{R}^d) \) and \( \text{sgn}(D) \) is its sign given by the functional calculus, then the quantized derivative of \( f \) is defined to be the commutator

\[
df := i[\text{sgn}(D), I \otimes M_f].
\]

It follows from work of S. Janson and T. H. Wolff in 1982 that \( df \in S^d \) if and only if \( f \) is a constant, while the property of the quantized derivative to be in the weak Schatten–von Neumann ideal \( S^{d,\infty} \) is achievable in nontrivial cases. In particular, if \( d > 1 \) and \( \delta \in L^\infty(\mathbb{R}^d) \), then \( df \in S^{d,\infty} \) if and only if \( \nabla f \in L^d(\mathbb{R}^d, C^d) \). Moreover, there exist constants \( c_d, C_d > 0 \) such that

\[
c_d \| \nabla f \|_{L^d(\mathbb{R}^d, C^d)} \leq \| df \|_{S^{d,\infty}} \leq C_d \| \nabla f \|_{L^d(\mathbb{R}^d, C^d)}.
\]

The latter result was proved by S. Lord, E. McDonald, F. Sukochev, and D. Zanin in [10] using the method of double operator integration. The appearance of OI in this problem is suggested by the formula \((5)\), which extends to the infinite-dimensional case under appropriate assumptions. This variant of \((5)\) gives

\[
[D(I + D^2)^{-1/2}, I \otimes M_f] = T_{g(t)}^{D,D}(\nabla f, I \otimes M_f),
\]

where \( g(t) = t(1 + t^2)^{-1/2} \) is the regularized sign function. Results for double operator integrals allow estimation of \( T_{g(t)}^{D,D} \) on \( S^1 \) and on \( B(C^{1/2}_{\tilde{\alpha}}(\mathbb{R}^2)) \). Then interpolation transfers this estimate to \( T_{g(t)}^{D,D} \) on \( S^{d,\infty} \). Along with a suitable analysis of the commutators \([D, I \otimes M_f]\) and \([\text{sgn}(D) - D(I + D^2)^{-1/2}, I \otimes M_f]\), it ultimately gives the stated upper bound.

**Smoothness of Banach Norms**

The study of smoothness properties in function Banach spaces was naturally lifted to a similar investigation in the noncommutative Haagerup \( L^p \) spaces \( L^p_{\text{Haag}}(M) \) associated with an arbitrary von Neumann algebra \( M \). This general setting includes the following cases: when \( M \) is semifinite, the space \( L^p_{\text{Haag}}(M) \) is isometric to the classical noncommutative \( L^p \)-space associated with \( M \), \( 1 \leq p < \infty \); when \( M \) is a type \( I \) von Neumann algebra, then \( L^p_{\text{Haag}}(M) \) contains an isometric copy of the sequence space \( \ell^p \); when \( M \) is not of type \( I \), \( L^p_{\text{Haag}}(M) \) contains an isometric copy of the function space \( L^p(0,1) \).

The first-order Fréchet differentiability of the norm power \( \| \cdot \|_{L^p_{\text{Haag}}}^p \), \( 1 < p < \infty \), was essentially established by H. Kosaki in 1984 and its higher-order Fréchet differentiability by D. Potapov, F. Sukochev, A. Tomskova, and D. Zanin in [17]. Namely, we have that \( \| \cdot \|_{L^p_{\text{Haag}}}^p \) is

(i) infinitely many times Taylor–Fréchet differentiable whenever \( p \) is an even integer;
(ii) \((p-1)\)-times Taylor–Fréchet differentiable whenever \( p \) is an odd integer;
(iii) \(|p|\)-times Taylor–Fréchet differentiable whenever \( p \) is not an integer.

More specifically, for self-adjoint \( A, X \in L^p_{\text{Haag}}(M) \),

\[
\|A + X\|_{L^p_{\text{Haag}}}^p = \|A\|_{L^p_{\text{Haag}}}^p + \sum_{k=1}^{[p-1]} \delta_k^A(X,\ldots,X) + O(\|X\|_{L^p_{\text{Haag}}}^p),
\]

where \( \delta_k^A \) is a symmetric \( k \)-linear bounded functional on \( L^p_{\text{Haag}}(M)^k \times \cdots \times L^p_{\text{Haag}}(M) \). When \( p \) is even, \( O(\|X\|_{L^p_{\text{Haag}}}^p) = \delta_p^A(X,\ldots,X) \). The differentials \( \delta_k^A \) are given in terms of multilinear operator integrals.

Since \( \|X\|_{L^p_{\text{Haag}}}^p = \psi(|X|^p) \), where \( \psi \) is a normalized trace on the weak noncommutative \( L^1 \)-space \( L^{1,\infty}(N) \) associated with a certain semifinite crossed product of von Neumann algebra \( N \), the analysis of smoothness properties of \( \| \cdot \|_{L^p_{\text{Haag}}}^p \) reduces to the analysis of the operator function arising from the scalar function \( f(t) = |t|^p \). The latter analysis is fulfilled by means of multilinear operator integration developed on weak noncommutative \( L^p \)-spaces \( L^{p,\infty}(N) \), which contain \( L^p_{\text{Haag}}(M) \) as a closed subspace. One of the features specific to this setting is involvement of Calderón-type operators in derivation of Hölder-type estimates for OI.

**Closing Remarks**

In this brief journey to operator integration we considered only the case of self-adjoint operators \( A_1,\ldots,A_{n+1} \). Multilinear operator integration has also been developed in the
cases when $A_1, \ldots, A_{n+1}$ are unitary, contractive, dissipative, or unbounded normal operators. The theory has also been extended to noncommutative weak $L^p$-spaces and, in the linear case, to general Banach spaces. The details can be found in [19].

While we demonstrated usefulness of operator integration on selected representative applications, the OI method has more to offer. Further general results based on this method along with their variants designed for specific models of mathematical physics and noncommutative geometry can be found in [19], thematic surveys, and recent and upcoming articles.

References

Anna Skripka
Credits
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The Early Career Section offers information and suggestions for graduate students, job seekers, early career academics of all types, and those who mentor them. Angela Gibney serves as the editor of this section. Next month’s theme will be communicating mathematics.

The Year to Come

In the Early Career

A. Gibney

In the first year of our series, more than sixty colleagues contributed ideas and advice on topics for early career mathematicians and those who mentor them. Some devoted people have written more than one piece, and others have generously shared their ideas for points to discuss. At the time I’m writing this, almost fifty people have signed on to write articles in year two. You can read the names of all past and (some of the) future contributors below. If you haven’t had a chance, I urge you to read what they have had to say: [bit.ly/2lEMeHAEarlyCareer](bit.ly/2lEMeHAEarlyCareer).

In the coming year we plan to take on new topics as well as treat different facets of some of the themes from our first year. In February we will focus on aspects of the communication of mathematics. In March we will talk about challenging issues and how to deal with them, especially prior to tenure. In April we will spotlight careers in business, industry, and government. Our themes in May are research, creativity, and productivity. For the June/July issue we will consider different types of academic jobs, and in August, getting ready for the academic job market. Matters surrounding working with students will be the focus of the September issue, and in October we will revisit the important theme of mentoring. Topics for November include strategies for evaluating applications for jobs and graduate school, and we will consider how to prepare for a successful interview. We are opening the floor to our readers so they may tell us about great mathematical activities to highlight in our December issue.

To nominate a program for the December 2020 Good Ideas issue, fill out a short form at [www.angelagibney.org/the-early-career](www.angelagibney.org/the-early-career). You may also let us know you’d like to contribute to the Early Career section, name a colleague to be invited to contribute, suggest a topic we
Finding Your Reward

Skip Garibaldi

What is it about doing math that people find really rewarding? Don’t answer too quickly. It’s easy to get wrapped up in your current interests and lose sight of other opportunities.

Before we get down to business, let’s get our terminology straight. Sometimes people conflate “math” with something like “things done by professors in a department of mathematics,” but I mean something more inclusive. When I look at my colleagues in business, industry, and government or faculty in departments of computer science or engineering, for example, I see some people who are not only interested in equations or applications but also care deeply about theorems and proofs. I think they are “doing math.” I mean math in that broader sense here.

So what are some of the things that people find rewarding about doing math? For some readers, the answer is simple: proving theorems or explaining exciting theorems to others are both pleasurable activities, and we should count ourselves lucky if someone is willing to pay us to do them. One of my friends does math for this reason; his aim is to prove theorems that he personally finds beautiful, and all other considerations, such as publication, are nearly irrelevant. There is an elemental appeal to this approach to mathematics.

There are also earthly rewards to the math life. For one, math can provide the admiration and respect of your peers, such as the prestige that comes with being a professor or receiving a prize. This is a completely normal motivation. (Recall the quote “Give me enough medals and I’ll win..."

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you any war” or look at the reward structure in almost any modern video game.) Another, smaller-scale reward is that the math life can be remarkably flexible when compared with other scientific endeavors; you can prove a theorem at a beach or coffee shop, but it would be tough to do most laboratory work there.

Also, many people find a reward in seeing the effect of their work on others. In education, you might look to the human flourishing you encourage (borrowing the language of [2]) or the successes of your former students. In research, you may write an influential paper or help defend the nation.

Having an effect on others is easier if you work as part of a team. Professors working in pure mathematics may not think of a new theorem as having concrete real-world consequences like saving lives, but it happens all the time when the mathematician proving the theorem is connected with a team to convert it into a more effective medical test or a way to prevent a terrorist attack. I know someone whose goal is for his work to save 10,000 lives; such a challenging goal is more achievable as part of a team. And there can be joy and pride in being a part, even a nameless part, in a big endeavor [1].

It’s worth trying to figure out which rewards speak to you. (I confess that I am inspired by all of the ones mentioned here.) Most of us are exposed to only a few of the possibilities in the natural course of our careers, so exploring your options might mean stepping outside your comfort zone, like taking a class in a neighboring subject or finding a project to work on with someone from a different discipline or taking an internship at a company or with the government. Figuring out what will be most fulfilling and getting to a place where that itch gets scratched usually requires planning and dedication. The first step is knowing what you want.

References

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Skip Garibaldi

Author photo is courtesy of the author.
Jean-Pierre Wintenberger

Pierre Colmez and Chandrashekhar Khare

A Brief Bio
Jean-Pierre Wintenberger was born in Neuilly-sur-Seine, near Paris, in 1954. His parents were scientists who transmitted to him their curiosity, interest, and passion for science and research. He entered the École Normale Supérieure de Paris in 1973, along with his friend Jean-Yves Mérindol, who later became an algebraic geometer and the president of Strasbourg University. Some of the other students who entered the ENS at the same time were Jean-Francois Mestre, Joseph Oesterlé, Guy Henniart (and Hélène Esnault in ENS Sèvres—for women: the two ENSs merged in 1985). Just the previous year Henri Carayol, Laurent Clozel, Étienne Fouvry, Gérard Laumon, Jean-Loup Waldspurger (and Colette Moeglin in ENS Sèvres) had entered the ENS, to be followed by Bernadette Perrin-Riou at ENS Sèvres in 1974. This was a spectacularly rich crop of students, forming a core group who went on to play a big role in the growth of the French school of number theory, particularly arithmetic geometry and automorphic forms.

Jean-Pierre got his first thesis in 1978 and his Thèse d’État (Habilitation) in 1984 in Grenoble, under the supervision of Jean-Marc Fontaine. He held the position of researcher in CNRS from 1978 to 1991, first in Grenoble, then in Orsay. He was a professor at the Université de Strasbourg from 1991 till he retired in 2017. He was a member of the Institut Universitaire de France, received the Prix Thérèse Gautier from the French Academy of Science in 2008, and was an invited speaker in the Number Theory Section at the International Congress of Mathematicians in 2010. He received the 2011 AMS Frank Nelson Cole Prize in Number Theory (jointly with Khare) for their proof of Serre’s modularity conjecture.

Jean-Pierre passed away on January 23, 2019. He is survived by his son, Olivier Wintenberger, who is an applied mathematics professor at Sorbonne Université, and his daughter, Claire Guillet, who is a doctor in Grenoble.

Grenoble

Pierre Colmez

Jean-Pierre Wintenberger did his thèse de 3-ième cycle and his thèse d’État in Grenoble, under the supervision of Jean-Marc Fontaine. This is where I met him for the first time. I spent two years (1985–87) in Grenoble for my PhD: my
thesis problem had been given to me by John Coates, who was then a professor at Orsay, but I had ended up in Grenoble, with Fontaine as an official adviser, by some bizarre twist. My first year there was rather miserable, as my thesis (about complex L-functions) had a big gap. The second year was much more fun: Fontaine had just returned from his year at Minneapolis, where he was collaborating with William Messing on their proof [6] of Fontaine’s $C_{cris}$ conjecture [3] on periods of $p$-adic algebraic varieties with good reduction, and everybody was speaking of $p$-adic periods (including me: I was fantasizing about a product formula for these numbers, analogous to the product formula for rational numbers, and most of what was being discussed found its way in the output [1] of my fantasies). Roland Gillard [8] had just proved, in the case of ordinary reduction, a $p$-adic analog of Shimura’s multiplicative relations between periods of CM abelian varieties (a vast generalization of the celebrated Chowla–Selberg formula expressing periods of elliptic curves with complex multiplicity in terms of values of the $\Gamma$-function at rational arguments, the simplest formula of this type being $\int_1^\infty \frac{dx}{\sqrt{x^3-2}} = \frac{\Gamma(1/4)\Gamma(1/2)}{2\Gamma(3/4)}$ using methods introduced by Gross [9] in his geometric proof of the Chowla–Selberg formula. Wintenberger had started to attack the general case, which is more difficult, as the periods do not live in $\mathbb{C}$ but it is functorial and provides a bridge between the absolute Galois group of $p$-adic fields and varieties defined over the rationals. Important examples of reasonable infinite extensions of $\mathbb{Q}_p$ are the cyclotomic extension $\mathbb{Q}_p(\mu_{p^\infty})$, the Kummer extension $\mathbb{Q}_p(p^{1/p^\infty})$, or extensions fixed by the kernel of representations $\rho : G_{\mathbb{Q}_p} \to \text{GL}_d(\mathbb{Q}_p)$ “coming from geometry” (i.e., from the étale cohomology of algebraic varieties defined over $\mathbb{Q}_p$ or its finite extensions). The case of the cyclotomic extension gives a dévissage of the absolute Galois group $G_{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ with the following shape: one has a natural exact sequence

$$1 \to G_{F_p((T))} \to G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times \to 1,$$

where $G_{F_p((T))}$ is the absolute Galois group of $F_p((T))$. A reasonable infinite extension of $\mathbb{Q}_p$ can be written as an increasing union of finite extensions $L_n$ of $\mathbb{Q}_p$, and $X(L)$ is the set of sequences $(x_n)_{n \in \mathbb{N}}$, with $x_n \in \mathbb{L}_n$ and $N_{L_{n+1}/L_n}(x_{n+1}) = x_n$ for all $n \in \mathbb{N}$. The set $X(L)$ is turned into a field of characteristic $p$ by setting $(x_n) + (y_n) = (s_n)$ and $(x_n)(y_n) = (t_n)$, with

$$t_n = x_n y_n \quad \text{and} \quad s_n = \lim_{k \to \infty} N_{L_{n+k}/L_n}(x_{n+k} + y_{n+k})$$

(that the limit exists is the nontrivial part of this construction and uses crucially the fact that the extension is reasonable).

Another striking contribution was his construction [16] of a natural splitting of the Hodge filtration for varieties over a $p$-adic field. If $X$ is a smooth projective algebraic variety of dimension $d$ defined over a characteristic 0 field
K, Grothendieck has defined its algebraic de Rham cohomology $H^*_\text{dR}(X/K)$ by means of algebraic differential forms. The $H^*_\text{dR}(X/K)$’s are finite-dimensional $K$-vector spaces that vanish for $i > 2d$ and are endowed with a decreasing filtration, the Hodge filtration, by sub-$K$-vector spaces. If $K$ is a subfield of $C$, then $C \otimes_K H^*_\text{dR}(X/K)$ is isomorphic to the de Rham cohomology of the $2d$-dimensional differentiable manifold $X(C)$, and Hodge theory provides a description of $C \otimes_K H^*_\text{dR}(X/K)$ in terms of harmonic forms, which, in turn, induces a canonical splitting of the Hodge filtration on $C \otimes_K H^*_\text{dR}(X/K)$ (but not on $H^*_\text{dR}(X/K)$ itself, as this splitting usually involves complex numbers that are transcendental over $K$).

Now, if $K$ is a finite extension of $Q_p$, there is nothing like harmonic forms at our disposal (at least, up to now). But Wintenberger managed to define a natural splitting of the Hodge filtration in the case where $X$ has “good reduction modulo $p^i$ and $K/Q_p$ is unramified. In that case the cohomology of $X$ is controlled by that of its reduction, and the morphism $x \mapsto x^p$ that exists in characteristic $p$, the Frobenius morphism, induces a morphism $\varphi$ on the $H^1_{\text{dR}}(X/K)$’s. Hence $H^1_{\text{dR}}(X/K)$ is what Fontaine calls a filtered $\varphi$-module (i.e., a $K$ vector space with a $\varphi$ and a filtration). Now, $p$-adic Hodge theory (nothing to do with harmonic forms) implies that this filtered $\varphi$-module has special properties: there exists an $O_K$-lattice $M$ (with $O_K$ the ring of integers of $K$) such that $\varphi$ is divisible by $p^i$ on $M \cap \mathfrak{Fil}^i$ and $M = \sum_i p^{-i} \varphi(M \cap \mathfrak{Fil}^i)$ (such a lattice is said to be strongly divisible). Wintenberger’s result is a linear algebra result concerning these filtered $\varphi$-modules admitting a strongly divisible lattice, and there is no geometry involved. This result has remained a mystery: is there a theory of $p$-adic harmonic forms that would explain the existence of this natural splitting? Does this splitting exist without assuming $K/Q_p$ to be unramified or $X$ to have good reduction?

Wintenberger was interested in this splitting for the construction [17] of special representations $\varphi : G_K \to \text{GL}_d(Q_p)$ of the absolute Galois group $G_K$ of $K$ with $\varphi(G_K)$ open in a given algebraic subgroup of $\text{GL}_d(Q_p)$ (some kind of inverse Galois problem for finite extensions of $Q_p$): Fontaine-Laffaille theory [5] allows us to translate the problem in terms of $\varphi$-modules admitting a strongly divisible lattice. This was not the last time that Wintenberger used this splitting for questions related to representations of Galois groups (see e.g. [19]).

I did not really follow very closely what he was doing later on after he took a position in Strasbourg, and I was amazed to discover at a conference that he organized in Strasbourg about Serre’s conjecture [14] on the modularity of mod $p$ representations of the absolute Galois group of $Q$ that he was actually proving, in collaboration with Chandrashekhar Khare [10–12], this very conjecture (a dream of quite a few number theorists at the time)! He had been thinking about a strategy to attack it for a long time.

Strasbourg

Chandrashekhar Khare

I got to know Jean-Pierre well through our work together on Serre’s modularity conjecture. We arrived at working on the conjecture by different paths. Jean-Pierre was interested in studying cases of the Mumford–Tate conjecture for abelian varieties defined over number fields and in particular in the first case of it that was still open: that of a 4-dimensional abelian variety defined over a number field. I came to it more directly, and since my PhD thesis in 1995 I had been interested in Serre’s conjecture. I studied congruences between modular forms in my thesis and then got interested in questions of lifting Galois representations. Thus we came from different mathematical backgrounds to Serre’s conjecture, and I think our collaboration benefited from this diversity of interest and training we brought to it.

Jean-Pierre invited me to visit him for a month in Strasbourg, and I visited him in the fall of 2004. Little did I expect that we would spend the month proving the first cases of Serre’s conjecture, which made no superfluous hypotheses on the residue characteristic and image of the representation! Combining observations we had each made independently, we had a result on Serre’s conjecture almost the day I arrived in Strasbourg, and then we spent my month’s stay making sure of the details. Jean-Pierre explained to me his beautiful idea of “killing ramification” in the first days of my visit. The idea is used to reduce the general case of Serre’s modularity conjecture to the level one case. It struck me as an idea that could perhaps naturally occur only to someone who thought very $p$-adically, and Jean-Pierre since his thesis with Jean-Marc Fontaine in 1979 had thought about the then emerging field of $p$-adic Hodge theory, proving foundational results and finding new applications of it.

We were happy to part at the end of my visit, when I returned to Mumbai, having convinced ourselves that we could show that there were no irreducible odd Serre-type...
Galois representations of certain low weights and levels as predicted by Serre and most satisfyingly that the only odd irreducible Serre-type representation of level 1 and Serre weight 12 was the one that arose from Ramanujan’s Δ-function. We also had a strategy to prove all of Serre’s conjecture assuming broad generalizations of the modularity lifting results pioneered by Wiles. At the end of the productive visit we celebrated by going for dinner, along with the number theorists at Strasbourg, to an elegant restaurant, La Casserole, in one of the small twisting alleys near the Cathédrale.

We wrote our first paper on Serre’s conjecture expecting that to prove the full conjecture using our strategy would require very elaborate developments of the modularity lifting machinery by specialists in the area. But within a few months we found a plausible path using a modification of our original strategy that made extensive and novel use of congruences between Galois representations and was consequently less demanding in terms of the modularity lifting theorems we would need. The topography of this path was reminiscent of the winding, intersecting paths surrounding the Cathédrale in Strasbourg that led to different views of its lone Gothic spire, its motif orienting a visitor’s meanderings around it.

It still took us almost five years (2004–09) to complete all the details and have the proof of the full conjecture published. We communicated throughout this period mainly via email, interspersed with meeting at conferences and short visits to Salt Lake City, Paris, Montréal, Strasbourg, Monte Veritá, ... Our work benefited enormously from the rigor and technical deftness Jean-Pierre brought to sorting through the niceties of proving modularity lifting theorems in delicate cases, like the 2-adic lifting theorems we needed for our strategy. I admired Jean-Pierre’s focus on what was central to mathematics and his wide mathematical culture, part of which seemed to be due to his education as a Normalien and then being trained in the formidable French school of arithmetic geometry. I also admired the sense of adventure in his mathematical work that led him to work on Serre’s modularity conjecture, a subject that was a little distant from the mathematics that had occupied him prior to our joint work on it.

Jean-Pierre passed away before he reached sixty-five. I had last seen him in the hospital Pitie-Salpêtrière in Paris in July 2018. He seemed mentally alert but physically worn out. I had hoped to see him again this year, but that was not to be. In his last years he suffered from Parkinson’s disease. Within a week of Jean-Pierre’s death, his advisor, Jean-Marc Fontaine, passed away, without whose foundational work on p-adic Hodge theory our proof of Serre’s conjecture would not have been possible.

I will miss Jean-Pierre’s friendship, unassuming nature, and keen intellect.

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Credits

Figures 1 and 3 are courtesy of Olivier Wintenberger. Figure 2 is courtesy of Ken Ribet.
Three Countries, Two Continents: Common Challenges and Opportunities in Teaching First-Year Mathematics

Andie Burazin, Veselin Jungić, and Miroslav Lovrić

Every year, for tens of thousands of first-year university students across Canada, taking a mathematics course is one of their initial university academic experiences. Serving as both a gatekeeper and a requirement for programs ranging from economics to science to engineering, first-year mathematics courses come with numerous challenges for students, instructors, and mathematics departments.

Present-day first-year students share many characteristics with incoming students from a generation ago: a good number of them lack the necessary mathematical knowledge and skills that they need to succeed and thus find themselves placed into various remedial environments; they sit in large classes and are exposed to mathematical pedagogies that no longer work; they do not see the value of what they are supposed to learn in their mathematics classes, do not seem to be engaged, and often do not attend lectures.

However, the two generations do differ in a significant way. Psychologist and college instructor Linda Bips [Bip10] wrote: “Many of today’s students lack resilience and at the first sign of difficulty are unable to summon strategies to cope. The hardship can be a failing grade on a test, a cut from the team, or a romantic breakup. At the first sign of trouble many become unable to function and persevere. Often they even anticipate difficulties and their anxiety alone paralyzes them.”

For a large number of incoming university students, taking a fast-paced, topic-packed first-year mathematics course presents a significant obstacle, and many of these students cannot successfully meet the expectations and standards of such courses and their instructors. This puts instructors teaching these courses in the position of having to reconcile two seemingly contradictory tasks. On the one hand, they have a responsibility to teach their course at the appropriate level of rigour and to be academically demanding. On the other hand, they play an important role in welcoming, encouraging, and supporting a heterogeneous group of students during one of the major transitions in their lives.

In addition, course instructors are expected to keep up with teaching and learning technologies: from handling ever-evolving learning management systems to creating and/or managing electronic learning resources, to using in-class technology for delivering or enhancing their lectures, to facing competition from a variety of freely available online resources.

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To stay relevant, mathematics departments are forced to reassess the ways of delivery and the content of their courses and academic programs. This includes introducing new pedagogies to meet the needs of the increasingly diverse (in every sense of the word) classes of incoming students and modifying existing courses and/or creating new ones that address the needs of their programs and the programs that they are expected to “serve.”

Ample evidence suggests that university mathematical teaching communities in Australia and the United States are facing similar challenges and opportunities as their Canadian colleagues.

In the last few years, we, the authors of this article, have organized and participated in a number of events that critically examined teaching and learning of mathematics and statistics at a national level in Canada. To share and inform others about our conversations, discussions, and decisions, we reached beyond our Canadian community of first-year instructors. The result is this article, whose purpose is to briefly summarize several recent initiatives in Australia, the United States, and Canada that have the common goal of constructively addressing aspects of the current situation and the future status of teaching first-year university mathematics courses. (We wish to emphasize that this is in no way a comprehensive overview but rather a limited selection of such initiatives.)

In Australia, between 2012 and 2014, Deborah King from the University of Melbourne led the First Year in Maths (FYiMaths) project. This project, funded by the Australian Government Office for Learning and Teaching, aimed to promote and support strategic changes and improvements in teaching first-year mathematics courses and enhancing the overall student-learning experience. The issues addressed by the project team ranged from discussing the challenges that students face during their transition from high school to university to arguing for a well-defined and supportive departmental teaching stream faculty position that would coordinate first-year mathematics courses.

One of the outcomes of the FYiMaths project was the establishment of the First Year Mathematics and Statistics Courses (FYiMaths Network, “a ‘community of practice’ providing an informal and supportive group for academics and educators to access information, exchange ideas, address shared challenges and collaborate on research and policy questions” [https://fyimaths.org.au]. Currently, the network includes academics and educators teaching undergraduate mathematics, individuals that provide additional mathematics support, and secondary mathematics teachers in Australia and New Zealand.

The FYiMaths project final report [Kin15] is quite revealing of the current state of first-year mathematics teaching in Australia. The report provided a list of challenges that Australian university instructors face when teaching first-year mathematics courses, which mirrors the challenges of their North American colleagues. These teaching challenges are further validated in a recent article in the Notices of the AMS [Buc19] in which the author summarizes the “most significant results, events, and developments in undergraduate mathematics education of the last decade” (p. 46).

In the United States, the Mathematical Association of America (MAA) coordinated two large studies of introductory mathematics courses, centered around calculus [Bre16, MAA10]; see also [Bre15]. In 2016, the conference Precalculus to Calculus: Insights and Innovations was held in St. Paul, Minnesota. The conference “shared insights gained from the MAA studies of calculus instruction, highlighted some of the many innovations now underway around the country, and facilitated networking of peer institutions that are seeking to improve student successes in precalculus and calculus” [MAA16].

In 2015, AMATYC, AMS, ASA, MAA, and SIAM jointly sponsored “The Common Vision” project with the ultimate goal to “galvanize the mathematical sciences community around a modern vision for undergraduate programs and to spur grassroots efforts within the community as a foundation for addressing the collective challenges we face” ([Sax16, p. vi]). One of the outcomes of this project was a far-reaching document titled A Common Vision for Undergraduate Mathematical Science Programs in 2025 [Sax16]. The consensus among all five organizations is that the status quo is unacceptable, and the document calls for significant action in the areas of “curricula, course structure, workforce preparation, and faculty development” (p. vi).

In 2018, a resource Guide to Evidence-Based Instructional Practices in Undergraduate Mathematics [Abe18] was published, with an aim to support faculty seeking to improve students’ learning outcomes. The authors of the guide summarized their work and findings in the initial section “Manifesto: A Declaration of Values,” a passionate call for a “community wide transformation toward improved learning experiences and equitable access to mathematics for all students.”

In Canada, the efforts to address at least some of the issues around teaching first-year mathematics courses have intensified, and recently the conversations and action have emerged as a genuine grassroots movement. In 2017, two of the coauthors of this article published their “Call for national dialogue: The present and future of teaching first year mathematics at Canadian universities” [Jun17]. The article called for instructors from across Canada to “communicate, share our experiences, coordinate our efforts and work together.”

The response of the Canadian mathematics teaching community was overwhelming. Since 2017, several meetings on the topic of teaching the first-year mathematics courses have been organized at a national level. In addition, in 2017 the First Year Mathematics and Statistics Courses Repository, a resource supporting this ongoing national dialogue, was created and introduced to our community of
first-year instructors. This shareable dynamic online database ([https://firstyearmath.ca](https://firstyearmath.ca)) contains extensive data about first-year mathematics and statistics courses collected from instructors across Canada. The database includes course content, information about resource and technology use, learning outcomes, modes of delivery, connections with other courses, as well as informal descriptions of various practices in teaching these courses.

The attendees at the national meetings represent a cross section of the Canadian mathematics and statistics teaching community. The audiences comprised postsecondary teaching practitioners from all walks of academic life: graduate students, contract/sessional faculty, teaching faculty, research faculty, and undergraduate associate chairs.

About a half of the attendees were female, which reflects the fact that at the university level a large group of female mathematicians and statisticians are involved in teaching lower-level courses. This implies that when we discuss the gender gap in our mathematics and statistics postsecondary community, then we—faculty, departments, as well as relevant professional organizations—need to recognize and value teaching (and teachers!) at a higher degree than presently. Of course, when we talk about “women in mathematics and statistics” we must celebrate all great achievements by “women researchers in mathematics and statistics,” and we have to continue pointing out that they are still significantly underrepresented in academia. What, in our view, is missing is the full recognition of the tremendous contributions that a large group of female mathematicians and statistics instructors make on a daily basis in their classrooms across Canada (and, we assume, in Australia and the USA). Besides being on the front line of postsecondary education, these talented, knowledgeable, and hard-working women are the true role models and inspiration for all their students.

The Canadian mathematics and statistics teaching community must advocate for a major shift in the way mathematics and statistics departments view and treat their teaching faculty, in particular the large number of instructors who are in precarious contract, part-time, or limited-term positions. And yet it is these instructors who are routinely tasked with teaching important first-year courses. Mathematics departments should create a respectful and supportive environment for the teaching faculty and offer full-time, permanent employment while allowing for a small number of emergency short-term contractual positions.

A further barrier to a significant change is the well-known gap between mathematics education research and teaching practice (see, for instance, [Art99], where the author articulates the reasons why this gap naturally emerges). Our meetings have demonstrated that the number of Canadian postsecondary mathematics and statistics instructors who are familiar with the relevant education research is rather small. In this light, thinking about evidence-based teaching, we are curious to know how the research findings could be identified (when and if they exist) and how to ensure that they reach those who are actually teaching. We believe that our conferences, the Repository, and the newsletter ([https://firstyearmath.ca/newsletter](https://firstyearmath.ca/newsletter)) we started publishing are small steps toward achieving this goal.

Our meetings have witnessed numerous calls for examining the unsustainably dominant role of calculus in the university mathematics curriculum. We have heard strong views suggesting that perhaps the time has come to abandon calculus and restructure first-year mathematics around a different paradigm, such as mathematical modeling, problem solving, or mathematical thinking.

Somewhat unexpectedly, it became apparent that in Canada responses to the set of common challenges in teaching first-year mathematics courses may depend on a number of external parameters, such as the size of the university (large versus small), institution’s mandate (research intense versus teaching intense), financial model (public versus private), and geographical location (have versus have-not provinces). Our meetings made us more aware that the outcomes of what we do in our classes may be directly affected by the dynamics of the economic, social, and political environments in which our students and we, their instructors, live and work.

In summary, recent documents and conversations related to teaching first-year courses across the three countries convey a sense of urgency and the need for significant changes. The challenges, problems, objectives, and strategies that have been discussed are definitely not new: they have been around for decades (see, for instance, [Com97]). As it seems that our previous efforts have not managed to alleviate any of the dominant problems, we have to ask ourselves: should we, postsecondary teaching practitioners and our students, hope that this time will be successful? Our answer is yes, and our optimism is based on the fact that there is a widespread consensus that “the status quo is unacceptable.” The recent initiatives indicate that there is growing awareness that we, the global mathematics and statistics teaching community, have to put additional effort into supporting our current and future students to learn mathematics and to achieve their academic goals. By using proven, evidence-based ways to transfer our knowledge of and passion for mathematics to our students, we can meet our share of the responsibility to ensure that the next generation of scholars and instructors holds high the torch of mathematics and statistics.
References


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Author photos are courtesy of the authors.

Andie Burazin
Veselin Jungić
Miroslav Lovrić
An Invitation to the Rogers–Ramanujan Identities
Reviewed by Krishnaswami Alladi

In the entire theory of partitions and \(q\)-hypergeometric series (\(q\)-series, for short), the two Rogers–Ramanujan identities are unmatched in simplicity of form, elegance, and depth:

\[
R_1(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{m=1}^{\infty} \frac{1}{(1-q^{5m-4})(1-q^{5m-1})},
\]

\[
R_2(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{m=1}^{\infty} \frac{1}{(1-q^{5m-3})(1-q^{5m-2})}
\]

for \(|q| < 1\). The partition interpretation of the identities is given below.

The identities have a fascinating history. The Indian mathematical genius Srinivasa Ramanujan (1887–1920) noticed that the ratio \(R_1(q)/R_2(q)\) admits a continued fraction expansion, which in view of (1) and (2) enjoys a product representation. More precisely, the Ramanujan continued fraction identity is

\[
\rho(q) := \frac{R_1(q)}{R_2(q)} = 1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \cdots}}} = \prod_{m=1}^{\infty} \frac{(1-q^{5m-3})(1-q^{5m-2})}{(1-q^{5m-4})(1-q^{5m-1})}
\]

for \(|q| < 1\). (3)

More importantly, Ramanujan realized that the ratio \(q^{1/5}/\rho(q)\) satisfies transformation properties and used them to make the incredible evaluation

\[
\frac{q^{-2\pi/\sqrt{5}}}{\rho(e^{-2\pi\sqrt{5}})} = \frac{\sqrt{5}}{1 + (5^{3/4}(\sqrt{5}-1)/2)^{1/5}} - \frac{\sqrt{5} + 1}{2}.
\]

He sent (4) along with dozens of startling results of great variety in his first of two letters in 1913 to G. H. Hardy (1877–1947) of Cambridge University (see [4, p. 8]). It was a result such as this that made Hardy realize that Ramanujan was a genius in the class of Euler or Jacobi for sheer manipulative ability.

Ramanujan's continued fraction plays a central role in the theory of modular forms and has links with the fundamental elliptic modular function \(j(\tau)\). We provide a brief description of this now.

The modular group \(\Gamma(1)\) is the set of \(2 \times 2\) matrices with integer entries having determinant 1, where a matrix \(A\) is identified with its negative \(-A\). This is because \(\Gamma(1)\) is
identified with the set of mappings

\[ \phi(\tau) = \frac{a\tau + b}{c\tau + d} \]

where \( \text{Im}(\tau) > 0 \) and \( a, b, c, d \) are integers satisfying \( ad - bc = 1 \). A modular function on \( \Gamma(1) \) is a meromorphic function \( f \) on the upper half-plane that satisfies

\[ f(\phi(\tau)) = f(\tau) \]

for all such mappings \( \phi \) and has, at worst, a pole at \( i\infty \).

The elliptic modular function \( j(\tau) \) is a modular function on \( \Gamma(1) \) and is called a \textit{Hauptmodul} for \( \Gamma(1) \) because every modular function on \( \Gamma(1) \) can be expressed as a rational function of \( j \). One can generalize this discussion by considering subgroups of the modular group, and an important class of subgroups comprises the principal congruence subgroups \( \Gamma(N) \), namely, the members of \( \Gamma(1) \) that are congruent to the identity modulo \( N \). It turns out that \( q^{1/5}/\rho(q) \) is a Hauptmodul for the congruence subgroup \( \Gamma(5) \), where \( |q| < 1 \) is represented as \( q = e^{2\pi i \tau} \), with \( \text{Im}(\tau) > 0 \).

Although (1) and (2) form the source for (3), when Ramanujan arrived in England in 1914 and Hardy asked for proofs of (1) and (2), Ramanujan could not provide them. Interestingly, in 1917, when Ramanujan was going through certain old issues of the \textit{Journal of the London Mathematical Society}, he accidentally came across a trilogy of papers dating back to 1893–95 of the British mathematician L. J. Rogers (1862–1933), in which (1), (2), and (3), and many identities similar to (1) and (2) were proved [6]. Rogers was a brilliant mathematician, who in Hardy’s own admission was a man of talents similar to Ramanujan (see [4, p. 91]), but his work was ignored by his British peers.

Indeed, it was Ramanujan’s discovery of Rogers’s work that led to Rogers’s belated recognition.

As this drama was playing out in England, Issai Schur (1875–1941) in Germany, cut off from England due to World War I, had simultaneously and independently discovered and proved the Rogers–Ramanujan identities and stated their partition (combinatorial) version as well. Actually Hardy’s colleague Major MacMahon (1854–1929) had stated the partition version of (1) and (2), but in 1915 since he was not aware of Rogers’s proof, he stated them as unproved conjectures in his monumental book on combinatorial analysis [5]. MacMahon was a wizard in computing, and he even assisted in numerically verifying the celebrated Hardy–Ramanujan asymptotic formula for partitions, but before becoming a mathematician he had a successful career in the British military, and that was how he was known as Major MacMahon even in the mathematical world.

Neither Rogers nor Ramanujan stated the combinatorial (partition) interpretation of (1) and (2), and so the following partition theorem is independently due to MacMahon and Schur:

**Partition version of the Rogers–Ramanujan identities.** For \( i = 1, 2 \), the number of partitions of an integer into parts that differ by \( \geq 2 \) and least part \( \geq i \) is equal to the number of partitions of that integer into parts \( \equiv \pm i \pmod{5} \).

Since Schur emphasized the partition version of (1) and (2) as above, he was able to discover the next partition theorem with parts in congruence classes \( \pm 1 \pmod{6} \) connecting to partitions with parts differing by \( \geq 3 \), but he needed the extra condition that there should be no
consecutive multiples of 3. And this is where the real story of Rogers–Ramanujan-type identities begins, which leads us to this book of Sills.

A Rogers–Ramanujan-type (R-R type) identity is an infinite $q$-series = infinite $q$-product identity, where the series is the generating function of partitions whose parts satisfy gap conditions, and the product is the generating function of partitions whose parts satisfy congruence conditions. The term R-R type derives from (1) and (2), which are the prototype. The very first partition theorem that is of this type is due to Euler, who founded the theory of partitions and $q$-series. Euler’s theorem states that the number of partitions of an integer into distinct parts is equal to the number of partitions of that integer into odd parts; this can be analytically expressed in the form

$$
\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{m=1}^{\infty} \frac{1}{(1-q^{2m-1})}.
$$

Schur viewed the distinct parts condition as gap $\geq 1$ between parts, and odd parts as those $\equiv \pm 1 \pmod{4}$. From this viewpoint, (1) is the “next level” partition theorem (but much deeper!) because the gap condition is $\geq 2$ and the congruence is $\equiv 1 \pmod{5}$. Schur’s theorem itself deals with gaps $\geq 3$, with the additional condition that there are no consecutive multiples of 3 connecting to the congruence condition for parts $\equiv 1 \pmod{6}$. As the modulus of the congruence increases, the gap conditions become more complex (see Andrews [1, Ch. 7] for a good discussion of R-R type identities).

In the 1940s, W. N. Bailey (1893–1961) found a mechanism [3], now called the Bailey chain, to generate R-R type identities. This was developed by his student Lucy Slater (1922–2008), who published a list [7] of more than 100 R-R type identities. The full potency of the Bailey chain was revealed by George Andrews, who in the 1960s and 1970s led the development of a systematic theory of R-R type identities both from a $q$-series and partition theoretic point of view (see [2, Ch. 3]). Andrews was also inspired by Basil Gordon’s lovely and important generalization of the Rogers–Ramanujan partition theorem to all odd moduli (1961). Today, R-R type identities are playing a central role in (i) the theory of modular forms, (ii) conformal field theory and statistical mechanics, and (iii) Lie algebras (vertex operators).

When the residue classes can be paired off as conjugates $(\pmod{M})$ in the product form of an R-R type identity, such an identity has links with the theory of modular forms, because the conjugacy of the residue classes means that these products can be represented in terms of theta functions by the use of Jacobi’s fundamental triple product identity:

$$
\sum_{n=-\infty}^{\infty} z^n q^{n^2} = \prod_{m=1}^{\infty} \frac{(1 - q^{2m})(1 + q z^{2m-1})(1 + z^{-1} q^{2m-1})}{(1 - q^{2m})}.
$$

(In the product above, if one replaces $q$ by $q^{M/2}$ and chooses $z = q^{(M-2a)/2}$, for some integer $a \in [1, M/2]$, then one would be dealing with conjugate residue classes modulo $M$ because of the presence of $z$ and $z^{-1}$.) A beautiful example is the pair of Gollnitz–Gordon identities,

$$
\sum_{n=0}^{\infty} \frac{q^{n^2}(1+q)(1+q^3)\cdots(1+q^{2n-1})}{(1-q^2)(1-q^4)\cdots(1-q^{2n})} = \prod_{m=1}^{\infty} \frac{1}{(1-q^{8m-7})(1-q^{8m-4})(1-q^{8m-1})}
$$

and

$$
\sum_{n=0}^{\infty} q^{n^2+2n}(1+q)(1+q^3)\cdots(1+q^{2n}) \frac{1}{(1-q^2)(1-q^4)\cdots(1-q^{2n})} = \prod_{m=1}^{\infty} \frac{1}{(1-q^{8m-5})(1-q^{8m-4})(1-q^{8m-3})},
$$

where the two products can be written as ratios of certain theta functions. These identities are to the modulus 8 what the Rogers–Ramanujan identities are to the modulus 5. L. J. Rogers found several such identities connected to the modulus 5 or its multiples, and many of his identities (together with (1) and (2)) turned out to play a central role in Rodney Baxter’s solution (1979) of the hard-hexagon model in statistical mechanics (see Andrews [2, Chs. 1

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Figure 3. Leonard James Rogers, FRS (1862–1933).
Thus the word “invitation” in the title is most appropriate, provides a substantial number of references that will lead both the student and the expert to some of the most important sources in the field. Especially valuable to the researcher is the list of 200 R-R type identities (extending the Slater list substantially), classifying them according to moduli in increasing size. Of historical value is the correspondence of Bailey with Slater and Freeman Dyson that Sills has presented in the appendices. The world-renowned physicist Dyson started out working on partitions and q-series as an undergraduate at Cambridge University, inspired by the work of Ramanujan. G. H. Hardy assigned the young Dyson, twenty years old then in 1943, to referee a certain paper of Bailey on his mechanism to generate Rogers–Ramanujan type identities. Dyson had some penetrating comments on Bailey’s work, and so even though he was the referee, Hardy informed Bailey as to who the referee was in sending the comments. A correspondence between Bailey and Dyson ensued that lasted until 1946.

Finally, I should say that, aptly, George Andrews, as a leader in the field and as the former PhD advisor of Andrew Sills, has written a fine foreword to the book.

In summary, this is a comprehensive and easily readable treatment of R-R type identities of appeal to both the expert and the potential entrant to the field. It will be a fine addition to both libraries and your personal book collection.

References


Book Review

January 2020 Notices of the American Mathematical Society 71

Credits

Photos of Srinivasa Ramanujan and G. H. Hardy are courtesy of the Archives of the Mathematisches Forschungsinstitut, Oberwolfach.

Photo of Leonard Rogers is courtesy of The Royal Society, England.

Photo of the author is courtesy of the author.
New and Noteworthy Titles on our Bookshelf January 2020

**Lost in Math: How Beauty Leads Physics Astray**
by Sabine Hossenfelder

The pursuit of mathematical beauty has derailed modern physics and ushered in an existential crisis that threatens to reshape the field. This is the opinion of Sabine Hossenfelder, a theoretical physicist who is deeply frustrated with the current state of affairs. Some aesthetically appealing theories, such as the existence of multiverses, appear difficult or impossible to verify experimentally. If it cannot be tested and has no hope of being falsified, is it really science? Shouldn’t the discipline honor definitive experiments over untestable theories? These are the sorts of questions that Hossenfelder grapples with.

Although the book is non-technical, the reader should probably have a healthy interest in modern physics. For a book with “math” in the title, its mathematical content may leave the typical Notices reader disappointed, since the mathematics is discussed mostly at a vague and superficial level. Nevertheless, there is no shortage of familiar names. For example, Alain Connes, Henri Poincaré, David Hilbert, Richard Borcherds, and Edward Witten make very brief appearances, as do Calabi–Yau manifolds, E8, and the Monster Group. However, the book’s main focus is physics, not mathematics. Many of the key players in particle physics and cosmology play prominent roles. Readers who are casual watchers of popular science programs will recognize Max Tegmark, Nima Arkani-Hamed, Steven Hawking, and several others.

Hossenfelder observes a few disturbing trends in her discipline that should resonate with mathematicians: increased specialization, lack of communication between subfields, the focus on grant writing, the rise in non-tenure-track positions, and a system that rewards work in “popular” areas at the expense of innovation and risk-taking.

There is no question that this book is interesting and eye opening. Despite its intense focus on the pursuit of mathematical elegance in foundational physics and the ramifications of this search, this is definitely not a “math book.” Nevertheless, it should be a light and enjoyable read for any mathematician with a sincere interest in foundational physics.

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**On Gravity: A Brief Tour of a Weighty Subject**
by A. Zee

Gravitational waves have received a lot of attention in the popular science media in recent years. There are probably a great many mathematicians who are curious about these exciting developments but who are unsure where to start. Mathematicians with no background in physics should find this book an appealing introduction to gravity. Zee describes its level as “slightly above a popular physics book and somewhat below a physics textbook.” This is an entirely accurate description. The book is light on mathematics, although a short appendix entitled “What Does Curved Spacetime Mean?” involves some multivariable calculus.

Zee is well known in the theoretical physics community for his “…in a Nutshell” books. He approaches gravity with his signature conversational and humorous style. The title sums up the book perfectly. It is a whirlwind tour of gravity from Newton and Einstein to Hawking radiation, gravitons, and LIGO. These are covered in nineteen chapters, each punctuated by short sections with thought-provoking titles such as “Gravity is absurdly weak,” “Let there be light! But wait, what is light?” and “Are inertial mass and gravitational mass really the same mass?” The book ends with the imagined responses of Aristotle, Newton, Einstein, and a yet-to-be determined quantum theorist of gravity to the question “Why do we all fall down?” Though mathematical physicists will find this book too lightweight, most Notices readers will find this book engaging, informative, and entertaining.
The AMS Book Program serves the mathematical community by publishing books that further mathematical research, awareness, education, and the profession while generating resources that support other Society programs and activities. As a professional society of mathematicians and one of the world’s leading publishers of mathematical literature, we publish books that meet the highest standards for their content and production. Visit bookstore.ams.org to explore the entire collection of AMS titles.

Topology Through Inquiry
Michael Starbird and Francis Su
MAA Textbook #58, 313 pp., $69

What properties make for a good IBL textbook? The whole point of IBL, Inquiry-Based Learning, is to ask good questions. Questions that motivate and inspire students to create their own knowledge. Questions that point in fruitful directions without divulging the essence. Questions that arise naturally, in a way that teaches where the question came from and how a student might ask his or her own good questions. It’s not enough to just list some axioms, definitions, and theorem statements and encourage the students to get started. The steps to the theorem have to be the right size, you have to nudge the students in productive directions with your questions and examples, even definitions should be motivated by appropriate examples. It takes years and many iterations to polish IBL course notes into a high-quality textbook.

Taken together these authors have approximately sixty-five years of experience teaching using inquiry. This experience shows. Topology Through Inquiry is, quite clearly, the product of decades of thinking about how to teach topology, and thinking about how to teach using inquiry, and, in a larger sense, teaching students how to think for themselves.

One other consequence of these authors’ extensive experience is the sheer quantity of material in this text. There is more than enough material here to fill an entire semester with just point-set topology. For your second semester you could cover the four chapters on geometric topology that provide (i) two different proofs of the classification of 2-manifolds, (ii) the fundamental group, and (iii) covering spaces. The last third of the book presents homology—first we get introduced to $Z_2$ simplicial homology, then the $Z$ version, and finally singular homology. Taking a student through the entirety of this book could easily keep him or her occupied for a year or two. He or she will learn an enormous amount of topology by the end and, more importantly, will have gained enormous insight into how to think like a mathematician.

A History of Mathematics in the United States and Canada (Volume 1): 1492–1900
David E. Zitarelli
MAA Spectrum #94, 474 pp., $120

The history of mathematics is not just the evolution of concepts and theories. At least, it is not only that in David Zitarelli’s fascinating A History of Mathematics in the United States and Canada. It also includes the story of journals and of colleges and universities and of professional societies and conferences and of textbooks and curricula. And, mostly, more than anything else really, it is the story of the people who lived mathematical lives. People like Isaac Greenwood, the alcoholic and slightly disreputable Hollis Professor of Mathematicks [sic] and Natural and Experimental Philosophy at Harvard in the years 1728–38, and David Rittenhouse, who in 1771 authored the first scientific paper published in the US. In the nineteenth century there appear characters whose work consisted entirely of mathematics—e.g., Benjamin Peirce and Nathaniel Bowditch—despite the absence of any US mathematical infrastructure. This infrastructure—journals, professional societies, graduate programs—comes into existence only after the founding of Johns Hopkins in 1876.

Zitarelli’s story is a rich tapestry of the ideas and personalities and institutions that formed mathematics in the US and Canada in the years 1492 to 1900. It paints a vivid picture of the establishment and growth of a mathematical enterprise in North America as the activity of people and tells the fascinating personal and scientific stories of those folks. Volume two, which covers the first half of the twentieth century, should appear in 2021.
Uncovering Lottery Shenanigans

Skip Garibaldi

For centuries, the lottery has been an interesting subject for mathematics. The combination of enormous amounts of money, minuscule probabilities, and repeated game play naturally leads to subtle probability problems that have attracted the attention of well-known mathematicians such as Leonhard Euler [6], Herman Chernoff [4], and Persi Diaconis and Frederick Mosteller [3].

The amounts of money are indeed enormous, as Americans spend around $80 billion each year on the lottery. That’s more than twice Microsoft’s worldwide revenue and is roughly what the United States spends annually on incarceration [1]. With that much money changing hands and with laws and the level of enthusiasm for enforcing those laws varying from state to state, it’s not surprising that there are shenanigans going on.

What kinds of shenanigans, you ask?

• Inside jobs, like the drawings allegedly rigged by Eddie Tipton [2] or the 1980 Pennsylvania Lottery scandal dramatized in the movie Lucky Numbers [8].
• Exploiting a design flaw in a particular lottery game, as Mohan Srivastava did for the Ontario Lottery with their scratcher games Tic-Tac-Toe (2003) and SuperBingo (2007); see [14], [10].
• People buying lots of tickets on the rare occasions when the expected rate of return was positive, such as:
  o the Massachusetts Cash WinFall game that ran 2005–12 [5], [13].
  o Stefan Mandel buying all (or almost all) of the tickets in the Virginia Lottery in 1992 and others [12].
• Whatever Joan Ginther, the so-called “Luckiest Woman on Earth,” is up to, as described in [7], [9].

Those are all interesting stories, and I’ve provided references in case you want to learn more about any of them, but this talk is about a different kind of shenanigan. The shenanigan in this talk doesn’t depend on the rules of any particular game. People are using the technique now in many states across the United States and have been for many years. In some places it is illegal, and everywhere it is practiced it diverts money from the poor and powerless. It has even been used by criminals to launder money.

The story begins with an unusually perceptive reporter, Lawrence Mower, who looked at the list of prizes given out by the Florida Lottery and noticed that some individuals had claimed staggering numbers of large prizes. On the one hand, something fishy was going on. On the other hand, there is a positive probability that those individuals legitimately won that many prizes with only modest spending—that they were phenomenally lucky. (That was effectively the theory presented by the Florida Lottery secretary, who said, “That’s what the lottery is all about. You can buy one ticket and you become a millionaire” [11].) Deciding between these two views is not as easy as you might guess, and the answer can depend on the details of the games played. For example, a pathological gambler who bets in the right way can indeed win many modestly large prizes while only losing their house.

To resolve the disagreement between these two views, Mower enlisted the help of three professors who set to work proving theorems and exploiting those theorems to make calculations. This talk will be about that work and the results of Mower’s subsequent on-the-ground investigations. The success of this work has led to further investigations by numerous reporters across the country, which in turn have led to arrests, changes in state policy, and legal disputes.

References
Credits
Author photo is courtesy of the author.
Different Problems, Common Threads: Computing the Difficulty of Mathematical Problems

Karen Lange

Mathematics is filled with existence theorems such as

Theorem 1. Every vector space has a basis.

Such statements do not address how one goes about finding the known-to-exist object. For example, Theorem 1 naturally leads to the “Basis Problem.”

Problem 2 (Basis Problem). Given a vector space, can we compute a basis for it?

It turns out that the answer to this question is no—from a computational viewpoint, bases may not exist for vector spaces. For this reason, we say that the Basis Problem is not a “computable” problem. We can, however, ask just how far from computable the Basis Problem is and what other problems have the same computational power. A natural way to compare the algorithmic difficulty of two problems is to determine whether having the ability to solve one allows a person to solve the other. In this case, the latter problem must be no “harder” than the former. Under this problem-reduction approach, two problems have the same computational power if they each can be used to solve the other.

In my talk, we will explore the key ideas behind taking a “computable” perspective on mathematics and how this compares with an “existence” one, as in the example above. We will also illustrate the problem-reduction approach using problems from across mathematics. Moreover, this approach has strong connections to proof theory, specifically, calibrating exactly which axioms of mathematics are needed to prove the original existence theorems.

My talk draws on ideas from computability theory (also known as recursion theory), a branch of logic. To learn more about what logic offers other areas of mathematics, check out the AMS-ASL Special Session, Logic Facing Outwards, Wednesday, January 15, 2:15PM–6PM, and Thursday, January 16, 8AM–12PM. These talks, like mine, will not assume a background in logic. The remainder of this article gives a taste of some of the ideas mentioned earlier.

To address the Basis Problem, one first needs a precise understanding of what it means to “compute.” Alan Turing gave the accepted definition [4] in terms of so-called “Turing Machines,” and his model of computation laid the theoretical framework for the invention of computers as we know them today. We will rely here on the useful (and accurate) anachronism of thinking of Turing’s model as being a computer and Turing Machines as being computer programs that use natural numbers for inputs and outputs. In short, a Turing Machine is a finite set of instructions that can be applied to a particular input in a step-by-step and completely determined fashion. Given some input n, the resulting procedure either terminates or “halts” after some finite number of steps and gives an output, or the computation continues to proceed forever. We can now define when a function with domain and codomain \( \mathbb{N} \) is computable.

Definition 3. A function \( f : \mathbb{N} \to \mathbb{N} \) is computable if there is a Turing Machine that, when run on input \( n \), halts and outputs \( f(n) \).

As long as a function’s domain and codomain can be encoded by natural numbers, we can also discuss the computability of the function relative to that encoding. For example, it is not hard to come up with an easy-to-compute bijection between \( \mathbb{N}^2 \) and \( \mathbb{N} \) that encodes pairs \((i,n)\) as single natural numbers \(i \cdot n\). Furthermore, a computer program written in a particular language can be viewed as a finite sequence of symbols from a fixed finite alphabet. By ordering all such finite sequences (say by length and then lexicographically), we obtain a list of programs \( P_i \) \( i \in \mathbb{N} \). Many of the programs \( P_i \) are nonsense (e.g., do not obey syntax rules), but certainly every possible program written in that language appears. Since there are only countably many programs, and thus Turing Machines, and uncountably many functions \( f : \mathbb{N} \to \mathbb{N} \), we know that there exist noncomputable functions. Even better, though, the encoding \( (P_i)_{i \in \mathbb{N}} \) allows us to write down a particular example of a function that is not computable.

Consider the Halting Problem:

Problem 4 (Halting Problem). If program \( P_i \) is run on input \( n \), does the computation halt?

In other words, the problem asks whether a particular program’s computation terminates when run on a particular input (see Figure 1). The Halting Problem is likely all too familiar. Computer users encounter it when they see the spinning "wheel" or "beachball of death" on their screen.
and have to decide whether to wait or restart the machine (see Figure 2).

We can encode a “solution” to the Halting Problem by the function $h : \mathbb{N} \to \mathbb{N}$ where

$$h(i, n) = \begin{cases} 1 & \text{if program } P_i \text{ run on input } n \text{ halts after finitely many steps,} \\ 0 & \text{otherwise.} \end{cases}$$

One can show that the function $h$, and thus the Halting Problem, is not computable.

Notice that the Halting Problem is an existence statement in disguise: the problem is asking whether there is some point in time when the computation is finished. In fact, the Halting Problem encodes all purely existential information in a sense that can be made precise and so plays a central role in computability theory.

Let us return to the problem of computing a basis for a given vector space. Although we have formalized what we mean by computation, the Basis Problem remains imprecise without making explicit the terms “given a vector space” and “compute a basis.” If we are to “compute a basis,” it seems reasonable that we must be able to compute information about the given vector space. As alluded to above, an object can only be computable if it can be encoded in terms of natural numbers. Hence, we will assume we are working with countable vector spaces, say over the countable field $\mathbb{Q}$ for concreteness. (This assumption is standard, but there are ways to discuss the computability of uncountable structures; see, e.g., [2].) Once an indexing $(\vec{u}_i)_{i \in \mathbb{N}}$ of a vector space $V$ is fixed, the vector addition and scalar multiplication by any given $q \in \mathbb{Q}$ on $V$ induce functions on the indices $i$. In other words, the induced functions, which we will call $i_+$ and $i_q$, describe the vector space operations in terms of the indexing (see Figure 3 for a careful definition of $i_+$ and $i_q$).

**Definition 5.** Let $V$ be a countable vector space over $\mathbb{Q}$.

1. A presentation of $V$ is an indexing $(\vec{u}_i)_{i \in \mathbb{N}}$ of the elements of $V$ together with the functions $i_+$ and $i_q$ for each $q \in \mathbb{Q}$.
2. A presentation $(\vec{u}_i)_{i \in \mathbb{N}}$ of $V$ is computable if the functions describing vector addition and scalar multiplication in terms of this indexing are computable; i.e., $i_+$ and the function $(q, n) \to i_q(n)$ are computable.

<table>
<thead>
<tr>
<th>Vector Addition</th>
<th>Scalar Multiplication</th>
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<tbody>
<tr>
<td>on Indices</td>
<td>by $q \in \mathbb{Q}$</td>
</tr>
<tr>
<td>$i_+ : \mathbb{N}^2 \to \mathbb{N}$</td>
<td>$i_q : \mathbb{N} \to \mathbb{N}$</td>
</tr>
<tr>
<td>$(i, j) \to k$ if $\vec{u}_i + \vec{u}_j = \vec{u}_k$</td>
<td>$n \to m$ if $q \vec{v}_n = \vec{v}_m$</td>
</tr>
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Thus, when we say “given a vector space” in the Basis Problem, we mean that we are handed a computable presentation $(\vec{u}_i)_{i \in \mathbb{N}}$ of a vector space. The phrase “compute a basis” is then synonymous with giving an algorithm for determining which $\vec{u}_i$ in $V$ are in the basis. In other words, a basis relative to a presentation $(\vec{u}_i)_{i \in \mathbb{N}}$ is computable if the characteristic function (in terms of the indices $i$) of the basis is computable. We can now formalize the Basis Problem as

**Problem 6.** Does every computable presentation of a (countable) vector space have a computable basis?

We mentioned earlier that the answer is no. One can construct such a vector space by building a vector space in which certain linear dependences do not become apparent until late in the construction. Although the Basis Problem is not computable, it is no harder than the Halting Problem. Given a computable vector space $V$, the ability to solve the Halting Problem allows one to compute a basis for $V$. Even more exciting, the Basis Problem can be used to solve the Halting Problem, in that there is a computable presentation of a vector space $V$ such that any basis of $V$ can be used to compute the function $h$. Therefore, the two problems are computationally the same [1]; see also [3, §III.4].

To fully understand the power of these problems, we need to compare them to others. A version of the Heine-
Borel theorem for the real closed unit interval (i.e., finding a finite subcovering of an open covering made of intervals) is a strictly weaker problem than the Halting/Basis Problems, whereas writing a countable abelian group as the direct sum of a divisible group and a reduced group is a strictly harder one [3, §§IV.1, VI.4]. The overall structure of problem difficulty is extremely rich, and understanding which problems have the same computational power illuminates what makes these problems “tick.” Please join me as we take a computable perspective on mathematics at the 2020 Joint Meetings. I also hope to see you at the AMS-ASL Special Session, Logic Facing Outwards, Wednesday, January 15, and Thursday, January 16.

References

Karen Lange

Credits
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Photo of the author is courtesy of Richard Howard.
Smooth Invariants of Four-Dimensional Manifolds and Quantum Field Theory

Gregory W. Moore

In the past fifty years there has been a very vibrant interaction between physicists who study quantum field theories and string theories on the one hand and mathematicians investigating a wide variety of topics on the other hand. The interactions with mathematicians who study various aspects of (algebraic) geometry and (low-dimensional) topology have been particularly strong. Important mathematical discoveries such as (homological) mirror symmetry and the topological field theory interpretations of knot polynomials and knot homologies have their roots in physics. This talk will focus on one paradigmatic example of such an interaction between physicists and mathematicians: the relation between mathematical invariants of four-dimensional smooth manifolds and the physics of supersymmetric quantum field theories.

From the mathematical viewpoint, the differential topology of four-dimensional manifolds presents many interesting unanswered questions. There is nothing even close to a useful classification of simply connected, compact, oriented four-manifolds. Two important examples of smooth invariants of four-manifolds are based on the study of the moduli space of solutions to partial differential equations for connections on principal bundles over a four-manifold $X$. The Donaldson invariants, discovered in the 1980s, are based on the study of the renowned instanton equations. These are equations for a connection on a principal bundle over $X$ that state that the curvature is an anti-self-dual 2-form. There is typically a moduli space of solutions to these equations, and Simon Donaldson showed that the intersection theory on this moduli space can be used to define differential topological invariants of $X$. The Seiberg–Witten invariants are equations associated to a spin-c structure on $X$. They are equations for a connection on a principal $U(1)$ bundle over $X$ (together with one other field). The moduli space of solutions to the Seiberg–Witten equations can also be used to define smooth invariants of $X$, known as Seiberg–Witten invariants. Since the moduli space of solutions to the Seiberg–Witten equations is much simpler than instanton moduli space (it is smooth and compact), it is easier to work with the Seiberg–Witten invariants. Happily, the Seiberg–Witten invariants turn out to be just as powerful, if not more powerful, than the Donaldson invariants.

In 1988, following some important questions and suggestions by Michael Atiyah, Edward Witten gave an interpretation of the Donaldson invariants as correlation functions of certain special operators in a certain quantum field theory known as “topologically twisted $N = 2$ supersymmetric Yang–Mills theory.” However, the path integral formulation of the invariants could not be used as a practical means of evaluating the Donaldson invariants until aspects of the low-energy dynamics of that quantum field theory were understood better. This deeper understanding was achieved in 1994 when Nathan Seiberg, together with Witten, understood the nature of the groundstates of the theory in detail. After that breakthrough there was rapid progress, culminating in the formulation of the Seiberg–Witten invariants.

In fact, the Donaldson invariants can be expressed in terms of the Seiberg–Witten invariants. The relation was conjectured by Witten, at least for the case when $X$ has $b_2^+ > 1$. The relation was derived using physical methods (path integrals and effective field theory) by G. Moore and Witten in 1997. A crucial step in that derivation involves the so-called $u$-plane integral. This is a very subtle, finite-dimensional integral over the complex plane, closely related to the theta-lifts used in the theory of automorphic forms. The main part of the talk will focus on some progress achieved in the past few years continuing the approach to four-manifold invariants using quantum field theory. Time permitting, at least three results will be explained.

First, the topological twisting procedure of Witten can be extended to arbitrary quantum field theories with $N = 2$ supersymmetry. Around 2008 many new supersymmetric $N = 2$ field theories were discovered. Many of the new theories have the intriguing property that there is no known Lagrangian description (and probably no Lagrangian description exists). It has been a long-standing and natural question to ask if these new theories lead to new four-manifold invariants. Work with I. Nidaiev shows that, at least in the simplest of these non-Lagrangian theories, the answer is negative. Nevertheless, in the process of answering the question we still learn nontrivial facts about the Seiberg–Witten invariants. (An example is the superconformal simple type property of the Seiberg–Witten invariants.)
Second, it was already noted long ago by Moore and Witten and by Malmendier and Ono that for the case where $X$ is $\mathbb{CP}^2$ and $S^2 \times S^2$ the $u$-plane integral can be evaluated exactly using the theory of mock modular forms. Recent work with G. Korpas, J. Manschot, and I. Nidaiev has gone further, demonstrating that in fact for all four-manifolds with $b_2^+ > 1$ the $u$-plane integral can be expressed succinctly as the constant term in a Fourier expansion of a mock modular form (or of a mock Jacobi form).

Third, it was already noted by Donaldson in the 1980s that the theory can be extended to families of four-dimensional manifolds. The Donaldson invariants are then no longer functions on the homology of the four-manifold $X$ but rather are functions on the homology valued in the cohomology of the classifying space $B\text{Diff}(X)$, where $\text{Diff}(X)$ is the group of orientation-preserving diffeomorphisms of $X$. The talk will present some results from ongoing work with J. Cushing showing that such invariants can also be given an interpretation in quantum field theory, thus generalizing the formulation of the Donaldson invariants provided by Witten.
A 2020 View of Fermat’s Last Theorem

Kenneth A. Ribet

Fermat’s Last Theorem (FLT) was formulated in the seventeenth century and proved only about twenty-five years ago. The theorem is a compelling topic because of the simplicity of its statement (and the complexity of its proof) and because it gave rise to entire subjects within mathematics as researchers probed the problem in previous centuries.

I spoke about this subject at the 1994 Joint Math Meetings in Cincinnati. As the audience gathered, there was palpable tension in the ballroom because of the gap in the proof of FLT that Andrew Wiles had announced in June 1993. After speaking for 30 minutes about the mathematics behind the proof, I projected a statement that Wiles had made at the end of 1993. Here is the key passage:

…the final calculation of a precise upper bound for the Selmer group in the semistable case (of the symmetric square representation associated to a modular form) is not yet complete as it stands. . . .

Although there was no guarantee in January 1994 that there would be a happy end to the story, the gap that Wiles alluded to in his statement was repaired by an article written by Richard Taylor and Andrew Wiles the following fall. The complementary manuscripts by Wiles [12] and Taylor–Wiles [11] were published together in the Annals of Mathematics in 1995, more or less at the same time that elements of the proof of FLT were being explained to large audiences at a conference at Boston University.1

It is important to recall that the full proof depended on hundreds (if not thousands) of pages of difficult prior work as well as the two new articles in the Annals. In addition to my 1990 article on Serre’s conjecture [7], the argument outlined by Wiles appealed to the main theorem of Langlands’s book Base Change for GL(2) [5], an irreducibility result in Barry Mazur’s “Eisenstein ideal” article [6], and much more. (My 1995 article [8] sketches some of the mathematical tools that were used in the proof.)

The work of Wiles and Taylor–Wiles established the modularity of elliptic curves over the field of rational numbers. (In [7], I had proved that FLT would follow from this modularity.) Their new techniques led to a series of spectacular developments, including Serre’s modularity conjecture [9], which was proved in 2009 by Khare and Wintenberger [2,3], and most of the conjecture of Fontaine and Mazur [1]. (For some cases of the proof, see, e.g., [4].)

It might be natural to guess that these and other developments in the Langlands program would allow for a vast overhaul of the proof that was completed in 1994. Indeed, there is no shortage of examples of major theorems whose initial proofs were simplified considerably by subsequent analysis. Are we now able to present a proof of Fermat’s Last Theorem that is substantially more efficient than the quarter-century old version?

Certainly it is possible to formulate what looks like a succinct argument: FLT is a direct consequence of Serre’s modularity conjecture [9] (which is now a theorem as mentioned above). Appealing to Serre’s conjecture in this way has the technical advantage that the auxiliary (Frey) elliptic curve used in Wiles’s argument disappears almost immediately after it is introduced. All we need to say is that if $a^p + b^p = c^p$ (with $a, b, c, p$ nonzero integers and $p$ a prime ≥5), then the mod $p$ Galois representation attached to the elliptic curve with equation $y^2 = x(x - a^p)(x + b^p)$ is an irreducible Galois representation that furnishes a counterexample to Serre’s conjecture.

This one-sentence proof is not a clean simplification of the argument that was presented over a full week at the 1995 Boston University conference. The irreducibility of the Galois representation still relies on Mazur’s theorem from [6]. More importantly, the proof of Serre’s conjecture uses all of the ingredients that went into the original proof of FLT, plus quite a few more. (In particular, Khare and Wintenberger used Taylor’s work on potential modularity [10] to establish Serre’s modularity conjecture.)

Thus the question remains: is the proof simpler in 2020 than it was in 1995? As one writes on social media, “it’s complicated.” I will detangle some of the issues in Denver.

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The reader will not be surprised to learn that 20/20 vision is referred to as 6/6 vision in countries that use the metric system.

1This proof is summarized on a t-shirt that one can obtain from https://promys.org/resources/fermats-tshirts.

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The Laplacian has different expressions in homogeneous media (e.g., the lake) versus inhomogeneous media (e.g., the mountain). Nevertheless, under mild assumptions on the domains, in both cases there exists a canonical measure supported on the boundary of the domain where the operators are studied. These canonical measures (harmonic measure in the homogeneous case and elliptic measure in the inhomogeneous case) are associated in a unique way to the domain and the operator and constitute the building blocks for all solutions to Laplace’s equation. In the domains considered in this talk there is another canonical measure supported on the boundary, namely, the surface measure. (In Figure 1 this measure corresponds to the area. It can tell us the size of the glacier on top of Mount Shuksan.) From the mathematical point of view my objective is to discuss how the relationship between the harmonic measure and the surface measure characterizes the geometry of a large class of domains. I will mention the challenges that arise in the inhomogeneous case and conclude with a complete characterization of the inhomogeneous media that behave like the homogeneous one under this lens. The relationship between two measures is described using notions that come from harmonic analysis. The geometry of the domain is described using notions arising in quantitative geometric measure theory.

Early on, the work of F. and M. Riesz [RR] put in evidence the deep connection that exists between the properties of the harmonic measure of a domain and the regularity of the boundary expressed in terms of its differentiability, that is, in terms of the existence of tangent planes almost everywhere with respect to surface measure. Recently, the relationship between the properties of the harmonic measure of a domain with respect to the surface measure of its boundary and the weakly differentiable properties of it has been an area of active inquiry. Here the weakly differentiable properties of the boundary refer to the existence of tangent planes almost everywhere (rectifiability) and a quantitative analogous notion (uniform rectifiability).

For instance, in the scale-invariant sense and under some quantitative topological assumptions, one can show that the \( A_\infty \) property (quantitative mutual absolute continuity) of harmonic measure with respect to surface measure characterizes the uniform rectifiability of the boundary or even to the nontangentially approximability of the exterior domain. This characterization is a team effort achieved over time by different groups: [Sem], [DJ], [HMUT], and [AHMNT].

One of the main motivations of my recent work has been to understand whether the elliptic measure of a variable-coefficient divergence form elliptic operator (the Laplacian in an inhomogeneous medium) distinguishes between a rectifiable and a purely unrectifiable boundary. Together with S. Hofmann, J. M. Martell, S. Mayboroda, and Z. Zhao, we embarked on this project a few years ago. We have found...
the optimal solution, namely, a large class of variable-coefficient elliptic operators for which the $A_\infty$ property of the associated elliptic measure with respect to surface measure guarantees that the boundary of the domain is uniformly rectifiable ([HMMTZ1] and [HMMTZ2]).

In this talk I will share some of the highlights of this journey. I hope to convey the excitement of obtaining an optimal result after working for several years on a challenging project.

References


Credits

Figure 1 is courtesy of Mariana Smit Vega García. Author photo is courtesy of the University of Washington.
Reimagining the JMM with Your Help

Jill Pipher and Catherine Roberts

Beginning in 2022, the AMS will be solely responsible for managing the JMM. We are taking this opportunity to invite your input on reimagining the meetings.

How can we help you participate?

- Propose a panel, workshop, etc. for consideration for JMM 2020
- Propose something for JMM 2021 and beyond
- Request informal space for a meetings or event
- Become an exhibitor or sponsor
- Share thoughts on reimagining the JMM

More than a year has passed since the announcement of upcoming changes to the Joint Mathematics Meetings (see November 2018 Notices). Starting with the January 2022 meetings in Seattle, the AMS will solely manage the JMM. Moreover, the Mathematical Association of America will be reducing the programming it currently organizes for the JMM. We are pleased to provide this update to help keep our community informed.

While there is still much to do as we prepare to take over management of the JMM in 2022, work is underway. This spring, the AMS Council and Board of Trustees endorsed the following statement to illustrate our commitment to ensuring these meetings remain relevant to our entire community:

The American Mathematical Society endorses the principle that the Joint Mathematics Meetings will strive to represent the full spectrum of interests of the mathematical community.

The new Joint Meetings Planning Committee (JMPC) is now at work. This ad hoc committee’s principal activities are “to oversee the planning for and conduct of the annual January mathematics meetings, including the overall structure of the program, the overall schedule, and the inclusion of sessions and events sponsored by other organizations.” Members are Secretary Carla D. Savage (Chair), Associate Secretaries Georgia Benkart and Steven H. Weintraub, Executive Director Catherine A. Roberts, Associate Executive Director T. Christine Stevens, and Director of Meetings Penny Pina. Our initial meetings this fall identified and classified what needs to happen and how we will proceed. The AMS is determining how to best incorporate programming from several other organizations into the JMM.

In the meantime, we have received hundreds of comments, suggestions, and offers from individuals and other mathematical societies. Much of this arrived via our JMM Reimagined portal, which continues to gather input and suggestions from our community at http://www.ams.org/about-us/jmm-reimagined. This summer’s MAA MathFest featured a Q&A session with the executive directors from both the AMS and MAA. Hopefully, this session reassured those who value teaching-related sessions and...
activities that attendees will continue to see these elements in the JMM Reimagined program. Indeed, the AMS welcomes your engagement—please reach out to propose something to help preserve and enhance the January meetings. Societies and math institutes are engaged now with internal discussions about ways in which they might increase or change their involvement with the JMM. What is especially exciting about this level of community engagement is that the importance of the January meetings is abundantly clear. We are extremely grateful to the many suggestions and ideas—you will be hearing from us soon to help realize many of them.

We have assurances from the MAA that the Project NExT activities, the AMS/MAA/SIAM Gerald and Judith Porter lecture, and one MAA/AMS joint invited address will continue. Many of MAA’s Special Interest Groups have deep connections with the JMM that they hope to preserve, so if you are part of a Special Interest Group please reach out to the AMS to discuss how we might ensure this. Other MAA activities will be carried on at the JMM. For example, Pi Mu Epsilon will run the undergraduate poster sessions and, together with the AMS, will seek funding to continue the undergraduate travel grants that the AMS picked up for JMM 2020. The AMS is already forming plans to continue the rich education and pedagogy content familiar to many who attend the JMM. We will continue something like the MAA mini-courses and introduce new professional development opportunities. Several people have asked if the Graduate Student Fair and the Employment Center will continue—the answer is YES!

If you visit [https://www.ams.org/meetings/national/national](https://www.ams.org/meetings/national/national) you will see that anyone can propose events (all sorts) for future January meetings. Anyone can propose sessions or events of any type, even though we have not yet established a procedure for evaluating and approving proposals. If your group typically holds a business meeting at the JMM, you may schedule those as well. We expect the registration fees to support providing conference rooms with standard AV to all participating groups for no additional charges, but groups will be responsible for catering or “extra” AV. Again, this is a work in progress, and more information will be forthcoming.

The AMS would like to hear from YOU. Please join Executive Director Catherine Roberts and other leaders at a town hall hosted by the AMS Committee on Meetings and Conferences on Friday, January 17, 2020 at our JMM in Denver. Remember, these changes won’t go into effect until JMM 2022 in Seattle. Now is the time to help us reimagine the January mathematics meetings!

We both look forward to seeing you in Denver at the Joint Mathematics Meetings. Please enjoy special events in remembrance of Maryam Mirzakhani, including the new AMS Invited Address named in her honor.

Credits
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My Summer as a Science Journalist

Leila Sloman

If you had told me when I was nine or ten years old that I would do a PhD in math, I would have been surprised—to say the least. Math and science, I thought, were boring. I preferred to spend my time reading novels, writing short stories, and doing acrobatics in the backyard.

As I grew older, I discovered the beauty of mathematical arguments. I grew addicted to the way everything clicked into place when I solved a hard math or physics problem. But I still loved my humanities classes. When I applied to university, I considered a range of paths: engineering, physics, psychology, journalism. In the end, I chose a joint major in math and physics, and it became clear my next step would be graduate school. But I always felt the tug of my other self—the one that had pursued writing instead of mathematics.

When my department at Stanford University advertised the AAAS Mass Media fellowship, it felt like an opportunity tailor-made for me. The fellowship places science and engineering graduate students and postdocs at news outlets around the United States, where they become science journalists for the summer. I had never written a news story before, but by some miracle I was awarded the fellowship. I spent the summer at Scientific American in New York, where I produced news stories on all kinds of scientific topics. I wrote about pigeons flocking, quantum computers, cosmology, and whether or not to use a fan on a hot day. And, of course, math.

The summer was a total change of pace from life as a graduate student. I learned to write a story the public would care about—to hook readers with an intriguing lede, to make concepts as concrete as possible. I struggled the most with mathematical stories; my background clouded my view of what a reader would need explained. (It turns out readers will not think of a “graph” as a collection of edges and vertices and may ask what function it’s associated to.) But in the end, it was immensely satisfying to take a complicated subject and translate it into a compelling story. It’s like a logic puzzle of its own: Using only words accessible to a Calculus I student, explain how a quantum computer works. (There may be infinitely many solutions.)

Science journalism also brought me back in touch with the reasons I fell in love with math and science in the first place. Instead of worrying about technical details, I was focusing on the human story behind scientific results. Journal articles on cosmology that I found incomprehensible came to life in interviews with the scientists, who reminded me of the awe-inspiring vastness of the universe. Even though my first instinct when I see a spider is to attack it with a broom, I found myself nodding along with insect vision researchers: yes, jumping spiders are, undeniably, charming and adorable.

During my time as a graduate student, I’ve often wondered about the importance and effectiveness of mathematical outreach programs. Most mathematicians are not bettering the world directly the way ecologists or social workers are. If we want people to use their talents to improve society, should they really go into math? Being exposed to science stories from a range of disciplines—through interviews and the work of other journalists at Scientific American—reminded me why mathematics and basic science are crucial to so much of the modern world. For my very first story, I spoke to a mathematician using the Ising model—originally conceived of as a model of magnetism—to describe how Arctic sea ice is melting and to a neuroscientist who applies it to what’s going on in our brains. When Wilhelm Lenz proposed the Ising model, he (probably) never imagined it would have applications in such diverse fields. But without it, we might not have the neural networks that scientists are now using to study patients with dementia.

I still have a lot to learn about writing (and about math). I don’t know if I’ll become a science journalist. But this summer has opened up a world of opportunities for
someone who never thought of herself as a math person but fell in love with math anyway.

For more information about the fellowship program, see https://www.ams.org/massmediafellow or https://www.aaas.org/programs/mass-media-fellowship. Email amsdc@ams.org with any questions. The deadline for 2020 fellowship applications is January 1, 2020.
As I first looked at my cohort of fellows and saw a variety of climate scientists, nuclear physicists, material scientists, and veterinarians, I wondered how desirable a theoretical mathematician might be for a congressional office. I am a number theorist with a background in computer science and experience in higher education and was interested in a more general position that would let me work on a range of topics. I quickly found out that there is a very high demand for fellows with a wide variety of technical backgrounds, as evidenced by there being requests for over ninety positions with only thirty-three fellows. After interviewing in around ten offices, I accepted an offer to work in the office of Senator Amy Klobuchar from Minnesota.

Throughout the year, my day-to-day work varied dramatically depending on what was happening in the news, on the Senate floor, or in the senator’s schedule. My portfolio consisted of topics related to education, health care, workforce development, and data privacy. Although my role did not require the direct use of any concrete mathematics, I saw how applicable many of the transferable skills I learned as a researcher and educator are in the halls of Congress. I used my capabilities as a mathematician to effectively problem solve, think analytically and logically, and use data to drive and support policy decisions. I would examine, analyze, and summarize legislation that was being introduced on the floor or oversight letters that were being sent out, and I got to brainstorm, research, and draft new legislation for the senator. I also used my talents as an educator to synthesize information and relay it in an understandable way to diverse audiences without jeopardizing accuracy. I would regularly meet with constituent and advocacy groups to relay their concerns, priorities, or stories to the senator. I also regularly helped edit or write background summaries, talking points, and memos on a variety of topics within my portfolio.

Last year, I had the honor and privilege of serving as the AMS Congressional Fellow as a part of the American Association for the Advancement of Science (AAAS) Science and Technology Policy Fellowship program. I was the only mathematician among thirty-three Congressional Fellows sponsored by a variety of scientific professional societies to work in Congress for the year. Broadly speaking, the program has two goals: 1. “Science for Policy”—to bring scientific and technical expertise to the legislative process, and 2. “Policy for Science”—to make sure more members of the math and science communities are engaged in the political process so that we can better advocate for such priorities as research funding or STEM education initiatives. Whether it is increasing funding for scientific and medical research, addressing climate change, or regulating the tech, oil, or health care industries, there is an increased demand for people with scientific and technical backgrounds to aid in navigating the intricacies involved with effective legislation and regulatory oversight. The 2018 election saw a laudable increase in members of Congress elected with STEM backgrounds, but still resulted in just under four percent of Congress having come from a science, engineering, or math profession. This underscores the important role the fellowship plays in providing the opportunity for more experts in a wide variety of technical fields to actively engage in the legislative process.

The fellowship begins with a two-week orientation run by the AAAS to help fellows transition from an academic or technical role to that of government policymaking. We learned about a wide variety of topics, ranging from effective science communication to an expedited summary of US political history to a detailed look at the federal budget process. At the conclusion of the orientation, the Congressional Fellows begin a matching process to find the office that is the best fit for their professional and personal interests.

From the Lecture Hall to the National Mall
My Year as the AMS Congressional Fellow

James Ricci
The fellowship provided a constant deluge of learning opportunities, which provided many important lessons. The following are a few of the ones I thought were particularly worth sharing:

1. **Politics, policy, and procedure all matter.** During our orientation, Judy Schneider, a congressional specialist who has been working at the Congressional Research Service since 1979, provided us with a heavily condensed version of the same training she gives to incoming new members of Congress. Her presentation was a treasure trove of information, but the part that stuck with me the most throughout the year is that there are three things needed to enact any piece of legislation: policy, politics, and procedure. Although you need all three to pass a bill into law, they are not always equal. In many ways, Congress was designed more to prevent bad legislation from happening than to enact good ideas. This was an important lesson which taught me that you can have an incredibly well-researched and thought-out piece of legislation in Congress, but if the political atmosphere isn’t right or there is not a well-thought-out procedure on which to pass the bill, it will likely go nowhere. Being flexible in how you frame and motivate an issue can therefore go a long way in getting it passed. Even then, it will take time and a lot of advocacy.

2. **Constituents have a lot of influence.** I took over a hundred meetings with constituents and advocacy groups over the course of the year. Although not all of the requests or priorities can possibly be acted upon, these meetings provide an excellent opportunity for constituents to voice their concern or support for particular issues and similarly allow staffers to put context and faces to policies. Members of Congress are first and foremost representatives for their district or state. They really care about what their constituents have to say, and hearing a lot about a particular issue can change a member’s mind or drive them to action. I have seen the influence that effective advocacy from groups can have on a member’s actions, and it underscores the importance of engaging in these types of meetings.

3. **There is a lot of bipartisanship going on.** Coming into the fellowship, I was a little worried about the increasing polarization of our political system. While partisanship certainly exists and can be particularly strong on certain topics, my experience in the Senate has opened my eyes to the impressive amount of bipartisanship that does still happen in Congress. Many of the senators and representatives are friends outside the floor and work together on a whole host of issues, gaining bipartisan support in the process. At a time when the perception of Congress tends to be filled with negative feelings and press coverage, it is important to remember that it is also teeming with genuinely good people that are motivated to serve their country and district.

4. **The fellowship is a great way to advance your career.** The fellowship also awarded me a multitude of opportunities to learn and expand upon my professional goals and network. Over the course of the year, I developed a meaningful working knowledge of how our legislative system works, gained a diverse professional network both on the Hill and in many different scientific and advocacy fields, learned what it is like working in a fast-paced high-stakes environment, and expanded my knowledge in a variety of new subject areas. I have also engaged with the mathematical communities through the AMS, AWM, and MAA on many different levels. I have extended my stay in DC for another year as an AAAS Executive Branch Fellow, and I am planning on returning to academia next year with a deeper understanding of the political process. These experiences in the world of policymaking will inform my teaching and allow me to more effectively engage with the mathematics and higher education communities to better advocate for policies I find important.

I have learned a great deal from my time as a fellow and am incredibly grateful to the AMS for this opportunity. It has been inspiring to work in such an amazing office with truly hard-working, smart, and motivated people, and I am grateful for everything they have taught me and helped me accomplish.

Congress will address issues impacting our community whether we are involved in the process or not, so it is imperative that we keep ensuring our voices are heard. For those interested in personally getting involved, the AMS funds one Congressional Fellow per year, with this year’s application deadline of **February 15, 2020**, and there are multiple other types of science and technology policy fellowships available to mathematicians. There will be an AMS Committee on Science Policy panel discussion titled “A Call to Action: Grassroots Advocacy for Our Profession” at the Joint Mathematics Meetings in Denver, Colorado, this year on Friday, January 17, at 2:30 pm, followed by an information session on the AMS Congressional Fellowship at 4:30 pm. I will be at both sessions, so please come and find out more about how to get involved!

James Ricci
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—Caleb McWhorter, Editor-in-Chief 2019–20

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Community Updates

2019 Trjitzinsky Awards

The AMS has made awards to eight undergraduate students through the Waldemar J. Trjitzinsky Memorial Fund. The fund is made possible by a bequest from the estate of Waldemar J., Barbara G., and Juliette Trjitzinsky. The will of Barbara Trjitzinsky stipulates that the income from the bequest should be used to establish a fund in honor of the memory of her husband to assist needy students in mathematics.

For the 2019 awards, the AMS chose eight geographically distributed schools to receive one-time awards of US$3,000 each. The mathematics departments at those schools then chose students to receive the funds to assist them in pursuit of careers in mathematics. The schools are selected in a random drawing from the pool of AMS institutional members.

Waldemar J. Trjitzinsky was born in Russia in 1901 and received his doctorate from the University of California, Berkeley, in 1926. He taught at a number of institutions before taking a position at the University of Illinois, Urbana-Champaign, where he remained for the rest of his professional life. He showed particular concern for students of mathematics and in some cases made personal efforts to ensure that financial considerations would not hinder their studies. Trjitzinsky was the author of about sixty mathematics papers, primarily on quasi-analytic functions and partial differential equations. A member of the AMS for forty-six years, he died in 1973.

Following are the names of the selected schools for 2019, the names of the students receiving the awards, and brief biographical sketches of the students.

University of Notre Dame: Yuxin Lin, originally from Guangzhou, China, is a third-year honors mathematics student at the University of Notre Dame. Yuxin has a deep appreciation for mathematics and has been involved with a number of mathematical extracurriculars even as a high school student. Before coming to Notre Dame, she participated in the Ross Mathematics Program and in 2018 returned as a junior counselor. She has also been active with the Notre Dame Putnam team, earning a score of 40 points in the 2018 exam. Her main mathematical interest has been in number theory, in which she has been active, with directed readings with Andrei Jorza, and in the summer of 2019, she participated in the University of Chicago REU. Yuxin has also been an active teaching assistant in Notre Dame’s “proofs” help room for math students who are taking proof-based courses. A quote from one of her professors is: “She is very mature mathematically as evidenced in her writing, thinking, and performance in the math classes. Her mathematical sophistication is well above her peers. She is very tenacious and focused on her clearly formulated goals.”

University of Idaho: Eli Smith, from Boise, Idaho, will be the first in his family to graduate college. From a young age, he was surrounded by mental illness, constantly facing obstacles at home. Despite these hardships, Eli is thankful for his beginnings and from them has learned that there are so many factors in the world that are out of our control, from our neurochemistry to our everyday behaviors. This perspective has opened his heart to empathy for behaviors resulting from mental illness, and he was drawn to math when he realized that he could model aspects of the human brain. He began research on the brain at the University of Idaho, and he is certain that some combination of mathematics and the mind will be in his future. In school, he tutors algebra through Calculus 2, and he takes part in PME. Eli absolutely loves learning and teaching math! This was reaffirmed when the faculty at the University of Idaho awarded him the Eugene and Osa Taylor scholarship in his junior and senior years. Eli states that he is beyond thankful for this scholarship and all of the other opportunities the University of Idaho has provided him.

Baylor University: Stephen Blake Allan is a senior studying mathematics and physics at Baylor University. After a start toward engineering, he transitioned into mathematics upon realizing that his interests had a more
abstract character. He is also an active member of Baylor’s Honors Residential College and has invested in student development and tutoring programs across campus. His particular interests lie in bridging rigorous and intuitive understandings of mathematical and physical phenomena, which attracts him to areas in mathematical physics and spectral theory. Blake is also indebted to the careful instruction and mentorship of the faculty at Baylor, particularly Fritz Gesztesy and Lance Littlejohn, for his introduction into professional mathematics. Due to his passion for both the art of research and the craft of teaching, Blake hopes to pursue graduate study and eventually a professorship in mathematics.

Texas Tech University: Claudia Munoz is a senior from Wichita Falls, Texas. She is majoring in mathematics and minoring in both English and Russian. She found her love for math as a young girl, being very intrigued by the beauty of numbers. At the age of six, Claudia learned how to play chess and soon became a national and international chess champion. Throughout her chess career, Claudia has represented the United States in eight different countries, in top-caliber tournaments such as the World Junior Chess Championship. Her unified love for math and chess encouraged her to choose Texas Tech University as her home university. At Texas Tech, she has had the opportunity to work with great professors who have molded and guided her into becoming the mathematician she aspires to be. She is also on the Texas Tech chess team and president of the Knight Raiders Chess Club. Claudia recently participated in the Math in Moscow program, where she was able to grow in her understanding of math. In the future, Claudia desires to open her own math and chess school and wants to be a part of teaching the next generation of mathematicians.

Florida International University: Juan Linares Cabrera, a senior in mathematics, was born (1994) and raised in Cuba. He moved to the United States following his mother, father, and brother five years ago. As a high school student in Cuba, Juan distinguished himself with his proficiency in mathematics. He was placed very high in the high school mathematics national team there, but for lack of money to cover the travel expenses, he never competed in an International Mathematical Olympiad. Juan’s parents are well educated and hold bachelor’s degrees in literature and economics, but, unfortunately, their degrees were not validated in the United States, and neither of them could practice their profession here. Like many new immigrants, Juan’s parents have modest financial means, but their children’s education is a top priority for them. Juan is considered a rising star among the math majors at Florida International. He is also pursuing a minor in computer science. Juan is a top student in challenging classes such as point set topology, advanced calculus, and algebraic structures. He currently has a 3.9 GPA and has made the dean’s list every semester at Florida International. Juan plans to attend a graduate school to study machine learning and cryptography as well, combining his love for mathematics and computers.

Wellesley College: Amy (Qing Hai) Li is a senior math major at Wellesley College. She took IB Math in high school but wasn’t interested in calculus until her AP Physics teacher showed her how those mathematical tools helped make sense of the world around her. At Wellesley, Amy continued to take math classes as a requirement for the physics major but didn’t feel that she was a “math person.” However, she was hooked by the proofs in her linear algebra class. Here was a game where, once the rules and assumptions were fixed, one could be sure of their conclusions! Amy enjoyed her analysis and algebra classes so much that she switched to the math major. She has done research in physics, applied analysis, and algebraic geometry. Amy is passionate about math research and communication and can’t wait to work on her senior thesis in topology.

Bucknell University: Jordan Kovacs, a senior at Bucknell University, will be graduating with a BS in mathematics with a statistics concentration and a minor in Italian studies. Ever since he was a child, he has loved the field of mathematics, and he has found comfort in applied fields of statistics. In Jordan’s first summers at Bucknell University, he participated in the STEM Scholars Research Program, where he first studied spatial health care accessibility for people in rural Colorado and, second, studied the factors of the spread of the Zika virus via sensitivity analysis. While this research experience was unique and interesting, Jordan is studying to become an actuary. He had an actuarial internship this summer with Nationwide Mutual Insurance, where he learned about annuity pricing and actuarial industry trends, and this internship furthered his desire to work in the field. Outside of academia and work, Jordan likes to watch movies, read, or play video games; he also likes to go hiking and kayaking and going to the beach. Regardless of the activity, you will find him
California State University, Dominguez Hills: Valeria Arredondo has always felt that mathematics chose her. When she first came to the United States, while she did not speak any English, she felt that math spoke a universal language that gave her both confidence and joy. Since then, she has been on a path toward achieving her degree in mathematics as a member of the class of 2021 at California State University, Dominguez Hills. Valeria is the first member of her family to go to college. She is active on campus as a spirited member of the Math Club and also serves as the event coordinator for the immigrant student alliance, Espiritu de Nuestro Futuro. She has been both a tutor and a supplemental instructor for several math classes at CSUDH. She is passionate about both mathematics and education and plans to earn her Single Subject Teaching Credential in Mathematics so that she can share her passion with others, and, as a female student with all male math teachers until her senior year of high school, she is excited to add representation to the field and inspire others.

—AMS Trjitzinsky Fund announcement

Erdős Memorial Lecture

The Erdős Memorial Lecture is an annual invited address named for the prolific mathematician Paul Erdős (1913–1996). The lectures are supported by a fund created by Andrew Beal, a Dallas banker and mathematics enthusiast. The Beal Prize Fund is being held by the AMS until it is awarded for a correct solution to the Beal Conjecture (see www.math.unt.edu/~mauldin/beal.html). At Mr. Beal’s request, the interest from the fund is used to support the Erdős Memorial Lecture.


Andrei Okounkov of Columbia University will deliver the 2020 Erdős Memorial Lecture at the 2020 Fall Western Sectional Meeting on October 24, 2020. See https://www.ams.org/meetings/lectures/meet-erdos-lecture for more information.

—AMS announcement

Deaths of AMS Members

Salvatore D. Bernardi, of Escondido, California, died on August 11, 2013. Born on April 12, 1919, he was a member of the Society for 67 years.

Marie R. J. Charpentier, of France, died in 1994. Born on October 29, 1903, she was a member of the Society for 46 years.

Myron A. Coler, of La Jolla, California, died on September 20, 2004. Born on March 30, 1913, he was a member of the Society for 63 years.

John L. D’Arcy, of Huntingdon Valley, Pennsylvania, died on May 23, 2009. Born on January 24, 1945, he was a member of the Society for 34 years.

Dan Laksov, professor, Royal Institute of Technology, died on October 25, 2013. Born on July 10, 1940, he was a member of the Society for 34 years.

Esther Seiden, professor, Hebrew University of Jerusalem, died on June 6, 2014. Born on March 3, 1908, she was a member of the Society for 64 years.

T. P. Srinivasan, professor, University of Kansas, died on June 8, 2013. Born on December 24, 1932, he was a member of the Society for 47 years.
Harper Awarded SASTRA Ramanujan Prize

Adam Harper of the University of Warwick has been awarded the 2019 SASTRA Ramanujan Prize for making “path-breaking contributions to analytic and probabilistic number theory by establishing a number of deep and surprising results.” The prize is awarded for outstanding contributions by mathematicians age thirty-two or younger in areas of mathematics influenced by Ramanujan.

The prize citation reads: "Adam Harper is awarded the 2019 SASTRA Ramanujan Prize for several outstanding contributions to analytic and probabilistic number theory. The prize recognizes his marvelous 2012 PhD thesis at Cambridge University and his paper of 2013 in Crelle’s Journal [Journal for Pure and Applied Mathematics], in which, among other things, he disproved a widely believed conjecture on the normal distribution of partial sums of random multiplicative functions by novel use of deep probabilistic methods. The prize also recognizes Harper’s recent brilliant proof of a conjecture of Helson that the partial sums have better than square-root cancellation using ideas on multiplicative chaos, and a related paper on higher moments of random multiplicative functions to appear in Algebra and Number Theory. The prize also recognizes Harper’s seminal work using the Riemann Hypothesis to determine the correct order upper bound for the higher moments of the Riemann zeta function on the critical line. The prize also notes that Harper proved the upper bound part of a conjecture of Fyodorov, Hiary, and Keating on the almost always maximum size of the Riemann zeta function in short intervals on the critical line. The prize recognizes Harper’s ingenious work on S-unit equations and related results for these equations over smooth numbers that significantly improves earlier important work of Konyagin and Soundararajan. The prize notes that in his 2012 Journal of Number Theory paper, Harper established certain conjectures of Soundararajan on the equidistribution of smooth numbers and more recently that Harper has established a powerful Bombieri–Vinogradov type theorem for smooth numbers. In addition, the prize recognizes Harper’s novel proof of a famous theorem of Halasz that led to two major joint papers on the ‘pretentious approach to number theory’ with Andrew Granville and Kannan Soundararajan in Compositio Mathematics (2019) and the Proceedings of the American Mathematical Society (2018). Finally, the prize recognizes that Harper’s fundamental research spans several areas, such as the large sieve, in joint work with Ben Green in Geometric and Functional Analysis (2014) and on prime number races in joint work with Youness Lamzouri in Probability and Related Fields (2018) and with Kevin Ford and Lamzouri, to appear in Mathematische Annalen. Harper’s pathbreaking results in analytic number theory using deep ideas from probability in novel ways is having a major impact on these fields and has resulted in an explosion of activity.”

Adam Harper was born in Lowestoft, United Kingdom. He completed his PhD in 2012 at Cambridge University under the guidance of Ben Green. He held a postdoctoral fellowship at CRM Montreal with Andrew Granville in 2012–2013. He was a research fellow at Cambridge University in 2013–2016.

The members of the prize committee for the 2019 SASTRA Ramanujan Prize were:

- Krishnaswami Alladi, Chair, University of Florida
- David Bressoud, Macalester College
- Kevin Ford, University of Illinois, Urbana
- Gerhard Frey, University of Essen
- Robert Tijdeman, University of Leiden
- Shouwu Zhang, Princeton University
- Maryna Viazovska, École Polytechnique, Lausanne

The previous winners of the SASTRA Ramanujan Prize are:

- Manjul Bhargava and Kannan Soundararajan (two full prizes), 2005
- Terence Tao, 2006
- Ben Green, 2007
- Akshay Venkatesh, 2008
- Kathrin Bringmann, 2009
- Wei Zhang, 2010
- Roman Holowinsky, 2011
- Zhiwei Yun, 2012
- Peter Scholze, 2013
- James Maynard, 2014
Sébastien Gouëzel of the University of Nantes has been awarded the eighth Michael Brin Prize in Dynamical Systems for his “groundbreaking and influential work on the spectral theory of transfer operators and on statistical properties of hyperbolic dynamical systems and random walks on hyperbolic groups.”

Gouëzel received his PhD in 2004 from Université Paris XI under the supervision of Viviane Baladi. He has held positions as CNRS “chargé de recherche” at the Universities of Rennes (2005–2015) and Nantes (2015–present). He was an invited speaker at the International Congress of Mathematicians in Rio de Janeiro, Brazil, in 2018.

The Michael Brin Prize in Dynamical Systems was endowed in 2008 by Michael Brin of the University of Maryland to recognize mathematicians who have made a substantial impact in dynamical systems and related fields at an early stage of their careers. It carries a cash award of US$18,000.

—Giovanni Forni, University of Maryland, College Park

Ambrosio Awarded 2019 Balzan Prize

Luigi Ambrosio of Scuola Normale Superiore di Pisa has been awarded the 2019 Balzan Prize for Theory of Partial Differential Equations. According to the prize citation, Ambrosio “is a remarkable mathematician whose astonishing capacity for synthesis has made it possible to create hitherto unimaginable bridges between partial differential equations and the calculus of variation. His influence on the analysis of very general spaces is exceptional.” Ambrosio was born in Piedmont, Italy, and received his PhD from Scuola Normale Superiore di Pisa in 1988 under the direction of Ennio De Giorgi. He has held positions at the University of Pisa, the University of Benevento, and the University of Pavia. His honors include the Caccioppoli Prize of the Italian Mathematical Union (1998) and the Fermat Prize (2003). He is managing editor of the journal Calculus of Variations and Partial Differential Equations. Ambrosio tells the Notices, “My first contact and excitement with mathematics came with my grandfather, showing to me tricky games with numbers. Mathematically, I grew up in the Scuola Normale and at Pisa University. It has been very exciting for me to move across different fields, from calculus of variations to geometric measure theory and, in the last few years, optimal transport and probability.”

Four Balzan Prizes are awarded annually by the International Balzan Prize Foundation in the broad categories of Literature, Moral Sciences, and the Arts; and the Physical, Mathematical, and Natural Sciences and Medicine. Two prizes in each category are awarded to scholars, artists, and scientists who have distinguished themselves in their fields on an international level. The Articles of the Balzan Foundation stipulate that half the prize money be donated to finance research projects preferably carried out by young scholars or scientists.

—From a Balzan Prize Foundation announcement

Wise Awarded 2019 Lobachevsky Medal

Daniel Wise of McGill University has been awarded the 2019 Lobachevsky Medal for his work in geometric group theory, metric spaces of nonpositive curvature, residually finite groups, subgroup separability, 3-dimensional manifolds, and coherence, as well as for his general research in theory of infinite groups with applications to geometry and topology. He received his PhD from Princeton University in 1996 under the direction of Martin Bridson. He received a National Science Foundation Postdoctoral Fellowship in 1996–1997 and has held positions at Princeton University, the University of California, Berkeley, Cornell University, and Brandeis University. He joined the faculty at McGill in 2001. His honors include the Veblen Prize in Geometry (2013), the Jeffery-Williams Prize (2016), the CRM-Fields-PlIMS Prize (2016), and a Guggenheim Fellowship (2016). He is a Fellow of the Royal Society and of the Royal Society of Canada. The Lobachevsky Medal is awarded for
contributions to geometry and fundamental and applied mathematics. It carries a cash award of US$75,000.

—Kazan State University announcement

2020 AWM Fellows Chosen

The Executive Committee of the Association for Women in Mathematics (AWM) established the AWM Fellows Program to recognize individuals who have demonstrated a sustained commitment to the support and advancement of women in the mathematical sciences, consistent with the AWM mission: “to encourage women and girls to study and to have active careers in the mathematical sciences, and to promote equal opportunity and the equal treatment of women and girls in the mathematical sciences.”

The 2020 class of AWM Fellows are researchers, mentors, and educators who are recognized by their peers and students for their commitment to supporting women in the mathematical sciences.

Following are the names and institutions of the 2020 AWM Fellows.

- Margaret Bayer, University of Kansas
- Joan S. Birman, Barnard College, Columbia University
- Petra Bonfert-Taylor, Thayer School of Engineering, Dartmouth College
- Susanne C. Brenner, Louisiana State University
- Jennifer Chayes, Microsoft Research
- Alissa S. Crans, Loyola Marymount University
- Donatella Danielli, Purdue University
- Sarah J. Greenwald, Appalachian State University
- Leslie Hogben, Iowa State University and American Institute of Mathematics
- Fern Y. Hunt, National Institute of Standards and Technology
- Michelle Manes, University of Hawaii and National Science Foundation
- Maura Mast, Fordham University
- Eileen L. Poiani, Saint Peter’s University
- Chi-Wang Shu, Brown University
- Karen E. Smith, University of Michigan
- Diane L. Souvaine, Tufts University
- Karen Keskulla Uhlenbeck (retired), University of Texas at Austin, Visitor Institute for Advanced Study
- Roselyn E. Williams, Florida Agricultural and Mechanical University

—From an AWM announcement

Credits

Photo of Luigi Ambrosio is courtesy of Giandonato Tartarelli, Scuola Normale Superiore di Pisa.
Photo of Sébastien Gouëzel is courtesy of Renaud Le Guernic.
Photo of Daniel Wise is courtesy of Yael Halevi-Wise.
Mathematics Opportunities

Listings for upcoming mathematics opportunities to appear in Notices may be submitted to notices@ams.org.

Early Career Opportunity

AMS-Simons Travel Grants

AMS-Simons Travel Grants provide early-career mathematicians with $2,500 funding for two years to be used for research-related travel. Up to seventy grants will be awarded. The application period is February 1–March 31, 2020. See the website www.ams.org/travel-grants/AMS-SimonsTG.

—From an AMS-Simons announcement

Simons Foundation Collaboration Grants for Mathematicians

The Simons Foundation invites applications for Collaboration Grants for Mathematicians (US$8,400 per year for five years). The application deadline is January 30, 2020. See tinyurl.com/zvoqm5t. Contact Elizabeth Roy, mps@simonsfoundation.org.

—From a Simons Foundation announcement

PIMS Education Prize

The Pacific Institute for the Mathematical Sciences (PIMS) annually awards a prize to a member of the PIMS community who has made a significant contribution to education in the mathematical sciences. The deadline for nominations is March 15, 2020. See www.pims.math.ca/pims-glance/prizes-awards.

—From a PIMS announcement

Early Career Opportunity

CAIMS/PIMS Early Career Award

The Canadian Applied and Industrial Mathematics Society (CAIMS) and the Pacific Institute for the Mathematical Sciences (PIMS) award the CAIMS/PIMS Early Career Award in Applied Mathematics for exceptional research in any area of applied mathematics. Nominations must be received by January 31, 2020. See www.pims.math.ca/pims-glance/prizes-awards.

—From a CAIMS/PIMS announcement

Early Career Opportunity

NRC Research Associateship Programs

The National Research Council (NRC) Research Associateship Programs (RAP) promote excellence in scientific and technological research conducted by the US government through the administration of programs offering graduate-, postdoctoral-, and senior-level research opportunities at sponsoring federal laboratories and affiliated institutions. The deadline for applications is February 1, 2020. For more information, see https://sites.nationalacademies.org/pga/rap/.

—From an NRC announcement

Early Career Opportunity

News from IPAM

The Institute for Pure and Applied Mathematics (IPAM) offers programs that encourage collaboration across disciplines and between two areas of mathematics. IPAM holds long programs (three months) and workshops (three to five days) throughout the academic year for junior and senior mathematicians and scientists who work in academia, research laboratories, and industry. In the summer, IPAM offers industrial research programs in multiple locations for undergraduate and graduate students, and a one- to three-week “summer school” for graduate students and recent PhDs.

IPAM seeks program proposals from the math and science communities. Please send your idea for a workshop,
Mathematics Opportunities

NEWS

Mathematics Opportunities

- Workshop IV: Social Dynamics beyond Vehicle Autonomy, November 30–December 4, 2020

Early Career Opportunity

Thesis-Writing Fellowship for Students Doing Extraordinary Teaching and Outreach

A $15,000 fellowship for PhD students graduating in the 2020–2021 academic year who have done extraordinary teaching and outreach during their time as graduate students, especially during summers, is being offered through Noah Snyder’s NSF CAREER Grant DMS-1454767. The deadline is February 12, 2020. Find more details at [pages.iu.edu/~nsnyder1/fellowship.html](http://pages.iu.edu/~nsnyder1/fellowship.html).

—Noah Snyder

long program, or summer school to director@ipam.ucla.edu by October 1 for consideration at the upcoming Science Advisory Board meeting. For more information, go to www.ipam.ucla.edu/propose-a-program.

IPAM’s upcoming programs are listed below. Please go to www.ipam.ucla.edu/programs for detailed information on each program and to find application and registration forms.

**Winter Workshops 2020.** Apply for travel support or register for each workshop online.

- Theory and Computation for 2D Materials, January 13–17, 2020
- Emerging Opportunities for Mathematics in the Microbiome, January 23–24, 2020
- Deep Learning and Medical Applications, January 27–31, 2020
- Asymptotic Algebraic Combinatorics, February 3–7, 2020
- Computational Psychiatry, February 18–21, 2020
- Intersections between Control, Learning, and Optimization, February 24–28, 2020

**High-Dimensional Hamilton-Jacobi PDEs, March 9–June 12, 2020.** Apply online for support to be a core participant for the entire program or to attend any of the following one-week workshops.

- High-Dimensional Hamilton-Jacobi PDEs Tutorials, March 10–13, 2020
- Workshop I: High-Dimensional Hamilton-Jacobi Methods in Control and Differential Games, March 30–April 3, 2020
- Workshop II: PDE and Inverse Problem Methods in Machine Learning, April 20–24, 2020
- Workshop III: Mean Field Games and Applications, May 4–8, 2020
- Workshop IV: Stochastic Analysis Related to Hamilton-Jacobi PDEs, May 18–22, 2020

**Graduate Summer School: Mathematics of Topological Phases of Matter, June 22–26, 2020.** As space is limited, we do not offer open registration for this summer school. Rather, all participants will need to apply online and be selected to participate by the organizers.

**Mathematical Challenges and Opportunities for Autonomous Vehicles, September 14–December 18, 2020.** Apply online for support to be a core participant for the entire program or to attend any of the following one-week workshops.

- Autonomous Vehicles Tutorials, September 15–18, 2020
- Workshop I: Individual Vehicle Autonomy: Perception and Control, October 5–9, 2020
- Workshop II: Safe Operation of Connected and Autonomous Vehicle Fleets, October 26–30, 2020
- Workshop III: Large Scale Autonomy: Connectivity and Mobility Networks, November 16–20, 2020
Classified Advertising

Employment Opportunities

KANSAS

University of Kansas
Department of Mathematics

Department of Mathematics at University of Kansas invites applications for a tenure-track, Assistant Professor in Statistics to begin August 18, 2020. Requirements include a PhD in Mathematics, Statistics or related fields and outstanding publication record in Statistics. For a complete announcement and to apply online, go to https://employment.ku.edu/academic/156398R. Initial review begins November 18, 2019. In a continuing effort to enrich its academic environment and provide equal educational and employment opportunities, the university actively encourages applications from members of underrepresented groups in higher education. KU is an EO/AAE. All qualified applicants will receive consideration for employment without regard to race, color, religion, sex (including pregnancy), age, national origin, disability, genetic information or protected Veteran status.

CHINA

Tianjin University, China
Tenured/Tenure-Track/Postdoctoral Positions at the Center for Applied Mathematics

Dozens of positions at all levels are available at the recently founded Center for Applied Mathematics, Tianjin University, China. We welcome applicants with backgrounds in pure mathematics, applied mathematics, statistics, computer science, bioinformatics, and other related fields. We also welcome applicants who are interested in practical projects with industries. Despite its name attached with an accent of applied mathematics, we also aim to create a strong presence of pure mathematics.

Light or no teaching load, adequate facilities, spacious office environment and strong research support. We are prepared to make quick and competitive offers to self-motivated hard workers, and to potential stars, rising stars, as well as shining stars.

The Center for Applied Mathematics, also known as the Tianjin Center for Applied Mathematics (TCAM), located by a lake in the central campus in a building protected as historical architecture, is jointly sponsored by the Tianjin municipal government and the university. The initiative to establish this center was taken by Professor S. S. Chern. Professor Molin Ge is the Honorary Director, Professor Zhiming Ma is the Director of the Advisory Board. Professor William Y. C. Chen serves as the Director.

TCAM plans to fill in fifty or more permanent faculty positions in the next few years. In addition, there are a

The Notices Classified Advertising section is devoted to listings of current employment opportunities. The publisher reserves the right to reject any listing not in keeping with the Society's standards. Acceptance shall not be construed as approval of the accuracy or the legality of any information therein. Advertisers are neither screened nor recommended by the publisher. The publisher is not responsible for agreements or transactions executed in part or in full based on classified advertisements.

The 2020 rate is $3.65 per word. Advertisements will be set with a minimum one-line headline, consisting of the institution name above body copy, unless additional headline copy is specified by the advertiser. Headlines will be centered in boldface at no extra charge. Ads will appear in the language in which they are submitted. There are no member discounts for classified ads. Dictation over the telephone will not be accepted for classified ads.


US laws prohibit discrimination in employment on the basis of color, age, sex, race, religion, or national origin. Advertisements from institutions outside the US cannot be published unless they are accompanied by a statement that the institution does not discriminate on these grounds whether or not it is subject to US laws.

Submission: Send email to classads@ams.org.
number of temporary and visiting positions. We look forward to receiving your application or inquiry at any time. There are no deadlines.

Please send your resume to mathjobs@tju.edu.cn. For more information, please visit [http://cam.tju.edu.cn](http://cam.tju.edu.cn) or contact Ms. Erica Liu at mathjobs@tju.edu.cn, telephone: 86-22-2740-6039.
New Books Offered by the AMS

Algebra and Algebraic Geometry

Introduction to Arithmetic Groups
Armand Borel

Fifty years after it made the transition from mimeographed lecture notes to a published book, Armand Borel’s *Introduction aux groupes arithmétiques* continues to be very important for the theory of arithmetic groups. In particular, Chapter III of the book remains the standard reference for fundamental results on reduction theory, which is crucial in the study of discrete subgroups of Lie groups and the corresponding homogeneous spaces.

The review of the original French version in *Mathematical Reviews* observes that “the style is concise and the proofs (in later sections) are often demanding of the reader.” To make the translation more approachable, numerous footnotes provide helpful comments.

*University Lecture Series*, Volume 73

bookstore.ams.org/ulect-73

An Introduction to Symmetric Functions and Their Combinatorics
Eric S. Egge, Carleton College, Northfield, MN

This book is a reader-friendly introduction to the theory of symmetric functions, and it includes fundamental topics such as the monomial, elementary, homogeneous, and Schur function bases; the skew Schur functions; the Jacobi–Trudi identities; the involution \( \omega \); the Hall inner product; Cauchy’s formula; the RSK correspondence and how to implement it with both insertion and growth diagrams; the Pieri rules; the Murnaghan–Nakayama rule; Knuth equivalence; *jeu de taquin*; and the Littlewood–Richardson rule. The book also includes glimpses of recent developments and active areas of research, including Grothendieck polynomials, dual stable Grothendieck polynomials, Stanley’s chromatic symmetric function, and Stanley’s chromatic tree conjecture. Written in a conversational style, the book contains many motivating and illustrative examples. Whenever possible it takes a combinatorial approach, using bijections, involutions, and combinatorial ideas to prove algebraic results.

The prerequisites for this book are minimal—familiarity with linear algebra, partitions, and generating functions is all one needs to get started. This makes the book accessible to a wide array of undergraduates interested in combinatorics.

*Student Mathematical Library*, Volume 91

bookstore.ams.org/stml-91
Differential Equations

Aiping Wang, North China Electric Power University, Beijing, China, and Anton Zettl, Northern Illinois University, DeKalb

In 1910 Herman Weyl published one of the most widely quoted papers of the 20th century in Analysis, which initiated the study of singular Sturm-Liouville problems. The work on the foundations of Quantum Mechanics in the 1920s and 1930s, including the proof of the spectral theorem for unbounded self-adjoint operators in Hilbert space by von Neumann and Stone, provided some of the motivation for the study of differential operators in Hilbert space with particular emphasis on self-adjoint operators and their spectrum. Since then the topic developed in several directions and many results and applications have been obtained.

In this monograph the authors summarize some of these directions discussing self-adjoint, symmetric, and dissipative operators in Hilbert and Symplectic Geometry spaces.

Ordinary Differential Operators

Outside's AMS

Mathematical Surveys and Monographs, Volume 245


bookstore.ams.org/surv-245

Hochschild Cohomology for Algebras

Sarah J. Witherspoon, Texas A&M University, College Station

This book gives a thorough and self-contained introduction to the theory of Hochschild cohomology for algebras and includes many examples and exercises. The book then explores Hochschild cohomology as a Gerstenhaber algebra in detail, the notions of smoothness and duality, algebraic deformation theory, infinity structures, support varieties, and connections to Hopf algebra cohomology. Useful homological algebra background is provided in an appendix. The book is designed both as an introduction for advanced graduate students and as a resource for mathematicians who use Hochschild cohomology in their work.

Graduate Studies in Mathematics, Volume 204


bookstore.ams.org/gsm-204
Math Education

Inspiring Mathematics
Lessons from the Navajo Nation Math Circles
Dave Auckly, Kansas State University, Manhattan, Bob Klein, Ohio University, Athens, Amanda Serenevy, Riverbend Community Math Center, South Bend, IN, and Tatiana Shubin, San Jose State University, CA, Editors

The people of the Navajo Nation know mathematics education for their children is essential. They were joined by mathematicians familiar with ways to deliver problems and a pedagogy that, through exploration, shows the art, joy and beauty in mathematics. This combined effort produced a series of Navajo Math Circles—interactive mathematical explorations—across the Navajo Reservation.

This book contains the mathematical details of that effort. Between its covers is a thematic rainbow of problem sets that were used in Math Circle sessions on the Reservation. The problem sets are good for puzzling over and exploring the mathematical ideas within. They will help nurture curiosity and confidence in students.

This book is a wonderful resource for any teacher wanting to enrich the mathematical lives of students and for anyone curious about mathematical thinking outside the box.

This item will also be of interest to those working in general interest.

Titles in this series are co-published with the Mathematical Sciences Research Institute (MSRI).

Mathematical Surveys and Monographs, Volume 244

bookstore.ams.org/surv-244

Nonlinear Dirac Equation
Spectral Stability of Solitary Waves
Nabile Boussaïd, Université de Franche-Comté, Besançon, France, and Andrew Comech, Texas A&M University, College Station, and Institute for Information Transmission Problems, Moscow, Russia

This monograph gives a comprehensive treatment of spectral (linear) stability of weakly relativistic solitary waves in the nonlinear Dirac equation. It turns out that the instability is not an intrinsic property of the Dirac equation that is only resolved in the framework of the second quantization with the Dirac sea hypothesis. Whereas general results about the Dirac-Maxwell and similar equations are not yet available, we can consider the Dirac equation with scalar self-interaction, the model first introduced in 1938. In this book we show that in particular cases solitary waves in this model may be spectrally stable (no linear instability). This result is the first step towards proving asymptotic stability of solitary waves.

The book presents the necessary overview of the functional analysis, spectral theory, and the existence and linear stability of solitary waves of the nonlinear Schrödinger equation. It also presents the necessary tools such as the limiting absorption principle and the Carleman estimates in the form applicable to the Dirac operator, and proves the general form of the Dirac-Pauli theorem. All of these results are used to prove the spectral stability of weakly relativistic solitary wave solutions of the nonlinear Dirac equation.

This item will also be of interest to those working in analysis.

Mathematical Surveys and Monographs, Volume 244

bookstore.ams.org/surv-244
Number Theory

The Distribution of Prime Numbers
Dimitris Koukoulopoulos, Université de Montréal, QC, Canada

Prime numbers have fascinated mathematicians since the time of Euclid. This book presents some of our best tools to capture the properties of these fundamental objects, beginning with the most basic notions of asymptotic estimates and arriving at the forefront of mathematical research.

Throughout, the emphasis has been placed on explaining the main ideas rather than the most general results available. As a result, several methods are presented in terms of concrete examples that simplify technical details, and theorems are stated in a form that facilitates the understanding of their proof at the cost of sacrificing some generality. Each chapter concludes with numerous exercises of various levels of difficulty aimed to exemplify the material, as well as to expose the readers to more advanced topics and point them to further reading sources.

Graduate Studies in Mathematics, Volume 203

bookstore.ams.org/gsm-203

New in Contemporary Mathematics

Probability and Statistics

Probabilistic Methods in Geometry, Topology and Spectral Theory
Yaiza Canzani, University of North Carolina, Chapel Hill, Linan Chen, McGill University, Montreal, Quebec, Canada, and Dmitry Jakobson, McGill University, Montreal, Quebec, Canada, Editors

This volume contains the proceedings of the CRM Workshops on Probabilistic Methods in Spectral Geometry and PDE, held from August 22–26, 2016 and Probabilistic Methods in Topology, held from November 14–18, 2016 at the Centre de Recherches Mathématiques, Université de Montréal, Montréal, Quebec, Canada.

This volume covers recent developments in several active research areas at the interface of Probability, Semiclassical Analysis, Mathematical Physics, Theory of Automorphic Forms and Graph Theory.

This item will also be of interest to those working in geometry and topology and differential equations.

Contemporary Mathematics, Volume 739
December 2019, 197 pages, Softcover, ISBN: 978-1-4704-4145-6, LC 2019023076, 2010 Mathematics Subject Classification: 05C80, 11F72, 33C55, 35P20, 58J51, 58J65, 60C05, 60G15, 60G60, 81Q50, List US$117, AMS members US$93.60, MAA members US$105.30, Order code CONM/739

bookstore.ams.org/conm-739
New AMS-Distributed Publications

Analysis

**Microlocal Analysis of Quantum Fields on Curved Spacetimes**

Christian Gérard, Université de Paris 11, Orsay, France

This monograph focuses on free fields and the corresponding quasi-free states, and, more precisely, on Klein–Gordon fields and Dirac fields.

This monograph is addressed to both mathematicians and mathematical physicists. Mathematicians will find the book useful as a rigorous exposition of free quantum fields on curved spacetimes and as an introduction to some interesting and physically important problems arising in this domain. Mathematical physicists may find this text a helpful introduction to the use of more advanced tools of microlocal analysis in this area of research.

This item will also be of interest to those working in mathematical physics.

A publication of the European Mathematical Society (EMS). Distributed within the Americas by the American Mathematical Society.

ESI Lectures in Mathematics and Physics, Volume 10

[bookstore.ams.org/emsesilec-10](bookstore.ams.org/emsesilec-10)

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**General Interest**

**Gösta Mittag-Leffler and Vito Volterra**

40 Years of Correspondence

Frédéric Jaëck, Ecole Normale Supérieure, Paris, France, Laurent Mazliak, Université Pierre et Marie Curie, Paris, France, Emma Sallent Del Colombo, Universitat de Barcelona, Spain, and Rossana Tazzioli, Université Lille 1, Villeneuve-d’Ascq, France, Editors

This vast set of letters affords the reader a general overview of mathematical life at the turn of the 19th century and an appreciation of the European intellectual spirit which came to an end, or at least suffered a drastic turn, when the Great War broke out.

Volterra and Mittag-Leffler’s exchanges illustrate how general analysis, especially functional analysis, gained a dramatic momentum during those years, and how Volterra became one of the major leaders of the field, opening the path for several fundamental developments over the following decades. Through the letters, the reader can follow the institutional career and scientific activity of both Volterra and Mittag-Leffler, who shared many details about their lives.

The four editors are all specialists in the history of mathematics of the considered period. An extensive general introduction to the correspondence explains the context and the conditions in which it was developed. Moreover, the original letters are annotated with a large number of footnotes which provide a broader cultural picture from these captivating documents.

A publication of the European Mathematical Society (EMS). Distributed within the Americas by the American Mathematical Society.

Heritage of European Mathematics, Volume 12

[bookstore.ams.org/emshem-12](bookstore.ams.org/emshem-12)
The AMS Mathematics Calendar is now an online-only product!

Please note that the Notices of the American Mathematical Society no longer includes these listings.

You can submit an entry to the Mathematics Calendar at www.ams.org/cgi-bin/mathcal/mathcal-submit.pl

Questions and answers regarding this page can be sent to mathcal@ams.org.
Meetings & Conferences of the AMS
January Table of Contents

The Meetings and Conferences section of the Notices gives information on all AMS meetings and conferences approved by press time for this issue. Please refer to the page numbers cited on this page for more detailed information on each event.

Invited Speakers and Special Sessions are listed as soon as they are approved by the cognizant program committee; the codes listed are needed for electronic abstract submission. For some meetings the list may be incomplete. Information in this issue may be dated.

The most up-to-date meeting and conference information can be found online at www.ams.org/meetings.

**Important Information About AMS Meetings:** Potential organizers, speakers, and hosts should refer to page 110 in the January 2020 issue of the Notices for general information regarding participation in AMS meetings and conferences.

**Abstracts:** Speakers should submit abstracts on the easy-to-use interactive Web form. No knowledge of \LaTeX\ is necessary to submit an electronic form, although those who use \LaTeX\ may submit abstracts with such coding, and all math displays and similarly coded material (such as accent marks in text) must be typeset in \LaTeX. Visit [www.ams.org/cgi-bin abstracts abstract.pl](http://www.ams.org/cgi-bin abstracts abstract.pl) Questions about abstracts may be sent to abs-info@ams.org. Close attention should be paid to specified deadlines in this issue. Unfortunately, late abstracts cannot be accommodated.

### Meetings in this Issue

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*See [www.ams.org/meetings](http://www.ams.org/meetings) for the most up-to-date information on the meetings and conferences that we offer.*
MEETINGS & CONFERENCES

General Information Regarding Meetings & Conferences of the AMS

Speakers and Organizers: The Council has decreed that no paper, whether invited or contributed, may be listed in the program of a meeting of the Society unless an abstract of the paper has been received in Providence prior to the deadline.

Special Sessions: The number of Special Sessions at AMS meetings is limited. Special Sessions at annual meetings are held under the supervision of the Program Committee for National Meetings and, for sectional meetings, under the supervision of each Section Program Committee. They are administered by the Associate Secretary in charge of that meeting with staff assistance from the Meetings and Conferences Department in Providence. (See the list of Associate Secretaries on page 109 of this issue.)

Each person selected to give an Invited Address is also invited to generate a Special Session, either by personally organizing one or by having it organized by others on their behalf. Proposals to organize a Special Session are sometimes solicited either by a program committee or by the Associate Secretary. Other proposals should be submitted to the Associate Secretary in charge of that meeting (who is an ex officio member of the program committee) at the address listed on page 109 of this issue. These proposals must be in the hands of the Associate Secretary at least seven months (for sectional meetings) or nine months (for national meetings) prior to the meeting at which the Special Session is to be held, in order that the committee may consider all the proposals for Special Sessions simultaneously. Special Sessions must be announced in the Notices in a timely fashion so that any Society member who so wishes may submit an abstract for consideration for presentation in the Special Session. Contributors should know that there is a limit to the size of a single Special Session, so sometimes all places are filled by invitation. An author may speak in more than one Special Session at the same meeting. However, multiple talks given by the same author at a meeting must be distinct and have distinct abstracts. Papers submitted for consideration for inclusion in Special Sessions but not accepted will receive consideration for a Contributed Paper Session, unless specific instructions to the contrary are given.

The Society reserves the right of first refusal for the publication of proceedings of any Special Session. If published by the AMS, these proceedings appear in the book series Contemporary Mathematics. For more detailed information on organizing a Special Session, see the website www.ams.org/meetings/meet-specialsessionmanual.html.

Contributed Papers: The Society also accepts abstracts for ten-minute contributed papers. These abstracts will be grouped by related Mathematical Reviews subject classifications into sessions to the extent possible. The title and author of each paper accepted and the time of presentation will be listed in the program of the meeting. Although an individual may present only one ten-minute contributed paper at a meeting, any combination of joint authorship may be accepted, provided no individual speaks more than once.

Other Sessions: In accordance with policy established by the AMS Committee on Meetings and Conferences, mathematicians interested in organizing a session (for either an annual or a sectional meeting) on employment opportunities inside or outside academia for young mathematicians should contact the Associate Secretary of the meeting with a proposal by the stated deadline. Also, potential organizers for Poster Sessions (for an annual meeting) on a topic of choice should contact the Associate Secretary before the deadline.
Abstracts: Abstracts for all papers must be received by a conference coordinator in Providence by the stated deadline. Unfortunately, late papers cannot be accommodated.

Submission Procedures: Visit the Meetings and Conferences homepage on the Web at www.ams.org/meetings and select “Submit Abstracts.”

Site Selection for Sectional Meetings
Sectional meeting sites are recommended by the Associate Secretary for the section and approved by the Secretariat. Recommendations are usually made eighteen to twenty-four months in advance. Host departments supply local information, twenty-five to thirty rooms equipped with computer projection systems and chalkboards or whiteboards for Contributed Paper Sessions and Special Sessions, a large auditorium equipped with a computer projection system and microphones for the Invited Addresses, space for registration activities, membership table, and an AMS book exhibit, and registration clerks. The Society will partially reimburse for expenses needed to run these meetings successfully.

For more information, contact the Associate Secretary for the section.
Your generous gifts help hundreds of mathematically talented youths receive immersive mathematics education each summer. Thank you for your donations!

**SUPPORT** AMS Epsilon Fund for Young Scholars Programs.

[www.ams.org/support](http://www.ams.org/support)

*Thank you!*

For questions or more information, contact the AMS Development office at 401.455.4111 or development@ams.org
Meetings & Conferences of the AMS

Important information regarding meetings programs: AMS sectional meeting programs do not appear in the print version of the Notices. However, comprehensive and continually updated meeting and program information with links to the abstract for each talk can be found on the AMS website. See https://www.ams.org/meetings.

Final programs for sectional meetings will be archived on the AMS website accessible from the stated URL.

Denver, Colorado
Colorado Convention Center

January 15–18, 2020
Wednesday – Saturday

Meeting #1154
Joint Mathematics Meetings, including the 126th Annual Meeting of the AMS, 103rd Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM)

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/national.html.

Joint Invited Addresses
Skip Garibaldi, IDA Center for Communications Research, La Jolla, Uncovering lottery shenanigans (AMS-MAA Invited Address).
Karen M. Lange, Wellesley College, Different problems, common threads: Computing the difficulty of mathematical problems (AMS-MAA Invited Address).
Rajiv Maheswaran, Second Spectrum, The fantastic intersection of math and sports: Where no one is afraid of a decimal point (MAA-AMS-SIAM Gerald and Judith Porter Public Lecture).
Birgit Speh, Cornell University, Branching laws for representations of a non compact orthogonal group (AWM-AMS Noether Lecture).

AMS Invited Addresses
Bonnie Berger, Massachusetts Institute of Technology, Biomedical data sharing and analysis at scale.
Ingrid Daubechies, Duke University, Mathematical Frameworks for Signal and Image Analysis.
Gregory W. Moore, Rutgers University, Smooth invariants of four-dimensional manifolds and quantum field theory.
Nancy Reid, University of Toronto, *In praise of small data: statistical and data science* (AMS Josiah Willard Gibbs Lecture).


Tatiana Toro, University of Washington, *Differential operators and the geometry of domains in Euclidean space* (AMS Maryam Mirzakhani Lecture).

Anthony Várilly-Alvarado, Rice University, *The geometric disposition of Diophantine Equations*.

### AMS Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at [http://jointmathematicsmeetings.org/meetings/abstracts/abstract.pl?type=jmm](http://jointmathematicsmeetings.org/meetings/abstracts/abstract.pl?type=jmm).

Some sessions are cosponsored with other organizations. These are noted within the parenthesis at the end of each listing, where applicable.

*Advances in Multivariable Operator Theory: Connections with Algebraic Geometry, Free Analysis, and Free Probability,* Joseph A. Ball, Virginia Tech, and Paul S. Muhly, University of Iowa.

*Advances in Operator Algebras,* Jan Charleworth, University of California, Berkeley, Brent Nelson, Michigan State University, Sarah Reznikoff, Kansas State University, and Lauren Ruth, Vanderbilt University.

*Algebraic Cycles in Arithmetic and Geometry,* Jeff Achter, Colorado State University, and Sebastian Casalaina-Martin, University of Colorado Boulder.

*Algebras and Algorithms,* Keith A. Kearnes, Peter Mayr, and Agnes Szendrei, University of Colorado, Boulder.

*Algorithms, Analysis, and Applications of Numerical PDEs,* Xiaoming He, Missouri University of Science and Technology, and Jianguo (James) Liu, Colorado State University.

*Algorithms, Experimentation, and Applications in Number Theory,* Beth Malmskog, Colorado College, and Christopher Rasmussen, Wesleyan University.

*Analysis and Differential Equations at Undergraduate Institutions,* William Green, Rose-Hulman Institute of Technology, and Katharine Ott, Bates College.

*Analysis of Nonlocal Models,* Giacomo Capodaglio, Florida State University, Marta D’Elia, Center for Computing Research, Sandia National Laboratories, and Max Gunzburger, Florida State University.

*Analytic and Probabilistic Combinatorics,* Miklós Bóna, University of Florida, and Jay Pantone, Marquette University.

*Analytic Theory of Automorphic Forms and L-Functions,* Amanda Folsom, Amherst College, Michael Griffin, Brigham Young University, Larry Rolen, Vanderbilt University, and Jesse Thorner, University of Florida.

*Applications and Computations in Knot Theory,* Harrison Chapman, Colorado State University, Heather A. Dye, McKendree University, and Jesse S.F. Levitt, University of Southern California.

*Applied Topology,* Henry Adams, Colorado State University, and Mikael Vejdemo-Johansson, CUNY College of Staten Island.


*Arithmetic Galois Actions,* Ozlem Ejder, Colorado State University, Jamie Juul, University of British Columbia, and Rachel Pries, Colorado State University.

*Aspects and Applications of Algebraic Combinatorics,* William J. Martin, Worcester Polytechnic Institute, and Jason Wilfliord, University of Wyoming.

*C*-Algebras, Dynamical Systems and Applications,* Robin Deeley, University of Colorado Boulder, and Zhuang Niu and Ping Zhong, University of Wyoming.

*Choiceless Set Theory and Related Areas,* Paul Larson, Miami University, and Jindrich Zapletal, University of Florida (AMS-ASL).

*Coding Theory and Applications,* Allison Beemer, New Jersey Institute of Technology, Ian F. Blake, University of British Columbia, Christine A. Kelley, University of Nebraska-Lincoln, and Felice Manganiello, Clemson University.

*Cohomological Field Theories and Wall Crossing,* Yefeng Shen, University of Oregon, and Mark Shoemaker, Colorado State University.

*Combinatorial Structures and Integrable Systems,* Maxim Arnold and Nathan Williams, University of Texas at Dallas.

*Commutative Algebra,* Patricia Klein, University of Kentucky, and Haydee Lindo, Williams College.

*Computational and Categorical Methods in Homotopy Theory,* Agnes Beaudry, University of Colorado Boulder, and Julie Bergner, University of Virginia.
Computational Biomedicine, Nek Valsou and Niels Halama, National Center for Tumor Diseases Heidelberg, German Cancer Research Center.


Differential and Difference Equations in Biological Dynamics, Xiang-Sheng Wang and Aijun Zhang, University of Louisiana at Lafayette.

Differential Geometry and Global Analysis, I, Honoring the Memory of Tadashi Nagano (1930-2017), Bang-Yen Chen, Michigan State University, Nicholas D. Brubaker and Thomas Murphy, California State University, Fullerton, Takashi Sakai, Tokyo Metropolitan University, Makiko Sumi Tanaka, Tokyo University of Sciences, Bogdan D. Suceava, California State University, Fullerton, and Mihaela B. Vajiac, Chapman University.

Evolution, Chris McCarthy and Johannes Hamilton, Borough of Manhattan Community College CUNY.


Explicit Methods in Arithmetic Geometry in Characteristic p, I (a Mathematics Research Communities Session), Vaidehee Thatte, Binghamton University, Sarah Arpin, University of Colorado Boulder, and Nicholas Triantafillou, University of Georgia.

Extremal and Probabilistic Combinatorics, Sean English and Emily Heath, University of Illinois Urbana Champaign, and Michael Tait, Villanova University.

Fractal Geometry, Dynamical Systems, and Applications, Andrea Arauza Rivera, California State University, East Bay, Robert Niemeyer, Metropolitan State University, and John Rock, California State Polytechnic University, Pomona.

Frames, Designs, and Optimal Spherical Configurations, Xuemei Chen, New Mexico State University, Alexey Glazyrin, University of Texas Rio Grande Valley, Kasso Okoudjou, University of Maryland, College Park, and Oleksandr Vlasiuk, Florida State University.

From STEM to STEAMS (Science, Technology, Engineering, AI, Mathematics, Statistics), Charles Chen, Applied Materials, and Mason Chen, Stanford OHS.

Future Directions in Theory & Applications of Nonlinear Reaction-Diffusion Equations, Jerome Goddard, II, Auburn University at Montgomery, Nsoki Mavinga, Swarthmore College, and Quinn Morris, Appalachian State University.

Geometric Representation Theory and Equivariant Elliptic Cohomology, I (a Mathematics Research Communities Session), Anne Dranowski, University of Toronto, Noah Arbesfeld, Imperial College London, and Dominic Culver, University of Illinois Urbana-Champaign.

Geometry of Differential Equations, Jeanne Clelland and Yuhao Hu, University of Colorado Boulder, and George Wilkens, University of Hawaii.

Getting Started in Undergraduate Research: Topics, Tools and Open Problems, Hannah Highlander, University of Portland, Pamela E. Harris, Williams College, Erik Insko, Florida Gulf Coast University, and Aaron Wootton, University of Portland (AMS-MAA).

Group Actions in Harmonic Analysis, Keri Kornelson, University of Oklahoma, and Emily J. King, University of Bremen.

Groups and Topological Dynamics, Constantine Medynets, United States Naval Academy, Volodymyr Nekrashevych, Texas A&M University, and Dmytro Savchuk, University of South Florida.

Hamiltonian Systems, Sean Gasior, University of Sydney, Gabriel Martins, California State University Sacramento, and Andres Perico, University of California Santa Cruz.

Harmonic Analysis, Taryn C. Flock, Macalester College, and Betsy Stovall, University of Wisconsin-Madison.

Highly Accurate and Structure-Preserving Numerical Methods for Nonlinear Partial Differential Equations, Qin Sheng, Baylor University, Jorge E. Macias-Diaz, Universidad Autonoma de Aguascalientes, and Joshua L. Padgett, Texas Tech University.

History of Mathematics, Jemma Lorena, Pitzer College, Sloan Despeaux, Western Carolina University, Daniel Otero, Xavier University, and Adrian Rice, Randolph-Macon College (AMS-MAA).

How to Discover and Train Gifted Students, Scott Annin, California State University, Fullerton, Cezar Lupu, Texas Tech University, Shoo Seto, University of California, Irvine, and Bogdan D. Suceava, California State University, Fullerton.

If You Build It They Will Come: Presentations by Scholars in the National Alliance for Doctoral Studies in the Mathematical Sciences, David Goldberg, Purdue University, and Phil Kutzko, University of Iowa.

Interactions Among Partitions, Basic Hypergeometric Series, and Modular Forms, Chris Jennings-Shaffer, University of Denver, and Frank Garvan, University of Florida.

Interactions between Combinatorics, Representation Theory, and Coding Theory, Manabu Hagiwara, Chiba University, and Richard Green, University of Colorado Boulder.

Interactions of Inverse Problems, Computational Harmonic Analysis, and Imaging, M. Zuhair Nashed, University of Central Florida, Will Freeden, University of Kaiserslautern, and Otmar Scherzer, University of Vienna.
Interfaces Between PDEs and Geometric Measure Theory, I (Associated with AMS Maryam Mirzakhani Invited Address Tatiana Toro), Robin Neumayer and Zihui Zhao, Institute for Advanced Study.

International Research Experience for Students (IRES), Asuman G. Aksoy, Claremont McKenna College, and Zair Ibragimov, California State University, Fullerton.

Iterative Methods for Large-Scale Data Analysis, Jamie Haddock, University of California Los Angeles, and Anna Ma, University of California San Diego.

Logic Facing Outward, I (Associated with Joint AMS-MAA Invited Address Karen Lange), Karen Lange, Wellesley College, and Russell Miller, Queens College & Graduate Center CUNY (AMS-ASL).


Mathematical Analysis in Data Science, I (Associated with AMS Colloquium Lectures Ingrid Daubechies), Radu Balan, University of Maryland, Tingran Gao, University of Chicago, Sinan Gunturk, New York University, and Ozgur Yilmaz, University of British Columbia.

Mathematical and Computational Research in Data Science, Linda Ness, DIMACS, Rutgers University, F. Patricia Medina, Yeshiva University, and Kathryn Leonard, Occidental College (AMS-AWM).

Mathematical Aspects of Conformal Field Theory, Shashank Kanade and Andrew Linshaw, University of Denver, and Robert McRae, Vanderbilt University.

Mathematical Information in the Digital Age of Science, Patrick Ion, IMKT & University of Michigan, Olaf Teschke, zb-Math, and Stephen Watt, University of Waterloo.

Mathematical Physics, Some Open Problems for the 21st Century, Michael Maroun.

Mathematical Programming and Combinatorial Optimization, Steffen Borgwardt, University of Colorado Denver, and Tamon Stephen, Simon Fraser University.

Mathematics and Motherhood, Della Dumbaugh, University of Richmond, Carrie Diaz Eaton, Bates College, and Emille Lawrence, University of San Francisco.

Matrices and Graphs, Leslie Hogben, Iowa State University and American Institute of Mathematics, and Bryan L. Shader, University of Wyoming.

Mean Field Games: Theory and Applications, François Delarue, University of Nice Sophia Antipolis.

Modeling Natural Resources, Shandelle M. Henson, Andrews University, and Julie Blackwood, Williams College.

Noncommutative Geometry and Applications, Frederic Latremoliere, University of Denver.

Novel Teaching Practices in Mathematics, David Weisbart, University of California, Riverside.

Outreach Strategies for Reaching Underrepresented Students at the Pre-College Level, Jacob Castaneda, The Art of Problem Solving/Bridge to Enter Advanced Mathematics (BEAM), Cory Colbert, Washington & Lee University, Li-Mei Lim, Boston University/PROMYS, Max Warshauer, Texas State University at San Marcos, and Daniel Zaharopol, The Art of Problem Solving Initiative/Bridge to Enter Advanced Mathematics (BEAM).

Partition Theory and q-Series, Madeline Locus Dawsey, The University of Texas at Tyler, Marie Jameson, University of Tennessee, Knoxville, and James Sellers, Pennsylvania State University.

Pedagogical Innovations That Lead to Successful Mathematics, Michael A. Radin, Rochester Institute of Technology, Natali Hritonenko, Prairie View A&M University, and Ellina Grigorieva, Texas Women’s University.

Quantization for Probability Distributions and Dynamical Systems, Mrinal Kanti Roychowdhury, University of Texas Rio Grande Valley.

Quantum Theory of Matter Meets Noncommutative Geometry and Topology, Masoud Khalkhali, University of Western Ontario, and Markus Pflaum, University of Colorado, Boulder.

Random Combinatorial Structures, Complex Analysis and Integrable Systems, Virgil U. Pierce, University of Northern Colorado, and Nicholas M. Ercolani, University of Arizona.

Random Matrices and Integrable Systems, I (a joint session with the SIAM Minisymposium on the same topic), Ken McLaughlin, Colorado State University, and Sean O’Rourke, University of Colorado Boulder (AMS-SIAM).


Recent Advances in Function and Operator Theory, Kelly Bickel, Bucknell University, Alberto Condori, Florida Gulf Coast University, William Ross, University of Richmond, and Alan Sola, Stockholm University.

Recent Advances in Time-Stepping Methods for Ocean Modeling, Sara Calandrini, Konstantin Pieper, and Max Gunzburger, Florida State University.

Recent Advances of Mathematical Modeling on Ecology and Epidemiology, Xi Huo, University of Miami, and Rongsong Liu, University of Wyoming.
Recent Developments in Numerical Methods for PDEs, Valeria Barra, University of Colorado Boulder, and Oana Marin, Argonne National Laboratory.

Recent Trends in Semigroup Theory, Michael Kinyon, University of Denver, and Ben Steinberg, City College of New York. Representations of Finite Groups and Related Structures, Mandi Schaeffer Fry, Metropolitan State University of Denver, and Nat Thiem, University of Colorado Boulder.

Representation Theory Inspired by the Langlands Conjectures, I (Associated with Joint AWM-AMS Noether Lecture Birgit Speh), Birgit Speh, Cornell University, and Peter Trapa, University of Utah (AMS-AWM).

Research from the Rocky Mountain-Great Plains Graduate Research Workshop in Combinatorics, Steve Butler, Iowa State University, Michael Ferrara, University of Colorado Denver, Jeremy Martin, University of Kansas, Tyrrell McAllister, University of Wyoming, and Jamie Radcliffe, University of Nebraska-Lincoln.

Research in Graph Theory and Combinatorics by Research Experience for Undergraduate Faculty (REUF) Alumni and Their Students, Katie Anders and Kassie Archer, University of Texas at Tyler, and Briana Foster-Greenwood, California State Polytechnic University-Pomona.

Research in Mathematics by Early Career Graduate Students, Marat Markin, Morgan Rodgers, and Khang Tran, California State University Fresno.

Research in Mathematics by Undergraduates and Students in Post-Baccalaureate Programs, Darren A. Narayan, Rochester Institute of Technology, Khang Tran, California State University Fresno, Mark David Ward, Purdue University, and John Wierman, The Johns Hopkins University (AMS-AWM).

Riemannian Foliations and Applications, Igor Prokhorenkov and Ken Richardson, Texas Christian University. Self-Distributive Structures, Knot Theory, and the Yang-Baxter Equation, Mohamed Elhamdadi, University of South Florida, Petr Vojtechovska, University of Denver, and David Stanovsky, Charles University in Prague.


Spectral and Transport Properties of Disordered Systems, Peter D. Hislop, University of Kentucky, and Jeffrey Schenker, Michigan State University.

Stochastic Analysis and Applications in Finance, Actuarial Science and Related Fields, Julius N. Esunge, University of Mary Washington, See Keong Lee, Universiti Sains Malaysia, and Aurel I. Stan, The Ohio State University.

Stochastic Differential Equations and Application of Mathematical Biology, Jianjun Paul Tian, New Mexico State University, Hai-Dang Nguyen, University of Alabama, Xianyi Zeng, University of Texas at El Paso, and Robert Smits, New Mexico State University.

Stochastic Spatial Models (a Mathematics Research Communities Session), Tobias Johnson, College of Staten Island, Erin Beckman, Duke University, and Katelynn Kochalski, SUNY Geneseo.

Symbolic Dynamics, Ronnie Pavlov, University of Denver, and Scott Schmieding, Northwestern University. The Geometry of Complex Polynomials and Rational Functions, Trevor Richards, Monmouth College, and Malik Younsi, University of Hawai'i.

The Kaczmarz Algorithm with Applications in Harmonic Analysis and Data Science, Xuemei Chen, New Mexico State University, Palle E.T. Jorgensen, University of Iowa, and Eric Weber, Iowa State University.

The Mathematics of Social Justice, Andrea Arauza Rivera, California State University, East Bay, Paige Helms, University of Washington, Ryan Moruzzi, Ithaca College, and Robin Wilson, California Poly Pomona.

Topological Measures of Complexity: Inverse Limits, Entropy, and Structure of Attractors, Lori Alvin, Furman University, Jan P. Boronski, National Supercomputing Centre IT4innovations, Joanna Furno, University of Houston, and Piotr Oprocha, AGH University of Science and Technology.

Utilizing Mathematical Models to Understand Tumor Heterogeneity and Drug Resistance, James Greene, Clarkson University, Hwayeon Ryu, Elon University, and Kamila Larripa, Humboldt State University.


Wall to Wall Modeling Activities in Differential Equations Courses, Janet Fierson, La Salle University, Therese Shelton, Southwestern University, and Brian Winkel, SIMIODE.

Women in Mathematical Biology, Christina Edholm, University of Tennessee, Amanda Laubmeier, University of Nebraska-Lincoln, Katharine Gurski, Howard University, and Heather Zinn Brooks, University of California Los Angeles (AMS-AWM).

Women in Symplectic and Contact Geometry, Morgan Weiler, Rice University, Catherine Cannizzo, Simons Center for Geometry and Physics, and Melissa Zhang, University of Georgia (AMS-AWM).

Charlottesville, Virginia

University of Virginia

March 13–15, 2020

Meeting #1155
Southeastern Section
Associate secretary: Brian D. Boe

Invited Addresses

Moon Duchin, Tufts University, How we divide ourselves up to vote, and why it matters (Einstein Public Lecture in Mathematics).

Laura Ann Miller, University of North Carolina, The fluid dynamics of nutrient exchange in organs and organisms at the mesoscale.

Betsy Stovall, University of Wisconsin-Madison, An inverse problems approach to some questions arising in harmonic analysis.

Yusu Wang, Ohio State University, Topological and geometric methods for graph analysis.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.

Advances in Difference, Differential, Fractional Differential and Dynamic Equations with Applications (Code: SS 2A), Muhammad Islam and Youssouf Raffoul, University of Dayton.

Advances in High and Infinite Dimensional Stochastic Analysis (Code: SS 36A), Juraj Foldes, University of Virginia, Nathan Glatt-Holtz, Tulane University, and Mouhamadou Sy, University of Virginia.

Advances in Infectious Disease Modeling: From Cells to Populations (Code: SS 4A), Lauren Childs, Stanca Ciupe, and Omar Saucedo, Virginia Tech.

Advances in Operator Algebras (Code: SS 22A), Ben Hayes and David Sherman, University of Virginia.

Algebraic Groups: Arithmetic and Geometry (Code: SS 10A), Raman Parimala, Emory University, Andrei Rapinchuk, University of Virginia, and Igor Rapinchuk, Michigan State University.

Categorical Representation Theory and Beyond (Code: SS 11A), You Qi and Liron Speyer, University of Virginia, and Joshua Sussan, CUNY Medgar Evers (AMS-AAAS).

Celebrating Diversity in Mathematics (Code: SS 20A), Lauren Childs, Virginia Tech, Sara Maloni, University of Virginia, and Rebecca R.G., George Mason University.

Combinatorial Methods in Geometric Group Theory (Code: SS 26A), Tarik Aougab, Haverford College, Marissza Loving, Georgia Institute of Technology, Priyam Patel, University of Utah, and Sunny Xiao, Brown University.

Combinatorics Related to Geometry and Representation Theory (Code: SS 34A), Heather M Russell, University of Richmond, and Rebecca Goldin, George Mason University.

Commutative Algebra (Code: SS 28A), Eloïsa Grifo, University of California, Riverside, and Sean Sather-Wagstaff, Clemson University.

Convexity and Probability in High Dimensions (Code: SS 31A), Steven Hoehner, Longwood University, and Mark Meckes and Elisabeth Werner, Case Western Reserve University.

Curves, Jacobians, and Abelian Varieties (Code: SS 1A), Andrew Obus, Baruch College (CUNY), Tony Shaska, Oakland University, and Padmavathi Srinivasan, Georgia Institute of Technology.
Cyber Defense and Cryptography in Undergraduate Education (Code: SS 23A), Lubjana Beshaj, West Point Military Academy, and Tony Shaska, Oakland University.

Homotopy Theory (Code: SS 15A), Julie Bergner and Nick Kuhn, University of Virginia.

Integral Probability (Code: SS 27A), Leonid Petrov, University of Virginia, and Axel Saenz.

Knots and Links in Low-Dimensional Topology (Code: SS 5A), Thomas Mark, University of Virginia, Allison Moore, University of California Davis, and Ziva Myer, Duke University.

Knot Theory and its Applications (Code: SS 25A), Hugh Howards and Jason Parsley, Wake Forest University, and Eric Rawdon, St. Thomas University.

Mathematical Modeling of Problems in Biological Fluid Dynamics (Code: SS 30A), Laura Miller, University of North Carolina at Chapel Hill, and Nick Battista, The College of New Jersey.

Mathematical String Theory (Code: SS 9A), Ilarion Melnikov, James Madison University, Eric Sharpe, Virginia Tech, and Diana Vaman, University of Virginia (AMS-AAAS).

Motivic Aspects of Topology and Geometry (Code: SS 16A), Kirsten Wickelgren, Duke University, and Inna Zakharevich, Cornell University.

Nonlocal PDEs and Applications (Code: SS 33A), Siming He, Duke University, and Changhui Tan, University of South Carolina.

Numerical Methods for Partial Differential Equations: A Session in Honor of Slimane Adjerid’s 65th Birthday (Code: SS 3A), Mahboub Baccouch, University of Nebraska at Omaha.

Probabilistic Methods in Geometry and Analysis (Code: SS 19A), Fabrice Baudoin and Li Chen, University of Connecticut.

Quantum Algebra and Geometry (Code: SS 24A), Marco Aldi, Virginia Commonwealth University, Michael Penn, Randolph College, and Nicola Tarasca and Juan Villarreal, Virginia Commonwealth University.

Recent advances in Graph Theory and Combinatorics (Code: SS 8A), Neal Bushaw, Virginia Commonwealth University, and Martin Rolek and Gexin Yu, College of William and Mary (AMS-AAAS).

Recent Advances in Harmonic Analysis (Code: SS 7A), Amalia Culiuc, Amherst College, Yen Do, University of Virginia, and Eyvindur Ari Palsson, Virginia Tech.

Recent Advances in Mathematical Biology (Code: SS 32A), Junping Shi, College of William & Mary, Zhisheng Shuai, University of Central Florida, and Xixiang Wu, Middle Tennessee State University.

Recent Combinatorial Advances in Representation Theory and Algebraic Geometry (Code: SS 29A), Jennifer Morse, University of Virginia, and Sarah Mason, Wake Forest University.

Recent Progress on Singular and Oscillatory Integrals (Code: SS 35A), Betsy Stovall and Joris Roos, University of Wisconsin-Madison.

Representation Theory of Algebraic Groups and Quantum Groups: A Tribute to the Work of Cline, Parshall and Scott (CPS) (Code: SS 13A), Chun-Ju Lai and Daniel K. Nakano, University of Georgia, and Weiqiang Wang, University of Virginia.

Special Sets of Integers in Modern Number Theory (Code: SS 14A), Cristina Ballantine, College of the Holy Cross, and Hester Graves, Center for the Computing Sciences.

Tensors and Complexity (Code: SS 17A), Visu Makam, Institute for Advanced Study, and Rafael Oliveira, University of Toronto.

The Mathematics of Redistricting (Code: SS 18A), Marion Campisi, San Jose State University, Thomas Ratliff, Wheaton College, and Ellen Veomett, Saint Mary’s College of California.

Trends in Teichmüller Theory (Code: SS 21A), Thomas Koberda and Sara Maloni, University of Virginia, and Giuseppe Martone, University of Michigan.

Youth and Enthusiasm in Arithmetic Geometry and Number Theory. (Code: SS 12A), Evangelia Gazaki and Ken Ono, University of Virginia.

Accommodations

Participants should make their own arrangements directly with the hotel of their choice. Special discounted rates were negotiated with the hotels listed below. Rates quoted do not include the Virginia state hotel tax (13.3%), local taxes and hotel fees may apply. Participants must state that they are with the American Mathematical Society’s (AMS) Spring Southeast Sectional Meeting to receive the discounted rate. The AMS is not responsible for rate changes or for the quality of the accommodations. Hotels have varying cancellation and early checkout penalties; be sure to ask for details.

The Graduate Charlottesville (.4 miles from Nau Hall), 1309 W. Main Street, Charlottesville, VA 22903; (434)220-0163; www.graduatehotels.com/charlottesville. Rates are US$134 per night for a room with one king bed or two queen
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beds. The hotel has valet parking which is currently US$16 per night. The Graduate also has complimentary bikes for guests' use as well as a 24-hour fitness center. The deadline for reservations at this rate is February 2, 2020.

Hampton Inn & Suites - At The University (.6 miles from Nau Hall), 900 W. Main Street, Charlottesville, VA 22903; (434)923-8600; www.hilton.com/en/hotels/chomsh-hampton-suites-charlottesville-at-the-university. Rates are US$149 per night for a room with one king bed or two queen beds which include free hot buffet breakfast. The Hampton Inn & Suites offers complimentary Wi-Fi and has a business center on property as well as a fitness center. Self parking is complimentary. The deadline for reservations at this rate is February 12, 2020.

Home 2 Suites by Hilton Charlottesville Downtown (1.7 miles from Nau Hall), 201 Monticello Avenue, Charlottesville, VA 22902; (434)295-0003; www.hilton.com/en/hotels/chodnht-home2-suites-charlottesville-downtown/?SEQ_id=GMB-HT-CHODNHT. Rates are US$129 on Thursday night for all room types and US$169 on Friday and Saturday nights for all room types. These rates include complimentary hot buffet breakfast. Home 2 Suites also offers complimentary parking, laundry and Wi-Fi. This property features a business center and a 24-hour fitness center with indoor pool. The deadline for reservations at this rate is February 12, 2020.

Residence Inn by Marriott Charlottesville (1.8 miles from Nau Hall), 111 Millmont Street, Charlottesville, VA 22903; (434)923-0300; www.marriott.com/hotels/travel/chori-residence-inn-charlottesville/?scid=bb1a189a-fec3-4d19-a255-54ba596fefe2. Rates are US$189 per night for a room with a queen bed and pull-out sofa bed. This room rate includes free hot buffet breakfast. The Residence Inn offers free parking, a business center as well as a fitness center with a pool. The deadline for reservations at this rate is February 3, 2020.

Country Inn & Suites - UVA (2.3 miles from Nau Hall), 1600 Emmit Street North, Charlottesville, VA 22901; (434)293-4600; www.radissonhotels.com/en-us/hotels/country-inn-charlottesville-va?cid=a%3Ae%20b%3Agmb%20c%3Aamer%20%3Alocat%200%3Acis%20%3Aus%20%3AUSACUVA&checkInDate=2019-11-18&checkOutDate=2019-11-19&adults%5B%5D=1&children%5B%5D=0&searchType=lowest&promotionCode=. Rates are US$139 per night for a room with one king bed or two queen beds. This rate is for the first night only. And people who find this offer tempting may hand over their credit card data, believing they have scored a great rate and their housing is a done deal. Unfortunately, this often turns out to be the start of a long, costly nightmare.

Note that some of these room poachers create fake websites on which they represent themselves as the organizers of the meeting and include links to book rooms, etc. The only official website for this meeting is ams.org and one that has the official AMS logo.

These housing bureaus are not affiliated with the American Mathematical Society or any of its meetings, in any way. The AMS would never call anyone to solicit reservations for a meeting. The only way to book a room at a rate negotiated for an AMS Sectional Meeting is via a listing on AMS Sectional Meetings pages or Notices of the AMS. The AMS cannot be responsible for any damages incurred as a result of hotel bookings made with unofficial housing bureaus.

Food Services

There are many dining options within half a mile of the meeting and Charlottesville offers options of all types of cuisine. For more information on dining throughout Charlottesville please visit www.visitcharlottesville.org/restaurants. General information for visiting Charlottesville can be found here: www.visitcharlottesville.org.

Some of the dining options near the meeting include:

- Fry's Spring Station, 2115 Jefferson Park Avenue, www.eatatfrys.com
- The Pigeon Hole, 11 Elliewood Avenue, www.thepigeonholeville.com
MEETINGS & CONFERENCES

- Farm Bell Kitchen, 1209 W. Main Street, www.farmbellkitchen.com
- White Spot, 1407 University Avenue, (434)295-9899
- Lemongrass, 104 14th Street NW, (434)244-8424
- Armando’s Mexican Restaurant, 105 14th Street NW, www.armandosmexicanrest.com
- Roots Natural Kitchen, 1329 W Main Street, www.rootsnaturalkitchen.com
- Box’d Kitchen, 909 W Main Street, www.boxdkitchen.com
- Anna’s Pizza No 5, 115 Maury Avenue, (434)977-6228

Registration and Meeting Information

Advance Registration: Advance registration for this meeting will open on January 22, 2020. Advance registration fees will be US$71 for AMS members, US$115 for nonmembers, US$13 for students and unemployed mathematicians, and US$15 for emeritus members. Fees will be payable by cash, check, or credit card. Participants may cancel registrations made in advance by emailing mmsb@ams.org. 100% refunds will be issued for any advance registrations canceled by the first day of the meeting. After this date, no refunds will be issued.

On-site Information and Registration: The registration desk will be located in the atrium on the first floor of Nau Hall. The AMS book exhibit and coffee service will also be located in the first floor atrium of Nau Hall. The Invited Address lectures will be located in Nau Hall, Room 101 and the Special Sessions and Contributed Paper Sessions will be held in Nau Hall, Gibson Hall, New Cabell Hall, Clark Hall, Maury Hall and Monroe Hall. Please look for additional information about specific session room locations on the web and in the printed program. For further information on building locations, a campus map is available at visitormap.virginia.edu/#!.

The registration desk will be open on Friday, March 13, 1:00 pm to 5:00 pm and Saturday, March 14, 7:30 am to 4:00 pm. The same fees listed above apply for on-site registration and are payable with cash, check, or credit card.

Program Books

In order to keep registration fees as low as possible, save on printing costs, and make the meetings more environmentally friendly, a small fee will be charged for receiving a program book.

If you want to receive a program book, please complete the section entitled “Printed Program” on the registration form and pay a nominal fee of US$3 for each copy. If you do not want to receive a program book, please skip that section and click “Next.” All purchased program books will be distributed at the registration desk at the meeting. No program books will be mailed before the meeting. A small quantity of program books may be available for purchase on-site at meetings, however supplies will be limited.

For your convenience, the following changes have been made to the website to make the program more accessible and mobile-friendly.

1. **Online Timetable:** The sectional meetings now have an online timetable display. It will provide a quick reference to where the sessions and rooms are located.

2. **Print-friendly Pages:** The pages of the program now have a "Print" button in the top right-hand corner. It is black and is depicted by a little printer. If you cannot see it, please maximize your window. The printer icon is sometimes bundled into the generic “share” icon when the window is re-sized. If you click the printer button, it will print the text of the page without additional webpage elements. If you go to the page of a special session and click the print icon, you will get a schedule of all the parts of that session. This can also be printed to pdf if you have a pdf printer installed.

3. **Room Locations:** The room locations are now more conspicuous in the web program when the program is scheduled.

Other Activities

**Book Sales:** Stop by the on-site AMS bookstore to review the newest publications and take advantage of exhibit discounts and free shipping on all on-site orders! AMS and MAA members receive 40% off list price. Nonmembers receive a 25% discount. Not a member? Ask a representative about the benefits of AMS membership.

**AMS Editorial Activity:** An acquisitions editor from the AMS book program will be present to speak with prospective authors. If you have a book project that you wish to discuss with the AMS, please stop by the book exhibit.

**Membership Activities:** During the meeting, stop by the AMS Membership Exhibit to learn about the benefits of AMS Membership. Members receive free shipping on purchases all year long and additional discounts on books purchased at meetings, subscriptions to *Notices* and *Bulletin*, discounted registration for world-class meetings and conferences and more! Complimentary refreshments will be served courtesy in part of the AMS Membership Department.

**AMS BIG Career Development and Job Search Workshop:** Join us to hear mathematicians who are employed in BIG jobs discuss their work and career paths. This workshop is geared for graduate students and anyone else who wants to
know more about careers in business, entrepreneurship, industry, government, and nonprofits ("BIG"). Friday, March 13 from 5:30–8:30 pm. Details on this and other BIG activities are accessible through https://www.ams.org/big.

**Child Care Grants**

The AMS will provide a limited number of reimbursement grants of US$125 per family to help with the cost of child care for registered participants at the meeting. The funds may be used for any form of child care that frees a parent to participate more fully in the meeting. Grants will be awarded on a first-come, first-served basis, one per family, and one per season (i.e. Spring or Fall), the latter depending on the amount of grants available. Registration for the meeting as well as membership in the AMS is required to apply for this program.

Information about applying for child care grants will be available prior to the opening of advance registration in January; watch the meeting website for details and instructions. Applications will be on Mathprograms.org and will be accepted on a first-come, first-served basis until January 21, 2020. Final decisions on recipients will be made on or before February 14, 2020. All grant funds will be provided in the form of a check which will be issued at the meeting.

**Special Needs**

It is the goal of the AMS to ensure that its conferences are accessible to all, regardless of disability. The AMS will strive, unless it is not practicable, to choose venues that are fully accessible to the physically handicapped.

If special needs accommodations are necessary in order for you to participate in an AMS Sectional Meeting, please communicate your needs in advance to the AMS Meetings Department by:

- Registering early for the meeting,
- Checking the appropriate box on the registration form, and
- Sending an email request to the AMS Meetings Department at mmsb@ams.org or meet@ams.org.

**AMS Policy on a Welcoming Environment**

The AMS strives to ensure that participants in its activities enjoy a welcoming environment. In all its activities, the AMS seeks to foster an atmosphere that encourages the free expression and exchange of ideas. The AMS supports equality of opportunity and treatment for all participants, regardless of gender, gender identity or expression, race, color, national or ethnic origin, religion or religious belief, age, marital status, sexual orientation, disabilities, or veteran status.

Harassment is a form of misconduct that undermines the integrity of AMS activities and mission. The AMS will make every effort to maintain an environment that is free of harassment, even though it does not control the behavior of third parties. A commitment to a welcoming environment is expected of all attendees at AMS activities, including mathematicians, students, guests, staff, contractors and exhibitors, and participants in scientific sessions and social events. To this end, the AMS will include a statement concerning its expectations towards maintaining a welcoming environment in registration materials for all its meetings, and has put in place a mechanism for reporting violations. Violations may be reported confidentially and anonymously to 855-282-5703 or at www.mathsociety.ethicspoint.com. The reporting mechanism ensures the respect of privacy while alerting the AMS to the situation. Violations may also be brought to the attention of the coordinator for the meeting (who is usually at the meeting registration desk), and that person can provide advice about how to proceed.

For AMS policy statements concerning discrimination and harassment, see the AMS Anti-Harassment Policy at www.ams.org/about-us/governance/policy-statements/anti-harassment-policy.

Questions about this welcoming environment policy should be directed to the AMS Secretary at https://www.ams.org/about-us/governance/sec-contact.

**Local Information and Maps**

This meeting will take place on the University of Virginia campus. A campus map can be found at visitormap.virginia.edu/#/. Information about the University of Virginia Department of Mathematics can be found at https://math.virginia.edu/. Please visit the university website at www.virginia.edu/ for additional information on the campus.

Please watch the AMS website at www.ams.org/meetings/sectional1/sectional1.html for additional information on this meeting.

**Parking**

On Friday, the campus parking garage which is most convenient to the meeting is the Central Grounds Garage at 400 Emmet Street (goo.gl/maps/gkLqwFztsUZZcAM9). Central Grounds Garage is about a 10 minute walk to the buildings where the meeting will take place. The fee to park at Central Grounds Garage is US$2/hour from 8:00 am–5:00 pm and
US$1/hour after 5:00 pm. On Saturday and Sunday, the campus parking garage most convenient to the meeting is the Stadium Lot on Stadium Road near Scott Stadium. South Lot is about a 10 minute walk to the buildings where the meeting will take place. It is free of charge to park at South Lot on Saturday and Sunday (goo.gl/maps/gkLqfWfztSUZzaMB9). For more information about visitor parking at the University of Virginia, please visit https://parking.virginia.edu/visitor-parking.

**Travel**

This meeting will take place on the campus of the University of Virginia located in Charlottesville, Virginia. Nau Hall is located at 1540 Jefferson Park Avenue, Charlottesville, VA 22903.

**By Air:**
The Charlottesville-Albemarle Airport (CHO) is located approximately eight miles from the University. CHO is a non-hub, commercial service airport offering 50 daily non-stop flights to and from Charlotte, Philadelphia, New York/LaGuardia, Washington/Dulles, Detroit and Atlanta. CHO is served by Delta Connection, United Express (Atlantic Coast Airlines) and US Airways Express (Piedmont Airlines). www.gocho.com

To reach the University Grounds or your hotel from the airport you can use Uber, Lyft or Yellow Cab ((434)295-4131) which has a flat rate of $25.

The Richmond International Airport is located approximately 85 miles from the University of Virginia campus. Please visit the airport website for a list of airlines and lists of cities with daily direct flights to RIC, flyrichmond.com.

**By Rail:**
The Charlottesville Amtrak Station (www.amtrak.com/stations/cvs) is located at 810 West Main Street, approximately two miles from the University of Virginia. www.amtrak.com

**By Car:**
To navigate to the Central Grounds Parking Garage by GPS, use the following address: 400 Emmet Street South, Charlottesville, VA 22903.

**From I-64:** Get off at exit 118B for the Route 29/250 Bypass. Go approximately 2 miles and take the off ramp for Route 250 East Business (Ivy Road). Turn right off the exit ramp and follow Ivy Road. At the fourth traffic light, turn right on Route 29 Business/Emmet Street. Continue a half block. Turn left into the University of Virginia Central Grounds Parking Garage (hourly fees), which is directly in front of Newcomb Hall.

**From I-66 via US Route 29 South:** Take I-66 from Northern Virginia past Manassas. Take the exit marked “Route 29 South - Gainesville.” Proceed to Charlottesville, where Route 29 becomes Emmet Street. Pass University Hall basketball arena on the right. Proceed straight through the traffic light at the intersection of Route 250 Business (University Avenue/Ivy Road). Continue a half block. Turn left into the University of Virginia Central Grounds Parking Garage (hourly fees), which is directly in front of Newcomb Hall.

**From the North via US Route 29 South:** Take US Route 29 South until you drive under the US 250 Bypass. Remain on 29 South Business (Emmet Street) and continue past the intersection with US Route 250 Business (University Avenue/Ivy Road). Continue a half block. Turn left into the University of Virginia Central Grounds Parking Garage (hourly fees), which is directly in front of Newcomb Hall.

**From the South via US Route 29 North:** Take US Route 29 North to the US 250 East Business exit. Turn right when coming off the exit ramp onto 250 East (Ivy Road). Turn right off the exit ramp and follow Ivy Road. At the fourth traffic light, turn right on Route 29 Business/Emmet Street. Continue a half block. Turn left into the University of Virginia Central Grounds Parking Garage (hourly fees), which is directly in front of Newcomb Hall.

**Car Rental:** Hertz is the official car rental company for the meeting. To make a reservation accessing our special meeting rates online at www.hertz.com, click on the box “I have a discount,” and type in our convention number CV#04N30010. You can also call Hertz directly at (800)654-2240 (US and Canada) or (405)749-4434 (other countries). At the time of your reservation, the meeting rates will be automatically compared to other Hertz rates and you will be quoted the best comparable rate available. Meeting rates include unlimited mileage and are subject to availability. Advance reservations are recommended, blackout dates may apply.

**Information for International Participants**

Visa regulations are continually changing for travel to the United States. Visa applications may take from three to four months to process and require a personal interview, as well as specific personal information. International participants should view the important information about traveling to the US found at https://travel.state.gov/content/travel/en.html. If you need a preliminary conference invitation in order to secure a visa, please send your request to epm@ams.org.
If you discover you do need a visa, the National Academies website (see above) provides these tips for successful visa applications:

* Visa applicants are expected to provide evidence that they are intending to return to their country of residence. Therefore, applicants should provide proof of “binding” or sufficient ties to their home country or permanent residence abroad. This may include documentation of the following:
  - family ties in home country or country of legal permanent residence
  - property ownership
  - bank accounts
  - employment contract or statement from employer stating that the position will continue when the employee returns;
* Visa applications are more likely to be successful if done in a visitor’s home country than in a third country;
* Applicants should present their entire trip itinerary, including travel to any countries other than the United States, at the time of their visa application;
* Include a letter of invitation from the meeting organizer or the US host, specifying the subject, location and dates of the activity, and how travel and local expenses will be covered;
* If travel plans will depend on early approval of the visa application, specify this at the time of the application;
* Provide proof of professional scientific and/or educational status (students should provide a university transcript).

This list is not to be considered complete. Please visit the websites above for the most up-to-date information.

**Medford, Massachusetts**

*Tufts University*

**March 21–22, 2020**

Saturday – Sunday

**Meeting #1156**

Eastern Section

Associate secretary: Steven H. Weintraub

Announcement issue of *Notices*: January 2020  
Program first available on AMS website: February 11, 2020  
Issue of *Abstracts*: Volume 41, Issue 2

**Deadlines**

For organizers: Expired

For abstracts: January 28, 2020

*The scientific information listed below may be dated. For the latest information, see [https://www.ams.org/amsmtgs/sectional.html](https://www.ams.org/amsmtgs/sectional.html).*

**Invited Addresses**

- Daniela De Silva, Columbia University, *A viscosity approach to the regularity of variational problems.*
- Enrique Pujals, City University of New York, *Fifty years of the stability conjecture.*
- Christopher T Woodward, Rutgers University, New Brunswick, *Lagrangian Floer theory in the tropics.*

**Special Sessions**

*If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at [https://www.ams.org/cgi-bin/abstracts/abstract.pl](https://www.ams.org/cgi-bin/abstracts/abstract.pl).*

- **Algebraic Geometry in Dynamics** (Code: SS 15A), Nguyen-Bac Dang, Stony Brook University, and Nicole Looper and Rohini Ramadas, Brown University.
  - Analysis on Homogeneous Spaces (Code: SS 6A), Jens Christensen, Colgate University, Matthew Dawson, CIMAT, Mérida, México, and Fulton Gonzalez, Tufts University.
  - Anomalous Diffusion Processes (Code: SS 3A), Christoph Borgers, Tufts University, and Claude Greengard, New York University and Foss Hill Partners.
  - Automorphisms of Riemann Surfaces, Subgroups of Mapping Class Groups and Related Topics (Code: SS 10A), S. Allen Broughton, Rose-Hulman Institute of Technology, Jen Paulhus, Grinnell College, and Aaron Wootton, University of Portland.
MEETINGS & CONFERENCES


Discrete and Convex Geometry (Code: SS 18A), Undine Leopold and Egon Schulte, Northeastern University, and Pablo Soberón, Baruch College, CUNY.

Equivariant Cohomology (Code: SS 9A), Jeffrey D. Carlson, The Fields Institute, and Loring Tu, Tufts University.

Geometric Dynamics and Billiards (Code: SS 4A), Boris Hasselblatt and Eunice Kim, Tufts University, Kathryn Lindsey, Boston College, and Zbigniew Nitecki, Tufts University.

Homological Methods in Commutative Algebra (Code: SS 12A), Janet Striuli, Fairfield University and National Science Foundation, and Oana Veliche, Northeastern University.

Inverse Problems and Their Applications (Code: SS 19A), Yousef Qranfal, Wentworth Institute of Technology.

Linear Algebraic Groups: their Structure, Representations, and Geometry (Code: SS 23A), George McNinch, Tufts University, and Eric Sommers, University of Massachusetts.


Mathematics of Data Science (Code: SS 5A), Vasileios Maroulas, University of Tennessee Knoxville, and James M. Murphy, Tufts University.

Mirror Symmetry and Enumerative Geometry (Code: SS 20A), Mandy Cheung, Harvard University, and Siu-Cheong Lau and Yu-Shen Lin, Boston University.

Modeling and Analysis of Partial Differential Equations in Fluid Dynamics and Related Fields: Geometric and Probabilistic Methods (Code: SS 1A), Geng Chen, University of Kansas, Siran Li, Rice University and Centre de Recherches Mathématiques, Université de Montréal, and Kun Zhao, Tulane University.

Moduli of Curves, Hilbert Schemes, and Tropical Geometry (Code: SS 17A), Ignacio Barros, Northeastern University, Noah Giansiracusa, Bentley University, and Rob Silversmith, Northeastern University.

Probability in Dynamical Systems of Physical Origin (Code: SS 13A), Alex Blumenthal, University of Maryland, and Peter Nandori, Yeshiva University.

Quantum Probability, Orthogonal Polynomials, and Special Functions (Code: SS 11A), Maxim Derevyagin and Ambar Sengupta, University of Connecticut.

Random Discrete Structures (Code: SS 22A), Xavier Pérez-Giménez, University of Nebraska, and Lutz P Warnke, Georgia Institute of Technology.

Recent Advances in Schubert Calculus and Related Topics (Code: SS 2A), Christian Lenart and Changlong Zhong, State University of New York at Albany.

Subgroups in Nonpositive Curvature (Code: SS 8A), Robert Kropholler, Kim Ruane, and Genevieve Walsh, Tufts University.

Symmetries of Polytopes, Maps, and Graphs (Code: SS 16A), Gabe Cunningham, University of Massachusetts Boston, and Mark Mixer, Wentworth Institute of Technology.

The Combinatorics and Geometry of Jordan Type and Commuting Varieties (Code: SS 14A), Peter Crooks and Anthony Iarrobino, Northeastern University, and Leila Khatami, Union College.

Accommodations

Participants should make their own arrangements directly with the hotel of their choice. Special discounted rates were negotiated with the hotels listed below. Rates quoted do not include state and local taxes (11.7%–14.45% depending on the city) and hotel fees may apply. Participants must state that they are with the American Mathematical Society’s (AMS) Spring Eastern Sectional Meeting to receive the discounted rate. The AMS is not responsible for rate changes or for the quality of the accommodations.

Hotels have varying cancellation and early checkout penalties; be sure to ask for details.

Holiday Inn Express and Suites Boston-Cambridge, 50 Monsignor O’Brien Highway, Cambridge; (617)354-1313; www.ihg.com/holidayinnexpress/hotels/us/en/cambridge/boscb/hoteldetail?cm_mmc=GoogleMaps-_EX_ -US-_BOSCB. Rates are US$139 per night for a room with one king bed or two queen beds. To reserve a room at this rate please contact the hotel by phone and speak with reservations; to look into availability, or book via email, please contact Fatima Cortez at fatima.chavarriacortez@hhlp.com; and to book online use this page: www.ihg.com/holidayinnexpress/hotels/us/en/cambridge/boscb/hoteldetail?fromRedirect=true&qSrt=sBR&qIta=99801505&icdv =99801505&qt1H=BOSCB&gropCd=AMS&setPMCookies=true&qSHBrC=EX&qDest=250%20Monsignor%20Highway,%20Cambridge,%20MA,%20US&srb_u=1. Once on the page, please enter the dates for the meeting to access the discounted rates. Amenities include complimentary breakfast, complimentary coffee in the lobby, in-room mini refrigerator and microwave, complimentary internet access, and business center. This property offers self-parking to guests in...
this block for a discounted rate of US$15 per night. Check-in is at 3:00 pm, check-out is at 11:00 am. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. This property is located approximately 3.5 miles from campus. The deadline for reservations at this rate is March 20, 2020.

The Row Hotel at Assembly Row, 360 Foley Street, Somerville; (617)628-1300; www.therowhotelandassemblyrow.com. Rates are US$179 per night for a room with one king bed or two queen beds. To reserve a room at this rate please contact the hotel and indicate that you are with the American Mathematical Society’s (AMS) Spring Eastern Sectional Meeting. This property requires a first night’s deposit. Amenities include complimentary wireless internet; fitness center; indoor pool; business center; Reflections on-site restaurant, bar, and lounge offering room service; a sun terrace; complimentary guest pantry; and complimentary coffee and espresso. This property offers overnight valet parking for a fee of US$40 per night. Check-in is at 4:00 pm, check-out is at 12:00 pm. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. This property is located approximately 2.5 miles from campus. The deadline for reservations at this rate is February 20, 2020.

Kimpton Marlowe Hotel, 25 Edwin H. Land Boulevard, Cambridge; (617)868-8000; www.hotelmarlowe.com. Rates are US$169 per night for a room with one king bed, double occupancy; additional daily fee of US$20 per person will be charged for a room with more than 2 people in it. To reserve a room at this rate please contact the hotel by phone at 1-800-825-7140 or use the Central Reservations Line at 1-800-KIMPTON. Guests in this block will have the US$25 per day amenities fee waived (please visit the hotel website for a list of which amenities are included, such as upgraded Internet access and credit for food and beverage). Standard amenities include complimentary basic internet access for IHG Rewards Club Members; fitness center; yoga mats in each room and PUBLIC bikes for guest use; business center; complimentary coffee and tea service 7:00 am–10:00 am Saturday and Sunday; Bambara Kitchen and Bar on-site restaurant serving break-fast, lunch, and dinner; and Cambridge Lobby Bar on-site cocktail lounge. Parking is available for US$30 per day and valet parking is available for a fee of US$43 per day. Check-in is at 4:00 pm, check-out is at 12:00 pm. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. This property is located approximately 5 miles from campus. The deadline for reservations at this rate is February 21, 2020.

Boston Marriott Burlington, One Burlington Mall Road, Burlington, MA, 01803; (781)229-6565; www.marriott.com/hotels/travel/bosbu-boston-marriott-burlington. Rates are US$143 per night for a deluxe king room or US$153 for a deluxe double room. To reserve a room at these rates please call the property directly or call Marriott Reservations at (800)228-9290 and identify your affiliation with the American Mathematical Society Spring Eastern Sectional Meeting Room Block. Reservations can also be made online here: www.marriott.com/event-reservations/reservation-link.mi?id=1569944278412&key=GRP&app=resvlink. Amenities include complimentary basic internet access and high-speed internet access for a fee; fitness center; indoor pool; business center; Chopps American Bar and Grill on-site restaurant serving breakfast, lunch, and dinner; sundry/convenience store; on-site barber/beauty shop; and Starbucks. Complimentary on-site parking is available, and valet parking is available for a fee of US$15 per day. Check-in is at 4:00 pm, check-out is at 12:00 pm. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. This property is located approximately 12 miles from campus. The deadline for reservations at this rate is February 20, 2020.

Hyatt Place Medford-Boston, 116 Riverside Avenue, Medford, MA, 02155; (781)395-8500; www.hyatt.com/en-US/hotel/massachusetts/hyatt-place-boston-medford/boszm. Rates are US$135 per night for a room with one king bed or two double beds. To reserve a room at these rates please call the property and identify your affiliation with the American Mathematical Society Spring Eastern Sectional Meeting at Tufts University. Amenities include complimentary Breakfast Bar for World of Hyatt members, in-room refrigerator, complimentary high-speed internet access, fitness center, indoor swimming pool, business center, The Bar on-site restaurant, The Market 24-hour convenience store, free shuttle within 5-mile radius between 7:00 am–9:00 pm, and US$10 parking in garage and adjacent lot. Check-in is at 3:00 pm, check-out is at 12:00 pm. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. This property is located approximately 1 mile from campus. The deadline for reservations at this rate is February 28, 2020.

Boston Marriott Cambridge, 50 Broadway Street, Cambridge, MA, 02142; (617)494-6600; www.marriott.com/hotels/travel/boscb-boston-marriott-cambridge. Rates are US$219 per night for a room with one king bed or two double beds. To reserve a room at these rates please call the property directly or call Marriott Reservations at (800)228-9290 and identify your affiliation with the American Mathematical Society Spring Eastern Sectional Meeting at Tufts University. Amenities include complimentary Breakfast Bar for World of Hyatt members, in-room refrigerator, complimentary high-speed internet access, fitness center, indoor swimming pool, business center, The Bar on-site restaurant, The Market 24-hour convenience store, free shuttle within 5-mile radius between 7:00 am–9:00 pm, and US$10 parking in garage and adjacent lot. Check-in is at 3:00 pm, check-out is at 12:00 pm. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. This property is located approximately 1 mile from campus. The deadline for reservations at this rate is February 28, 2020.
US$45 per day, and valet parking is available for a fee of US$50 per day. Check-in is at 4:00 pm, check-out is at 12:00 pm. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. This property is located approximately 7 miles from campus. The deadline for reservations at this rate is February 21, 2020.

**Housing Warning**

Please beware of aggressive housing bureaus that target potential attendees of a meeting. They are sometimes called “room poachers” or “room-block pirates” and these companies generally position themselves as a meeting’s housing bureau, convincing attendees to unknowingly book outside the official room block. They call people who they think will more likely than not attend a meeting and lure them with room rates that are significantly less than the published group rate—for a limited time only. And people who find this offer tempting may hand over their credit card data, believing they have scored a great rate and their housing is a done deal. Unfortunately, this often turns out to be the start of a long, costly nightmare.

Note that some of these room poachers create fake websites on which they represent themselves as the organizers of the meeting and include links to book rooms, etc. The only official website for this meeting is ams.org and one that has the official AMS logo.

These housing bureaus are not affiliated with the American Mathematical Society or any of its meetings, in any way. The AMS would never call anyone to solicit reservations for a meeting. The only way to book a room at a rate negotiated for an AMS Sectional Meeting is via a listing on AMS Sectional Meetings pages or Notices of the AMS. The AMS cannot be responsible for any damages incurred as a result of hotel bookings made with unofficial housing bureaus.

**Food Services**

This meeting will take place over the spring recess for Tufts University, so all dining options on campus will be closed during the dates of this meeting.

There are many dining options in Davis Square, and the towns of Medford and Somerville offer a wide variety of cuisines. For more information on dining and attractions throughout Medford please visit www.bostonusa.com/about-boston/greater-boston-regions/medford.

Some of the dining options in the area include:

- **Ball Square Cafe Breakfast and Lunch**, 708 Broadway, Somerville; popular local establishment serving breakfast and lunch all day. Saturday and Sunday 6:30 am–3:00 pm; ballsquarecafe.com.

- **Boston Burger**, 37 Davis Square, Somerville; burgers, shakes and American cuisine. Saturday 11:00 am–11:00 pm and Sunday 12:00 pm–8:00 pm; www.bostonburgercompany.com.

- **The Chicken and Rice Guys**, 64 Salem Street, Medford; casual Mediterranean rice bowls. Saturday and Sunday 11:30 am–9:00 pm; cnrguys.com.

- **Dave’s Fresh Pasta**, 1 Holland Street, Davis Square, Somerville; specialty market, wine and cheese shop with a sandwich counter. Saturday 11:00 am–6:00 pm, closed Sunday; davesfreshpasta.com/index.html.

- **Flatbread Company**, 45 Day Street, Davis Square, Somerville; organic flatbreads served in a candlepin bowling alley. Saturday 11:00 am–11:30 pm, Sunday 11:00 am–10:30 pm; saccobowl.com.

- **Magnificent Muffin and Bagel Shoppe**, 1118 Broadway, Somerville; breakfast spot serving bagels and baked goods. Saturday and Sunday 7:00 am–12:00 pm; magnificentmuffinandbagelshoppesomerville.cafecityguide.website.

- **Modern Pastry**, 20 Salem Street, Medford; bakery serving authentic Italian pastries and coffee. Saturday 8:00 am–9:00 pm and Sunday 8:00 am–2:00 pm; www.modernpastry.com.

- **Pikaichi**, 123 Boston Avenue, Medford; casual restaurant serving ramen. Saturday 11:30 am–3:00 pm and 5:00 pm–8:45 pm, and Sunday 11:30 am–3:00 pm; www.pikaichiramen.com.

- **Opa Greek Yeeros**, 378 Highland Avenue, Davis Square, Somerville; casual counter-service Greek cuisine. Saturday and Sunday 11:00 am–9:00 pm; www.opagreekyeerossomerville.com/#/.

- **Pokeworks**, 261 Elm Street, Davis Square; casual counter-service chain serving poke, salad, and sushi. Saturday 11:30 am–10:00 pm and Sunday 11:00 am–10:30 pm; www.pokeworks.com/somerville.

- **The Porch Southern Fare & Juke Joint**, 75 Rivers Edge Drive, Medford; southern fare, BBQ, and live music. Saturday and Sunday 11:30 am–4:00 pm and 5:00 pm–9:30 pm, Sunday brunch 10:30 am–4:00 pm; www.theporchesouthern.com.

- **Semolina Kitchen and Bar**, 572 Boston Avenue, Medford; serving handmade pasta and pizza. Saturday 11:00 am–11:00 pm, closed Sunday; semolinakitchen.com/index.html.

- **Tasty Cafe**, 24 Riverside Avenue, Medford; serving breakfast, sandwiches, salads, baked goods, and smoothies. Saturday and Sunday 7:00 am–3:00 pm; www.tastycafeboston.com.
• **Tenoch Mexican**, 21 Boston Avenue, Medford; casual Mexican dining featuring tacos and tortas. Saturday and Sunday 8:00 am–9:00 pm; www.tenochmexican.com.

• **Zam Zam**, 42 Riverside Avenue, Medford; serving Indian, Halal and Pakistani cuisine. Saturday and Sunday 11:00 am–10:30 pm; zamzammedford.com.

**Registration and Meeting Information**

**Advance Registration:** Advance registration for this meeting will open on **January 22, 2020**. Advance registration fees will be US$71 for AMS members, US$115 for nonmembers, US$13 for students and unemployed mathematicians, and US$15 for emeritus members. Fees will be payable by cash, check, or credit card. Participants may cancel registrations made in advance by emailing mmsb@ams.org. 100% refunds will be issued for any advance registrations canceled by the first day of the meeting. After this date, no refunds will be issued.

**On-site Information and Registration:** Registration and the book exhibit will be held in the Alumnae Lounge in Aidekman Hall. The Invited Address lectures will be located in Distler Auditorium in Granoff Hall. The Special Sessions and Contributed Paper Sessions will be held in Bromfield-Pearson, Anderson Hall, Aidekman Arts Center/Jackson Gym, Aidekman Arts Center, Pearson Chemical Laboratory, Granoff Music Center, Sophia Gordon Hall, and 574 Building. There will be a reception Saturday evening held on the fourth floor of the 574 Building. Please look for additional information about specific session room locations on the web and in the printed program. For further information on building locations, a campus map is available at campusmaps.tufts.edu/medford.

The registration desk will be open on Saturday, March 21 from 7:30 am–4:00 pm and on Sunday, March 22 from 8:00 am–12 pm. The same fees listed above apply for on-site registration and are payable with cash, check or credit card.

**Program Books**

In order to keep registration fees as low as possible, save on printing costs, and make the meetings more environmentally friendly, a small fee will be charged for receiving a program book.

If you want to receive a program book, please complete the section entitled "Printed Program” on the registration form and pay a nominal fee of US$3 for each copy. If you do not want to receive a program book, please skip that section and click “Next.” All purchased program books will be distributed at the registration desk at the meeting. No program books will be mailed before the meeting. A small quantity of program books may be available for purchase on-site at meetings, however supplies will be limited.

For your convenience, the following changes have been made to the website to make the program more accessible and mobile-friendly.

1. **Online Timetable:** The sectional meetings now have an online timetable display. It will provide a quick reference to where the sessions and rooms are located.

2. **Print-friendly Pages:** The pages of the program now have a “Print” button in the top right-hand corner. It is black and is depicted by a little printer. If you cannot see it, please maximize your window. The printer icon is sometimes bundled into the generic “share” icon when the window is re-sized. If you click the printer button, it will print the text of the page without additional webpage elements. If you go to the page of a special session and click the print icon, you will get a schedule of all the parts of that session. This can also be printed to pdf if you have a pdf printer installed.

3. **Room Locations:** The room locations are now more conspicuous in the web program when the program is scheduled.

**Other Activities**

**Book Sales:** Stop by the on-site AMS bookstore to review the newest publications and take advantage of exhibit discounts and free shipping on all on-site orders! AMS and MAA members receive 40% off list price. Nonmembers receive a 25% discount. Not a member? Ask a representative about the benefits of AMS membership.

**AMS Editorial Activity:** An acquisitions editor from the AMS book program will be present to speak with prospective authors. If you have a book project that you wish to discuss with the AMS, please stop by the book exhibit.

**Membership Activities:** During the meeting, stop by the AMS Membership Exhibit to learn about the benefits of AMS Membership. Members receive free shipping on purchases all year long and additional discounts on books purchased at meetings, subscriptions to Notices and Bulletin, discounted registration for world-class meetings and conferences and more! Complimentary refreshments will be served courtesy in part of the AMS Membership Department.

**Child Care Grants**

The AMS will provide a limited number of reimbursement grants of US$125 per family to help with the cost of child care for registered participants at the meeting. The funds may be used for any form of child care that frees a parent to
participate more fully in the meeting. Grants will be awarded on a first-come, first-served basis, one per family, and one per season (i.e. Spring or Fall), the latter depending on the amount of grants available. Registration for the meeting as well as membership in the AMS is required to apply for this program.

Information about applying for child care grants will be available prior to the opening of advance registration in January; watch the meeting website for details and instructions. Applications will be on Mathprograms.org and will be accepted on a first-come, first-served basis until January 28, 2020. Final decisions on recipients will be made on or before February 21, 2020. All grant funds will be provided in the form of a check which will be issued at the meeting.

Special Needs
It is the goal of the AMS to ensure that its conferences are accessible to all, regardless of disability. The AMS will strive, unless it is not practicable, to choose venues that are fully accessible to the physically handicapped.

If special needs accommodations are necessary in order for you to participate in an AMS Sectional Meeting, please communicate your needs in advance to the AMS Meetings Department by:

- Registering early for the meeting,
- Checking the appropriate box on the registration form, and
- Sending an email request to the AMS Meetings Department at mmsb@ams.org or meet@ams.org.

AMS Policy on a Welcoming Environment
The AMS strives to ensure that participants in its activities enjoy a welcoming environment. In all its activities, the AMS seeks to foster an atmosphere that encourages the free expression and exchange of ideas. The AMS supports equality of opportunity and treatment for all participants, regardless of gender, gender identity or expression, race, color, national or ethnic origin, religion or religious belief, age, marital status, sexual orientation, disabilities, or veteran status.

Harassment is a form of misconduct that undermines the integrity of AMS activities and mission.

The AMS will make every effort to maintain an environment that is free of harassment, even though it does not control the behavior of third parties. A commitment to a welcoming environment is expected of all attendees at AMS activities, including mathematicians, students, guests, staff, contractors and exhibitors, and participants in scientific sessions and social events. To this end, the AMS will include a statement concerning its expectations towards maintaining a welcoming environment in registration materials for all its meetings, and has put in place a mechanism for reporting violations. Violations may be reported confidentially and anonymously to (855)282-5703 or at www.mathsociety.ethicspoint.com. The reporting mechanism ensures the respect of privacy while alerting the AMS to the situation. Violations may also be brought to the attention of the coordinator for the meeting (who is usually at the meeting registration desk), and that person can provide advice about how to proceed.

For AMS policy statements concerning discrimination and harassment, see the AMS Anti-Harassment Policy at www.ams.org/about-us/governance/policy-statements/anti-harassment-policy.

Questions about this welcoming environment policy should be directed to the AMS Secretary at https://www.ams.org/about-us/governance/sec-contact.

Local Information and Maps
This meeting will take place on the Tufts University Medford/Somerville campus located at 419 Boston Avenue, Medford, Massachusetts, 02155 (this is the address of the Dowling Hall Parking Garage). A campus map can be found at campusmaps.tufts.edu/medford. Information about the Tufts University Department of Mathematics can be found at math.tufts.edu. Please visit the university website at www.tufts.edu for additional information on the campus.

For information on gender neutral restrooms on campus please visit students.tufts.edu/sites/default/files/genderNeutralRestrooms.pdf. Any nursing person who is a member of the Tufts community (facult, staff, students, and visitors) can use the lactation rooms provided on campus. Please contact Melissa Colton at mac@ams.org to make arrangements to access these rooms.

Please watch the AMS website at https://www.ams.org/meetings/sectional/sectional.html for additional information on this meeting.

Parking
Parking is available for visitors to campus in two locations: Dowling Hall Garage or Cousens Parking Lot. The Dowling Hall Garage is located at 419 Boston Avenue, Medford. Enter the parking garage at Dowling Hall, 419 Boston Avenue. Gate will automatically open upon entrance. When you exit your vehicle or have concluded your visit, please pay the day rate of US$8 at the kiosk on the 3rd, 5th or 7th floor. You will need to know your license plate. Kiosk accepts VISA,
Mastercard, and Discover. When exiting the garage, a camera will read your license plate and raise the gate arm. The Cousins Parking Lot is located at 161 College Avenue, Medford and parking is available for a fee of US$8 per day. There is no overnight parking in these locations; all vehicles must exit these locations by midnight. No cash is accepted in these locations; VISA/MasterCard/Discover Card are the only methods of payment accepted. Parking permits may be purchased online in advance by visiting the Tufts Parking Portal website located here: tufts.t2hosted.com/Account/Portal.

Travel
Tufts University Medford/Somerville Campus is located 5.6 miles outside of the city of Boston. Logan International Airport is the closest airport to the University. The most common types of transportation used from the airport are rental cars, taxis, and the public transportation system called the “T.”

By Air:
Logan International Airport (BOS) is the closest airport to Tufts University Medford/Somerville Campus. Logan Airport is located two miles from the city center and approximately 9 miles from the University. Most Boston and Cambridge hotels are within a 5 mile drive from the airport.

Taxis can be found at the ground transportation area outside each terminal. Fare to Tufts University will be approximately US$40–45 depending on traffic and the time of day.

The campus can also be reached via mass transit on the MBTA “T” system. Shuttle buses are available to pick up passengers in front of the airline terminals and bring them to the airport subway stop on the Silver Line. Board the Silver Line Waterfront Bus at the front of your airport terminal. Ride to South Station (15–20 minutes) and get off the bus. At South Station, board the Red Line (subway) to Davis Square, about a 15–20 minute ride.

By Train:
The Boston region is served by Amtrak and by the MBTA commuter trains. Reservations can be made on Amtrak at www.amtrak.com. Information about regional trains on the MBTA commuter lines can be found at www.mbta.com/schedules/commuter-rail. If traveling by rail, you will arrive at South Station, located at Summer Street & Atlantic Avenue, or North Station, located at 135 Causeway Street. From South Station you can you can board the MBTA “T” Red Line and take it to the Davis Square Station. Exit the Davis Square Station on the College Avenue side. You may take the #94 or #96 bus (which which leave from the Davis Square T entrance directly across from the Tedeschi convenience store, NOT the entrance next to Somerville Theatre) or you may walk 15 minutes along College Avenue to the Tufts campus.

If you board the bus, you will get off at the stop titled “Boston Avenue at Tufts Garage.” The bus stop is across the street from a large, seven-story parking garage.

If you walk, turn right out of the subway station and follow College Avenue to the Powderhouse Square traffic circle. Cross Powderhouse Circle to continue on College Avenue. As you continue on College Avenue, the Tufts Campus will be on your left.

By Bus:
The Boston region is served several regional bus lines including BoltBus, Greyhound, and Peter Pan. All intercity/interstate buses depart from and arrive at South Station. Ticket counters are located on the third level of the Transportation Center. For information, call the South Station Bus Terminal at (617)737-8040 or visit www.south-station.net/regional-bus. The MBTA's Red and Silver Lines stop at South Station; board the MBTA “T” Red Line and take it to the Davis Square Station. Exit the Davis Square Station on the College Avenue side.

Another bus option for some attendees traveling on the east coast may be Go Buses Bus Line. This bus line travels between Manhattan and Washington, DC and several cities on the east coast including Cambridge, MA. For more information on GoBuses please visit www.gobuses.com.

By Car:
All directions to Tufts will direct drivers to a final destination of the Dowling Hall Parking Garage. Visit this address (www.google.com/maps/dir//42.408651,-71.118093/@42.4085158,-71.1883128,12z) and type in your starting address for step-by-step directions to the Tufts campus.

When using a GPS please be certain to enter 419 Boston Avenue, Medford, MA 02155 as the final destination.

Local Transportation
Bus and Subway Service: The Massachusetts Bay Transportation Authority (MBTA, known as the “T” in Boston) offers access to the city of Medford. The meeting will be held on the Medford/Somerville Campus of Tufts University. This campus is accessible by bus services on Routes 80, 94, and 96. A one-way Charlie ticket costs US$2.00, a one-day pass
costs US$12.75, and a seven-day pass costs US$22.50. For additional information about bus fares and schedules please visit www.mbta.com/schedules/bus.

This campus is accessible by subway via the Red Line of the MBTA and Davis Square Station. A one-way Charlie ticket costs US$2.90, a one-day pass costs US$12.75, and a seven-day pass costs US$22.50.

Any amount of money for one-way fares or passes (one-day, seven-day) can be purchased as a CharlieCard or CharlieTicket at fare vending machines found at all subway stations. CharlieCards and CharlieTickets can be used for subway and bus fares.

Commuter rail lines connect with the Red Line at South Station. More information can be found here: www.mbta.com.

Taxi Service: Licensed, metered taxis are available in Medford, MA. Both Lyft and Uber also operate in the Medford area.

Weather
The average high temperature in Medford for March is in the mid-40s Fahrenheit, and the average low is in the low 30s Fahrenheit. Visitors should be prepared for inclement weather and check weather forecasts in advance of their arrival.

Social Networking
Attendees and speakers are encouraged to tweet about the meeting using the hashtag #AMSmtg.

Information for International Participants
Visa regulations are continually changing for travel to the United States. Visa applications may take from three to four months to process and require a personal interview, as well as specific personal information. International participants should view the important information about traveling to the US found at travel.state.gov/content/travel/en.html. If you need a preliminary conference invitation in order to secure a visa, please send your request to mac@ams.org.

If you discover you do need a visa, the National Academies website (see above) provides these tips for successful visa applications:

* Visa applicants are expected to provide evidence that they are intending to return to their country of residence. Therefore, applicants should provide proof of “binding” or sufficient ties to their home country or permanent residence abroad. This may include documentation of the following:
  - family ties in home country or country of legal permanent residence
  - property ownership
  - bank accounts
  - employment contract or statement from employer stating that the position will continue when the employee returns;

* Visa applications are more likely to be successful if done in a visitor’s home country than in a third country;

* Applicants should present their entire trip itinerary, including travel to any countries other than the United States, at the time of their visa application;

* Include a letter of invitation from the meeting organizer or the US host, specifying the subject, location and dates of the activity, and how travel and local expenses will be covered;

* If travel plans will depend on early approval of the visa application, specify this at the time of the application;

* Provide proof of professional scientific and/or educational status (students should provide a university transcript).

This list is not to be considered complete. Please visit the websites above for the most up-to-date information.

West Lafayette, Indiana
Purdue University

April 4–5, 2020
Saturday – Sunday

Meeting #1157
Central Section
Associate secretary: Georgia Benkart

Announcement issue of Notices: February 2020
Program first available on AMS website: February 18, 2020
Issue of Abstracts: Volume 41, Issue 2

Deadlines
For organizers: Expired
For abstracts: February 4, 2020
MEETINGS & CONFERENCES

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Christine Berkesch, University of Minnesota, Title to be announced.
Matthew Hedden, Michigan State University, Title to be announced.
Brian Street, University of Wisconsin-Madison, Title to be announced.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.

Advances in Mathematical Modeling, Analysis and Numerical Simulation of Particulate Suspensions and Related Multiphase Flows (Code: SS 32A), Abhinandan Chowdhury, Savannah State University, and Ivan Christov, Purdue University.
Analysis and Probability in Sub-Riemannian Geometry (Code: SS 5A), Jeremy Tyson, University of Illinois Urbana-Champaign, and Jing Wang, Purdue University.
Coding and Cryptography (Code: SS 34A), Ryann Cartor, Clemson University, Neville Fogarty, Christopher Newport University, and Gretchen Matthews and Dane Skabelund, Virginia Tech.
Combinatorial Algebra and Geometry (Code: SS 22A), Christine Berkesch, University of Minnesota, and Laura Matusevich and Aleksandra Sobieska, Texas A&M University.
Combinatorial Techniques in Commutative Algebra (Code: SS 42A), Giulio Caviglia, Purdue University, and Jay Schweig, Oklahoma State University.
Commutative Algebra and Connections with Algebraic Geometry (Code: SS 36A), Claudia Polini, University of Notre Dame, and Bernd Ulrich, Purdue University.
Complex Geometry (Code: SS 29A), Laszlo Lempert, Chi Li, and Sai-Kee Yeung, Purdue University, and Yuan Yuan, Syracuse University.
Computational Aspects of Symplectic Topology (Code: SS 30A), Olguta Buse, Indiana University-Purdue University Indianapolis, Richard Hind, University of Notre Dame, and Jun Li, University of Michigan.
Contemporary Applications of Gradient Flows and Variational Methods (Code: SS 12A), Tao Luo and Nung Kwan (Aaron) Yip, Purdue University.

Gaussian and non-Gaussian Stochastic Analysis (Code: SS 16A), Cheng Ouyang, University of Illinois at Chicago, and Takashi Owada and Samy Tindel, Purdue University.
Geometric Topology in the Middle Dimensions (Code: SS 43A), James F. Davis, Indiana University, and Mark Powell, Durham University.
Group Theory and Logic (Code: SS 13A), Meng-Che (Turbo) He, Purdue University, Julia F. Knight, University of Notre Dame, and D.B. McReynolds and Thomas Sinclair, Purdue University.
Harmonic Analysis (Code: SS 2A), Brian Street and Shaoming Guo, University of Wisconsin-Madison.
Higher Structures in Topology, Geometry and Physics (Code: SS 24A), Ralph Kaufmann, Purdue University, Martin Markl, Institute of Mathematics of the Czech Academy of Sciences, and Sasha Voronov, University of Minnesota.
Integrability, Symmetry and Physics (Code: SS 26A), E. Birgit Kaufmann, Purdue University, and Oleksandr Tsymbaliuk, Yale University.
Knots and Links in 3-Manifolds (Code: SS 31A), Micah Chrisman and Sujoy Mukherjee, The Ohio State University, and Robert Todd, Mount Mercy University.
Low-dimensional Topology (Code: SS 4A), Matthew Hedden, Katherine Raoux, and Lev Tovstopyat-Nelip, Michigan State University.
Mathematical Finance and Actuarial Sciences (Code: SS 11A), Kiseop Lee and Jianxi Su, Purdue University, and Jose Figueira-Lopez, Washington University, St. Louis.
Mathematical Methods for Inverse Problems (Code: SS 3A), Isaac Harris and Peijun Li, Purdue University.
Modeling, Analysis and Simulation of Complex Fluid Systems in Physics and Biology (Code: SS 33A), Carme Calderer, University of Minnesota, Chun Liu, Illinois Institute of Technology, and Pei Liu, University of Minnesota.
Model Theory and its Applications (Code: SS 41A), Saugata Basu, Purdue University, Philipp Hieronymi, University of Illinois at Urbana-Champaign, and Margaret E.M. Thomas, Purdue University.
MEETINGS & CONFERENCES

Multiplicative Ideal Theory in honor of the career of William Heinzer (Code: SS 8A), Evan Houston, University of North Carolina, Charlotte, and Alan Loper, Ohio State University.

Network Science (Code: SS 20A), Nicole Eikmeier, Grinnell College, and David F. Gleich, Purdue University.

Nonlinear Partial Differential Equations from Variational Problems and Fluid Equations (Code: SS 9A), Tao Huang, Wayne State University, and Changyou Wang, Purdue University.

Numerical Linear Algebra (Code: SS 18A), Jianlin Xia and Xuefeng Xu, Purdue University.

Optimization and Algebraic Geometry (Code: SS 40A), Jonathan Hauenstein, University of Notre Dame, and Ali Mohammad Nezhad, Purdue University.

Optimization for Discrete Geometry (Code: SS 19A), Mark Magsino and Hans Parshall, The Ohio State University.

p-adic Galois Representations, Modularity, and Related Topics (Code: SS 39A), Patrick Allen, University of Illinois at Urbana-Champaign, Andrei Jorza, University of Notre Dame, and Tong Liu, Purdue University.

Quantum Algebra and Quantum Topology (Code: SS 10A), Shawn Cui, Purdue University, Julia Plavnik, Indiana University, and Tian Yang, Texas A&M University.

Recent Advances in Adaptive Mesh Refinement and A Posteriori Error Estimation (Code: SS 38A), Shuhao Cao, University of California, Irvine, and Zhiguo Yang, Purdue University.

Recent Advances in Modeling, Computational Methods and Simulations of Physical/Biological Systems (Code: SS 27A), Suchuan Steven Dong, Jie Shen, and Zhiguo Yang, Purdue University.

Recent Developments in Automorphic Forms and Representations of p-adic Groups (Code: SS 7A), David Goldberg, Baiying Liu, and Freydoon Shahidi, Purdue University.

Recent Developments in Commutative Algebra (Code: SS 6A), Jennifer Kenkel, University of Kentucky, and Liquan Ma and Ulrich Walther, Purdue University.

Recent Developments in High Order Numerical Methods for Partial Differential Equations (Code: SS 21A), Zheng Sun, The Ohio State University, and Xiangxiong Zhang, Purdue University.

Rigidity Theory, Distance Geometry and Applications (Code: SS 17A), Mireille Boutin, Purdue University, Gregor Kemper, Technische Universität München, and Jessica Sidman, Mount Holyoke College.

Scientific Machine Learning (Code: SS 23A), Tong Qin and Dongbin Xiu, The Ohio State University.


Stability in Topology, Arithmetic, and Representation Theory (Code: SS 37A), Jeremy Miller and Peter Patzt, Purdue University, and Andrew Putman, University of Notre Dame.

Stochastic Processes in Random Environments (Code: SS 14A), Jonathon Peterson, Purdue University, and Atilla Yilmaz, Temple University.

Structure Preserving Numerical Methods for Hyperbolic and Kinetic Equations (Code: SS 28A), Jingwei Hu, Purdue University.

The Interface of Harmonic Analysis and Analytic Number Theory (Code: SS 1A), Theresa Anderson, Purdue University, Robert Lemke Oliver, Tufts University, and Eyvindur Palsson, Virginia Tech University.

Theory and Algorithms for Data Science (Code: SS 25A), Tingran Gao, University of Chicago, and Haizhao Yang, Purdue University.

Fresno, California
California State University, Fresno

May 2–3, 2020
Saturday – Sunday

Meeting #1158
Western Section
Associate secretary: Michel L. Lapidus

Announcement issue of Notices: March 2020
Program first available on AMS website: March 19, 2020
Issue of Abstracts: Volume 41, Issue 2

Deadlines
For organizers: Expired
For abstracts: March 3, 2020
MEETINGS & CONFERENCES

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Sami Assaf, University of Southern California, Los Angeles, Combinatorics of Schubert Calculus.
Natalia Komarova, University of California, Irvine, Title to be announced.
Joseph Teran, University of Southern California, Los Angeles, Title to be announced.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.

Advances by Scholars in the Pacific Math Alliance (Code: SS 22A), Andrea Arauza Rivera, California State University, East Bay, Mario Banuelos, California State University, Fresno, Jessica De Silva, California State University, Stanislaus, and John Rock, California Polytechnic University, Pomona.

Advances in Functional Analysis and Operator Theory (Code: SS 6A), Yuri Latushkin, University of Missouri, Columbia, Marat Markin, California State University, Fresno, Igor Nikolaev, St. John’s University, and Ilya Spitkovsky, New York University, Abu Dhabi.

Algebraic geometry in statistics and machine learning (Code: SS 25A), Robert Krone, University of California, Jose Rodriguez, University of Wisconsin, and Tingting Tang, Notre Dame University.

Algebraic Structures in Knot Theory (Code: SS 4A), Carmen Caprau, California State University, Fresno, and Sam Nelson, Claremont McKenna College.

Algorithms in the study of hyperbolic 3-manifolds (Code: SS 26A), Robert Haraway, III, Oklahoma State University, and Maria Trnkova, University of California, Davis.

Analysis of Fractional Differential and Difference Equations with its Application (Code: SS 20A), Bhuvaneswari Sambandham, Dixie State University, and Aghalaya S. Vatsala, University of Louisiana at Lafayette.

Artin-Schelter regular algebras and related topics (Code: SS 27A), Ellen Kirkman, Wake Forest University, and James Zhang, University of Washington.

Combinatorics Arising from Representations (associated with the Invited Address by Sami Assaf) (Code: SS 16A), Sami Assaf, University of Southern California, Nicolle Gonzalez, University of California, Los Angeles, and Brendan Pawloski, University of Southern California.

Combinatorics of Reduced Decompositions of eElements of Coxeter Groups and Related Topics (Code: SS 17A), Samantha Dahlberg and Jennifer Elder, Arizona State University.

Complexity in Low-Dimensional Topology (Code: SS 14A), Jennifer Schultens, University of California, Davis, and Eric Sedgwick, DePaul University.

Data Analysis and Predictive Modeling (Code: SS 8A), Earvin Balderama, California State University, Fresno, and Adriano Zambom, California State University, Northridge.

DG Methods in Commutative Algebra and Representation Theory (Code: SS 2A), Benjamin Briggs, Janina Letz, and Josh Pollitz, University of Utah.

Discrete Geometry and Combinatorial Structures (Code: SS 23A), Morgan Rodgers, California State University, and Oscar Vega.

How to Solve It? Heuristics and Inquiry Based Learning (Code: SS 18A), Mario Banuelos, California State University, Fresno, Andrew G. Benedek, Research Centre for the Humanities, Hungary, and Agnes Tuska, California State University, Fresno.

Inverse Problems (Code: SS 5A), Hanna Makaruk, Los Alamos National Laboratory, and Robert Owczarek, University of New Mexico, Albuquerque and University of New Mexico, Los Alamos.

Math Circle Games and Puzzles that Teach Deep Mathematics (Code: SS 13A), Maria Nogin, Adnan Sabuwala, and Agnes Tuska, California State University, Fresno.

Mathematical Biology: Confronting Models with Data (Code: SS 21A), Erica Rutter, University of California, Merced.

Mathematical Methods in Evolution and Medicine (associated with the Invited Address by Natalia Komarova) (Code: SS 1A), Natalia Komarova and Jesse Kreger, University of California, Irvine.

Methods in Non-Semisimple Representation Categories (Code: SS 11A), Eric Friedlander, University of Southern California, Los Angeles, Julia Pevtsova, University of Washington, Seattle, and Paul Sobaje, Georgia Southern University, Statesboro.

Numerical Semigroups and Applications (Code: SS 3A), Elie Alhajji, West Point Military Academy, and Christopher O’Neill, San Diego State University.
Recent Advances in Mathematical Biology, Ecology, Epidemiology, and Evolution (Code: SS 10A), Lale Asik, Texas Tech University, Khanh Phuong Nguyen, University of Houston, and Angela Peace, Texas Tech University.

Research in Mathematics by Early Career Graduate Students (Code: SS 7A), Doreen De Leon, Marat Markin, and Khang Tran, California State University, Fresno.

Research in Mathematics Education (Code: SS 15A), Ravi Somayajulu and Jenna Tague, Clovis Community College.

Scientific Computing (Code: SS 19A), Changho Kim and Roummel Marcia, University of California, Merced.

Special Functions in Number Theory (Code: SS 24A), Cezar Lupu and Dermot McCarthy, Texas Tech University.

Women in Mathematics (Code: SS 12A), Doreen De Leon, Katherine Kelm, and Oscar Vega, California State University, Fresno.

Zero Distribution of Entire Functions (Code: SS 9A), Khang Tran and Tamás Forgács, California State University, Fresno.

El Paso, Texas
University of Texas at El Paso

September 12–13, 2020
Saturday – Sunday

Meeting #1159
Central Section
Associate secretary: Georgia Benkart

Announcement issue of Notices: June 2020
Program first available on AMS website: July 28, 2020
Issue of Abstracts: Volume 41, Issue 3

Deadlines
For organizers: February 20, 2020
For abstracts: July 14, 2020

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/sectional.html.

Invited Addresses
Caroline Klivans, Brown University, Title to be announced.
Brisa Sanchez, Drexel University, Title to be announced.
Alejandra Sorto, Texas State University, Title to be announced.

Special Sessions
If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.

Algebraic, Geometric and Topological Combinatorics (Code: SS 6A), Art Duval, University of Texas at El Paso, Caroline Klivans, Brown University, and Jeremy Martin, University of Kansas.


High-Frequency Data Analysis and Applications (Code: SS 1A), Maria Christina Mariani and Michael Pokojovy, University of Texas at El Paso, and Ambar Sengupta, University of Connecticut.

Leibniz Algebras and Related Topics (Code: SS 7A), Guy Biyogmam, Georgia College and State University, and Jerry Lodder, New Mexico State University.

Low-dimensional Topology and Knot Theory (Code: SS 4A), Mohamed Ait Nouh and Luis Valdez-Sanchez, University of Texas at El Paso.

Methods and Applications in Data Science (Code: SS 9A), Sangjin Kim, Ming-Ying Leung, Xiaogang Su, and Amy Wagler, The University of Texas at El Paso.

Nonlinear Analysis and Optimization (Code: SS 2A), Behzad Djafari-Rouhani, University of Texas at El Paso, and Akhtar A. Khan, Rochester Institute of Technology.


Statistical Methodology and Applications (Code: SS 8A), Ori Rosen and Suneel Chatla, University of Texas at El Paso.
State College, Pennsylvania

*Pennsylvania State University, University Park Campus*

**October 3–4, 2020**
Saturday – Sunday

**Meeting #1160**
Eastern Section
Associate secretary: Steven H. Weintraub

Announcement issue of *Notices*: August 2020
Program first available on AMS website: August 25, 2020
Issue of *Abstracts*: Volume 41, Issue 3

**Deadlines**
For organizers: March 3, 2020
For abstracts: August 11, 2020

The scientific information listed below may be dated. For the latest information, see [https://www.ams.org/amsmtgs/sectional.html](https://www.ams.org/amsmtgs/sectional.html).

**Invited Addresses**
- Melody Chan, Brown University, *Title to be announced*.
- Steven J. Miller, Williams College, *Title to be announced*.
- Tadashi Tokieda, Stanford University, *Title to be announced*.

Chattanooga, Tennessee

*University of Tennessee at Chattanooga*

**October 10–11, 2020**
Saturday – Sunday

**Meeting #1161**
Southeastern Section
Associate secretary: Brian D. Boe

Announcement issue of *Notices*: August 2020
Program first available on AMS website: September 1, 2020
Issue of *Abstracts*: Volume 41, Issue 4

**Deadlines**
For organizers: March 10, 2020
For abstracts: August 18, 2020

The scientific information listed below may be dated. For the latest information, see [https://www.ams.org/amsmtgs/sectional.html](https://www.ams.org/amsmtgs/sectional.html).

**Invited Addresses**
- Giulia Saccà, Columbia University, *Title to be announced*.
- Chad Topaz, Williams College, *Title to be announced*.
- Xingxing Yu, Georgia Institute of Technology, *Title to be announced*.

**Special Sessions**
If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at [https://www.ams.org/cgi-bin/abstracts/abstract.pl](https://www.ams.org/cgi-bin/abstracts/abstract.pl).

- **Applied Knot Theory** (Code: SS 4A), Jason Cantarella, University of Georgia, Eleni Panagiotou, University of Tennessee at Chattanooga, and Eric Rawdon, University of St Thomas.
- **Boundary Value Problems for Differential, Difference, and Fractional Equations** (Code: SS 2A), John R Graef and Lingju Kong, University of Tennessee at Chattanooga, and Min Wang, Kennesaw State University.
- **Commutative Algebra** (Code: SS 1A), Simplice Tchamna, Georgia College, and Lokendra Paudel, University of South Carolina, Salkehatchie.
- **Structural and Extremal Graph Theory** (Code: SS 3A), Hao Huang, Emory University, and Xingxing Yu, Georgia Institute of Technology.
Salt Lake City, Utah

University of Utah

October 24–25, 2020
Saturday – Sunday

Meeting #1162
Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of Notices: August 2020

Program first available on AMS website: September 17, 2020
Issue of Abstracts: Volume 41, Issue 4

Deadlines
For organizers: March 24, 2020
For abstracts: September 1, 2020

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Bhargav Bhatt, University of Michigan, Ann Arbor, Title to be announced.
Jonathan Brundan, University of Oregon, Eugene, Title to be announced.
Andrei Okounkov, Columbia University, Title to be announced (Erdős Memorial Lecture).
Mariel Vazquez, University of California, Davis, Title to be announced.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.

Extremal Graph Theory (Code: SS 1A), József Balogh, University of Illinois, and Bernard Lidicky, Iowa State University.
Monoidal Categories in Representation Theory (associated with the Invited Address by Jon Brudan) (Code: SS 2A), Jonathan Brundan, Ben Elias, and Victor Ostrik, University of Oregon.

Washington, District of Columbia

Walter E. Washington Convention Center

January 6–9, 2021
Wednesday – Saturday

Meeting #1163
Joint Mathematics Meetings, including the 127th Annual Meeting of the AMS, 104th Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).

Associate secretary: Brian D. Boe
Announcement issue of Notices: October 2020
Program first available on AMS website: November 1, 2020
Issue of Abstracts: To be announced

Deadlines
For organizers: April 2, 2020
For abstracts: To be announced
MEETINGS & CONFERENCES

Providence, Rhode Island
Brown University

March 20–21, 2021
Saturday – Sunday
Eastern Section
Associate secretary: Steven H. Weintraub
Announcement issue of Notices: To be announced

Program first available on AMS website: To be announced
Issue of Abstracts: To be announced

Deadlines
For organizers: To be announced
For abstracts: To be announced

San Francisco, California
San Francisco State University

May 1–2, 2021
Saturday – Sunday
Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of Notices: To be announced

Program first available on AMS website: To be announced
Issue of Abstracts: To be announced

Deadlines
For organizers: To be announced
For abstracts: To be announced

Grenoble, France
Université de Grenoble-Alpes

July 5–9, 2021
Monday – Friday
Associate secretary: Michel L. Lapidus
Announcement issue of Notices: To be announced
Program first available on AMS website: To be announced

Issue of Abstracts: To be announced

Deadlines
For organizers: To be announced
For abstracts: To be announced

Buenos Aires, Argentina
The University of Buenos Aires

July 19–23, 2021
Monday – Friday
Associate secretary: Steven H. Weintraub
Announcement issue of Notices: To be announced
Program first available on AMS website: To be announced

Issue of Abstracts: To be announced

Deadlines
For organizers: To be announced
For abstracts: To be announced

Omaha, Nebraska
Creighton University

October 9–10, 2021
Saturday – Sunday
Central Section
Associate secretary: Georgia Benkart
Announcement issue of Notices: To be announced

Program first available on AMS website: To be announced
Issue of Abstracts: To be announced

Deadlines
For organizers: To be announced
For abstracts: To be announced
Albuquerque, New Mexico
University of New Mexico

October 23–24, 2021
Saturday – Sunday
Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of Notices: To be announced

Program first available on AMS website: To be announced
Issue of Abstracts: To be announced

Deadlines
For organizers: To be announced
For abstracts: To be announced

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/sectional.html.

Special Sessions
If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.

Inverse Problems (Code: SS 1A), Hanna Makaruk, Los Alamos National Laboratory, and Robert Owczarek, University of New Mexico.

Seattle, Washington
Washington State Convention Center and the Sheraton Seattle Hotel

January 5–8, 2022
Wednesday – Saturday
Associate secretary: Georgia Benkart
Announcement issue of Notices: October 2021
Program first available on AMS website: To be announced

Issue of Abstracts: To be announced

Deadlines
For organizers: To be announced
For abstracts: To be announced

Boston, Massachusetts
John B. Hynes Veterans Memorial Convention Center, Boston Marriott Hotel, and Boston Sheraton Hotel

January 4–7, 2023
Wednesday – Saturday
Associate secretary: Steven H. Weintraub
Announcement issue of Notices: October 2022
Program first available on AMS website: To be announced

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Deadlines
For organizers: To be announced
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