The Laplacian has different expressions in homogeneous media (e.g., the lake) versus inhomogeneous media (e.g., the mountain). Nevertheless, under mild assumptions on the domains, in both cases there exists a canonical measure supported on the boundary of the domain where the operators are studied. These canonical measures (harmonic measure in the homogeneous case and elliptic measure in the inhomogeneous case) are associated in a unique way to the domain and the operator and constitute the building blocks for all solutions to Laplace’s equation. In the domains considered in this talk there is another canonical measure supported on the boundary, namely, the surface measure. (In Figure 1 this measure corresponds to the area. It can tell us the size of the glacier on top of Mount Shuksan.) From the mathematical point of view my objective is to discuss how the relationship between the harmonic measure and the surface measure characterizes the geometry of a large class of domains. I will mention the challenges that arise in the inhomogeneous case and conclude with a complete characterization of the inhomogeneous media that behave like the homogeneous one under this lens. The relationship between two measures is described using notions that come from harmonic analysis. The geometry of the domain is described using notions arising in quantitative geometric measure theory.

Early on, the work of F. and M. Riesz [RR] put in evidence the deep connection that exists between the properties of the harmonic measure of a domain and the regularity of the boundary expressed in terms of its differentiability, that is, in terms of the existence of tangent planes almost everywhere with respect to surface measure. Recently, the relationship between the properties of the harmonic measure of a domain with respect to the surface measure of its boundary and the weakly differentiable properties of it has been an area of active inquiry. Here the weakly differentiable properties of the boundary refer to the existence of tangent planes almost everywhere (rectifiability) and a quantitative analogous notion (uniform rectifiability).

For instance, in the scale-invariant sense and under some quantitative topological assumptions, one can show that the $A_\infty$ property (quantitative mutual absolute continuity) of harmonic measure with respect to surface measure characterizes the uniform rectifiability of the boundary or even to the non-tangentially approximable character of the exterior domain. This characterization is a team effort achieved over time by different groups: [Sem], [DJ], [HMUT], and [AHMNT].

One of the main motivations of my recent work has been to understand whether the elliptic measure of a variable-coefficient divergence form elliptic operator (the Laplacian in an inhomogeneous medium) distinguishes between a rectifiable and a purely unrectifiable boundary. Together with S. Hofmann, J. M. Martell, S. Mayboroda, and Z. Zhao, we embarked on this project a few years ago. We have found
the optimal solution, namely, a large class of variable-coefficient elliptic operators for which the $A_\infty$ property of the associated elliptic measure with respect to surface measure guarantees that the boundary of the domain is uniformly rectifiable ([HMMTZ1] and [HMMTZ2]).

In this talk I will share some of the highlights of this journey. I hope to convey the excitement of obtaining an optimal result after working for several years on a challenging project.

References


Credits

Figure 1 is courtesy of Mariana Smit Vega García. Author photo is courtesy of the University of Washington.

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