The cover design is based on imagery from “Model Selection for Optimal Prediction in Statistical Machine Learning,” page 155.
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A WORD FROM...

Robin Wilson

Black history is American history, and the history of Black mathematicians in the United States is a part of the history of the American Mathematical Society. As with the history of the United States, the history of the AMS has not always been one of inclusion. With this special issue in honor of Black History Month, we shine light on some of that history, as well as uplift the efforts of mathematicians and institutions to redirect this tide of history and create equity in the field. In particular, in this issue, we have collected a broad yet impactful collection of feature articles, opinion pieces, and historical articles written by Black scholars. In addition, we present history and communication pieces that reflect on the past exclusion of Black scholars from the mathematics field and explore some new programs that might bring greater diversity to the mathematics research community. Many of these articles challenge the traditional narrative, both past and present, about contributions of Black people to the mathematical sciences in particular, and their impact on the larger mathematics community.

Our feature articles include Surface Bundles in Topology, Algebraic Geometry, and Group Theory by Nick Salter (Columbia University) and Bena Tshishiku (Brown University), Model Selection for Optimal Prediction in Statistical Machine Learning by Ernest Fokoué (Rochester Institute of Technology), and Demographic Population Cycles in Infectious Salmon Anemia Models by Abdul-Aziz Yakubu (Howard University). In Reflections of a Mathematics Teacher Educator: Considerations for Mathematicians who Teach Teachers, Christina Eu-banks-Turner (Loyola Marymount University) shares her insights on best practices for mathematicians who teach future mathematics teachers.

Communications include Fostering Diversity in Top-Rated Pure Mathematics Graduate Programs by Jana Gevertz (The College of New Jersey) and Joanna Wares (The University of Richmond) about promising practices that top PhD-granting institutions in pure mathematics have implemented to increase diversity in their graduate programs; Gift from Uhlenbeck Funds Karen EDGE Fellowship by Sophia Merow about a new fellowship for underrepresented mid-career mathematicians; and Institute for the Quantitative Study of Inclusion, Diversity, and Equity (QSIDE) about a cross-institutional collaborative using quantitative techniques and community partnerships to promote inclusion, diversity, and equity in our society.

This issue also includes the Opinion article Mathematics: The Key to Empowering Tomorrow’s Workforce by Tanya Moore (Intersecting Lines) about the importance of outreach in giving the next generation the quantitative skills they need to tackle tomorrow’s problems. We also include an insightful historical contribution by Jesse Kass (University of South Carolina), titled James L. Solomon and the End of Segregation at the University of South Carolina about the role of a mathematics graduate student in the desegregation of the University of South Carolina.

It has been an honor to work with these authors and read their contributions to the Notices. I hope that readers of the Notices will find these articles as engaging and thought provoking as I have.

Robin Wilson is a professor of mathematics at Cal Poly Pomona and an associate editor of Notices. His email address is robinwilson@cpp.edu. The opinions expressed here are not necessarily those of the Notices or the AMS.
February 2020

Cover Credit: The image used in the cover design appears in “Model Selection for Optimal Prediction in Statistical Machine Learning” by Ernest Fokoué, p. 155.

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Inspiring Mathematics
Lessons from the Navajo Nation Math Circles

Dave Auckly, Kansas State University, Manhattan, Bob Klein, Ohio University, Athens, Amanda Serenevy, Riverbend Community Math Center, South Bend, IN, and Tatiana Shubin, San Jose State University, CA, Editors

The people of the Navajo Nation know mathematics education for their children is essential. They were joined by mathematicians familiar with ways to deliver problems and a pedagogy that, through exploration, shows the art, joy and beauty in mathematics. This combined effort produced a series of Navajo Math Circles—interactive mathematical explorations—across the Navajo Reservation.

This book contains the mathematical details of that effort. Between its covers is a thematic rainbow of problem sets that were used in Math Circle sessions on the Reservation. The problem sets are good for puzzling over and exploring the mathematical ideas within. They will help nurture curiosity and confidence in students.

The problems come with suggestions for pacing, for adjusting the problems to be more or less challenging, and for different approaches to solving them. This book is a wonderful resource for any teacher wanting to enrich the mathematical lives of students and for anyone curious about mathematical thinking outside the box.

Titles in this series are co-published with the Mathematical Sciences Research Institute (MSRI).

Subscription prices for Volume 67 (2020) are US$720 list; US$576 institutional member; US$432 individual member; US$648 corporate member. (The subscription price for members is included in the annual dues.) A late charge of 10% of the subscription price will be imposed upon orders received from non-members after January 1 of the subscription year. Add for postage: Domestic—US$6; International—US$25. Surface delivery outside the US and India—US$27; in India—US$40; expedited delivery to destinations in North America—US$35; elsewhere—US$120. Subscriptions and orders for AMS publications should be addressed to the American Mathematical Society, PO Box 845904, Boston, MA 02284-5904 USA. All orders must be prepaid.

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LETTERS TO THE EDITOR

Open Access Publishing in Mathematics

The last few years have borne witness to a number of important changes within the scholarly communications sphere that have the potential to radically disrupt research at large, and mathematics in particular.

Researcher and institutional dissatisfaction with traditional journal subscriptions and so-called “Big Deals” have taken the Open Access movement from grassroots activism to politically mandated regulation. The announcement in 2018 of Plan S [https://www.coalition-s.org](https://www.coalition-s.org), an initiative launched by an international consortium of research funders, including the European Commission and the European Research Council, with the explicit aim of making all publicly funded research Open Access, has rapidly accelerated publisher plans to embrace Open Access models of publication.

The European Mathematical Society, in association with a number of other learned societies in the field of mathematics, supports this transition to Open Research. Indeed mathematics as a discipline has a longstanding culture of liberal Green Open Access policies. We are committed to developing sustainable models for Open Access publishing as we believe these will come to dominate publishing models in the coming years.

However, this is a transition that should not be taken without due caution and consideration. In many countries in the Northern Hemisphere there is a tendency to focus on Gold Open Access as the publishing model of choice—a model that requires authors to pay an article processing charge (APC) for publication. This may result in a number of unintended consequences, for example:

- Researchers without funding may be denied access to publish in their journal of choice. This may include researchers from developing economies, or those publishing in underfunded or niche areas of research.
- Journal profitability may become tied to published output, resulting in an inflation in the number of published articles, accompanied by a reduction in the quality of said research.
- APCs are often presented without a breakdown of where costs are incurred in the publication process, which can leave authors and their institutions unsure of the value a publishing house adds to the finished article.

Society-based publishing houses are not immune to these concerns, nor is the discipline of mathematics as a whole. For this reason, it is important that as the publishing houses of learned societies, we outline a set of guiding principles for our Open Access publishing models to ensure that quality and fairness remain at the heart of our publication programmes. At the same time, publishing houses play an important role in the curation and dissemination of research, so any publishing model must allow for the long term sustainability of the organisation. We therefore propose the following criteria:

- The quality of publications is paramount and beyond compromise.
- Publications shall be accessible and available in perpetuity.
- Pricing models shall be transparent and fair.
- The publishing house serves the mathematics community, and commits surplus funds to community initiatives.
- The publishing house commits to collaborative relationships with other stakeholders within the mathematics community.

It is with these principles in mind that the publishing house of the European Mathematical Society has begun to investigate sustainable Open Access models for its journal and book portfolio. Over the coming months we will be presenting our findings and inviting feedback from our community of editors, as well as librarians and researchers in the field.

—Volker Mehrmann
President
European Mathematical Society

—André Gaul
Managing Director
European Mathematical Society Publishing House

—Laura Simonite
Head of Business Development
European Mathematical Society Publishing House

Press information: for enquiries contact Laura Simonite at simonite@ems.press.

(Received November 7, 2019)
Surface Bundles in Topology, Algebraic Geometry, and Group Theory

Nick Salter and Bena Tshishiku

Surface Bundles
A surface is one of the most basic objects in topology, but the mathematics of surfaces spills out far beyond its source, penetrating deeply into fields as diverse as algebraic geometry, complex analysis, dynamics, hyperbolic geometry, geometric group theory, etc. In this article we focus on the mathematics of families of surfaces: surface bundles. While the basics belong to the study of fiber bundles, we hope to illustrate how the theory of surface bundles comes into close contact with a broad range of mathematical ideas. We focus here on connections with three areas—algebraic topology, algebraic geometry, and geometric group theory—and see how the notion of a surface bundle provides a meeting ground for these fields to interact in beautiful and unexpected ways.

What is a surface bundle? A surface bundle is a fiber bundle \( \pi : E \to B \) whose fiber is a 2-dimensional manifold \( S \) and whose structure group is the group \( \text{Diff}(S) \) of diffeomorphisms of \( S \). In particular, \( B \) is covered by open sets \( \{U_\alpha\} \) on which the bundle is trivial \( \pi^{-1}(U_\alpha) \cong U_\alpha \times S \), and local trivializations are glued by transition functions \( U_\alpha \cap U_\beta \to \text{Diff}(S) \).

Although the bundle is locally trivial, any nontrivial bundle is globally twisted, similar in spirit to the Möbius strip (Figure 1). This twisting is recorded in an invariant called the monodromy representation to be discussed in the section “Monodromy.”

Figure 1. The Möbius strip is the total space of a bundle over \( S^1 \) whose fibers are diffeomorphic to \([0, 1]\).
A surface bundle $E \to B$ with fiber $S$ is also called an $S$-bundle over $B$, and $E$ is called the total space. Informally, one thinks of $E$ as a family of surfaces parameterized by $B$; i.e., for each $b \in B$, there is a surface $\pi^{-1}(b) \cong S$.

**Surface bundles in nature.** Surface bundles arise naturally across mathematics. The most basic source of $S$-bundles comes from the mapping torus construction. Given $f \in \text{Diff}(S)$, define $E_f$ as the quotient of $[0,1] \times S$ by identifying $\{0\} \times S$ with $\{1\} \times S$ by $f$; then $E_f$ is the total space of an $S$-bundle over the circle $E_f \to S^1$. See Figure 2. Surprisingly, this simple-minded construction is ubiquitous in the classification of 3-manifolds and in particular hyperbolic 3-manifolds. Thurston proved that if $f$ admits a hyperbolic structure, i.e., a Riemannian metric with sectional curvature $K \equiv -1$. Furthermore, by work of Agol, Wise, and Kahn–Markovic, every closed hyperbolic 3-manifold $M$ has a finite cover of the form $E_f \to M$ for some $f : S \to S$ [Ago13].

Surface bundles also figure prominently in 4-manifold theory. Donaldson [Don98] proved that every symplectic 4-manifold $M$ admits a *Lefschetz fibration* $M \to \mathbb{C}P^1$, which can be viewed as a surface bundle where finitely many fibers are allowed to acquire singularities of a simple form (so-called *nodes*).

Surface bundles appear in algebraic geometry, where they are more commonly known as *families of curves*. Special examples can be obtained by simply writing down families of equations. For instance, let $B$ be the space of tuples $b = (b_1, \ldots, b_n)$ of distinct points in $\mathbb{C}$, fix $d \geq 2$, and for $b \in B$, consider the surface

$$S(b) = \{(x, y) \in \mathbb{C}^2 : y^d = (x - b_1) \cdots (x - b_n)\}. \quad (1)$$

Then $E = \{(x, y, b) \mid (x, y) \in S(b)\}$ is the total space of an $S$-bundle over $B$ under the projection map $(x, y, b) \mapsto b$. Here $B$ is the *configuration space* of $n$ (ordered) points in $\mathbb{C}$. The study of this single $S$-bundle is already incredibly rich, with connections to representations of braid groups and geometric structures on moduli spaces of Riemann surfaces [McM13].

Vector bundles are also a source of surface bundles: given a rank-$3$ real vector bundle, the associated unit-sphere bundle is an $S^2$-bundle. In fact, any $S^2$-bundle is obtained from this construction because, by a theorem of Smale, $\text{Diff}(S^2)$ is homotopy equivalent to the orthogonal group $O(3)$ (this homotopy equivalence implies the bundle statement by the theory of classifying spaces discussed in "The Classification Problem"). On the other hand, if $S^g$ is a closed oriented surface of genus $g \geq 1$, then $\text{Diff}(S^g)$ is not homotopy equivalent to a compact Lie group. As such, the study of $S^g$-bundles for $g \geq 1$ is the first instance of a *nonlinear bundle theory*. There are many analogies between the theory of vector bundles and surface bundles, but there are also many new phenomena, connections, and open questions.

**Conventions.** For the remainder of this article we assume, for simplicity, that $S = S^g$ is a closed, oriented surface of genus $g \geq 1$ (and at times $g \geq 2$). Working with oriented surfaces, we consider only orientation-preserving diffeomorphisms; for brevity, we suppress this from the notation and will not mention it further.

**The mapping class group.** Given the wealth of examples of surface bundles described above, we need a good way to tell different surface bundles apart. We’ll discuss two approaches to this—classifying spaces and monodromy—in "The Classification Problem" and "Monodromy." Monodromy is a special feature for $S^g$-bundles compared to other bundle theories, and it is where the mapping class group plays a prominent role.

To explain this, consider the mapping torus construction discussed above (Figure 2). If $f$ is isotopic to the identity (i.e., there is a path from $f$ to id in $\text{Diff}(S^g)$), then $E_f$ is just the product bundle $S^1 \times S^g$. More generally, for any $f \in \text{Diff}(S^g)$, the bundle $E_f$ is unchanged if $f$ is changed by an isotopy. Therefore, if we want to understand the different bundles obtained as mapping tori, we should start by considering the quotient $\text{Mod}(S^g) = \text{Diff}(S^g)/\text{Diff}_0(S^g)$ by the (normal) subgroup of diffeomorphisms isotopic to the identity. The group $\text{Mod}(S^g)$ is called the *mapping class group*. It is isomorphic to the group $\pi_0 \text{Diff}(S^g)$ of path components of $\text{Diff}(S^g)$.

For example, $\text{Mod}(T^2) \cong \text{SL}_2(\mathbb{Z})$. Any $A \in \text{SL}_2(\mathbb{Z})$ acts linearly on $\mathbb{R}^2$ and descends to $T^2$, and, conversely, up to homotopy or isotopy, a diffeomorphism of $T^2$ is determined by its action on $\pi_1(T^2) \cong \mathbb{Z}^2$. For $g \geq 1$, $\text{Mod}(S^g)$
is an infinite, finitely presented group. In “Monodromy” we explain how $\text{Mod}(S_g)$ plays a central role, not only for $S_g$-bundles over $S^1$, but for $S_g$-bundles over any base.

The Classification Problem

In this section we describe the basic tools and framework from algebraic topology for studying $S$-bundles. As mentioned above, we focus on the case $S = S_g$.

Two bundles $E \to B$ and $E' \to B$ are isomorphic if there is a diffeomorphism $E \to E'$ that sends fibers to fibers and covers the identity map on $B$.

Optimistically, one would like to solve the classification problem: for a given $B$, determine the set of isomorphism classes of $S_g$-bundles $E \to B$. This problem can be translated to a homotopy-theoretic problem via classifying space theory.

Usually the classification problem is too difficult to solve completely. In practice one wants a rich collection of invariants that (i) measure topological properties of $S_g$-bundles and (ii) enable us to distinguish $S_g$-bundles found in nature. In the study of vector bundles, a primary role is played by characteristic classes, but as we explain, these are fairly coarse invariants.

Classifying space for surface bundles. For a CW-complex $B$, let $\text{Bun}_{S_g}(B)$ be the set of isomorphism classes of $S_g$-bundles over $B$. For each $g \geq 0$, there is a space $\text{BDiff}(S_g)$ and a bijection

$$\text{Bun}_{S_g}(B) \cong [B, \text{BDiff}(S_g)]$$

where the right-hand side is the set of homotopy classes of maps $B \to \text{BDiff}(S_g)$. The space $\text{BDiff}(S_g)$ is called the classifying space for $S_g$-bundles. In the language of homotopy theory, the functor $B \mapsto \text{Bun}_{S_g}(B)$ is represented, and $\text{BDiff}(S_g)$ is the universal element.

The space $\text{BDiff}(S_g)$ is defined uniquely up to homotopy by the property that there is a principal $\text{Diff}(S_g)$-bundle $P \to \text{BDiff}(S_g)$ with $P$ contractible. In the bijection (2), given a map $B \to \text{BDiff}(S_g)$, the corresponding $S_g$-bundle $E \to B$ is obtained by pullback:

$$\begin{array}{ccc}
E & \xrightarrow{\text{pr}_S} & \text{BDiff}(S_g) \\
\downarrow & & \\
B & \xrightarrow{\text{BDiff}(S_g)} & \text{BDiff}(S_g)
\end{array}$$

The bundle on the right is known as the universal $S_g$-bundle. See [Mor01] for more details.

We want to understand the homotopy type of $\text{BDiff}(S_g)$. As mentioned above, there is a fibration $\text{Diff}(S_g) \to P \to \text{BDiff}(S_g)$ where $P$ is contractible. Hence the homotopy types of $\text{Diff}(S_g)$ and $\text{BDiff}(S_g)$ are closely related; indeed by the long exact sequence of homotopy groups, $\pi_i(\text{BDiff}(S_g)) \cong \pi_{i-1}(\text{Diff}(S_g))$. When $g \geq 2$, the homotopy type of $\text{Diff}(S_g)$ is as simple as possible.

Theorem 1 (Earle–Eells). If $g \geq 2$, then the identity component $\text{Diff}_0(S_g) < \text{Diff}(S_g)$ is contractible. Consequently, the surjection $\text{Diff}(S_g) \to \text{Mod}(S_g)$ is a homotopy equivalence.

The homotopy type of $\text{Diff}(S_g)$ for $g = 0, 1$ is also known: $\text{Diff}(S^2)$ is homotopy equivalent to $O(3)$, and $\text{Diff}(T^2)$ is homotopy equivalent to $T^2 \times \text{SL}_2(\mathbb{Z})$; see e.g. [Mor01]. Theorem 1 was originally proved using complex analysis (Teichmüller theory) and PDE; a purely topological proof was given by Gramain (see [Hat]).

By Theorem 1, $\text{BDiff}(S_g)$ is homotopy equivalent to $\text{BMod}(S_g)$ for $g \geq 2$. Since $\text{Mod}(S_g)$ is a discrete group, its classifying space is an Eilenberg–Mac Lane space $\text{BMod}(S_g) \cong K(\text{Mod}(S_g), 1)$. Observe that a map $f : B \to \text{BDiff}(S_g)$ induces a homomorphism $\pi_1(B) \to \text{Mod}(S_g)$. This is a fundamental invariant of the bundle associated to $f$, known as the monodromy; we discuss it further in “Monodromy.”

In practice, it can be useful to have a concrete model for $\text{BDiff}(S_g)$. From the point of view of homotopy theory (as in [MW05, Hat]), the most useful model is the “Grassmannian” of surfaces embedded in $\mathbb{R}^{2g}$. Unfortunately it would be too much of a detour to dwell on this further; see [Hat].

A second model for $\text{BDiff}(S_g)$ is known as moduli space $\mathcal{M}_g$. Using Theorem 1 it suffices to give a model for $\text{BMod}(S_g)$. For this we need a contractible space with a free, properly discontinuous action of $\text{Mod}(S_g)$. To this end, consider the space $\mathcal{H}$ of hyperbolic metrics on $S_g$. The group $\text{Diff}(S_g)$ acts by pullback of metrics, and $\text{Diff}_0(S_g)$ acts freely. Miraculously, the Teichmüller space $\mathcal{T} : = \mathcal{H}/\text{Diff}_0(S_g)$ is finite-dimensional and contractible: $\mathcal{T} \cong \mathbb{R}^{6g-6}$. There is a natural action of $\text{Mod}(S_g)$ on $\mathcal{T}$, and the quotient $\mathcal{M}_g : = \mathcal{T}/\text{Mod}(S_g)$ is the moduli space of hyperbolic metrics on $S_g$.

We would like to say that $\mathcal{M}_g$ is a model for $\text{BMod}(S_g)$, but this is not true because $\text{Mod}(S_g)$ does not act freely on $\mathcal{T}$. Indeed, the stabilizer of $[\mu] \in \mathcal{T}$ is the isometry group $\text{Isom}(S_g, \mu)$, which is finite but not necessarily trivial. To circumvent this issue, we use the fact that $\text{Mod}(S_g)$ contains many finite-index, torsion-free subgroups $\Gamma \leq \text{Mod}(S_g)$. For such a group, $\mathcal{T}/\Gamma$ is a genuine $K(\Gamma, 1)$, and there is a finite covering $\mathcal{T}/\Gamma \to \mathcal{M}_g$ of orbifolds. For this reason, we call $\mathcal{M}_g$ a virtual classifying space for $\text{Mod}(S_g)$. This is adequate for many purposes; e.g., there is an isomorphism

$$H^*(\text{BMod}(S_g); \mathbb{Q}) \cong H^*(\mathcal{M}_g; \mathbb{Q}).$$

The moduli space $\mathcal{M}_g$ is many things at once. In addition to the set of hyperbolic metrics up to isometry, it is the set of algebraic curves up to isomorphism and the set of Riemann surfaces up to biholomorphism. This brings
the study of $S_g$-bundles into close contact with hyperbolic geometry, complex analysis, and algebraic geometry.

Characteristic classes. There are very few spaces $B$ for which $\text{Bun}_{S_g}(B) \cong [B, \text{BDiff}(S_g)]$ has been computed completely. Instead one can ask for invariants that distinguish different elements of $[B, \text{BDiff}(S_g)]$.

A characteristic class for $S_g$-bundles is a function $c$ that assigns to each $S_g$-bundle $E \to B$ a cohomology class $c(E) \in H^*(B)$. In order to be useful, this function should be natural with respect to bundle pullbacks: given a pullback square

$$
\begin{array}{ccc}
\phi^*(E) & \rightarrow & E \\
\downarrow & & \downarrow \\
B' & \rightarrow & B
\end{array}
$$

we require $c(\phi^*(E)) = \phi^*(c(E))$ in $H^*(B')$. Equivalently, a characteristic class is a natural transformation $c : \text{Bun}_{S_g}(\cdot) \to H^*(\cdot)$.

Since every $S_g$-bundle $E \to B$ is obtained by pullback from the universal $S_g$-bundle over $\text{BDiff}(S_g)$, any cohomology class $c \in H^*(\text{BDiff}(S_g))$ defines a characteristic class; conversely, every characteristic class is of this form (evaluate on the universal bundle). In other words, $H^*(\text{BDiff}(S_g))$ is the set (or ring) of all characteristic classes of $S_g$-bundles.

Computing $H^*(\text{BDiff}(S_g))$ is of fundamental importance for studying $S_g$-bundles, but it is also of interest in other fields. By the preceding discussion,

$$
H^*(\text{BDiff}(S_g); \mathbb{Q}) \cong H^*(\text{BMod}(S_g); \mathbb{Q}) \\
\cong H^*(\text{M}_g; \mathbb{Q}).
$$

For our purpose, it is noteworthy that elements in the cohomology of $\text{Mod}(S_g)$ and $\mathbb{M}_g$ give characteristic classes of $S_g$-bundles.

Observe that the space $\text{BDiff}(S_g)$, the group $\text{Mod}(S_g)$, and the moduli space $\mathbb{M}_g$ are mostly naturally objects of algebraic topology, geometric group theory, and algebraic geometry, respectively. There has been a fertile exchange of ideas, tools, and techniques between these areas. To show this interaction, we briefly mention some of what is known about $H^*(\text{BDiff}(S_g); \mathbb{Q})$. Much of this is discussed in [Mor01] and references therein. The groups $H^*(\text{BMod}(S_g))$ satisfy homological stability, meaning that for each $i \geq 0$, $H^i(\text{BMod}(S_g))$ is independent of $g$ when $g \gg i$. This was proved by Harer in the early 1980s. Around the same time, Morita and Miller defined certain characteristic classes $e_i \in H^{2i}(\text{BDiff}(S_g))$, and Mumford defined analogous classes in the Chow ring of $\mathbb{M}_g$. Collectively these are known as MMM classes or as $\kappa$ classes. Mumford conjectured that these classes generate the cohomology in degrees $i \ll g$, and this was proved in 2002 by Madsen–Weiss, who determined the homotopy type of $\text{BDiff}(S_g)$ “in the limit” as $g \to \infty$ [MW05].

Despite all of this progress, $H^*(\text{BDiff}(S_g); \mathbb{Q})$ is still mostly unknown. By an Euler characteristic computation for $\mathbb{M}_g$ by Harer–Zagier, the MMM classes account for a small fraction of the total cohomology. We have only scratched the surface.

We conclude this section with a simple geometric argument that shows $H^1(\text{BMod}(S_g); \mathbb{Z}) = 0$ for $g \geq 3$. Recalling that $H_1(BG)$ is the abelianization $G_{ab}$, it suffices to show $\text{Mod}(S_g)_{ab} = 0$. Dehn proved that $\text{Mod}(S_g)$ is generated by mapping classes known as Dehn twists that are supported on an annulus in $S_g$ whose complement is connected. Any two such Dehn twists are conjugate. Therefore, $\text{Mod}(S_g)_{ab}$ is a quotient of $\mathbb{Z}$, generated by the image of any Dehn twist $A$. There is a relation $ABC = DFE$ between seven Dehn twists known as the lantern relation (Figure 3). For $g \geq 3$ all seven annuli can be chosen to have connected complement, so that the image of this relation in $\text{Mod}(S_g)_{ab}$ proves $3A = 4A$ or $A = 0$. This concludes the proof. See also [FM12, §5.1].

![Figure 3. Dehn twists about these curves satisfy the lantern relation $ABC = DFE$.](image)

**Monodromy**

In "Surface Bundles" we saw that the mapping torus construction provides a rich supply of $S_g$-bundles over $S^1$, but the argument of the preceding paragraph shows that none of these bundles are distinguished by characteristic classes! In this section we discuss the monodromy representation of an $S_g$-bundle. We will see that this is a complete invariant, so that in some sense we face the opposite problem: the challenge is to distill practical, computable information from the monodromy.

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2To see this, cut $S_g$ along either annulus. By the classification of surfaces, the cut-open surfaces are seen to be homeomorphic. This homeomorphism can be extended via the identity across the annuli, yielding a map of $S_g$ taking one annulus to the other.
Throughout this section we assume \( g \geq 2 \). By the bijection (2), associated to an \( S_g \)-bundle \( E \to B \), there is a map \( B \to \text{BDiff}(S_g) \), unique up to homotopy. The induced map on fundamental groups
\[
\pi_1(B) \to \pi_1(\text{BDiff}(S_g)) \cong \pi_0(\text{Diff}(S_g)) \cong \text{Mod}(S_g)
\]
is called the monodromy representation of \( E \to B \).

The monodromy representation can be described concretely as follows: given \([\gamma] \in \pi_1(B)\) represented by \( \gamma : S^1 \to B \), consider the pullback \( \gamma^*(E) \to S^1 \). Any bundle over the circle is obtained from the mapping torus construction (remove one fiber to get a bundle over the interval, which is trivial because any map \([0,1] \to \text{BDiff}(S_g)\) is null-homotopic), so \( \gamma^*(E) \cong E_{f_{\gamma}} \) for some \( f_{\gamma} \in \text{Diff}(S_g) \) whose isotopy class \([f_{\gamma}] \in \text{Mod}(S_g)\) is independent of the choice of representative of \([\gamma]\). The monodromy representation is the map \( [\gamma] \mapsto [f_{\gamma}] \). It measures how the “picture” of the fiber changes under the transition maps along the loop \( \gamma \).

Monodromy as a complete invariant. By equation (2) and Theorem 1 for \( g \geq 2 \),
\[
\text{Bun}_{S_g}(B) \cong [B, \text{BDiff}(S_g)] \cong [B, B\text{Mod}(S_g)].
\]

From \( K(\pi,1)\)-theory, a map to \( B\text{Mod}(S_g) \) is determined by the induced map on \( \pi_1 \), up to based homotopy. Hence \([B, B\text{Mod}(S_g)]\) is isomorphic to the quotient of \( \text{Hom}(\pi_1(B), \text{Mod}(S_g)) \) by the action of \( \text{Mod}(S_g) \) by conjugation.

In summary, for \( g \geq 2 \) the isomorphism class of an \( S_g \)-bundle is determined uniquely by its monodromy representation. The monodromy is a complete invariant! Next we give examples of \( B \) where this can be used to completely determine \( \text{Bun}_{S_g}(B) \).

As a trivial example, if \( \pi_1(B) = 0 \), then the only \( S_g \)-bundle over \( B \) is the trivial bundle \( B \times S_g \). (Here it is important to remember that \( g \geq 2 \).) This illustrates a stark difference between \( S_g \)-bundles and vector bundles; for example, there are many nontrivial vector bundles over spheres \( S_k \) with \( k \geq 2 \).

As a second example, for \( B = S^1 \), isomorphism classes of \( S_g \)-bundles over \( S^1 \) are in bijection with homomorphisms \( \mathbb{Z} \to \text{Mod}(S_g) \) up to conjugation, i.e., with conjugacy classes of elements of \( \text{Mod}(S_g) \). Here we clearly see why conjugation is relevant: to identify \( E \to S^1 \) with \( E_{f_{\gamma}} \), we must first choose a homeomorphism between the fiber over the basepoint and \( S_g \). Different choices change \( f \) by conjugation.

The surprising part of the statement “monodromy is a complete invariant” is that for any homomorphism \( \rho : \pi_1(B) \to \text{Mod}(S_g) \), there is a bundle \( E(\rho) \to B \) whose monodromy is \( \rho \). It’s not at all obvious how to explicitly construct \( E(\rho) \) from \( \rho \). This is the power of Theorem 1. We note however that the monodromy is not a complete invariant of the total space up to homeomorphism, since a given 3-manifold may fiber as an \( S_g \)-bundle in more than one way. See \([\text{Thu}86]\).

The monodromy-topology dictionary. Let’s think more about the bijection
\[
\text{Bun}_{S_g}(B) \cong \text{Hom}(\pi_1(B), \text{Mod}(S_g))/\text{conjugation}.
\] (3)

In the previous section we gave examples where the left-hand side could be explicitly computed using the right-hand side, but usually this is an unreasonable task. Even when \( B = S_h \) is also a closed surface, there is no known classification of homomorphisms \( \pi_1(S_h) \to \text{Mod}(S_g) \).

We would like to emphasize a different perspective on (3) that leads to interesting problems. Observe that the left-hand side of (3) is topological, while the right-hand side is group-theoretic. Understanding how geometric or topological properties of \( S_g \)-bundles translate to properties of the monodromy and vice versa leads to a dictionary. Below we mention a couple of entries of this dictionary.

Geometric classification of mapping tori. The precise conjugacy classification of elements of \( \text{Mod}(S_g) \) is well known. According to the Nielsen–Thurston classification, there are three types of conjugacy classes: \textit{periodic, reducible}, and \textit{pseudo-Anosov}. “Periodic” is synonymous with “finite-order”; a reducible element preserves (setwise) some finite collection of curves up to isotopy. Thus a pseudo-Anosov element is simply any element with neither of these special properties. The miracle of the Nielsen–Thurston classification is that every pseudo-Anosov element nevertheless has a very tightly controlled form; see \([\text{FM}12, \S 13]\).

Thurston used this classification to describe the geometry of mapping tori.

**Theorem 2 (Thurston).** Fix \( g \geq 2 \) and \( \text{fix } [f] \in \text{Mod}(S_g) \). Then \([f]\) is

(a) \textit{periodic if and only if } \text{E}_f \textit{ admits a Riemannian metric locally isometric to } \mathbb{H}^2 \times \mathbb{R};

(b) \textit{reducible if and only if } \text{E}_f \textit{ contains an incompressible torus;}

(c) \textit{pseudo-Anosov if and only if } \text{E}_f \textit{ admits a hyperbolic metric.}

A geometric restriction on the bundle gives an algebraic restriction on the monodromy and vice versa. The most striking and difficult part of the theorem is: if \([f]\) is pseudo-Anosov, then \( E_f \) is hyperbolic. We remark that a mapping class can be both periodic and reducible, so (a) and (b) are not mutually exclusive.

Given Thurston’s theorem, it is natural to ask for conditions on the monodromy of a bundle \( E \to B \) with \( \dim B \geq 2 \) that guarantee that \( E \) has negative curvature. This seems to be a subtle question. It is not hard to see that it is necessary for every nontrivial element of the monodromy group
to be pseudo-Anosov [FM02], but the converse is not generally known. It is a well-known open question whether or not there exists a homomorphism $\pi_1(S_h) \to \text{Mod}(S_g)$ such that the image of every nontrivial element is pseudo-Anosov.

Complex structures on $S_g$-bundles over surfaces. When $B = S_h$ is a closed surface, the total space $E$ of any $S_g$-bundle over $B$ is a compact 4-manifold and thus can potentially be diffeomorphic to a complex surface. Furthermore, it is possible for the bundle projection $E \to B$ to be holomorphic with respect to some complex structure on $B$. Since the monodromy $\rho$ of $E \to B$ determines the topology of $E$, this information is encoded inside $\rho$, albeit in a highly nontrivial way. In “Sections of $S_g$-bundles,” we will discuss the geometric Shafarevich problem, which shows that holomorphic families are exceedingly rare. Here, we mention some entries in the monodromy-topology dictionary concerned with the (non)existence of a complex structure on $E$.

Hodge theory provides one major source of obstructions. This is at its most powerful when the space under study is Kähler and not merely complex. It follows quickly from the Enriques–Kodaira classification that if $E$ is a compact complex surface that fibers over a surface, then $E$ is of general type and hence Kähler. Thus the basic “Kähler package” imposes nontrivial constraints on the cohomology algebra of $E$. By (3), the structure of $H^*(E; \mathbb{Z})$ (as a ring) can be obtained from $\rho$. In fact, the cup product structure on an $S_g$-bundle is encoded as a certain family of characteristic classes with “twisted coefficients”; see [Sal18]. Another Hodge-theoretic obstruction is provided by Deligne’s semisimplicity theorem, which places strong restrictions on how $\rho$ can act on the homology of the fiber [Del87].

To close this discussion we mention a theorem of Shiga [Shi97] providing another constraint on the monodromy of a holomorphic $S_g$-bundle $E \to B$ over a compact Riemann surface $B$. Shiga’s theorem asserts that in this setting, either all the fibers are biholomorphic or else the monodromy is geometrically irreducible, meaning that there is no simple closed curve globally fixed by the monodromy.

To further illuminate the themes under development (especially the monodromy-topology dictionary and interactions with algebraic geometry), in the final two sections we take a closer look at two topics: sections of $S_g$-bundles and $S_g$-bundles over surfaces.

Sections of $S_g$-bundles

A basic notion in any fiber bundle theory is that of a section: if $p : E \to B$ is a bundle map, then $s : B \to E$ is called a section if $pos = \text{id}$. In other words, a section is a continuously varying choice of distinguished point in each fiber. Given an $S_g$-bundle $p : E \to B$ with corresponding monodromy representation $\rho : \pi_1(B) \to \text{Mod}(S_g)$, there is a simple characterization of the homotopy classes of sections of $p$. Such sections are in correspondence with liftings $\tilde{\rho}$ of $\rho$ as encoded in the diagram below:

$$\begin{array}{ccc}
\pi_1(B) & \xrightarrow{\tilde{\rho}} & \text{Mod}(S_g) \\
\downarrow & & \\
\text{Mod}(S_g, \ast) & \xrightarrow{\ast} & \text{Mod}(S_g)
\end{array}$$

Here $\text{Mod}(S_g, \ast)$ is the based mapping class group, defined as the group of diffeomorphisms fixing a distinguished point $\ast \in S_g$, modulo isotopies fixing $\ast$.

Sections of $S_g$-bundles in algebraic geometry and number theory. Before we discuss some of the tools used to construct and obstruct sections of $S_g$-bundles, it is worthwhile to mention some applications. Sections of $S_g$-bundles are often of interest in problems of an algebraic-geometric flavor. One notable instance of this concerns the geometric Mordell problem. Loosely speaking, this asks for an enumeration of holomorphic sections of $S_g$-bundles over surfaces in the case where the total space has a complex structure. Arakelov and Parshin showed that the number of such sections is always finite. In fact, this is obtained from the geometric Shafarevich problem alluded to in “Monodromy.” For simplicity we state the version obtained by Parshin; Arakelov treats the more general case when $B$ is a compact Riemann surface with finitely many points removed. See e.g. [McM00].

Theorem 3 (Geometric Shafarevich). Let $B$ be a compact Riemann surface. For $g \geq 2$, there are only finitely many truly varying families $p : E \to B$ of Riemann surfaces of genus $g$.

A truly varying family $p : E \to B$ is an $S_g$-bundle where $E$ has a complex structure, $p$ is holomorphic, and the fibers are not all biholomorphic. The geometric Mordell problem follows from geometric Shafarevich by way of the “Parshin trick.” The idea is that each section $s : B \to E$ of a truly varying family can be used to construct a new truly varying family over $B$ by constructing a branched cover of $E$ branched along $s(B)$. This construction will be discussed further in “Bundles and Branched Covers” in the context of Atiyah–Kodaira bundles. Moreover, the genus of the fibers in the new family depends only on the genus of the original. Finiteness of families over $B$ (Shafarevich) then implies finiteness of sections (Mordell).

As explained by McMullen in [McM00], the geometric Mordell problem is actually the complex-geometric analogue of Faltings’s theorem in number theory. Faltings’s theorem concerns Diophantine equations $F(x, y, z)$ such as $x^n + y^n + z^n = 0$ ($n \geq 3$) whose complex points determine a Riemann surface of genus at least 2; it asserts that such an equation has only finitely many rational solutions. Scheme-theoretically, one can view such a
Diophantine equation as a “surface bundle” over $^3\text{Spec}(\mathbb{Z})$, where the “fibers” consist of the reductions of $F \bmod p$. From this point of view, a rational solution $(x, y, z)$ of $F$ determines a section of this bundle by assigning the distinguished point $(x, y, z) \pmod{p}$ to the fiber $F \pmod{p}$ over $p \in \text{Spec}(\mathbb{Z})$. McMullen explains how Faltings’s arguments have direct analogues in the setting of complex geometry, leading to the proof of the geometric Shafarevich problem given by Imayoshi–Shiga. In fact, the connections between $S_g$-bundles and number theory go beyond mere analogies. Recently, Lawrence–Venkatesh [LV] gave a new proof of Faltings’s theorem that involves a topological analysis of the monodromy of certain $S_g$-bundles over surfaces.

Sections of tautological bundles. Another application of the theory of sections of $S_g$-bundles occurs in studying the existence and classification of sections of “naturally occurring” $S_g$-bundles. The most “natural” of all such bundles is the universal curve $M_{g,*} \to M_g$ whose fiber over a point $x \in M_g$ is the Riemann surface corresponding to $x$. The section question in this case simply asks if there is a way to continuously choose a distinguished point on all Riemann surfaces simultaneously. Unsurprisingly, $M_{g,*} \to M_g$ does not have a section for $g \geq 2$. However, it is possible to choose a continuously varying family of six everywhere-distinct points on the universal curve in genus 2, furnished by the so-called Weierstrass points (Figure 4).

Thus a more sophisticated version of the section question asks if it is possible to choose, for any $n \geq 1$, a “multi-section” of $n$ everywhere-distinct points. If one restricts attention to holomorphic multisections, work of Hubbard [Hub76] shows that this is impossible, but this does not preclude the possibility that some merely continuous multisection could exist. For the universal curve $M_{g,*}$, it was only recently shown that no continuous multisection exists for $g \geq 4$ by L. Chen and the first author [CS], building off ideas of Mess. The basic tool is the theory of canonical reduction systems, described below, which can be viewed as a version of the Jordan normal form for mapping classes.

Sections: Toolkit. The study of sections of $S_g$-bundles again incorporates themes and tools from a variety of mathematical disciplines. A first question is whether a given bundle admits any sections at all. Unlike in the theory of vector bundles, where the “zero-section” provides a quick affirmative answer to this question, an $S_g$-bundle may or may not admit a section. This is similar to the situation one encounters when studying nowhere-vanishing sections of vector bundles. The standard machinery in the latter setting is obstruction theory, which manufactures cohomological invariants that obstruct the existence of sections. However, obstruction theory breaks down when the fibers are $K(\pi, 1)$ spaces with $\pi$ a group with trivial center, as is the case for $S_g$-bundles. Thus, by and large, the study of sections of $S_g$-bundles takes on a quintessentially geometric-group-theoretic flavor governed by the study of liftings $\beta$ as in (4).

Given $\rho : \pi_1(B) \to \text{Mod}(S_g)$, how could one obstruct or classify the lifts $\beta : \pi_1(B) \to \text{Mod}(S_g, \ast)$? The theory of canonical reduction systems provides one approach. Here we provide only a casual overview of how arguments using these ideas work; for a more precise discussion (including an actual definition of a canonical reduction system), see e.g. [FM12, §13.2]. In keeping with the basic philosophy of geometric group theory, the method is to consider the action of $\pi_1(B)$ on the set of simple closed curves on $S_g$ afforded by the monodromy $\rho$. If one finds an ample supply of “simple” elements in the image of $\rho$ (e.g. elements with large centralizers in $\text{Mod}(S_g)$), one can profitably understand the dynamics of this group action from the point of view of how $\pi_1(B)$ shuffles around simple closed curves on the surface. This information can be used to classify and obstruct sections: one asks where a distinguished point could be placed in relation to the simple closed curves under study, and in favorable circumstances one can see (e.g. by exploiting relations in $\pi_1(B)$ and/or $\text{Mod}(S_g)$) that there is simply no place to put a distinguished point that is compatible with the known dynamics of the action.

One shortcoming of this approach is that current techniques apply only when $B$ has a fundamental group with certain properties. In many common situations (e.g. when $B = S_h$ is itself a surface), there are not enough commuting elements of $\pi_1(B)$ to be able to implement the above ideas. Our knowledge of sections of $S_g$-bundles over surfaces is extremely limited—in fact, the question of Mess [Kir78, Problem 2.17] from 1990 asking if every $S_g$-bundle over a surface admits a multisection is still open.

Bundles and Branched Covers

The main goal of this section is to describe a construction due to Atiyah and Kodaira of $S_g$-bundles over surfaces obtained by branched coverings. In contrast to the effortless way that $S_g$-bundles over $S^1$ are constructed (Figure 2), constructing interesting $S_g$-bundles over surfaces takes work,
and the branched covering constructions we discuss here have many interesting applications.

Before we begin, we mention that the bundle \( E \to B \) over the configuration space from (1) in “Surface Bundles” is obtained via branched covers: the map \( S(b) \ni (x, y) \mapsto x \in \mathbb{C} \) is a \( d \)-fold cover branched over \( b_1, \ldots, b_n \). Thus \( B \) is parameterizing a family of branched covers of \( \mathbb{C} \) with moving branched points. The Atiyah–Kodaira construction works similarly.

**Atiyah–Kodaira bundles.** We start with the basics of the construction, which we explain in one of the simplest cases. Consider the surface \( S_3 \), and let \( \sigma : S_3 \to S_3 \) be a free involution (Figure 5). The product \( S_3 \times S_3 \) contains a (disconnected) surface \( \Sigma \), defined as the union of the graphs of the identity and \( \sigma \). We would like to take a 2-fold cover \( E \to S_3 \times S_3 \), branched over \( \Sigma \).

**Figure 5.** Free involution on surface of genus 3. Cut along the dotted line and double to obtain a 2-fold branched cover \( S_6 \to S_3 \).

Before explaining more details, let’s skip ahead to the output: the construction produces an \( S_6 \)-bundle \( E \to S_{129} \).

Where do these numbers come from? For the fiber \( S_6 \), first observe that \( \Sigma \subset S_3 \times S_3 \) meets \( \{x\} \times S_3 \) in two points \((x, x)\) and \((x, \sigma(x))\), so under a double cover \( E \to S_3 \times S_3 \) branched over \( \Sigma \), the preimage of \( \{x\} \times S_3 \) is a 2-fold cover \( S_6 \to S_3 \) branched over two points.

Now we explain the base \( S_{129} \). The issue is that the branched cover \( E \to S_3 \times S_3 \) is not guaranteed to exist. A sufficient condition for the existence is that the homology class \([\Sigma] \in H_2(S_3 \times S_3)\) is even. Unfortunately, \([\Sigma]\) is not even. To fix this, we first pass to the \( 2^k \)-sheeted cover \( S_{129} \to S_3 \) with deck group \( H_1(S_3; \mathbb{Z}/2\mathbb{Z}) \). The preimage of \( \Sigma \) under \( S_{129} \times S_3 \to S_3 \times S_3 \) determines an even homology class, and \( E \) is defined as a branched cover of \( S_{129} \times S_3 \).

This construction can be done very generally: given a surface bundle \( E \to B \) over a manifold and a multisection—viewed as a codimension-2 submanifold \( \Sigma \subset E \) that projects to \( B \) as a covering space—after replacing \( B \) with a finite cover, there is a cover \( E' \to E \) branched along \( \Sigma \). If \( E \to B \) and \( \Sigma \subset E \) are both holomorphic, then the resulting bundle is also holomorphic. This is the essence of the Parshin trick discussed in “Sections of \( S_g \)-bundles.”

The Atiyah–Kodaira examples exhibit many interesting phenomena, and they appear in surprisingly many situations. A variant of the Atiyah–Kodaira construction appears in the work of Lawrence–Venkatesh [LV] mentioned in “Sections of \( S_g \)-bundles.” We close by mentioning a sampling of other applications of the construction.

**Signature.** The total space \( E \) of an Atiyah–Kodaira bundle is a closed, oriented 4-manifold and therefore has a signature \( \text{sig}(E) \), defined as the signature of the intersection form \( H_2(E) \times H_2(E) \to \mathbb{Z} \). Under a branched cover, the signature is multiplied by the degree of the cover with a correction term that is proportional to the self-intersection number of the branching locus. Thus, although \( \text{sig}(S_3 \times S_{129}) = 0 \), we have \( \text{sig}(E) = 256 \). These were the first examples constructed of \( S_g \)-bundles over surfaces with nonzero signature.

Consequently, the MMM class \( e_1 \in H^2(\text{BDiff}(S_g)) \) is nontrivial for \( g = 6 \) (and hence for \( g \geq 6 \) by Harer stability). To see this, we remark that the function that assigns \( \text{sig}(S_3 \times S_{129}) = 0 \) to \( S_6 \) is a characteristic class. Specifically, there is a homomorphism \( H_2(\text{BDiff}(S_g)) \to \mathbb{Q} \) that sends a cycle represented by \( S_6 \to \text{BDiff}(S_g) \) to the signature of the associated bundle. This is well defined because signature is a cobordism invariant. From the Atiyah–Kodaira construction \([\text{sig}] \neq 0 \) in \( H^2(\text{BDiff}(S_g); \mathbb{Q}) \), and since \( e_1 = 3 \cdot [\text{sig}] \) (by Hirzebruch’s signature theorem), we conclude \( e_1 \neq 0 \). In fact, the class \([\text{sig}] \in H^2(\text{BDiff}(S_g); \mathbb{Q}) \) is nontrivial and generates this group when \( g \geq 3 \).

**Nontriviality of MMM classes.** Morita generalized the preceding argument to prove that all the MMM classes are nontrivial. More precisely, for each fixed \( i \), there is \( g \geq i \) so that \( e_i \neq 0 \in H^2(\text{BDiff}(S_g); \mathbb{Q}) \). He proved this by iterating the Atiyah–Kodaira construction: for example, given the Atiyah–Kodaira bundle \( p : E \to S_{129} \), consider the pullback to an \( S_6 \)-bundle \( p^*(E) \to E \). This bundle has a tautological section over which one can branch; in this way Morita obtained a bundle over a finite cover of \( E \) with \( e_2 \neq 0 \). See [Mor01, §4.4] for more details.

In other directions, the Atiyah–Kodaira construction has also been used to give examples of inequivalent symplectic structures on 4-manifolds [Leb00] and examples of CAT(0) metrics with no Riemannian smoothings [Sta15]. A variant of the construction has also been used to study the geography problem for symplectic 4-manifolds [BNOP19].
Conclusion. There are many ways to arrive at the theory of surface bundles: as a nonlinear bundle theory (algebraic topology), as a source for interesting 3- and 4-dimensional manifolds (low-dimensional topology and geometric group theory), or as objects naturally arising from moduli of Riemann surfaces (algebraic geometry). Each area brings to surface-bundle theory its own collection of ideas and techniques. This leads to a rich interaction where questions in one area motivate results in another. The interactions that we have discussed above represent only a small fraction of what is known and what is left to be discovered.

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References


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Credits

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Model Selection for Optimal Prediction in Statistical Machine Learning

Introduction
At the core of all our modern-day advances in artificial intelligence is the emerging field of statistical machine learning (SML). From a very general perspective, SML can be thought of as a field of mathematical sciences that combines mathematics, probability, statistics, and computer science with several ideas from cognitive neuroscience and psychology to inspire the creation, invention, and discovery of abstract models that attempt to learn and extract patterns from the data. One could think of SML as a field of science dedicated to building models endowed with the ability to learn from the data in ways similar to the ways humans learn, with the ultimate goal of understanding and then mastering our complex world well enough to predict its unfolding as accurately as possible. One of the earliest applications of statistical machine learning centered around the now ubiquitous MNIST benchmark task, which consists of building statistical models (also

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known as learning machines) that automatically learn and accurately recognize handwritten digits from the United States Postal Service (USPS). A typical deployment of an artificial intelligence solution to a real-life problem would have several components touching several aspects of the taxonomy of statistical machine learning. For instance, when artificial intelligence is used for the task of automated sorting of USPS letters, at least one component of the whole system deals with recognizing the recipient of a given letter as accurately as (or even better than) a human operator. This would mean that the statistical machine learning model can ideally recognize handwritten digits regardless of the various ways in which those digits are written.

1. **Theoretical Foundations**

It is typical in statistical machine learning that a given problem will be solved in a wide variety of different ways. As a result, it is a central element in SML, both within each paradigm and among paradigms, to come up with good criteria for deciding and determining which learning machine or statistical model is the best for the given task at hand. To better explain this quintessential task of model selection, we consider a typical statistical machine learning setting, with two sets \( \mathcal{X} \) and \( \mathcal{Y} \), along with their Cartesian product \( \mathcal{X} \times \mathcal{Y} \). We further define \( \mathcal{Z} = \mathcal{X} \times \mathcal{Y} \) to be the n-fold Cartesian product of \( \mathcal{Z} \). We assume that \( \mathcal{Z} \) is equipped with a probability measure \( \psi \), albeit assumed unknown throughout this paper. Let \( \mathbf{Z} \in \mathcal{Z}^n \), with \( \mathbf{Z} = ((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)) \), denote a realization of a random sample of \( n \) examples, where each example \( z_i = (x_i, y_i) \) is independently drawn according to the above probability measure \( \psi \) on the product space \( \mathcal{X} \times \mathcal{Y} \). For practical reasons, and in keeping with the data science and artificial intelligence lexicon, we shall quite often refer to the random sample \( \mathbf{Z} \) as the data set, and will use the compact and comprehensive notation

\[
\mathcal{D}_n = \{(x_i, y_i) \sim p_{xy}(x, y), \ i = 1, \ldots, n\},
\]

where all pairs \( (x_i, y_i) \in \mathcal{X} \times \mathcal{Y} \), and \( p_{xy}(x, y) \) is the probability density function associated with the probability measure \( \psi \) on \( \mathcal{Z} \). Given a random sample \( \mathbf{Z} = ((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)) \), one of the most pervading goals in both theoretical and applied statistical machine learning is to find the function \( f^* : \mathcal{X} \rightarrow \mathcal{Y} \) that best captures the dependencies between the \( x_i \)'s and the \( y_i \)'s in such a way that, given a new random (unseen) observation \( z_{\text{new}} = (x_{\text{new}}, y_{\text{new}}) \sim p_{xy}(x, y) \) with \( z_{\text{new}} \notin \mathcal{D}_n \), the image \( f^*(x_{\text{new}}) \) of \( x_{\text{new}} \sim p_x(x) \) provides a prediction of \( y_{\text{new}} \) that is as accurate and precise as possible, in the sense of yielding the smallest possible discrepancy between \( y_{\text{new}} \) and \( f^*(x_{\text{new}}) \).

This setting, where one seeks to build functions of the type \( f : \mathcal{X} \rightarrow \mathcal{Y} \), is the foundational setting of machine learning in general and statistical machine learning in particular. Throughout this paper, we shall refer to \( \mathcal{X} \) as the input space and to \( \mathcal{Y} \) as the output space. For simplicity, we shall assume that \( \mathcal{X} \subseteq \mathbb{R}^p \) for our methodological and theoretical derivations and explanations, but will allow \( \mathcal{X} \) to be more general in practical demonstrations and examples. We will consider both regression learning corresponding to \( \mathcal{Y} = \mathbb{R} \) and multiclass classification learning (pattern recognition) corresponding to output spaces of the form \( \mathcal{Y} = \{1, 2, \ldots, G\} \), where \( G \) is the number of categories.

**Definition 1.** A loss function \( \mathcal{L}(\cdot, \cdot) \) is a nonnegative bivariate function \( \mathcal{L} : \mathcal{Y} \times \mathcal{Y} \rightarrow \mathbb{R}_+ \), such that given \( a, b \in \mathcal{Y} \), the value of \( \mathcal{L}(a, b) \) measures the discrepancy between \( a \) and \( b \).
and $b$, or the deviance of $a$ from $b$, or the loss incurred from using $b$ in place of $a$. For instance, $\mathcal{L}(y, f(x)) = \mathcal{L}(f(x), y)$ will be used throughout this paper to quantify the discrepancy between $y$ and $f(x)$ with the finality of choosing the best $f$, the optimal $f$, the $f$ that minimizes expected discrepancy over the entire $\mathcal{X}$. The loss function plays a central role in statistical learning theory as it allows an unambiguous measure and quantification of optimality.

Definition 2. The theoretical risk or generalization error or true error of any function $f \in \mathcal{Y}^X$ is given by

$$R(f) = \mathbb{E}[\mathcal{L}(Y, f(X))]$$

$$= \int_{\mathcal{X} \times \mathcal{Y}} \mathcal{L}(y, f(x)) p_{xy}(x, y) dx dy$$

and can be interpreted as the expected discrepancy between $f(X)$ and $Y$, and indeed as a measure of the predictive strength of $f$. Ideally, one seeks to find the minimizer $f^*$ of $R(f)$ over all measurable functions $f \in \mathcal{Y}^X$, specifically,

$$f^* = \arg\inf_{f \in \mathcal{Y}^X} \{R(f)\} = \arg\inf_{f \in \mathcal{Y}^X} \{\mathbb{E}[\mathcal{L}(Y, f(X))]\}$$

whose corresponding theoretical risk $R^*$ serves as the gold standard and is given by

$$R^* = R(f^*) = \inf_{f \in \mathcal{Y}^X} \{R(f)\}.$$  

If we reconsider our overarching goal stated earlier, then the smallest risk (expected loss) in the prediction of $Y_{\text{new}}$ given $X_{\text{new}}$ is achieved with the $f^*$ of (3), and that theoretical optimal risk is $R^*$ of (4), namely, $\mathbb{E}[\mathcal{L}(Y_{\text{new}}, f^*(X_{\text{new}}))] = R^*$. The theoretical optimal predictive model is therefore $f^*$, although we must recognize that it is of no practical use as it cannot be computed. For instance, when we consider both classification and regression, the theoretical optimal predictive model $f^*$ can be elicited and derived for some well-known foundational loss functions. For classification, an intuitive and indeed theoretical than practical importance, it turns out to provide multiple results featuring learning machines whose performance is compared to the performance of the Bayes classifier [7] and [19]. Although this result is of more theoretical than practical importance, it turns out to provide a framework of reference for building more practical classification learning machines. Although we do not know the true density $p_{xy}(\cdot, \cdot)$, we can assume a wide variety of possible densities in special cases, and then attempt the
construction of the Bayes classifier under those distributional assumptions. It is found in practice that when the assumptions are met (or almost met), the ensuing learning machine tends to exhibit superior predictive performances. For instance, under the assumption of multivariate Gaussian class conditional densities with equal covariance matrices in binary classification, one can derive the population Bayes Gaussian linear discriminant analysis classifier, whose estimator from the corresponding data yields the best predictive performance over all other learning machines. It bears repeating that this superior performance presupposes that the assumed multivariate Gaussianity is plausible. Every single aspect of optimal predictive model selection we have mentioned so far is strongly tied to the distributional characteristics of the space under consideration. In the case of superior predictive performances inherited from the correct assumption of the generator of the data, it must be said that practical data sets often arise from rather complex distributions that are often far too difficult to estimate. One could even consider estimating the density and then estimating the corresponding classifier. Unfortunately, the task of probability density estimation in complex high-dimensional spaces turns out to be a treacherous task, often more complex (statistically and computationally) than the classification task one would be intending to use density estimation for. Some researchers have resorted to semiparametric solutions like the use of mixtures of Gaussians (or mixtures of other parametric densities) to model their class conditional densities, and have done so with great success, although the analysis of mixtures is fraught with challenges, to the point that having to deal with those along with the main task of classification may render their use unattractive and not viable in this context. For this reason, practitioners and methodological and theoretical researchers tend to focus on more realizable goals than the hunt for the universal best learning machines. The approach consists of assuming that the function underlying the data (the decision boundary in the context of classification) is a member of a class of functions with some specific (sometimes desirable) properties. Of course, the very fact of choosing a specific function space automatically comes at the potential price of incurring an approximation error. In the example given earlier, assuming Gaussian class conditional probability densities with equal covariance matrices led to the derivation of a classifier belonging to the space of linear learning machines. In this case, the ensuing function space was implicit in the distributional choice. We will see later that the choice of the function space is often quite explicit and typically motivated by experience or pure convenience. Before we delve into the search for optimal predictive models in specific function spaces, it is useful to point out that fundamental statistical learning results exist in regression analysis that are similar to the ones presented earlier in the context of classification learning.

**Theorem 2.** Consider functions \( f : \mathbb{R}^p \to \mathbb{R} \) and the squared theoretical risk functional

\[
R(f) = \mathbb{E}[(Y - f(X))^2] = \int_{\mathcal{X} \times \mathcal{Y}} (y - f(x))^2 p_{xy}(x, y) dx dy. \tag{11}
\]

Then the best function \( f^* = \arg\inf_{f} \{R(f)\} \) is given by the conditional expectation of \( Y \) given \( X \); i.e., \( \forall x \in \mathcal{X}, f^*(x) = \mathbb{E}[Y | X = x] = \int_{\mathcal{Y}} y p(y | x) dy. \tag{12} \)

Theorem 2 provides the basic foundation of all regression analysis under the squared error loss. Clearly, the conditional expectation of \( Y \) given that \( X = x \) given in equation (12) is the theoretical optimal predictive function in regression, with a corresponding theoretical risk that is the baseline.

**Theorem 3.** For every \( f : \mathcal{X} \to \mathcal{Y}, R(f) = \int_{\mathcal{X}} (f(x) - f^*(x))^2 \psi(x) + \sigma_x^2, \) where

\[
\sigma_x^2 = R^* = R(f^*) = \int_{\mathcal{X} \times \mathcal{Y}} (y - \mathbb{E}[Y | X = x])^2 p_{xy}(x, y) dx dy. \tag{13}
\]

Since the conditional density \( p(y | x) \) of \( Y \) given \( x \), which is the main ingredient of \( f^* \), is not known in practice, the optimum remains a theoretical one and serves as a gold standard and reference when the squared error loss is used, as is often the case. In an effort to realize an estimator of the optimum with the data, one can consider the traditional nonparametric regression machinery. In one dimension, nonparametric regression works very well, but it unfortunately suffers from the curse of dimensionality. Just as with classification learning, one could relax the generality of \( p(y | x) \) by assuming, for instance, a specific distribution. An example of this is the ubiquitous assumption of Gaussianity by which \( p(y | x) = \phi(y; h(x), \sigma^2) \), where \( h \in \mathcal{H} \) is a function with certain properties, taken from a function space \( \mathcal{H} \). The function space \( \mathcal{H} \) could be anything from the space of linear functions in the \( p \)-dimensional Euclidean space \( \mathbb{R}^p \) to a space of certain nonlinear functions to reproducing kernel Hilbert spaces (RKHS) anchored by a suitably chosen kernel (similarity measure). We will seek to solve the more reasonable problem of choosing from a function space \( \mathcal{H} \subset \mathcal{Y} \) the function \( f^* \in \mathcal{H} \) that best estimates the dependencies between \( x \) and \( y \). As stated earlier, trying to find \( f^* \) is hopeless. One needs to select a function space \( \mathcal{H} \subset \mathcal{Y} \), then choose the best function \( f_{\mathcal{H}}^* \) from \( \mathcal{H} \), i.e.,

\[
f_{\mathcal{H}}^* = \arg\inf_{f \in \mathcal{H}} \left\{ \mathbb{E}[\mathcal{L}(Y, f(X))] \right\}. \tag{14}
\]
so that
\[ R(f^*) = R^*_f = \inf_{f \in \mathcal{F}} R(f). \]

For notational simplicity, we will simply use \( f^* \) and \( R^* \) in place of \( f^*_\mathcal{F} \) and \( R^*_\mathcal{F} \), respectively. For the regression learning task, for instance, one could assume that the input space \( \mathcal{X} \) is a closed and bounded interval of \( \mathbb{R} \), i.e., \( \mathcal{X} = [a, b] \), and then consider estimating the dependencies between \( x \) and \( y \) from within the space \( \mathcal{H} \) of all bounded functions on \( \mathcal{X} = [a, b] \), i.e.,
\[ \mathcal{H} = \{ f : \mathcal{X} \rightarrow \mathbb{R} \mid \exists \mathcal{B} \geq 0 \text{ such that } |f(x)| \leq B \}. \]

One could further make the functions of the above \( \mathcal{H} \) continuous, so that the space to search becomes
\[ \mathcal{H} = \{ f : [a, b] \rightarrow \mathbb{R} \mid f \text{ is continuous} \} = C([a, b]), \]
which is the well-known space of all continuous functions on a closed and bounded interval \([a, b]\). This is indeed a very important function space. In fact, polynomial regression consists of searching our learning machine from a function space that is a subspace of \( C([a, b]) \). In other words, in polynomial regression learning, we are searching the space
\[ \mathcal{P}([a, b]) = \{ f \in C([a, b]) \mid f \text{ is a polynomial in } \mathbb{R} \}. \]

Interestingly, Weierstrass did prove that \( \mathcal{P}([a, b]) \) is dense in \( C([a, b]) \). One considers the space of all polynomials of some degree \( p \), i.e.,
\[ \mathcal{H} = \mathcal{P}^p([a, b]) = \left\{ f \in C([a, b]) \mid \exists \mathcal{B} \in \mathbb{R}^{p+1} \right\}, \]
\[ f(x) = \sum_{j=0}^{p} \beta^j x^j, \quad \forall x \in [a, b]. \]

Similarly, for the classification learning task of binary pattern recognition with \( \mathcal{Y} = \{-1, +1\} \), one may consider finding the best linear separating hyperplane, so that the corresponding function space is
\[ \mathcal{H} = \left\{ f : \mathcal{X} \rightarrow \mathcal{Y} \mid \exists \mathcal{w}_0 \in \mathbb{R}, \mathcal{w} \in \mathbb{R}^{p} : \forall x \in \mathcal{X}, \right. \]
\[ f(x) = \text{sign} \left( \mathcal{w}^T x + \mathcal{w}_0 \right), \quad (15) \]
or even a more complex function space capable of modelling and representing nonlinear decision boundaries like
\[ \mathcal{H}(\Phi) = \left\{ f : \mathcal{X} \rightarrow \mathcal{Y} \mid \exists \mathcal{w}_0 \in \mathbb{R}, \mathcal{w} \in \mathcal{F} : \forall x \in \mathcal{X}, \right. \]
\[ f(x) = \text{sign} \left( \langle \mathcal{w}, \Phi(x) \rangle + \mathcal{w}_0 \right), \quad (16) \]

where \( \Phi : \mathcal{X} \rightarrow \mathcal{F} \) is a mapping that projects each input \( x \) up to a high-dimensional feature space \( \mathcal{F} \), thereby allowing the corresponding machine the capacity to capture nonlinear decision boundaries.

**Empirical Foundations**

Throughout the previous section, we explored some basic aspects of the theoretical foundations of optimal prediction model selection. It turns out that \( f^* \in \mathcal{H} \), just like \( f^* \), cannot be computed because \( p_{\mathcal{E}}(x, y) \) is never known in practice. What does happen in practice is that, given the data set \( \mathcal{D}_n \) along with the chosen loss function \( \mathcal{L}(\cdot, \cdot) \), the empirical risk \( \hat{R}(f) \) is defined as an estimator of the theoretical risk \( R(f) \). From a practical perspective, given a data set \( \mathcal{D}_n \), empirical risk minimization is used in place of theoretical risk minimization to construct estimators of \( f^* \), namely,
\[ \hat{f} = \hat{f}_{\mathcal{H},n} = \hat{f}_n = \arg\min_{f \in \mathcal{H}} \{ \hat{R}_n(f) \} \]
\[ = \arg\min_{f \in \mathcal{H}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \mathcal{L}(y_i, f(x_i)) \right\}. \quad (17) \]

Although the zero-one loss function allows us to theoretically define what constitutes the universal best optimal classifier, it cannot be used in any given function space to construct an estimated learning machine, because its use inherently implies an untenable combinatorial exploration. Fortunately, many other loss functions have been typically used in the search for optimal predictive models in statistical machine learning. With \( f : \mathcal{X} \rightarrow \{-1, +1\} \), and \( h \in \mathcal{H} \) such that \( f(x) = \text{sign}(h(x)) \), some frequently used loss functions for binary classification include: (a) Zero-one \((0/1)\) loss: \( \mathcal{L}(y, f(x)) = \frac{1}{y \neq h(x)} \), (b) Hinge loss: \( \mathcal{L}(y, f(x)) = \max(1 - y h(x), 0) \), (c) Logistic loss: \( \mathcal{L}(y, f(x)) = \log(1 + \exp(-y h(x))) \), and (d) Exponential loss: \( \mathcal{L}(y, f(x)) = \exp(-y h(x)) \). With \( f : \mathcal{X} \rightarrow \mathbb{R} \) and \( f \in \mathcal{H} \), some loss functions for regression include: (a) \( L_1 \) loss: \( \mathcal{L}(y, f(x)) = |y - f(x)| \), (b) \( L_2 \) loss: \( \mathcal{L}(y, f(x)) = |y - f(x)|^2 \), (c) \( \epsilon \)-insensitive \( L_1 \) loss: \( \mathcal{L}(y, f(x)) = |y - f(x) - \epsilon| \), and (d) \( \epsilon \)-insensitive \( L_2 \) loss: \( \mathcal{L}(y, f(x)) = |y - f(x)|^2 - \epsilon \). Other loss functions exist.

Although the empirical risk minimization principle provides an effective practical framework for learning patterns underlying the data, the estimator \( \hat{f}_{\mathcal{H},n} \) derived from it must be handled with great care and caution for a wide variety of reasons, which we now make clear. With the definitions of \( f^* \), \( f^* \), and now \( \hat{f}_{\mathcal{H},n} \) in hand, a natural and almost quintessential yet somewhat audacious question would be to assess the difference between \( \hat{f}_{\mathcal{H},n} \) and \( f^* \), maybe by some suitably defined norm, say \( \|\hat{f}_{\mathcal{H},n} - f^*\| \), maybe using probabilistic measures like \( \Pr[||\hat{f}_{\mathcal{H},n} - f^*||] \) or even \( E[||\hat{f}_{\mathcal{H},n} - f^*||] \), though it might not be trivial at all how to properly define such a norm, let alone the corresponding probability distribution. A difference like
\[ \|\hat{f}_{\mathcal{H},n} - f^*\|_{\mathcal{H}} \text{ might be easier, although itself neither easy nor even practically realizable. The typical approach is to deal with the utility of the function like } \mathcal{R}(f) \text{ rather than the function itself. Now, the relationship between } \mathcal{R}(\hat{f}_{\mathcal{H},n}) \text{ and the other theoretical risks is captured by the following cascade of inequalities, namely,} \]

\[ \mathcal{R}(f^*) \leq \mathcal{R}(f^*) \leq \mathcal{R}((\hat{f}_{\mathcal{H},n}). \]

The true risk \( \mathcal{R}(\hat{f}_{\mathcal{H},n}) \) of the realized estimator \( \hat{f}_{\mathcal{H},n} \) is clearly and unsurprisingly the largest of the three. Since \( \mathcal{R}^* \) is unrealizable in practice, the natural goal should at least be: Out of all the functions in \( \mathcal{H} \) generated using the data \( \mathcal{D}_n \), choose the one that best imitates \( f^* \), which means choose \( \hat{f}_{\mathcal{H},n} \in \mathcal{H} \) such that \( \mathbb{E}[\mathcal{R}(\hat{f}_{\mathcal{H},n})] - \mathcal{R}(f^*) \) is smallest.

If one could directly (or even indirectly) construct \( \hat{f}^{(\text{opt})}_{\mathcal{H},n} \in \mathcal{H} \) such that

\[ \hat{f}^{(\text{opt})}_{\mathcal{H},n} = \arg\min \mathbb{E}[\mathcal{R}(\hat{f}_{\mathcal{H},n})] - \mathcal{R}(f^*), \]

then \( \hat{f}^{(\text{opt})}_{\mathcal{H},n} \) would be the optimal predictive model. Unfortunately, such a function cannot be directly constructed in practice because its objective function is purely theoretical. The so-called excess risk, \( \mathbb{E}[\mathcal{R}(\hat{f}_{\mathcal{H},n}) - \mathcal{R}^*] \), defined as the expected value of the difference between the true risk \( \mathcal{R}(\hat{f}_{\mathcal{H},n}) \) associated with \( \hat{f}_n \) and the overall minimum risk \( \mathcal{R}^* \), can be decomposed to explore in greater detail the source of error in the function estimation process:

\[ \mathbb{E}[\mathcal{R}(\hat{f}_n) - \mathcal{R}^*] = \mathbb{E}[\mathcal{R}(\hat{f}_n) - \mathcal{R}(f^*)] + \mathbb{E}[\mathcal{R}(f^*) - \mathcal{R}^*]. \]

Making the excess risk small is tricky because of the following dilemma: If the approximation error is made small, typically by making the function space \( \mathcal{H} \) larger and more complex so that the members of \( \mathcal{H} \) approximate \( f^* \) very well, then the corresponding estimation error tends to get undesirably large. Many authors have written excessively on methods for achieving desirable trade-offs with favorable predictive benefits. The empirical risk \( \hat{\mathcal{R}}_n(\hat{f}_n) \) on \( \hat{f}_n \) can be made arbitrarily very small by making \( \mathcal{H} \) very complex, leading to a phenomenon known as overfitting. It must be emphasized that such a function has very little to do with being optimal predictive, because the theoretical (true) risk \( \mathcal{R}(\hat{f}_n) \) of such an \( \hat{f}_n \) is undesirably large. Indeed, when it comes to optimal prediction, it is crucial for the estimator \( \hat{f}_{\mathcal{H},n} \) to have an empirical risk \( \hat{\mathcal{R}}_n(\hat{f}_n) \) that is as close as possible to the true risk \( \mathcal{R}(\hat{f}_n) \). Now, it is well known among practitioners that almost all statistical machine learning problems are inherently inverse problems, in the sense that learning methods seek to optimally estimate an unknown generating function using empirical observations assumed to be generated by it. As a result, statistical machine learning problems are inherently ill-posed, in the sense that they typically violate at least one of Hadamard’s three well-posedness conditions. For clarity, according to Hadamard a problem is well-posed if it fulfills the following three conditions: (a) a solution exists; (b) the solution is unique; and (c) the solution is stable, i.e., does not change drastically under small perturbations. For many machine learning problems, the first condition of well-posedness, namely, existence, is fulfilled. However, the solution is either not unique or not stable. With large \( p \) small \( n \) for instance, not only is there a multiplicity of solutions but also the instability thereof, due to the singularities resulting from the fact that \( n \ll p \). Typically, the regularization framework is used to isolate a feasible and optimal (in some sense) solution. TikhoNov’s regularization is the one most commonly resorted to and typically amounts to a Lagrangian formulation of a constrained version of the initial problem, the constraints being the objects used to isolate a unique and stable solution.

**Effect of Model Complexity**

To gain deeper insights into the properties and challenges inherent in optimal predictive model selection, we now consider a practical exploration of univariate regression learning using the polynomial function space, namely,

\[ \mathcal{H} = \left\{ f \in C([a, b]) : \exists \theta_0, \theta_1, ..., \theta_p \in \mathbb{R} \right\} \]

Having chosen our function space \( \mathcal{H} \) along with the squared error loss, our statistical learning task consists of finding the minimizer of the empirical counterpart of the average squared errors (ASE), i.e.,

\[ \hat{f}_{\mathcal{H},n} = \hat{f} = \arg\min_{f \in \mathcal{H}} \mathbb{E}[\text{ASE}(f)] = \arg\min_{f \in \mathcal{H}} \left\{ \frac{1}{n} \sum_{i=1}^{n} (Y_i - f(X_i; \Theta))^2 \right\}. \]
solution to problem (20) is given by
\[
\hat{\theta} = \arg \min_{\theta \in \Theta} \left\{ \frac{1}{n} \sum_{i=1}^{n} \left( Y_i - \sum_{j=0}^{p} \theta_j x_i^j \right)^2 \right\}
\]
\[
= \arg \min_{\theta \in \Theta} \left\{ (Y - X\theta)^T(Y - X\theta) \right\}
\]
\[
= (X^T X)^{-1} X^T Y. \tag{21}
\]

The estimator in equation (21) has many quintessential properties and applications. It is therefore important to dissect and unpack those key aspects of statistical learning.

(a) Stochastic nature of the estimator. First and foremost, the estimate \( \hat{\theta} = (\hat{\theta}_0, \hat{\theta}_1, \ldots, \hat{\theta}_p)^T \) of \( \theta = (\theta_0, \theta_1, \ldots, \theta_p)^T \) is a random variable, and as a result the estimate \( \hat{f}(x) = \hat{f}(x; \hat{\theta}) = \hat{\theta}_0 + \hat{\theta}_1 x + \hat{\theta}_2 x^2 + \cdots + \hat{\theta}_p x^p \) of \( f^*(x) \) is also a random variable. We therefore have to be mindful whenever \( \hat{f} \) is used, that it is inherently a random entity whose handling is best done with the powerful machineries of probability and statistics.

(b) Bias and variance. Since \( \hat{f}(x) \) is a random variable, we must compute important aspects like its bias \( B[\hat{f}(x)] = \mathbb{E}[\hat{f}(x)] - f^*(x) \), which measures how far our chosen class of models is from the true generator of the data, and its variance \( \mathbb{V}[\hat{f}(x)] =\mathbb{E}[(\hat{f}(x) - \mathbb{E}[\hat{f}(x)])^2] \), which, as the name says, tells us relatively how stable the constructed estimator is.

(c) Model complexity and temptation to overfit. Since our goal expressed through the objective function is to find the member of the class \( \mathcal{H} \) that minimizes the empirical risk, it is very tempting at first to use the data at hand to build the \( \hat{f} \) that makes \( \text{ASE}(\hat{f}) \) the smallest. For instance, the higher the value of \( p \), the smaller \( \text{ASE}(\hat{f}(\cdot)) \) will get. In fact, in the most extreme of scenarios, one could simply make \( \text{ASE}(\hat{f}(\cdot)) = 0 \) by specifying \( \hat{f}(x) = y_i \forall i = 1, ..., n \).

In a sense, we have a dilemma: If we make \( \hat{f} \) complex (large \( p \)), we make the bias small, but the variance is increased. If we make \( \hat{f} \) simple (small \( p \)), we make the bias large, but the variance is decreased. In this case, the degree \( p \) of the polynomial represents the complexity of the corresponding model. In the end, we will have to come up with various criteria for estimating the optimal complexity, in the sense of the one that leads to low prediction error. To help gain deeper insights into this fundamental statistical machine learning phenomenon, let’s consider the synthetic (artificial) task of learning a univariate polynomial regression from the data. We simulate the data using the function \( f^*(x) = -x + \sqrt{2} \sin(\pi^{3/2} x^2) \) for \( x \in [-1, +1] \), with a noise variance \( \sigma^2 = 0.3^2 \).

Figure 1 helps us gain insights into the basics of bias-variance trade-off. The polynomial of degree 1, which happens to be the model with lowest nonzero complexity, performs poorly, as does the perfect memorizer whose complexity is virtually infinite since it simply connects all the points. The solid line model does a great job learning the underlying function. The low complexity models attempt to avoid a large estimation variance but then pay a price in the form of an increased bias, resulting in a large prediction error. The high complexity models attempt to fit too well, literally memorizing the data in the extreme case, and thereby learning both the noise and the signal, resulting in a large variance as the price paid for low bias, ultimately yielding another high prediction error. The optimal fit depicted by the solid line model is achieved by settling for a trade-off between bias and variance. The task dedicated to determining that optimal complexity, which results in the optimal predictive performance, occupies a central place in statistical machine learning and will be further discussed throughout this paper. The phenomenon of bias-variance trade-off is of fundamental importance and can be further explained in the context of regression learning by the so-called bias-variance decomposition of the theoretical risk on \( \hat{f}(\cdot) \) under the squared error loss. Let’s consider the data set \( \mathcal{D}_n \). Let’s also assume that \( Y_i = f^*(x_i) + \epsilon_i \), where the \( \epsilon_i \)’s are i.i.d. from some distribution with \( \text{mean}(\epsilon) = 0 \) and \( \text{variance}(\epsilon) = \sigma^2 \).

Let \( \hat{f} \) be our estimator of \( f^* \) built using the random sample provided. Let \( x \in \mathcal{X} \). The pointwise bias-variance decomposition of the expected squared error is given by \( R(\hat{f}) = \mathbb{E}[(Y - \hat{f}(x))^2] = \sigma^2 + \text{Bias}^2(\hat{f}(x)) + \text{variance}(\hat{f}(x)) \), where \( \sigma^2 = \text{variance}(\epsilon) \) is the variance of the noise term but essentially represents the irreducible learning error, that is, the error inherent in the structure of the population, one that cannot be changed by any learning machine.
is easy to verify that this is the smallest possible error, i.e., 
\[ R^* = R(f^*) = E[(Y - f^*(x))^2] = \text{variance}(\epsilon) = \sigma^2. \]
Interestingly, the bias-variance phenomenon depicted in Figure 3 and Figure 1 in the context of regression learning is also present in classification learning. A detailed account of the same type of decomposition for the 0/1 loss used in classification can be found in [12] and [8]. The optimal decision boundary seen in Figure 2 is obtained using cross validation on the \( k \)-Nearest Neighbors learning machine (22) for various values of \( k \). Clearly, \( k \) does indeed control the complexity of the underlying model, namely, the decision boundary. Although the decision boundary in this case cannot be explicitly written or learned as the optimal of some explicit objective function, one can still use cross validation to determine the optimal value of \( k \) (optimal neighborhood size). This tremendous flexibility of the cross validation principle is certainly one of its greatest strengths, which makes it very appealing and widely applicable in statistical machine learning. For classification, \( \mathcal{Y} = \{1, \ldots, G\} \),
\[
\hat{f}^{(kNN)}(x) = \arg\max_{g \in \mathcal{Y}} \left\{ \frac{1}{k} \sum_{i=1}^{n} 1(y_i = g)1(x_i \in V_k(x)) \right\}. \tag{22}
\]
For regression, \( \mathcal{Y} = \mathbb{R} \),
\[
\hat{f}^{(kNN)}(x) = \frac{1}{k} \sum_{i=1}^{n} \gamma_i 1(x_i \in V_k(x)). \tag{23}
\]

**Figure 2.** Optimal kNN decision boundary.

**Figure 3.** Bias-variance trade-off and model complexity.

**Elements of Model Identification**

Once a specific function space is chosen for our learning task, like we did earlier with our choice of the space of univariate real-valued polynomials, it is not enough to know the highest polynomial degree for our particular regression learning task. Indeed, we also need to know which of the coefficients are nonzero. In other words, we need a clear and unambiguous way, like an index, to distinguish the members of \( \mathcal{H} \) so that we can identify and then select specific ones. To help clarify that, we can think of the function space \( \mathcal{H} \) in this case as a vector space with the monomials \( \{x, x^2, \ldots, x^p\} \) as the basis vectors or atoms of the expansion that help span the space. In general, one considers a basis set \( \{B_1(x), B_2(x), \ldots, B_p(x)\} \), so that for polynomial regression, \( B_j(x) = x^j \). Using the basis set, a member \( f \in \mathcal{H} \) can then be specified by simply indicating which of the monomials are combined together to form its representation. For our space of univariate real-valued polynomials of degree at most \( p \), we could use one of the key building blocks of the parametric model selection machinery, namely, a vector of indicator variables. With the \( p \) original atoms, there are \( 2^p - 1 \) nonempty models, each corresponding to a subset of the provided atoms. We shall use a vector \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_p)^T \) to denote the index of a given model, with each \( \gamma_j \) being an indicator of the atom’s presence in the model under consideration, namely, \( \gamma_j = 1 \) (atom \( B_j(x) \) appears in model \( M_\gamma \)).

For simplicity we shall assume no intercept; i.e., \( \beta_0 = 0 \). Here, \( \gamma = (1, 1, \ldots, 1)^T \) corresponds to the full model \( M_f \), while \( \gamma = (0, 0, \ldots, 0)^T \) corresponds to the empty model, also referred to as the null model, and given by \( M_0 : \ Y = \epsilon \) (pure zero-mean noise).

Equipped with this index, \( |M_\gamma| = |f_\gamma| = p_\gamma = \sum_{j=1}^{p} \gamma_j \) is the number of atoms in model \( M_\gamma \), and \( \bar{\gamma}_\gamma \in \mathbb{R}^{p_\gamma} \) is the subset of \( \bar{\theta} \in \mathbb{R}^p \) made up of only the \( \gamma_j \)’s picked up by \( \gamma \), that is, \( \bar{\theta}_{\gamma_j} = \gamma_j \bar{\theta}_j \). Finally, \( X_\gamma \) is the submatrix of \( X \) whose columns are only those \( p_\gamma \) columns of \( X \) picked up by \( \gamma \), so that \( X_\gamma \) is really an \( n \times p_\gamma \) matrix, and the corresponding
model \( M_\gamma \) is given by

\[
M_\gamma : \quad Y = X_\gamma \beta_\gamma + \varepsilon. \tag{24}
\]

Putting everything together, we define a function space \( \mathcal{H} \) as the hypothesis space containing the pattern underlying our data, but in a sense, using the language of models, we are somewhat dealing with a model space \( \mathcal{M} \). Having now defined the useful concept of index (indicator vector) of a given model, we can unambiguously specify members \( f_\gamma \in \mathcal{H} \) or \( M_\gamma \in \mathcal{M} \), using the vector \( \gamma \in \Gamma = \{0, 1\}^p \), which represents the indexing of that specific member of the model space \( \mathcal{M} \) or, equivalently, the function space \( \mathcal{H} \). Clearly, \( \Gamma \) is made up of the \( 2^p \) models. For our polynomial regression task, we are in the presence of the so-called parametric family of models, in the sense that the choice of a member \( M_\gamma \) of the model space \( \mathcal{M} \) through its index vector \( \gamma \) maps to the corresponding collection of parameters contained in \( \theta_\gamma \). In such a parametric context, the unambiguous specification of a model or the corresponding function thereof typically indicates both the model \( M_\gamma \) and the corresponding parameter vector \( \theta_\gamma \). Let \( \hat{x} = (B_1(x), B_2(x), \ldots, B_p(x))^T \) and \( V_\gamma \in [0, 1]^{p \times p} \), such that \( V_\gamma[j,k] = \gamma_j \delta_{jk} \), \( j = 1, \ldots, p \), \( k = 1, \ldots, p \). Any member \( f_\gamma = f_\gamma(x|\theta_\gamma, M_\gamma) \in \mathcal{H} \) can be fully specified as

\[
f_\gamma(x|\theta_\gamma, M_\gamma) = \hat{x}^T V_\gamma \theta_\gamma = \sum_{j=1}^{p} \gamma_j \theta_\gamma B_j(x). \tag{25}
\]

For any \( M_\gamma \in \mathcal{M} \), the ordinary least squares (OLS) estimate encountered earlier in equation (21) is now given by

\[
\hat{\theta}_\gamma^{\text{OLS}} = \sum_{j=1}^{p} \gamma_j \theta_\gamma B_j(x). \tag{26}
\]

It is easy to see that the ordinary least squares prediction of average response at \( x \) is given by

\[
f_\gamma^{\text{OLS}}(x) = f_\gamma^{\text{OLS}}(x|\theta_\gamma^{\text{OLS}}, M_\gamma) = \hat{x}^T V_\gamma \hat{\theta}_\gamma = \sum_{j=1}^{p} \gamma_j \hat{\theta}_\gamma B_j(x). \tag{27}
\]

It is important to note that the identifier of functions need not be a vector as in the above parametric modelling scenario. In nonparametric univariate regression learning, for instance, the identifier of a member of the Nadaraya–Watson space of estimators is simply a real scalar, which is the bandwidth of the kernel used in the estimation

\[
f_\gamma^{(NW)}(x) = \sum_{i=1}^{n} \gamma_i K \left( \frac{x - x_i}{\gamma} \right) / \sum_{i=1}^{n} K \left( \frac{x - x_i}{\gamma} \right). \tag{28}
\]

For this nonparametric scenario, the model index \( \gamma \) suffices to fully specify the model, as there are no parameters in the traditional sense of a finite collection of model coefficients. Here \( \gamma \in \Gamma \subseteq \mathbb{R}_+^* \), which means that our model space search is done on an infinite subset of the right-hand side of the real number line. For the \( k \)-Nearest Neighbors learning machine, the complexity of the implicit underlying model is measured by \( k \), the size of the neighborhood, which is a discrete number from 1 to \( n \). Therefore, for \( k \text{NN} \), \( \gamma \in \Gamma = \{1, 2, \ldots, n\} \). In practice, this is truncated to a reasonable maximum number of neighbors.

**Model Selection Criteria**

When it comes to model selection for optimal prediction both Bayesian statistics and non-Bayesian statistics have contributed richly. Essentially, one can identify four main ways to address the quest for optimal prediction: namely, (a) Selection, (b) Compression, (c) Regularization, and (d) Aggregation. The first three approaches operate under the strong assumption that a single member of function space \( \mathcal{H} \) exists with optimal predictive properties, and all the methods and techniques seek to find that unique member. All the existing criteria are carefully created, designed, and developed to help yield that member of \( \mathcal{H} \). On the other hand, aggregation, also known as ensemble learning or model averaging or model combination, takes the view that a single optimum might not exist. Aggregation operates on the assumption that many decent candidate models exist, and instead of needlessly wasting time to seek a unique optimum that one may never find, it is better to combine the good candidates in some fashion to yield an overall lower prediction (generalization) error. Over the years, aggregation techniques like Bayesian Model Averaging (BMA) [2,11], Bootstrap Aggregating (Bagging) [3], Random Forest [4], Random Subspace Learning, Stacking, and certainly Adaptive Boosting [14] and Gradient Boosting have emerged and continue to be developed. Interestingly, these so-called ensemble learning methods tend to yield the best predictive performances in practical applications.

**Likelihood based selection.** In the presence of a multiplicity of potential models competing to fit the data, and considering that the estimators of those models are based on random samples with inherently built-in uncertainty, it makes sense to assume that any choice of a model consequently has built-in uncertainty. Before the data is collected and the model built, \( p(M_\gamma) \) represents its prior probability. Once the data is collected, the posterior probability \( p(M_\gamma|\mathcal{D}_n) \) of model \( M_\gamma \) provides a reasonable mechanism for assessing and measuring the uncertainty attached to its selection. Now, using \( m_\gamma(\mathcal{D}_n) = p(\mathcal{D}_n|M_\gamma) = \int_{\Theta} p(\mathcal{D}_n|\theta_\gamma, M_\gamma)p(\theta_\gamma)d\theta_\gamma \), we can write

\[
p(M_\gamma|\mathcal{D}_n) = \frac{p(M_\gamma)m_\gamma(\mathcal{D}_n)}{\sum_{\gamma} p(M_\gamma)m_\gamma(\mathcal{D}_n)} = \frac{p(\mathcal{D}_n|M_\gamma)p(M_\gamma)}{p(\mathcal{D}_n)}. \tag{29}
\]
In a parametric context like the one introduced in “Elements of Model Identification,” the Bayesian estimator of the parameter vector \( \theta \) for model \( M_\gamma \in \mathcal{M} \) is given by
\[
\hat{\theta}_{\gamma}^{(\text{Bayes})} = \hat{\theta} = \mathbb{E}[\theta | M_\gamma, \mathcal{D}_n] = \int \theta p(\theta | M_\gamma, \mathcal{D}_n) d\theta.
\]
(30)

From a Bayesian perspective, if model \( M_\gamma \) is selected, then the predictor of the response \( Y \) given \( X \) is given by
\[
\hat{f}_{\gamma}^{(\text{Bayes})}(x) = x^T V_\gamma \mathbb{E}[\theta | M_\gamma, \mathcal{D}_n] = x^T V_\gamma \hat{\theta} = \sum_{j=1}^{p} \eta_B j(x) \hat{\theta}_j.
\]
(31)

Under the squared error loss, the Bayesian Model Averaging (BMA) predictor provides the optimal predictor [2, 11], whose corresponding prediction function is given by
\[
\hat{f}_{\gamma}^{(\text{BMA})}(x) = \sum_{\gamma \in \Gamma} \sum_{j=1}^{p} \eta j p(M_\gamma | \mathcal{D}_n) B_j(x) \hat{\theta}_j.
\]
(32)

The median probability model index vector is given by
\[
\gamma^{(\text{ned})} \in \Gamma = \{0,1\}^p, \text{ where } \gamma^{(\text{ned})} = 1 \left( \text{PIP}_j \geq \frac{1}{2} \right).
\]

The median probability model is the model made up of atoms appearing in at least half of the models in the model space. The main limitations of the median probability model lies in the fact that the model does not always exist, mainly due to the rigidity of the threshold. In [9] I remedied this limitation by suggesting a flexibility and adaptive approach for optimal predictive atom selection in the general basis function expansion framework. An alternative to the median probability model is the highest posterior model, whose model index vector is given by
\[
\gamma^{(\text{HPP})} = \arg\max_{\gamma \in \Gamma} \{ p(M_\gamma | \mathcal{D}_n) \}.
\]

Recall also that given a model \( M_\gamma \in \mathcal{M} \), along with the corresponding \( \theta_\gamma \in \mathbb{R}^p \), the likelihood of \( \theta_\gamma \) is
\[
L(\theta_\gamma | M_\gamma, \mathcal{D}_n) = p(\mathcal{D}_n | f_\gamma(X) \theta_\gamma, M_\gamma) = \prod_{i=1}^{n} p(y_i | f_\gamma(x_i | \theta_\gamma, M_\gamma)),
\]
(34)

and the maximum likelihood estimator of \( \theta_\gamma \) is
\[
\hat{\theta}_\gamma^{(\text{MLE})} = \arg\max_{\theta_\gamma \in \mathbb{R}^p} \{ \log L(\theta_\gamma | M_\gamma, \mathcal{D}_n) \}.
\]
(35)

The Schwarz Bayesian Information Criterion (BIC) [15], although very prevalent in non-Bayesian settings, just happens, as its name suggests, to have a Bayesian origin. The model index \( \gamma^{(\text{BIC})} \) of a model \( M_\gamma \in \mathcal{M} \) is given by
\[
\gamma^{(\text{BIC})} = \arg\min_{\gamma \in \Gamma} \{ \text{BIC}_n(M_\gamma) \},
\]
(36)

where the score \( \text{BIC}_n(M_\gamma) \) of model \( M_\gamma \in \mathcal{M} \) is
\[
\text{BIC}_n(M_\gamma) = -2 \log L(\hat{\theta}_\gamma | M_\gamma; \mathcal{D}_n) + |M_\gamma| \log n.
\]
(37)

The Akaike Information Criterion (AIC) [1], where the score \( \text{AIC}_n(M_\gamma) \) of model \( M_\gamma \in \mathcal{M} \) is defined as
\[
\text{AIC}_n(M_\gamma) = -2 \log L(\hat{\theta}_\gamma | M_\gamma; \mathcal{D}_n) + 2|M_\gamma|,
\]
(38)

predates BIC, and while BIC is regarded as the chief selection criterion, AIC has enjoyed the distinct property of yielding typically better predictive performances.

Elements of cross validation. A more universally applicable model selection score is the ubiquitous cross validation score. In its most general formulation, the \( V \)-fold cross validation score proceeds by deterministically dividing the data set \( \mathcal{D}_n \) into \( V \) chunks (folds) of almost equal sizes, such that \( \mathcal{D}_n = \bigcup_{v=1}^{V} \mathcal{D}_v \) and \( n = \sum_{v=1}^{V} |\mathcal{D}_v| \). The cross validation score is given by
\[
\text{CV}(\hat{f}) = \frac{1}{V} \sum_{v=1}^{V} \hat{\epsilon}_v, \tag{39}
\]

where
\[
\hat{\epsilon}_v = \frac{1}{|\mathcal{D}_v|} \sum_{i=1}^{n} 1(z_i \in \mathcal{D}_v) \mathcal{L}(y_i, \hat{f}(-\mathcal{D}_v)(x_i)),
\]

and \( \hat{f}(-\mathcal{D}_v)(\cdot) \) is the estimator of \( f \) constructed without the \( v \)th chunk \( \mathcal{D}_v \) of \( \mathcal{D}_n \). An algorithmic (pseudo-code) description is given below in Algorithm 1 to help build an intuitive understanding of this most general model selection scores. In practice, the data is often randomly shuffled prior to the deterministic splitting into chunks. The oldest incarnation of the cross validation principle is leave one out cross validation, which corresponds to \( V = n \). It is important to mention here that cross validation is one of the most used approaches to model selection for optimal prediction in statistical machine learning. From its earliest days with M. Stone’s [17] seminal paper, along with its wide variety of extensions and adaptations, like [16], the cross validation principle has continually played a central role in the selection of various types of model hyperparameters. In virtually all the model spaces considered in this paper, cross validation is the default approach for empirical intraspace model comparison and model selection. When classification and regression trees are used as the function space, their pruning is done via cross validation. Cross validation is also used as one way to estimate
the number of base learners in ensemble learning methods like Bagging [3] or Random Forest [4] or even adaptive boosting [14]. Cross validation also plays a central role in support vector machine classification and support vector regression learning, as well as in ridge regression [10] and the famous LASSO [18] and its extension. In short, cross validation is central to non-Bayesian regularization. One of the greatest appeals of the cross validation principle lies in its generality, its flexibility, and its wide applicability. Cross validation is typically used for determining the optimal complexity in both parametric and nonparametric function spaces, but also crucially for selecting the specific member of the function space that achieves the lowest prediction error, provided such a unique member exists. It is important to know that there are learning machines, and very good ones at that, that are constructed purely algorithmically. While it is difficult or even at times impossible to use some of the other optimal predictive model selection criteria on purely algorithmic machines like k-Nearest Neighbors learning machines of equations (22) and (23), it is straightforward to use cross validation on them, as long as the error is well defined. Cross validation applies nicely to the most interpretable learning machines, namely, classification and regression trees, which are built purely algorithmically but still benefit from the predictive power and flexibility of the cross validation principle.

Algorithm 1: V-fold Cross Validation

Input: Training data \( D_n = \{ z_i = (x_i, y_i)^T, \ i = 1, \ldots, n \} \)
where \( x_i \in \mathcal{X} \) and \( y_i \in \mathcal{Y} \), and the function of interest is denoted by \( f \), sample size \( n \), number of folds \( V \)

Output: Cross Validation score \( CV(\hat{f}) \)

for \( v = 1 \) to \( V \) do

Extract the validation set \( D_v = \{ z_i \in D_n : \ i \in [1 + (v-1) \times m, v \times m] \} \)

Extract the training set \( D_v^c = D_n \backslash D_v \)

Build the estimator \( \hat{f}(\cdot|D_v^c) \) using \( D_v^c \)

Compute predictions \( \hat{f}(\cdot|D_v^c)(x_i) \) for \( z_i \in D_v \)

Compute the validation error for the \( v \)th chunk
\[ \hat{\varepsilon}_v = \frac{1}{|D_v|} \sum_{i=1}^{n} (z_i \in D_v)(\mathcal{L}(y_i, \hat{f}(\cdot|D_v^c)(x_i))) \]

Compute the CV score \( CV(\hat{f}) = \frac{1}{V} \sum_{v=1}^{V} \hat{\varepsilon}_v \)

Regularized risk minimization. One of the fundamental results in statistical learning theory has to do with the fact that the minimizer of the empirical risk could turn out to be overly optimistic and lead to poor generalization performance. It is indeed the case that by making our estimated classifier very complex, it can adapt too well to the data at hand, meaning very low in-sample error rate, but yield very high out-of-sample error rates due to overfitting, the estimated classifier having learned both the signal and the noise. In technical terms, this is referred to as the bias-variance dilemma, in the sense that by increasing the complexity of the estimated learning machine, the bias is reduced (good fit all the way to the point of overfitting) (see Figure 2), but the variance of that estimator is increased. On the other hand, considering much simpler estimators leads to less variance but higher bias (due to underfitting, model not rich enough to fit the data well). This phenomenon of the bias variance dilemma is particularly potent with massive data when the number of predictor variables \( p \) is much larger than the sample size \( n \). One of the main tools in the modern machine learning arsenal for dealing with this is the so-called regularization framework, whereby instead of using the empirical risk alone, a constrained version of it, also known as the regularized or penalized version, is used. Indeed, within a selected space \( \mathcal{H} \) of potential learning machines, one typically chooses some loss function \( \mathcal{L}(.,. \cdot) \) with some desirable properties like smoothness or convexity (this is because one needs at least to be able to build the desired classifier), and then finds the minimizer of its regularized version, i.e.,

\[
\hat{f}_{\mathcal{H}, \lambda, n} = \arg\min_{f \in \mathcal{H}} \left\{ \mathcal{R}(f) + \lambda \Omega_{\mathcal{H}}(f) \right\}, \quad (40)
\]

where \( \lambda \) controls the bias-variance trade-off. Typically, \( \lambda > 0 \) and is determined by cross validation. Cross validation for determining \( \lambda \) proceeds by defining a grid \( \Lambda \subset \mathbb{R}_+ = (0, +\infty) \) of possible values of \( \lambda \). Sometimes, based on intuition or experience, it could just be \( \Lambda = [\lambda_{\text{min}}, \lambda_{\text{max}}] \); then

\[
\hat{\lambda}_{(\text{opt})} = \arg\min_{\lambda \in \Lambda} \left\{ CV(f_{\lambda}) \right\}. \quad (41)
\]

\( \hat{f}_{\mathcal{H}_{\lambda_{(\text{opt})}}, n} \) is clearly far better than \( \hat{f}_{\mathcal{H}, n} \) from equation (17). By inherent design, the cross validation mechanism endows \( \hat{f}_{\mathcal{H}_{\lambda_{(\text{opt})}}, n} \) with some predictive power, making it an estimator with the potential for predictive optimality. As long as the loss function \( \mathcal{L}(\cdot, \cdot) \) and the penalty function \( \Omega_{\mathcal{H}}(\cdot) \) have desirable mathematical and statistical properties like convexity and differentiability and boundedness to allow the search of the function space \( \mathcal{H} \) to be performed by optimization, \( \hat{f}_{\mathcal{H}_{\lambda_{(\text{opt})}}, n} \) thanks to the cross validation mechanism, provides a practical framework for potentially selecting the optimal predictive member of \( \mathcal{H} \). It is important to note that finding \( \hat{f}_{\mathcal{H}_{\lambda_{(\text{opt})}}, n} \in \mathcal{H} \) does not in any way guarantee that the true risk \( R(\hat{f}_{\mathcal{H}_{\lambda_{(\text{opt})}}, n}) \) on \( \hat{f}_{\mathcal{H}_{\lambda_{(\text{opt})}}, n} \) is close to \( R^* = R(f^*) \). In other words, \( \hat{f}_{\mathcal{H}_{\lambda_{(\text{opt})}}, n} \)
is the best in \( \mathcal{H} \), but there is no guarantee that it is anywhere near \( f^\ast \). \( \hat{f}_{\mathcal{H},n}[\cdot] \) is what we refer to here as the intraspace optimal predictive model, since it is the cross validated best estimator within the function space \( \mathcal{H} \).

Logistic regression is arguably one of the most widely used statistical learning machines, even enjoying a direct and strong relationship with artificial neural networks. Using the traditional \([0,1] \) labelling on the response variable \( Y \), we have \( P[Y_i = 1|x_i; \theta_r, M_r] = \pi(x_i; \theta_l) \) where \( \pi(x_i; \theta_l) = \frac{1}{1+e^{-\theta_l x_i}} \). The likelihood function is \( L(\theta_r; M_r, \mathcal{D}_n) = \prod_{i=1}^{n} \left[ \pi_i(\theta_r)^{y_i} (1 - \pi_i(\theta_r))^{1-y_i} \right] \).

The corresponding regularized empirical risk for the binary multiple linear regression model is given by

\[
\hat{R}_\lambda(\theta_r, M_r) = -\log L(\theta_r; M_r, \mathcal{D}_n) + \lambda \| \theta_r \|_{\mathcal{H}}. \tag{43}
\]

Now, the celebrated support vector machine \([19] \) for binary classification with response variable taking values in \([-1, +1] \) is a solution to the regularized empirical hinge risk functional, namely,

\[
\tilde{\mathbf{w}} = \arg \min_{\mathbf{w} \in \mathcal{F}} \left\{ \frac{1}{2} \sum_{i=1}^{n} (1 - y_i (\mathbf{w} \cdot \Phi(x_i)))^+ + \lambda \| \mathbf{w} \|_{\mathcal{H}} \right\}.
\]

Using quadratic programming on the dual formulation of this problem with \( \alpha_i \) as the Lagrangian multipliers, we get \( \tilde{\mathbf{w}} = \sum_{i=1}^{n} \tilde{\alpha}_i \Phi(x_i) \), and the corresponding estimated prediction function is

\[
\hat{f}_{\text{svm}}(x) = \text{sign} \left( \sum_{i=1}^{n} y_i \tilde{\alpha}_i \mathcal{K}(x, x_i) \right),
\]

where the nonzero \( \tilde{\alpha}_i \)s correspond to the so-called support vectors, and \( \mathcal{K}(x, x_i) = \langle \Phi(x), \Phi(x_i) \rangle \) is an incarnation of the so-called kernel trick that makes SVM immensely practical. Here, \( \mathcal{K}(\cdot, \cdot) \) is a bivariate function called kernel defined on \( \mathcal{X} \times \mathcal{X} \), used to measure the similarity between two points in an observation space. One of the most commonly used kernels in statistical machine learning is the Gaussian radial basis function kernel given by

\[
\mathcal{K}(x, x_i) = \exp \left( -\frac{1}{2\tau^2} \| x - x_i \|_2^2 \right).
\]

There are many other kernels and kernel methods like Gaussian processes \([5, 6] \).

### Computational Model Selection

Before \( \hat{f}_{\mathcal{H},n}[\cdot] \) can be deemed good from a predictive perspective, its complexity must be controlled in order to endow it with good generalization properties, i.e., small prediction error on out-of-sample observations. This focus on the “generalizability” of \( \hat{f}_{\mathcal{H},n} \) is incredibly central to statistical learning when optimal prediction is the primary goal. Let \( \mathcal{D}_n = \{ Z_1, Z_2, ..., Z_n \sim p_Z(z) \} \) where \( Z_i = (X_i, Y_i) \in \mathcal{X} \times \mathcal{Y} \).

Consider random splits of \( \mathcal{D}_n \) into a training and a test set such that \( \mathcal{D}_n = \mathcal{D}_{tr} \cup \mathcal{D}_{te} \) such that \( n = |\mathcal{D}_{tr}| + |\mathcal{D}_{te}| \). Consider mappings \( f: \mathcal{X} \rightarrow \mathcal{Y} \) and a loss function \( \ell(\cdot, \cdot) \). Then the training and test errors are given by

\[
\hat{R}_{tr}(f) = \frac{1}{|\mathcal{D}_{tr}|} \sum_{i=1}^{n} \ell(Y_i, f(X_i))I(Z_i \in \mathcal{D}_{tr}) \tag{44}
\]

and

\[
\hat{R}_{te}(f) = \frac{1}{|\mathcal{D}_{te}|} \sum_{i=1}^{n} \ell(Y_i, f(X_i))I(Z_i \in \mathcal{D}_{te}). \tag{45}
\]

If \( \hat{f} = \arg \inf_{f \in \mathcal{H}} \{ \hat{R}_{te}(f) \} \), then \( E(\hat{R}_{te}(\hat{f})) \leq E(\hat{R}_{te}(\hat{f})) \).

The so-called optimism of the training error is given by \( \text{Opt}(\hat{R}_{tr}(\hat{f})) = E(\hat{R}_{tr}(\hat{f})) - E(\hat{R}_{tr}(\hat{f})) \) and represents the amount by which the training error (empirical risk) underestimates (hence the term optimism) the test error (generalization error). Indeed, when the function is made more and more complex, the empirical risk gets lower and lower and farther from the true error, as seen in Figure 3. This is an instance of bias-variance dilemma that happens to be at the heart of methodological, theoretical, practical, computational, and epistemological aspects of statistical machine learning. The result of (3) highlights the reason why (17), the minimizer of the empirical risk, does not possess the predictive power needed, in the sense that it does not generalize well. In our quest for optimal predictive models, we will therefore not rely on the empirical risk alone, but instead will resort to score functions with inherent built-in mechanisms for selecting models that generalize well, i.e., produce lower prediction errors. Practically speaking, if the data \( \mathcal{D}_n \) is randomly split \( S \) times, so that for each replication \( s \), the randomly shuffled (permuted) version \( \mathcal{D}_{n}^{(s)} \) admits the decomposition \( \mathcal{D}_{n}^{(s)} = \mathcal{D}_{tr}^{(s)} \cup \mathcal{D}_{te}^{(s)} \), then the \( s \)th replication of the test error is given by

\[
e_{te}^{(s)} = \text{te}(\hat{f}(\mathcal{D}_{te}^{(s)})) = \frac{1}{|\mathcal{D}_{te}^{(s)}|} \sum_{i=1}^{n} \ell(Y_i^{(s)}, \hat{f}(\mathcal{D}_{te}^{(s)})(X_i^{(s)})), \tag{46}
\]

where \( \hat{f}(\mathcal{D}_{te}^{(s)})(\cdot) \) is the instance of \( \hat{f} \) obtained using the \( s \)th random replication of the training set. Clearly, one has \( S \) realizations of the test error, and \( \{ e_{te}^{(1)}, ..., e_{te}^{(s)}, ..., e_{te}^{(S)} \} \) can be regarded as a sample of size \( S \) from the distribution of the true test error. One of the quantities often computed from the \( S \) realizations of the test error is the corresponding average test error

\[
\frac{1}{S} \sum_{s=1}^{S} \text{te}(\hat{f}(\mathcal{D}_{te}^{(s)})). \tag{47}
\]
It is important to note that \( \hat{f}(\varphi_{te}(\cdot)) \) should be internally optimized using its own internal intraspace optimality search criterion (like cross validation). This assumption is made with the finality of making sure that the interspace model comparison operates on the best of each considered model space. Let \( \mathcal{C} \) be a collection of models, ideally with each from a different function space or a different method of estimation (learning). For instance, \( \mathcal{C} = \{\hat{f}_{\text{DA}}, \hat{f}_{\text{SVM}}, \hat{f}_{\text{CART}}, \hat{f}_{\text{RF}}, \hat{f}_{\text{GPR}}, \hat{f}_{\text{KNN}}, \hat{f}_{\text{Boost}}, \hat{f}_{\text{Logit}}, \hat{f}_{\text{RDA}}\} \).

**Algorithm 2: Stochastic Hold-Out for Generalization**

**Input:** Training data \( \mathcal{D}_n = \{z_i = (x_i, y_i), \ i = 1, ..., n\} \), where \( x_i \in \mathcal{X} \) and \( y_i \in \mathcal{Y} \), and list of learning machines to be evaluated, sample size \( n \), number of random splits \( S \), number of learning machines \( M \), Proportion \( \tau \in (1/2, 1) \) of observations in training set

**Output:** Matrix \( E = (E_{sm}) = \hat{R}_{te}(\hat{f}_{m}(\cdot)) \) of test error values for several learning machines

for \( s = 1 \) to \( S \) do

Generate the \( s \)th random split of the data set \( \mathcal{D}_n \) into training set \( \mathcal{D}_{tr}^{(s)} \) and test set \( \mathcal{D}_{te}^{(s)} \) such that \( \mathcal{D}_n = \mathcal{D}_{tr}^{(s)} \cup \mathcal{D}_{te}^{(s)} \) and \( n = |\mathcal{D}| = |\mathcal{D}_{tr}^{(s)}| + (1 - \tau)|\mathcal{D}_{te}^{(s)}| \)

for \( m = 1 \) to \( M \) do

Build and refine the \( m \)th learning machine \( \hat{f}_{m}(\varphi_{tr}^{(s)})(\cdot) \) using \( \varphi_{tr}^{(s)} \)

Compute predictions \( \hat{f}_{m}(\varphi_{te}^{(s)})(x_i) \) for \( z_i \in \mathcal{D}_{te}^{(s)} \)

Compute the test error for the \( m \)th learning machine

\[
\hat{\epsilon}_{sm} = \hat{R}_{te}(\hat{f}_{m}^{(s)}) = \frac{1}{|\mathcal{D}_{te}^{(s)}|} \sum_{i=1}^{n} \mathbb{1}(z_i \in \mathcal{D}_{te})L(y_i, \hat{f}_{m}^{(s)}(x_i))
\]

Given a data set \( \mathcal{D}_n \) and a collection of potential function spaces like \( \mathcal{C} \), one defines

\[
E = (E_{sm}) = \hat{R}_{te}(\hat{f}_{m}^{(s)}) = \text{te}(\hat{f}_{m}(\varphi_{tr}^{(s)}))
\]

\[
= \text{Error of } \hat{f}_{m}(\varphi_{tr}^{(s)})(\cdot) \text{ on } \varphi_{te}^{(s)}
\]

Then one proceeds to generate the matrix \( E_{te} \) containing \( S \) realized values of the test error for each hypothesis space. For classification, \( E_{te} \in [0,1]^{S \times M} \), and for regression \( E_{te} \in \mathbb{R}^{S \times M} \). Once \( E_{te} \) is generated, an interspace predictive model comparison is performed. The practical empirical optimal predictive model is given by

\[
\hat{f}(\varphi_{te}^{(opt)}(\cdot)) = \arg \min_{f \in \mathcal{C}} \{\text{AVTE}(f)\}.
\]

It important to note that the median can also be used in place of the mean. Besides, the replications allow various statistical analyses on the predictive performances of each function space. A typical way to explore empirical interspace model comparison is to generate comparative boxplots of the replicated test errors, which can be done using the stochastic hold-out scheme described in Algorithm 2. Figure 4 depicts the results for the famous Crabs leptograp-sus benchmark data set, and Figure 5 does the same for the Ionosphere data set, which is another benchmark data set. Both data sets can be obtained from R.

![Figure 4. Predictive performances on the Crabs data.](image)

As a matter of fact, each optimal classifier from a given space \( \mathcal{H} \) will typically perform well if the data at hand and the generator from which it came somewhat accord with the properties of the space \( \mathcal{H} \). This remark is probably what prompted the famous so-called no free lunch theorem, herein stated informally. (No Free Lunch). There is no learning method that is universally superior to all other methods on all data sets. In other words, if a learning method is presented with a data set whose inherent patterns violate its assumptions, then that learning method will underperform. Indeed, it is very humbling to see that some of the methods deemed somewhat simple sometimes hugely outperform the most sophisticated ones when compared on the basis of average out-of-sample (test) error.

**Discussion and Conclusion**

Modern data science and artificial intelligence greatly value the creation and construction of statistical learning
machines endowed with an inherent capability to predict accurately and precisely. In this paper, we have explored the niceties and subtleties of such a goal and have demonstrated that it requires a hefty dose of care and caution and definitely calls upon a solid theoretical understanding of learnability along with a lot of artlike practical common sense. Anyone who has done practical data science knows beyond a shadow of a doubt that data has a mind of its own, and tends to resist the temptation to seek a holy grail or a unified field, or any paradigm that works perfectly all the time. Practical data science almost always forces the practitioner to solve the problem at hand as thoroughly and as idiosyncratically as possible rather than seek a one-size-fits-all method that works well everywhere. At the heart of what we suggested throughout this paper is the theoretical result known as the no free lunch theorem, which reveals, both implicitly and explicitly, that the theoretical bounds studied extensively by experts do not really help much when it comes to practically selecting the optimal predictive model. Optimal predictive modelling is, and may always be, both a science and an art, requiring both mathematical and statistical rigor along with practical computational common sense.

References

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Demographic Population Cycles in Infectious Salmon Anemia Models

Abdul-Aziz Yakubu

Introduction

Salmon, for example Atlantic and Pacific salmon, is the common name of several species of the Salmonidae family of fish. Trout, for example brown and seawater trout, is the name of some of the others in the family. Salmon are typically anadromous. That is, they are born in fresh water, migrate to the ocean, then return to fresh water to reproduce.

Several of the salmon species are available from both wild and farmed sources. In the past forty years, the farmed salmon industry has grown substantially. Today, approximately 60 percent of the world’s salmon production is farmed. In 2015, more than 2,200,000 tons of farmed salmon were produced, while in comparison around 880,000 tons of wild salmon were caught [7], [13]. The majority of salmon farms consist of a collection of net cages, known as “open net-cages,” that are simply suspended in the water with no barrier between the farm and the surrounding environment. Typically, the open net-cages are located in areas along the coast where ocean currents deliver oxygen to the farmed salmon while their

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wastes (feces and feed) disperse to the environment [12], [17], [18]. Due to salmon’s high content of protein and omega-3 fatty acids, its consumption reduces the risk of cardiovascular diseases. Also, salmon is a good source of minerals and vitamins. The European Union, the United States, and China make up over 70 percent of the global market for Atlantic salmon, and in each of these places consumption of both farmed and wild salmon is increasing [7].

Infectious salmon anemia virus (ISA v) is the cause of infectious salmon anemia (ISA), a serious viral fish disease that affects mostly farmed Atlantic salmon (Salmo salar) in several areas of the world. This highly contagious disease can be insidious, with an initially low mortality rate. However, the cumulative ISA mortality rate can sometimes exceed 90 percent if it remains unchecked. ISA has caused significant mortality with severe implications for production economics among salmon farms in Northern Europe, Canada, Maine, and Chile [6]. [8]. In 2000, ISA devastated the salmon industry of the Faroe Islands, and in 1998–99 an epizootic in Scotland cost an estimated $32,000,000 to eliminate, and millions of fish were culled in ISA control efforts. ISA is a major threat not only to the viability of farmed Atlantic salmon but also to dwindling stocks of wild Atlantic salmon. Production losses, loss of export markets, and the associated social impacts make elimination and control of the ISA v infection a priority for the Atlantic salmon industry. Currently, there is no treatment for fish infected with ISA v. Furthermore, our current understanding of the epidemiology of ISA is still incomplete, and this complicates its control. ISA-infected salmon pose no human health threat.

Mathematical models of infectious disease transmission dynamics are ubiquitous in the literature. These models can play important roles in helping to quantify possible infectious disease control and mitigation strategies. The models allow us to first test a variety of control and mitigation strategies via computer simulations before actual implementation on real populations. An important epidemiological threshold parameter that quantifies disease invasion or extinction in a population is the basic reproduction number or basic reproduction ratio, denoted by \( R_0 \) [1] and [2]. \( R_0 \) is defined as the average number of secondary cases produced by a single infectious individual introduced into a totally susceptible population. Consequently, values of \( R_0 < 1 \) imply that the number of infections will decrease and the disease eventually dies out as the chain of transmission cannot be maintained. However, values of \( R_0 > 1 \) imply that the number of infections will increase, the disease invades, and an epidemic occurs in the population. For example, it was estimated that \( R_0 \) of measles is between 12 and 18. That is, each individual infected with measles would, on average, transmit the measles infection to 12–18 other individuals in a totally susceptible population. To help quantify possible ISA disease control and mitigation strategies, in this paper, we introduce discrete-time mathematical models of ISA disease transmission dynamics in salmon fish farms and wild fisheries in close proximity. For these models, we compute \( R_0 \), the threshold parameter for ISA disease elimination or invasion.

Epidemiological studies suggest that horizontal transmission of ISA v occurs readily within salmon open net-cages. The infection also takes place, although more slowly, between salmon in different nets at a site, as well as between farms. ISA v probably infects fish through the gills, but ingestion has not been ruled out. Unlike farmed salmon, wild salmon live and breed in their native bodies of water, and humans have no control over their breeding, feeding, or health. Wild salmon are known to swim over long distances with no restriction. The locations of salmon farms among wild salmon migratory routes are known to raise risk of ISA infections in nearby wild salmon populations [9], [10], [14]. Determining the possible role wild salmon play in ISA v transfer to nearby salmon farms is an interesting question.

In a 2016 study of ISA disease transmission dynamics, Milliken and Pilyugin used a deterministic two-patch diffusion linked continuous-time system of ordinary differential equations with the logistic (noncyclic) salmon growth function to model ISA v infections in farmed salmon populations that are in close proximity to a wild salmon migratory route [12]. The ODE model of Milliken and Pilyugin assumed that, in the absence of the ISA infection, the salmon population is at rest on a static equilibrium point. However, salmon populations are known to exhibit density-dependent population cycles [15], [18]. A striking example of oscillations in salmon populations is the 4-year cycle in the number of spawning sockeye salmon (Oncorhynchus nerka) that return to their native streams. Also, pink salmon (Oncorhynchus gorbuscha) exhibits enormous variations in existence and amplitude of 2-year cycles in abundance throughout its range in the Pacific Ocean. Simple single species nonlinear discrete-time population models, such as one hump maps, are capable of generating such period k or k-years population cycles and chaotic dynamics, even without environmental stochasticity. The Ricker model, a discrete-time nonlinear population model first used by Ricker in 1954 to study population cycles in fish, is capable of generating density-dependent period k population cycles in a demographic equation (in the absence of a disease; see [11] and [15]). In this article, we develop a discrete-time six-dimensional viral dispersal-linked farmed-wild salmon ISA model with Ricker demographic dynamics. Others have studied discrete-time infectious disease models; for example see [1] and [2]. As in [12], we consider a farmed-wild salmon system that...
We also assume that the ISA virus in the water environment via the dispersion or diffusion of the virus between the open net-cages and the nearby open body of water. To capture the salmon demographic population cycles, unlike in [12], we assume that susceptible farmed and wild salmon populations exhibit Ricker growth or recruitment (or birth) functions, and the proportion of ISA that disperses from the open net-cages to the open body of water is not equal to the proportion of ISA that disperses from the nearby open water to the cages. Furthermore, the mortality rates in the farmed and wild salmon populations are not identical. We also assume that the ISA-infected salmon populations cannot reproduce, and there are no vertical ISA transmissions. That is, we assume that all newborn salmon populations are susceptible.

In a recent paper [18], van den Driessche and Yakubu introduced a discrete-time model of ISA without dispersion. Several authors have studied population models that allow for the interaction of density-dependent demographic dynamics and dispersion with or without explicit disease dynamics (for example, see [3]). In this article, our emphasis is on exploring the effects of ISA dispersion on the invasion or elimination of the ISA disease infection and the basic reproduction number, \( R_0 \), where the salmon demographic dynamics is periodic. When \( R_0 < 1 \) and the demographic equation has a locally asymptotically stable period \( k \) population cycle, we prove the local asymptotic stability of the dispersal-linked farmed-wild salmon ISA disease-free period \( k \) population cycle, and the disease dies out in the farmed and wild salmon populations. Also, under the same period \( k \) demographic assumption, we prove that the ISA disease-free period \( k \) population cycle is unstable and the disease invades the farmed and wild salmon populations when \( R_0 > 1 \). In addition, we use simulations to explore the relationships between high and low proportion of viral dispersals from the farmed cages to the nearby open body of water on ISA invasion or elimination.

**Farmed-Wild Salmon Viral Dispersal Linked ISA Model**

To introduce a farmed-wild salmon discrete-time ISA model with viral dispersion, we assume that at each time \( t \in \{0,1,2,\ldots\} \), each live salmon is either susceptible farmed, \( S_F^t \), or wild, \( S_W^t \), or infectious farmed, \( I_F^t \), or wild, \( I_W^t \). That is, we let \( S_F^t, I_F^t, \) and \( N_F^t = S_F^t + I_F^t \), respectively, denote the population size of susceptible, infectious, and total population of fish (respectively, wild) salmon at each time \( t \) when \( \Lambda = F \) (respectively, when \( \Lambda = W \)). Once infected, salmon do not recover from the ISA disease. At each time \( t \in \{0,1,2,\ldots\} \), we denote the virus population size in the open net-cages by \( V_F^t \), and the nearby virus population size in the open body of water by \( V_W^t \). Thus, we use an SI farmed-wild ISA epidemic model with virus dispersion and no recovery class to describe the salmon populations.

Per each unit time interval, \( d_F \in (0,1) \) (respectively, \( d_W \in (0,1) \)) is the fraction of farmed (respectively, wild) salmon that die “naturally,” \( \hat{d}_F = (1 - d_F) \) (respectively, \( \hat{d}_W = (1 - d_W) \)) is the fraction of farmed (respectively, wild) salmon that survives, \( \mu_F \in (0,1) \) (respectively, \( \mu_W \in (0,1) \)) is the farmed (respectively, wild) constant ISA-induced mortality, \( \hat{\mu}_F = (1 - \mu_F) \) (respectively, \( \hat{\mu}_W = (1 - \mu_W) \)) is the fraction of ISA infectious farmed (respectively, wild) salmon that survives the infection, \( d_{V_F}^t \in (0,1) \) (respectively, \( d_{V_W}^t \in (0,1) \)) is the constant fraction of virus that is cleared in the open net-cages (respectively, open body of water), and \( \hat{d}_{V_F}^t = (1 - d_{V_F}^t) \) (respectively, \( \hat{d}_{V_W}^t = (1 - d_{V_W}^t) \)) is the fraction of virus that survives in the open net-cages (respectively, open body of water). Salmon have high annual mortality rates at sea compared to other marine fisheries. For Atlantic salmon at sea, it is estimated that \( 65\% \leq d_w \leq 85\% \). Furthermore, Atlantic salmon at sea have more predators, such as larger piscivorous fish (striped bass, cod, haddock), birds, and mammals. Thus, \( d_F < d_W \). Since ISA is a viral disease of farmed Atlantic salmon, \( \mu_W < \mu_F \) and \( d_{V_W}^t \geq d_{V_F}^t \).

In each unit interval, we assume that a fraction of susceptible farmed salmon, \( \vartheta_F \in (0,1) \), becomes infected from direct contact with infectious farmed salmon with probability function \( \varphi_F(t) = (1 - \vartheta_F(t)) \), and the remaining fraction, \( (1 - \vartheta_F) \in (0,1) \), becomes infected via contact with the ISA in the open net-cages with probability function \( \varphi_{F,V}^t(V_F^t) = (1 - \vartheta_V^t(V_F^t)) \) for each time \( t \in \{0,1,2,\ldots\} \). Similarly, we assume that a fraction of wild susceptible salmon, \( \vartheta_W \in (0,1) \), becomes infected from direct contact with infectious wild salmon with probability function \( \varphi_W(t) = (1 - \vartheta_W(t)) \) and the remaining fraction, \( (1 - \vartheta_W) \in (0,1) \), becomes infected via contact with the ISA virus in the open body of water with probability function \( \varphi_{V}^t(V_W^t) = (1 - \varphi_V^t(V_W^t)) \), where the “escape” functions

\[
\varphi_F, \varphi_W, \varphi_{F,V}^t, \varphi_{V}^t : \mathbb{R}_+ \rightarrow [0,1]
\]

are nonlinear decreasing smooth concave up functions with \( \varphi_F(0) = \varphi_W(0) = \varphi_{F,V}^t(0) = \varphi_V^t(0) = 1 \), \( \varphi_F < 0 \), \( \varphi_W < 0 \), \( (\varphi_F)' < 0 \), \( (\varphi_W)' < 0 \), \( \varphi_F > 0 \), \( \varphi_W > 0 \), \( \varphi_{F,V}^t > 0 \), \( \varphi_V^t > 0 \), and \( (\varphi_{F,V}^t)' > 0 \). Thus, in each unit time interval, we assume that the probability of exposure of an individual salmon to multiple ISA infection pathways is very low and so can be ignored.
To introduce specific examples of the escape functions, \( \varphi_\Lambda \) and \( \varphi_\Lambda^\dagger \) for each \( \Lambda \in \{F, W\} \), we assume throughout that disease infections are modeled as Poisson processes, and for a single infectious individual fish, we let \( \beta_\Lambda \) or \( \beta_\Lambda^\dagger \) be the mean number of occurrences of infections in the unit interval for each \( \Lambda \). Then

\[
\varphi_\Lambda (I^\Lambda) = \exp ( -\beta_\Lambda I^\Lambda)
\]

(respectively, \( \varphi_\Lambda^\dagger (V^\Lambda) = \exp ( -\beta_\Lambda^\dagger V^\Lambda) \))
equals the fraction of susceptible salmon that escapes from infection after coming in contact with \( I^\Lambda \) (respectively, \( V^\Lambda \)) infectious salmon (respectively, ISA\( \nu \)) in the unit time interval.

We account for virus shedding by the infectious farmed and wild salmon populations. During each unit time interval, \( \delta_F I^F_1 \) and \( \delta_W I^W_1 \) are respectively the population of virus shed by the infectious farmed salmon into the open net-cages and nearby wild salmon populations into the open body of water, where \( \delta_F, \delta_W > 0 \). In each unit time interval, a constant fraction of ISA\( \nu \) in the open net-cages, \( m_F \in (0, 1) \), disperses to the nearby open body of water, while the constant fraction of ISA\( \nu \) in the nearby open body of water, \( m_W \in (0, 1) \), disperses to the open net-cages.

To allow cyclic salmon demographic dynamics, we let

\[
g_F, g_W : \mathbb{R}_+ \rightarrow \mathbb{R}_+
\]
denote the nonlinear differentiable Ricker recruitment (or birth) function of susceptible farmed (respectively, wild) salmon into the susceptible farmed (respectively, wild) salmon class per unit time interval, where

\[
g_F(S^F_i) = r_F S^F_i \exp (-b_F S^F_i)
\]

and

\[
g_W(S^W_i) = r_W S^W_i \exp (-b_W S^W_i).
\]

For the farmed (respectively, wild) salmon, the parameter \( r_F \) (respectively, \( r_W \)) is the density-independent probability of survival from egg to age 1, and \( b_F \) (respectively, \( b_W \)) is the coefficient of density-dependent mortality [15]. The Ricker model is an example of simple spawner-recruitment and per capita growth function that has played fundamental roles in fishery science for many years.

Our farmed-wild salmon ISA model implicitly assumes three distinct temporal phases. In both the open net-cages and nearby open body of water, at the end of each unit time interval, susceptible salmon populations become infectious, a fraction of infectious salmon die from ISA, viral dispersion and shedding occur; a fraction of each salmon class is removed (natural death and virus clearing); and susceptible salmon populations reproduce into the susceptible class. Typically, continuous-time differential equation models with similar well-defined distinct temporal phases are nonautonomous. Taking into account the temporal ordering of events in both farmed and wild salmon populations, we derive our ISA model with viral dispersal in the following three steps.

1. **ISA transmission, ISA-induced death, viral shedding, and viral dispersion:**

\[
\begin{aligned}
S^F_{i+1} &= S^F_i \left( \varphi_F S_F (I^F_i) + (1 - \varphi_F) \varphi^\dagger_F (V^F_i) \right) \\
I^F_{i+1} &= \hat{d}_F \left( \varphi_F \varphi_F (I^F_i) + (1 - \varphi_F) \varphi_F^\dagger (V^F_i) \right) + \hat{\mu}_F I^F_i \\
V^F_{i+1} &= \hat{d}_F \left( (1 - m_F) V^F_i + \delta_F I^F_i + m_W V^W_i \right) \\
S^W_{i+1} &= g_W (S^W_i) + \hat{d}_W \left( \varphi_W S_W (I^W_i) + (1 - \varphi_W) \varphi^\dagger_W (V^W_i) \right) \\
I^W_{i+1} &= \hat{d}_W \left( (1 - m_W) I^W_i + \delta_W I^W_i + m_W V^W_i \right) + \hat{\mu}_W I^W_i \\
V^W_{i+1} &= \hat{d}_W \left( (1 - m_W) V^W_i + \delta_W I^W_i + m_W V^W_i \right)
\end{aligned}
\]

In the open net-cages, after ISA transmission, ISA-induced death, viral shedding, and viral dispersion, \( S^F(1) \) and \( I^F(1) \) denote the population sizes of susceptible and infectious farmed salmon, respectively, while \( V^F(1) \) denotes the population size of the ISA\( \nu \).

2. **Natural death (survival):**

\[
\begin{aligned}
S^F_{i+1} &= \hat{a}_F S^F_i \\
I^F_{i+1} &= \hat{a}_F I^F_i \\
V^F_{i+1} &= \hat{a}_F V^F_i
\end{aligned}
\]

In the open net-cages, after ISA transmission, ISA-induced death, viral shedding, viral dispersion, and natural death, \( S^F(2) \) and \( I^F(2) \) denote the population sizes of susceptible and infectious farmed salmon, respectively, while \( V^F(2) \) denotes the population size of the ISA\( \nu \).

3. **Reproduction (S into S):**

\[
\begin{aligned}
S^F_{i+1} &= g_F (S^F_i) + S^F_{i+1} \\
S^W_{i+1} &= S^W_i + S^W_{i+1}
\end{aligned}
\]

In the open net-cages, after ISA transmission, disease-induced death, viral shedding, viral dispersion, natural death, and reproduction, \( S^F(3) \) and \( I^F(3) \) denote the population sizes of susceptible and infectious farmed salmon, respectively, while \( V^F(3) \) denotes the population size of the ISA\( \nu \).

The corresponding equations for the wild populations are similar. Our assumptions and notation lead to the following discrete-time farmed-wild salmon ISA model with viral dispersion:

\[
\begin{aligned}
S^F_{i+1} &= g_F (S^F_i) + \hat{a}_F S^F_i \left( \varphi_F \varphi_F (I^F_i) + (1 - \varphi_F) \varphi_F^\dagger (V^F_i) \right) \\
I^F_{i+1} &= \hat{a}_F \left( \varphi_F \varphi_F (I^F_i) + (1 - \varphi_F) \varphi_F^\dagger (V^F_i) \right) + \hat{\mu}_F I^F_i \\
V^F_{i+1} &= \hat{a}_F \left( (1 - m_F) V^F_i + \delta_F I^F_i + m_W V^W_i \right) \\
S^W_{i+1} &= g_W (S^W_i) + \hat{a}_W \left( \varphi_W S_W (I^W_i) + (1 - \varphi_W) \varphi_W^\dagger (V^W_i) \right) \\
I^W_{i+1} &= \hat{a}_W \left( (1 - m_W) I^W_i + \delta_W I^W_i + m_W V^W_i \right) + \hat{\mu}_W I^W_i \\
V^W_{i+1} &= \hat{a}_W \left( (1 - m_W) V^W_i + \delta_W I^W_i + m_W V^W_i \right)
\end{aligned}
\]
where \( t = 0,1,2, \ldots \). We study Model (1) with initial conditions
\[
(S_0^F, I_0^F, V_0^F, S_0^W, I_0^W, V_0^W) \in \mathbb{R}_+^6.
\]

Model (1) predicts the vectors of population sizes in the farm, \((S^F_t, I^F_t, V^F_t, S^W_t, I^W_t, V^W_t)\), and nearby open water, \((S^W_{t+1}, I^W_{t+1}, V^W_{t+1})\), at time \((t+1)\) from knowledge of the vectors of population sizes in the farm and open water, \((S^F_t, I^F_t, V^F_t)\) and \((S^W_t, I^W_t, V^W_t)\), at time \(t\), where \( t = 0, 1, 2, \ldots \). The unit of time depends on the specific salmon application. For simplicity, we take the unit of time in Model (1) to be a year. The basic reproduction number, \( R_0 \), the lifetime production of ISA infections produced per ISA infectious salmon, is independent of the model’s time scale. We will use Model (1) to compute \( R_0 \). Model (1), a well-posed model, exhibits no unbounded salmon or virus population growth.

**Discrete-Time ISA Model without Dispersal**

The open net-cages and nearby open body of water of Model (1) are linked only by viral dispersion. When there are no dispersals and \( m_\Lambda = 0 \) for each \( \Lambda \in \{F, W\} \), the two subsystems decouple into two identical subsystems, except for the choice of parameters, and are given by the following single patch ISA model of the farmed (respectively, wild) salmon population in isolation when \( \Lambda = F \) (respectively, when \( \Lambda = W \)):

\[
\begin{align*}
S^\Lambda_{t+1} &= g_\Lambda(S^\Lambda_t) + \tilde{d}_\Lambda S^\Lambda_t \varphi_\Lambda(I^\Lambda_t)
+ (1 - \vartheta_\Lambda) \tilde{\varphi}_\Lambda(V^\Lambda_t) \\
I^\Lambda_{t+1} &= \hat{d}_\Lambda S^\Lambda_t \varphi_\Lambda(I^\Lambda_t) + (1 - \vartheta_\Lambda) \tilde{\varphi}_\Lambda(V^\Lambda_t) + \beta_\Lambda I^\Lambda_t \\
V^\Lambda_{t+1} &= \tilde{\varphi}_\Lambda(V^\Lambda_t + \delta_\Lambda I^\Lambda_t),
\end{align*}
\]

where \( \Lambda \in \{F, W\} \). In [18], van den Driessche and Yakubu studied Model (2) with \( \Lambda = F \).

Here, we consider the simplified demographic equation of Model (2) with no ISA disease \( I^\Lambda_t = V^\Lambda_t = 0 \) for each \( \Lambda \in \{F, W\} \), the following iterative Ricker equation:

\[
S^\Lambda_{t+1} = r_{\Lambda} S^\Lambda_t \exp(-b_{\Lambda} S^\Lambda_t) + (1 - d_{\Lambda}) S^\Lambda_t.
\]

To illustrate salmon population cycles in Model (3) with \( \Lambda = F \), in the farmed salmon population we choose the probability of natural death to be less than the probability of staying alive per unit time interval and set \( d_F = 0.4 \). To scale the farmed salmon population sizes, we choose the scaling parameter, \( b_F \), to equal 0.1. Figure 1 shows period-doubling bifurcations in Model (3) as the farmed salmon intrinsic growth rate, \( r_F \), is varied between 20 and 470. Figure 1 shows windows of farmed salmon population cycles of period 1, 2, 4, 8, \ldots. and the range of the values of the parameter \( r_F \) for the local asymptotic stability of the periodic farmed salmon population cycles. A period \( k \) population cycle is a \( k \)-years population cycle, and Figure 1 shows simulations of the 2-years population cycle of pink salmon when \( r_F \in (70, 260) \) and the 4-years population cycle of the sockeye salmon when \( r_F \in (270, 370) \).

![Figure 1.](image)

The basic reproduction number \( R_0^{\Lambda,3} \). Assuming ISA shedding is not a new infection [4], [5], in [18], van den Driessche and Yakubu used the next generation matrix method to compute the basic reproduction number, \( R_0^{\Lambda,3} \), for the three-dimensional Model (2) for \( \Lambda = F \) or \( \Lambda = W \), where the salmon demographic threshold \( R_d^{\Lambda} > 1 \) and equation (3) has a unique positive locally asymptotically stable period \( k \) salmon population cycle. Thus, when the farmed (respectively, wild) salmon population is in isolation and \( R_d^{\Lambda} > 1 \), then the ISA disease-free period \( k \) isolated farmed (respectively, wild) salmon population cycle of Model (2) is locally asymptotically stable, and the ISA disease epidemic is eliminated in the population when \( R_0^{\Lambda,3} < 1 \), where \( \Lambda = F \) (respectively, \( \Lambda = W \)).

However, the ISA disease-free period \( k \) isolated farmed (respectively, wild) salmon population cycle is unstable and ISA invades the isolated population of the three-dimensional Model (2) when \( R_0^{\Lambda,3} > 1 \).

**Illustrative examples in single patch model without dispersion.** To illustrate ISA elimination and endemicity in Model (2), we assume that the ISA infections are modeled as Poisson processes and let

\[
\Lambda = F, \quad \varphi_\Lambda(I^\Lambda) = \exp(-\beta_\Lambda I^\Lambda),
\quad \tilde{\varphi}_\Lambda(V^\Lambda) = \exp(-\beta_\Lambda V^\Lambda)
\]

where

\[
b_F = 0.1, \quad d_F = \delta_F = 0.4, \quad d_F = 0.2, \quad r_F = 72.
\]
With our choice of parameters, \( R_d^F = \frac{r_F}{d_F} = 180 > e^{2\frac{\beta_F}{d_F}} = e^5 \approx 148.4 \), and Model (3), the ISA disease-free equation has a unique positive asymptotically stable period 2 farmed salmon population cycle (see Figures 1 and 2). Figure 2 shows that \( R_0^{F,3} < 1 \) (ISA is eliminated), and the ISA disease-free period 2 farmed population cycle is asymptotically stable.

To illustrate the impact of higher values of the ISA infection rate, \( \beta_F \), on Figure 2, we increase \( \beta_F \) to \( \beta_F = 0.04 \) and keep all the other parameters fixed at their current values in Figure 2. In this case, Figure 3 shows that \( R_0^{F,3} > 1 \) (ISA disease is endemic), the ISA disease-free period 2 farmed population cycle is unstable, and Model (2) has an ISA endemic period 4 population cycle. It is interesting to note that, in this example, the 2-years period of the demographic farmed salmon population cycle is not equal to the 4-years period of the ISA endemic disease dynamics. How do ISA infections influence the 4-years cycle of sockeye salmon?
ISA Model with Only Farmed Salmon and Viral Dispersal

When the farmed salmon population is present in the open net-cages, and the wild salmon population is missing in the nearby open body of water, then $S^W_t = I^W_t = 0$, and Model (1) reduces to the following four-dimensional ISA model with only farmed salmon that are confined to the open net-cages, and with dispersion of ISA viruses between the open net-cages and the nearby open body of water:

$$
\begin{align*}
S^F_{t+1} &= g_F(S^F_t) + d_F S^F_t (\theta_F \varphi_F (I^F_t) + (1 - \theta_F) \varphi_F (V^F_t)) \\
I^F_{t+1} &= \hat{d}_F S^F_t (\theta_F \varphi_F (I^F_t) + (1 - \theta_F) \varphi_F (V^F_t)) \\
V^F_{t+1} &= \hat{d}_V (1 - m_F) V^F_t + \delta_F I^F_t + m_W V^W_t \\
V^W_{t+1} &= \hat{d}_V (1 - m_W) V^W_t + m_F V^F_t
\end{align*}
$$

(4)

In Model (4), the shedding of ISA virus by the infected farm salmon makes the open net-cages a suitable environment for the reproduction of the ISA. However, the absence of wild salmon makes the nearby open water an unsuitable barren environment that is devoid of resources for the reproduction of the ISA. What is the relationship between the basic reproduction number of Model (4), $\mathcal{R}_0^{F,4}$, and that of Model (3), $\mathcal{R}_0^{F,3}$? In the absence of wild salmon in close proximity to an ISA-infected salmon farm, can ISA dispersion eliminate the ISA from the farm? To answer this question, we first note that Models (2) and (4) share the same ISA disease-free $(I^F_t = V^F_t = V^W_t = 0)$ demographic equation, Model (3). Next, we compute the ISA disease-free equilibrium points.

**DFE for ISA model with only farmed salmon and viral dispersal.** Model (4) has only the trivial ISA disease-free equilibrium (DFE),

$$(S^F_0, I^F_0, V^F_0, V^W_0) = (0, 0, 0, 0)$$

when $\mathcal{R}_d^F < 1$. As in Model (2), Model (4) has two ISA DFES, the trivial ISA DFE and

$$(S^F_0, I^F_0, V^F_0, V^W_0) = \left( \frac{\ln \mathcal{R}_d^F}{b_F}, 0, 0, 0 \right),$$

when $1 < \mathcal{R}_d^F$. However, when $\mathcal{R}_d^F > e^{\frac{2}{a_F}}$ it is possible for Model (4) to have an asymptotically positive period $k > 1$ population cycle ISA DFE. To study Model (4) with persistent fixed or cyclic farmed salmon populations, we assume that $\mathcal{R}_d^F > 1$ and the ISA disease-free equation has a unique positive asymptotically stable period $k$ farmed salmon population cycle,

$$\left\{ (s^F_1, s^F_2, \ldots, s^F_k) \right\},$$

where $k \in \{1, 2, \ldots \}$. In the next section, we compute $\mathcal{R}_0^{F,4}$, the basic reproduction number for the four-dimensional Model (4) with only farmed salmon and viral dispersion. The basic reproduction number $\mathcal{R}_0^{F,4}$. Proceeding as in [18] we obtain $\mathcal{R}_0^{F,4}$, the basic reproduction number for the four-dimensional Model (4) with only farmed salmon and viral dispersion. That is, when $\mathcal{R}_d^F > 1$ and the salmon population is present only in the open net-cages, then the ISA disease-free period $k$ farmed salmon population cycle of Model (4),

$$\left\{ (s^F_1, 0, 0, 0), (s^F_2, 0, 0, 0), \ldots, (s^F_k, 0, 0, 0) \right\},$$

is locally asymptotically stable, and the ISA epidemic is eliminated in the farmed salmon population when $\mathcal{R}_0^{F,4} < 1$, where there is viral dispersion. However, $\mathcal{R}_0^{F,4}$ is unstable, and the ISA disease persists in the farmed salmon population of the four-dimensional Model (4) when $\mathcal{R}_0^{F,4} > 1$.

When $1 < \mathcal{R}_d^F < e^{\frac{2}{a_F}}$, assuming that virus shedding is not a new infection, and proceeding exactly as in the case with no viral dispersion, we obtain that

$$\mathcal{R}_0^{F,4} > \mathcal{R}_0^{F,3} = \mathcal{R}_{0d}^{F,3} + \mathcal{R}_{0V}^{F,3}.$$

That is, unlike in the ODE model of [12], in Model (4), $\mathcal{R}_0^{F,4} > 1$ whenever $\mathcal{R}_0^{F,3} > 1$. Thus, in the absence of the wild salmon, linear dispersion of the virus cannot eliminate ISA from an infected farm. However, in agreement with [12], we obtain that $\mathcal{R}_0^{F,4} < 1$ implies $\mathcal{R}_0^{F,3} < 1$. That is, in the absence of wild salmon, the susceptible farmed salmon population in the open net-cages is not a rich enough resource to overcome the deleterious effects of being coupled to the barren nearby open body of water. Mathematically, it is possible to have $\mathcal{R}_0^{F,4} > 1$ while $\mathcal{R}_0^{F,3} < 1$.

$\mathcal{R}_0^{F,4}$ versus $\mathcal{R}_0^{F,3}$. Can viral dispersion stabilize the endemic period 4 population cycle of Figure 3? To study the impact of viral dispersion on Figure 3, in Model (4) we let

$$d^W_F = 0.4, \quad m_F = 0.6 > m_W = 0.001$$

and keep all the other parameters fixed at their current values in Figure 3. In this case, the proportion of ISA virus that disperses from the farmed cages far exceeds the proportion that disperses from the open water to the cages, and Figure 4 shows that the virus clearing in the nearby open water helps stabilize the 4-years ISA endemic cycle of Figure 3 to a 2-years ISA endemic cycle. That is, higher proportion of viral dispersion from the farmed cages is capable of stabilizing ISA virus dynamics via what appears to be a period-doubling bifurcation reversal. To explore the effect of higher proportion viral dispersion from the open water to the ISA infected farm of Figure 3, we let

$$m_F = 0.001 < m_W = 0.6$$
Figure 4. $\mathcal{R}_F^{d,4} > 1$ and Model (4) has an asymptotically stable endemic period 4 population cycle, where

$$d_W = 0.4, m_F = 0.6, m_W = 0.001$$

and all the other parameters are kept fixed at their current values in Figure 3.

Figure 5. $\mathcal{R}_F^{d,4} > 1$ and Model (4) has an asymptotically stable endemic period 4 population cycle, where

$$d_W = 0.4, m_F = 0.001, m_W = 0.6$$

and all the other parameters are kept fixed at their current values in Figure 3.

and keep all the other parameters fixed at their current values in Figure 3. In this case, the proportion of ISA$_V$ that disperses from the farmed cages is much lower than the proportion that disperses from the open water to the cages, and Figure 5 shows that lower proportion of virus dispersion from the open water preserves the 4-years ISA endemic cycle of Figure 3.

**DFE for ISA$_V$ Farmed-Wild Salmon Viral Dispersal Model**

Model (1) has only the trivial ISA disease-free equilibrium (DFE),

$$\Sigma_{00} = (S^F, I^F, V^F, S^W, I^W, V^W) = (0, 0, 0, 0, 0, 0)$$

when $\mathcal{R}_d^F < 1$ and $\mathcal{R}_d^W < 1$. Recall that $\mathcal{R}_d^F < 1$ (respectively, $\mathcal{R}_d^W < 1$) implies the extinction of the farmed (respectively, wild) salmon population in the disease-free equation. Hence, $\Sigma_{00}$ is globally asymptotically stable, and the farmed-wild salmon system collapses in Model (1) when $\mathcal{R}_d^F < 1$ and $\mathcal{R}_d^W < 1$. When $\mathcal{R}_d^F > 1$ and $\mathcal{R}_d^W < 1$, then the wild salmon population goes extinct in the nearby water while the farmed salmon population persists in the open cages. Similarly, when $\mathcal{R}_d^F < 1$ and $\mathcal{R}_d^W > 1$, then the farmed salmon population goes extinct in the open cages while the wild salmon population persists in the nearby water. Consequently, Model (4) is the “limiting” system of Model (1).
whenever either $\mathcal{R}_d^F > 1$ and $\mathcal{R}_d^W < 1$ or $\mathcal{R}_d^F < 1$ and $\mathcal{R}_d^W > 1$. Thus, when $\mathcal{R}_d^F > 1$ and $\mathcal{R}_d^W < 1$, in addition to $\Sigma_{00}$,

$$\Sigma_{F0} = (S^F, I^F, V^F, S^W, I^W, V^W) = \left(\frac{\ln \mathcal{R}_d^F}{b_F}, 0, 0, 0, 0, 0\right)$$

is an ISA DFE of Model (1) with susceptible farmed salmon. Similarly, when $\mathcal{R}_d^F < 1$ and $\mathcal{R}_d^W > 1$, in addition to $\Sigma_{00}$,

$$\Sigma_{0W} = (S^F, I^F, V^F, S^W, I^W, V^W) = \left(0, 0, \frac{\ln \mathcal{R}_d^W}{b_W}, 0, 0\right)$$

is an ISA DFE of Model (1) with susceptible wild salmon. In either case, $\Sigma_{00}$ is unstable. When the salmon population persists on a fixed point or a period $k$ population cycle attractor in the disease-free equation, $\mathcal{R}_0^{F,4}$ determines the stability of $\Sigma_{F0}$ while $\mathcal{R}_0^{W,4}$ determines the stability of $\Sigma_{0W}$.

When $\mathcal{R}_d^F > 1$ and $\mathcal{R}_d^W > 1$, in addition to the unstable trivial ISA DFE, $\Sigma_{00}$, Model (1) has the ISA DFE with susceptible farmed and wild salmon populations,

$$\Sigma_{FW} = (S^F, I^F, V^F, S^W, I^W, V^W) = \left(\frac{\ln \mathcal{R}_d^F}{b_F}, 0, 0, \frac{\ln \mathcal{R}_d^W}{b_W}, 0, 0\right).$$

To study Model (1) with persistent fixed or periodic farmed and wild salmon populations, we assume that $\mathcal{R}_d^F > 1$ and $\mathcal{R}_d^W > 1$, and the disease-free system of two equations has a unique positive asymptotically stable period $k$ farmed and wild salmon population cycle,

$$\{(\bar{s}_1^F, \bar{s}_1^W), (\bar{s}_2^F, \bar{s}_2^W), \ldots, (\bar{s}_k^F, \bar{s}_k^W)\},$$

where $k \in \{1, 2, \ldots\}$. In the next section, we compute $\mathcal{R}_0^6$, the basic reproduction number for the full six-dimensional farmed and wild salmon Model (1) with dispersion.

The basic reproduction number $\mathcal{R}_0^6$. Proceeding as in the previous sections we obtain $\mathcal{R}_0^6$, the basic reproduction number for the six-dimensional Model (1) with both farmed and wild salmon populations and viral dispersion. When $\mathcal{R}_d^F > 1$, $\mathcal{R}_d^W > 1$, and the salmon populations persist in the open net-cages and the nearby open water, the ISA disease-free period $k$ farmed and wild salmon population cycle of Model (1),

$$\{(\bar{s}_1^F, 0, 0, \bar{s}_1^W, 0, 0), (\bar{s}_2^F, 0, 0, \bar{s}_2^W, 0, 0), \ldots, (\bar{s}_k^F, 0, 0, \bar{s}_k^W, 0, 0)\},$$

is locally asymptotically stable, and the ISA epidemic is eradicated in both farmed and wild salmon populations when $\mathcal{R}_0^6 < 1$. However,

$$\{(\bar{s}_1^F, 0, 0, \bar{s}_1^W, 0, 0), (\bar{s}_2^F, 0, 0, \bar{s}_2^W, 0, 0), \ldots, (\bar{s}_k^F, 0, 0, \bar{s}_k^W, 0, 0)\}$$

is unstable and the disease invades the salmon populations when $\mathcal{R}_0^6 > 1$. In the next section, we study the impact of the presence of the wild salmon population in the nearby body of water on the endemic period 2 ISA population cycle of Figure 4.

Illustrative examples in farmed-wild salmon model with dispersion. To illustrate the impact of viral dispersion in the presence of wild salmon in close proximity to the ISA-infected salmon farm of Figure 4, in Model (1), we assume that for each $\Lambda \in \{F, W\}$, ISA infections are modeled as Poisson processes, and let

$$\beta_W = \beta_V^W = 0.01, \quad d_V^W = 0.7,$$

$$\delta_W = 0.01, \quad b_W = 0.1, \quad d_W = 0.8,$$

$$\mu_W = 0.1, \quad \delta_W = 0.01, \quad$$

and $m_F = 0.6 > m_W = 0.001$, where all the other model parameters are kept fixed at their current values in Figure 4. With our choice of parameters,

$$\mathcal{R}_d^F > e^{\frac{\beta_W}{\delta_W}}; \quad \mathcal{R}_d^W = \frac{r_W}{d_W} > e^{\frac{\beta_W}{\delta_W}}, \quad \text{and the ISA disease-free equation for the farmed salmon population is a unique positive asymptotically stable period 2 population cycle (see Figure 1), while that of the wild salmon is a unique positive asymptotically stable period 8 population cycle (not shown).}

ISA has caused high death rates in marine farmed Atlantic salmon, but the virus has not been associated with die-offs in wild salmon populations. We capture this with our choice of parameters, $\mathcal{R}_0^{F,3} > 1$, and Figure 4 shows that the pre-dispersion dynamics of the isolated farmed salmon is an ISA endemic period 2 population cycle. However, without viral dispersion, ISA is cleared in the isolated wild salmon population and $\mathcal{R}_0^{W,3} < 1$ (not shown). When the proportion of ISA that disperses from the ISA enzootic farmed cages far exceeds the proportion that disperses from the open water to the cages ($m_F = 0.6 > m_W = 0.001$), Figure 6 shows that it is possible for the viral-linked full system to change from an ISA-free wild salmon population without viral dispersion to an ISA-infected wild salmon population with dispersion, where $\mathcal{R}_0^6 > 1$. However, when we set $m_F = 0.001 < m_W = 0.6$ and keep all the other parameters fixed at their current values in Figure 6, then the proportion of ISA that disperses from the farmed cages is much lower than the proportion that disperses from the open water to the cages. In this case, dispersion does not alter the pre-dispersion source-sink dynamics (not shown). That is,
in the event of an ISA outbreak in a farmed salmon population, a higher proportion of ISA viral dispersion from the infected farm could put the nearby wild salmon population at risk of the ISA disease.

Figure 6. \( R_0^v > 1 \) and Model (1) has an asymptotically stable ISA endemic period 6 in both farmed and wild salmon populations, where \( \beta_W = \beta^v_W = 0.01 \), \( d^v_W = 0.7 \), \( \delta_W = 0.01 \), \( b_W = 0.1 \), \( d_W = 0.8 \), \( n_V = 60 \), \( \mu_W = 0.1 \), \( \delta_W = 0.01 \), \( m_F = 0.6 > m^v_W = 0.001 \), and all the other model parameters are kept fixed at their current values in Figure 4.

Concluding Remarks
Salmon populations are known to exhibit patterns of cyclic dominance of subpopulations. To study the impact of ISA viral disease and the patterns of cyclic dynamics in salmon populations, we introduce a viral dispersal-linked farmed-wild salmon ISA discrete-time infectious disease model with intrinsically generated demographic population cycles (cyclic disease-free dynamics). Unlike the logistic growth dynamics in the ODE model of [12], the susceptible farmed and wild salmon populations in our model exhibit cyclic dynamics via the Ricker growth or recruitment (or birth) functions. Furthermore, in our ISA model, the proportion of ISA virus that disperses from the farmed cages to the open water is not equal to the proportion that disperses from the open water to the cages.

We compute the basic reproduction number, \( R_0 \), for ISA disease elimination or invasion. When \( R_0 < 1 \) and the Ricker demographic equation has a locally asymptotically stable \( k \)-years population cycle, we prove the local asymptotic stability of the dispersal-linked farmed-wild salmon ISA disease-free \( k \)-years cycle (with and without the wild salmon present). That is, the ISA is eliminated in the farmed and wild salmon populations whenever \( R_0 < 1 \). Also, under the same \( k \)-years Ricker demographic assumption, we prove that the ISA disease-free \( k \)-years population cycle is unstable and the ISA invades the farmed and wild salmon populations when \( R_0 > 1 \).

In the absence of wild salmon in close proximity to farmed salmon cages, we illustrate in Figures 3 and 4 that higher viral dispersion from ISA-infected farmed cages is capable of stabilizing ISA cyclic disease dynamics via what appears to be a period-doubling bifurcation reversal, while lower viral dispersion from the cages preserves the predispersion ISA enzootic cyclic dynamics (see Figure 5). Furthermore, in Figure 6, we illustrate that higher proportion of ISA viral dispersion from an ISA-infected salmon farm could put the nearby wild salmon population at risk of the ISA disease. Whenever there is an ISA disease outbreak on a salmon farm and data for estimating the model parameters is available, our computed \( R_0 \) can be used to determine when intervention measures will have the ISA disease outbreak under control. Theoretical results on the relationship between the period of the Ricker demographic ISA disease-free period \( k \) salmon population cycles and the period of the endemic cycles of the viral dispersal-linked farmed-wild salmon ISA disease models are open.

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References


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Credits

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The Early Career Section offers information and suggestions for graduate students, job seekers, early career academics of all types, and those who mentor them. Angela Gibney serves as the editor of this section. Next month’s theme will be dealing with challenging issues.

Communicating Mathematics
Math Under a Minute

Izzet Coskun

The party was rowdy, lousy with graduate students. I found myself in a corner next to a guy from the business school and his somewhat silent sidekick. He asked me what I do. You know how it goes: “I’m in math.” “What kind of math?” “Algebraic geometry—I solve polynomial equations using geometry and symmetry.” “Is that useful for anything?”

After mention of physics, economics, computer science, ... “Yes, but what do you study?” he demanded.

I didn’t think about this at the time, but of course a person you meet at a party in grad school may end up with a job where they play a role in deciding on funding for the National Science Foundation, or have an impact on mathematics in some other capacity. Maybe they’ll have a talented kid and think about encouraging them to study mathematics.

I tried to get through it. I told him that I study the geometry of spaces with lots of symmetries. That a good example is the sphere. Under rotation, any point on the sphere looks like any other point. I pantomimed rotating a sphere. “Homogeneous spaces are higher dimensional analogues that similarly have a large symmetry group so that every point looks like every other point. I study the geometry of these spaces.” He leaned closer. I could smell the alcohol on his breath. “What do you want to know about these spaces?” I continued with my account: “I count spheres on these spaces that satisfy constraints like passing through a set of points.” He didn’t really believe me—he asked me why I wouldn’t tell him what I really do. “Do you think I won’t understand?” I gave in, admitting the truth: “I study the genus zero Gromov-Witten invariants of homogeneous varieties like Grassmannians using degenerations of scrolls.” I could see his arm muscles tense, fists ball up. I thought he would lunge at me. His sidekick, quiet before, intervened to suggest they go and get a beer, ending my misery.

It turned out OK that night; I didn’t get punched. There is, of course, no irony in that years later I find myself giving advice on how to communicate mathematics quickly in informal settings.

Whether before a promotion committee, at a party where one might meet future politicians or future parents of future colleagues, in the elevator on the way up to tea, or in the dean’s office at a job interview, we often have the opportunity to explain our work to a general audience. The time we have is usually short (even my parents will edge their way to the dance floor after listening to me for fifteen minutes on higher codimension cycles). Our audience will not be familiar with our terminology. Communicating mathematics in such settings is challenging.

Try to be engaging and present your material at an accessible level. Avoid jargon and give the basic ideas with technical details appropriate to the situation. To explain specializations in algebraic geometry at a holiday party at my wife’s law firm, I might say something like

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“Specialization is the art of simplifying a problem until you solve it.” If pressed I may add that you can vary polynomial systems by changing the coefficients of the defining polynomials. “The system might be easier to solve for some special coefficients—for example, if many of the terms of the polynomials are zero. I study how geometric properties of a complicated system can be deduced by studying simpler systems.” I avoid any mention of flatness or schemes.

A simple example often conveys what we do more effectively than our latest technical progress. I once gave a one-minute presentation on my work at a Kavli Symposium of the National Academy of Sciences to a group of accomplished young scientists. I spoke about the problem of using specialization to find the number of lines intersecting four general lines in space. I asked the audience to visualize the problem and then I showed them a movie of the specialization. When two of four lines intersect, finding the two solutions is relatively easy and the audience was comfortable with it. By the time I said that I study higher dimensional analogues of this problem, my minute was up.

Pictures are potent communication devices. We can all be envious of colleagues who study complex dynamics and can show beautiful renditions of Mandelbrot or Julia sets, especially ones with names like Basilica, Rabbit, Airplane...

Canny phrases can be effective in describing one’s work. These evoke the main idea of the proof or the essence of an object at hand. “The dog on the leash theorem” or “train tracks” are good examples. Such slogans call to mind an idea, avoiding the accompanying long explanation.

Even if we find it hard to describe our most recent results, we can try alternatives that convey aspects of it. Applications, even if not implied directly by our own work, can be illustrative. We can explain a simple toy problem, even when this problem does not exhibit the main technical difficulties we face.

It can be fun to bring up certain problems, especially those you know may elicit strong reactions or whose solutions are counterintuitive. I have had spirited arguments over the Monty Hall problem. When people get upset, try not to lose your cool, acknowledge the solution may appear unreasonable at first glance, discuss more intuitive variants.

It is of course always important to be respectful of people. Most will have had a brief and often fraught exposure to mathematics. Nevertheless, they are often genuinely interested in hearing about the work that we do and will make a good faith effort to understand what we tell them.

Do not appear arrogant, condescending, or not willing to try. While sometimes we find ourselves at a party with a drunk jerk who isn’t going to remember what we tell them anyway, most of the time there is an opportunity for honest communication. With some forethought, there is no need to switch to physics to do a reasonable job!
social media presence informed by the comments of my followers.

Consuming and sharing mathematics. When I look at my Twitter feed, I see mathematical discussions, links to research articles, links to mathematics in the news, information about books on mathematics, problems and puzzles, photos of mathematical artwork, pedagogical conversations, career advice, conversations about equity in the profession, and real-time glimpses of conferences and workshops.

One can get a lot out of social media just by being a consumer of information—being a “lurker,” as they say. But to get the most out of social media, it is important to engage in conversations. That is the “social” part of social media. So, make a Twitter account, start following people, and begin interacting. Post photos of found mathematics on Instagram. Make a video or podcast. Start asking and answering questions on one of the math-related Stack Exchange sites—MathOverflow, Mathematics, Mathematics Educators, and TeX.

For those who are so inclined, I strongly recommend starting a blog. My blog is available for anyone to read, but my main audience is myself. In the January 2019 Early Career column, Robert Lazarsfeld wrote about the benefit of keeping a mathematical journal ([Laz19]). That is great advice, and I treat my blog as a public journal. When I learn some new piece of mathematics or have a teaching experience that I think went particularly well, I write about it on my blog. Then the page is there for me the next time I need the information, and I get to share the idea with others. The commenters often provide ways to take my idea even further or point me toward resources I did not know about.

In his March 2019 Early Career column, Jordan Ellenberg encouraged mathematicians to write outward-facing mathematics ([Ell19]). Writing books, articles, and editorials for the general public is one way to do that. But social media is a much easier way to communicate mathematics to people who might not be mathematicians.

For years, the web was not conducive to presenting mathematics because LaTeX was unavailable. This is no longer a barrier. There are many ways to include beautiful-looking mathematics on the web. In fact, one of the great strengths of sharing mathematics online is that it opens up so many possibilities that are unavailable in print: videos, interactive graphing programs, animations, applets, downloadable files (such as 3D-printing files and data), and so on. Space is not nearly the issue online that it is in print. Plus, publication happens instantly.

Blatant self-promotion is awkward, but we would like people to read our articles and books. Social media provides a natural way of building a network of people who share your interests. They will follow along as you conduct your research, and when it is finished, they will be interested to hear about it. It happens organically. Also, social media provides an easy way to promote the work of your friends, colleagues, collaborators, and others.

One objection to blogging or tweeting about your scholarship is that someone might steal your work. Here, I suggest using common sense. Getting people excited about your work, getting feedback from other scholars, and finding potential collaborators are great benefits of writing about your work online. But it may not be wise to share work that is the key to a future publication.

Professional development. Like many mathematicians, I care deeply about teaching. I recognize that there are many ways to teach effectively, and there is always room for improvement. I follow people online who also care about and have thoughts about teaching. I have witnessed conversations about inquiry-based learning, standards-based grading, flipped classrooms, teaching using primary sources, and more. I have been prompted to rethink the way I teach certain material, and I have acquired new examples to use in my classes that either wow the students or target a known problem-point for new learners. I have been introduced to new technology that can be used to teach more effectively and efficiently. I have debated which textbooks to use (or not use) in a particular class. I have also been the recipient of the generous sharing of teaching materials.

There have been important conversations about diversity and equity in mathematics and in academia more generally that have gotten me to rethink how I interact with my colleagues and my students.

The best cocktail party. The internet is a place where we can rub shoulders with Fields medalists, best-selling authors, award-winning teachers, and experts in every area of mathematics. It is exciting to eavesdrop on their conversations and to correspond with them directly. But there are many other interesting mathematicians online: professors at research universities, teaching institutions, liberal arts colleges, and community colleges; high school and middle school teachers; students at all levels; mathematical artists; applied mathematicians; pure mathematicians; statisticians; data scientists; mathematics enthusiasts; historians of mathematics; textbook authors; popular book authors; puzzle makers; journalists; people from every corner of the world; engineers; physicists; computer scientists; and people in the business world. There are women, men, people of color, members of the LGBTQ community, introverts, and extroverts.

Find your community, whether it is people in your research area, women in mathematics, people who enjoy discussing pedagogy, members of the LGBTQ community, or mathematicians with your religious or political views. Or better yet, build a diverse community so you can hear all of these voices. Social media allows us to interact with people at various stages in their careers, in different geographic locations, with different backgrounds. For instance, even though I teach at the college level, I find it very interesting and inspiring to follow high school mathematics teachers.
One criticism of the modern web is that it creates silos—filter bubbles that prevent people from seeing outside their own worldview. I’m sure that is true in some cases. But in the mathematics community, I see the opposite. It is easy to see why a graduate student might think that mathematics happens only at R1 research universities and that anyone who does not end up at such an institution is not a mathematician. But through social media we see the true diversity of mathematics and of mathematicians. We can be successful in many more ways than by solving a famous open problem or by producing PhD students. I have been inspired by and have learned from middle school teachers, parents who run math circles, mathematical artists, and other people not considered to be “mathematicians.” The sense of community is real and powerful. In fact, these social media platforms humanize the superstars. We see that they live ordinary lives, they make mistakes, they have gaps in their knowledge, and they care about students, the profession, and the world.

Social media is a great resource for isolated academics—mathematicians who do not have access to a local research community. Many research collaborations emerge out of social media connections. Social media duplicates some of the benefits of attending conferences, which are becoming increasingly more difficult for many academics to attend, whether it is for financial, personal, or geographical reasons.

Words of warning. The online world can be toxic, and we make ourselves vulnerable by putting ourselves out there. It can be especially bad for women, people of color, members of the LGBTQ community, and individuals in other underrepresented groups. Before writing this article, I reached out to friends who are members of these groups. They reported many more positive experiences in mathematical social media than negative ones. The one exception was MathOverflow, in which the ability to downvote questions and answers, to close conversations, and to comment on responses make it an unwelcome place for some of them.

My personal golden rule for social media is that it should not be a new source of stress. If a Twitter user tries to start an argument, I don’t reply or I block the user. There’s a familiar internet warning: “Don’t read the comments.” I would not offer that advice in mathematical social media; some of the best ideas come out of online conversations. However, comments can be hurtful, especially when a blog post goes viral and trolls come out of the woodwork. In such cases, ignore the comments, delete them, block the commenters, or turn off the commenting function for the page altogether.

There is also the possibility that a social media user could damage his or her personal or professional life by posting an inappropriate joke, photo, or comment. We warn our children to be careful about what they post online, but it is important for us to remember this as well. Do not post something on social media—regardless of what you think your privacy settings are—that you would not want to be in the New York Times, in the newsfeed of all of your students, or in the hands of your provost.

Who to follow. I have avoided recommending specific blogs to read, people to follow on Twitter, or YouTube channels to watch. There are many good options, and each person should build his or her own circles. But as a start, one might visit truesci phi . org, which has lists of mathematicians on Twitter and mathematicians’ favorite Twitter feeds. Also, mathematicians may enjoy the Twitter hashtags #mathchat and #mtbos (“math Twitter blogosphere”). For blogs, one might peruse the AMS’s “Blog on Mathematical Blogs” (blogs.ams.org/blogonmathblogs), which is updated regularly. Or, for a one-stop-shop, the MathFeed app.

References

Greg Kuperberg
Using the arXiv

Although these days virtually everyone in the mathematics profession knows something about the arXiv, a small introduction may still make sense before turning to advice about using it. (First of all, it is officially just “arXiv,” but many people like to say “the arXiv.”) I am in the latter camp.

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I’ll start with the most general point that comes to mind. Since the arXiv has an extensive online help system, that capacity, I was invited to contribute this Early Career piece. Since the arXiv has an extensive online help system, I will focus on advice rather than instructions for using it. I’ll start with the most general point that comes to mind.

Practicalities should come before opinions. In its twenty-eight years of existence, the arXiv has always attracted many opinions. People have ideas about endorsing the arXiv, or extending it, or criticizing it. Or more radically, some people boycott the arXiv, while others dream about supplanting it with something better. All of this is perfectly valid for discussion, but I recommend first considering how the arXiv can be useful to you as a mathematician. Particularly in the early days, the arXiv’s user interface was on the forbidding side. Although that has improved, the arXiv has grown relentlessly because people need it, not because they like it. (Although many people like it a lot.)

Reading the arXiv

With that general point out of the way, here are a few thoughts about reading the arXiv.

Subscribing to the arXiv can be excellent for your development... Whether by email subscription or by browsing the daily announcements, recent arXiv listings are a great way to experience the drama of new research in mathematics. This advice might only be fresh for graduate students. If you are a graduate student, you might want to try it. Although it’s not really all the research in the world in any category, it’s a comprehensive portion, and it is more likely to have the best new results. It is impossible to understand everything posted, but it’s still very exciting.

...but don’t overdose on announcements. Although there are roughly thirty arXiv categories in mathematics, the average math category still gets more than 1,000 submissions per year. Again, each category is especially likely to carry the best results in the field. If you have subscribed for a while, and especially if you are on the job market, it’s best to calm down and not feel that you have to keep up with everything. Of course you don’t.

Egogoogling has its benefits. If you already have a publication record, then I see nothing wrong with looking for your own name in the arXiv from time to time. I think that getting cited or mentioned is a reasonable form of encouragement, and people who cite you are often doing research that is relevant to yours. The arXiv has a full-text search facility; or you can look in Google Scholar; or you can do a Google search of the form "bourbaki inurl:pdf/1901 site:arxiv.org."

Submitting e-prints

Here are rather more pointers about submitting e-prints to the arXiv.

There are many good reasons to submit to the arXiv... The main criteria for suitability of an arXiv submission are fairly simple: Does the submission have the basic format of a research document, and is it plausibly interesting for a research audience in its category? This criterion is similar to publishability in a journal, although it is not exactly the same; for instance, it includes PhD theses. It also doesn’t have to be a final draft (although good drafts are generally preferable). The main consequence of submitting an e-print is also straightforward: People will see it and read it! It is not uncommon to get unsolicited email expressing interest in your work after you submit to the arXiv. It is also particularly convenient to submit an e-print before giving a talk, since interested listeners will retrieve it afterwards (or even during your talk in some cases).

Even if submitting to the arXiv isn’t traditional for your advisor, or your coauthor, or your department (all of which still sometimes happen), it is still a valuable method of scholarly communication.

...but don’t be cavalier. Although the submission standards for the arXiv are more flexible than for journals, the arXiv is not a free-for-all for stunts, would-be blog posts, and wild experiments. Moreover, although revising an arXiv submission is a perfectly normal practice, all public versions are permanent. The best principle is simply to respect the audience, as you would when giving an invited seminar.

Limited pre-arXiv circulation can be sensible. Okay, you think that you proved an important theorem; but is it possibly too good to be true? Instead of rushing to post it
before your excitement fades, you can consider limited cir-
cumstance, the best step is to document the shared material
pages in common with your PhD thesis. In any such cir-
exceptions; for instance, you may have a journal article with
reason that the arXiv has text overlap labels. But there are
from another paper by you or anyone else. This is one
course, it is usually poor scholarship to reuse a lot of text
should't change it for small reasons.
e-print carefully before you submit it to the arXiv, and you
same document. So you should choose the title of your
when two documents are meant to be two versions of the
calculation among close colleagues. I don’t think that anyone
be outright scared of submitting to the arXiv; every-
one accepts that to err is human, and arXiv submissions
can be revised. Moreover, papers with mistakes are often
partially correct or otherwise still valuable. Rather, you
should remember that you don’t need to wait for the for-
mal referee process to get some useful independent review.

Use arXiv identifiers for both published and unpublish-
ed references. Many people see it as traditional to
give the arXiv number only for references that are not yet
published in journals and thus only the journal citation for
published papers. For various reasons, it’s better to include
the arXiv number for every bibliography entry that has
one. It makes your bibliography more useful. You should
also follow the standard format of an arXiv identifier,

Use the daily deadline as a breather, not a race. As
the arXiv instructions explain, the submission deadline for
the immediate next announcement is always 2pm, Eastern
time, excluding weekends and holidays. Although you may
feel eager to rush the submission before this deadline, this is
usually a mistake. It’s usually wiser to finalize a submission
after the cutoff rather than immediately before it to give
yourself about a day to check for mistakes.

Take your time with the metadata and the preview. In
the old days, the arXiv had a nerve-racking interface with
no preview page, but now fortunately it has one. The sub-
mission system lets you double-check everything, and you
really should. It is easier than you think to rush forward
with a missing bibliography, with a spelling mistake in
your title, etc.

Choose categories according to readership, not taxa-
onomy. The arXiv is not just an automated digital library,
but also a living social network. You should choose the
classification for submission based on who would be in-
 terested in your e-print (with some restraint), rather than
which concepts it mentions. For instance, just because you
use groups in your paper, that does not necessarily make it
a group theory paper.

Keep your paper titles in sync. For all of the work that
goes into document identifiers and bibliographic rec-
ords, a stable title for an e-print or a paper matters more
than ever. It matters for both people and software to decide
when two documents are meant to be two versions of the
same document. So you should choose the title of your
e-print carefully before you submit it to the arXiv, and you
shouldn’t change it for small reasons.

Notate any major reuse of text in the comments. Of
course, it is usually poor scholarship to reuse a lot of text
from another paper by you or anyone else. This is one
reason that the arXiv has text overlap labels. But there are
exceptions; for instance, you may have a journal article with
pages in common with your PhD thesis. In any such cir-
cumstance, the best step is to document the shared material
yourself in the comment field. If you put a standard arXiv
identifier here, it will get hyperlinked to the other e-print.

The TeX source is public. It is good practice to clean
up your LaTeX and remove vestiges that you don’t want
other people to see. You are also always free to download
the source of any arXiv e-print, if you want to work from
examples to better understand LaTeX document classes,
style files, TikZ, commutative diagrams, etc.

You can include code and data as extra files. If your
work is computer-assisted and you have supporting code or
data, then it can be a good idea to include it with the TeX
source of the submission. (Within reason—don’t do this
if it runs into gigabytes.) In fact, attached files are better
than appendices with huge, unusable data tables. You can
include a special file, “00README.XXX,” to get the auto-
compiler to ignore these files.

Seek solutions before blaming the system. Although it
has gotten much better, submitting to the arXiv is still a bit
forbidding, above all because it has a tiny staff in relation
to its enormous user base. If you feel stymied while trying
to submit something reasonable, then there is probably a
workaround. Studying the help system and asking other
arXiv users is a good idea. If that’s not sufficient, you can
send help email to the Cornell staff; the only problem is
that the staff is small.

Share the password with coauthors, and add pub-
ication information. It may seem that your e-print no
longer needs your attention after you submit it and it has
appeared, but there are some side errands to keep in mind.
For one thing, you may want to revise it later, but a couple
of other steps are also good practice. You should share
the password with your coauthors so that every interested
coauthor can share ownership. Also, if you later publish
the e-print in a journal, it is good practice to add a journal
reference in the journal-ref field. (These extras might one
day be more automated; for now, they are usually but not
quite always on you.)

Revise your arXiv e-prints, but sparingly. The ability to
revise an arXiv e-print is an important part of the system.
It is silly to imagine that the first arXiv version needs to be
perfect. At the other extreme, it’s equally spurious to inter-
pret the arXiv version as just a draft and the journal version
as "official." You help the readership when you revise your
arXiv e-prints. In fact, if you find a mistake after your paper
has been published, you can make the arXiv version more
accurate than the journal version. You should just be careful
to plan ahead and collect corrections before revising an
arXiv e-print. Only the first five versions are listed in the
daily announcements, although the most recent version is
always the default one on the web. Also, the first version
usually gets the most attention in the short term.

Other Thoughts

Although one of my points is that practicalities should
come before opinions, in the end there is also time for
opinions, activism, and thoughts about the future. I certainly think that the arXiv in general and the math section in particular are as important as ever. If you want to actively support the arXiv, then one valuable form of participation right now is to serve as a math category moderator. (If you have early career concerns such as getting a job or getting tenure, those would ordinarily take precedence over serving as an arXiv moderator.) Looking to the future, I see the math arXiv as an unfinished effort, no longer mainly because participation is less than 100 percent, but above all because the journal publication system is still roughly the same as it was in the twentieth century. (Journal articles are now submitted and published online, but other basics such as journal titles and paid subscriptions are still traditional.) I think that the peer-reviewed layer of mathematical communication will be modernized in an effort parallel to the arXiv, although not necessarily directly as part of the arXiv. However, this will be a major reform and it remains to be seen how it will happen.

Where to Submit Your Paper

Chuck Weibel

If you are early in your career, and have just finished writing a paper, you will want to get it published. However, you probably don't have a handle on where to submit your masterpiece. This is a very important decision, since your nascent career probably depends heavily on accepted publications to get jobs and get promoted.

Here is a list of dos and don'ts, based upon the assumption that you don't have tenure and are within five years of your PhD.

Do ask for advice! The best advice I can give you is to talk with a senior faculty member about which journal to submit your article to, and take advantage of their experience. Your advisor is not the only person you should ask; it helps (but isn’t crucial) if that person knows your field.

When I was a graduate student, I had a woefully bad understanding of where to publish my first paper. Thankfully, after hearing some of my ideas about this, a sympathetic faculty member (Paul Sally) sat me down and helped me decide where to submit it. He knew nothing about the subject area, but he had experience with submitting papers.

Do tell what you’ve done on page one! One of the worst mistakes I see authors make is to postpone telling the reader what the punchline is until page 3 of their paper. Don't begin with a long history of the context of your main result—tell the reader what you’ve proven, and only then explain why the reader cares. It is even better if you announce this in the first fifteen lines. If your result uses special terminology, explain the terminology immediately after stating your result. You can put your result into context after the reader knows what it is.

Editors choose referees, and make accept/reject decisions, based on how well the paper sells itself. Since they frequently only read the introduction, and often only the first page, that has to be where they see what is great about your paper. (My apologies to any diligent editors reading this. I’m speaking in general terms about human behavior.)

Follow the crowd. Do think about which journals have published similar papers in the same subject. The “Citations” link for reviews of these papers (and other papers by their authors) in MathSciNet is a very useful tool for getting a list of journals that may be appropriate for your paper. In many cases, you may want to submit to a “niche” journal like the Journal of X. (X can be Algebra, Combinatorics, Topology, Functional Analysis, Linear Algebra, Differential Equations, etc.)

Don’t go for broke! Do not submit your paper to a top journal unless you have solved a really famous outstanding problem. Although you might get lucky with a quick decision, which is always a rejection, the more common result is a rejection after eight months or more. At that point you will have to revisit the “where?” problem.

Delaying the time before you get credit for your work can have real-world negative consequences for you. In boxing terminology, when you submit above the weight class of your paper, you hurt your career.

How to relocate. Suppose that your paper is rejected. Now you have to go through the process all over again. But don’t be discouraged! If you are lucky, the referee will propose a more appropriate journal for your rejected paper, and the editor may pass along this recommendation with a promise to share the referee report (and sometimes the referee’s identity) with editors of the new journal. This is great for you, because you don’t have to wait very long for a referee report, and it is great for the community of referees, because it avoids duplication of effort. This referral process
is especially common in general-purpose journals, which often redirect your paper to a “niche” journal.

Some publishers are now pushing the idea of internally “cascading” peer-review of papers. This means that authors are encouraged to submit their rejected papers to another journal owned by the same publisher. For you, this has the advantage of speeding up acceptance of your paper; for the publisher, this has the advantage of capturing manuscripts within their portfolios. The disadvantage for you is that your paper may be redirected to an inappropriate journal. If you think this has happened to you, ask a colleague if they think it is an appropriate journal.

Don’t use impact factors! Journals with high impact factors are not necessarily considered “top journals” for promotion purposes; like any metric, impact factors are often “gamed” by publishers.

Certain young researchers (not in the US!) are paid by the sum of their publications, weighted by impact factor. This is a horrible system as it results in papers being sequentially submitted to a chain of journals, spiraling down until they reach the appropriate level. If this applies to you, you have my sympathy. My advice to you is simple: get advice from a senior colleague!

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Credits
Author photo is courtesy of the author.

Journal Refereeing: Merge with the Collective Mind

Ken Ono and Robert Schneider

Imagine you are a medieval alchemist. You devote your life to uncovering hidden truths, expressed in a poetry of esoteric symbols and terminology. Today a manuscript of new discoveries has made its way into your hands from an unknown author. Will this delicate codex unlock the enigmas that drive your work, or spark an explosion of ideas in other seekers? Who is this anonymous soul-mate sharing your own rare passion? It now becomes your quest—and your honor—to decipher the mysterious treatise.

At this point you may think the authors have played one too many a Dungeons & Dragons campaign, or recently binge-watched fantasy B-movies online (call it research for this article). But if you replace “medieval alchemist” with the word “mathematician” in the opening sentence, the paragraph now describes you yourself receiving a paper to referee. We offer here our thoughts about navigating this singular scenario, in which your judgment may shape the future history of your field. Like the alchemist in the opening paragraph, and just as romantically, as a journal referee you are in the position of secretly knowing new theorems (even entire theories) years before they officially enter the literature. Moreover, you are invited to help shape the literature of your era.

What is the job of the research journal referee? In a nutshell, you will:

• Check the work to verify the ideas and equations are correct.
• Offer advice and raise questions to help clarify or strengthen the arguments (if a result is promising, one should give authors the opportunity to revise).
• Offer suggestions for improving exposition and overall presentation of the piece.
• Finally, write a referee report including a summary of the paper, a list of corrections and suggestions for the author(s), and an evaluation of appropriateness for publication in the journal (we note that referees do not make final editorial decisions, merely recommendations).

The first three bullet points fall somewhere between editorial work and collaboration; we caution that the last can be misinterpreted as the charge to be a guard or gatekeeper. We urge you to lean in a different direction: we should encourage each other in our work.

As to how one should evaluate a new result, we offer solid advice from two of our heroes. G. H. Hardy is well known to have instructed referees for the Proceedings of the London Mathematical Society to use the following guidelines.

Hardy’s criteria for refereeing. One should ask three questions of the result:

• Is it new?
• Is it true?
• Is it interesting?

1Preprint servers, such as the arXiv, serve an important “unofficial” role in mathematical publishing.

Certainly these seem self-evident as minimal standards for publishing any piece of nonfiction. Equally succinct and perhaps more inspiring are criteria that former AMS President G. E. Andrews has been known to mention privately (for instance, once to the second author) as his own rules-of-thumb.

Andrews’s criteria for refereeing. Does the result satisfy at least two of the three questions:

- Is it surprising?
- Is it elegant?
- Is it useful?

These idealistic rules preserve the practical simplicity of Hardy’s criteria, yet place aesthetics in the foreground. We would like to posit that if a paper satisfies even one of Andrews’s criteria, it soars above almost all other instances of human activity, and is worthy of praise.

There is further advice for referees in the great article by Arend Bayer in the March 2019 issue of these Notices. As supplementary rules-of-thumb, we suggest that referees:

- Always first seek the beauty and importance in a paper.
- Try to really understand the author’s goals—one cannot review a paper without first seeing these.
- Give every paper a fair chance.
- Remember that authors work hard to conceive and write a paper, so never write a report after a single glance.

Refereeing is usually discussed in the context of service to the mathematics community (which is very thankful to you, on that note). But in addition to giving up one’s own time, we gain a lot from our refereeing work, too. Refereeing a paper shouldn’t be taken on like a homework assignment. This is an opportunity, not just to serve your community, but to merge with the collective mind of mathematics and participate in its creative process.

Speaking personally, the authors experience anticipation with every new paper we agree to review: we have felt the excitement of recognizing a groundbreaking result (surprisingly, not always right away), the sense of revelation, and the eventual pride of having played a nurturing role when it debuts. And along with the whole mathematics community, we are swept up in the wave of new research and enthusiasm triggered by new advances. So of course, we are on the lookout for the next potential mind-blower to show up by email!

Then like our imagined alchemist, be curious of the contents. Approach each submission in the sincere hope that it may eventually appear in print. For the sake of others in your field, you have the responsibility to see what is valuable in it, and to help revise it so its value will be readily recognized. Read carefully and comment respectfully. And if you should not recommend to accept a paper, be clear that the piece is just not appropriate for the present journal—not that it is being rejected from the literature.

It may be through you alone that important work will find its audience.

Ken Ono
Robert Schneider

Credits
Ono author photo is courtesy of Ken Ono. Schneider author photo is by Mike Colletta.

Design and Construction of Mathematical Posters

Anya Michaelsen

When making a poster, or any kind of presentation for that matter, it’s important to keep in mind that your primary goal is to communicate. This one guiding principle can help inform decisions from “How do I structure it?” to “Should I include this equation?” and “What should my title be?” So with this in mind, there are three key components to creating an effective poster:

- **Structure**—appearances and organization.
- **Content**—details to include and wording.
- **Logistics**—the nitty-gritty, putting it together.

**Structure**

Despite the saying, people do tend to judge books by their covers, which is why the structure of your poster, its “first impression,” is one of the most important aspects of your poster. The first aspect that anyone will notice about your poster is its structure and overall appearance. If they see a wall of text or poorly formatted equations, that’s an immediate turnoff. You want the first glance at your poster to invite someone to come up and read it or talk to you, not scare them off. After all, you can’t communicate if no one comes to talk to you. How do you do this?

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Organize your poster into sections. Just like a paper or presentation, you want your poster to have clear and defined sections to guide your reader, which is usually done with boxes and titles. Typically you want the content to flow top to bottom and then left to right. Some possible sections (in order) could be:

- **Introduction or Motivation**—set up the question or problem and provide relevant history.
- **Background (general and/or specific)**—present needed terms and theorems.
- **Results and Justification**—state your results and any supporting proofs, derivations, or data.
- **Discussion or Further Research**—discuss implications or pose future questions.

Include white space. It may seem counterintuitive, but what you don’t put on your poster can be as important as what you do. If you have text, keep it short; a long dense paragraph is daunting and unlikely to be read. Use bullet points or lists, where possible, to make the information easy to find and interpret. Use headings and subheadings if applicable. Include space between titles, text, equations, and figures so information isn’t too densely packed.

Use color, bold, italics, etc. Use color¹ and text formatting to draw the audience to your key points. Highlight terms your reader needs to know and use visual indicators to set these apart. In a definition or construction, bold the term being defined. If you have a lot of theorems and definitions, consider color coding so a reader can quickly find the highlights.

Include figures or pictures if possible. If there is a diagram, plot, or other figure that can be used to communicate or illustrate some part of your work, include it in your poster. Figures are a great way to break up text and give your audience a visual framework to understand your result. You can potentially construct a flow chart or diagram to communicate some of the ideas or relationships between concepts. If you can’t construct a relevant graphic, don’t worry, use more equations or a photo of the researchers!

Use the structure to convey content. When you are starting out, think about any internal structure or organization of your work; the organizing boxes are a great example. You can also use structure in other ways to communicate content. Do you have two similar proofs or constructions? Consider using parallel boxes for them, lining up corresponding pieces visually on your poster. Does your proof have three major parts? Try separating them into three columns within a box, or even into three separate boxes. Do you have two big lemmas and then a theorem bringing them together? Put the lemmas into parallel boxes, with the theorem in one big box below them, mirroring the way your lemmas funnel into your main result. Using these and similar ideas to structure your poster can further guide your reader or audience’s understanding.

Content

Now that you have some general ideas for how to structure your poster, it’s time to figure out what actually goes in it. Before you start sketching/typing/drawing, sit down and ask yourself:

- What are the key ideas/results I want to convey?
- What math is needed to understand my results?
- Who is my audience? What is their background?

Know your audience. Who are you talking to determines a lot of how you will talk to them. So it’s important to know the intended audience of your poster. Who is your intended audience? What background will they have? This will inform how much background you should aim to cover.

Will your audience have any specific knowledge of your field? If so, your background section can be more tailored to your research. If you have a mixed audience or a more general audience, you could split the background into an “Advanced/Specific Background” section and a “General Background” section that could be skipped as needed. Will they be looking at your poster without you or will you always be there to explain it? If your poster needs to stand on its own, make sure it’s as self-contained as possible.

Communication is key. At the end of the day, it’s important to remember that the purpose of your poster and your presentation of it is to communicate. Using accessible language, having the appropriate background, and knowing your audience are all tools to facilitate communication.

When presenting your work it is also helpful to have several “pitches” prepared. In addition to your main poster presentation, which you should have prepared, you should have a one-minute mini-summary that you can give at any time. Having this ready means you’ll always be able to share your research, whether it’s in the elevator or in line for coffee.

Logistics

Editing. When it comes down to actually putting your poster together, many people use LaTeX or PowerPoint. LaTeX has more functionality for typesetting mathematics

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¹Preferably colorblind-safe colors and combinations.

**Figure 1.** A photo of the author presenting a poster.
whereas PowerPoint has a more intuitive visual editor. Both PowerPoint and LaTeX have many templates available online to get you started, but you should check that these are the correct size for your presentation.

An article with links to PowerPoint templates as well as further poster design recommendations can be found at the URL in the footnote. For LaTeX there are several common packages used for posters. You can also browse poster templates on Overleaf for other ideas.

**Printing.** For printing your poster you must first decide on the material: paper or cloth. Paper is standard, and most conferences provide a board and fasteners for mounting your poster, but you should check. Paper is likely the easiest to print and your institution may do it for you; otherwise you can print commercially. When flying, poster tubes can typically be placed in overhead bins, although may count as your carry-on.

Next there’s the cloth or fabric poster, which can be easily folded and stored for travel. A fabric poster takes up less space and has comparable resolution to paper. The downsides are the cost and potential time to order online since you may not find local fabric printing.

### Concluding Remarks

A poster can feel like a big undertaking, especially under a time crunch, and just getting started is sometimes the hardest part. However, getting a first draft is also the most important part. In his *New York Times* article, Tim Herrera emphasizes the importance of getting tasks done, even (or especially) if they aren’t done perfectly at first. This applies equally to making posters and to writing articles about making posters. So when you are starting out, don’t worry about making everything perfect and just focus on getting a draft done. Once you have a first draft, you can revise and edit to improve, but you have to start first.

It is my hope that this advice helps you in creating your own research poster or in advising others to do so. Whether specific suggestions fit your preferences and research will ultimately be up to you to use as you see fit. Ultimately, however, you should always return to the guiding principle of communication and ask yourself: does this help me communicate my work? If the answer is yes, then you are on the right track.

I am far from the first to offer advice on posters. For further reading on creating and presenting research posters, see the references section below. You can also find other resources online or ask mentors.

### References


[Str] Strong DR. *Designing communications for a poster fair*, Penn State University.

### Credits

Figure 1 and author photo are courtesy of the author.
James L. Solomon and the End of Segregation at the University of South Carolina

Jesse Leo Kass

Last year, the University of South Carolina Board of Trustees approved the installation of a plaque honoring James L. Solomon, Jr., a former math graduate student and one of the first three African American students to desegregate the university in 1963.¹

This article provides an overview of segregation, its impact on mathematics, and James Solomon’s role in bringing it to an end. It also includes speeches from an April 22, 2019 ceremony where a model of the plaque was unveiled: Henrie Monteith Treadwell (one of the other students to desegregate the university) contributes a letter about her friendship with Solomon, Nathaniel Knox talks about his experience as a PhD student in mathematics at the university in the 1970s, and Carl L. Solomon (James’s son) talks about his father.

The Origins of Segregation at the University of South Carolina

Immediately after the Civil War, African Americans in South Carolina had unprecedented educational opportunities. An 1868 reform to the state constitution made publicly funded schools—including the University of South Carolina—have an unprecedented opportunity to provide education to African American students. The university administration, however, was slow to integrate its student body. In 1870, the university admitted 25 African American students to a separate class on the university’s campus. The class was later moved to a separate building, and the students were required to attend classes at a separate location.

¹In this article, the term “segregation” is used to refer to de jure segregation that prevented African American student enrollment. It should be noted that some of the universities mentioned in this article were only partially segregated. For example, in footnote 6, Vanderbilt University is described as being segregated in 1960, although some schools within the university admitted African Americans as early as 1953. A detailed treatment of desegregation at Vanderbilt is given in [Kea08] and an overview of desegregation at state universities in the South is given in [WHM09]. It should also be noted that some schools that are not usually described as “segregated” had exclusionary policies regarding access to amenities like housing. For example, The Ohio State University started admitting African American students in 1889 but maintained an unwritten policy of prohibiting them from campus housing until the 1940s; see e.g. [Him72, p. 27].
olina—open to all students regardless of race. Starting in 1873, the first African American students began enrolling in the university, making it the only integrated state-supported university in the South. African Americans also held high-level positions at the university: the faculty included Richard Greener (the first African American graduate of Harvard), and the board of trustees was half African American. However, the racial integration of the university did not last long. This measure and the broader platform of reforms that it represented were swept away in the 1876 election, which brought to office the former Confederate general Wade Hampton as Democratic governor. One of the first resolutions passed by the new government declared:

Be it resolved...to...devise plans for the organization...of one university...for the white and one for the colored youth of the State, which...shall be kept separate and apart, but shall forever enjoy precisely the same privileges.... [Car77, No. 37 Joint Resolution]

This resolution was followed by legal and social restrictions enforcing racial separation. These measures limited the options available to an African American student in South Carolina to attending an HBCU (or Historically Black College or University) or to moving north to attend one of the land-grant or private universities that admitted African Americans.

**Segregation and Its Impact on Mathematics**

The professional trajectories of African American mathematicians were profoundly shaped by legalized segregation and other exclusionary policies. Not only did such measures make it difficult for African Americans to obtain a college education, but those who persevered and wanted to work as professional mathematicians faced limited job opportunities. While HBCUs employed largely African American faculty, many other universities had formal or informal policies against hiring African Americans. Moreover, those who did secure academic positions still struggled to participate fully in academic culture. The career of William Claytor vividly illustrates these challenges.

Claytor was the third African American to receive a PhD in mathematics (from the University of Pennsylvania in 1933). Early in his career, Claytor went to the 1936 AMS meeting at Duke University. At the meeting, he was forced to stay at a private residence because he was barred from the (whites-only) hotel reserved for conference participants. Even though his talk was well received by mathematicians such as Lefschetz, Claytor was deeply discouraged by his experience and hesitated to attend future professional meet-

Despite the barriers they faced, African Americans made remarkable mathematical achievements during the twentieth century. Their achievements are nicely surveyed in Edray Goins’s 2019 MAA Invited Address [Goi19].
The response of the MAA was stronger, although only partially successful. The MAA Board of Governors passed a resolution affirming “its steady intention to conduct the scientific meetings of the Association...so as to promote the interests of Mathematics without discrimination...” They delegated the implementation of the resolution to individual Section officers. This gave the organizers of regional MAA meetings considerable autonomy, and there was significant variation in how they applied the resolution. Organizers in the MD-DC-VA and Louisiana-Mississippi regions made significant changes to make meetings more accessible to African Americans. For example, meetings in Mississippi were held at desegregated resort hotels.\textsuperscript{7} By contrast, there were problems with implementing the resolution in the Southeastern region. This was highlighted in 1960 when a group of African American mathematicians from Atlanta University found themselves repeating the earlier experience of the Fisk mathematicians. That year a regional meeting was held at the University of South Carolina. The university was still closed to African Americans, so to accommodate them, the meeting was held at a nearby hotel. However, the hotel was “whites-only,” and while the Atlanta mathematicians were allowed to attend the meetings, they were forbidden from staying there. They left the meeting in protest. In a striking symbol of the persistence of exclusionary policies, the hotel in question was the Hotel Wade Hampton, named after the very governor who had ushered in segregation.

**Higher Education before Desegregation**

In protesting the MAA meeting, the Atlanta mathematicians were part of a growing group of people challenging the segregation of the University of South Carolina. African Americans had tried to enroll at the university since the 1930s, but until 1954, university officials could block their efforts by simply complying with state laws. However, this changed when those laws were declared unconstitutional by the US Supreme Court’s *Brown* decision.\textsuperscript{8}

As the Atlanta mathematicians’ 1960 experience in South Carolina demonstrated, the *Brown* decision did not immediately open the university to African Americans. Rather than desegregate, university officials and state politicians responded by setting up barriers to implementing the *Brown* decision. Ten days after the decision was issued, the faculty at the University of South Carolina voted in favor of requiring applicants to submit standardized test scores, a policy designed, in part, to legally restrict the admission of African Americans.\textsuperscript{9} Faculty, staff, and students were pressured not to challenge segregation. University administrators fired a dean and an assistant professor who openly criticized segregation [Les01, pp. 123–128], and administrators at nearby Allen University and Benedict College, under pressure from the governor, dismissed seven faculty [SBM60].

African American students who tried to desegregate universities faced even more serious threats. No African American students were admitted to the University of South Carolina in the 1950s, but students in other Deep South states were. Their experiences were an indication of what could be expected. One of the first students to try to use the *Brown* decision to gain admission to a university was Autherine Lucy. In 1956, she secured a court order that forced the University of Alabama to admit her, and she began attending classes. During her first week, riots in

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\textsuperscript{5}This motion is not well documented. For example, Appendix 2 of [New80] does not record any response of the AMS. The only reference to the AMS motion the author could find is [Par16, p. 235], which cites a 1952 letter from the AMS secretary to David Blackwell. [Par16] quotes that motion as directing the secretary “to obtain, as a condition of holding a meeting, assurances that at any event scheduled in the program there will be no discrimination as to race, color, religion, or nationality, and that when accommodations and other facilities are provided these shall be provided to all attending the meeting.”

\textsuperscript{6}Old issues of the Notices and the Bulletin show that meetings were held at Tulane University in 1951, Auburn University in 1951, North Carolina State University in 1952, Wofford College in 1953, University of Alabama in 1954, Rice University in 1955, the University of Miami in 1957, Duke University in 1958, Wake Forest University in 1959, and Vanderbilt University in 1960. At the time, these institutions were segregated. Additionally, whites and African Americans were offered separate hotel accommodations at a 1956 meeting at the University of Kentucky (which was not segregated).

\textsuperscript{7}A description of how meetings were made accessible and welcoming to African Americans is given in [NB95].

\textsuperscript{8}The *Brown* v. Board of Education of Topeka decision declared state laws segregating public schools unconstitutional. The case had its origins in South Carolina. The Supreme Court combined the suit from Topeka, Kansas with five other suits. The first of these was *Briggs v. Elliot*, a case challenging the segregation of schools in Summerton, SC.

\textsuperscript{9}The committee that developed this policy was headed by the chair of the math department. At the time, requiring test scores was innovative. The University of South Carolina was the first public university in the South to require SAT scores for undergraduate admissions. This policy was part of a broader program in which state officials maintained exclusionary policies through the use of national standardized tests. For more details, see [Les01, pp. 120–122] and especially [Bak06, pp. 129–135].
It was after returning to teach at Morris College that Solomon became interested in desegregation and the civil rights movement through discussions with student leaders. Enrolling in the University of South Carolina's newly established graduate program in mathematics was a way to get involved in the movement while also gaining useful professional development.

The symbolic power of Solomon's involvement in desegregation cannot be emphasized enough. Morris College was founded to provide education for African Americans who had been largely excluded from higher education by law. The college founder, James J. Durham, was a freed slave who had started his formal education at the University of South Carolina but had been unable to complete his degree due to exclusionary laws passed in 1877. Solomon was attacking those very laws in 1963, and he was doing so with the support of Morris College students and faculty. Solomon discussed applying to U of SC with Morris College President Reuben. Reuben encouraged him (which was no small matter, as applying would put the college at risk of political attacks like those experienced at Allen University and Benedict College in the 1950s), and he decided to apply.

Solomon's decision made the news. The Charlotte Observer announced his application under the headline “Third Negro Seeks USC Entry in Fall” [AP63b], and The State newspaper featured a photo of him arriving on campus to take the GREs [AP63a]. Solomon recalled being concerned that the whole world would know if he didn’t score well enough on the test to be admitted. He evidently did not protest of her attendance broke out on campus. Civil unrest proved an effective means for blocking desegregation when legal tools failed. Citing safety concerns, university officials suspended Lucy, and by the end of the month, they had expelled her. For the next seven years, no African American students attended the university.

African Americans had greater success in desegregating universities in the Deep South in the 1960s: the University of Georgia was desegregated in 1961 and the University of Mississippi in 1962. As in Alabama, the enrollment of African American students was met with fierce resistance with riots breaking out during the first month of classes. The riots in Mississippi were especially violent, with two killed and many injured. To maintain order, army troops were stationed on the campus for the academic year. Remarkably, the African American students at Georgia and Mississippi completed their degrees despite the violence on campus. The desegregation of the University of Mississippi left South Carolina as the only state to have avoided desegregating its public universities.10

James Solomon and the Desegregation of the University of South Carolina

By July 1963, the University of South Carolina had exhausted its legal options for preventing desegregation. Henrie Monteith Treadwell, an African American who had been denied admission, filed a lawsuit, and in response, a federal court ordered the university to admit all qualified African American applicants.11 James Solomon, an applicant to the graduate program in mathematics, was one of the first three African American students admitted under the court order.

At the time, Solomon was teaching math at Morris College. He had been a student there and then gone to Atlanta University (AU) as a graduate student. At AU, he wrote an MS thesis, “Lectures in the theory of functions of a complex variable Part III,” which presents the theory of residue calculus and conformal representations. The thesis was submitted only a few months after the 1960 MAA meeting in South Carolina, and the thesis drew on lectures of Lonnie Cross,12 one of the mathematicians who had left the meeting in protest.

10Desegregation of private universities had occurred earlier. The first university to desegregate was the private HBCU Allen University which enrolled Andre Toth, a white student from Hungary, in early September, 1957. As a private university, the state laws segregating higher education did not apply, although that year, the university found itself in conflict with the state government. This conflict is detailed in [SBM60], while Toth’s experience is described in [Por16, Chapter 6].

11This was not the first court order to desegregate a public university in South Carolina. A federal court had ordered Clemson University to desegregate in January 1963, and the first African American student, Harvey Gantt, enrolled later that month.

12Lonnie Cross later changed his name to Abdulalim A. Shabazz.

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well as Solomon and the other applicants, Robert Anderson and Henrie Monteith Treadwell, were admitted and entered the university on September 11, 1963.14

In a recent speech, U of SC President Pastides described the students’ experiences as “mixed.” While there was none of the violence and disorder that was seen at other Deep South schools, the students also were not fully welcomed on campus. Robert Anderson had a particularly hard time. He lived in a dorm room on campus and was regularly harassed throughout the night by students bouncing basketballs outside his room and shouting obscenities. To provide support, Solomon would occasionally stay with Anderson, and he remembered the experience as frightening. The harassment followed Anderson elsewhere. Solomon recalled walking across campus with Anderson and having students stand by their windows and shout racist insults at them. Solomon himself was not constantly exposed to the campus environment the way the other two students were because he was married with kids and living in Sumter, South Carolina.

In later interviews, Solomon had positive things to say about his time in the math department. He said that he received a warm welcome from many people. He praised the department chair Wyman Williams, the only person he mentioned by name, for helping him obtain NSF support for his studies. Solomon said that although some of the professors had a problem with racism, they were never unfair to him with grading. Summarizing the experience, he said, “I must truthfully say that I encountered no real problems” [Hay89].

After his time at the University of South Carolina came to an end, Solomon continued to teach at Morris College for many years and then served in various positions in state government including as commissioner of the Department of Social Services. Desegregation was not, however, the end of Solomon’s work reforming higher education. In 1981, the federal government declared South Carolina’s higher educational system to be noncompliant with the Civil Rights Act, and Solomon served on a commission that implemented major reforms that brought the system into compliance. While higher education in South Carolina remains marked by persistent racial disparities, the speeches below by Nathaniel Knox and Carl Solomon attest to how Solomon created opportunities that had been impossible during segregation.

Solomon is an honored figure in the community. The year after desegregation, Morris College students dedicated the school yearbook to him (see Figure 6). For his public service, Governors Richard Riley and Carroll Campbell both awarded him the Order of the Palmetto, the highest award that can be given to a resident of the state. Solomon truly embodies Morris College’s motto: “Enter to Learn, Depart to Serve.”

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14The day after they registered, another African American student, James Hollins, registered for classes at the University of South Carolina Beaufort.
I really enjoyed my time here. Standing on the shoulders of Mr. Solomon, there were no trails to blaze. So the only things that mattered for me were the mathematics and the goals I had set for myself.

I will share with you two of the many experiences I recall from my time here. The first is this: One day as I walked in a LeConte hallway, I suddenly understood the concept of a function. I had been dealing with functions since ninth grade Algebra I, but after about fifteen years I saw the beauty and power of the concept. It could be a sledge hammer or a scalpel.

The second relates to my dissertation work with Dr. Scheiblich. We met, I think, at 11:00 a.m. on Thursday, starting in September. We’d meet and discuss my progress or lack thereof. In either case, there was a lot of learning on my part. Then, one day in April as we discussed my work over the past week, he smiled and I knew I had succeeded. However, my thoughts were “I’ve got him, I’ve got him.” The rest is history.

While I have the floor, I must acknowledge another set of strong shoulders on which I stand. Our nation is indebted to Dr. Johnny Houston for the work he has done in shepherding the National Association of Mathematicians.
(NAM) through most of its existence. Through NAM programs, my students and faculty, and those of other institutions serving students of color, were afforded opportunities to exhibit the quality of their work in arenas that matter. As a result, the country is reaping the benefits of the work of these individuals as mathematicians in various government agencies and high tech organizations, as university faculty, and as entrepreneurs. So, on behalf of our country, I thank you, Johnny. I also thank Dr. Goins and the current Board for carrying on the work.

I commend the organizing committee and the University for acknowledging Mr. Solomon’s achievement as an important milestone in the development of the U of SC Mathematics Department. Again, I thank you for allowing me to speak.

Speech of Carl L. Solomon

Carl L. Solomon is the son of James Solomon. He received a JD from U of SC in 1994 and runs the Solomon Law Group. He contributed the following speech about his father at the April commemoration:

His story started in 1942 or 43 depending on who’s telling the story. And it started in McDonough, Georgia when my grandmother Tessie told my grandfather, “My sons are too smart to pick tomatoes.” Because at his age and at that time, when you got old enough you left school and you went out into the fields. So my grandfather put them all on the wagon and went to Atlanta so my father could continue in school. And in Atlanta my grandfather who didn’t have anything and kind of needed a vocation that wasn’t related to the field started learning how to preach. And part of what he did was he had to teach the next generation of preachers. My dad didn’t want to be a preacher, but my grandfather said, “Boy I have to have a job, so somebody’s got to come to church on Sunday and learn from me.” And my daddy went because it was required. At that time, my father had never considered going to college. So one day they were sitting in the pews and another person close to his age went over to him and said, “Where are you going to school?” And that was a question he had never Answered because my grandmother wanted him to be the first Solomon to graduate high school. But he’s now around people that are second generation, not only business owners, but people that had graduated college. He said, “I’m going to Morris Brown.” He didn’t know about Morris Brown College. He had seen the name, and he didn’t want to be embarrassed. So he graduated at 16 and he went to Morris Brown because that’s what he set his sights on sitting in that pew.

At 18 he got drafted into the Korean War and after he ended up in little old Sumter, South Carolina where he met my mother. And he met her and told her he wanted to impress her and he was broke. And he told her, “I know I don’t have a lot to offer you, but I have had two years of college. And if you marry me, we’ll go back to school. I’ll pay for it.” He’s broke now, but he laid floors and worked in factories and they both graduated and if you ask my mom, she’ll say her grades were better than his every semester. So he gets a degree at Morris College and he wants to do something and they gave him a little job at a school but to be a professor you needed another degree. So he ended up at Clark Atlanta University and got a mathematics degree.

So my dad is in Sumter and he’s teaching math at Morris College and he had a saying that he developed there that I learned to hate until I met people that he used to teach. His saying was, “You not only have to get it right, you have to get it right the first time.” Now what does that mean when you’re his son and you have to learn algebra? It means you have to use a pen to do your math, which I thought was the most absurd thing in the world and I did not always get it right the first time.

So my father is there, he’s starting to teach, and he meets some people: Ruben L. Gray, Ernest Finney, and later, Matthew Perry. And this was at a time when no African Americans had been elected to anything. And in my dad’s words: “We just couldn’t understand, having learned math, with the numbers of diversity in Sumter, how we couldn’t get elected.” They didn’t run in parties, they ran in slates and if just the African Americans voted for them they would fall in the middle of the slates whether they got any white votes or not. So they ran. He credited math for giving him the courage to run for the

15Based on a transcript provided by Candace Bethea.

16Morris Brown College is an HBCU in Atlanta, GA and is different from Morris College, the college that Solomon taught at.

17The university was called Atlanta University when Solomon attended. The name was changed in 1988 when the university merged with Clark College.
Thank you so much for this opportunity. I will leave you with this: my father and mother couldn’t be here today. They’re 88 and 86 years of age. One has Alzheimer’s and one has a slight vascular dementia. They remember great things from the 1950s, not too many things from last week. I will tell you that being in love and dancing when you’re 30 and when you’re 40 is great, but being together and dancing when you’re 80 and all you can ever remember is being together is even better.

Installing a plaque commemorating James Solomon is an important and necessary step in honoring his legacy. Yet the plaque should serve not just as a reminder of Solomon and his actions but also as a call to continue challenging racial disparities in higher education. For AMS members, a fitting way to do this would be to work with Morris College and other HBCUs in the Deep South. The substantial support these schools provided Solomon helped guarantee that African American mathematicians cannot be formally excluded from professional meetings as they had been in the 1950s, but these benefits have not fully reached the communities the schools serve. African American undergraduate enrollment at U of SC, for example, is lower today than it was in 1981 (the year Solomon began to serve on an education reform commission). We encourage readers to learn about ways in which the AMS is collaborating with HBCUs and to think of ways for further collaboration.

The Legacy of Desegregation

Installing a plaque commemorating James Solomon is an important and necessary step in honoring his legacy. Yet the plaque should serve not just as a reminder of Solomon and his actions but also as a call to continue challenging racial disparities in higher education. For AMS members, a fitting way to do this would be to work with Morris College and other HBCUs in the Deep South. The substantial support these schools provided Solomon helped guarantee that African American mathematicians cannot be formally excluded from professional meetings as they had been in the 1950s, but these benefits have not fully reached the communities the schools serve. African American undergraduate enrollment at U of SC, for example, is lower today than it was in 1981 (the year Solomon began to serve on an education reform commission). We encourage readers to learn about ways in which the AMS is collaborating with HBCUs and to think of ways for further collaboration.

Figure 9. James Solomon in the 1970 Morris College yearbook.

Figure 10. Participants at the April 22 ceremony; l to r: Edray Goins, Candace Bethea, Tracey Weldon-Stewart, Jesse Leo Kass, Carl L. Solomon, Lacy K. Ford, Johnny L. Houston, Linyuan Lu, Matthew Boylan, Paula Vasquez.
ACKNOWLEDGMENTS. The author would like to thank Bobby Donaldson for his support, in particular for his suggestion to install a plaque in honor of James Solomon; Carl L. Solomon and the Solomon family, Nathaniel Knox and the Knox family, as well as Henrie Monteith Treadwell who all participated in the commemoration ceremony and who granted me permission to share their speeches; Johnny L. Houston and Edray Goins for participating in the ceremony and for their insights about the experiences of African American mathematicians in the 1950s; Matthew Boylan, Tracey Weldon-Stewart, and Paula Vasquez for their help with organizing the ceremony and for their useful feedback on drafts of this paper; Laura Najim and Trudie Wierts for their help in organizing the ceremony; Lacy K. Ford and Linyuan Lu for supporting the ceremony; Candace Bethea for providing a transcript of Solomon’s speech; Louisa McClintock for generous help editing this paper; the two anonymous referees for providing useful feedback; Graham Duncan and Elizabeth Cassidy West (at the South Caroliniana Library) and Janet Smith Clayton (at the Learning Resources Center at Morris College) for their help in locating materials on Solomon and for securing permission to reproduce photos; Ronda Sanders for photographing the ceremony and allowing me to reproduce her photos.

References

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Figures 6 and 9 are courtesy of Morris College, Learning Resources Center—Coleman Collection Archives, Sumter, SC.
Author photo is courtesy of Louisa M. McClintock.
Reflections of a Mathematics Teacher Educator: Considerations for Mathematicians Who Teach Teachers

Christina Eubanks-Turner

Early on in my career, I knew I wanted to be involved in training mathematics teachers. In graduate school, as I worked toward a doctorate in mathematics, focusing on the study of commutative algebra, I always had a passion for using my knowledge and skills to impact K–12 education. I was fortunate to study in a mathematics department that allowed interested graduate students to be involved with professional development programs to train current teachers and teach mathematics courses for future teachers. While that was over fifteen years ago, I still have a desire to be a change agent in education and look for meaningful ways to work with current and future mathematics teachers.

Recent developments in K–12 mathematics education, such as the implementation of the Common Core standards and the call for the inclusion of equitable mathematics teaching practices (NCTM, 2014), require mathematics teachers at every level to reflect on, and in many cases alter, their teaching practice. To keep abreast of these needs, mathematics teacher educators have begun to “define, research, and refine the characteristics of effective teachers of mathematics” (AMTE, 2017, p. xvi). As there is a well-established knowledge base on K–12 mathematics teacher education (Sanchez, 2011), there is a growing need for research on the knowledge that mathematics teacher educators need in order to prepare mathematics teachers (Jaworski & Huang, 2014).

Not only does mathematics teaching require knowledge of mathematics, but also a knowledge of teaching and learning. The MET II report states that “the mathematical knowledge needed by teachers at all levels is substantial yet quite different from that required in other mathematical professions” (MET, 2012). An example given by the Elementary Math Project (2019) states, “people with common knowledge of mathematics can divide accurately and solve division word problems, while teachers with specialized content knowledge possess common knowledge of division as well as the knowledge to explain why division procedures work, how division can be interpreted using equal sharing, measurement and missing factor interpretations, how remainders can be interpreted, and under what circumstances quotients are smaller, equal to, or larger than the dividend.” This knowledge, called Mathematical Knowledge for Teaching, is a specialized content knowledge that is specific to teaching mathematics (Ball et al., 2008). Although there have been national recommendations that mathematics content courses for teachers include content that develops mathematical knowledge for teaching, there has been minimal consideration given to the development of future teachers’ mathematical knowledge for teaching in...
mathematics content courses for teachers (Hart, Oesterle, & Swars, 2013). This realization has garnered attention from both the mathematics and mathematics education communities. Recently, the Association for Mathematics Teacher Educators released Standards for Preparing Teachers of Mathematics (AMTE, 2017), which gives a comprehensive guide for various stakeholders of the national vision for teacher preparation, including standards for mathematicians who teach future teachers.

Researchers have begun to investigate the knowledge that educators of current and future K–12 teachers (Mathematics Teacher Educators) need to teach teachers (Zopf, 2010; Masingila, Olanoff, & Kimani, 2017). Similar to the conceptualization of Mathematical Knowledge for Teaching, Zopf defined the Mathematical Knowledge for Teaching Teachers as “the mathematical knowledge used by mathematics teacher educators in the work of teaching mathematics to teachers” (p. 11). Just as K–12 teachers utilize specialized mathematical knowledge to teach children, mathematics teacher educators use specialized knowledge to teach teachers, which differs in some aspects from the knowledge needed to teach children. For instance, mathematics teacher educators teach adults, not children, and so this requires mathematics teacher educators to help teachers relearn mathematics (Superfine et al., forthcoming).

While mathematicians are trained to deeply understand the complexities of mathematics in content courses for teachers, they typically receive no training or development to be an educator (Bass, 1997). The MET II recommends that mathematics and statistics departments offer instructors support for professional development to increase expertise in teaching teachers (MET II, 2012). As a mathematician who has worked in teacher education since 2005, some of my mathematical knowledge for teaching teachers came from my experiences teaching current teachers in graduate courses and professional development and teaching future elementary and secondary teachers. Additionally, my study of educational research, collaborating with math educators to better understand teacher education, and many years of presenting and participating in various workshops, conferences, and seminars have enhanced my mathematical knowledge for teaching teachers. Below I give recommendations mathematicians should consider when they teach mathematics courses for teachers to support the development of teachers’ mathematical knowledge for teaching. These recommendations are for mathematics teacher educators who teach current and/or future teachers.

1. Teachers need to see clear connections between the mathematics that they are learning and the mathematics that they will teach. This recommendation is especially relevant for those teaching secondary school teachers. As mathematics teacher educators we should strive to help teachers see how most, if not all, topics in mathematics content courses for teachers connect to the mathematics they are expected to teach, and if this is not possible, we should carefully consider the purpose of those topics in these courses. The latter does not imply that mathematics teacher educators should teach K–12 math only: as the MET II states, “coursework for prospective teachers should examine the mathematics they will teach in-depth, from a teacher’s perspective” (p. 17), but there should be clear, unambiguous discussions about how the coursework they are learning relates to the elementary or secondary concepts they will teach. Among other connections, teachers need to see a progression of how elementary or secondary concepts are concrete representations of concepts they are learning in their content courses, which gives teachers an understanding of the vertical alignment of the concept, deepening their content knowledge and illustrating the mathematical relevance of these topics.

As an example, I regularly teach an abstract algebra course for secondary teachers, where I explain in explicit detail and work with teachers through examples and exercises to demonstrate how the zero product property connects to the abstract algebra concept of integral domains. This example serves as a relevant connection for secondary teachers, as the zero product property is viewed by many secondary teachers as an important concept related to solving for roots of polynomials, which is a focus of secondary algebra. Also, mathematics teacher educators should make sure teachers understand how the knowledge they are gaining affects their understanding and teaching of the mathematics they teach. The latter requires mathematics teacher educators to have some knowledge of the mathematics currently being taught in elementary and secondary schools.

2. Mathematics teacher educators should ensure that teachers experience highly effective teaching practices in their mathematics content courses. Also, mathematics teacher educators should be transparent about instructional practices they are utilizing while teaching these courses so teachers can understand why they are modeling a particular practice and how it impacts learning. For mathematicians who are just starting as mathematics teacher educators and want to find where to begin searching for resources on research-based effective mathematical teaching practices, I would suggest they start by asking a colleague who has expertise in education. Various professional societies that focus on teaching mathematics at various levels have resources related to best practices in mathematics teaching (NCTM, 2014; MAA, 2017; AMTE, 2017). Note that while many resources focus on K–12 mathematics teaching, many practices can be used or modified for use in higher education.

Also, if mathematics teacher educators are working with mathematics teachers who are currently in the classroom,
mathematics teacher educators can view their work with those teachers as a partnership and learn effective pedagogical practices from them. Some instructional strategies I routinely use in all my courses and explicitly model in my courses for teachers are to introduce a mathematical concept by having teachers solve and discuss low floor/high ceiling tasks, that is, tasks that all students can access but that can extend to high levels (Boaler, 2015); orchestrate and sequence productive mathematical discussion in class (Smith et al., 2011); and have students engage in paired board work to solve problems (MAA, 2017).

3.
Beginning teachers need opportunities to rehearse mathematics teaching in a low-stakes environment. Not only should teachers have rehearsals in their education courses (McDonald et al., 2013), there is still a need for rehearsals in their mathematical content courses, where teachers will get the opportunity to not only teach K–12 mathematics that connects to the mathematics they are learning, but also teach the mathematics they are being taught in their mathematical content courses. Rehearsing mathematics would allow teachers the opportunity to demonstrate their knowledge in practice. Specifically, mathematics teacher educators should have students engage in approximations of practice (Grossman et al., 2009). Approximations of practice allow beginning teachers to simulate different components of teaching. These approximations of practice should not replace real teaching experiences, but allow beginning teachers to act out teaching in a safe environment where they can get feedback from their mathematics teacher educator and peers (Grossman et al., 2009).

McDonald, Kazemi, and Kavanagh (2013) give a learning cycle related to approximations of practice to enact core teaching practices. In this cycle, teachers learn a particular practice from their mathematics teacher educator. Then the teacher prepares and rehearses the practice with their mathematics teacher educator and peers. After, the teacher enacts the practice with real students, which is then followed by an analysis of the enactment. Approximations of practice I have used in my content courses for teachers include teachers crafting questions that could be posed to students after watching videos of mathematical teaching in a 6–12 classroom, analyzing errors in authentic student work, and rehearsing and simulating teaching to peers.

4.
Teachers should have some understanding of the cultural relevance of the mathematics they learn and are called to teach. Ultimately, teachers are the greatest “advertisers/sellers” of mathematics to future generations, and so they should be able to communicate the significance of mathematics in society in order to convey that significance to their students. Mathematics teacher educators should make sure to communicate how K–12 mathematics is a quantitative language used to describe science, technology, and society. Mathematics teacher educators should be sure to communicate to all students, especially teachers, that many different cultures made and continue to make impactful contributions to mathematics by emphasizing contributions made by people from cultures that are underrepresented in mathematics to go beyond Eurocentric views of mathematical invention.

“Why is this important?” is a common question asked by students at all levels. Having mathematics teacher educators being able to communicate how various mathematical topics relate to our present and future world and demonstrate to students the usefulness of mathematics through application can help resolve such questions. Recently, there have been efforts to humanize/rehumanize mathematics and show that “mathematical thinking is influenced by a diversity of human environments and their elements, which include language, religion, mores, economics, social, and political activities” (Rosa & Orey, 2016, p. 4). This focus on humanizing/rehumanizing mathematics is critical for those who have been marginalized as “schooling often creates structures, policies and rituals that convince people they are no longer mathematical” (Goffney & Gutiérrez, 2018). Thus, when teaching teachers, mathematics teacher educators should communicate the importance and model inclusive practices that reduce barriers of access for all students.

In all my courses for teachers, I give historical information about the development of the mathematical topics of the course and show how many diverse people have contributed to mathematics. I also stress that many mathematical discoveries came out of a need to solve societal issues, which helps illustrate that mathematics is not a collection of abstract concepts that dropped out of the sky and that abstraction is a human innovation. Further, this helps dispel beliefs that mathematics is accessible only to certain people who are “smart enough to get it.” Additionally, I stress the importance of metacognition (thinking about one’s thinking) and process documentation in problem solving to value all mathematical thinking in class and not just traditional problem solving.

As a mathematician who is a mathematics teacher educator, I have the important task of not only helping teachers gain in-depth knowledge of mathematics but also helping to develop their Mathematical Knowledge for Teaching. As many teacher preparation programs teach pedagogy courses separate from the content courses, this can leave teachers with the complicated task of trying to merge these two knowledge bases (Ball, 2008). Therefore, to consolidate teachers’ knowledge, instructors in both pedagogy and content courses need to work to enhance teachers’ Mathematical Knowledge for Teaching.

Regardless of who our students are and what they major in, as mathematicians, familiarizing ourselves with
research-based, best instructional practices would benefit them all. Furthermore, mathematicians who teach teachers should continually strive to improve our own Mathematical Knowledge for Teaching Teachers to more effectively teach so that teachers have the potential to positively impact K–12 student learning.

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Author photo is by Amari Turner.
Mathematics: The Key to Empowering Tomorrow’s Workforce

Tanya Moore

In the African-American tradition there is a phrase, Each One, Reach One, that reflects the value of bringing along others once you have acquired a certain level of knowledge or success. In the context of the mathematics community, this value is often reflected in the math-related activities and events that happen outside the classroom to prepare the next generation for their chosen educational and career paths. As technology promises to change the way we work by altering the landscape of the labor market, mathematics will take on a new level of importance. The role of service and outreach and the willingness for Each One to Reach One to increase mathematical engagement will matter even more.

“We will always have STEM with us. Some things will drop out of the public eye and will go away, but there will always be science, engineering, and technology. And there will always, always be mathematics.” —Katherine Johnson, NASA research mathematician [5]

Math Has a Central Role in the Future Labor Market

Advances in automation and artificial intelligence have resulted in dramatic alterations in the way we work, and mathematical thinking and knowledge have already become increasingly critical to jobs of the future. Statistical and quantitative skills have emerged as increasingly high priorities in workplace competencies [11]. In a recent survey of more than three hundred top executives in companies around the country, 72 percent rated critical thinking/problem solving as one of the top four important skills they look for in employees [7]. Managers and leaders will be expected to use data to inform decision making and possess a strong business intelligence to support a thriving company. The other looming change relates to the critical role postsecondary credentials will have in determining economic options for individuals. In 2020, two-thirds of jobs will require postsecondary certificates and degrees [3]. Math proficiency serves as an important gatekeeper to higher education, given the role played by standardized tests (like the SATs) and math course requirements beyond elementary algebra for college admission. People with strong math skills will have viability and resiliency in the new economy.

“Math literacy and economic access are how we are going to give hope to the young generation.” —Robert Moses, civil rights activist and author [10]
The New Economy Will Impact Demographic Groups Differently

The new labor market is predicted to have a disproportionately negative impact on communities of color [1,2,11]. Increases in automation will not affect the number of jobs but, rather, the types of jobs available in the workplace [9]. In particular, these newer types of job opportunities will require innovation and productivity that rely on advanced levels of education. Recent reports on the future of work cite the likelihood of unemployment or long-term displacement for vulnerable groups of workers, particularly those with lower levels of education and skills [8,9,11]. For example, the transportation industry highlights this pending disruption. Millions of truck drivers, one of the few middle-income jobs that do not require formal postsecondary education, will have their jobs altered or replaced by self-driving vehicles [1]. Education beyond high school will be critical for access to quality jobs. “Given educational disparities, Hispanic and African-American workers may be hit the hardest, with 12 million displaced,” states a 2019 McKinsey Report [8]. Additionally, despite the desirability of competitive math skills, the American adult population has fallen behind their international peers in math literacy [7], and persistent, albeit declining, racial/ethnic gaps remain in math achievement [4].

“Our parents and teachers preached over and over again that education is the vehicle to a productive life, and through diligent study and application we could succeed at whatever we attempted to do.” —Evelyn Boyd Granville, second African American woman to earn a doctorate in mathematics [6]

Service as a Strategy to Increase Access to Mathematics

The classroom remains the traditional and predominant setting for learning math. For some students, this environment is an intimidating and ineffective way to understand and connect to the discipline. Outreach and service projects are valuable strategies that mathematicians can use to support math achievement in the next generation. They can provide a personalized and contextualized experience of mathematics, reinforce core competencies, expose students to research, and identify connections between math and everyday life. Math camps, summer REUs, after-school tutoring, Math Circles, special workshops, and conferences are all examples of the types of projects where mathematicians share what they know and love about mathematics outside the traditional classroom environment. Service can also be informal—there are mathematicians who facilitate math workshops for their children and their children’s friends or those that tutor math in local schools. In Margot Shetterly’s *Hidden Figures*, we learn that Mary Jackson helped girls in her Girl Scout troop with their algebra homework [12]. For interested community members not knowing where to start with these types of initiatives there are resources like GirlsGetMath from ICERM or examples to follow like BEAM (Bridge to Enter Advancement Mathematics) or YPP (The Young People’s Project), programs focused on reaching underrepresented groups. All these activities provide a platform for increasing math awareness and providing an experience that makes math accessible and fun. Service programs can also facilitate a sense of belonging within the larger math community. Sometimes these activities are grant-funded, count towards tenure, or result in additional compensation, but more often than not, it is done by mathematicians as volunteers in response to a need, an action arising from a belief of Each One, Reach One.

Most of my professional career has been spent outside academia. Ironically, it has taught me how important math instruction is to accessing educational and economic opportunities across many different sectors. While the impact of math-focused outreach and service projects that support non-math majors or non-traditional careers in mathematics is less well known, what I know for sure is that math matters. I’ve known high school students who walked out of math placement exams because they feared failing, managers who avoided budgets because of their discomfort with creating formulas in Excel, and community college students who took multiple years to complete their AA degree because they were stuck in a cycle of remedial math courses. What types of opportunities will these students have in the future workforce?

The Power of Each One, Reach One

But, I also know there is hope. Through service, I’ve seen middle school girls take pride in creating math tutorial videos in a summer recreation camp and teens at a continuation high school feel empowered when they learned statistics to design and analyze a survey focused on understanding youth violence in their community. I’ve witnessed the delight in unemployed job seekers after they learned how to use a binary number system in an IT class. I’ve watched a young man who turned his life around follow his dreams of becoming a chef and a boy who, after volunteering at a STEM summer camp, saw a change in his behavior and ultimately today is working with youth violence in their community. I’ve witnessed the power of Each One, Reach One to make a positive difference.

Strong economies can hide inequities in participation. Broadening participation in mathematics through service helps to strengthen and diversify the pipeline into math-related careers and increase economic opportunity for the labor force’s next generation. The power of math lies not only in its marketable skills, but also in its ability to create confidence and a strategic process for problem solving. Math permeates so many aspects of our lives, no matter whether you identify as a mathematician or approach the subject with caution. It also serves as a critical component of educational advancement and a key skill for success in the new economy. With this in mind, we need to figure out how to truly broaden participation in mathematics and
promote the expectation of math literacy for all. Nearly two decades ago, Bob Moses, the educator, civil rights activist, and author of the book *Radical Equations: Civil Rights from Mississippi to the Algebra Project*, was prescient when he flagged the critical role algebra would play in gaining access to a new economy fueled by technology. He called it the new civil rights issue of our time. Mathematical experiences outside the classroom help prepare a broader population for success in the workforce. We need to pursue a multitude of avenues to improve mathematical literacy and create equitable employment opportunities for all. Tomorrow’s workforce depends on it.

References


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Simplicity: Ideals of Practice in Mathematics and the Arts

Reviewed by Douglas Norton

Simplicity: Ideals of Practice in Mathematics and the Arts
Edited by Roman Kossak and Philip Ording

Simplify this fraction. Simplify the expression. Simplify your answer. We certainly present simplicity to our students as a desired goal, sometimes to the extent of conflating in significance the path to a solution and the form of the solution. On the research side of our mathematical lives, embedded in our own reference to a proof as “elegant” is the idea of a proof demonstrating some sort of simplicity.

One hundred years after David Hilbert (Figure 1) presented his famous list of unsolved problems at the International Congress of Mathematicians in 1900 [1], historian of mathematics Rüdiger Thiele discovered another problem buried away in Hilbert’s mathematical notebooks: “The 24th problem in my Paris lecture was to be: Criteria of simplicity, or proof of the greatest simplicity of certain proofs” [2]. While Hilbert’s list of problems inspired and challenged the mathematical community throughout the twentieth century, his 24th problem never appeared in the literature until this relatively recent discovery. Nevertheless, the ideas appear independently as formal threads in twentieth-century proof theory, model theory, and algorithmic information theory.

The book Simplicity: Ideals of Practice in Mathematics and the Arts addresses ideas of simplicity in mathematical proof in a general and philosophical way that requires no previous grounding in the specialty theories of the preceding paragraph while providing both subtle and fascinating insights into the questions raised.

The volume presents selected lectures and additional contributions from a conference also titled Simplicity, held at the Graduate Center of the City University of New York in April of 2013. (See the conference poster in Figure 2.)

Why mathematics and the arts? Mathematical proportions proposed by the Greek sculptor Polykleitus in the fifth century BCE, perspective in Renaissance Italian painting, symmetry in Islamic tilings, geometry in the paintings of Piet Mondrian and the De Stijl school, and tessellations in M. C. Escher are all examples of specific mathematical tools utilized by artists. The past two decades have found both a broadening of the content and a widening of the appeal of the crossover between mathematics and the arts. The Bridges Organization works to “foster research, practice, and new interest in mathematical connections to art, music, architecture, education, and culture” through its annual Bridges Conferences [3]. The Journal of Mathematics and the Arts [4] is a peer-reviewed journal that focuses on connections between mathematics and the arts. An impressive juried exhibition of mathematical art has become a regular...
feature at the Joint Mathematics Meetings [5]. Educators at all levels have begun to advocate for the inclusion of the arts in the push for science, technology, engineering, and mathematics education, with STEM evolving into STEAM [6].

Organizers of the Simplicity conference were Juliette Kennedy (University of Helsinki), Roman Kossak (the Graduate Center of CUNY), and Philip Ording (then at Medgar Evers College of CUNY, now at Sarah Lawrence College), all mathematicians with crossover interests in logic and philosophy, model theory, and mathematics and the arts, respectively. In their preface to the book, editors Kossak and Ording provide the following:

That mathematicians attribute aesthetic qualities to theorems or proofs is well known. The question that interests us here is to what extent aesthetic sensibilities inform mathematical practice itself. When one looks at various aspects of mathematics from this perspective, it is hard not to notice analogies with other areas of creative endeavor—in particular, the arts.… [W]e find that a more profound connection between art and mathematics than any formal similarity is a similarity in method. For this reason the conference emphasized ideals of practice [pp. viii–ix].

They see, and the conference participants explore, simplicity as the essence of the similarity of method and the ideal of practice common to twentieth-century Western art and Hilbert’s quest for consistency, efficiency, and rigor in proofs. The papers gathered in this volume present a fascinating peek at what the interactions among the mathematicians, artists, and philosophers gathered at the conference were like. Talks at the conference (as in Figure 3) were complemented by panel discussions across disciplinary boundaries; see Figure 4. The observations below are intended to follow a few threads that wend their way through the text rather than providing a sequential stroll through the papers in the collection.

Juliet Floyd, philosopher of mathematics and of language at Boston University, opens her piece “The Fluidity of Simplicity: Philosophy, Mathematics, Art” with the line: “Simplicity is not simple” [p. 155]. Is there a definition on which we can agree in the mathematical context? Is there one in the arts? Are they mirror images, funhouse mirror images, or completely unrelated? Andrés Villaveces, professor of mathematics at the National University of Colombia, Bogotá, observes in his piece “Simplicity via Complexity: Sandboxes, Reading Novalis”:

The simplicity question—the quest for the simplest proof or the simplest design, line, or resolution of architectural space or rhyme or melody…draws a tenuous but intriguing connection between mathematics and various other disciplines (architecture, physics, design, chemistry, music, etc.) [p. 192].

Let us first consider what the authors have to say in the mathematical arena.

Étienne Ghys, mathematician at the École Normale Supérieure in Lyon, presented the first address at the conference and the first paper in the collection, entitled “Inner Simplicity vs. Outer Simplicity.” In these, he demonstrates why he was the inaugural recipient of the Clay Mathematics Institute Award for Dissemination of Mathematical
Kolmogorov complexity of an object is the length of the shortest computer algorithm that produces the object as output. A more general usage would be that the complexity of an object is the length of the shortest description of the object. Ghys contrasts high and low Kolmogorov complexity through two pictures. A square with a random distribution of yellow and orange dots (Figure 5) would require a long sentence for a complete dot-by-dot description, while just a few short lines of code can generate the Mandelbrot set (Figure 6). This brief description renders the Mandelbrot set “simple” from the outer simplicity perspective, but Ghys finds this unsatisfactory; it is not simple in terms of inner simplicity.

Ghys provides another example with proofs. He presents a single sentence from a number theory book by Jean-Pierre Serre that he recalls and describes as follows:

I spent two days on this one sentence. It’s only one sentence, but looking back at this sentence, I see now that it is just perfect. There is nothing to change in it; every single word, even the smallest, is important in its own way… Serre’s language is so efficient, so elegant, so simple. It is so simple that I don’t understand it…. Everything, every single word is fundamental. Yet, from the Kolmogorov point of view, this is very simple…. Finally, at the end of the second day, all of a sudden, I grasped it and I was so happy that I could understand it. From Kolmogorov’s point of view, it’s simple, and yet for me—and, I imagine many students—it’s not simple [p. 6].

He provides counterpoint to this with a delightful meander through networks, density, the Internet, and a theorem by Endre Szemerédi, with the following conclusion: “[It’s an example of a theorem for which the published proof is complicated, but nevertheless I understand it. For me it’s simple. I think I will never forget the proof because I understand it. And this is the exact opposite of the one-line by Jean-Pierre Serre, which was so short that it took me days to understand it” [p. 14].

Many of the views of mathematical simplicity expressed throughout the volume are a conflation of these two, always about a basic interaction with mathematical ideas and proofs. Dennis Sullivan, professor of mathematics and Einstein Chair holder at the Graduate Center of CUNY, reiterates this connection in his final essay of the collection, entitled “Simplicity Is the Point”: “Understanding is more important to me than proofs…. So, proof and understanding are intimately tied, but understanding is, for me, the primary goal, and simplicity plays a role in that” [p. 269].

Marjorie Senechal addresses simplicity in science in her piece entitled “The Simplicity Postulate.” Senechal is the Louise Wolff Kahn Professor Emerita in Mathematics and History of Science and Technology at Smith College, as well as editor-in-chief of The Mathematical Intelligencer.
What do we mean by simplicity in proofs? Senechal proposes *ease, unpretentious, and minimal* as words that come to mind [p. 79]. She adds *probable* to the list and tells the story of Dorothy Wrinch, Harold Jeffreys, and the Simplicity Postulate, a scientific version of the classic Occam’s Razor. Occam’s Razor is the principle that in solving a problem, the solution with the fewest assumptions—the simplest—tends to be the correct one. Wrinch and Jeffreys applied this reasoning to claim that scientists choose from among competing theories to explain a given physical phenomenon by selecting equations with smaller order, degrees, and magnitudes of coefficients. This Simplicity Postulate did not last long in scientific circles, and yet it had an impact on the wider acceptance of Bayesian statistics. (This story contains one of my favorite lines in a book full of interesting lines: Senechal says that Wrinch “crossed disciplinary boundaries easily, without glancing on forcoming trains” [p. 80].) While Senechal’s discussion centers on selection between competing scientific theories rather than forms of mathematical proof, the linguistic distinctions are helpful. Reducing a proof to some sort of minimalism may be quite difficult, sacrificing one type of simplicity for another, just as Serre’s sentence in the Ghyss paper may have simultaneously minimized length of description and ease of comprehension.

On the other end of the spectrum is the approach of proof theory, the branch of mathematical logic that studies proofs themselves as formal mathematical objects. One may define the simplicity of a proof in terms of minimizing some sort of measure: the number of symbols in some reduced version of a proof or the number of references to a class of topics. Andrew Arana, associate professor of philosophy at l’Université Paris 1 Panthéon-Sorbonne, addresses this approach in “On the Alleged Simplicity of Impure Proof.” Arana calls a proof *pure* if it draws only on “what is ‘close’ or ‘intrinsic’ to that theorem” [p. 207]. By what metric do we define “closeness”? He mentions two: “A proof is *elementally close* to a theorem if the proof draws only on what is more elementary or simpler than the theorem.... A proof is *topically close* to a theorem if the proof draws only on what belongs to the content of the theorem, or what we have called the *topic* of the theorem” [p. 208].

These metrics may bring to mind some contrasting proofs in number theory. Zagier provides a one-sentence proof that every prime congruent to 1 mod 4 can be written as the sum of two squares [7]. The involution and fixed-point results required for the proof are sophisticated terms for the simple checking of algebraic identities, no less elementary than the ideas of prime numbers and squares of whole numbers. They are both simple and lie within the mathematical neighborhood of the topic of the theorem. By either metric, this proof may be simple in terms of length as well as in terms of structural content. Compare this with the Wiles proof of Fermat’s Last Theorem: a theorem similarly simple to state, with a proof extremely long and distant from the statement by either of Arana’s metrics.

In a similar vein, Rosalie Iemhoff, philosopher at Utrecht University in the area of proof theory, claims in “Remarks on Simple Proofs” that a proof should “not contain reasoning about geometric objects when the conclusion of the proof is a statement about the natural numbers” [p. 147]. This constraint, or Arana’s “topically close” constraint above, would render many of the entries in the Mathematics Magazine series “Proofs Without Words” impure or not simple, as the proof of Nicomachus’s theorem in Figure 7.

On the other hand, such proofs certainly score high (for simplicity) on the Kolmogorov complexity version of the test of how many words it takes to carry out the proof!

Maryanthe Malliaris and Assaf Peretz approach the question from a different direction in “What Simplicity Is Not.” Malliaris is a mathematician at the University of Chicago, and Peretz is in the interdisciplinary Group in Logic at Berkeley. Their list includes the following: simplicity is not outside existence, is not totally subjective, is not necessarily timeless, is not necessarily functional, is not necessary, does not equal perfection, is not necessarily, does not last long in scientific circles, and yet it had an impact on the wider acceptance of Bayesian statistics. (This story contains one of my favorite lines in a book full of interesting lines: Senechal says that Wrinch “crossed disciplinary boundaries easily, without glancing on forcoming trains” [p. 80].) While Senechal’s discussion centers on selection between competing scientific theories rather than forms of mathematical proof, the linguistic distinctions are helpful. Reducing a proof to some sort of minimalism may be quite difficult, sacrificing one type of simplicity for another, just as Serre’s sentence in the Ghyss paper may have simultaneously minimized length of description and ease of comprehension.

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**Figure 7.** Proof without words of the Nicomachus theorem (squared triangular numbers).
difficult or not simple when introduced, but we’ve grown accustomed to their faces. Are they actually simpler, or do they just seem that way? This may be akin to something I tell my students in Advanced Calculus class when introducing a particular maneuver: “The first time you see this, it is a trick. The next time you see it, it is a technique. The next time, it is an old friend.” It is not just the passage of time but the acquired familiarity that can provide a shift toward “simple.”

Jan Zwicky, professor emerita at the University of Victoria, considers an appropriate opposite for simple in “The Experience of Meaning.” Zwicky channels environmental philosopher Arne Næss to distinguish between complicated and complex. Unsurprisingly, Næss sees an ecosystem as complex—intricate, interrelated, and structured—while finding one’s way through a huge unfamiliar city without a map is complicated—disorganized, even messy. “By definition, a complex thing cannot be simple in the sense of having no parts or divisions. It will have multiple aspects, and there are often many different relations between these aspects. But complexity is uncluttered. Everything fits” [p. 94]. This echoes the etymological lesson in the opening article by Ghys. The roots of the words “simple” and “complex” are both related to the French word plier for fold. Something simple has one fold; something complex has many folds. To explain something is to unfold it.

Zwicky continues by evoking the great twentieth-century mathematician Paul Erdős, who claimed that, as Ghys describes it, “somewhere in heaven there is THE BOOK and in it are some jewels, some wonderful proofs and we should work toward these beautiful proofs, simple proofs, elegant proofs” [p. 10]. The Erdős version of simplicity suggests a proof that utilizes just what it needs, with an economy of thought and a style of presentation. Martin Aigner and Günter Ziegler’s book Proofs from THE BOOK [8] is now in its fifth edition, suggesting a continuing interest in this idea of the beauty and simplicity of proofs. Erdős inspired Aigner and Ziegler to write the book and assisted them as they began writing it. The volume begins with Euclid’s proof of the infinitude of primes. Zwicky quotes Marjorie Senechal:

Though no one has seen the book or ever will, all mathematicians know that Euclid’s proof of the infinitude of primes is in it, and no mathematician doubts that computer-generated proofs, the kind that methodically check case after case, are not. The proofs in God’s book are elegant. They surprise. In other words, they are light, quick, exact, and visible [p. 95].

Zwicky continues: “In other words, the proofs in God’s book...are potent with meaning. They may be complex, but they are not complicated; there is no clutter” [p. 95]. And yet although Ghys finds most of the proofs in Aigner and Ziegler “wonderful,” he finds that he does not remember them. They fail the “understanding means remembering” test of his inner simplicity.

As an aside, I find it interesting that there are six different proofs of the infinitude of primes to lead off the book about THE BOOK. I wonder if some are simpler than others by the various definitions of simplicity we see in this collection on Simplicity. On a related note, one of the editors of this volume, Philip Ording, has a new book entitled 99 Variations on a Proof [9].

In his essay referenced above, Andrés Villaveces posits that complexity often provides a key step in the simplification process. There may even be “spiraling, back-and-forth movement between simplification and complexification” along the way [p. 191]. I confess that I first read the title as “Simplicity vs. Complexity,” but the actual title makes more sense. Villaveces’s example of “Simplicity via Complexity” is the combination of Gödel’s completeness and incompleteness theorems. He proposes that one could interpret the incompleteness result as motivation to expand the terrain or “play in a larger sandbox,” as he puts it, leading to the more satisfying result of completeness in a larger setting. I propose that the expansion of our number systems across the centuries, from natural numbers to fractions and negatives and irrationals and complex numbers, added a level of complexity at each stage that simplified by broadening the validity of the sentence “Every equation of this type has a solution.” Here is a more quotidian example. In the midst of a big cleanup project around the house or at the office, I often refer to “the storm before the calm,” flipping the usual phrase: sometimes you need to pull everything out and make a bigger mess in order to reconfigure in a better result. The same holds with proofs and complexity, I guess.

Villaveces sees the question of simplicity in proof as three problems:

The notion itself of a simplest proof is the first and perhaps trickiest problem; the existence of such a simplest proof is a second, independent issue; finally, the question of how to provide such a simplest proof—provided it exists—is a third problem [p. 193].

These are all reasonable concerns. Is the idea of a simplest proof a valid one, and if so, what do we mean by it? Does a simplest proof of a given theorem exist, or is the process of simplification an asymptotic one? And then, of course, the practical question: if the first two questions are answered in the affirmative, how in the world do we do it? Iemhoff, the philosopher referenced above, adds the following: “Mathematics is the science par excellence that can be simple and complex at the same time.... In contrast with the use of the word in daily life, in mathematics, a simple argument does not necessarily mean that it is easy to find” [p. 145].

The authors address all of these questions in careful and multifaceted ways; this summary provides only brief peeks...
at their observations. What of the notion of simplicity in art, and how does it relate to that in mathematics?

Juliette Kennedy swaps her organizer hat for a presenter one; she explores the writings and philosophy of twentieth-century sculptor Fred Sandback in her essay “Kant, Co-Production, Actuality, and Pedestrian Space: Remarks on the Philosophical Writings of Fred Sandback.” (A photo of one of Sandback’s works appears on the cover of the book, reproduced in Figure 8.) Sandback’s pieces are made with acrylic yarn stretched across walls and corners in what he termed “habitable drawings” [p. 39]. He describes some of his work:

Around 1968, a friend and I coined the term “pedestrian space,” which seemed to fit the work we were doing at the time… Pedestrian space was literal, flat-footed, and everyday. The idea was to have the work right there along with everything else in the world, not up on a spatial pedestal [10].

In this sense, art can be something larger than ourselves as long as it successfully meets us where we are: personal and yet accessibly personal for each viewer in a different way. This compares with Arana’s idea of topical closeness above. A simple proof in this sense uses mathematical material that is nearby, pedestrian, not as lacking in inspiration or excitement but as within walking distance. Sandback’s pedestrian space mines the immediacy of nearby space for artistic expression; its topical closeness is an experiential closeness.

These ideas are consonant with the writings of the late twentieth-century American conceptual artist Sol LeWitt:

Conceptual art is not necessarily logical…. Some ideas are logical in conception and illogical perceptually. The ideas need not be complex. Most ideas that are successful are ludicrously simple. Successful ideas generally have the appearance of simplicity because they seem inevitable [11].

His use of the term inevitable suggests that simplicity is related to accessibility. Accessibility may be a key to artistic expression as well as appreciation, just as it can be key to both understanding and communicating mathematics.

Art, then, can be a reduction, in the cooking sense: the process of thickening or intensifying the flavor of a liquid mixture such as a sauce by evaporation. In his essay “The Complexity of Simplicity: The Inner Structure of the Artistic Image,” Finnish architect and former dean of the Helsinki University of Technology Juhani Pallasmaa quotes master twentieth-century Finnish architect Alvar Aalto: “Almost every formal assignment involves dozens often hundreds, sometimes thousands of conflicting elements that can be forced into functional harmony only by an act of will. This harmony cannot be achieved by any other means than art” [p. 18]. As Pallasmaa says, “Instead of analyzing and separating things, art is fundamentally engaged in merging and fusing opposites” [p. 18]. It is more than a merging of opposites: “The ultimate ideal of all art (and an impossibility, we must admit) is to fuse the complexity of human experiences into a singular image…” [p. 21], much as the William Blake poem calls us “[t]o see a World in a Grain of Sand” [12]. It is this simultaneous universality of meaning and the individual process of association and interpretation that provide the challenge and the evocative richness of art, through which "simplicity turns into labyrinthian complexity" [p. 22]. In a similar fashion, the mathematician evaporates out the unnecessary in a reduction process to turn what is simply a proof into a simple proof, even when the ideas involved are larger than the simplicity of the argument may suggest.

So what do the mathematical and artistic claims on simplicity have in common? The authors and editors make no attempt at tying it all together with a neat bow at the end. We end up where we began: “Simplicity is not simple.” Perhaps it is the reduction of the broader to the essential. In mathematics, it is both the culling of the unnecessary and retaining focus on the core meaning of the concept at hand, ultimately toward comprehensibility and even memorability by the general reader. In art, it can be distancing from the subjectivity of the artist toward universality. As Pallasmaa quotes Balthus, twentieth-century Polish-French modern artist: “Great painting has to have a universal

Figure 8. Fred Sandback, Untitled (Fourth of Ten Corner Constructions, Sculptural Study, Yellow Version), c. 1981/2007. Gold acrylic yarn 97 1/8 x 70 x 70 inches (246.7 x 177.8 x 177.8 cm).
meaning…. I want to give painting back its lost anonymity, because the more anonymous painting is, the more real it is” [p. 23]. In both art and mathematics, there is a certain irony in how the universality of a proof or an artistic piece confers personal meaning or individual relatability to the reader or the viewer or anyone experiencing the art, or the art in the proof, on their own terms.

Singers have made a subtle change to the words to the old Shaker song “Simple Gifts” over the years, replacing the article “the” by “a” in the first lines. In the original, the song begins:

‘Tis the gift to be simple, ‘tis the gift to be free,
‘Tis the gift to come down, where we ought to be… [13].

Here, simplicity is seen not as one of many gifts but as THE gift, from Divine Providence, from the Muses, from the Keeper of THE BOOK. The contributors to this volume give us fascinating evidence that they have plumbed the depths of this simple and complex topic. Their sharing is a gift to us.

Simplicity is the first volume in a new Springer series on Mathematics, Culture, and the Arts. If this volume is any indication of the intellectual richness and payoff to come, I look forward to reading Great Circles: The Transits of Mathematics and Poetry and Africa and Mathematics: From Colonial Findings Back to the Ishango Rods, the next books in the series.

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**New and Noteworthy Titles on our Bookshelf**

**February 2020**

**Mind and Matter**
*A Life in Math and Football*
by John Urschel and Louisa Thomas

This twenty-eight-chapter autobiography alternates between the mathematical and athletic experiences of John Urschel, the MIT mathematics graduate student and former offensive lineman for the Baltimore Ravens professional football team. These two universes are almost completely disjoint to Urschel, who describes himself as extremely adept at mental and emotional compartmentalization. For example, the anticipation of the next big game and the stress of finding a mistake in a proof occupy completely different tracks in his mind. Although one might expect a budding young mathematician to have taken a deeply cerebral approach to football, analyzing stances and movements in terms of forces and vectors, Urschel explains that football was a more instinctive activity for him, a physical outlet apart from his rapidly developing mathematical life.

Urschel credits his success in the classroom and on the field largely to hard work, determination, and his supportive family. Urschel, the son of a lawyer and a surgeon, received unwavering parental support for his academic and athletic endeavors. He gives much credit for his success to his upbringing. For example, he recounts how, from an early age, his mother assigned him logic puzzles and mathematics problems in hopes that he might one day become an aerospace engineer. He also recounts many seminal mathematical experiences from his collegiate days: opportunities for undergraduate research and thoughtful mentoring chief among them.

We learn quite a bit about Urschel’s personal heroes. For example, he holds John von Neumann in the highest regard: “few people have been behind so many revolutions… Every American kid, I believe, should know his name.” Throughout the book, Urschel provides frank accounts of and opinions on topics ranging from a frightening concussion that temporarily diminished his mathematical abilities, the threat of chronic traumatic encephalopathy (CTE), and the Jerry Sandusky molestation scandal that rocked Penn State several years ago. Since *Mind and Matter* is aimed at a mass-market audience, the mathematics is kept to a minimum. Overall, the book provides an introspective and deeply personal look at a rising star in the mathematical world.

**The Prime Number Conspiracy**
*The Biggest Ideas in Math from Quanta*
Edited by Thomas Lin

This book assembles thirty-seven of the best mathematics-related articles that appeared in the online magazine *Quanta* ([https://www.quantamagazine.org](https://www.quantamagazine.org)) since its inception in 2012. These highly readable pieces are aimed at the intellectually curious layperson. Students and mathematicians alike will be able to catch up on cutting-edge research and acquaint themselves with rising stars and famous figures, who are frequently profiled, interviewed, and quoted.

Despite the title and cover art, prime numbers are not the only focus of the book. *The Prime Number Conspiracy* covers large swaths of mathematics, from number theory and combinatorics to dynamical systems and probability theory. The articles are grouped into seven chapters: “What’s so special about prime numbers?” (four articles), “Is math the universal language of nature?” (seven articles), “How do surprising proofs discovered?” (eight articles), “How do the best mathematical minds work?” (eight articles), “What can or can’t computers do?” (four articles), “What is infinity?” (three articles), and “Is mathematics good for you?” (three articles).

One can easily imagine that this would be the first book a mathematician who has slept through the last six years would want to read upon waking up.
The AMS Book Program serves the mathematical community by publishing books that further mathematical research, awareness, education, and the profession while generating resources that support other Society programs and activities. As a professional society of mathematicians and one of the world’s leading publishers of mathematical literature, we publish books that meet the highest standards for their content and production. Visit bookstore.ams.org to explore the entire collection of AMS titles.

Women Who Count: Honoring African American Women Mathematicians
By Shelly M. Jones
Illustrated by Veronica Martins

Women Who Count is a mathematical activity book for children interspersed with short biographies of 28 African American women mathematicians from the late 1800s to the present. The first three are about the first three African American women to earn a mathematics PhD in the United States. They are Martha Euphemia Lofton Haynes (Catholic University of America, PhD 1943), Evelyn Boyd Granville (Yale University, PhD 1949), and Marjorie Lee Browne (University of Michigan, PhD 1950). These three women mentored and influenced many of the mathematicians featured in the next chapter, who in turn were active in outreach and education, and took on leadership roles in their departments (some of which were newly desegregated at the time). A chapter of the book is devoted to four mathematicians who contributed to the space race of the 1950s and 60s, recently made famous by Margot Lee Shetterly’s popular book Hidden Figures. The final chapter relates stories of contemporary mathematicians at or near the beginnings of their careers.

Jones is an energetic advocate for exposing African American children to exemplary mathematical role models whom they can relate to and be inspired by. She has given a TEDx talk on Culturally Relevant Pedagogy in Mathematics: A Critical Need, and has presented at teachers conferences and professional development workshops around the country. She is a contributing author for The Brilliance of Black Children in Mathematics: Beyond the Numbers and Toward New Discourse.

As this book is geared toward young children, the puzzles are simple—such as coloring pages, word searches, magic squares, and simple codes—and the biographies are brief and upbeat, but there is more to this book than a way to keep children occupied. By juxtaposing activities with stories of real people, the book succeeds in building a positive and inspiring narrative of human achievement and possibilities.

100 Years of Math Milestones: The Pi Mu Epsilon Centennial Collection
By Stephan Ramon Garcia and Steven J. Miller

This hefty book brings together a collection of 100 problems in celebration of the 100th anniversary of the math honor society Pi Mu Epsilon in 2012. The problems were originally published in four different issues in the Pi Mu Epsilon journal, and then later collected together to form the chapters of this book, one for each year between 1912 and 2012. Each chapter opens with a mathematical event or idea loosely related to the year, followed by the statement of the problem, and a discussion of the surrounding mathematics.

The chapters are wide-ranging and varied both in topic and in level of sophistication. Some chapters center on famous mathematical figures, like Paul Erdős, Georg Cantor, Alan Turing, Julia Robinson, and Martin Gardner, while others focus on seminal math problems (some solved, some still open) like the Riemann hypothesis, Fermat’s Last Theorem, the Poincaré conjecture, and the Langlands program. There are also chapters that discuss well-known puzzles like the Rubik’s cube; centers for mathematical outreach like AIM and the National Museum of Mathematics; and tech age phenomena like RSA encryption, TeX, Mathematica, and arXiv.

In short, this book is a collection of engagingly written snapshots of 100 topics that will perk up the ears of undergraduate math majors and graduate students eager to familiarize themselves with the world of mathematics.
The Present State of Affairs

A number of intractable aspects of the United States’ educational system contribute to the continued underrepresentation of particular groups in graduate mathematics programs. These groups include, but are not limited to, women, people of color, sexual minorities, and gender minorities. Discussing the issue of increasing the number of women and, in particular, women of color in STEM, Dr. Dandrielle Lewis, associate professor of mathematics and director of liberal studies at the University of Wisconsin–Eau Claire states, “Contributing factors to the low representation include lack of positive and engaging STEM experiences, negative classroom experiences, lack of self-confidence in their mathematics skills, and ‘chilly’ campus and office climates” [2]. These factors result in low graduate mathematics application rates from historically underrepresented groups (HUGs).

Reports from the interviewed universities generally indicate a shortage of applicants from HUGs. Columbia University found that in its 2017–2018 graduate mathematics application pool, only 12 percent of applicants were women. Berkeley reported that, since 2016, approximately 18 percent of the graduate mathematics program applicants have been women. Chicago estimated that around 15 percent of its graduate mathematics applicants are women. Regarding underrepresented people of color, the institutions surveyed reported that rates of application and matriculation are significantly lower than their representation in the US population, but did not report exact numbers.

Low application rates yield low rates of admission and matriculation which, in turn, leads to individuals from HUGs earning mathematics doctorates at rates far below their demographic representation in society. With regard to gender diversity, the American Mathematical Society’s
(AMS’s) Report on the 2016–2017 New Doctoral Recipients reveals that women who are US citizens were awarded 24.4 percent of pure math doctorates [3, Table D.4, II and III]. Moreover, US women fared worse at large private institutions, earning only 20 percent of mathematics PhDs awarded to US citizens [3, Table D.4, VII].

Correspondingly, the AMS report found that from 2016–2017, African American students received only 2.1 percent of the pure mathematics PhDs earned by US citizens (14 of 664) [3, Table D.4, II and III]. Large private institutions reported granting no pure mathematics doctorates to African American students [3, Table D.4, VII]. Hispanic/Latinx US citizens garnered less than 3 percent of pure mathematics PhDs (19 of 664) [3, Table D.4, II and III], with only two such degrees being granted from large private institutions [3, Table D.4, VII]. Hawaiian or Pacific Islanders earned less than 1 percent of such degrees (3 of 664) [3, Table D.4, II and III], and students of American Indian or Native Alaskan descent also earned less than 1 percent (2 of 664) [3, Table D.4, II and III]. Large private institutions did not award any PhDs to students of these backgrounds [3, Table D.4, VII].

Low rates of earned PhDs have a feedback effect. A lack of doctorates among HUGs leads to low rates of faculty hiring from HUGs, resulting in mathematics faculties that are fairly homogeneous. Dr. Helen Grundman, AMS’s former director of education and diversity, has noted that having fewer relatable faculty yields fewer mentors, a probable factor in the depletion of diversity as you move up the ranks [4].

Mechanisms for Change

Initiatives for fostering diversity and inclusivity abound in academia, including at the top-rated mathematics graduate programs that we engaged. These initiatives can be broken down into three stages: the pre-application period, the admission process, and the post-acceptance departmental culture.

The Pre-Application Period

Effecting change before graduate school application time can be a powerful tool for recruiting a diverse class. Organizations like the National Association of Mathematicians (NAM) are focused on promoting the mathematical development of underrepresented minorities, and are a valuable resource for those looking to diversify their applicant pool. Dr. Edray Goins, professor of mathematics at Pomona College and president of NAM, is a fervent advocate of early outreach and recruitment. In his view, building relationships with historically black colleges and universities (HBCUs) is an important step in recruitment that many universities should utilize to increase diversity. Dr. Goins notes that although only around 10 percent of African American students attend HBCUs, nearly 40 percent of mathematics degrees granted to African American undergraduates come from those institutions.

In our discussions with Dr. Goins, he highlighted two ongoing programs focused on strengthening relationships between the community of mathematical scholars and historically underrepresented students: NAM’s Undergraduate MATHFest [5] and the Field of Dreams Conference [6]. Funded by NAM, the Undergraduate MATHFest is an annual fall meeting held around the country on a rotating basis [5]. Although open to all students with aspirations of attending graduate school in the mathematical sciences, the focus of the symposium is on undergraduates from HBCUs. The meeting serves dual purposes: it provides attending students with a platform to present the results of their undergraduate research, and it provides attending universities a forum for meeting students and promoting their graduate programs. Dr. Goins believes that attending NAM’s Undergraduate MATHFest is an excellent way for a university to send a strong message to African American students that they are welcome and wanted in their program.

Dr. Goins also highlighted the annual Field of Dreams Conference, sponsored by the Math Alliance [6]. The Math Alliance has a broad list of goals aimed at improving the diversity of doctoral degree holders in the mathematical sciences, and this conference represents just one of their initiatives. The Field of Dreams Conference focuses on first-generation students and those traditionally underrepresented in mathematics. Participants are informed about graduate programs in mathematics, meet with faculty members, and are advised on the application process, preparation for graduate level studies, and career opportunities. Graduate programs can recruit at the conference and invite student attendees to apply to their institutions. In addition, graduate programs that attend are given access to the names, contact information, CVs, and transcripts of all student participants. Actively inviting historically underrepresented students to apply sends a strong message of welcome to such students, and would likely help increase the diversity and strength of the applicant pool.

In addition to these conferences, the National Science Foundation funds a large number of REU programs to support undergraduate research at various host institutions. Dr. Goins views REUs that bring students from HUGs to campus as another useful recruitment tool, one that allows students to become familiar with particular institutions and programs before considering applying for advanced degrees.

We found that some of the surveyed institutions are using pre-application recruitment tools to increase the diversity of their applicant pool. For example, the University of Chicago as a whole recently instituted the Discover UChicago program. The program offers talented US students from historically underrepresented groups an expense-paid weekend at the university. There they can
attend graduate admissions workshops, speak with the faculty and graduate students, and explore the graduate school itself [7]. The mathematics program successfully recruited one student who attended Discover UChicago to their 2019–2020 entering class. Chicago’s mathematics department also uses an REU to recruit students from HUGs. Additionally, graduate students at the university have organized GRIT (Graduate Recruitment Initiative Team) to encourage undergraduates from HUGs at other universities to apply for graduate school at Chicago [8]. The GRIT program originated in biology, and mathematics graduate students are beginning to participate.

Similarly, MIT seeks to recruit a diverse student body through on-campus research opportunities. The university-wide MIT Summer Research Program [9] endeavors to increase the number of underrepresented minorities and underserved (e.g., low socio-economic background, first generation) students in research enterprises. Their Summer Program in Undergraduate Research (SPUR) in mathematics pairs one or two selected students with a graduate student mentor to work on a research project over a six-week period. To increase the diversity of students participating in that program, in 2017 they launched SPUR+ to specifically target students from HUGs who may not have the research background expected of SPUR participants [10]. SPUR+ begins three weeks prior to SPUR in order to provide guided reading in preparation for research.

To build a pipeline for admissions, MIT’s and Stanford University’s mathematics programs both report engaging in outreach efforts at early educational levels. For instance, Stanford University runs Math Circles at the elementary, middle, and high school levels, as well as a summer camp for high school students called SUMaC. Stanford reports that SUMaC has been an effective tool for recruiting outstanding talent, with a number of their top female mathematics majors having attended the camp. MIT has PRIMES (Program for Research in Mathematics, Engineering and Science) Circle, a free after-school math enrichment program for talented high school students living within commuting distance of Boston. The goal of this program is to increase diversity in the mathematical community by helping strong students from HUGs pursue their interest in mathematics and setting them on a path toward pursuing a math-based major in college. PRIMES Circle has about fifteen students per year, 80–90 percent of whom are women and 25 percent of whom are African American or Hispanic. MIT also has MathRoots, a two-week residential summer math camp for promising high school students from HUGs, selected from across the US. MathRoots’ goal is to engage enthusiastic young mathematics students in creative problem solving by immersing them in a community of like-minded peers and mentors. The program has enrolled twenty students from HUGs annually over the past five years, with almost all of these students attending a four-year college and more than a quarter enrolling at MIT for their undergraduate studies.

Consistent with best practices encouraged by Dr. Goins, some of the top-rated mathematics programs actively recruit at local and national conferences that support undergraduate research and professional development for HUGs. Both MIT and Princeton report working with institutional partners to staff booths at the annual SACNAS (Society for Advancing Chicanos, Hispanics and Native Americans in Science) National Diversity in STEM Conference [11]. MIT uses this conference as an opportunity to invite students to apply to the MIT Summer Research Program.

The Admissions Process

With a more diverse applicant pool, the next step to consider when seeking to diversify a graduate cohort is how to evaluate applicants during the admissions process. Our conversations with university representatives revealed that the top-rated mathematics graduate programs are exploring how their admissions processes influence the diversity of their classes. Berkeley has altered the ranking of admission files by including broader impact scores in candidate rankings and de-emphasizing GREs. The university now uses a more holistic score: one that considers grades, letters of recommendation, and research experiences. It also educates its admission committee members about implicit bias before they review applications. The University of Chicago and Princeton University have also de-emphasized GRE scores and instead rely on holistic measures that account for student background when reviewing applications.

Of course, just because a program admits a diverse group of students does not mean that those students will choose to attend. The nature of the offer from the university plays a critical role in the student’s decision-making process. Many of the interviewed programs emphasized the importance of making competitive offers to students from HUGs. For example, the math department at Berkeley is able to nominate fifteen students each year for a campus-wide internal competition for Berkeley and Chancellor Fellowships. These both come with two full years of fellowship support. Typically, 12–15 of the students nominated by the math department are offered the fellowship by the university, with 8–12 being offered to underrepresented students. Columbia University utilizes their Provost Diversity Fellowship, a one-time award of $8,000 added to an offer, to attract students from HUGs.

Departmental Culture

In addition to the particulars of an offer, the culture of a university’s mathematics graduate program and the success of its underrepresented students will bear on whether an offer of attendance will be accepted. Considerations of culture and success may well create a feedback mechanism in the acceptance of offers for graduate study. A program
with a track record of progress in diversifying and graduating underrepresented students is likely to have an edge in appealing to students from HUGs. For example, Stanford’s mathematics graduate student body is composed of 30 percent women, and they have attracted six of the last eight recipients of the Alice T. Schafer Mathematics Prize for Excellence in Mathematics by an Undergraduate Woman. Representatives from Stanford attribute these numbers to their progressive recruiting efforts.

As efforts to encourage HUGs to apply and matriculate start achieving positive results, mathematics graduate programs can also work to create an environment in which underrepresented students thrive. Representatives from many of the top-rated mathematics programs that we engaged mentioned that they are aware of, and actively addressing, issues of departmental culture. Some of the initiatives to help students from HUGs thrive begin as the students are preparing to embark on their graduate education. As examples, the University of California, Berkeley and Harvard University both have pre-first-year summer programs for HUGs. The program at Harvard provides admitted students coaching over the summer to prepare them for qualifying exams taken upon entering graduate school. Harvard reports that this program has worked well, with almost all of their students passing the qualifying exams at the beginning of their first year.

Dr. Ami Radunskaya, professor at Pomona College and past president of the Association for Women in Mathematics, has years of experience in the EDGE (Enhancing Diversity in Graduate Education) program. The EDGE program “seeks to create, identify and disseminate programs and strategies that improve the persistence of women and minority graduate students, and contribute to the development of a diverse mathematical community” [12]. From her discussions with students, she has learned the importance of fostering a collaborative culture among graduate students. Further, recognizing that students have a diversity of mathematical backgrounds, programs that offer opportunities to fill curricular gaps without stigma also foster the sort of supportive environment in which all students can thrive.

Several of the universities surveyed have programs at the departmental and institutional levels aimed at fostering an inclusive culture. At the department level, Columbia University has an active chapter of the Association of Women in Mathematics. MIT hosts a Women in Math Lunch Seminar series that brings together women graduate students, postdocs, faculty, and even some undergraduates to hear a senior female mathematician talk about her career and research. Berkeley and Stanford have active Noetherian Ring groups. Berkeley also described several other structural initiatives, including the formation of an Equity & Inclusion Committee two years ago to evaluate its efforts to support a diverse student body. The mathematics department at Berkeley has also created an in-house sexual harassment and violence prevention training workshop for new graduate students, run by graduate students. Stanford University has several departmental mentoring programs organized by graduate students and overseen by faculty. For instance, one of their programs pairs an incoming student with a more experienced graduate student for mentorship. Stanford’s department also has the Stanford Women in Mathematics Mentoring program and a more general teaching-assistant mentoring program.

Several programs that support HUGs are institutional, rather than departmental-level, initiatives. Columbia University offers a Navigating Academia from the Margins workshop series through its Graduate School of Arts and Science’s Office of Academic Diversity and Inclusion. Stanford University participates in the pilot Wellness Information Network for Graduate Students program, which includes wellness ambassadors who actively work with graduate students to ensure their general well-being. Princeton University’s Graduate Scholars Program is designed to enhance and support the academic, social, and community development of HUGs during their first year of graduate studies. To ensure that the institution follows up with these students after their first year, the Access, Diversity, and Inclusion (ADI) team offers workshops on academic skills, time management, self-care, and financial planning, as well as informal social activities to support HUGs across academic disciplines. The deans of the ADI advise graduate students, supporting them through key milestones and guiding them toward completion and life beyond Princeton.

In Summary

Overall, our conversations with representatives from seven of the top-rated pure mathematics graduate departments revealed that the diversity of their graduate student body is important to each of them. We found that their initiatives operate on many levels: in how they shape the applicant pool through outreach initiatives, in developing more holistic admissions criteria, and in reshaping the culture of their mathematics department to be more inclusive of HUGs. With continued attention, we can work toward building a mathematics community that more accurately reflects the demographics of the nation.

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References

Credits
Photo of Jana L. Gevertz is courtesy of The College of New Jersey.
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Institute for the Quantitative Study of Inclusion, Diversity, and Equity (QSIDE)

Chad M. Topaz, Maria-Veronica Ciocanel, Phoebe Cohen, Miles Ott, and Nancy Rodríguez

Introduction to QSIDE
Consider the following vital questions: Are Black and Latinx artists represented proportionally in US museums? Why are some professional fields dominated by men and others by women? How can judges achieve equitable outcomes for defendants? The mathematical sciences have the potential to address these questions at a level of detail undreamt of a decade ago. The Institute for the Quantitative Study of Inclusion, Diversity, and Equity, Inc. (QSIDE) brings quantitative expertise together with expertise from the social sciences, humanities, and arts to discover the impact and scope of injustices, and to build solutions to remedy them.

Established as an independent nonprofit 501(c)3 corporation in 2019, QSIDE is a multi-institutional network of scholars and practitioners. Our vision is inspired by civil rights leaders such as educator and investigative journalist Ida B. Wells, who said that “the way to right wrongs is to shine the light of truth upon them.” For QSIDE, quantitative methods are one form of this light of truth. QSIDE has a four-pronged approach to achieving our vision. First, we incubate and conduct cross-disciplinary research that leverages quantitative tools. Second, we partner with people and organizations empowered to use our discoveries. Third, we build community among researchers and practitioners working on social justice issues through virtual and in-person networking and events. Finally, we increase capacity for work aligned with our vision by equipping those researchers and practitioners with appropriate methodologies and tools.

With this communication, we are reaching out to the mathematical sciences community. Our goal is to emphasize the need for social justice research and to highlight ways in which our community is uniquely positioned to contribute to this work. To motivate the QSIDE research agenda, we begin by introducing our framework for the concepts of inclusion, diversity, and equity, and by discussing the need for quantitative approaches to social justice.

Inclusion, Diversity, and Equity
Any potential advancement of inclusion, diversity, and equity (IDE) depends on a shared understanding of these values; we now describe QSIDE’s qualitative framework for them. Diversity refers to all of the differences that make individuals unique. Axes of diversity include, but are not limited to, gender, gender expression, sex, sexual orientation, race, ethnicity, national origin, age, religion, disability status, family status, and socioeconomic background. Empirical research in psychology and organizational theory has documented that diversity can drive creativity, good
decision making, and a positive feedback resulting in more diversity, among other benefits (Markus and Nurius 1986; Galinsky et al. 2015).

In professional and academic settings, we often discuss the diversity of groups. But merely accounting for the identities and characteristics represented in a group is limiting. The notion of equity expands on diversity. In considering equity, we ask these questions: Who wants to be in the group but cannot be? What is stopping them, and how can those barriers be removed? It is important to distinguish between equity and another notion, namely, equality. There is a popular cartoon that tries to explain the difference between these concepts through a drawing of three people of different heights trying to look over a tall fence by standing on boxes (Craig 2016). Equality is depicted as treating everyone equally: the boxes are all the same height, so the shortest person can’t see. Equity is depicted with boxes of differing heights, allowing all three people to look over the fence. One aspect of this cartoon is misleading, though, because it implies that it is a characteristic of the shortest person, namely their height, that requires a solution. It has been pointed out that inequities are not due to individuals themselves, but rather, are due to conditions that society creates. A more apt cartoon might picture three people of similar heights on a sloping ground, so inequitable access is properly attributed to external conditions (Sippin the EquiTEA 2018).

Even with diverse groups and equitable access, not all members of a group may feel welcome or heard. This is a challenge of inclusion. To make any community more fair and just requires a deep consideration of all three dimensions: inclusion, diversity, and equity. For instance, without attention to inclusion, initiatives aimed at expanding diversity often fail (Purity et al. 2017; Diversity Doesn’t Stick Without Inclusion 2017).

Another crucial IDE-related idea is intersectionality, first proposed by Kimberlé Williams Crenshaw in 1989 to describe the unique challenges that Black women face due to the intersecting identities of their gender and race (Crenshaw 1989). Crenshaw and others have broadened the term to describe the specific challenges of groups with multiple intersecting identities including LGBTQ status, gender, race, ethnicity, and disability. An intersectional approach is one that recognizes the unique challenges faced by those with multiple minoritized identities.

**Quantitative Approaches to Social Justice**

From the women’s suffrage movement to the battle for civil rights and in many other struggles, people have long sought justice by protesting in the streets, fasting, inspiring others to speak up, and a variety of other approaches (Morris 2001; Crawford 2003). While social justice efforts continue to progress, two steps forward are often followed by one step back, as with the recent rollback of the Voting Rights Act (The Voting Rights Act n.d.). At present, we live in a world with unprecedented computational power and access to data, potentially providing the opportunity to “step forward” by studying information about patterns and trends in society, hypothesizing causes, and critically examining problems and solutions.

The use of data to understand injustice is particularly important during a time of misinformation. While the globalization of information has brought many benefits, there are, nonetheless, well-documented cases of purposeful misinformation with negative consequences. For example, misinformation has supported the rhetoric that a wave of invading immigrants has brought increased crime to the US (Becker 2019).

Even in the absence of purposeful misinformation, data can address misunderstandings. One study of the perception gap between voters of political parties, for instance, asked Democrats to estimate what percentage of Republicans believe “properly controlled immigration can strengthen America” (Mounk 2019; Yudkin, Hawkins, and Dixon 2019). The Democrats estimated 52%, while in reality, 85% of Republicans in the study agreed with the statement. Conversely, Republicans estimated that 54% of Democrats are “proud to be American,” whereas the measured value was 82%.

QSIDE is a community of scholars and practitioners working at the intersection of IDE, data, mathematics, and computing. We hope that our work will address challenges that have yet to be recognized as well as those that are already mainstream but poorly understood. We strive to provide mechanisms for scholars of diverse backgrounds, experiences, and interests to network and collaborate, and to reap the benefits of advancing their research while making the world a more just place. With deep roots in the mathematical sciences community and a research-into-action philosophy, QSIDE also depends pivotally on the collaboration of humanists and social scientists who are trained to think critically about social justice issues. Their participation ensures that research is correctly motivated, designed, executed, and interpreted, and that quantitative frameworks are transparent and minimize bias; see, e.g., Weapons of Math Destruction (O’Neil 2016).

QSIDE develops projects in two ways. First, potentially in collaboration with individuals and organizations empowered to influence IDE, the director conceives projects and recruits participants from the roster of QSIDE-affiliated scholars. Second, scholars who are currently conducting research that fits within QSIDE’s vision may bring projects under the umbrella of the institute, whether or not they are already affiliates. While QSIDE’s uniqueness derives from its focus on IDE, its community ties, and its collaborative and interdisciplinary approach, the organization exists alongside other groups concerned with quantitative approaches to challenges in society: Data Science for
Previous and Ongoing Work

From STEM to the arts to the justice system and more, QSIDE affiliates are active on a range of projects. We briefly describe several of these to show the variety of approaches that can provide social justice insights, including data mining, statistical analysis, mathematical modeling, analysis, and simulations.

A 2016 data mining study introduced a large-scale gender inference method in order to assess gender representation on 435 journal editorial boards in the mathematical sciences (Topaz et al. 2016). This study, which was conducted by an applied mathematician and computer scientist who consulted with gender studies scholars, relied heavily on Amazon Mechanical Turk (MTurk). MTurk is an online crowdsourcing labor platform that allows requesters to hire workers to complete Human Intelligence Tasks, that is, tasks that a computer cannot be easily programmed to do. In (Topaz and Sen 2016), workers retrieved editorial board rosters from the internet and made gender inferences for each editorial board member. These inferences were validated by multiple workers and a random subset was validated by experts. In 2016, women made up approximately 16% of tenure-stream faculty positions in doctoral-granting mathematical sciences departments in the US. This study found that of the 13,000 editorships examined, only 8.9% were held by women, suggesting severe underrepresentation. Statistical tests revealed specific journals, subfields, publishers, and countries that significantly exceeded or fell short of the average.

A study of artists in major US museums (Topaz et al. 2019) brought together mathematicians, applied mathematicians, statisticians, and art scholars to expand the methodologies described above. This work inferred the gender, ethnicity, birth year, and national origin of over 10,000 artists whose works are held in the collections of eighteen museums. Of these, 85% were white and 87% were men. One might be tempted to attribute the high representation of white men to the collection mission of a museum, that is, its particular geographic and historical foci, as measured through national origins and birth years of artists. A statistical clustering analysis revealed that this attribution is not correct. More specifically, museums were clustered in two different ways: first, in the multidimensional space describing the overall gender and ethnicity profiles of their artists, and second, in the space describing their national origins and birth years. These two clustering schemes had essentially no correlation. That is to say, museums with very similar collection mission profiles can display markedly different levels of representation of women and non-whites, suggesting that a museum wishing to increase diversity in its collection might do so without changing its geographic or temporal emphasis. As a result of the publication of this work, members of the research team were invited to discuss diversity issues with the staff of one museum in the study and were invited to study the internal proprietary data of a second museum to provide further insights on diversity.

In the realm of mathematical modeling, (Clifton et al. 2019) considered a minimal model for the roles of bias and homophily (self-segregation) in representation of women in professional hierarchies. The model consists of a system of differential equations for the fraction of women at each hierarchical level. The system includes parameters accounting for gender bias in the hiring process and for homophily, which in this context means self-selection to apply for a position or for promotion. Using dynamical systems analysis and parameter fitting based on data sets from real professional hierarchies, the investigators predicted fields that may have a stronger bias against women (e.g., mathematics and chemistry) and fields that indicate stronger homophily (e.g., engineering and nursing). In addition, the model provides a prediction of the time to achieve gender parity in various fields, and may be used to suggest the most effective interventions to reach this goal.

Adopting a stochastic outlook, (Maes 2018) constructed Markov chain models to study affirmative action. The undergraduate author of this work spent one year conducting background research on policy, history, and law, and spent one year performing quantitative modeling. Affirmative action policies are hiring and recruiting practices designed to combat discrimination against members of certain groups. These policies have been the subject of frequent litigation over the past forty years. Affirmative action policies are typically assessed through retrospective data analysis. That is to say, after a policy is put in place, subsequent shifts in demographics are measured statistically. In contrast, the Markov chain model of (Maes 2018) was developed for ex ante (based on a forecast) assessment of race-based affirmative action policies in US undergraduate college admissions and was calibrated on data from the University of California, Berkeley. The model predicts how affirmative action admissions policies will affect student body demographics, and it also suggests other interventions that might diversify a student body without the legal risks associated with affirmative action.

QSIDE researchers have ongoing projects in a range of applications related to social justice. One project applies network analysis and random process modeling to understand mass shooting events. A more theoretical project involves a sociologist and strives to build a quantitative theory of group diversity in which individuals are modeled as elements of a Boolean semiring on a multiplex network. To better understand equity in the justice system, QSIDE has partnered with volunteers from the Berkshire County...
Sign up for the QSIDE mailing list at our website, qsdeinstitute.org

• Support QSIDE financially through our website. As QSIDE’s mission falls outside the scope of most research funding opportunities, we depend on private citizens and organizations for support.

• Get involved with quantitative IDE research. If you are interested in becoming part of our network, please contact us through the website.

ACKNOWLEDGMENTS. We are deeply indebted to Andrew Bernoff, Sara Clifton, Manuel Lladser, Karen Saxe, and two anonymous reviewers for helpful feedback on this article.

Bibliography


The Future of QSIDE

Beyond continuing the interdisciplinary QSIDE research agenda, we have several goals in mind. We plan to organize research workshops in order to highlight quantitative IDE research and to bring interested scholars together to connect. In order to increase capacity for the type of research we do, we hope to run training boot camps. A data science boot camp will provide scholars from the humanities and social sciences with sufficient background to be able to collaborate effectively with quantitative scientists. This boot camp will be based on a current boot camp that two QSIDE affiliates currently run at Williams College. Conversely, a social sciences and humanities boot camp will train quantitative scholars in some fundamentals of inquiry in those fields. Then, in order to incubate research, QSIDE will offer research grant opportunities that will support multidisciplinary work. To further effect change, we plan to develop a toolkit for researchers doing QSIDE-aligned work to help them influence policy. Ongoing support of participants in all of the aforementioned initiatives will be critical for their long-term effectiveness.

Get Involved

For some mathematical scientists, social justice work is a labor done separately from research. A central message of QSIDE’s work is that this separation need not always exist. As a member of the mathematical sciences community, you have a powerful set of skills at your disposal and these can be used to address social justice. Social justice issues are challenging, complex, and benefit from the careful, abstract, quantitative thinking that we, as mathematicians, are trained to do. For those readers seeking to support a closer connection between mathematical sciences research and social justice, here are ways to contribute:

• Sign up for the QSIDE mailing list at our website, qsdeinstitute.org

• Share our website on social media and in your other networks.

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Credits
Topaz author photo is courtesy of Macalester College.
Ciocanel author photo is courtesy of the AMS.
Cohen author photo is by Ashley Cart.
Ott and Rodriguez author photos are courtesy of the authors.

Applications are invited for:-

**Department of Mathematics**
**Associate Professors / Assistant Professors / Research Assistant Professors** (Ref. 190002DO)

Founded in 1963, The Chinese University of Hong Kong (CUHK) is a forward-looking comprehensive research university with a global vision and a mission to combine tradition with modernity, and to bring together China and the West. The Department of Mathematics in CUHK has developed a strong reputation in teaching and research. Many of the faculty members are internationally renowned and are recipients of prestigious awards and honours. The graduates are successful in both academia and industry. The Department is highly ranked internationally. According to the latest rankings, the Department is 28th in the QS World University Rankings, and 31st in the US News Rankings.

The Department is now inviting applications for substantiable-track faculty posts at Associate Professor / Assistant Professor / Research Assistant Professor levels in all areas of Mathematics. Applicants should have (i) a relevant PhD degree; and (ii) good potential for research and teaching. Those with outstanding research accomplishments are encouraged to apply. Research Assistant Professor is required to teach only one course each academic year.

Appointments will normally be made on contract basis for up to three years initially commencing August 2020, which, subject to mutual agreement, may lead to longer-term appointment or substantiation later. (Substantiation is not applicable to Research Assistant Professorship.)

Applications will be considered on a continuing basis until the posts are filled.

**Application Procedure**
The University only accepts and considers applications submitted online for the posts above. For more information and to apply online, please visit http://career.cuhk.edu.hk.
Gift from Uhlenbeck Funds Karen EDGE Fellowship

Sophia D. Merow

Hours after Karen Uhlenbeck learned she had been awarded the 2019 Abel Prize, a journalist asked her a question she was unprepared to answer: What would she do with the prize money?¹

“It had not crossed my mind,” Uhlenbeck later recalled. By the time a congratulatory email from Rhonda Hughes arrived, however, Uhlenbeck had come to the beginnings of a decision.

“Even though she was in the middle of a media storm,” Hughes remembers, “she replied and said that she wanted to talk about how to use the Prize money. She was thinking of using it to support mathematicians who were underrepresented minorities.”

Sophia D. Merow is special projects editor and Notices assistant. Her email address is merow.notices@gmail.com.

¹The Abel Prize comes with a monetary award of six million Norwegian kroner.

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DOI: https://dx.doi.org/10.1090/noti2017

Uhlenbeck credits her husband, fellow mathematician Robert Williams, and the February 2019 New York Times article² about Edray Goins with influencing her thought process.

“The description of the difficulties of minority mathematicians in being accepted rang a bell,” she says of the latter. Uhlenbeck thinks that while the environment for women in research mathematics has greatly improved in her lifetime, there’s still work to be done to ensure would-be mathematicians of all demographics feel welcome. “It seemed timely and appropriate to turn attention onto groups that are still not comfortable and at home in mathematics,” she says.

With a philanthropic direction in mind, Uhlenbeck then did what she had, “at a ripe old age, learned is the right thing to do.” She sought help from people whose work in the relevant area she respected and opened up a conversation with Hughes.

In 1998, Hughes and Sylvia Bozeman, of Bryn Mawr and Spelman Colleges, respectively, founded the Enhancing Diversity in Graduate Education (EDGE) Program with the goal of increasing the number of women, particularly those from underrepresented groups, who earn PhDs in the mathematical sciences. Uhlenbeck was aware of EDGE from its inception, but her “first close encounter” (Bozeman’s words) with the program came in 2008, when EDGE sponsored the Promoting Diversity at the Graduate Level in Mathematics forum at MSRI. She served on the conference’s organizing committee and gave one of the opening addresses.

Uhlenbeck saw in EDGE, which began as a summer bridge program, something missing from her own efforts to address the gender imbalance in mathematics. Uhlenbeck and Chuu-Lian Terng cofounded Women and Mathematics (WAM), an outreach program first connected with

²“For a Black Mathematician, What It’s Like to Be the ‘Only One’” (https://nyti.ms/2tpBwW4)
Many mathematicians are sympathetic to colleagues just starting out, but by mid-career, Uhlenbeck observes, “the expectation is that you will become a mentor and encourage younger people, while demands from family and society as a whole intensify. I think it is at mid-career that one enters the firing line.”

The Karen EDGE Fellowship will “support and enhance” the research programs and collaborations of mid-career mathematicians with funds—$8,000 per year for three years—to offset the cost of travel and supplies associated with a proposed research project. Through a partnership with the Institute, fellows will enjoy one expenses-paid trip per year to IAS, where an annual conference will encourage contact between awardees and senior members of the research community. While the fellows will no doubt benefit from their time in Princeton, Uhlenbeck points out that the Institute will profit too. “The Women and Mathematics Program at IAS introduced many young women to IAS,” she says, “and paved the way to improving attitudes and the environment for women at IAS.” Uhlenbeck also looks forward to personally meeting researchers with whom she otherwise might not cross paths: “Some good math will surely come out of it!”

Members of the mathematics community interested in supporting the “good math” of a diverse cohort of up-and-comers can be a part of the Karen EDGE Fellowship effort. Uhlenbeck could have earmarked her donation to endow one or two fellowships, but she opted instead to directly fund many. “I am hoping that my gift will inspire other mathematicians to donate money for endowments,” she says, “but I am too old and impatient to wait indefinitely to see something happen.”

EDGE thus has Uhlenbeck to thank for both her financial contribution and the example she has set for her peers. “I applaud the generosity of spirit which Karen Uhlenbeck exhibited in this gift and in her life work, always ‘lifting as she climbs,’” says Bozeman of Uhlenbeck’s donation. “We truly hope that others will see this as an area of need and want to contribute to a meaningful enterprise where they can help make a difference in the diversity of the mathematics community at its highest levels.”
COMMUNICATION

A Word from Karen. I would like to encourage mathematicians to consider making donations to the EDGE Foundation for the endowment for the fellowships. Also to get to know personally some of the remarkable men and women who are succeeding at mathematics who should not remain at the edges of the community. We all experience what it is to be outside the community. The first day at a new job, the first visit to a foreign math department where everybody knows everybody and nobody talks to us. Even worse, trying to switch fields and having nobody willing to explain anything. Actually, I think the very best mathematicians are good at accepting everybody who is interested in mathematics. We are natural teachers and sometimes good listeners. Just remember this!

See https://www.edgeforwomen.org/karen-edge-fellowship-program for additional details of the Karen EDGE Fellowship.
For recent Notices coverage of Uhlenbeck’s mathematics, see https://www.ams.org/journals/notices/201903/rnoti-p303.pdf.

Sophia D. Merow

Credits
Photo of Abel Prize ceremony is courtesy of Elizabeth Boluch Wood, IAS.
Karen EDGE Fellowship logo design is by Tarah Paul, Tai-Danae Bradley, and John de Pillis.
Author photo is by David Gabel.
Vice President or Member at Large
One position of vice president and member of the Council *ex officio* for a term of three years is to be filled in the election of 2020. The Council intends to nominate at least two candidates, among whom may be candidates nominated by petition as described in the rules and procedures.

Five positions of member at large of the Council for a term of three years are to be filled in the same election. The Council intends to nominate at least ten candidates, among whom may be candidates nominated by petition in the manner described in the rules and procedures.

Petitions are presented to the Council, which, according to Section 2 of Article VII of the bylaws, makes the nominations.

Prior to presentation to the Council, petitions in support of a candidate for the position of vice president or of member at large of the Council must have at least fifty valid signatures and must conform to several rules and procedures, which are described below.

Editorial Boards Committee
Two places on the Editorial Boards Committee will be filled by election. There will be four continuing members of the Editorial Boards Committee.

The President will name at least four candidates for these two places, among whom may be candidates nominated by petition in the manner described in the rules and procedures.

The candidate’s assent and petitions bearing at least 100 valid signatures are required for a name to be placed on the ballot. In addition, several other rules and procedures, described below, should be followed.

Nominating Committee
Three places on the Nominating Committee will be filled by election. There will be six continuing members of the Nominating Committee.

The President will name at least six candidates for these three places, among whom may be candidates nominated by petition in the manner described in the rules and procedures.

The candidate’s assent and petitions bearing at least 100 valid signatures are required for a name to be placed on the ballot. In addition, several other rules and procedures, described below, should be followed.

Rules and Procedures
Use separate copies of the form for each candidate for vice president, member at large, member of the Nominating or Editorial Boards Committees.

1. To be considered, petitions must be addressed to Carla D. Savage, Secretary, American Mathematical Society, 201 Charles Street, Providence, RI 02904-2213 USA, and must arrive by 24 February 2020.

2. The name of the candidate must be given as it appears in the American Mathematical Society’s membership records and must be accompanied by the member code. If the member code is not known by the candidate, it may be obtained by the candidate contacting the AMS headquarters in Providence (amsmem@ams.org).

3. The petition for a single candidate may consist of several sheets each bearing the statement of the petition, including the name of the position, and signatures. The name of the candidate must be exactly the same on all sheets.

4. On the next page is a sample form for petitions. Petitioners may make and use photocopies or reasonable facsimiles.

5. A signature is valid when it is clearly that of the member whose name and address is given in the left-hand column.

6. When a petition meeting these various requirements appears, the secretary will ask the candidate to indicate willingness to be included on the ballot. Petitioners can facilitate the procedure by accompanying the petitions with a signed statement from the candidate giving consent.
Nominations by Petition

The undersigned members of the American Mathematical Society propose the name of
_________________________________________________ as a candidate for the position of (check one):

☐ Vice President (term beginning 02/01/2021)
☐ Member at Large of the Council (term beginning 02/01/2021)
☐ Member of the Nominating Committee (term beginning 01/01/2021)
☐ Member of the Editorial Boards Committee (term beginning 02/01/2021)

of the American Mathematical Society.

Return petitions by February 24, 2020 to:
Secretary, AMS, 201 Charles Street, Providence, RI 02904-2213 USA

Name and address (printed or typed)
Calls for Nominations

AMS Prizes & Awards

Leroy P. Steele Prize for Lifetime Achievement

About the Prize
The Steele Prize for Lifetime Achievement is awarded for the cumulative influence of the total mathematical work of the recipient, high level of research over a period of time, particular influence on the development of a field, and influence on mathematics through PhD students. The amount of this prize is US$10,000.

Next Prize: January 2021
Nomination Deadline: March 31, 2020
Nomination Procedure: https://www.ams.org/steele-prize

Nominations can be submitted between February 1 and March 31. Nominations for the Steele Prize for Mathematical Exposition should include a letter of nomination, a complete bibliographic citation for the work being nominated, and a brief citation to be used in the event that the nomination is successful. Nominations will remain active and receive consideration for three consecutive years.

Leroy P. Steele Prize for Seminal Contribution to Research

About the Prize
The Steele Prize for Seminal Contribution to Research is awarded for a paper, whether recent or not, that has proved to be of fundamental or lasting importance in its field, or a model of important research.

Special Note: The Steele Prize for Seminal Contribution to Research is awarded according to the following six-year rotation of subject areas:
1. Open (2025)
2. Analysis/Probability (2026)
3. Algebra/Number Theory (2021)
4. Applied Mathematics (2022)
5. Geometry/Topology (2023)
6. Discrete Mathematics/Logic (2024)

Next Prize: January 2021
Nomination Deadline: March 31, 2020
Nomination Procedure: https://www.ams.org/steele-prize

Nominations for the Steele Prize for Seminal Contribution to Research is awarded according to the following six-year rotation of subject areas:

Leroy P. Steele Prize for Mathematical Exposition

About the Prize
The Steele Prize for Mathematical Exposition is awarded for a book or substantial survey or expository research paper. The amount of this prize is US$5,000.

Next Prize: January 2021
Nomination Deadline: March 31, 2020
Nomination Procedure: https://www.ams.org/steele-prize
Calls for Nominations
FROM THE AMS SECRETARY

Nominations can be submitted between February 1 and March 31. Nominations for the Steele Prize for Seminal Contribution to Research should include a letter of nomination, a complete bibliographic citation for the work being nominated, and a brief citation to be used in the event that the nomination is successful.

Fellows of the American Mathematical Society

The Fellows of the American Mathematical Society program recognizes members who have made outstanding contributions to the creation, exposition, advancement, communication, and utilization of mathematics.

AMS members may be nominated for this honor during the nomination period which occurs in February and March each year. Selection of new Fellows (from among those nominated) is managed by the AMS's Selection Committee, comprised of twelve members of the AMS who are also Fellows. Those selected are subsequently invited to become Fellows, and the new class of Fellows is publicly announced each year on November 1.

Learn more about the qualifications and process for nomination at [https://www.ams.org/profession/ams-fellows](https://www.ams.org/profession/ams-fellows).
Mathematical and Statistical Sciences Annual Survey

ANNUAL SURVEY

Report on 2017–2018
Academic Recruitment, Hiring, and Attrition

Amanda L. Golbeck, Thomas H. Barr, and Colleen A. Rose

Each year in academic mathematical sciences departments around the United States, new full-time faculty are recruited, and a subset of those positions are filled. The hiring infuses into the faculty a new cohort of mathematical scientists actively engaged in research and teaching. At the same time, others retire, take jobs elsewhere, or die, and this process removes a segment of the population of mathematical scientists. This report provides a snapshot of that process to aid in understanding the current status of such variables as: hiring rates, gender distribution, position type, and prior experience. Along with current data the report provides historical context to aid the reader in discerning trends and patterns. For further details, including all tables generated to prepare this report, please see www.ams.org/annual-survey.

A total of 692 mathematical sciences departments participated in this Recruitment, Hiring, and Attrition survey. This report is based on the completed questionnaires received from the 403 departments that reported recruiting to fill doctoral tenure-track and non-tenure-track positions during the academic year 2017–18 for employment beginning in the fall of 2018.

Overview of Recruitment
During the 2017–18 academic year, the estimated number of full-time positions under recruitment in mathematical sciences departments was 2,126 (SE = 58). This figure breaks down as follows: 849 tenure-track mathematics positions, 1,035 non-tenure-track mathematics positions, 136 tenure-track Stats positions, and 106 non-tenure-track Stats positions. See Figure R.1 for comparisons with earlier years. In the period from 2013 to 2018, the overall percentage of positions under recruitment that were tenure-track ranged from 46% to 52%, with the highest percentages in 2013–14 and 2016–17 of this range of time.

Figure R.1. Positions Under Recruitment in Mathematical Sciences

Amanda L. Golbeck is Associate Dean for Academic Affairs and Professor of Biostatistics in the Fay W. Boozman College of Public Health at University of Arkansas for Medical Sciences. Thomas H. Barr is AMS special projects officer. Colleen A. Rose is AMS survey analyst.
ANNUAL SURVEY

Mathematical and Statistical Sciences Annual Survey

Positions Filled

A total of 1,866 full-time positions in Mathematical Sciences were filled during the 2017–18 academic cycle, 1,674 from Mathematics Departments and 192 from Statistics or Biostatistics. Figure F.1 gives a breakdown for the most recent five years. The total for Math is down 8% from the 2007–08 cycle. For Stats, the number of filled positions is up 56% from 2007–08. One interesting feature implicit in these data is that the success rate for filling mathematical sciences tenure-track positions over the period 2013–18 is about 80%, whereas the success rate for non-tenure-track is about 95%.

Figure F.2 gives a breakdown on hiring by gender and department grouping. Percentages generally are obtained by comparison with Figure R.1. Here are further highlights and comparisons:

• Overall features in the 2017–18 cycle:
  ○ The estimated number of positions under recruitment was 2,126; this figure represents a slight increase from the 2016–17 estimate of 1,999 positions.
  ○ Women accounted for 31% of those hired, the same as in 2016–17.
  ○ Since 2010 recruitment has increased 75% in all Mathematical Sciences, 74% in Math, and 83% in Stats.

• Tenure-track positions under recruitment:
  ○ The number of open tenure-track positions was essentially the same as in 2016–17.
  ○ 46% (985) of all positions under recruitment were tenure-track. Of these 985 positions, 86% (851) were open to new PhDs, and 22% (214) were at the rank of associate/full professor.

• Non-tenure-track positions under recruitment:
  ○ Non-tenure-track positions increased 12% overall, to 1,141 from 1,023 in 2016–17.
  ○ 54% (1,141) of all positions under recruitment were non-tenure-track.

In Math the number of positions under recruitment (1,884) in 2017–18 was up 7% from the previous year (1,768). See Figure R.2 for a longer-term comparison. Over the period since 2007–08 recruitment has increased in Doctoral departments by 30% and decreased in both Masters departments and Bachelors departments by 27% and 25%, respectively. In the same ten-year period, the net number of mathematics positions under recruitment has decreased by 6%.

In Stats, the number of positions under recruitment in 2017–18 was 242, a 5% increase over 2016–17. Since 2013–14 the number of positions under recruitment has fluctuated between 220 and 242.

Figure R.2. Positions Under Recruitment in Mathematics Departments by Highest Mathematical Sciences Degree Offered

- Doctoral Math Tenure/tenure-track
- Masters Tenure/tenure-track
- Bachelors Tenure/tenure-track
- Doctoral Math Non-tenure-track
- Masters Non-tenure-track
- Bachelors Non-tenure-track

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  ○ 54% (1,141) of all positions under recruitment were non-tenure-track.
ANNUAL SURVEY

Mathematical and Statistical Sciences Annual Survey

• Non-tenure-track hires (continued)
  ○ Of non-tenure-track hires, 13% (137) were filled by non-doctoral faculty; 55% of these non-doctoral hires were women. Fifty-three percent of these non-doctoral, non-tenure-track hires were in Bachelors departments.
  ○ Of non-tenure-track hires, 22% (237) were temporary (one-year); 38% of these temporary hires were women. Forty percent of all temporary hires were in Bachelors departments.
  ○ Of non-tenure-track hires, 41% (445) were in postdoctoral positions; 27% of these postdocs were women.

• Women hires (see Figure F.2):
  ○ Of all hires, 31% (585) were women; of these women, Bachelors departments hired 38%, and Doctoral Math departments hired 40%.
  ○ In the Doctoral Math Group, women hires increased by 18% to 235.
  ○ All groups reported increases in the number of women hires over last year except Math Private Small (44%), Biostatistics (41%), and Masters (41%).
  ○ The number of women hired into tenure-track positions decreased slightly to 232 from 241; the

Figure F.1. Positions Filled in Mathematical Sciences

Figure F.2. Gender of Tenure-track and Non-tenure-track Hires by Department Grouping, Fall 2018 Employment

(439%), Math Public Small (45%), Math Private Small (44%), and Biostatistics (419%).
  ○ Of the 192 (25%) of tenure-track hires who were new PhDs, 34% were women.
  ○ Of tenure-track hires, 18% (141) had a non-tenure-track position in 2016–17; of these individuals, 30% were women.
  ○ Of tenure-track hires, 34% (261) held a postdoc in 2016–17, and 25% of these postdocs were women.

○ Of the 1,141 non-tenure-track positions under recruitment, 96% were filled. In comparison to 2016–17, all groups reported increased hiring of non-tenure-track faculty except Applied Math (430%) and Biostatistics (424%).
  ○ Of non-tenure-track hires, 36% (389) were filled by PhD faculty who were not new PhDs; 31% of these doctoral faculty hires were women.
  ○ Of non-tenure-track hires, 87% (954) were PhDs; 59% of these PhDs were new PhDs and 28% of these were women.
number hired into non-tenure-track positions increased by 22% to 353.
- Women accounted for 30% of all tenure-track and 32% of all non-tenure track hires; in 2016–17 these percentages were, respectively, 32% and 31%.

**Faculty Attrition**

Figure A.1 shows the variation in attrition from deaths and retirements among full-time faculty for the academic years 2013–14 through 2017–18. On average over the period shown, the percentage of faculty in doctoral departments retiring or dying each year is about 1.9%, and in Masters and Bachelors departments that percentage is about 2.7%.

During the same period, in the respective groups, the percentages of tenured faculty who retired averaged 3.4% for Doctoral Math departments, 4.6% for Bachelors and Masters, and 3.6% for Stats. As reported in previous years, departments continue to report the majority of those retiring as members of the tenured faculty. For instance, for 2015–17 approximately 83%, 82%, and 92% of retirees have been tenured.

Here are a few other highlights from the attrition data from the 2017–18 cycle:
- **Overall retirements by tenured faculty decreased by 9% to 531.**
- **Deaths and retirements increased by 3% to 614.**
- **Overall deaths and retirements break down by departmental grouping as follows:**
  - 43% (261) were from Bachelors
  - 31% (189) were from Doctoral Math
  - 20% (123) were from Masters
  - 7% (41) were from Stat

**Department Groupings**

In this report, *Mathematical and Statistical Sciences* departments are those in four-year institutions in the US that refer to themselves with a name that incorporates (with a few exceptions) “Mathematics” or “Statistics” in some form. For instance, the term includes, but is not limited to, departments of “Mathematics,” “Mathematical Sciences,” “Mathematics and Statistics,” “Mathematics and Computer Science,” “Applied Mathematics,” “Statistics,” and “Biostatistics.” Also, *Mathematics (Math)* refers to departments that (with exceptions) have “mathematics” in the name; *Stat/Biostat* refers to departments that incorporate (again, with exceptions) “statistics” or “biostatistics” in the name but do not use “mathematics.”

Listings of the actual departments that comprise these groups are available on the AMS website at [www.ams.org](http://www.ams.org/annual-survey/groupings).

<table>
<thead>
<tr>
<th>A department is in Group...</th>
<th>...when its subject area, highest degree offered, and PhD production rate* p</th>
</tr>
</thead>
<tbody>
<tr>
<td>Math Public Large</td>
<td>Math PhD, 7.0 ≤ p</td>
</tr>
<tr>
<td>Math Public Medium</td>
<td>Math PhD, 3.9 ≤ p &lt; 7.0</td>
</tr>
<tr>
<td>Math Public Small</td>
<td>Math PhD, p &lt; 3.9</td>
</tr>
<tr>
<td>Math Private Large</td>
<td>Math PhD, 3.9 ≤ p</td>
</tr>
<tr>
<td>Math Private Small</td>
<td>Math PhD, p &lt; 3.9</td>
</tr>
<tr>
<td>Applied Math</td>
<td>Applied mathematics, PhD</td>
</tr>
<tr>
<td>Statistics</td>
<td>Statistics, PhD</td>
</tr>
<tr>
<td>Biostatistics</td>
<td>Biostatistics, PhD</td>
</tr>
<tr>
<td>Masters</td>
<td>Math, master’s</td>
</tr>
<tr>
<td>Bachelors</td>
<td>Math, bachelor’s</td>
</tr>
<tr>
<td>Doctoral Math</td>
<td>Math Public, Math Private, &amp; Applied Math</td>
</tr>
<tr>
<td>Stat/Biostat or Stats</td>
<td>Statistics &amp; Biostatistics</td>
</tr>
<tr>
<td>Math</td>
<td>All groups except Statistics &amp; Biostatistics</td>
</tr>
</tbody>
</table>

*The doctorate-granting departments of mathematics PhD production rates are based on the size of the PhD program as reflected in the number of PhDs awarded (as reported by departments to the Annual Survey) during the ten years from July 1, 2000, through June 30, 2010. Since there are some departments that have not reported their PhDs for every Annual Survey during this time, the average annual number of PhDs awarded was used to compare the departments.*

**Survey Response Rates by Grouping**

**Recruitment, Hiring, and Attrition Response Rates**

<table>
<thead>
<tr>
<th>Group</th>
<th>Number</th>
<th>Percent</th>
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</thead>
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<tr>
<td>Math Public Large</td>
<td>23 of 26</td>
<td>88%</td>
</tr>
<tr>
<td>Math Public Medium</td>
<td>35 of 40</td>
<td>88%</td>
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<tr>
<td>Math Public Small</td>
<td>52 of 73</td>
<td>71%</td>
</tr>
<tr>
<td>Math Private Large</td>
<td>19 of 24</td>
<td>79%</td>
</tr>
<tr>
<td>Math Private Small</td>
<td>18 of 28</td>
<td>64%</td>
</tr>
<tr>
<td>Applied Math</td>
<td>16 of 23</td>
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</tr>
<tr>
<td>Statistics</td>
<td>39 of 60</td>
<td>65%</td>
</tr>
<tr>
<td>Biostatistics</td>
<td>30 of 45</td>
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</tr>
<tr>
<td>Masters</td>
<td>80 of 169</td>
<td>47%</td>
</tr>
<tr>
<td>Bachelors</td>
<td>380 of 1,014</td>
<td>37%</td>
</tr>
<tr>
<td>Total</td>
<td>692 of 1,502</td>
<td>46%</td>
</tr>
</tbody>
</table>

*The populations for Applied Math and Biostatistics are slightly less than for the Doctorates Granted Survey because some programs do not formally “house” faculty, teach undergraduate courses, or award undergraduate degrees.*
Acknowledgments
The Annual Survey attempts to provide an accurate appraisal and analysis of various aspects of the academic mathematical sciences scene for the use and benefit of the community and for filling the information needs of the professional organizations. Every year, college and university departments in the United States are invited to respond. The Annual Survey relies heavily on the conscientious efforts of the dedicated staff members of these departments for the quality of its information. On behalf of the Data Committee and the Annual Survey Staff, we thank the many secretarial and administrative staff members in the mathematical sciences departments for their cooperation and assistance in responding to the survey questionnaires.
2018–2019 Faculty Salaries Report

Amanda L. Golbeck, Thomas H. Barr, and Colleen A. Rose

This salary report is one part of the Annual Survey of Mathematical Sciences, a nation-wide survey administered by the AMS on behalf of the American Statistical Association (ASA), the Institute for Mathematical Statistics (IMS), the Mathematical Association of America (MAA), and the Society for Industrial and Applied Mathematics (SIAM). It provides a look at the salaries of faculty in the Mathematical Sciences in the US by rank in several different department groupings based on discipline, highest degree offered, and graduate counts. The graphs here are identified by those group names, and the group definitions are given at the end of the report.

Departments were asked to report for each rank the number of tenured and tenure-track, non-tenure-track, and part-time faculty whose 2018–19 academic-year salaries fell within given salary intervals. Reporting salary data in this fashion ensures confidentiality of individual responses, though it does mean that the quartiles reported in the tables are approximations. The quartiles reported have been estimated assuming that the density over each interval is uniform.

Note: In the graph for all faculty salaries on this page, the percentage scale ranges from 0 to 50, while the scale for all other graphs is 0 to 100. Salaries for non-tenure-track, part-time faculty and pay by the course were gathered for the first time starting with the 2018–19 academic year and will be reported separately.

Faculty Salary Reports from prior years are at https://www.ams.org/annual-survey/salaries. Interpretation of historical trends should be made with care. For instance, one factor influencing year to year changes in the mean reported salaries may be differences in the set of responding departments within the groups. Response rates are noted on the tables.

The first graph below provides a coarse comparative view of faculty salaries among four broad groups: departments whose highest degree is a (1) PhD in mathematics (including applied mathematics departments), (2) PhD in statistics or biostatistics, (3) masters degree in mathematics, and (4) bachelors degree in mathematics. In the remainder of this report, salary distributions are broken down within finer departmental categories and by faculty rank.

Figure 1. Full-time Faculty Salaries by Department Grouping, Fall 2018 (Tenured, Tenure-track, and Non-tenured)

Amanda L. Golbeck is Associate Dean for Academic Affairs and Professor of Biostatistics in the Fay W. Boozman College of Public Health at University of Arkansas for Medical Sciences. Thomas H. Barr is AMS special projects officer. Colleen A. Rose is AMS survey analyst.
## Math Public Large Group Faculty Salaries$^1$

24 responses out of 26 departments (92%)

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<tr>
<th>Rank</th>
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<th>Gender</th>
<th>No. Reported</th>
<th>Q1</th>
<th>Median</th>
<th>Q3</th>
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<th>Mean</th>
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<tbody>
<tr>
<td>New Asst Professors</td>
<td></td>
<td></td>
<td>All</td>
<td>40</td>
<td>87,500</td>
<td>92,500</td>
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<td>92,500</td>
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<td>92,500</td>
<td>94,470</td>
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<td>-</td>
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<td>62,500</td>
<td>60,930</td>
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<td>Men</td>
<td>23</td>
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<td>-</td>
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<td>-</td>
<td>-</td>
<td>36,810</td>
<td>-</td>
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</tbody>
</table>

$^1$ Wherever a quartile is not reported, that value is less than the lowest left endpoint of the bin of $55,000 for PhD-granting departments.

$^2$ Includes newly hired assistant professors.

---

### Figure 2. Math Public Large Group—Percentage of Faculty Salaries within each Salary Range by Category

*Includes newly hired assistant professors.*
### Math Public Medium Group Faculty Salaries

35 responses out of 40 departments (88%)

<table>
<thead>
<tr>
<th>Rank</th>
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</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Gender</td>
<td>No. Reported</td>
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<tr>
<td>New Asst Professors</td>
<td>All</td>
<td>38</td>
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<td>29</td>
</tr>
<tr>
<td></td>
<td>Women</td>
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<tr>
<td>Assistant Professor²</td>
<td>All</td>
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<td></td>
<td>Men</td>
<td>168</td>
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<tr>
<td></td>
<td>Women</td>
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<td></td>
<td>Women</td>
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<td>Full Professor</td>
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<td>Women</td>
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<td></td>
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<td></td>
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1. Wherever a quartile is not reported, that value is less than the lowest left endpoint of the bin of $55,000 for PhD-granting departments.
2. Includes newly hired assistant professors.

### Figure 3. Math Public Medium Group—Percentage of Faculty Salaries within each Salary Range by Category

*Includes newly hired assistant professors.
# Mathematical and Statistical Sciences Annual Survey

## ANNUAL SURVEY

### Mathematical and Statistical Sciences Annual Survey

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### Figure 4. Math Public Small Group—Percentage of Faculty Salaries within each Salary Range by Category

![Figure 4](image)

*Includes newly hired assistant professors.

---

### Table: Math Public Small Group Faculty Salaries

<table>
<thead>
<tr>
<th>Rank</th>
<th>Gender</th>
<th>No. Reported</th>
<th>Q1</th>
<th>Median</th>
<th>Q3</th>
<th>Mean</th>
<th>Mean</th>
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</thead>
<tbody>
<tr>
<td>New Asst Professors</td>
<td>All</td>
<td>55</td>
<td>67,500</td>
<td>67,500</td>
<td>77,500</td>
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<td>38</td>
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<td></td>
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<td>67,500</td>
<td>72,500</td>
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<td>72,500</td>
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<td></td>
<td>Women</td>
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<td>Men</td>
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<tr>
<td>Part-time Faculty</td>
<td>All</td>
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<td>-</td>
<td>-</td>
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<td>-</td>
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<td>18,620</td>
<td>-</td>
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</tbody>
</table>

1. Wherever a quartile is not reported, that value is less than the lowest left endpoint of the bin of $55,000 for PhD-granting departments.
2. Includes newly hired assistant professors.
### ANNUAL SURVEY

**Math Private Large Group Faculty Salaries**

14 responses out of 24 departments (58%)

<table>
<thead>
<tr>
<th>Rank</th>
<th>Gender</th>
<th>No. Reported</th>
<th>Q1</th>
<th>Median</th>
<th>Q3</th>
<th>Mean</th>
<th>Mean</th>
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</thead>
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<td>All</td>
<td>57</td>
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<td>125,000</td>
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<td>155,000</td>
<td>205,000</td>
<td>182,170</td>
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<td>72,500</td>
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<td>Men</td>
<td>166</td>
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<td>62,500</td>
<td>72,500</td>
<td>71,480</td>
<td>-</td>
</tr>
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<td></td>
<td>Women</td>
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<td>-</td>
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<td></td>
<td></td>
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<td></td>
<td>-</td>
</tr>
</tbody>
</table>

1 Wherever a quartile is not reported, that value is less than the lowest left endpoint of the bin of $55,000 for PhD-granting departments.

2 Includes newly hired assistant professors.

3 Too few to report.

---

**Figure 5. Math Private Large Group—Percentage of Faculty Salaries within each Salary Range by Category**

*Includes newly hired assistant professors.*
### Math Private Small Group Faculty Salaries

17 responses out of 28 departments (61%)

<table>
<thead>
<tr>
<th>Rank</th>
<th>Gender</th>
<th>No. Reported</th>
<th>2018-19</th>
<th>2017-18</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Q1</td>
<td>Median</td>
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<tr>
<td>New Asst Professors</td>
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<td>82,500</td>
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<tr>
<td></td>
<td>Men</td>
<td>9</td>
<td>72,500</td>
<td>77,500</td>
</tr>
<tr>
<td></td>
<td>Women</td>
<td>4</td>
<td>72,500</td>
<td>82,500</td>
</tr>
<tr>
<td>Assistant Professor&lt;sup&gt;3&lt;/sup&gt;</td>
<td>All</td>
<td>63</td>
<td>77,500</td>
<td>82,500</td>
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<tr>
<td></td>
<td>Men</td>
<td>45</td>
<td>77,500</td>
<td>82,500</td>
</tr>
<tr>
<td></td>
<td>Women</td>
<td>18</td>
<td>77,500</td>
<td>82,500</td>
</tr>
<tr>
<td>Associate Professor</td>
<td>All</td>
<td>94</td>
<td>82,500</td>
<td>92,500</td>
</tr>
<tr>
<td></td>
<td>Men</td>
<td>78</td>
<td>82,500</td>
<td>92,500</td>
</tr>
<tr>
<td></td>
<td>Women</td>
<td>16</td>
<td>82,500</td>
<td>97,500</td>
</tr>
<tr>
<td>Full Professor</td>
<td>All</td>
<td>190</td>
<td>105,000</td>
<td>125,000</td>
</tr>
<tr>
<td></td>
<td>Men</td>
<td>173</td>
<td>105,000</td>
<td>125,000</td>
</tr>
<tr>
<td></td>
<td>Women</td>
<td>17</td>
<td>105,000</td>
<td>125,000</td>
</tr>
<tr>
<td>Non-tenure-track faculty</td>
<td>All</td>
<td>141</td>
<td>40,000</td>
<td>57,500</td>
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<tr>
<td></td>
<td>Men</td>
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<td>40,000</td>
<td>62,500</td>
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<tr>
<td></td>
<td>Women</td>
<td>56</td>
<td>40,000</td>
<td>57,500</td>
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<tr>
<td>Part-time Faculty</td>
<td>All</td>
<td>32</td>
<td>-</td>
<td>-</td>
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<tr>
<td></td>
<td>Men</td>
<td>25</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>Women</td>
<td>7</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

1. Wherever a quartile is not reported, that value is less than the lowest left endpoint of the bin of $55,000 for PhD-granting departments.
2. Too few to report.
3. Includes newly hired assistant professors.

### Figure 6. Math Private Small Group—Percentage of Faculty Salaries within each Salary Range by Category

- New-Hire Asst
- Assistant<sup>*</sup>
- Associate
- Full
- Non-tenure-track Faculty
- Part-time Faculty

<sup>*</sup> Includes newly hired assistant professors.
### Applied Mathematics Group Faculty Salaries

18 responses out of 23 departments (78%)

<table>
<thead>
<tr>
<th>Rank</th>
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<th>2017-18</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Gender Reported</td>
<td>Q1</td>
</tr>
<tr>
<td>New Asst Professors²</td>
<td></td>
<td></td>
</tr>
<tr>
<td>All</td>
<td>Men</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>Women</td>
<td>1</td>
</tr>
<tr>
<td>Assistant Professor³</td>
<td></td>
<td></td>
</tr>
<tr>
<td>All</td>
<td>Men</td>
<td>56</td>
</tr>
<tr>
<td></td>
<td>Women</td>
<td>15</td>
</tr>
<tr>
<td>Associate Professor</td>
<td></td>
<td></td>
</tr>
<tr>
<td>All</td>
<td>Men</td>
<td>83</td>
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<tr>
<td></td>
<td>Women</td>
<td>15</td>
</tr>
<tr>
<td>Full Professor</td>
<td></td>
<td></td>
</tr>
<tr>
<td>All</td>
<td>Men</td>
<td>135</td>
</tr>
<tr>
<td></td>
<td>Women</td>
<td>15</td>
</tr>
<tr>
<td>Non-tenure-track faculty</td>
<td></td>
<td></td>
</tr>
<tr>
<td>All</td>
<td>Men</td>
<td>125</td>
</tr>
<tr>
<td></td>
<td>Women</td>
<td>49</td>
</tr>
<tr>
<td>Part-time Faculty</td>
<td></td>
<td></td>
</tr>
<tr>
<td>All</td>
<td>Men</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td>Women</td>
<td>2</td>
</tr>
</tbody>
</table>

¹ Wherever a quartile is not reported, that value is less than the lowest left endpoint of the bin of $55,000 for PhD-granting departments.

² Too few to report.

³ Includes newly hired assistant professors.

---

### Figure 7. Applied Math Group—Percentage of Faculty Salaries within each Salary Range by Category

- New-Hire Asst
- Assistant* (Includes newly hired assistant professors)
- Associate
- Full
- Non-tenure-track Faculty
- Part-time Faculty

---

* Includes newly hired assistant professors.
### Statistics Group Faculty Salaries

15 responses out of 59 departments (25%)

<table>
<thead>
<tr>
<th>Rank</th>
<th>2018-19</th>
<th>2017-18</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Gender</td>
<td>No. Reported</td>
</tr>
<tr>
<td>New Asst Professors</td>
<td>All</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>Men</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>Women*</td>
<td>4</td>
</tr>
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<td>Assistant Professor²</td>
<td>All</td>
<td>65</td>
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<tr>
<td></td>
<td>Men</td>
<td>46</td>
</tr>
<tr>
<td></td>
<td>Women</td>
<td>19</td>
</tr>
<tr>
<td>Associate Professor</td>
<td>All</td>
<td>63</td>
</tr>
<tr>
<td></td>
<td>Men</td>
<td>41</td>
</tr>
<tr>
<td></td>
<td>Women</td>
<td>22</td>
</tr>
<tr>
<td>Full Professor</td>
<td>All</td>
<td>98</td>
</tr>
<tr>
<td></td>
<td>Men</td>
<td>78</td>
</tr>
<tr>
<td></td>
<td>Women</td>
<td>20</td>
</tr>
<tr>
<td>Non-tenure-track Faculty</td>
<td>All</td>
<td>79</td>
</tr>
<tr>
<td></td>
<td>Men</td>
<td>42</td>
</tr>
<tr>
<td></td>
<td>Women</td>
<td>37</td>
</tr>
<tr>
<td>Part-time Faculty</td>
<td>All</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Men</td>
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</tr>
<tr>
<td></td>
<td>Women</td>
<td>0</td>
</tr>
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</table>

¹ Wherever a quartile is not reported, that value is less than the lowest left endpoint of the bin of $55,000 for PhD-granting departments.
² Too few to report.
³ Includes newly hired assistant professors.

---

**Figure 8. Statistics Group—Percentage of Faculty Salaries within each Salary Range by Category**

*Includes newly hired assistant professors.*
### Biostatistics Group Faculty Salaries

17 responses out of 49 departments (35%)

<table>
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<td></td>
<td>Men 4</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>Women 2</td>
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</tr>
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<td>Assistant Professor</td>
<td>All 50</td>
<td>82,500</td>
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<tr>
<td></td>
<td>Men 33</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>Women 17</td>
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</tr>
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<td>Associate Professor</td>
<td>All 50</td>
<td>97,500</td>
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<tr>
<td></td>
<td>Men 37</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>Women 13</td>
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</tr>
<tr>
<td>Full Professor</td>
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<tr>
<td></td>
<td>Men 69</td>
<td>-</td>
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<td></td>
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<td>All 134</td>
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<tr>
<td></td>
<td>Men 52</td>
<td>-</td>
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<td></td>
<td>Women 82</td>
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</tr>
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</tr>
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<td></td>
<td>Men 0</td>
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<tr>
<td></td>
<td>Women 0</td>
<td>-</td>
</tr>
</tbody>
</table>

1 Wherever a quartile is not reported, that value is less than the lowest left endpoint of the bin of $55,000 for PhD-granting departments.

2 Too few to report.

3 Includes newly hired assistant professors.

---

**Figure 9.** Biostatistics Group—Percentage of Faculty Salaries within each Salary Range by Category

*Includes newly hired assistant professors.*
### Masters Group Faculty Salaries

<table>
<thead>
<tr>
<th>Rank</th>
<th>Gender</th>
<th>No. Reported</th>
<th>Q1</th>
<th>Median</th>
<th>Q3</th>
<th>Mean</th>
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<td>72,500</td>
<td>68,990</td>
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<td>57,500</td>
<td>67,500</td>
<td>72,500</td>
<td>69,450</td>
<td>69,026</td>
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<td>67,500</td>
<td>72,500</td>
<td>67,990</td>
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<td>All</td>
<td>429</td>
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<td>62,500</td>
<td>72,500</td>
<td>67,640</td>
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<td>277</td>
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<td>72,500</td>
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<td>77,500</td>
<td>75,850</td>
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<td>97,500</td>
<td>95,410</td>
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<td>97,500</td>
<td>150,130</td>
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<td></td>
</tr>
<tr>
<td>Part-time Faculty</td>
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<td>47,500</td>
<td>41,570</td>
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</tr>
<tr>
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<td>Men</td>
<td>135</td>
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<td>42,500</td>
<td>47,500</td>
<td>41,570</td>
<td>-</td>
<td></td>
</tr>
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<td></td>
<td>Women</td>
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<td>-</td>
<td>32,500</td>
<td>47,500</td>
<td>36,430</td>
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</tbody>
</table>

1. Wherever a quartile is not reported, that value is less than the lowest left endpoint of the bin of $25,000 for Masters-granting departments.
2. Includes newly hired assistant professors.

---

**Figure 10. Masters Group — Percentage of Faculty Salaries within each Salary Range by Category**

- **New-Hire Asst**
- **Assistant* (includes newly hired assistant professors)**
- **Associate**
- **Full**
- **Non-tenure-track Faculty**
- **Part-time Faculty**

---

* Includes newly hired assistant professors.
### Bachelors Group Faculty Salaries

312 responses out of 1018 departments (31%)

<table>
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</tr>
</thead>
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<td>Gender</td>
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</tr>
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<td>No. Reported</td>
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</tr>
<tr>
<td>New Asst Professors</td>
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</tr>
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</tr>
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</tr>
<tr>
<td></td>
<td>All</td>
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<td>Men</td>
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</tr>
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<td></td>
<td>Women</td>
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<tr>
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<td>Women</td>
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</tr>
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</tr>
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<td>403</td>
</tr>
<tr>
<td></td>
<td>Women</td>
<td>365</td>
</tr>
<tr>
<td>Non-tenure-track faculty</td>
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<td>181</td>
</tr>
<tr>
<td></td>
<td>Men</td>
<td>96</td>
</tr>
<tr>
<td></td>
<td>Women</td>
<td>85</td>
</tr>
</tbody>
</table>

Wherever a quartile is not reported, that value is less than the lowest left endpoint of the bin of $25,000 for Bachelors-granting departments.

*Includes newly hired assistant professors.

---

**Figure 11. Bachelors Group—Percentage of Faculty Salaries within each Salary Range by Category**

*Includes newly hired assistant professors.*
**Departmental Groupings**

In this report, *Mathematical and Statistical Sciences* departments are those in four-year institutions in the US that refer to themselves with a name that incorporates (with a few exceptions) “Mathematics” or “Statistics” in some form. For instance, the term includes, but is not limited to, departments of “Mathematics,” “Mathematical Sciences,” “Mathematics and Statistics,” “Mathematics and Computer Science,” “Applied Mathematics,” “Statistics,” and “Biostatistics.” Also, *Mathematics (Math)* refers to departments that (with exceptions) have “mathematics” in the name; *Stat/Biostat (Stats)* refers to departments that incorporate (again, with exceptions) “statistics” or “biostatistics” in the name but do not use “mathematics.”

Listings of the actual departments that comprise these groups are available on the AMS website at [https://www.ams.org/annual-survey/groupings](https://www.ams.org/annual-survey/groupings).

<table>
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<th>A department is in Group...</th>
<th>...when its subject area, highest degree offered, and PhD production rate* p</th>
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<td>Math Public Medium</td>
<td>Math PhD, 3.9 ≤ p &lt; 7.0</td>
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<tr>
<td>Math</td>
<td>All groups except Statistics &amp; Biostatistics</td>
</tr>
</tbody>
</table>

* The doctorate-granting departments of mathematics PhD production rates are based on the size of the PhD program as reflected in the number of PhDs awarded (as reported by departments to the Annual Survey) during the ten years from July 1, 2000, through June 30, 2010. Since there are some departments that did not report their PhDs for every Annual Survey during this time, the average of their reported annual number of PhDs awarded was used to compare the departments.

**Obtain a Special Faculty Salaries Analysis**

Each year the AMS provides a limited number of special faculty salary analyses to departments requesting them. These reports are based on data gathered through the Survey and provide more nuanced comparisons with similar institutions than is possible with the Faculty Salaries Report. In order to receive a special analysis, your department must have responded to the most recent Faculty Survey.

Send a list of your peer institutions (a minimum of 12 institutions is required) to ams-survey@ams.org along with the date by which the analysis is needed. (If not enough of your peer group have responded to the salary survey, you will be asked to provide additional institutions.) A minimum of two weeks is needed to complete a special analysis.

The analysis produced includes a listing of your peer group institutions along with their salary survey response status; a summary table including the rank (assistant, associate, and full professor); the number reported in each rank; the 1st quartile, median, 3rd quartile, and mean salaries for each along with bar graphs.

**Acknowledgments**

The Annual Survey attempts to provide an accurate appraisal and analysis of various aspects of the academic mathematical sciences scene for the use and benefit of the community and for filling the information needs of the professional organizations. Every year, college and university departments in the United States are invited to respond. The Annual Survey relies heavily on the conscientious efforts of the dedicated staff members of these departments for the quality of its information. On behalf of the Joint Data Committee and the Annual Survey Staff, we thank the many secretarial and administrative staff members in the mathematical sciences departments for their cooperation and assistance in responding to the survey questionnaires.
How do AMS Graduate Student Chapters support the mathematical community and beyond?

Sam Houston State University | Hurricane Harvey Goods Drive
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www.ams.org/studentchapters
The authors of this piece are organizers of the AMS 2020 Mathematics Research Communities summer conference Analysis in Metric Spaces, one of five topical research conferences offered this year that are focused on collaborative research and professional development for early-career mathematicians. Additional information can be found at https://www.ams.org/programs/research-communities/2020MRC-Metspace. Applications are open until February 15, 2020.

The subject of analysis, more specifically, first-order calculus, in metric measure spaces provides a unifying framework for ideas and questions from many different fields of mathematics. One of the earliest motivations and applications of this theory arose in Mostow’s work [Mos73], in which he extended his celebrated rigidity theorem for hyperbolic manifolds to the more general framework of manifolds locally modeled on negatively curved symmetric spaces of rank one. In his proof, Mostow used the theory of quasiconformal mappings on the visual boundaries of rank-one symmetric spaces. These visual boundaries are equipped with a sub-Riemannian structure that is locally non-Euclidean and has a fractal nature. Mostow’s study of quasiconformal maps on such boundaries motivated Heinonen and Koskela [HK98] to axiomatize several aspects of Euclidean quasiconformal geometry in the setting of metric measure spaces and thereby extend Mostow’s work beyond the sub-Riemannian setting. The groundbreaking work [HK98] initiated the modern theory of analysis on metric spaces.

Analysis on metric spaces is nowadays an active and independent field, bringing together researchers from different parts of the mathematical spectrum. It has far-reaching applications to areas as diverse as geometric group theory, nonlinear PDEs, and even theoretical computer science. As a further sign of recognition, analysis on metric spaces has been included in the 2010 MSC classification as a category (30L: Analysis on metric spaces). In this short survey, we can discuss only a small fraction of areas into which analysis on metric spaces has expanded. For more comprehensive introductions to various aspects of the subject, we invite the reader to consult the monographs [Hei01, HK00, HKST15, BB11, MT10, AGS08, BS07, Hei07].

Poincaré inequalities in metric spaces. Inspired by the fundamental theorem of calculus, Heinonen and Koskela proposed the notion of upper gradient as a substitute for the derivative of a function on a metric measure space $(X,d,\mu)$. More precisely, $g \geq 0$ is an upper gradient for a real-valued function $u$ on $X$ if

$$
|u(\gamma(1)) - u(\gamma(0))| \leq \int_{\gamma} g \, ds
$$

for each path $\gamma : [0,1] \to X$ of finite length.

Upper gradients are not unique, but if a function $u$ has an upper gradient $g \in L^p(\mu)$, then there is a unique $p$-weak upper gradient $g_u$ with minimal $L^p$-norm for which the preceding inequality holds for “almost every” curve $\gamma$. The metric measure space $X$ is said to support a $p$-Poincaré inequality for some $p \geq 1$ if constants $C > 0$ and $\lambda \geq 1$ exist...
so that for every ball $B = B(x, R) \subset X$, the inequality
\[ \int_B |u - u_B| d\mu \leq CR \left( \int_{\lambda B} g_p^\mu d\mu \right)^{1/p} \]
holds for all function-upper gradient pairs $(u, g_u)$. Here $u_B = f_B u d\mu$ and $\lambda B = B(x, \lambda R)$.

Over the past twenty years, many aspects of first-order calculus have been systematically developed in the setting of PI spaces, that is, metric measure spaces equipped with a doubling measure and supporting a Poincaré inequality. For example, for PI spaces we now have a rich theory of Sobolev functions which in turn lies at the foundation of the theory of quasiconformal mappings and nonlinear potential theory.

A wealth of interesting and important examples of non-Euclidean PI spaces exist, including sub-Riemannian manifolds such as the Heisenberg group, Gromov-Hausdorff limits of manifolds with lower Ricci curvature bounds, visual boundaries of certain hyperbolic buildings, and fractal spaces that are homeomorphic to the Menger curve. The scope of the theory, however, is not fully explored.

**Quasiconformal maps and nonlinear potential theory in metric spaces.** A homeomorphism between metric spaces is said to be quasiconformal if it distorts the geometry of infinitesimal balls in a controlled fashion. Conformal maps form a special subclass for which infinitesimal balls are mapped to infinitesimal balls. Since the only conformal maps between higher-dimensional Euclidean spaces are Möbius transformations, quasiconformal homeomorphisms form a more flexible class for geometric mapping problems. For quasiconformal maps on PI spaces, we now have a well-developed theory that features many of the aspects of the Euclidean theory, such as Sobolev regularity, preservation of sets of measures zero, and global distortion estimates, among other things.

A function $u$ on a domain $\Omega$ in a metric measure space $(X, d, \mu)$ is said to be $p$-quasiharmonic for $p \geq 1$ if a constant $Q \geq 1$ exists so that the inequality
\[ \int_{\text{spt } \varphi} g_u^Q d\mu \leq Q \int_{\text{spt } \varphi} g_{u+\varphi}^Q d\mu \]
holds whenever $\varphi$ is a Lipschitz function with compact support $\text{spt } \varphi$ in $\Omega$. In case $Q = 1$, we say that $u$ is $p$-harmonic; this coincides with the classical Euclidean notion of a $p$-harmonic function, defined as a weak solution to the $p$-Laplace equation
\[ \text{div}(|Vu|^{p-2}Vu) = 0. \]

Quasiharmonic functions are useful in the study of quasiconformal mappings. For example, one can characterize quasiconformal homeomorphisms between $n$-dimensional Euclidean domains as those homeomorphisms that preserve the class of $n$-quasiharmonic functions. A similar statement is also true for PI spaces. This generalizes the well-known fact that planar conformal mappings are precisely the orientation-preserving homeomorphisms that preserve harmonic functions under pullback.

The further development of potential theory in the setting of metric measure spaces leads to a classification of spaces as either $p$-parabolic or $p$-hyperbolic. This dichotomy can be seen as a nonlinear analog of the recurrence/transience dichotomy in the theory of Brownian motion. This classification is helpful in the development of a quasiconformal uniformization theory or for a deeper understanding of the links between the geometry of hyperbolic spaces and the analysis on their boundaries at infinity.

**Differentiability of Lipschitz functions.** The notion of upper gradient generalizes to metric spaces the norm of the gradient of a $C^1$-function. It is a priori unclear how to formulate a notion of the gradient itself (or of the differential of a function) in the absence of a linear structure. Cheeger [Che99] introduced a linear differential structure for real-valued functions on metric measure spaces and established a version of Rademacher’s theorem for Lipschitz functions defined on PI spaces. This differential structure gives rise to a finite-dimensional measurable vector bundle, the generalized cotangent bundle, over the metric space: to each real-valued Lipschitz function $u$ corresponds an $L^\infty$-section $du$ of this bundle. Moreover, the pointwise Euclidean norm $|du|$ is comparable to the minimal upper gradient $g_u$ almost everywhere. This structure can be used in turn to investigate second-order PDEs in divergence form as a basis for a theory of differential currents in metric spaces and for many other purposes.

**Bi-Lipschitz embedding theorems.** An earlier version of Rademacher’s differentiation theorem for Lipschitz maps between Carnot groups was proved by Pansu [Pan89]. Semmes observed that the Pansu--Rademacher theorem implies that nonabelian Carnot groups do not admit bi-Lipschitz copies in finite-dimensional Euclidean spaces. Moreover, such spaces do not bi-Lipschitz embed into Hilbert space or even into any Banach space with the Radon--Nikodým property (RNP). Indeed, the algebraic features of sub-Riemannian geometry have direct implications for metric questions such as bi-Lipschitz equivalence or embeddability.

The bi-Lipschitz embedding problem is intimately related to the existence of suitable differentiation theories for Lipschitz functions and maps. Roughly speaking, this relationship proceeds via incompatibility between the
geometry of the cotangent bundles of the source and target spaces. In view of Cheeger’s differentiation theorem, one can allow arbitrary PI space as source spaces here and take RNP Banach spaces as targets, for example. On the other hand, there is no effective differentiation theory for maps into $\ell^\infty$, because according to the Fréchet embedding theorem, every separable metric space embeds isometrically into $\ell^\infty$.

The target space $L^1$ presents an interesting intermediate case. It is not an RNP Banach space, yet deep bi-Lipschitz nonembedding theorems are available for this target. In particular, Cheeger and Kleiner [CK10] showed that the Heisenberg group does not bi-Lipschitz embed into $L^1$. In concert with results of Lee and Naor, this fact exhibits the Heisenberg group as a geometrically natural example relevant for algorithmic questions in computer science. There has been significant additional quantitative work along these lines, culminating in Naor and Young’s sharp lower bound for the integrity gap of the Goemans–Linial semidefinite program for the Sparsest Cut Problem [NY18]. For further information, see Naor’s ICM lecture [Nao10].

Geometric measure theory on metric spaces. In yet another direction, Kirchheim’s proof of the almost everywhere metric differentiability of Lipschitz mappings into metric spaces led to far-reaching generalizations of the area and co-area formulas. Subsequently, Ambrosio and Kirchheim [AK00] developed an extension of the Federer–Fleming theory of currents in complete metric spaces, thus opening a new chapter in geometric measure theory leading to a study of (quantitative) rectifiability in metric spaces. This notion has been further developed in sub-Riemannian settings, especially in the Heisenberg group, but many questions remain open. These theories have been relevant, for example, in the work of Naor and Young [NY18] mentioned above, where quantitative rectifiability of surfaces in the Heisenberg group is prominently featured.

Dynamics and analysis on metric spaces. An interesting source of examples of spaces that can be studied with the methods of quasi-geometric and analysis on metric spaces is fractals that arise from self-similar or dynamical constructions such as limit sets of Kleinian groups, Julia sets of rational maps, or attractors of iterated function systems. Often the geometry of these spaces is too “rough” to expect finer analytic properties such as the Poincaré inequality to hold. However, if these spaces admit a good first-order calculus, then striking consequences often emerge. For example, Cannon’s well-known conjecture in geometric group theory predicts that a Gromov hyperbolic group $G$ admits a geometric action on hyperbolic 3-space if its boundary at infinity $\partial_\infty G$ is a topological 2-sphere. While the conjecture is still open, one can show that the desired conclusion is true if $\partial_\infty G$ (equipped with a visual metric) has good analytic properties, say, if it is quasisymmetrically equivalent to a PI space. For more information see the ICM lectures [Bon06] and [Kle06].

The problem of deciding when a metric space is quasisymmetrically equivalent to a space with “better” analytic properties can be seen as a generalization of classical uniformization theorems in complex analysis, at least for low-dimensional fractals such as Sierpiński carpets or fractal 2-spheres. In order to study such problems, one often employs concepts from classical complex analysis in a metric space setting. For example, the modulus of a path family, originally introduced by Ahlfors and Beurling in the complex plane, now plays a prominent role in much recent work on mapping theory in general, abstract metric spaces.

Conclusion. This brief note barely hints at the breadth and the depth of the problems of current concern in the theory of analysis on metric spaces. The 2020 AMS MRC Analysis in Metric Spaces will address a number of questions that have been the subject of much recent investigation but are far from being completely understood.

**ACKNOWLEDGMENTS.** The second author is supported by the Simons Foundation, and the other authors by the National Science Foundation.

**References**


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<th>Graduate/Undergraduate Student Rate</th>
<th>Introductory Rate*</th>
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The Inverse Eigenvalue Problem of a Graph, Zero Forcing, and Related Parameters

Shaun M. Fallat, Leslie Hogben, Jephian C.-H. Lin, and Bryan L. Shader

Overview

The dynamics of many physical systems can be distilled from the eigenvalues and eigenfunctions of a corresponding operator. For example, possible vibrations of a thin membrane can be described in terms of the eigenvalues and eigenfunctions of the Laplace operator on the membrane. Kac’s famous question “Can you hear the shape of a drum?” is a type of inverse eigenvalue problem, that is, a problem that asks what are the properties of the system if the eigenvalues of the corresponding operator are known. For example, the eigenvalues of the Laplacian determine the area of the membrane but don’t (uniquely) determine the shape of the membrane (up to isometry). In this context, we can view the inverse eigenvalue problem of a graph \( G \) as, “What possible collection of sounds (that is, eigenvalues) can a drum of your shape, that is, a matrix whose off-diagonal nonzero pattern is described by the edges of \( G \), make?”

Ever since the development of the Perron–Frobenius theory for nonnegative matrices, there has been an interest in understanding how the combinatorial structure of a matrix is related to eigenvalues of the matrix. The graph of the \( n \times n \) symmetric matrix \( A = [a_{ij}] \) has vertex set \( 1, 2, \ldots, n \) and the edge joining \( i \) and \( j \) if and only if \( i \neq j \) and \( a_{ij} \neq 0 \). Given a graph \( G \) with vertex set \( 1, \ldots, n \), \( S(G) \) denotes the set of all symmetric \( n \times n \) matrices whose graph is \( G \). For example, if \( P_n \) denotes the path with edges \( \{1, 2\}, \{2, 3\}, \ldots, \{(n - 1), n\} \), then \( S(P_n) \) denotes all matrices of the
form shown in Figure 1. It is known that a set of \( n \) real numbers is the spectrum of a matrix in \( S(P_n) \) if and only if these numbers are distinct [5].

\[
A = \begin{bmatrix}
a_1 & b_1 \\
b_1 & a_2 & b_2 \\
& \ddots & \ddots & \ddots \\
& & b_{n-1} & a_n \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
\end{bmatrix}
\]

\[\mathcal{S}(A) = P_n\]

Figure 1. An irreducible \( n \times n \) tridiagonal matrix and its graph.

The inverse eigenvalue problem for \( G \) (IEP-\( G \)) asks us to determine all multisets of \( n \) real numbers that are the spectrum of some matrix \( S(G) \). A specific instance of the IEP-\( G \) would be: Is the multiset \( \{\lambda_1, \ldots, \lambda_n\} \) the spectrum of some matrix in \( S(G) \)?

This note concerns two related classes of problems, the IEP-\( G \) and zero forcing processes and parameters. Zero forcing was introduced independently in several different areas of mathematics and its applications, including in the study of the IEP-\( G \).

Zero forcing is a coloring game on a graph, where initially each vertex is colored blue\(^1\) or white, and the goal is to color all the vertices blue by applying a color change rule. For (standard) zero forcing, the color change rule is: A blue vertex \( u \) can change the color of a white vertex \( w \) to blue if \( w \) is the unique white neighbor of \( u \). The minimum number of blue vertices needed to color all the vertices of \( G \) blue is the zero forcing number of \( G \) and is denoted by \( Z(G) \). The process of forcing vertices blue models forcing zero entries in a null vector of a matrix in \( S(G) \), and \( Z(G) \) is an upper bound for the maximum multiplicity of an eigenvalue of any matrix in \( S(G) \). The process of applying the color change rule to a grid graph is illustrated in Figure 2.

There are numerous variations and applications of zero forcing. Each variant is determined by its color change rule, which defines when a vertex can change color from white to blue. The interpretation of a blue vertex varies with the application, such as a zero in a null vector of a matrix, a node in an electrical network that can be monitored by phasor measurement units (PMUs) placed at the initially blue vertices, a part of a graph that has been searched for an adversary, or a person who has heard a rumor in a social network.

The next two sections address new tools for the IEP-\( G \) and new work on processes related to zero forcing, including propagation and throttling. Earlier background on the IEP-\( G \) and zero forcing can be found in [4] and the extensive reference list therein. The new methods for the IEP-\( G \) build on the ideas of Colin de Verdière, who proved an analogous result for maximum nullity. The maximum multiplicity of an eigenvalue in \( S(G) \) is equal to the maximum nullity over all matrices in \( S(G) \), and there has been extensive work on the problem of determining the maximum nullity of a matrix in \( S(G) \), partly fueled by Colin de Verdière-type parameters. After the introduction of the zero forcing number as an upper bound for maximum nullity in [1] and in control of quantum systems, most of the initial research on the subject focused on the zero forcing number \( Z(G) \) (minimum number of blue vertices needed to color the entire graph blue). More recently there has been considerable interest in the process by which the graph is colored blue, including the speed of propagation (using an initial set of blue vertices of minimum cardinality) or minimizing a combination of the resources (number of initially blue vertices) used to accomplish a task (coloring all vertices blue) and the time it takes to color the whole graph blue.

**The Strong Spectral Property**

Solving specific instances of the IEP-\( G \) is often difficult, much like finding a needle in a haystack. However, recently developed theories based on manifolds have transformed this area of research by showing that if one finds a sufficiently “nice” solution to the IEP-\( G \) problem, then one is guaranteed a solution for any supergraph of \( G \). The theory of transversal intersections of manifolds generalizes the implicit function theorem and asserts that if \( P \) lies in the intersection of the manifolds \( M_1 \) and \( M_2 \), and the vector sum of the tangent space to \( M_1 \) at \( P \) and the tangent space to \( M_2 \) at \( P \) spans the entire ambient space, then any sufficiently small perturbations of \( M_1 \) and \( M_2 \) intersect at a point near \( P \).

A particular example of this phenomenon occurred for the case of distinct eigenvalues. Classically it was known that any set of distinct real numbers can be realized as the

\(^{1}\)Most early papers color the vertices black and some very recent work refers to blue vertices as filled vertices.
spectrum of a matrix $A$ in $S(T)$ for any tree $T$. Since any connected graph contains a spanning tree, it was shown by Monfared and Shader in 2015 that any distinct set of real numbers can occur as a spectrum of a matrix in $S(G)$ for any connected graph $G$ by first determining a “nice” matrix $B$ realizing this set of eigenvalues for the noted spanning tree, which is then perturbed to produce a desired matrix $A$ in $S(G)$. The proof relies on treating $S(G)$ and the set of symmetric matrices with the same spectrum as manifolds.

Given a multiset $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ of real numbers, we define $E_\Lambda$ to be the set of real symmetric $n \times n$ matrices with spectrum $\Lambda$. It is known that $E_\Lambda$ is a submanifold of the manifold of real symmetric $n \times n$ matrices, as is $S(G)$, and that there is a matrix in $S(G)$ with spectrum $\Lambda$ if and only if these two manifolds have nonempty intersection. Having manifolds intersect transversally is illustrated in the next very simple example.

Example 1. Let $S_2(\mathbb{R})$ be the space of all $2 \times 2$ real symmetric matrices. Each matrix in $S_2(\mathbb{R})$ can be written as $\begin{bmatrix} x & z \\ z & y \end{bmatrix}$, so $S_2(\mathbb{R})$ is isomorphic to $\mathbb{R}^3$ and each matrix in $S_2(\mathbb{R})$ can be represented as a point in $\mathbb{R}^3$.

Let $\Lambda = \{1, 3\}$ and define $E_\Lambda = \{M \in S_2(\mathbb{R}) : \text{spec}(M) = \Lambda\}$. For any $h \in \mathbb{R}$, define $M_h = \{M \in S_2(\mathbb{R}) : M_{12} = M_{21} = h\}$. In Figure 3, the blue ellipse is $E_\Lambda$, the plane is $M_0$, and they intersect transversally at $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$.

Using classical matrix theory results and taking orthogonal complements, one can show that we have a transversal intersection if and only if the only symmetric matrix $X$ such that $I \circ X = O$, $A \circ X = O$, and $AX =XA$ is $X = O$ (where $\circ$ denotes the entrywise product). This is called the strong spectral property (SSP). Properties of the SSP immediately imply that any set of distinct real numbers can be realized as the spectrum of a matrix $A$ in $S(G)$ for any graph $G$, since a diagonal matrix with distinct eigenvalues has the SSP [3].

Suppose $G$ has $n$ vertices and $A \in S(G)$ is a matrix with the SSP. Then the following powerful consequences are known (see, e.g., [6]). For any supergraph $H$ of $G$ with the same order, there is a matrix $A' \in S(H)$ with $\text{spec}(A') = \text{spec}(A)$. For any supergraph $H$ of $G$ on $m$ vertices, there is a matrix $A' \in S(H)$ such that $\text{spec}(A')$ is the disjoint union of $\text{spec}(A)$ and a set of distinct $m - n$ real numbers. The previous two statements were then used to characterize graphs $G$ with $q(G) = n - 1$ [3]. They are also used to solve the IEP-$G$ for graphs of order at most 5 [2].

The spectra of matrices with the SSP also respect the graph minor operation. If $G$ can be obtained from some graph $H$ by contracting an edge, then there is a matrix $A' \in S(H)$ such that $\text{spec}(A') = \text{spec}(A) \cup \{\lambda\}$ for some $\lambda$ sufficiently large [2].

Propagating properties together with the supergraph properties, we say the collection of ordered multiplicity lists reached by matrices in $S(G)$ with the SSP is minor monotone. As evidenced by these various results, strong properties like the SSP provide a generic way to construct new matrices with prescribed spectral properties.

**Propagation Time and Throttling for Zero Forcing**

There are many processes that propagate through networks and model real-world systems. A rumor can spread through a social network. A computer virus can spread across the Internet. In many applications, the time needed for the process to complete starting with a minimum set is
of interest, or it may be better to speed up the process by using a larger initial set while minimizing a combination of resources (initial blue vertices) and time. These two questions have attracted considerable interest recently for zero forcing and related graph-searching parameters.

Propagation time is the number of time steps needed for a minimum zero forcing set to color all the vertices blue, performing all possible independent forces at each time step. More precisely, start with $B[0] = B$ as the set of initial blue vertices. Define $B[t]$ to be the set of blue vertices in $G$ after the color change rule is applied to every white vertex independently using $B[t-1]$ as the set of blue vertices. The propagation time of $B$ in $G$, $pt(G; B)$, is the least $t$ such that $B[t] = V(G)$ (or infinity if $B$ is not a zero forcing set of $G$). An examination of Figure 2 shows that $pt(G_{4 \times 7}; B) = 6$ for the $4 \times 7$ grid graph $G_{4 \times 7}$ and initial blue set $B$ shown there. The animation at https://aimath.org/~hogben/4x7gridanimation.gif shows the blue vertices propagating across the graph. The *propagation time* of a graph $G$ is

$$pt(G) = \min_{B \subseteq V(G)} \{ pt(G; B) : |B| = Z(G) \}.$$ 

In fact, the initial blue set in Figure 2 realizes the propagation time of $G_{4 \times 7}$, so $pt(G_{4 \times 7}) = 6$.

Throttling minimizes the sum of the number of blue vertices and the propagation time. Specifically, for a subset $B$ of vertices, the throttling number of $B$ in $G$ is $th(G; B) = |B| + pt(G; B)$. For the $4 \times 7$ grid graph $G_{4 \times 7}$ and initial blue set $B$ shown in Figure 2, $th(G_{4 \times 7}; B) = 4 + 6 = 10$. The *throttling number* of a graph $G$ is

$$th(G) = \min_{B \subseteq V(G)} \{ th(G; B) \}.$$ 

Since it is known that $th(G) \geq \lceil 2\sqrt{n} - 1 \rceil$ for any graph $G$ of order $n$, and since $2\sqrt{28} - 1 \approx 9.583$, $th(G_{4 \times 7}; B) = 10$. However, a set $B$ that realizes the throttling number is not necessarily a minimum zero forcing set. For example, a path on $n$ vertices has a minimum zero forcing set consisting of one vertex, but throttling is achieved by choosing approximately $\sqrt{n}$ initially blue vertices.

In addition to zero forcing, propagation time and throttling have been studied for other graph parameters such as Cops and Robbers; for more information, see [6] and the references therein.

**Want to Learn More about IEPG-$G$ and Zero Forcing?**

We are organizing a Mathematics Research Community on the inverse eigenvalue problem for graphs, zero forcing, and related parameters, including propagation and throttling, which will take place June 14–20, 2020, at the Whistling Pines Conference Center in Rhode Island. A key objective of this MRC is to gather together early-career researchers with interests in matrix theory and discrete mathematics, and we encourage such researchers to apply to this MRC to enhance and contribute to the collaborative advances in this area.

Pre-workshop activities are planned, including a reading list of background on various topics associated with the core subject matter of this workshop and a series of online tutorials that will be delivered by designated junior experts in this discipline. To find out more information about this MRC, please consult the website www.ams.org/programs/research-communities/2020MRC-Haystacks.

We look forward to welcoming a new group of energetic researchers to offer different and exciting perspectives on the topics proposed in our workshop.

**References**


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Combinatorial Adventures in Analysis, Algebra, and Topology

Stephen Melczer, Marni Mishna, and Robin Pemantle

Capturing large-scale behavior of combinatorial objects can be difficult but very revealing. Consider Figure 1, for instance, which illustrates the behavior of random objects of large size in two combinatorial classes inspired by quantum computing and statistical mechanics.

Figure 1. ACSV has been used to determine the shape of large random quantum random walks [3] (left) and cube groves [5] (right).

Perhaps surprisingly, complex analysis, algebraic geometry, topology, and computer algebra are important ingredients in such results, combining to form the relatively new field of *analytic combinatorics in several variables* (ACSV). ACSV builds on classic generating function techniques in combinatorics to provide new tools for the study of sequences and discrete objects. Applications previously studied in the literature include queuing systems, bioinformatics, data structures, random tilings, special functions, and integrable systems. Ongoing research efforts from the combinatorial side concern an evolving study of lattice paths [8] and objects from representation theory and algebraic combinatorics [14].

The purpose of this note is to give a brief overview of ACSV and its history to drive interest among those in any of the diverse areas touched by the topic. A large collection of worked examples can be found in [15] and a detailed treatment of ACSV in [16].
Analytic Combinatorics

The use of analytic techniques in combinatorics and probability theory has a long history, dating back to the eighteenth-century work of Laplace [12] and contemporaries such as Stirling [18] and de Moivre [9]. In modern times, the use of real and complex analysis to derive asymptotic behavior is the domain of analytic combinatorics [11], a field which finds application in many areas of mathematics, computer science, and the natural sciences. The key idea behind such results is that important properties of a sequence \((f_n) = (f_0, f_1, \ldots)\) of real or complex numbers are easily deduced from the generating function

\[
F(z) = \sum_{n \geq 0} f_n z^n = f_0 + f_1 z + \cdots.
\]

Although much can be deduced treating \(F\) only as a formal series [20], when \(f_n\) grows at most exponentially as \(n\) goes to infinity then \(F\) defines a complex-analytic function at the origin. The powerful methods of complex analysis can then be used to study properties of \(f_n\), including its asymptotic behavior.

Indeed, the Cauchy integral formula implies that for every natural number \(n\) the coefficient \(f_n\) can be represented by a contour integral

\[
f_n = \frac{1}{2\pi i} \int_{\mathcal{C}} F(z) z^{-n-1} dz,
\]

where \(\mathcal{C}\) is a positively oriented circle sufficiently close to the origin. The singularities of the generating function play a crucial role in determining sequence asymptotics: the domain of integration \(\mathcal{C}\) can be deformed without changing the value of the Cauchy integral as long it does not cross any singularities of \(F\). Setting some technicalities aside, each singularity of \(F\) gives a “contribution” to asymptotics of \(f_n\) depending on its location in the complex plane and the local behavior of \(F\) near the singularity. This approach is powerful enough to be completely automated for large classes of generating functions, such as rational and algebraic functions, and is flexible enough to extend as required by applications. A thorough development of this univariate theory and a survey of its applications can be found in the fantastic text of Flajolet and Sedgewick [11].

ACSV

In contrast to the well-developed univariate theory, until recently there was no general framework for the study of multivariate generating functions. To rectify this gap in the literature, over the last two decades the theory of ACSV has been developed [16]. Here we focus on a rational function \(F(z) = F(z_1, \ldots, z_d)\) in \(d > 1\) variables with power series expansion

\[
F(z) = \sum_{\mathbf{i} \in \mathbb{N}^d} f_{\mathbf{i}} z_1^{i_1} \cdots z_d^{i_d}.
\]

Given \(\mathbf{r} \in \mathbb{N}^d\), the \(\mathbf{r}\)-diagonal of \(F(z)\) is the sequence \((f_{\mathbf{i}})\). Many mathematical sequences arise naturally as diagonals of explicit rational functions. Furthermore, coefficient series for all algebraic functions may be embedded as generalized diagonals of rational functions [17]. It is conjectured that the same is true of all globally bounded D-finite functions [7, Conjecture 4].

Example 1. A key step in Apéry’s proof [19] of the irrationality of \(\zeta(3)\) is determining asymptotics of the sequence

\[
b_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2.
\]

Because it is a sum of nonnegative terms, saddle point approximations will produce the desired asymptotics.

Such sums with mixed signs are notoriously difficult to estimate. However, any binomial sum sequence regardless of signs can automatically be written as the 1-diagonal of a rational function [6]. For instance, Apéry’s sum is the 1-diagonal of the four-variable rational function

\[
\frac{1}{1-t(1+x)(1+y)(1+z)(1+y+z+y+z+x+y+z+x+y+z+x+y+z+x+y+z+x+y+z)}.
\]

The methods of ACSV determine asymptotics of \(b_n\) completely automatically [13].

Although the \(\mathbf{r}\)-diagonal of a fixed rational function is defined only when \(\mathbf{r}\) has integer coordinates, ACSV shows that asymptotics along most directions \(\mathbf{r} \in \mathbb{R}^d_{>0}\) can be well defined by a limiting procedure.

Multivariate Analysis and Singularity Theory

As in the univariate setting, the methods of ACSV start from a Cauchy integral formula

\[
f_{\mathbf{i}} = \frac{1}{(2\pi i)^d} \int_{\mathcal{C}} F(z) z^{-\mathbf{i} \cdot 1} dz_1 \cdots dz_d,
\]

where \(\mathcal{C}\) is now a product of positively oriented circles sufficiently close to the origin. The singularities of \(F(z)\) again play a crucial role in asymptotics. If \(F(z) = G(z)/H(z)\) is a rational function with coprime numerator \(G\) and denominator \(H\), then the singularities of \(F\) form the set \(\mathcal{V} = \{z \in \mathbb{C}^d : H(z) = 0\}\).

Once the dimension is greater than 1, the singular set \(\mathcal{V}\) is no longer discrete but is an algebraic variety of positive dimension. In the geometrically simplest case, when \(\mathcal{V}\)

\footnote{Asymptotics of the \(\mathbf{r}\)-diagonal vary smoothly with \(\mathbf{r}\) inside open cones of \(\mathbb{R}^d_{>0}\). The boundaries of these cones form \((d-1)\)-dimensional sets where asymptotics can sharply change.}
forms a smooth manifold, one searches for a (generically finite) set of critical points near which the Cauchy integral can be approximated by saddle-point integrals which are easy to estimate asymptotically. When some of these critical points lie on the boundary of the generating function domain of convergence and other mild conditions hold, asymptotics can be computed using explicit formulas. In the absence of critical points on the boundary of convergence or when the geometry of this domain is difficult to establish, topological techniques may be applied to determine which critical points are responsible for the coefficient asymptotics. This was carried out in two dimensions in [10]; generalizing to higher dimensions is an open problem.

Without restricting the geometry of $\mathcal{V}$, recent work [4] has shown how stratified Morse theory helps determine the deformations of $c$ in $C^d \setminus \mathcal{V}$ which yield integrals of the type analyzed in classic works on singularity theory [1, 2].

The techniques of ACSV thus rely on an interesting mix of computer algebra, singularity theory, algebra, geometry, and topology. By asking different questions, ACSV promotes new work in these areas. For example, any advances in the automatic computation of singular integrals can be applied back to the wave-like partial differential equations for which purpose singularity theory was developed in the 1970s and 1980s. Indeed this development may already be ripe for a resurgence as it couples with modern computer algebra methods.

**AMS Math Research Community**

Being a new and rapidly growing field which incorporates techniques from diverse corners of mathematics, ACSV has a rich collection of problems accessible to early researchers in many areas. Some of these problems are computational in nature, including the development of computer algebra tools for asymptotics under varying assumptions. Others are topological at heart, such as computing intersection and linking numbers of attachment cycles which arise in the Morse theoretic constructions discussed above. Many relate to asymptotic computations of saddle-point asymptotics in degenerate settings which appear in combinatorial applications. Of course, after developing such tools there is also a need for combinatorialists to apply them to problems of differing scope. Most of the advances in ACSV have come from combinatorial problems just beyond the reach of existing techniques.

We thus encourage early-career mathematicians with experience in any one of the areas of combinatorics, computational algebra, analysis, algebraic topology, singularity theory, or hyperbolic partial differential equations to apply to our upcoming Math Research Community. Our hope is to have participants from a wide variety of fields interacting with each other to develop this exciting area of mathematics.

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Photo of Stephen Melczer was taken by Celia Laur.

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2020 Breakthrough Prize in Mathematics

Alex Eskin of the University of Chicago has been awarded the 2020 Breakthrough Prize in Mathematics “for revolutionary discoveries in the dynamics and geometry of moduli spaces of Abelian differentials, including the proof of the ‘magic wand theorem’ with Maryam Mirzakhani.” A description of this work reads: “Eskin teamed with famed Iranian mathematician and Fields Medalist Maryam Mirzakhani to prove a theorem about dynamics on moduli spaces. Their tour de force, published in 2013 after five years of labor, is a result with many consequences. One addresses the long-standing problem: If a beam of light from a point source bounces around a mirrored room, will it eventually reach the entire room—or will some parts remain forever dark? After translating the problem to a highly abstract multidimensional setting, the two mathematicians were able to show that for polygonal rooms with angles which are fractions of whole numbers, only a finite number of points would remain unlit.” Mirzakhani passed away in 2017.

Alex Eskin was born in Moscow, USSR, in 1965. He received his PhD from Princeton University in 1993 under the direction of Peter Sarnak. He was a member of the Institute for Advanced Study from 1993 to 1994 before joining the faculty of the University of Chicago, where he has been Arthur Holly Compton Distinguished Service Professor since 2012. He has been the recipient of Alfred P. Sloan and NSF Postdoctoral Research Fellowships and of a Packard Fellowship for 1997–2002. He received the Clay Research Award in 2007 and a Simons Investigator Award in 2014. He has given invited talks at International Congresses of Mathematicians in Berlin in 1998 and in Hyderabad in 2010. He is a Fellow of the AMS and a member of the National Academy of Sciences and the American Academy of Arts and Sciences.

The Breakthrough Prizes honor important, primarily recent, achievements in the categories of fundamental physics, life sciences, and mathematics. The prizes are sponsored by Sergey Brin, Priscilla Chan and Mark Zuckerberg, Pony Ma, Yuri and Julia Milner, and Anne Wojcicki. The prize carries a cash award of US$3 million.

In addition, three New Horizons in Mathematics Prizes were awarded for 2020 to promising early-career researchers. The prizes carry a cash award of US$100,000. The prizes were awarded to the following.

Tim Austin of the University of California, Los Angeles, was honored for “multiple contributions to ergodic theory, most notably the solution of the weak Pinsker conjecture.” Austin received his PhD from the University of California, Los Angeles, in 2010 under the direction of Terence Tao. Austin was named a Clay Research Fellow for the years 2010–2015. He held a visiting position at Brown University from 2010 to 2012, progressed from assistant to associate professor at the Courant Institute of Mathematical Sciences from 2012 to 2017, and joined the UCLA faculty in 2017. He has given a large number of talks, including the Bernoulli Lecture in Lausanne in 2019.

Emmy Murphy of Northwestern University received a New Horizons Prize for her “contributions to symplectic and contact geometry, in particular the introduction of notions of loose Legendrian submanifolds and, with Matthew Strom Borman and Yakov Eliashberg, overtwisted contact structures in higher dimensions.” Murphy received her PhD from Stanford University in 2012 and was a member of the faculty of the Massachusetts Institute of Technology from 2012 to 2017. She joined Northwestern in 2016 and

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has been an associate professor since 2018. She received a Von Neumann Fellowship at the Institute for Advanced Study for 2019–2020. She was awarded an Alfred P. Sloan Research Fellowship in 2015 and a Radcliffe Institute for Advanced Study Fellowship in 2016–2017. She received the AWM Birman Research Prize in Topology and Geometry in 2017 and was an invited speaker at the 2018 International Congress of Mathematicians.

Xinwen Zhu of the California Institute of Technology was awarded a New Horizons Prize “for work in arithmetic algebraic geometry including applications to the theory of Shimura varieties and the Riemann–Hilbert problem for $p$-adic varieties.” He received his PhD from the University of California, Los Angeles, in 2009 under Edward Frenkel. He held positions at Harvard University and Northwestern University before joining Caltech in 2014. He was awarded an AMS Centennial Fellowship in 2013 and an Alfred P. Sloan Fellowship in 2015. In 2019 he was awarded the Morningside Gold Medal of Mathematics, given to exceptional mathematicians of Chinese descent under the age of forty-five.

—Elaine Kehoe from Breakthrough Prize announcements

Credits
Photo of Tim Austin is courtesy of Katherine Smith.
**Mathematics People**

**Naor Awarded Ostrowski Prize**

Assaf Naor of Princeton University has been awarded the 2019 Ostrowski Prize “for his groundbreaking work in areas in the meeting point of the geometry of Banach spaces, the structure of metric spaces, and algorithms.” The prize citation reads in part: “The nature of his contribution is threefold: Solutions of hard problems, setting a significant research direction for himself and others to follow, and finding deep connections between pure mathematics and computer science.

“Since the mid-nineties, geometric methods have played an influential role toward designing algorithms for computational problems that a priori have little connection to geometry. Assaf Naor is the world leader on this topic, building a long-term, cohesive research program. He has discovered and applied deep results from the theory of Banach spaces and quantitative metric geometry to solve long-standing algorithmic questions and, in turn, has solved long-standing questions in analysis via techniques that are sometimes motivated by algorithmic applications. This has often led to development of new theories, e.g., the nonlinear spectral calculus and understanding of the geometry of the Heisenberg group.

“One particular focus of his research is on computing the ‘sparsest cut’ in graphs, i.e., to cut an $n$-vertex graph into two parts such that the number of edges across the two parts is minimized while requiring the two parts to be ‘balanced.’ This is an NP-hard problem, so the goal is to compute an approximate sparse cut. A specific algorithm is based on linear programming relaxation. Its approximation factor is the same as the distortion needed to embed a corresponding class of $n$-point metrics into $L_1$. Assaf Naor proved that a ball of radius $n$ in the Heisenberg group does not Lipschitz embed into $L_1$ with distortion better than $\sqrt{\log n}$. As a consequence, the semidefinite program for the sparsest cut problem on inputs of size $n$ is at least of order $\sqrt{\log n}$, matching the known upper bound.”

—Helmut Harbrecht, Universität Basel

**Phạm Awarded 2019 ICTP-IMU Ramanujan Prize**

Hoàng Hiệp Phạm of the Institute of Mathematics, Vietnam Academy of Science and Technology, has been awarded the 2019 ICTP-IMU Ramanujan Prize, given by the International Centre for Theoretical Physics (ICTP), the International Mathematical Union (IMUI), and the government of India.

The prize citation reads: “The prize is in recognition of his outstanding contributions to the field of complex analysis, and in particular to pluripotential theory, where he obtained an important result on the singularities of plurisubharmonic functions; complex Monge-Ampère equations and log canonical thresholds, which have important applications in algebraic and complex Kähler geometry. The prize is also in recognition of Dr. Phạm’s important organizational role in the advancement of mathematics in his home country, Vietnam.” The selection committee consisted of Alicia Dickenstein (University of Buenos Aires), Lothar Goettsche (ICTP, chair), Kapil Hari Paranjape (Indian Institute of Science
Jitomirskaya Awarded Heineman Prize

Svetlana Jitomirskaya of the University of California, Irvine, has been awarded the 2020 Dannie Heineman Prize for Mathematical Physics “for work on the spectral theory of almost periodic Schrödinger operators and related questions in dynamical systems—in particular, for her role in the solution of the Ten Martini problem, concerning the Cantor set nature of the spectrum of all almost Mathieu operators and in the development of the fundamental mathematical aspects of the localization and metal-insulator transition phenomena.” According to the citation, her “main accomplishments are in the area of quasiperiodic operators, where she is best known for developing the first nonperturbative methods of study of small denominators, that have influenced the future development of this field. She has also been involved, by herself and with collaborators, in the solution of several long-standing problems related to the almost Mathieu operators, also known as Harper’s, Aubry-Andre, or Azbel-Hofstadter model.” Jitomirskaya received her PhD in 1991 from Moscow State University. She became a lecturer at UC Irvine in 1991 and is currently Distinguished Professor of Mathematics. She received the AMS Satter Prize in 2005. She has been a recipient of Sloan and Simons Foundation Fellowships and is a member of the American Academy of Arts and Sciences. Jitomirskaya tells the Notices: “My mother was a prominent mathematician, yet she actively discouraged me from going into math, saying, in particular, ‘it’s not a good job for a girl.’ I ended up following her example rather than her advice, and haven’t regretted it. This is a job that has allowed me to spend a lot of time with each of my three children, while continuing to grow in my work.”

The Heineman Prize is awarded annually by the American Institute of Physics (AIP) and the American Physical Society (APS) in recognition of outstanding publications in mathematical physics. It carries a cash award of US$10,000.

Plan Awarded Aisenstadt Prize

Yaniv Plan of the University of British Columbia has been awarded the 2019 André Aisenstadt Prize in Mathematics by the Centre de Recherches Mathématiques (CRM). According to the citation, his research “is in the general area of mathematics of information that interacts with various fields, including high-dimensional data analysis, machine learning, harmonic analysis, probability, signal processing, and information theory.” His contributions include a theory of compressed sensing, low-rank matrix completion, one-bit compressed sensing, and high-dimensional data analysis. He delivered the Aisenstadt Prize Lecture, “The Role of Random Models in Compressive Sensing and Matrix Completion,” at CRM in November 2019. Plan obtained his PhD from the California Institute of Technology in 2011 under the supervision of Emmanuel Candes. He was an NSF Postdoctoral Fellow (2011–14) and was on the faculty of the University of Michigan, Ann Arbor, before joining the University of British Columbia. He was the recipient of an NSF Mathematical Sciences Postdoctoral Research Fellowship in 2011, and he received the Faculty Award of the UBC Mathematics and Pacific Institute for the Mathematical Sciences in 2016. In his free time, Plan enjoys fantasy role-playing games with his children and likes to wrestle with them. He enjoys swing music and once performed on a swing dance team. His favorite sport is basketball, which he plays regularly.

The Aisenstadt Prize recognizes outstanding research by a young Canadian mathematician.

—from a CRM announcement
Gomis Receives CAP/CRM Prize

Jaume Gomis of the Perimeter Institute for Theoretical Physics and the University of Waterloo has been awarded the 2019 CAP-CRM Prize in Theoretical and Mathematical Physics. Gomis was recognized “for his broad range of important contributions to string theory and strongly coupled gauge theories, including the pioneering use of nonlocal observables, the exact computation of physical quantities in quantum field theory, and the unraveling of the nonperturbative dynamics of gauge theories.” The citation continues: “Over the last 15 years, Dr. Gomis has pioneered novel methods for exploring strongly coupled gauge theories through nonlocal variables, and by studying these theories in curved spacetime. This has allowed him to generate physical insights into these theories and to carry out first-of-their-kind exact computations for key observables in quantum field theory. The computational tools Dr. Gomis has developed in pursuit of this research have also found applications in various areas of pure mathematics, including enumerative geometry, differential geometry, and mirror symmetry. His ongoing work continues to open new frontiers that will fuel new discoveries in theoretical and mathematical physics.”

The prize is awarded by the Canadian Association of Physicists (CAP) and the Centre de Recherches Mathématiques (CRM) and recognizes exceptional achievements in theoretical and mathematical physics.

—From a CAP-CRM announcement

Serra Awarded Rubio de Francia Prize

Joaquim Serra of ETH Zürich has been awarded the 2018 Rubio de Francia Prize of the Royal Spanish Mathematical Society (RSME) “for important contributions to analysis and partial differential equations, including his work on nonlinear integro-differential equations, regularity for minimal surfaces, and free boundary regularity in the obstacle problem.” Serra received his PhD in 2014 from the Universitat Politècnica de Catalunya (UPC Barcelona) under the direction of Xavier Cabré, and his collaborators have included Luis Caffarelli and Alessio Figalli. He has held postdoctoral positions at UPC Barcelona and the Weierstrass Institute for Applied Analysis and Stochastics. He is currently Swiss National Science Foundation (SNF) Ambizione Fellow at ETH Zurich. The prize honors the memory of renowned Spanish analyst J. L. Rubio de Francia and is awarded annually to a mathematician from Spain or who has received a PhD from a university in Spain, and who is at most thirty-two years of age, for high-caliber contributions to any area of pure or applied mathematics. The prize committee consisted of Pavel Exner, Charles Fefferman, Regina Liu, Rosa Maria Miró-Roig, María Pe Pereira, and Francisco Santos Leal (chair).

—Alvaro Pelayo, University of California, San Diego

Hrushovski Awarded Hopf Prize

Ehud Hrushovski of the University of Oxford and the Hebrew University of Jerusalem has been awarded the 2019 Heinz Hopf Prize “for his outstanding contributions to model theory and their application to algebra and geometry.” He gave the Heinz Hopf Lectures on “Logic and Geometry: The Model Theory of Finite Fields and Difference Fields” in October 2019. Hrushovski received his PhD in 1986 from the University of California, Berkeley, under the direction of Leo Harrington. He is a Fellow of the American Academy of Arts and Sciences and the Israel Academy of Sciences and Humanities. He received the Karp Prize of the Association for Symbolic Logic in 1993 (with Alex Wilkie) and was honored with the prize again in 1998. He was also awarded the Erdős Prize of the Israel Mathematical Union (1994) and the Rothschild Prize (1998). He has been an invited speaker (1990) and a plenary speaker (1998) at International Congresses of Mathematicians. Hrushovski tells the Notices: “I think best when running in the outdoors. My maternal grandparents came from a similar German-Jewish milieu to Heinz Hopf. My grandmother was born, like him, in a suburb of today’s Wroclaw; my grandfather, like him, was a student at Breslau University, was wounded in the First World War, and discharged in 1918 with an Iron Cross. They were separated by a few years of age, and I assume they never met.”

The Heinz Hopf Prize is awarded every two years for outstanding scientific achievements in the field of pure mathematics.

—From a Hopf Prize announcement
Logunov Receives 2019 Packard Fellowship

Aleksandr Logunov of Princeton University has been awarded a 2019 Packard Fellowship by the David and Lucile Packard Foundation. The citation reads, “In the nineteenth century Napoleon set a prize for the best mathematical explanation of Chladni’s resonance experiments. Nodal geometry studies the zeroes of solutions of elliptic differential equations such as the visible curves that appear in these physical experiments. Logunov’s research focuses on problems in nodal geometry, harmonic analysis, partial differential equations and geometrical analysis.” Logunov received his PhD from St. Petersburg State University in 2015 under the supervision of Viktor Havin. After two years as a postdoctoral fellow at Tel Aviv University, he joined the faculty at Princeton. In 2017 he received the Clay Research Award, jointly with Eugenia Malinnikova, for their introduction of novel geometric-combinatorial methods for the study of elliptic eigenvalue problems. He received a Clay Research Fellowship for the years 2018–20 and was an invited speaker at the International Congress of Mathematicians in 2018. He was awarded the Salem Prize in 2018.

Packard Fellows receive US$875,000 over five years to pursue their research. The Fellowships are designed to allow maximum flexibility in how the funding is used.

—From a Packard Foundation announcement

Ooguri Awarded Medal of Honor of Japan

Hirosi Ooguri of the University of Tokyo and the California Institute of Technology has been awarded a Medal of Honor with Purple Ribbon by the emperor of Japan, which recognizes “individuals who have contributed to academic and artistic discoveries, inventions, and innovations.” Ooguri was honored for “his many accomplishments and contributions to science.” He is director of the Kavli Institute for the Physics and Mathematics of the Universe at the University of Tokyo and the Fred Kavli Professor of Theoretical Physics and Mathematics at Caltech. He received his doctor of science degree from the University of Tokyo in 1989. He is a Fellow of the AMS and the American Academy of Arts and Sciences. His honors include the AMS Eisenbud Prize for Mathematics and Physics in 2008, the Chunichi Cultural Prize in 2016, and the Joachim Herz Foundation Hamburg Prize in 2018.

—Kavli Institute for the Physics and Mathematics of the Universe

2019 NSF CAREER Awards

The National Science Foundation (NSF) has named a number of recipients of 2019 Faculty Early Career Development (CAREER) Awards. The awards support early-career faculty members who have the potential to serve as academic role models in research and education and to lead advances in the mission of their departments or organizations. Following are the names, institutions, and proposal titles of the awardees selected by the NSF Division of Mathematical Sciences (DMS).

- Benjamin Bakker, University of Georgia: Hodge theory and moduli
- Melody Chan, Brown University: Algebraic curves and their moduli: Degenerations and combinatorics
- Hao Chen, University of California, Davis: New change-point problems in analyzing high-dimensional and non-Euclidean data
- Tamas Darvas, University of Maryland, College Park: Geometric potential theory
- Chao Gao, University of Chicago: Computational and theoretical investigations of variational inference
- Paul Hand, Northeastern University: Signal recovery from generative priors
- Matthew Hirn, Michigan State University: Understanding invariant convolutional neural networks through many particle physics
- Wei Ho, University of Michigan, Ann Arbor: Theory, heuristics, and data for arithmetic invariants
- Mihaela Ifrim, University of Wisconsin, Madison: Quasilinear dispersive evolutions in fluid dynamics
- Pierre Jacob, Harvard University: Unbiased estimation with faithful Markov chains for scalable statistical inference
- Adel Javanmard, University of Southern California: Valid and scalable inference for high-dimensional statistical models
- Benjamin Jaye, Clemson University: Analysis of operators on rough sets
- Ilya Kachkovskiy, Michigan State University: Quantum systems with deterministic disorder
- Karin Leiderman, Colorado School of Mines: Mathematical modeling to identify new regulatory mechanisms of blood clotting
- Xiaodong Li, University of California, Davis: Statistical analysis of nonconvex optimization in unsupervised learning
• **Xingjie Li**, University of North Carolina at Charlotte: A multiscale framework for crystalline defects in 2-dimensional materials
• **Baiying Liu**, Purdue University: Automorphic forms and the Langlands program
• **Yifei Lou**, University of Texas at Dallas: Mathematical modeling from data to insights and beyond
• **Sara Maloni**, University of Virginia: Geometric structures, character varieties, and higher Teichmüller theory
• **Kathryn Mann**, Cornell University: Geometric and topological approaches to group actions in low dimensions
• **Kevin McGoff**, University of North Carolina at Charlotte: Stochastic forward and inverse problems involving dynamical systems
• **Karola Meszaros**, Cornell University: Integer point transforms of polytopes
• **Oleksii Mostovyi**, University of Connecticut: An approach to pricing, hedging, stability, and asymptotic analysis in financial markets
• **Ronen Mukamel**, William Marsh Rice University: Totally geodesic varieties in moduli space: Arithmetic and classification
• **Akil Narayan**, University of Utah: Optimal approximation algorithms in high dimensions
• **Andrei Negut**, Massachusetts Institute of Technology: Higher enumerative geometry via representation theory and mathematical physics
• **Sigal Nitzan**, Georgia Institute of Technology: Bases in Hilbert function spaces and some of their applications
• **Megan Owen**, Lehman College, City University of New York: Statistical and geometric analysis for tree-shaped data
• **David Papp**, North Carolina State University: Large-scale optimization problems with applications in emerging radiotherapy modalities
• **William Perkins**, University of Illinois at Chicago: Phase transitions in algorithms, complexity, and geometry
• **Leif Ristroph**, New York University: Mathematical modeling, physical experiments, and biological data for understanding flow interactions in collective locomotion
• **Arvind Saibaba**, North Carolina State University: Fast and accurate algorithms for uncertainty quantification in large-scale inverse problems
• **Benjamin Shaby**, Colorado State University: Hierarchical models for spatial extremes
• **Pierre Simon**, University of California, Berkeley: Model theory and homogeneous structures
• **Weijie Su**, University of Pennsylvania: A statistical inferential framework for online learning algorithms
• **Nike Sun**, Massachusetts Institute of Technology: Phase transitions in randomized combinatorial search and optimization problems

• **Molei Tao**, Georgia Institute of Technology: Multiscale control of mechanical systems: Theory, computation and applications
• **Hung Tran**, University of Wisconsin, Madison: Front propagations and viscosity solutions
• **Thomas Trogdon**, University of Washington: Numerical linear algebra, random matrix theory and applications
• **Li Wang**, University of Minnesota, Twin Cities: Computational methods for multiscale kinetic systems: Uncertainty, non-locality, and variational formulation
• **Yi Wang**, Johns Hopkins University: Conformal geometry and Monge-Ampère type equations
• **Zhenqi Wang**, Michigan State University: New methods and applications for smooth rigidity of algebraic actions
• **Melanie Wood**, University of California, Berkeley: Randomness in number theory and beyond
• **Yao Yao**, Georgia Institute of Technology: Transport equations in fluids and biology: Singularity, dynamics, and mixing
• **Inna Zakharevich**, Cornell University: Constructing K-theoretic invariants for geometric objects
• **Ting Zhang**, Boston University: Statistical inference of tail dependent time series

—National Science Foundation

**Credits**

Photo of Assaf Naor is courtesy of The Simons Foundation.
Photo of Aleksandr Logunov is courtesy of Clay Math Institute.
Photo of Svetlana Jitomirskaya is courtesy of UCI Physical Sciences Communications.
Community Updates

Maryam Mirzakhani Lectures

The AMS Council established this Lecture in 2018 to honor Maryam Mirzakhani (1977–2017), the first woman and the first Iranian to win a Fields Medal. Mirzakhani was a professor at Stanford University and a highly original mathematician who made a host of striking and influential contributions to hyperbolic geometry, complex analysis, topology, and dynamics.

Tatiana Toro of the University of Washington delivered the inaugural Mirzakhani Lecture at the 2020 Joint Mathematics Meetings in Denver, Colorado, in January 2020. The title of her lecture was “Differential Operators and the Geometry of Domains in Euclidean Space.”

—AMS announcement

Arnold Ross Lectures

Noam D. Elkies of Harvard University will deliver the 2020 Arnold Ross Lecture at the University of Iowa on May 29, 2020. See https://www.ams.org/programs/students/ross-lectures/ross-lectures

Bjorn Poonen of the Massachusetts Institute of Technology gave the 2019 Ross Lecture, “Elliptic Curves,” at Pennsylvania State University in May 2019. Slides from his lecture can be viewed at math.mit.edu/~poonen/slides/elliptic.pdf

The lecture series for talented high school mathematics students is intended to stimulate their interest in mathematics beyond the traditional classroom and to show them the tremendous opportunities for careers in mathematics.

—From AMS announcements

Making Connections with ICERM

“Those who cannot learn from the past are condemned to repeat it.”

—Benjamin Disraeli

The headquarters of AMS and of ICERM (the Institute for Computational and Experimental Research in Mathematics) are both located in Providence, Rhode Island. This fall ICERM and the AMS got together for two events. The AMS Public Awareness Office ran a line-drawing activity for the general public at the Big Bang Waterfire, held at ICERM, and hosted a morning coffee at the AMS for attendees of the ICERM Illustrating Mathematics program.

—Annette Emerson and Mike Breen
AMS Public Awareness Officers
paoffice@ams.org

Credits
Photos are courtesy of Annette Emerson.
Mathematics Opportunities

Listings for upcoming mathematics opportunities to appear in Notices may be submitted to notices@ams.org.

Early Career Opportunity

AWM Birman Research Prize

The Association for Women in Mathematics (AWM) Joan and Joseph Birman Research Prize in Topology and Geometry recognizes exceptional research in topology/geometry by a woman within ten years of receipt of the PhD or who has not yet received tenure. The deadline for nominations is February 1, 2020. See https://awm-math.org/awards/awm-birman-research-prize

—From an AWM announcement

Early Career Opportunity

Call for Nominations for André Aisenstadt Prize

The Centre de Recherches Mathématiques (CRM) solicits nominations for the André Aisenstadt Mathematics Prize, awarded to recognize talented young Canadian mathematicians. The deadline for nominations is March 1, 2020. See the website www.crm.umontreal.ca/prix/prixAndreAisenstadt/prix_attributionAA_an.shtml

—From a CRM announcement

NRC Research Associateship Programs

The National Research Council (NRC) Research Associateship Programs promote excellence in scientific and technological research conducted by the US government through the administration of programs offering graduate-, postdoctoral-, and senior-level research opportunities at sponsoring federal laboratories and affiliated institutions. Application deadlines are February 1, May 1, August 1, and November 1, 2020. For details, see sites.nationalacademies.org/pga/rap/

—National Research Council announcement

Call for Nominations for Traub Prize

The Joseph F. Traub Prize is given for outstanding achievement in information-based complexity. The deadline for nominations is March 31, 2020. Nominations should be sent to erich.novak@uni-jena.de. See https://www.journals.elsevier.com/journal-of-complexity/awards/joseph-f-traub-prize

—Traub Prize Award Committee
Doctoral Conference on Mathematics Education

A national conference on doctoral programs in mathematics education will be held September 23–25, 2020, in Las Vegas, NV. The three-day conference is funded by the National Science Foundation (No. 1932697) and will bring together faculty members actively involved in doctoral programs in mathematics education to discuss issues relevant to designing and implementing doctoral preparation. The two specific goals of the conference are to:

1. Examine, discuss, and update the Principles to Guide the Design and Implementation of Doctoral Programs in Mathematics Education (https://amte.net/sites/all/themes/amte/resources/publications/DoctoralProgramsinMathematicsEducation.pdf) that were initially published by AMTE and NCTM (2002).
2. Collect/distribute current data related to niches in doctoral programs in mathematics education and identify ways this information can be used throughout the mathematics education community.

An application to the Conference is available at tinyurl.com/2020MathDocConference. The conference will fund three nights at the conference hotel and up to $500 for airfare. Every effort will be made to have a diverse representation of faculty members from a wide range of doctoral programs in mathematics education, although multiple participants from the same institution may be accepted. The deadline for completing and submitting applications is February 15, 2020. People with specific questions about the conference should contact Jeff Shih, UNLV, at jshih@unlv.nevada.edu or Robert Reys, University of Missouri, at reysr@missouri.edu.

—Robert Reys
The Notices Classified Advertising section is devoted to listings of current employment opportunities. The publisher reserves the right to reject any listing not in keeping with the Society's standards. Acceptance shall not be construed as approval of the accuracy or the legality of any information therein. Advertisers are neither screened nor recommended by the publisher. The publisher is not responsible for agreements or transactions executed in part or in full based on classified advertisements.

The 2020 rate is $3.65 per word. Advertisements will be set with a minimum one-line headline, consisting of the institution name above body copy, unless additional headline copy is specified by the advertiser. Headlines will be centered in boldface at no extra charge. Ads will appear in the language in which they are submitted. There are no member discounts for classified ads. Dictation over the telephone will not be accepted for classified ads.


US laws prohibit discrimination in employment on the basis of color, age, sex, race, religion, or national origin. Advertisements from institutions outside the US cannot be published unless they are accompanied by a statement that the institution does not discriminate on these grounds whether or not it is subject to US laws.

Submission: Send email to classads@ams.org.

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**KANSAS**

University of Kansas
Department of Mathematics

The Department of Mathematics, University of Kansas invites applications for three non-tenure-track Visiting Assistant Professor positions, one in algebra, one in geometry, and one in statistics to begin on August 18, 2020. Preference will be given to candidates whose research meshes well with current faculty interests in the designated areas. PhD or ABD in math, statistics or a related field is expected by the start date of the appointment. For a complete announcement and to apply online, go to [https://employment.ku.edu](https://employment.ku.edu) (and search for algebra 15931BR, geometry 15934BR, and statistics 15932BR). A complete on-line application includes: CV, cover letter, research and teaching statements, and the names and contact information for four references. At least four recommendation letters should be submitted electronically to [https://www.mathjobs.org](https://www.mathjobs.org) and search for positions for University of Kansas. Initial review of applications will begin December 2, 2019. In a continuing effort to enrich its academic environment and provide equal educational and employment opportunities, the university actively encourages applications from members of underrepresented groups in higher education. KU is an EO/AAE. All qualified applicants will receive consideration for employment without regard to race, color, religion, sex (including pregnancy), age, national origin, disability, genetic information or protected Veteran status.

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**CHINA**

Tianjin University, China
Tenured/Tenure-Track/Postdoctoral Positions at the Center for Applied Mathematics

Dozens of positions at all levels are available at the recently founded Center for Applied Mathematics, Tianjin University, China. We welcome applicants with backgrounds in pure mathematics, applied mathematics, statistics, computer science, bioinformatics, and other related fields. We also welcome applicants who are interested in practical projects with industries. Despite its name attached with an accent of applied mathematics, we also aim to create a strong presence of pure mathematics.

Light or no teaching load, adequate facilities, spacious office environment and strong research support. We are prepared to make quick and competitive offers to self-motivated hard workers, and to potential stars, rising stars, as well as shining stars.

The Center for Applied Mathematics, also known as the Tianjin Center for Applied Mathematics (TCAM), located by a lake in the central campus in a building protected as historical architecture, is jointly sponsored by the Tianjin municipal government and the university. The initiative to establish this center was taken by Professor S. S. Chern. Professor Molin Ge is the Honorary Director, Professor Zhiming Ma is the Director of the Advisory Board. Professor William Y. C. Chen serves as the Director.
TCAM plans to fill in fifty or more permanent faculty positions in the next few years. In addition, there are a number of temporary and visiting positions. We look forward to receiving your application or inquiry at any time. There are no deadlines.

Please send your resume to mathjobs@tju.edu.cn.
For more information, please visit http://cam.tju.edu.cn or contact Ms. Erica Liu at mathjobs@tju.edu.cn, telephone: 86-22-2740-6039.
FAN CHINA EXCHANGE PROGRAM

• Gives eminent mathematicians from the US and Canada an opportunity to travel to China and interact with fellow researchers in the mathematical sciences community.

• Allows Chinese scientists in the early stages of their careers to come to the US and Canada for collaborative opportunities.

Applications received before March 15 will be considered for the following academic year.

For more information on the Fan China Exchange Program and application process see www.ams.org/china-exchange or contact the AMS Professional Programs Department:

TELEPHONE: 800.321.4267, ext. 4105 (US & Canada)
401.455.4105 (worldwide)

EMAIL: chinaexchange@ams.org.
New Books Offered by the AMS

General Interest

Meeting under the Integral Sign?
The Oslo Congress of Mathematicians on the Eve of the Second World War
Christopher D. Hollings, Mathematical Institute and Queen’s College, University of Oxford, UK, and Reinhard Siegmund-Schultze, University of Agder, Kristiansand, Norway

This book examines the historically unique conditions under which the International Congress of Mathematicians took place in Oslo in 1936. Relying heavily on unpublished archival sources, the authors consider the different goals of the various participants in the Congress, most notably those of the Norwegian organizers and the Nazi-led German delegation. They also investigate the reasons for the absence of the proposed Soviet and Italian delegations.

History of Mathematics, Volume 44

bookstore.ams.org/hmath-44

Geometry and Topology

A Cornucopia of Quadrilaterals
Claudi Alsina, Universitat Politècnica de Catalunya, Barcelona, Spain, and Roger B. Nelsen, Lewis & Clark College, Portland, OR

A Cornucopia of Quadrilaterals collects and organizes hundreds of beautiful and surprising results about four-sided figures. There are results dating back to Euclid: the side-lengths of a pentagon, a hexagon, and a decagon inscribed in a circle can be assembled into a right triangle (the proof uses a quadrilateral and circumscribing circle); and results dating to Erdős: from any point in a triangle the sum of the distances to the vertices is at least twice as large as the sum of the distances to the sides.

Dolciani Mathematical Expositions, Volume 55
Discrete Mathematics and Combinatorics

Fundamentals of Graph Theory
Allan Bickle, Pennsylvania State University Altoona

The goal of this textbook is to present the fundamentals of graph theory to a wide range of readers. The book contains many significant recent results in graph theory presented using up-to-date notation. The author included the shortest, most elegant, most intuitive proofs for modern and classic results while frequently presenting them in new ways.


bookstore.ams.org/amstext-43

New in Contemporary Mathematics

Algebra and Algebraic Geometry

Analytic Methods in Arithmetic Geometry
Alina Bucur, University of San Diego, La Jolla, CA, and David Zureick-Brown, Emory University, Atlanta, GA, Editors

This volume contains the proceedings of the Arizona Winter School 2016 which was held from March 12–16, 2016 at the University of Arizona, Tucson, AZ. The School provided a unique opportunity to introduce graduate students to analytic methods in arithmetic geometry.

This item will also be of interest to those working in number theory.


bookstore.ams.org/conm-740

A Panorama of Singularities
Francisco-Jesús Castro-Jiménez, Universidad de Sevilla, Spain, David Bradley Massey, Northeastern University, Boston, MA, Bernard Teissier, Institut de Mathématiques de Jussieu-Paris Rive Gauche, France, and Meral Tosun, Galatasaray University, Istanbul, Turkey, Editors

This volume contains the proceedings of the conference A Panorama on Singular Varieties, celebrating the 70th birthday of Lê Dũng Tráng held from February 7–10, 2017 at the University of Seville, IMUS, Seville, Spain.


bookstore.ams.org/conm-742
NEW BOOKS

Mathematical Physics

Analytic Trends in Mathematical Physics
Houssam Abdul-Rahman, Robert Sims, and Amanda Young, all of University of Arizona, Tucson, Editors

This volume contains the proceedings of the Arizona School of Analysis and Mathematical Physics held from March 5–9, 2018 at the University of Arizona, Tucson, Arizona. The articles in this volume reflect recent progress and innovative techniques developed within mathematical physics.

This item will also be of interest to those working in analysis.

Contemporary Mathematics, Volume 741

New in Memoirs of the AMS

Algebra and Algebraic Geometry

An Elementary Recursive Bound for Effective Positivstellensatz and Hilbert’s 17th Problem
Henri Lombardi, Université de Franche-Comté, Besançon, France, Daniel Perrucci, Universidad de Buenos Aires, Argentina, and Marie-Françoise Roy, Université de Rennes, France

Memoirs of the American Mathematical Society, Volume 263, Number 1277
NEW BOOKS

Mathematical Physics

Geometric Optics for Surface Waves in Nonlinear Elasticity
Jean-François Coulombel, Université de Nantes, France, and Mark Williams, University of North Carolina, Chapel Hill

Memoirs of the American Mathematical Society, Volume 263, Number 1271

Probability and Statistics

The Triangle-Free Process and the Ramsey Number \( R(3,k) \)

This item will also be of interest to those working in discrete mathematics and combinatorics.

Memoirs of the American Mathematical Society, Volume 263, Number 1274

Number Theory

Sums of Reciprocals of Fractional Parts and Multiplicative Diophantine Approximation
Victor Beresnevich, University of York, United Kingdom, Alan Haynes, University of Houston, Texas, and Sanju Velani, University of York, United Kingdom

Memoirs of the American Mathematical Society, Volume 263, Number 1276

New AMS-Distributed Publications

Analysis

Asymptotic Analysis for Nonlinear Dispersive and Wave Equations
Keiichi Kato, Tokyo University of Science, Japan, Takayoshi Ogawa, Tohoku University, Japan, and Tohru Ozawa, Waseda University, Japan, Editors

This volume contains 18 papers related to the asymptotic analysis and qualitative research paper concerning the problems of nonlinear wave equations and nonlinear dispersive equations, such as nonlinear Schrödinger equations, the...
Hartree equation, the Camassa-Holm equation, and the Ginzburg-Landau equations. In one of the papers, the outstanding method developed by Professor Hayashi and his collaborators is introduced by one of his main collaborators, P.I. Naumkin.

Published for the Mathematical Society of Japan by Kinokuniya, Tokyo, and distributed worldwide, except in Japan, by the AMS.

**Advanced Studies in Pure Mathematics**, Volume 81

Hartree equation, the Camassa-Holm equation, and the Ginzburg-Landau equations. In one of the papers, the outstanding method developed by Professor Hayashi and his collaborators is introduced by one of his main collaborators, P.I. Naumkin.

**Hyperbolic Flows**
Todd Fisher, Brigham Young University, Provo, UT, and Boris Hasselblatt, Tufts University, Medford, MA

This book presents the theory of flows from the topological, smooth, and measurable points of view. The first part introduces the general topological and ergodic theory of flows, and the second part presents the core theory of hyperbolic flows as well as a range of recent developments.

This item will also be of interest to those working in differential equations.

A publication of the European Mathematical Society (EMS). Distributed within the Americas by the American Mathematical Society.

**Zurich Lectures in Advanced Mathematics**, Volume 25

**Regular Poisson Manifolds of Compact Types**
Marius Crainic, Utrecht University, The Netherlands, Rui Loja Fernandes, University of Illinois at Urbana-Champaign, and David Martínez Torres, Pontifical Catholic University, Rio de Janeiro, Brazil

This is the second paper of a series dedicated to the study of Poisson structures of compact types (PMCTs). In this paper, the authors focus on regular PMCTs, exhibiting a rich transverse geometry. They show that their leaf spaces are integral affine orbifolds.

This item will also be of interest to those working in geometry and topology.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the U.S., Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

**Astérisque**, Number 413

This item will also be of interest to those working in geometry and topology.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the U.S., Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.
The Meetings and Conferences section of the Notices gives information on all AMS meetings and conferences approved by press time for this issue. Please refer to the page numbers cited on this page for more detailed information on each event.

Invited Speakers and Special Sessions are listed as soon as they are approved by the cognizant program committee; the codes listed are needed for electronic abstract submission. For some meetings the list may be incomplete. Information in this issue may be dated.

The most up-to-date meeting and conference information can be found online at www.ams.org/meetings.

Important Information About AMS Meetings: Potential organizers, speakers, and hosts should refer to page 110 in the January 2020 issue of the Notices for general information regarding participation in AMS meetings and conferences.

Abstracts: Speakers should submit abstracts on the easy-to-use interactive Web form. No knowledge of \LaTeX is necessary to submit an electronic form, although those who use \LaTeX may submit abstracts with such coding, and all math displays and similarly coded material (such as accent marks in text) must be typeset in \LaTeX. Visit www.ams.org/cgi-bin/abstracts/abstract.pl. Questions about abstracts may be sent to abs-info@ams.org. Close attention should be paid to specified deadlines in this issue. Unfortunately, late abstracts cannot be accommodated.

### Meetings in this Issue

**2020**

- **January 15–18**: Denver, Colorado p. 285
- **March 13–15**: Charlottesville, Virginia p. 290
- **March 21–22**: Medford, Massachusetts p. 292
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- **October 10–11**: Chattanooga, Tennessee p. 304
- **October 24–25**: Salt Lake City, Utah p. 304

**2021**

- **January 6–9**: Washington, DC p. 305
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- **October 23–24**: Albuquerque, NM p. 308

**2022**

- **January 5–8**: Seattle, Washington p. 308

**2023**

- **January 4–7**: Boston, Massachusetts p. 308

See www.ams.org/meetings for the most up-to-date information on the meetings and conferences that we offer.

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**Associate Secretaries of the AMS**

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**Southeastern Section**: Brian D. Boe, Department of Mathematics, University of Georgia, 220 D W Brooks Drive, Athens, GA 30602-7403; email: brian@math.uga.edu; telephone: 706-542-2547.

**Western Section**: Michel L. Lapidus, Department of Mathematics, University of California, Surge Bldg., Riverside, CA 92521-0135; email: lapidus@math.ucr.edu; telephone: 951-827-5910.
Meetings & Conferences of the AMS

Denver, Colorado

Colorado Convention Center

January 15–18, 2020

Wednesday – Saturday

Meeting #1154

Joint Mathematics Meetings, including the 126th Annual Meeting of the AMS, 103rd Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM)

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/national.html.

Joint Invited Addresses

Skip Garibaldi, IDA Center for Communications Research, La Jolla, Uncovering lottery shenanigans (AMS-MAA Invited Address).

Karen M. Lange, Wellesley College, Different problems, common threads: Computing the difficulty of mathematical problems (AMS-MAA Invited Address).

Rajiv Maheswaran, Second Spectrum, The fantastic intersection of math and sports: Where no one is afraid of a decimal point (MAA-AMS-SIAM Gerald and Judith Porter Public Lecture).

Birgit Speh, Cornell University, Branching laws for representations of a non compact orthogonal group (AWM-AMS Noether Lecture).

AMS Invited Addresses

Bonnie Berger, Massachusetts Institute of Technology, Biomedical data sharing and analysis at scale.

Ingrid Daubechies, Duke University, Mathematical frameworks for signal and image analysis.

Gregory W. Moore, Rutgers University, Smooth invariants of four-dimensional manifolds and quantum field theory.
MEETINGS & CONFERENCES

Nancy Reid, University of Toronto, *In praise of small data: statistical and data science* (AMS Josiah Willard Gibbs Lecture).


Tatiana Toro, University of Washington, *Differential operators and the geometry of domains in Euclidean space* (AMS Maryam Mirzakhani Lecture).

Anthony Várilly-Alvarado, Rice University, *The geometric disposition of Diophantine equations.*

AMS Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at http://jointmathematicsmeetings.org/meetings/abstracts/abstract.pl?type=jmm.

Some sessions are cosponsored with other organizations. These are noted within the parenthesis at the end of each listing, where applicable.

**Advances in Multivariable Operator Theory: Connections with Algebraic Geometry, Free Analysis, and Free Probability,** Joseph A. Ball, Virginia Tech, and Paul S. Muhly, University of Iowa.

**Advances in Operator Algebras,** Ian Charlesworth, University of California, Berkeley, Brent Nelson, Michigan State University, Sarah Reznikoff, Kansas State University, and Lauren Ruth, Vanderbilt University.

**Algebraic Cycles in Arithmetic and Geometry,** Jeff Achter, Colorado State University, and Sebastian Casalaina-Martin, University of Colorado Boulder.

**Algebras and Algorithms,** Keith A. Kearnes, Peter Mayr, and Agnes Szendrei, University of Colorado, Boulder.

**Algorithms, Analysis, and Applications of Numerical PDEs,** Xiaoming He, Missouri University of Science and Technology, and Jiangguo (James) Liu, Colorado State University.

**Algorithms, Experimentation, and Applications in Number Theory,** Beth Malmskog, Colorado College, and Christopher Rasmussen, Wesleyan University.

**Analysis and Differential Equations at Undergraduate Institutions,** William Green, Rose-Hulman Institute of Technology, and Katharine Ott, Bates College.

**Analysis of Nonlocal Models,** Giacomo Capodaglio, Florida State University, Marta D’Elia, Center for Computing Research, Sandia National Laboratories, and Max Gunzburger, Florida State University.

**Analytic and Probabilistic Combinatorics,** Miklós Bóna, University of Florida, and Jay Pantone, Marquette University.

**Analytic Theory of Automorphic Forms and L-Functions,** Amanda Folsom, Amherst College, Michael Griffin, Brigham Young University, Larry Rolen, Vanderbilt University, and Jesse Thorner, Colorado State University.

**Applications and Computations in Knot Theory,** Harrison Chapman, Colorado State University, Heather A. Dye, McKendree University, and Jesse S.F. Levitt, University of Southern California.

**Applied Topology,** Henry Adams, Colorado State University, and Mikael Vejdemo-Johansson, CUNY College of Staten Island.

**Arithmetic Dynamics,** I (Associated with AMS Retiring Presidential Address Kenneth A. Ribet), Rafe Jones, Carleton College.

**Arithmetic Galois Actions,** Ozlem Ejder, Colorado State University, McKenna Hendricks, Colorado State University, and William Green, Rose-Hulman Institute of Technology.

**Aspects and Applications of Algebraic Combinatorics,** William J. Martin, Worcester Polytechnic Institute, and Jason Wilford, University of Wyoming.

**C*-Algebras, Dynamical Systems and Applications,** Robin Deeley, University of Colorado Boulder, and Zhuang Niu and Ping Zhong, University of Wyoming.

**Choiceless Set Theory and Related Areas,** Paul Larson, Miami University, and Jindrich Zapletal, University of Florida (AMS-ASL).

**Coding Theory and Applications,** Allison Beemer, New Jersey Institute of Technology, Ian F. Blake, University of British Columbia, Christine A. Kelley, University of Nebraska-Lincoln, and Felice Manganiello, Clemson University.

**Cohomological Field Theories and Wall Crossing,** Yefeng Shen, University of Oregon, and Mark Shoemaker, Colorado State University.

**Combinatorial Structures and Integrable Systems,** Maxim Arnold and Nathan Williams, University of Texas at Dallas.

**Commutative Algebra,** Patricia Klein, University of Kentucky, and Haydee Lindo, Williams College.

**Computational and Categorical Methods in Homotopy Theory,** Agnes Beaudry, University of Colorado Boulder, and Julie Bergner, University of Virginia.
MEETINGS & CONFERENCES

Computational Biomedicine, Nek Valous and Niels Halama, National Center for Tumor Diseases Heidelberg, German Cancer Research Center.


Differential and Difference Equations in Biological Dynamics, Xiang-Sheng Wang and Aijun Zhang, University of Louisiana at Lafayette.

Differential Geometry and Global Analysis, I, Honoring the Memory of Tadashi Nagano (1930–2017), Bang-Yen Chen, Michigan State University, Nicholas D. Brubaker and Thomas Murphy, California State University, Fullerton, Takashi Sakai, Tokyo Metropolitan University, Makiko Sumi Tanaka, Tokyo University of Sciences, Bogdan D. Suceava, California State University, Fullerton, and Mihaela B. Vajiac, Chapman University.

Evolution, Chris McCarthy and Johannes Hamilton, Borough of Manhattan Community College CUNY.


Explicit Methods in Arithmetic Geometry in Characteristic \(p\), I (a Mathematics Research Communities Session), Vaidehee Thatte, Binghamton University, Sarah Arpin, University of Colorado Boulder, and Nicholas Triantafillou, University of Georgia.

Extremal and Probabilistic Combinatorics, Sean English and Emily Heath, University of Illinois Urbana Champaign, and Michael Tait, Villanova University.

Fractal Geometry, Dynamical Systems, and Applications, Andrea Arauza Rivera, California State University, East Bay, Robert Niemeyer, Metropolitan State University, and John Rock, California State Polytechnic University, Pomona.

Frames, Designs, and Optimal Spherical Configurations, Xuemei Chen, New Mexico State University, Alexey Glazyrin, University of Texas Rio Grande Valley, Kasso Okoudjou, University of Maryland, College Park, and Oleksandr Vlasiuk, Florida State University.

From STEM to STEAMS (Science, Technology, Engineering, AI, Mathematics, Statistics), Charles Chen, Applied Materials, and Mason Chen, Stanford OHS.

Future Directions in Theory & Applications of Nonlinear Reaction-Diffusion Equations, Jerome Goddard, II, Auburn University at Montgomery, Nsoki Mavinga, Swarthmore College, and Quinn Morris, Appalachian State University.

Geometric Representation Theory and Equivariant Elliptic Cohomology, I (a Mathematics Research Communities Session), Anne Dranowski, University of Toronto, Noah Arbesfeld, Imperial College London, and Dominic Culver, University of Illinois Urbana-Champaign.

Geometry of Differential Equations, Jeanne Clelland and Yuhao Hu, University of Colorado Boulder, and George Wilkens, University of Hawaii.

Getting Started in Undergraduate Research: Topics, Tools and Open Problems, Hannah Highlander, University of Portland, Pamela E. Harris, Williams College, Erik Insko, Florida Gulf Coast University, and Aaron Wootton, University of Portland (AMS-MAA).

Group Actions in Harmonic Analysis, Keri Kornelson, University of Oklahoma, and Emily J. King, University of Bremen.

Groups and Topological Dynamics, Constantine Medynets, United States Naval Academy, Volodymyr Nekrashevych, Texas A&M University, and Dmytro Savchuk, University of South Florida.

Hamiltonian Systems, Sean Gasiorek, University of Sydney, Gabriel Martins, California State University Sacramento, and Andres Perico, University of California Santa Cruz.

Harmonic Analysis, Taryn C. Flock, Macalester College, and Betsy Stovall, University of Wisconsin-Madison.

Highly Accurate and Structure-Preserving Numerical Methods for Nonlinear Partial Differential Equations, Qin Sheng, Baylor University, Jorge E. Macias-Diaz, Universidad Autonoma de Aguascalientes, and Joshua L. Padgett, Texas Tech University.

History of Mathematics, Jemma Lorenat, Pitzer College, Sloan Despeaux, Western Carolina University, Daniel Otero, Xavier University, and Adrian Rice, Randolph-Macon College (AMS-MAA).

How to Discover and Train Gifted Students, Scott Annin, California State University, Fullerton, Cezar Lupu, Texas Tech University, Shoo Seto, University of California, Irvine, and Bogdan D. Suceava, California State University, Fullerton.

If You Build It They Will Come: Presentations by Scholars in the National Alliance for Doctoral Studies in the Mathematical Sciences, David Goldberg, Purdue University, and Phil Kutzko, University of Iowa.

Interactions Among Partitions, Basic Hypergeometric Series, and Modular Forms, Chris Jennings-Shaffer, University of Denver, and Frank Garvan, University of Florida.

Interactions between Combinatorics, Representation Theory, and Coding Theory, Manabu Hagiwara, Chiba University, and Richard Green, University of Colorado Boulder.

Interactions of Inverse Problems, Computational Harmonic Analysis, and Imaging, M. Zuhair Nashed, University of Central Florida, Willi Freeden, University of Kaiserslautern, and Otmar Scherzer, University of Vienna.
MEETINGS & CONFERENCES

Interfaces Between PDEs and Geometric Measure Theory, I (Associated with AMS Maryam Mirzakhani Invited Address Tatiana Toro), Robin Neumayer and Zihui Zhao, Institute for Advanced Study.

International Research Experience for Students (IRES), Asuman G. Aksoy, Claremont McKenna College, and Zair Ibragimov, California State University, Fullerton.

Iterative Methods for Large-Scale Data Analysis, Jamie Haddock, University of California Los Angeles, and Anna Ma, University of California San Diego.

Logic Facing Outward, I (Associated with Joint AMS-MAA Invited Address Karen Lange), Karen Lange, Wellesley College, and Russell Miller, Queens College & Graduate Center CUNY (AMS-ASL).


Mathematical Analysis in Data Science, I (Associated with AMS Colloquium Lectures Ingrid Daubechies), Radu Balan, University of Maryland, Tingran Gao, University of Chicago, Sinan Gunter, New York University, and Ozgur Yilmaz, University of British Columbia.

Mathematical and Computational Research in Data Science, Linda Ness, DIMACS, Rutgers University, F. Patricia Medina, Yeshiva University, and Kathryn Leonard, Occidental College (AMS-AWM).

Mathematical Aspects of Conformal Field Theory, Shashank Kanade and Andrew Linshaw, University of Denver, and Robert McRae, Vanderbilt University.

Mathematical Information in the Digital Age of Science, Patrick Ion, IMKT & University of Michigan, Olaf Teschke, zb-Math, and Stephen Watt, University of Waterloo.

Mathematical Physics, Some Open Problems for the 21st Century, Michael Maroun.

Mathematical Programming and Combinatorial Optimization, Steffen Borgwardt, University of Colorado Denver, and Tamon Stephen, Simon Fraser University.

Mathematics and Motherhood, Della Dumbaugh, University of Richmond, Carrie Diaz Eaton, Bates College, and Emille Lawrence, University of San Francisco.

Matrices and Graphs, Leslie Hogben, Iowa State University and American Institute of Mathematics, and Bryan L. Shader, University of Wyoming.

Mean Field Games: Theory and Applications, François Delarue, University of Nice Sophia Antipolis.

Modeling Natural Resources, Shandelle M. Henson, Andrews University, and Julie Blackwood, Williams College.

Noncommutative Geometry and Applications, Frederic Latremoliere, University of Denver.

Novel Teaching Practices in Mathematics, David Weisbart, University of California, Riverside.

Outreach Strategies for Reaching Underrepresented Students at the Pre-College Level, Jacob Castaneda, The Art of Problem Solving/Bridge to Enter Advanced Mathematics (BEAM), Cory Colbert, Washington & Lee University, Li-Mei Lim, Boston University/PROMYS, Max Warshauer, Texas State University at San Marcos, and Daniel Zaharovpol, The Art of Problem Solving Initiative/Bridge to Enter Advanced Mathematics (BEAM).

Partition Theory and $q$-Series, Madeline Locus Dawsey, The University of Texas at Tyler, Marie Jameson, University of Tennessee, Knoxville, and James Sellers, Pennsylvania State University.

Pedagogical Innovations That Lead to Successful Mathematics, Michael A. Radin, Rochester Institute of Technology, Natali Hritonenko, Prairie View A&M University, and Ellina Grigorieva, Texas Women's University.

Quantization for Probability Distributions and Dynamical Systems, Mrinal Kanti Roychowdhury, University of Texas Rio Grande Valley.

Quantum Theory of Matter Meets Noncommutative Geometry and Topology, Masoud Khalkhali, University of Western Ontario, and Markus Pflaum, University of Colorado, Boulder.

Random Combinatorial Structures, Complex Analysis and Integrable Systems, Virgil U. Pierce, University of Northern Colorado, and Nicholas M. Ercolani, University of Arizona.

Random Matrices and Integrable Systems, I (a joint session with the SIAM Minisymposium on the same topic), Ken McLaughlin, Colorado State University, and Sean O'Rourke, University of Colorado Boulder (AMS-SIAM).


Recent Advances in Function and Operator Theory, Kelly Bickel, Bucknell University, Alberto Condori, Florida Gulf Coast University, William Ross, University of Richmond, and Alan Sola, Stockholm University.

Recent Advances in Time-Stepping Methods for Ocean Modeling, Sara Calandrini, Konstantin Pieper, and Max Gunzburger, Florida State University.

Recent Advances of Mathematical Modeling on Ecology and Epidemiology, Xi Huo, University of Miami, and Rongsong Liu, University of Wyoming.
Recent Developments in Numerical Methods for PDEs, Valeria Barra, University of Colorado Boulder, and Oana Marin, Argonne National Laboratory.

Recent Trends in Semigroup Theory, Michael Kinyon, University of Denver, and Ben Steinberg, City College of New York.

Representations of Finite Groups and Related Structures, Mandi Schaeffer Fry, Metropolitan State University of Denver, and Nat Thiem, University of Colorado Boulder.

Representation Theory Inspired by the Langlands Conjectures, I (Associated with Joint AWM-AMS Noether Lecture Birgit Speh), Birgit Speh, Cornell University, and Peter Trapa, University of Utah (AMS-AWM).

Research from the Rocky Mountain-Great Plains Graduate Research Workshop in Combinatorics, Steve Butler, Iowa State University, Michael Ferrara, University of Colorado Denver, Jeremy Martin, University of Kansas, Tyrrell McAllister, University of Wyoming, and Jamie Radcliffe, University of Nebraska-Lincoln.

Research in Graph Theory and Combinatorics by Research Experience for Undergraduate Faculty (REUF) Alumni and Their Students, Katie Anders and Kassie Archer, University of Texas at Tyler, and Briana Foster-Greenwood, California State Polytechnic University-Pomona.

Research in Mathematics by Early Career Graduate Students, Marat Markin, Morgan Rodgers, and Khang Tran, California State University Fresno.

Research in Mathematics by Undergraduates and Students in Post-Baccalaureate Programs, Darren A. Narayan, Rochester Institute of Technology, Khang Tran, California State University Fresno, Mark David Ward, Purdue University, and John Wierman, The Johns Hopkins University (AMS-MAA-SIAM).

Riemannian Foliations and Applications, Igor Prokhorenkov and Ken Richardson, Texas Christian University.

Self-Distributive Structures, Knot Theory, and the Yang-Baxter Equation, Mohamed Elhamdadi, University of South Florida, Petr Vojtechovsky, University of Denver, and David Stanovský, Charles University in Prague.

Sequences, Words, and Automata, Eric Rowland and Manon Stipulanti, Hofstra University.

Set-Valued and Fuzzy-Valued Analysis with Applications, Vira Babenko, Drake University.

Singularities and Characteristic Classes, Paolo Aluffi, Florida State University, and Leonardo Mihalcea, Virginia Tech.

Spectral and Transport Properties of Disordered Systems, Peter D. Hislop, University of Kentucky, and Jeffrey Schenker, Michigan State University.

Stochastic Analysis and Applications in Finance, Actuarial Science and Related Fields, Julius N. Esunge, University of Mary Washington, See Keong Lee, Universiti Sains Malaysia, and Aurel I. Stan, The Ohio State University.

Stochastic Differential Equations and Application of Mathematical Biology, Jianjun Paul Tian, New Mexico State University, Hai-Dang Nguyen, University of Alabama, Xianyi Zeng, University of Texas at El Paso, and Robert Smits, New Mexico State University.

Stochastic Spatial Models (a Mathematics Research Communities Session), Tobias Johnson, College of Staten Island, Erin Beckman, Duke University, and Katelynn Kochalski, SUNY Geneseo.

Symbolic Dynamics, Ronnie Pavlov, University of Denver, and Scott Schmieding, Northwestern University.

The Geometry of Complex Polynomials and Rational Functions, Trevor Richards, Monmouth College, and Malik Younsi, University of Hawaii.

The Kaczmarz Algorithm with Applications in Harmonic Analysis and Data Science, Xuemei Chen, New Mexico State University, Palle E.T. Jorgensen, University of Iowa, and Eric Weber, Iowa State University.

The Mathematics of Social Justice, Andrea Arauza Rivera, California State University, East Bay, Paige Helms, University of Washington, Ryan Moruzzi, Ithaca College, and Robin Wilson, California Poly Pomona.

Topological Measures of Complexity: Inverse Limits, Entropy, and Structure of Attractors, Lori Alvin, Furman University, Jan P. Boronski, National Supercomputing Centre IT4innovations, Joanna Furno, University of Houston, and Piotr Oprocha, AGH University of Science and Technology.

Utilizing Mathematical Models to Understand Tumor Heterogeneity and Drug Resistance, James Greene, Clarkson University, Hwayeon Ryu, Elon University, and Kamila Larripa, Humboldt State University.


Wall to Wall Modeling Activities in Differential Equations Courses, Janet Fierson, La Salle University, Therese Shelton, Southwestern University, and Brian Winkel, SIMIODE.

Women in Mathematical Biology, Christina Edholm, University of Tennessee, Amanda Laubmeier, University of Nebraska-Lincoln, Katharine Gurski, Howard University, and Heather Zinn Brooks, University of California Los Angeles (AMS-AWM).

Women in Symplectic and Contact Geometry, Morgan Weiler, Rice University, Catherine Cannizzo, Simons Center for Geometry and Physics, and Melissa Zhang, University of Georgia (AMS-AWM).

Charlottesville, Virginia
University of Virginia

March 13–15, 2020
Friday – Sunday

Meeting #1155
Southeastern Section
Associate secretary: Brian D. Boe

Announcement issue of Notices: January 2020
Program first available on AMS website: February 4, 2020
Issue of Abstracts: Volume 41, Issue 2

Deadlines
For organizers: Expired
For abstracts: January 21, 2020

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/sectional.html.

Invited Addresses
Moon Duchin, Tufts University, How we divide ourselves up to vote, and why it matters (Einstein Public Lecture in Mathematics).
Laura Ann Miller, University of North Carolina, The fluid dynamics of nutrient exchange in organs and organisms at the mesoscale.
Betsy Stovall, University of Wisconsin-Madison, An inverse problems approach to some questions arising in harmonic analysis.
Yusu Wang, Ohio State University, Topological and geometric methods for graph analysis.

Special Sessions
If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.

Advances in Difference, Differential, Fractional Differential and Dynamic Equations with Applications (Code: SS 2A), Muhammad Islam and Yousef Raffoul, University of Dayton.
Advances in High and Infinite Dimensional Stochastic Analysis (Code: SS 36A), Juraj Foldes, University of Virginia, Nathan Glatt-Holtz, Tulane University, and Mouhamadou Sy, University of Virginia.
Advances in Infectious Disease Modeling: From Cells to Populations (Code: SS 4A), Lauren Childs, Stanca Ciupe, and Omar Saucedo, Virginia Tech.
Advances in Operator Algebras (Code: SS 22A), Ben Hayes and David Sherman, University of Virginia.
Algebraic Groups: Arithmetic and Geometry (Code: SS 10A), Raman Parimala, Emory University, Andrei Rapinchuk, University of Virginia, and Igor Rapinchuk, Michigan State University.
Categorical Representation Theory and Beyond (Code: SS 11A), You Qi and Liron Speyer, University of Virginia, and Joshua Sussan, CUNY Medgar Evers (AMS-AAAS).
Celebrating Diversity in Mathematics (Code: SS 20A), Lauren Childs, Virginia Tech, Sara Maloni, University of Virginia, and Rebecca R.G., George Mason University.
Combinatorial Methods in Geometric Group Theory (Code: SS 26A), Tarik Aougab, Haverford College, Marrissa Loving, Georgia Institute of Technology, Priyam Patel, University of Utah, and Sunny Xiao, Brown University.
Combinatorics Related to Geometry and Representation Theory (Code: SS 34A), Heather M Russell, University of Richmond, and Rebecca Goldin, George Mason University.
Commutative Algebra (Code: SS 28A), Eloisa Grifo, University of California, Riverside, and Sean Sather-Wagstaff, Clemson University.
Convexity and Probability in High Dimensions (Code: SS 31A), Steven Hoehner, Longwood University, and Mark Meckes and Elisabeth Werner, Case Western Reserve University.
Curves, Jacobians, and Abelian Varieties (Code: SS 1A), Andrew Obus, Baruch College (CUNY), Tony Shaska, Oakland University, and Padmavathi Srinivasan, Georgia Institute of Technology.
Cyber Defense and Cryptography in Undergraduate Education (Code: SS 23A), Lubjana Beshaj, West Point Military Academy, and Tony Shaska, Oakland University.

Homotopy Theory (Code: SS 15A), Julie Bergner and Nick Kuhn, University of Virginia.

Integrable Probability (Code: SS 27A), Leonid Petrov, University of Virginia, and Axel Saenz.

Knots and Links in Low-Dimensional Topology (Code: SS 5A), Thomas Mark, University of Virginia, Allison Moore, University of California Davis, and Ziva Myer, Duke University.

Knot Theory and its Applications (Code: SS 25A), Hugh Howards and Jason Parsley, Wake Forest University, and Eric Rawdon, St. Thomas University.

Mathematical Modeling of Problems in Biological Fluid Dynamics (Code: SS 30A), Laura Miller, University of North Carolina at Chapel Hill, and Nick Battista, The College of New Jersey.

Mathematical String Theory (Code: SS 9A), Ilarion Melnikov, James Madison University, Eric Sharpe, Virginia Tech, and Diana Vaman, University of Virginia (AMS-AAAS).

Motivic Aspects of Topology and Geometry (Code: SS 16A), Kirsten Wickelgren, Duke University, and Inna Zakharevich, Cornell University.

Nonlocal PDEs and Applications (Code: SS 33A), Siming He, Duke University, and Changhui Tan, University of South Carolina.

Numerical Methods for Partial Differential Equations: A Session in Honor of Slimane Adjerid's 65th Birthday (Code: SS 3A), Mahboub Baccouch, University of Nebraska at Omaha.

Probabilistic Methods in Geometry and Analysis (Code: SS 19A), Fabrice Baudoin and Li Chen, University of Connecticut.

Quantum Algebra and Geometry (Code: SS 24A), Marco Aldi, Virginia Commonwealth University, Michael Penn, Randolph College, and Nicola Tarasca and Juan Villarreal, Virginia Commonwealth University.

Recent Advances in Graph Theory and Combinatorics (Code: SS 8A), Neal Bushaw, Virginia Commonwealth University, and Martin Rolek and Gexin Yu, College of William and Mary (AMS-AAAS).

Recent Advances in Harmonic Analysis (Code: SS 7A), Amalia Culiuc, Amherst College, Yen Do, University of Virginia, and Eyvindur Ari Palsson, Virginia Tech.

Recent Advances in Mathematical Biology (Code: SS 32A), Junping Shi, College of William & Mary, Zhisheng Shuai, University of Central Florida, and Yixiang Wu, Middle Tennessee State University.

Recent Combinatorial Advances in Representation Theory and Algebraic Geometry (Code: SS 29A), Jennifer Morse, University of Virginia, and Sarah Mason, Wake Forest University.

Recent Progress on Singular and Oscillatory Integrals (Code: SS 35A), Betsy Stovall and Joris Roos, University of Wisconsin-Madison.

Representation Theory of Algebraic Groups and Quantum Groups: A Tribute to the Work of Cline, Parshall and Scott (CPS) (Code: SS 13A), Chun-Ju Lai and Daniel K. Nakano, University of Georgia, and Weiqiang Wang, University of Virginia.

Special Sets of Integers in Modern Number Theory (Code: SS 14A), Cristina Ballantine, College of the Holy Cross, and Hester Graves, Center for the Computing Sciences.

Tensors and Complexity (Code: SS 17A), Visu Makam, Institute for Advanced Study, and Rafael Oliveira, University of Toronto.

The Mathematics of Redistricting (Code: SS 18A), Marion Campisi, San Jose State University, Thomas Ratliff, Wheaton College, and Ellen Veomett, Saint Mary's College of California.

Trends in Teichmüller Theory (Code: SS 21A), Thomas Koberda and Sara Maloni, University of Virginia, and Giuseppe Martone, University of Michigan.

Youth and Enthusiasm in Arithmetic Geometry and Number Theory (Code: SS 12A), Evangelia Gazaki and Ken Ono, University of Virginia.
MEETINGS & CONFERENCES

Medford, Massachusetts
Tufts University

March 21–22, 2020
Saturday – Sunday

Meeting #1156
Eastern Section
Associate secretary: Steven H. Weintraub

MEETINGS & CONFERENCES

Announcement issue of Notices: January 2020
Program first available on AMS website: February 11, 2020
Issue of Abstracts: Volume 41, Issue 2

Deadlines
For organizers: Expired
For abstracts: January 28, 2020

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Daniela De Silva, Columbia University, A viscosity approach to the regularity of variational problems.
Enrique R Pujals, Graduate Center, CUNY, Fifty years of the stability conjecture.
Christopher T Woodward, Rutgers University, New Brunswick, Lagrangian Floer theory in the tropics.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.

Algebraic Geometry in Dynamics (Code: SS 15A), Nguyen-Bac Dang, Stony Brook University, and Nicole Looper and Rohini Ramadas, Brown University.
Analysis on Homogeneous Spaces (Code: SS 6A), Jens Christensen, Colgate University, Matthew Dawson, CIMAT, Mérida, México, and Fulton Gonzalez, Tufts University.
Anomalous Diffusion Processes (Code: SS 3A), Christoph Borgers, Tufts University, and Claude Greengard, New York University and Foss Hill Partners.
Applied Combinatorics (Code: SS 21A), Carina Curto, Pennsylvania State University, and Pedro Felzenszwalb and Caroline Klivans, Brown University.
Automorphisms of Riemann Surfaces, Subgroups of Mapping Class Groups and Related Topics (Code: SS 10A), S. Allen Broughton, Rose-Hulman Institute of Technology, Jen Paulhus, Grinnell College, and Aaron Wootton, University of Portland.
Discrete and Convex Geometry (Code: SS 18A), Undine Leopold and Egon Schulte, Northeastern University, and Pablo Soberón, Baruch College, CUNY.
Equivariant Cohomology (Code: SS 9A), Jeffrey D. Carlson, The Fields Institute, and Loring Tu, Tufts University.
Geometric Dynamics and Billiards (Code: SS 4A), Boris Hasselblatt and Eunice Kim, Tufts University, Kathryn Lindsey, Boston College, and Zbigniew Nitecki, Tufts University.
Homological Methods in Commutative Algebra (Code: SS 12A), Janet Striuli, Fairfield University and National Science Foundation, and Oana Veliche, Northeastern University.
Inverse Problems and Their Applications (Code: SS 19A), Youssef Qranfal, Wentworth Institute of Technology.
Linear Algebraic Groups: their Structure, Representations, and Geometry (Code: SS 23A), George McNinch, Tufts University, and Eric Sommers, University of Massachusetts.
Mathematics of Data Science (Code: SS 5A), Vasileios Maroulas, University of Tennessee Knoxville, and James M. Murphy, Tufts University.
Mirror Symmetry and Enumerative Geometry (Code: SS 20A), Mandy Cheung, Harvard University, and Siu-Cheong Lau and Yu-Shen Lin, Boston University.
MEETINGS & CONFERENCES

Modeling and Analysis of Partial Differential Equations in Fluid Dynamics and Related Fields: Geometric and Probabilistic Methods (Code: SS 1A), Geng Chen, University of Kansas, Siran Li, Rice University and Centre de Recherches Mathématiques, Université de Montréal, and Kun Zhao, Tulane University.

Moduli of Curves, Hilbert Schemes, and Tropical Geometry (Code: SS 17A), Ignacio Barros, Northeastern University, Noah Giansiracusa, Bentley University, and Rob Silversmith, Northeastern University.

Probability in Dynamical Systems of Physical Origin (Code: SS 13A), Alex Blumenthal, University of Maryland, and Peter Nandori, Yeshiva University.

Quantum Probability, Orthogonal Polynomials, and Special Functions (Code: SS 11A), Maxim Derevyagin and Ambar Sengupta, University of Connecticut.

Random Discrete Structures (Code: SS 22A), Xavier Pérez-Giménez, University of Nebraska, and Lutz P Warnke, Georgia Institute of Technology.

Recent Advances in Schubert Calculus and Related Topics (Code: SS 2A), Christian Lenart and Changlong Zhong, State University of New York at Albany.

Symmetries of Polytopes, Maps, and Graphs (Code: SS 16A), Gabe Cunningham, University of Massachusetts Boston, and Mark Mixer, Wentworth Institute of Technology.

The Combinatorics and Geometry of Jordan Type and Commuting Varieties (Code: SS 14A), Peter Crooks and Anthony Iarrobino, Northeastern University, and Leila Khatami, Union College.

West Lafayette, Indiana

Purdue University

April 4–5, 2020

Saturday – Sunday

Meeting #1157

Central Section

Associate secretary: Georgia Benkart

Announcement issue of Notices: February 2020
Program first available on AMS website: February 18, 2020
Issue of Abstracts: Volume 41, Issue 2

Deadlines

For organizers: Expired
For abstracts: February 4, 2020

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Christine Berkesch, University of Minnesota, Title to be announced.
Matthew Hedden, Michigan State University, Title to be announced.
Brian Street, University of Wisconsin-Madison, Title to be announced.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.

Advances in Mathematical Modeling, Analysis and Numerical Simulation of Particulate Suspensions and Related Multiphase Flows (Code: SS 32A), Abhinandan Chowdhury, Savannah State University, and Ivan Christov, Purdue University.

Analysis and Probability in Sub-Riemannian Geometry (Code: SS 5A), Jeremy Tyson, University of Illinois Urbana-Champaign, and Jing Wang, Purdue University.


Coding and Cryptography (Code: SS 34A), Ryann Carter, Clemson University, Neville Fogarty, Christopher Newport University, and Gretchen Matthews and Dane Skabelund, Virginia Tech.

Combinatorial Algebra and Geometry (Code: SS 22A), Christine Berkesch, University of Minnesota, and Laura Matusevich and Aleksandra Sobieska, Texas A&M University.
MEETINGS & CONFERENCES

Combinatorial Techniques in Commutative Algebra (Code: SS 42A), Giulio Caviglia, Purdue University, and Jay Schweig, Oklahoma State University.

Commutative Algebra and Connections with Algebraic Geometry (Code: SS 36A), Claudia Polini, University of Notre Dame, and Bernd Ulrich, Purdue University.

Complex Geometry (Code: SS 29A), Laszlo Lempert, Chi Li, and Sai-Kee Yeung, Purdue University, and Yuan Yuan, Syracuse University.

Computational Aspects of Symplectic Topology (Code: SS 30A), Olguta Buse, Indiana University-Purdue University Indianapolis, Richard Hind, University of Notre Dame, and Jun Li, University of Michigan.

Contemporary Applications of Gradient Flows and Variational Methods (Code: SS 12A), Tao Luo and Nung Kwan (Aaron) Yip, Purdue University.

Gaussian and non-Gaussian Stochastic Analysis (Code: SS 16A), Cheng Ouyang, University of Illinois at Chicago, and Takashi Owada and Samy Tindel, Purdue University.

Geometric Topology in the Middle Dimensions (Code: SS 43A), James F. Davis, Indiana University, and Mark Powell, Durham University.

Group Theory and Logic (Code: SS 13A), Meng-Che (Turbo) He, Purdue University, Julia F. Knight, University of Notre Dame, and D.B. McReynolds and Thomas Sinclair, Purdue University.

Harmonic Analysis (Code: SS 2A), Brian Street and Shaoming Guo, University of Wisconsin-Madison.

Higher Structures in Topology, Geometry and Physics (Code: SS 24A), Ralph Kaufmann, Purdue University, Martin Markl, Institute of Mathematics of the Czech Academy of Sciences, and Sasha Voronov, University of Minnesota.

Integrability, Symmetry and Physics (Code: SS 26A), E. Birgit Kaufmann, Purdue University, and Oleksandr Tsymbaliuk, Yale University.

Knots and Links in 3-Manifolds (Code: SS 31A), Micah Chrisman and Sujoy Mukherjee, The Ohio State University, and Robert Todd, Mount Mercy University.

Low-dimensional Topology (Code: SS 4A), Matthew Hedden, Katherine Raoux, and Lev Tovstopyat-Nelip, Michigan State University.

Mathematical Finance and Actuarial Sciences (Code: SS 11A), Kiseop Lee and Jianxi Su, Purdue University, and Jose Figueiredo-Lopez, Washington University, St. Louis.

Mathematical Methods for Inverse Problems (Code: SS 3A), Isaac Harris and Peijun Li, Purdue University.

Modeling, Analysis and Simulation of Complex Fluid Systems in Physics and Biology (Code: SS 33A), Carme Calderer, University of Minnesota, Chun Liu, Illinois Institute of Technology, and Pei Liu, University of Minnesota.

Model Theory and its Applications (Code: SS 41A), Saugata Basu, Purdue University, Philipp Hieronymi, University of Illinois at Urbana-Champaign, and Margaret E.M. Thomas, Purdue University.

Multiplicative Ideal Theory in honor of the career of William Heinzer (Code: SS 8A), Evan Houston, University of North Carolina, Charlotte, and Alan Loper, Ohio State University.

Network Science (Code: SS 20A), Nicole Eikmeier, Grinnell College, and David F. Gleich, Purdue University.

Nonlinear Partial Differential Equations from Variational Problems and Fluid Equations (Code: SS 9A), Tao Huang, Wayne State University, and Changyou Wang, Purdue University.

Numerical Linear Algebra (Code: SS 18A), Jianlin Xia and Xuefeng Xu, Purdue University.

Optimization and Algebraic Geometry (Code: SS 40A), Jonathan Hauenstein, University of Notre Dame, and Ali Mohammad Nezhad, Purdue University.

Optimization for Discrete Geometry (Code: SS 19A), Mark Magsino and Hans Parshall, The Ohio State University.

p-adic Galois Representations, Modularity, and Related Topics (Code: SS 39A), Patrick Allen, University of Illinois at Urbana-Champaign, Andrei Jorza, University of Notre Dame, and Tong Liu, Purdue University.

Quantum Algebra and Quantum Topology (Code: SS 10A), Shawn Cui, Purdue University, Julia Plavnik, Indiana University, and Tian Yang, Texas A&M University.

Recent Advances in Adaptive Mesh Refinement and A Posteriori Error Estimation (Code: SS 38A), Shuhao Cao, University of California, Irvine, and Zhiquiang Cai, Purdue University.

Recent Advances in Modeling, Computational Methods and Simulations of Physical/Biological Systems (Code: SS 27A), Sichuan Steven Dong, Jie Shen, and Zhiguo Yang, Purdue University.

Recent Developments in Automorphic Forms and Representations of p-adic Groups (Code: SS 7A), David Goldberg, Baiying Liu, and Freydoon Shahidi, Purdue University.

Recent Developments in Commutative Algebra (Code: SS 6A), Jennifer Kenkel, University of Kentucky, and Liquan Ma and Uli Walther, Purdue University.
Accommodations

Participants should make their own arrangements directly with the hotel of their choice. Special discounted rates were negotiated with the hotels listed below. Rates quoted do not include state and local taxes (7% state, 5% innkeepers, various local cities and towns) and hotel fees may apply. Participants must state that they are with the American Mathematical Society’s (AMS) Spring Central Sectional Meeting to receive the discounted rate or use code provided. The AMS is not responsible for rate changes or for the quality of the accommodations. Hotels have varying cancellation and early checkout penalties; be sure to ask for details.

Courtyard by Marriott, 150 Fairington Avenue, Lafayette, IN, 47905; (765) 449-4800 ext. 4932; www.marriott.com/event-reservations/reservation-link.mi?id=1565701033092&key=GRP&app=resvlink. Rate is US$109 per night for a room. To make a reservation at this rate please call the hotel directly or call toll-free global reservations at 1-888-236-2427. Please use group code/type G3794/ASSOC when making all reservations. A major credit card or advance deposit is required to guarantee all individual reservations. Amenities at this property include complimentary high-speed internet; indoor pool; fitness center; The Bistro serving breakfast, dinner, room service, and cocktails; a business center; airport shuttle; and complimentary parking. Check-in is at 5:00 pm, check-out is at 11:00 am. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. This property is located less than 5 miles from campus. The deadline for reservations at this rate is March 1, 2020.

Hilton Garden Inn, 356 East State Street, West Lafayette, IN; (765) 743-2100; www.hiltongardeninn.hilton.com/en/gi/groups/personalized/L/LAFWLGI-AMS-20200403/index.jhtml. Rate is US$129 per night for a room with 1 king bed. To make a reservation at this rate please call the hotel directly and request the rate for the American Mathematical Society (AMS) Meeting. A credit card or advanced deposit is required to guarantee individual reservations. This property has a cancellation policy requiring cancellation 48 hours in advance to avoid being charged for (1) night’s room and tax. Amenities at this property include complimentary high-speed internet; indoor pool; fitness center; The Garden Grill and Bar serving breakfast, room service, and cocktails; a business center; complimentary outdoor parking; and complimentary shuttle service to campus. Check-in is at 3:00 pm, check-out is at 11:00 am. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. This property is located approximately 1 mile from campus. The deadline for reservations at this rate is March 13, 2020.

Homewood Suites, 3939 South Street, Lafayette, IN; (765) 448-9700; www.homewoodsuites3.hilton.com/en/hotels/indiana/homewood-suites-by-hilton-Lafayette-LAFINWH/accommodations/index.html. Rate is US$129 per night for a room with standard king bed with pull-out sofa. To reserve a room at this rate please contact the hotel by phone directly or at 1-800-760-7718. Group rate supersedes any other discounts and/or promotions; 14-day cancellation policy exists, please ask for details when reserving a room. This property is pet-friendly. Amenities at this property include complimentary hot breakfast; complimentary high-speed internet; outdoor pool; fitness center; a business center; airport shuttle; and complimentary parking. Check-in is at 3:00 pm, check-out is at 12:00 pm. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. This property is located approximately 5 miles from campus. The deadline for reservations at this rate is March 2, 2020.

Four Points, 1600 Cumberland Avenue, West Lafayette, IN, 47906; 765-463-5511; www.marriott.com/hotels/travel/laffp-four-points-west-lafayette/. Rate is US$99 per night for a standard king or a room with two double beds. To make a reservation at this rate please call the hotel directly and request the rate for the AMS Meeting. A credit card or advanced deposit is required to guarantee all individual reservations. Amenities at this property include complimentary hot breakfast; complimentary high-speed internet; outdoor pool; fitness center; a business center; airport shuttle; and complimentary parking. Check-in is at 3:00 pm, check-out is at 12:00 pm. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. This property is located less than 5 miles from campus. The deadline for reservations at this rate is March 2, 2020.
beds. To reserve a room at this rate please contact the hotel by phone directly at (765) 463-5511. Individual sleeping room cancellations must be made by 3:00 pm (3) days prior to the arrival date or a (1) night room and tax fee will be charged. No-show reservations will be charged the amount of room and tax for the entire duration of the original reservation. This property is pet-friendly. Amenities at this property include kitchenette in-room; complimentary high-speed internet; indoor and outdoor pool; fitness center; a business center; Tailgate Grille and Bar serving dinner; and complimentary parking. Check-in is at 3:00 pm, check-out is at 11:00 am. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. This property is located approximately 3.4 miles from campus. The deadline for reservations at this rate is March 3, 2020.

Holiday Inn City Centre, 515 South Street, Lafayette, IN 47901; (765) 423-1000; www.hicclaf.com. Rate is US$129 per night, per room for king or two-queen room. Discounts such as AAA, AARP, etc., will not apply to group rates. To reserve a room at this rate please contact reservations at 1-800-HOLIDAY or directly at (765) 423-1000 and ask for the American Mathematical Society rate. You may also make your reservations online at www.hicclaf.com using the code AMS. Amenities at this property include internet access; microwaves and refrigerators available for rental; indoor pool; fitness center; a business center; Lobby Bar and Grill serving breakfast, dinner, and dessert; and complimentary parking. Check-in is at 3:00 pm, check-out is at 12:00 pm. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. This property is located approximately 1.5 miles from campus. The deadline for reservations at this rate is March 3, 2020.

DoubleTree by Hilton Lafayette East, 155 Progress Drive, Lafayette, IN 47905; (765) 447-0575; www.doubletree3.hilton.com/en/hotels/indiana/doubletree-by-hilton-lafayette-east-LAFDTDT/index.html?SEO_id=GMB-DT-LAFDTDT. Rate is US$119 per night for a room with one king bed or two double beds. To reserve a room at this rate individuals may call the hotel directly at (765) 447-0575 and press “O” for the hotel reservation specialist. Amenities at this property include complimentary Wi-Fi, mini-fridge, coffeemaker available in the guest rooms, a heated indoor pool, and a 24-hour fitness center. Breakfast and dinner are available at Made Market, with menus that focus on fresh and healthy American fare. Check-in is at 3:00 pm, check-out is at 12:00 pm. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. This property is located approximately 4.8 miles from campus. The deadline for reservations at this rate is March 4, 2020.

Fairfield Inn and Suites, 4000 South Street, Lafayette, IN 47905; (765) 449-0083; www.marriott.com/hotels /travel/laffi-fairfield-inn-and-suites-lafayette/?scid=bb1a189a-fec3-4d19-a255-54ba596f6be2. Rate is US$114 per night for a room. To reserve a room at this rate individuals may call the hotel directly at 765-446-0900 and press “O” for the hotel reservation specialist. Amenities at this property include complimentary Wi-Fi, heated indoor pool, 24-hour fitness center, business center, and complimentary hot breakfast. Check-in is at 3:00 pm, check-out is at 12:00 pm. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. This property is pet friendly with a maximum of 2 pets per guest room with a non-refundable fee of $75.00. Amenities at this property include complimentary Wi-Fi, an indoor pool, a fitness center, a business center, complimentary hot breakfast, cribs and high chairs available, coin laundry, and free parking. Check-in is at 3:00 pm, check-out is at 11:00 am. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. This property is located approximately 4.7 miles from campus. The deadline for reservations at this rate is March 2, 2020.

Home2 Suites, 3838 Grace Lane, Lafayette, IN 47905; (765) 771-7575; www.hilton.com/en/hotels/laflhtht -home2-suites-lafayette/. Rate is US$139 per night for a King Studio Suite with a pullout sofa sleeper. To reserve a room at this rate individuals may call the hotel directly at (765) 771-7575. This property is pet friendly with a $75.00 non-refundable fee. Amenities at this property include complimentary Wi-Fi, an indoor pool, a fitness center, a business center, complimentary hot breakfast, cribs and high chairs available, coin laundry, and free parking. Check-in is at 3:00 pm, check-out is at 11:00 am. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. This property is located approximately 4.7 miles from campus. The deadline for reservations at this rate is March 2, 2020.

Residence Inn, 3834 Grace Lane, Lafayette, IN 47905; (765) 479-7208; www.marriott.com/hotels/travel/lafri -residence-inn-lafayette/?scid=bb1a189a-fec3-4d19-a255-54ba596f6be2. Rate is US$149 per night for a King Studio Suite. To reserve a room at this rate individuals may call the hotel directly at (765) 479-7208. This property is pet friendly with a maximum of 2 pets per guest room with a non-refundable fee of $75.00. Please contact hotel for details. Amenities at this property include complimentary high-speed Wi-Fi, guest room with fully equipped kitchen, TV with cable and Netflix, 24-hour fitness center, and complimentary full American breakfast. Check-in is at 4:00 pm, check-out is at 11:00 am. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. This property is located approximately 4.5 miles from campus. The deadline for reservations at this rate is March 3, 2020.

Hampton Inn Lafayette, 3941 South Street, Lafayette, IN 47905; (765) 447-1600; www.hilton.com/en/hotels/laflhhhx -hampton-lafayette/?SEO_id=GMB-HP-LAFHHHX. Rate is US$115 per night for a room with two queen beds. To reserve a room at this rate individuals may call the hotel directly at (765) 447-1600. Amenities at this property include com-
Complimentary Wi-Fi, indoor pool, fitness center, complimentary hot breakfast, and free parking. Check-in is at 3:00 pm, check-out is at 12:00 pm. Cancellation and early check-out policies vary and penalties exist at this property; be sure to check when you make your reservation. This property is located approximately 4.6 miles from campus. The deadline for reservations at this rate is March 2, 2020.

**Quality Inn and Suites**, 4221 South Street, Lafayette, IN 47905; (765) 447-9460; www.choicehotels.com/indiana/lafayette/quality-inn-hotels/in267?source=gyxt. Rate is US$115 per night for a room with two queen beds. To reserve a room at this rate individuals may call the hotel directly at (765) 447-9460. This property is pet friendly; please contact hotel directly for more information. Amenities at this property include free coffee, free deluxe continental breakfast served daily from 6:00 am–9:00 am, coffee maker in guest rooms, free Wi-Fi, 24-hour fitness center, 24-hour laundry facility, and free parking. Check-in is at 3:00 pm, check-out is at 12:00 pm. Cancellation cutoff is March 20, 2019 by 5:00 pm local time - 14 days prior to date of arrival. There will be a one-night room fee plus tax if canceling with 14 days or less notice. This property is located approximately 4.9 miles from campus. The deadline for reservations at this rate is March 20, 2020.

**Housing Warning**

Please beware of aggressive housing bureaus that target potential attendees of a meeting. They are sometimes called “room poachers” or “room-block pirates” and these companies generally position themselves as a meeting’s housing bureau, convincing attendees to unknowingly book outside the official room block. They call people who they think will more likely than not attend a meeting and lure them with room rates that are significantly less than the published group rate—for a limited time only. And people who find this offer tempting may hand over their credit card data, believing they have scored a great rate and their housing is a done deal. Unfortunately, this often turns out to be the start of a long, costly nightmare.

Note that some of these room poachers create fake websites on which they represent themselves as the organizers of the meeting and include links to book rooms, etc. The only official website for this meeting is ams.org and one that has the official AMS logo.

These housing bureaus are not affiliated with the American Mathematical Society or any of its meetings, in any way. The AMS would never call anyone to solicit reservations for a meeting. The only way to book a room at a rate negotiated for an AMS Sectional Meeting is via a listing on AMS Sectional Meetings pages or Notices of the AMS. The AMS cannot be responsible for any damages incurred as a result of hotel bookings made with unofficial housing bureaus.

**Food Services**

At the time of the publication of this announcement, the University anticipates that a variety of dining options will be available on campus.

There are many dining options on Purdue’s campus and throughout the city of West Lafayette offering a variety of cuisines. For more information about dining on campus please visit www.dining.purdue.edu/CampusDining/index.html.

Some dining options in the area include:

- **1869 Tap Room**, Purdue Memorial Union, 101 North Grant Street, West Lafayette, IN 47906, (765) 494-8989, www.union.purdue.edu/dine/taproom.html; named for the year Purdue University was founded, serving pub fare. Monday–Sunday, 6:00 am–12:00 am, on campus.
- **Town and Gown**, 119 N River Road, West Lafayette, IN 47906, (765) 202-3425, www.townandgownbistro.com; serving crepes and all-day breakfast, lunch, and dinner at an affordable price. Monday–Friday, 7:00 am–10:00 pm, Saturday–Sunday, 8:00 am–10:00 pm, 0.7 miles from campus.
- **Triple XXX**, 2 N Salisbury Street, West Lafayette, IN; (765) 743-5373, www.triplexxxfamilyrestaurant.com; serving breakfast anytime, America’s first and oldest drive-in. Monday–Saturday, 5:30 am–11:00 pm, Sunday, 5:30 am–10:00 pm, 0.3 miles from campus.
- **Khana Khazana**, 108 Northwestern Avenue, West Lafayette, IN 47906, (765) 743-1223, www.facebook.com/KhanaKhazana2000; serving Indian cuisine. Saturday and Sunday, 12:00 pm–3:00 pm and 5:30 pm–9:00 pm, on campus.
- **Cosi**, 1101 Third Street, West Lafayette, IN 47906, (765) 494-8035, www.dining.purdue.edu//CampusDining//Restaurants/cosi.html; serving soups, sandwiches, and salads. Friday, 7:30 am–8:00 pm, Saturday, 11:00 am–3:00 pm, Sunday, 10:30 am–10:00 pm, on campus.
- **Blue Nile**, 117 Northwestern Ave #2, West Lafayette, IN 47906, (765) 269-998; serving Mediterranean cuisine. Friday, 7:30 am–8:00 pm, Saturday, 11:00 am–3:00 pm, Sunday, 10:30 am–10:00 pm, on campus.
MEETINGS & CONFERENCES

• Greyhouse, 100 Northwestern Avenue, West Lafayette, IN 47906, (765) 743-5316, www.greyhousecoffee.com; serving coffee and light fare. Friday–Sunday, 7:00 am–9:00 pm, 0.1 mile from campus.
• Pappy’s Sweet Shop, 101 Grant Street, West Lafayette, IN 47906, (765) 494-8948, www.dining.purdue.edu/CampusDining/Restaurants/pappys.html; serving American fare. Friday, 7:00–11:00 pm, Saturday and Sunday, 7:30 am–9:00 pm, on campus.

Registration and Meeting Information

Advance Registration: Advance registration for this meeting will open on January 22, 2020. Advance registration fees will be US$71 for AMS members, US$115 for nonmembers, US$13 for students and unemployed mathematicians, and US$15 for emeritus members. Fees will be payable by credit card. Participants may cancel registrations made in advance by emailing mmsb@ams.org. 100% refunds will be issued for any advance registrations canceled by the first day of the meeting. After this date, no refunds will be issued.

On-site Information and Registration: Registration and the book exhibit will be held in the Lawson Computer Science Building. The Invited Address lectures will be located in The Class of 50 Lecture Hall. The Special Sessions and Contributed Paper Sessions will be held in various buildings on campus. There will be a reception on Saturday evening. Please look for additional information about specific session room locations on the web and in the printed program. For further information on building locations, a campus map is available at www.purdue.edu/campus_map.

The registration desk will be open on Saturday, April 4 from 7:30 am to 4:00 pm and on Sunday, April 5 from 8:00 am to 12:00 pm. The same fees listed above apply for on-site registration and are payable with cash, check, or credit card.

Program Books

In order to keep registration fees as low as possible, save on printing costs, and make the meetings more environmentally friendly, a small fee will be charged for receiving a program book.

If you want to receive a program book, please complete the section entitled “Printed Program” on the registration form and pay a nominal fee of US$3 for each copy. If you do not want to receive a program book, please skip that section and click “Next.” All purchased program books will be distributed at the registration desk at the meeting. No program books will be mailed before the meeting. A small quantity of program books may be available for purchase on-site at meetings, however supplies will be limited.

For your convenience, the following changes have been made to the website to make the program more accessible and mobile-friendly.

1. Online Timetable: The sectional meetings now have an online timetable display. It will provide a quick reference to where the sessions and rooms are located.
2. Print-friendly Pages: The pages of the program now have a “Print” button in the top right-hand corner. It is black and is depicted by a little printer. If you cannot see it, please maximize your window. The printer icon is sometimes bundled into the generic “share” icon when the window is re-sized. If you click the printer button, it will print the text of the page without additional webpage elements. If you go to the page of a special session and click the print icon, you will get a schedule of all the parts of that session. This can also be printed to pdf if you have a pdf printer installed.
3. Room Locations: The room locations are now more conspicuous in the web program when the program is scheduled.

Other Activities

Book Sales: Stop by the on-site AMS bookstore to review the newest publications and take advantage of exhibit discounts and free shipping on all on-site orders! AMS and MAA members receive 40% off list price. Nonmembers receive a 25% discount. Not a member? Ask a representative about the benefits of AMS membership.

AMS Editorial Activity: An acquisitions editor from the AMS book program will be present to speak with prospective authors. If you have a book project that you wish to discuss with the AMS, please stop by the book exhibit.

Membership Activities: During the meeting, stop by the AMS Membership Exhibit to learn about the benefits of AMS Membership. Members receive free shipping on purchases all year long and additional discounts on books purchased at meetings, subscriptions to Notices and Bulletin, discounted registration for world-class meetings and conferences, and more!

Complimentary refreshments will be served courtesy in part of the AMS Membership Department.

Child Care Grants

The AMS will provide a limited number of reimbursement grants of US$125 per family to help with the cost of child care for registered participants at the meeting. The funds may be used for any form of child care that frees a parent to
participate more fully in the meeting. Grants will be awarded on a first-come, first-served basis, one per family, and one per season (i.e. Spring or Fall), the latter depending on the amount of grants available. Registration for the meeting as well as membership in the AMS is required to apply for this program.

Information about applying for child care grants will be available prior to the opening of advance registration in January; watch the meeting website for details and instructions. Applications will be on Mathprograms.org and will be accepted on a first-come, first-served basis until February 4, 2020. Final decisions on recipients will be made on or before March 6, 2020. All grant funds will be provided in the form of a check which will be issued at the meeting.

**Special Needs**

It is the goal of the AMS to ensure that its conferences are accessible to all, regardless of disability. The AMS will strive, unless it is not practicable, to choose venues that are fully accessible to the physically handicapped.

If special needs accommodations are necessary in order for you to participate in an AMS Sectional Meeting, please communicate your needs in advance to the AMS Meetings Department by:

- Registering early for the meeting,
- Checking the appropriate box on the registration form, and
- Sending an email request to the AMS Meetings Department at mmsb@ams.org or meet@ams.org.

**AMS Policy on a Welcoming Environment**

The AMS strives to ensure that participants in its activities enjoy a welcoming environment. In all its activities, the AMS seeks to foster an atmosphere that encourages the free expression and exchange of ideas. The AMS supports equality of opportunity and treatment for all participants, regardless of gender, gender identity or expression, race, color, national or ethnic origin, religion or religious belief, age, marital status, sexual orientation, disabilities, or veteran status.

Harassment is a form of misconduct that undermines the integrity of AMS activities and mission.

The AMS will make every effort to maintain an environment that is free of harassment, even though it does not control the behavior of third parties. A commitment to a welcoming environment is expected of all attendees at AMS activities, including mathematicians, students, guests, staff, contractors and exhibitors, and participants in scientific sessions and social events. To this end, the AMS will include a statement concerning its expectations towards maintaining a welcoming environment in registration materials for all its meetings, and has put in place a mechanism for reporting violations. Violations may be reported confidentially and anonymously to 855-282-5703 or at www.mathsociety.ethicspoint.com. The reporting mechanism ensures the respect of privacy while alerting the AMS to the situation. Violations may also be brought to the attention of the coordinator for the meeting (who is usually at the meeting registration desk), and that person can provide advice about how to proceed.

For AMS policy statements concerning discrimination and harassment, see the AMS Anti-Harassment Policy at https://www.ams.org/about-us/governance/policy-statements/anti-harassment-policy.

Questions about this welcoming environment policy should be directed to the AMS Secretary at https://www.ams.org/about-us/governance/sec-contact.

**Local Information and Maps**

This meeting will take place on the Purdue campus located at 610 Purdue Mall, West Lafayette, IN 47907. A campus map can be found at www.purdue.edu/campus_map. Information about the Purdue Mathematics Department can be found at www.math.purdue.edu. Please visit the university website www.purdue.edu for additional information on the campus.

For information on Gender Neutral restrooms on campus please visit www.purdue.edu/lgbtq/resources/gender-inclusive_restrooms.php. Any nursing person who is a member of the Purdue community (faculty, staff, students, and visitors) can use the lactation rooms provided on campus. Please contact Heather Butler at hpb@ams.org to make arrangements to access these rooms.

Please watch the AMS website at https://www.ams.org/meetings/sectional/sectional.html for additional information on this meeting.

**Parking**

Parking is available for visitors to campus. The closest lot to the meeting spaces is the garage on University Street (PGU) across from the Lawson Computer Science Building. Parking at this lot is free on weekends and after 5 pm on weekdays. For more information about parking on campus please visit www.purdue.edu/visit/getting-here/parking.php.
Travel

Purdue University Campus is located 65 miles outside of the city of Indianapolis, Indiana. Indianapolis International Airport is the closest airport to the University. Chicago O’Hare International Airport (ORD) is approximately 143 miles from campus. The most common types of transportation used from the airport are rental cars and shuttle services.

By Air:

**Indianapolis International Airport (IND)** is the closest airport to the Purdue University Campus. Indianapolis International Airport is located approximately 70.3 miles from the University.

Shuttles should be contacted ahead of time. Shuttles to Purdue University are serviced by Reindeer (serving IND and ORD), (765) 637-5124, www.reindeershuttle.com, and Express Air Coach (serving only ORD), (765) 743-3120, www.expressaircoach.com. Please contact the service providers for fees.

By Train:

The Lafayette region is served by Amtrak. Reservations can be made on Amtrak at www.amtrak.com/stations/laf. If traveling by rail, you will arrive at 200 North Second Street, Lafayette, IN 47901-1238. The train station, called the Big Four Depot, is located 1.2 miles from campus.

By Bus:

The Greater Lafayette CityBus offers access around the cities of Lafayette and West Lafayette, the home of Purdue University. The meeting will be held on the Purdue University Campus. This campus is accessible by bus services; a one-way ride ticket costs US$1.00, a one-day pass costs US$2.00. For additional information about bus fares and schedules please visit www.gocitybus.com.

By Car:

All directions to Purdue will direct drivers to a final destination of the University Street Parking Garage. When using a GPS please be certain to enter 305 N University St, West Lafayette, IN 47907 as the final destination for Lawson Hall which is across from the parking garage.

Local Transportation

**Bus and Subway Service:**

The Greater Lafayette CityBus offers access around the cities of Lafayette and West Lafayette, the home of Purdue University. The meeting will be held on the Purdue University Campus. This campus is accessible by bus services; a one-way ride ticket costs US$1.00, a one-day pass costs US$2.00. For additional information about bus fares and schedules please visit www.gocitybus.com.

**Taxi Service:**

Licensed, metered taxis serve the area. Local taxi service is available from several vendors including City Cab, (765) 477-1234 or 765-429-TAXI, as well as Four Star taxi service, (765) 448-6150. Both Lyft and Uber also operate in West Lafayette.

Weather

The average high temperature in West Lafayette for April is in the mid-50s Fahrenheit, and the average low is in the low 40s Fahrenheit. Visitors should be prepared for inclement weather and check weather forecasts in advance of their arrival.

Social Networking

Attendees and speakers are encouraged to tweet about the meeting using the hashtag #AMSmtg.

Information for International Participants

Visa regulations are continually changing for travel to the United States. Visa applications may take from three to four months to process and require a personal interview, as well as specific personal information. International participants should view the important information about traveling to the US found at travel.state.gov/content/travel/en.html. If you need a preliminary conference invitation in order to secure a visa, please send your request to hpb@ams.org.

If you discover you do need a visa, the National Academies website (see above) provides these tips for successful visa applications:

* Visa applicants are expected to provide evidence that they are intending to return to their country of residence. Therefore, applicants should provide proof of “binding” or sufficient ties to their home country or permanent residence abroad. This may include documentation of the following:
  * family ties in home country or country of legal permanent residence
• property ownership
• bank accounts
• employment contract or statement from employer stating that the position will continue when the employee returns;

* Visa applications are more likely to be successful if done in a visitor’s home country than in a third country;
* Applicants should present their entire trip itinerary, including travel to any countries other than the United States, at the time of their visa application;
* Include a letter of invitation from the meeting organizer or the US host, specifying the subject, location and dates of the activity, and how travel and local expenses will be covered;
* If travel plans will depend on early approval of the visa application, specify this at the time of the application;
* Provide proof of professional scientific and/or educational status (students should provide a university transcript).

This list is not to be considered complete. Please visit the websites above for the most up-to-date information.

Fresno, California
California State University, Fresno

May 2–3, 2020
Saturday – Sunday

Meeting #1158
Western Section
Associate secretary: Michel L. Lapidus

Announcement issue of Notices: March 2020
Program first available on AMS website: March 19, 2020
Issue of Abstracts: Volume 41, Issue 2

Deadlines
For organizers: Expired
For abstracts: March 3, 2020

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Sami Assaf, University of Southern California, Los Angeles, Combinatorics of Schubert Calculus.
Natalia Komarova, University of California, Irvine, Title to be announced.
Joseph Teran, University of Southern California, Los Angeles, Title to be announced.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.

Advances by Scholars in the Pacific Math Alliance (Code: SS 22A), Andrea Arauza Rivera, California State University, East Bay, Mario Banuelos, California State University, Fresno, Jessica De Silva, California State University, Stanislaus, and John Rock, California Polytechnic University, Pomona.

Advances in Functional Analysis and Operator Theory (Code: SS 6A), Yuri Latushkin, University of Missouri, Columbia, Marat Markin, California State University, Fresno, Igor Nikolaev, St. John’s University, and Ilya Spitkovsky, New York University, Abu Dhabi.

Algebraic geometry in statistics and machine learning (Code: SS 25A), Robert Krone, University of California, Jose Rodriguez, University of Wisconsin, and Tingting Tang, Notre Dame University.

Algebraic Structures in Knot Theory (Code: SS 4A), Carmen Caprau, California State University, Fresno, and Sam Nelson, Claremont McKenna College.

Algorithms in the study of hyperbolic 3-manifolds (Code: SS 26A), Robert Haraway, III, Oklahoma State University, and Maria Trnkova, University of California, Davis.

Analysis of Fractional Differential and Difference Equations with its Application (Code: SS 20A), Bhuvaneswari Sambandham, Dixie State University, and Aghalaya S. Vatsala, University of Louisiana at Lafayette.

Artin-Schelter regular algebras and related topics (Code: SS 27A), Ellen Kirkman, Wake Forest University, and James Zhang, University of Washington.
MEETINGS & CONFERENCES

Combinatorics Arising from Representations (associated with the Invited Address by Sami Assaf) (Code: SS 16A), Sami Assaf, University of Southern California, Nicolet Gonzalez, University of California, Los Angeles, and Brendan Pawloski, University of Southern California.

Combinatorics of Reduced Decompositions of Elements of Coxeter Groups and Related Topics (Code: SS 17A), Samantha Dahlberg and Jennifer Elder, Arizona State University.

Complexity in Low-Dimensional Topology (Code: SS 14A), Jennifer Schultens, University of California, Davis, and Eric Sedgwick, DePaul University.

Data Analysis and Predictive Modeling (Code: SS 8A), Earvin Balderama, California State University, Fresno, and Adriano Zambom, California State University, Northridge.

DG Methods in Commutative Algebra and Representation Theory (Code: SS 2A), Benjamin Briggs, Janina Letz, and Josh Pollitz, University of Utah.

Discrete Geometry and Combinatorial Structures (Code: SS 23A), Morgan Rodgers, California State University, and Oscar Vega.

How to Solve It? Heuristics and Inquiry Based Learning (Code: SS 18A), Mario Banuelos, California State University, Fresno, Andrew G. Benedek, Research Centre for the Humanities, Hungary, and Agnes Tuska, California State University, Fresno.

Inverse Problems (Code: SS 5A), Hanna Makaruk, Los Alamos National Laboratory, and Robert Owczarek, University of New Mexico, Albuquerque and University of New Mexico, Los Alamos.

Math Circle Games and Puzzles that Teach Deep Mathematics (Code: SS 13A), Maria Nogin, Adnan Sabuwala, and Agnes Tuska, California State University, Fresno.

Mathematical Biology: Confronting Models with Data (Code: SS 21A), Erica Rutter, University of California, Merced.

Mathematical Methods in Evolution and Medicine (associated with the Invited Address by Natalia Kamarova) (Code: SS 1A), Natalia Komarova and Jesse Kreeger, University of California, Irvine.

Methods in Non-Semisimple Representation Categories (Code: SS 11A), Eric Friedlander, University of Southern California, Los Angeles, Julia Pevtsova, University of Washington, Seattle, and Paul Sobaje, Georgia Southern University, Statesboro.


Recent Advances in Mathematical Biology, Ecology, Epidemiology, and Evolution (Code: SS 10A), Lale Asik, Texas Tech University, Khanh Phuong Nguyen, University of Houston, and Angela Peace, Texas Tech University.

Research in Mathematics by Early Career Graduate Students (Code: SS 7A), Doreen De Leon, Marat Markin, and Khang Tran, California State University, Fresno.

Research in Mathematics Education (Code: SS 15A), Ravi Somayajulu and Jenna Tague, Clovis Community College.

Scientific Computing (Code: SS 19A), Changho Kim and Roummel Marcia, University of California, Merced.

Special Functions in Number Theory (Code: SS 24A), Cezar Lupu and Dermot McCarthy, Texas Tech University.

Women in Mathematics (Code: SS 12A), Doreen De Leon, Katherine Kelm, and Oscar Vega, California State University, Fresno.

Zero Distribution of Entire Functions (Code: SS 9A), Khang Tran and Tamás Forgács, California State University, Fresno.

El Paso, Texas

University of Texas at El Paso

September 12–13, 2020
Saturday – Sunday

Meeting #1159
Central Section
Associate secretary: Georgia Benkart

Announcement issue of Notices: June 2020
Program first available on AMS website: July 28, 2020
Issue of Abstracts: Volume 41, Issue 3

Deadlines
For organizers: February 20, 2020
For abstracts: July 14, 2020

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/sectional.html.

Invited Addresses
Caroline Klivans, Brown University, Title to be announced.
**Brisa Sanchez**, Drexel University, *Title to be announced.*

**Alejandra Sorto**, Texas State University, *Title to be announced.*

**Special Sessions**

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at [https://www.ams.org/cgi-bin/abstracts/abstract.pl](https://www.ams.org/cgi-bin/abstracts/abstract.pl).

- **Algebraic, Geometric and Topological Combinatorics**, *Art Duval*, University of Texas at El Paso, *Caroline Klivans*, Brown University, and *Jeremy Martin*, University of Kansas.
- **Algebraic Structures in Topology, Logic, and Arithmetic**, *John Harding*, New Mexico State University, and *Emil Schwab*, The University of Texas at El Paso.
- **High-Frequency Data Analysis and Applications**, *Maria Christina Mariani* and *Michael Pokojovy*, University of Texas at El Paso, and *Ambar Sengupta*, University of Connecticut.
- **Leibniz Algebras and Related Topics**, *Guy Biyogmam*, Georgia College and State University, and *Jerry Lodder*, New Mexico State University.
- **Low-dimensional Topology and Knot Theory**, *Mohamed Ait Nouh* and *Luis Valdez-Sanchez*, University of Texas at El Paso.
- **Methods and Applications in Data Science**, *Sangjin Kim*, Ming-Ying Leung, *Xiaogang Su*, and *Amy Wagler*, The University of Texas at El Paso.
- **Nonlinear Analysis and Optimization**, *Behzad Djafari-Rouhani*, University of Texas at El Paso, and *Akhtar A. Khan*, Rochester Institute of Technology.
- **Numerical Partial Differential Equations and Applications**, *Son-Young Yi* and *Xianyi Zeng*, The University of Texas at El Paso.
- **Statistical Methodology and Applications**, *Ori Rosen* and *Suneel Chatla*, University of Texas at El Paso.

**State College, Pennsylvania**

*Pennsylvania State University, University Park Campus*

**October 3–4, 2020**

*Saturday – Sunday*

**Meeting #1160**

Eastern Section

Associate secretary: Steven H. Weintraub

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Announcement issue of *Notices*: August 2020

Program first available on AMS website: August 25, 2020

Issue of Abstracts: Volume 41, Issue 3

**Deadlines**

For organizers: March 3, 2020

For abstracts: August 11, 2020

The scientific information listed below may be dated. For the latest information, see [https://www.ams.org/amsmtgs/sectional.html](https://www.ams.org/amsmtgs/sectional.html).

**Invited Addresses**

- **Melody Chan**, Brown University, *Title to be announced.*
- **Steven J. Miller**, Williams College, *Title to be announced.*
- **Tadashi Tokieda**, Stanford University, *Title to be announced.*
**Chattanooga, Tennessee**

*University of Tennessee at Chattanooga*

**October 10–11, 2020**

*Saturday – Sunday*

**Meeting #1161**

Southeastern Section

Associate secretary: Brian D. Boe

Announcement issue of *Notices*: August 2020

Program first available on AMS website: September 1, 2020

Issue of *Abstracts*: Volume 41, Issue 4

**Invited Addresses**

- Giulia Saccà, Columbia University, *Title to be announced.*
- Chad Topaz, Williams College, *Title to be announced.*
- Xingxing Yu, Georgia Institute of Technology, *Title to be announced.*

**Special Sessions**

*If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.*

- *Advances in Graph Theory*, Xiaofeng Gu, University of West Georgia, and Dong Ye, Middle Tennessee State University.
- *Applied Knot Theory*, Jason Cantarella, University of Georgia, Eleni Panagiotou, University of Tennessee at Chattanooga, and Eric Rawdon, University of St Thomas.
- *Boundary Value Problems for Differential, Difference, and Fractional Equations*, John R Graef and Lingju Kong, University of Tennessee at Chattanooga, and Min Wang, Kennesaw State University.
- *Commutative Algebra*, Simplice Tchamna, Georgia College, and Lokendra Paudel, University of South Carolina, Salkehatchie.
- *Structural and Extremal Graph Theory*, Hao Huang, Emory University, and Xingxing Yu, Georgia Institute of Technology.

**Salt Lake City, Utah**

*University of Utah*

**October 24–25, 2020**

*Saturday – Sunday*

**Meeting #1162**

Western Section

Associate secretary: Michel L. Lapidus

Announcement issue of *Notices*: August 2020

Program first available on AMS website: September 17, 2020

Issue of *Abstracts*: Volume 41, Issue 4

**Deadlines**

For organizers: March 24, 2020

For abstracts: September 1, 2020

**Invited Addresses**

- Bhargav Bhatt, University of Michigan, Ann Arbor, *Title to be announced.*
- Jonathan Brundan, University of Oregon, Eugene, *Title to be announced.*
- Andrei Okounkov, Columbia University, *Title to be announced* (Erdős Memorial Lecture).
- Mariel Vazquez, University of California, Davis, *Title to be announced.*
Special Sessions
If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.

**Extremal Graph Theory**, József Balogh, University of Illinois, and Bernard Lidicky, Iowa State University.

Monoidal Categories in Representation Theory (associated with the Invited Address by Jon Brudan), Jonathan Brundan, Ben Elias, and Victor Ostrik, University of Oregon.

**Washington, District of Columbia**

Walter E. Washington Convention Center

**January 6–9, 2021**

Wednesday – Saturday

**Meeting #1163**

Joint Mathematics Meetings, including the 127th Annual Meeting of the AMS, 104th Annual Meeting of the Mathematical Association of America (MAA), annual meetings of the Association for Women in Mathematics (AWM) and the National Association of Mathematicians (NAM), and the winter meeting of the Association of Symbolic Logic (ASL), with sessions contributed by the Society for Industrial and Applied Mathematics (SIAM).

Associate secretary: Brian D. Boe
Announcement issue of Notices: October 2020
Program first available on AMS website: November 1, 2020
Issue of Abstracts: To be announced

**Deadlines**

For organizers: April 1, 2020
For abstracts: To be announced

**Call for Proposals for AMS Special Sessions for the Joint Mathematics Meetings in Washington, DC, January 6–9, 2021**

In his responsibilities for the AMS program at the 2021 Joint Mathematics Meetings (JMM), Associate Secretary Brian D. Boe solicits your proposals for AMS Special Sessions. A Special Session is a collection of papers devoted to a single topic or area of mathematics. The 2021 JMM will be held from Wednesday, January 6 through Saturday, January 9, 2021 in Washington, DC. Each Special Session proposal must include:

1. the title of the proposed Special Session;
2. the 2-digit primary Math Subject Classification Number which most closely matches the topic of the proposed Session;
3. the name, affiliation, and email address of each organizer, with one organizer designated as the contact person for all communications about the Session;
4. a brief (one or two paragraph) description of the topic of the proposed Special Session;
5. a sample list of speakers (along with their institutions) whom the organizers plan to invite. (The prospective speakers should not be invited before approval of the Session proposal.)

Organizers are strongly encouraged to consult the AMS Manual for Special Session Organizers at: https://www.ams.org/meetings/specialsessionmanual.html, in its entirety.

Special Sessions will, in general, be allotted between 5 and 10 hours in which to schedule speakers. To enable maximum movement of participants between sessions, organizers must schedule each speaker for either a) a 20-minute talk, 5-minute discussion, and 5-minute break; or b) a 45-minute talk, 10-minute discussion, and 5-minute break. Any combination of 20-minute and 45-minute talks is permitted, but all talks should begin and end at the scheduled time. In particular, all the talks should start on the hour or half-hour, except on the first afternoon when Special Sessions must begin at 2:15 pm and hence, talks will start on the quarter or third-quarter hour.

Proposals for AMS Special Sessions should be sent by email to Professor Boe (brian@math.uga.edu), and must be received by April 2, 2020. Late proposals will not be considered. No decisions will be made on Special Session proposals until after the submission deadline has passed.

The number of Special Sessions on the AMS program is limited and not all proposals can be accepted. A large number of high-quality proposals is expected. Therefore, please be sure to submit as informative and convincing a proposal as possible for review by the Program Committee. We aim to notify organizers whether their proposal has been accepted
by May 15, 2020. Following that deadline, specific additional instructions will be given to the contact persons of the accepted Special Sessions.

Atlanta, Georgia
Geography Institute of Technology

**March 13–14, 2021**
Saturday – Sunday

**Meeting #1164**
Southeastern Section
Associate secretary: Brian D. Boe

Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

**Deadlines**
For organizers: To be announced
For abstracts: To be announced

*The scientific information listed below may be dated. For the latest information, see [https://www.ams.org/amsmtgs/sectional.html](https://www.ams.org/amsmtgs/sectional.html).*

**Special Sessions**
If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at [https://www.ams.org/cgi-bin/abstracts/abstract.pl](https://www.ams.org/cgi-bin/abstracts/abstract.pl).

* Differential Graded Methods in Commutative Algebra, Saeed Nasseh, Georgia Southern University, and Adela Vraciu, University of South Carolina–Columbia.

Providence, Rhode Island
Brown University

**March 20–21, 2021**
Saturday – Sunday

**Meeting #1165**
Eastern Section
Associate secretary: Steven H. Weintraub

Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

**Deadlines**
For organizers: To be announced
For abstracts: To be announced

Cincinnati, Ohio
University of Cincinnati

**April 17–18, 2021**
Saturday – Sunday

**Meeting #1166**
Central Section
Associate secretary: Georgia Benkart

Announcement issue of *Notices*: To be announced
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

**Deadlines**
For organizers: To be announced
For abstracts: To be announced
San Francisco, California  
San Francisco State University  

**May 1–2, 2021**  
Saturday – Sunday  

**Meeting #1167**  
Western Section  
Associate secretary: Michel L. Lapidus  

Announcement issue of *Notices*: To be announced  
Program first available on AMS website: To be announced  
Issue of *Abstracts*: To be announced  

**Deadlines**  
For organizers: To be announced  
For abstracts: To be announced  

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Grenoble, France  
Université de Grenoble-Alpes  

**July 5–9, 2021**  
Monday – Friday  

**Meeting #1168**  
Associate secretary: Michel L. Lapidus  
Announcement issue of *Notices*: To be announced  

Program first available on AMS website: Not applicable  
Issue of *Abstracts*: Not applicable  

**Deadlines**  
For organizers: To be announced  
For abstracts: To be announced  

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Buenos Aires, Argentina  
The University of Buenos Aires  

**July 19–23, 2021**  
Monday – Friday  

**Meeting #1169**  
Associate secretary: Steven H. Weintraub  
Announcement issue of *Notices*: To be announced  

Program first available on AMS website: Not applicable  
Issue of *Abstracts*: Not applicable  

**Deadlines**  
For organizers: To be announced  
For abstracts: To be announced  

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Omaha, Nebraska  
Creighton University  

**October 9–10, 2021**  
Saturday – Sunday  

Central Section  
Associate secretary: Georgia Benkart  
Announcement issue of *Notices*: To be announced  

Program first available on AMS website: To be announced  
Issue of *Abstracts*: To be announced  

**Deadlines**  
For organizers: To be announced  
For abstracts: To be announced
MEETINGS & CONFERENCES

Albuquerque, New Mexico
University of New Mexico

October 23–24, 2021
Saturday – Sunday
Western Section
Associate secretary: Michel L. Lapidus
Announcement issue of Notices: To be announced

Program first available on AMS website: To be announced
Issue of Abstracts: To be announced

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs
/sectional.html.

Deadlines
For organizers: To be announced
For abstracts: To be announced

Special Sessions
If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.

Inverse Problems, Hanna Makaruk, Los Alamos National Laboratory, and Robert Owczarek, University of New Mexico.

Seattle, Washington
Washington State Convention Center and the Sheraton Seattle Hotel

January 5–8, 2022
Wednesday – Saturday
Associate secretary: Georgia Benkart
Announcement issue of Notices: October 2021
Program first available on AMS website: To be announced

Issue of Abstracts: To be announced

Deadlines
For organizers: To be announced
For abstracts: To be announced

Boston, Massachusetts
John B. Hynes Veterans Memorial Convention Center, Boston Marriott Hotel, and Boston Sheraton Hotel

January 4–7, 2023
Wednesday – Saturday
Associate secretary: Steven H. Weintraub
Announcement issue of Notices: October 2022
Program first available on AMS website: To be announced

Issue of Abstracts: To be announced

Deadlines
For organizers: To be announced
For abstracts: To be announced
You’re probably thinking, “Steve Buscemi doesn’t usually wear red when he goes sleeveless, does he?” Indeed, this image is not real. It’s from a computer-generated video known as a deepfake. Due to the increase in computing power and improvements in machine-learning, deepfake videos are now, unfortunately, both easier to make and harder to identify. All is not lost, though. Just as computers, with human guidance, create deepfakes, together they can detect them, too. Current approaches use many techniques, including geometry (of head and lip movements), linear algebra (to detect discrepancies that arise from transforming one face to another), and probability (to measure the chance that a video isn’t real) to identify fake videos. Yet the most important weapon in this battle against fraud may be not taking everything at face value.
The Distribution of Prime Numbers
Dimitris Koukoulopoulos
Université de Montréal, QC, Canada
Prime numbers have fascinated mathematicians since the time of Euclid. This book presents some of our best tools to capture the properties of these fundamental objects, beginning with the most basic notions of asymptotic estimates and arriving at the forefront of mathematical research.


Hochschild Cohomology for Algebras
Sarah J. Witherspoon
Texas A&M University, College Station, TX
This book gives a thorough and self-contained introduction to the theory of Hochschild cohomology for algebras. It includes many examples, exercises, and an appendix with useful homological algebra background.

Graduate Studies in Mathematics, Volume 204; 2019; 264 pages; Hardcover; ISBN: 978-1-4704-4931-5; List US$76; AMS members US$61; MAA members US$68.50; Order code GSM/204

Inspiring Mathematics
Lessons from the Navajo Nation Math Circles
Dave Auckly, Kansas State University, Manhattan, KS, Bob Klein, Ohio University, Athens, OH, Amanda Sereney, Riverbend Community Math Center, South Bend, IN, and Tatiana Shubin, San Jose State University, CA, Editors
This book contains a thematic rainbow of problem sets that were used in Navajo Math Circles—interactive mathematical explorations—across the Navajo Reservation. These problems will help nurture curiosity and confidence in students, and show the art, joy and beauty in mathematics.

Titles in this series are co-published with the Mathematical Sciences Research Institute (MSRI).


An Introduction to Symmetric Functions and Their Combinatorics
Eric S. Egge
Carleton College, Northfield, MN
This book is a reader-friendly introduction to the theory of symmetric functions. Whenever possible it takes a combinatorial approach, using bijections, involutions, and combinatorial ideas to prove algebraic results.

Student Mathematical Library, Volume 91; 2019; 342 pages; Softcover; ISBN: 978-1-4704-4899-8; List US$55; All individuals US$44; Order code STML/91

Free shipping for members in the USA (including Puerto Rico) and Canada.

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