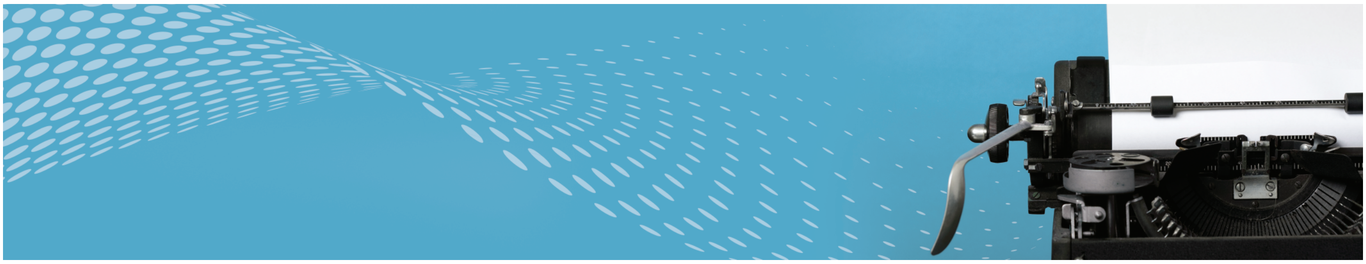

Angenent's Shrinking Doughnuts

Danny Calegari

Take a smooth surface F in $\mathbb{R}^3$. There's a best linear approximation at every point—the tangent plane. If the surface is C^2 we can do better. Translate and rotate the surface so that the given point is at the origin and the tangent plane is horizontal. Then, near the origin, the surface looks (to second order) like the graph of a quadratic form. The *mean curvature vector* H is ν times the trace of this quadratic form, where ν is the unit positive normal. For example, if F is the boundary of a convex region, the mean curvature H points always to the interior where it doesn't vanish.

Positive (resp., negative) eigenvalues of the quadratic form are directions in which the surface curves up (resp., down). When the trace is positive, the positive normals focus more than they spread apart, so if we move the surface infinitesimally in the normal direction the area decreases to first order, and in fact $-H \cdot \nu$ can be interpreted as the derivative of the area form. A critical point for area—i.e., a surface with $H = 0$ everywhere—is called a *minimal surface*, and the gradient flow for area in the space of all surfaces is achieved by deforming the surface by the vector field H.

This gradient flow is called the *mean curvature flow* (hereafter MCF), and we say that a family $F(t)$ of surfaces evolves by MCF if it satisfies $\partial_t F = H$. If we don't care about the parameterization of the evolving surfaces, we can consider more generally $\partial_t F = H + X$, where at each time t the vector field $X(t)$ is tangent to $F(t)$. Scaling a surface by λ multiplies the mean curvature by $1/\lambda$ so the rescaled surface evolves by MCF proportionally to the original at a rate in which time has been rescaled by λ^2; this is called *parabolic rescaling* of MCF.

The simplest example of MCF is a family of shrinking concentric round spheres. A sphere of radius r has mean curvature of length $2/r$ pointing radially to the interior. Thus when a round sphere evolves by MCF, the radius evolves by $r' = -2/r$ so that $r(t) = (r(0)^2 - 4t)^{1/2}$. Another example is a family of shrinking round cylinders. A cylinder of radius r has constant mean curvature $1/r$ so it shrinks like $r(t) = (r(0)^2 - 2t)^{1/2}$. These are examples of *self-shrinkers*: the surface $F(t)$ is the same shape as $F(0)$, just smaller. It continues to shrink, maintaining its overall shape, until it becomes singular. By translation and parabolic rescaling, any self-shrinker can be "normalized" to satisfy $F(t) = \sqrt{1-t}\,F(0)$. The round sphere of radius 2 centered at the origin is normalized, as is the round cylinder of radius $\sqrt{2}$.

An embedded surface evolving by MCF stays embedded, at least until it becomes singular. There's also a maximum principle for MCF: two disjoint surfaces evolving simultaneously stay disjoint. So every closed surface F must become singular in finite time. To see this, put F inside a round ball and apply MCF. By the maximum principle, F

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must become singular before the boundary of the ball (a round sphere) shrinks to a point.

A surface is *mean convex* if H always points into the interior where it doesn't vanish. The property of being mean convex is preserved by MCF. Let's see why. Start with a surface $F(0)$ that is mean convex, and evolve it by the flow. H points into the interior, so $F(t)$ starts to foliate some collar neighborhood of the interior of $F(0)$. Suppose beyond some first time t_0 we stop being mean convex, and H starts to point to the exterior. Then $F(t)$ for $t > t_0$ starts to move "outward" at that point and will bump into $F(s)$ for $s < t_0$, contradicting the maximum principle.

A more subtle theorem, due to Huisken, is that MCF preserves *convexity*. A strictly convex surface is topologically a sphere. Huisken shows that under MCF, a convex surface shrinks to a "round point" in finite time. This means that if one takes the evolving surface and rescales it homothetically for each t to have constant area, the rescaled surfaces converge smoothly to a round sphere.

A surface that is not convex might become singular somewhere while staying smooth elsewhere. Here's an example. A dumbell is two round spheres (the "bells") joined by a narrow neck. If the neck is thin enough, the mean curvature of the neck will be much bigger than that of the spheres, so it will pinch off before the spheres collapse (we'll see a rigorous proof of this soon). See Figure 1.

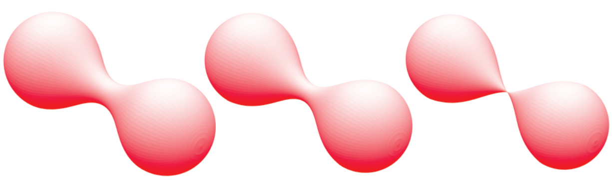

Figure 1. A shrinking dumbell pinches off a neck.

To analyze a singularity we perform parabolic rescalings of the MCF $F(t)$ by bigger and bigger amounts to get a sequence of MCFs that look like the result of "zooming in" on $F(t)$ near the singularity. It turns out that this sequence of MCFs has some subsequence that converges to a limiting MCF, and this limit is always a self-shrinker. This is rather analogous to zooming in at a point on a smooth surface to obtain a scale-invariant object: the tangent space. And in fact a limit of rescalings of an MCF is called a *tangent flow*. Thus, understanding and classifying self-shrinkers is a crucial part of understanding singularities of MCF.

The tangent flow of a convex MCF at the final singularity is a shrinking round sphere, and the tangent flow of the dumbell singularity is a shrinking round cylinder. If the initial surface is mean convex, it turns out that these are the only tangent flows that can arise at singularities.

But for more general initial $F(0)$ other singularities are possible. Even the topology of a self-shrinker can be complicated (actually, the topology *has* to be complicated: Simon Brendle recently showed that the only properly embedded genus zero self-shrinkers in three dimensions are the sphere, the cylinder, and the plane). We now give a rather surprising example due to Sigurd Angenent of a smooth torus whose evolution under MCF is a self-shrinker: a *shrinking doughnut*.

First of all, it's possible to give a variational characterization of being a self-shrinker. Let's look for a normalized self-shrinker, i.e., a surface F in $\mathbb{R}^3$ satisfying $F(t) = \sqrt{1-t}\,F(0)$. Since $\partial_t F = H + X$ with X tangent to F, the surface $F(0)$ is a normalized self-shrinker if and only if it satisfies the equation $\langle H + F/2, \nu\rangle = 0$.

For any function ϕ on $\mathbb{R}^3$ there is an associated functional S on surfaces in $\mathbb{R}^3$ defined by $S(F) := \int_F \phi\,\mathrm{darea}$. If we vary F by moving it infinitesimally in the normal direction by $f\nu$ for some smooth function f, then

$$\begin{aligned}\delta S &= \int_F \delta\phi\,\mathrm{darea} + \int_F \phi\,\delta\mathrm{darea}\\ &= \int_F \langle f\nu, \nabla\phi\rangle + \phi\langle f\nu, -H\rangle \mathrm{darea}.\end{aligned}$$

The function $\phi(x) = e^{-|x^2|/4}$ has gradient $\nabla\phi(x) = -(x/2)\phi$, so for such a ϕ,

$$\delta S = \int_F \phi\langle f\nu, -x/2 - H\rangle \mathrm{darea},$$

which vanishes identically for all f if and only if F is a self-shrinker. Another way to say this is that (normalized) self-shrinkers are *minimal surfaces* for the conformally rescaled metric $e^{-|x^2|/4}dx^2$ on $\mathbb{R}^3$.

Now for some doughnuts. To narrow the search let's look for a doughnut of revolution, obtained by taking a circle γ in the xz half-plane $z > 0$ and rotating it about the x-axis. For F to be critical for area for the metric $e^{-|x^2|/4}dx^2$ on $\mathbb{R}^3$ is for γ to be critical for *length* in the xz half-plane for the metric $ds^2 = z^2 e^{-(x^2+z^2)/4}(dx^2 + dz^2)$—i.e., for it to be a geodesic in this metric.

This metric is conformally Euclidean and incomplete. Its curvature is positive everywhere and blows up along the x-axis. There's a "bubble" where the curvature is relatively small, centered around $(0, 2)$. A geodesic will tend to "bounce" off a region of high curvature, and we can try to look for a suitable γ trapped in the bubble.

Since we only care about the image of the geodesic and not its parameterization, we can consider the geodesic flow in this metric with time reparameterized to give a flow on the (usual) unit tangent bundle of $\mathbb{R}^2$ with coordinates (x, z, θ). The equations for the flow can be computed from the metric by Liouville's formula; in this case we have

$$\dot{x} = \cos\theta, \quad \dot{z} = \sin\theta, \quad \dot{\theta} = \frac{x}{2}\sin\theta + \left(\frac{1}{z} - \frac{z}{2}\right)\cos\theta.$$

Take an initial value $\gamma(0) = (0, \rho, 0)$ and evolve by the reparameterized geodesic flow until the first time $t_0 := t_0(\rho)$

runs back into the z-axis (if it ever does) at $(0, z(t_0), \theta(t_0))$. Some trajectories (computed numerically) for initial values ρ between 1 and 4 are illustrated in Figure 2. Note that the initial value $\rho = 2$ (in blue) traces out an arc of a round circle perpendicular to the x-axis; the associated surface of revolution is the round sphere of radius 2. Likewise, the initial value $\rho = \sqrt{2}$ (in green) traces out a horizontal segment, which rotates to give the round cylinder of radius $\sqrt{2}$.

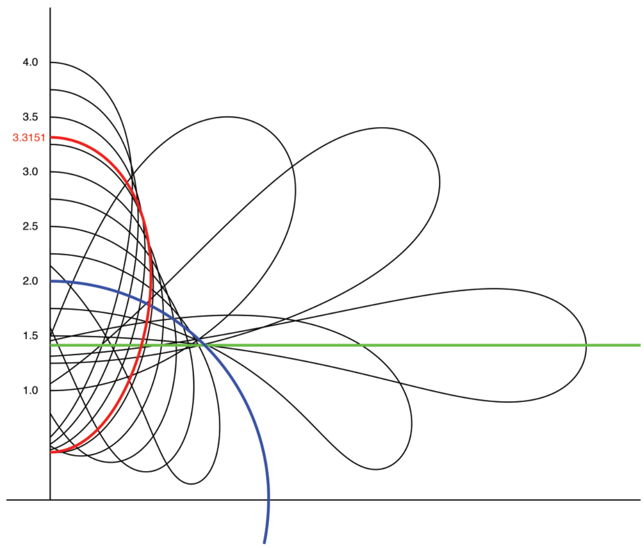

Figure 2. Geodesic segments in the metric $ds^2 = z^2 e^{-(x^2+z^2)/4}(dx^2 + dz^2)$ that start perpendicular to the z-axis.

If $\theta(t_0) = -\pi$ and γ is simple on $[0, t_0]$ we get an embedded segment perpendicular to the z-axis at its endpoints. Since the metric is invariant under $x \to -x$, reflecting this segment in the z-axis gives a simple closed geodesic, which can be revolved around the x-axis to produce a self-shrinking doughnut.

Some numerical experimentation gives $t_0(3.31) \sim -3.14924$ and $t_0(3.32) \sim -3.13434$, and there is a suitable intermediate value $\rho \sim 3.3151$ (in red in the figure) that does the trick.

Now here's Angenent's proof that the dumbell pinches a neck. The proof is an application of the maximum principle. In each of the two "bells" we can put shrinking round spheres. These give a lower bound on the time before the entire surface can collapse. Around the neck we can put an Angenent doughnut as a collar, which gives an *upper* bound on the time before the first singularity. Therefore if the neck is thin enough, it must pinch before the entire surface can collapse.

Author's Note: Angenent shrinks his doughnuts in "Shrinking Doughnuts," in *Nonlinear Diffusion Equations and Their Equilibrium States, 3 (Gregynog, 1989)*, Progr. Nonlinear Differential Equations Appl., 7, Birkhäuser, Boston, MA, pp. 21–38. MR1167827

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