The Geometry of Toric Syzygies

Christine Berkesch

To study the invariants of an algebraic variety $Y \subset \mathbb{P}^m$ over an algebraically closed field $\Bbbk$, one can pass to the ideal I_Y in $R = \Bbbk[x_0, x_1, \dots, x_m]$ consisting of polynomials that vanish on the points in Y. Computing invariants of such an ideal necessitates a widening of scope to any finitely generated module M over R which can be presented in terms of generators and relations. These relations again have relations, and so on. By successively approximating M in this way, and minimally so, we obtain an acyclic complex, called the *minimal free resolution* of M,

$$F_\bullet : \quad F_0 \xleftarrow{\partial_1} F_1 \xleftarrow{\partial_2} \cdots \xleftarrow{\partial_m} F_{m+1} \leftarrow 0,$$

where the F_i are free modules, which are simply direct sums of the polynomial ring R itself. Note that the article "the" is appropriate here, as a minimal free resolution is unique up to isomorphism.

If M is graded so it has generators and relations that are homogeneous, then after fixing bases, each ∂_i can be represented as a matrix of homogeneous forms. One keeps track of the degrees of these forms by decorating the free modules with parenthetical integers, which are negated by convention. For example, with $R = \Bbbk[x_0, x_1]$,

$$R \xleftarrow{\begin{bmatrix} x_0^2 - x_1^2 & x_0x_1^2 & x_1^3 \end{bmatrix}} \begin{matrix} R^1(-2) \\ \bigoplus \\ R^2(-3) \end{matrix} \xleftarrow{\begin{bmatrix} x_1^2 & 0 \\ -x_0 & x_1 \\ x_1 & -x_0 \end{bmatrix}} R^2(-4) \leftarrow 0$$

is the minimal free resolution of $R/\langle x_0^2 - x_1^2, x_0x_1^2, x_1^3 \rangle$. By construction, a minimal free resolution is acyclic or has no higher homology. Gröbner basis techniques are quite effective for computing (graded) minimal free resolutions, and this has been implemented in a variety of computer algebra software systems.

What geometric information can be recovered from a minimal free resolution $F_\bullet$? To begin, there is the celebrated *Hilbert Syzygy Theorem*. Over the polynomial ring, $F_\bullet$ has finite length: above, we see that $F_i = 0$ for $i > m + 1$. In fact, if M comes from geometry, meaning that it is not supported at the graded maximal ideal, then this bound can be lowered by one, so that its minimal free resolution has length at most $m = \dim(\mathbb{P}^m)$.

Given a variety $Y \subset \mathbb{P}^m$, the dimension of R/I_Y in degree d, or Hilbert function of R/I_Y at d, is known to eventually be a polynomial in d, called the *Hilbert polynomial* of Y. The leading coefficient of this polynomial encodes the degree of Y, which is the number of points obtained by intersecting Y with a linear subspace of complementary dimension. And this is only the beginning. For instance, if $C \subset \mathbb{P}^m$ is a curve, then the Hilbert polynomial of C is

$$(\text{degree } C)d + (1 - \text{genus } C),$$

whose coefficients topologically classify the embedded curve.

Finally, detecting when a minimal free resolution involves only linear forms is related to *Castelnuovo–Mumford regularity*. This invariant measures the vanishing of sheaf cohomology, as well as controlling when the Hilbert polynomial and Hilbert function coincide.

With so many spectacular examples of geometry coming from syzygies for projective space, one might look for an analogous theory for spaces that have a similar ideal-variety correspondence. However, even for a product of projective spaces, minimal free resolutions do not as closely reflect geometry. For instance, a minimal free resolution can be much longer than the dimension of the ambient space. Seeking to remedy this situation, Daniel Erman, Gregory G. Smith, and I began developing the theory of *virtual resolutions* for smooth complete toric varieties. These are complexes that, while they are no longer acyclic or unique up to isomorphism, better reflect the geometric theory of syzygies that mirrors the story for projective space.


Christine Berkesch

Credits

Author photo is courtesy of Bonny Fleming.

Christine Berkesch is an associate professor of mathematics at the University of Minnesota. Her email address is cberkesc@umn.edu.

For permission to reprint this article, please contact: reprint-permission@ams.org.

DOI: https://doi.org/10.1090/noti2068