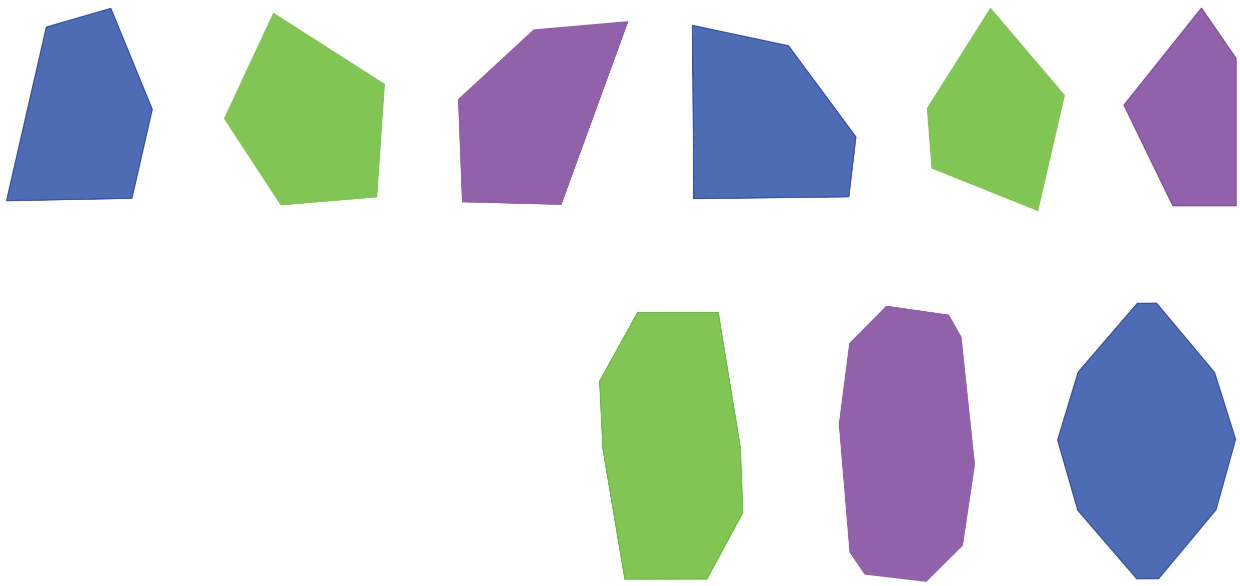
Can You Pave the Plane with Identical Tiles?



Chuanming Zong

Everybody knows that identical regular triangles or squares can tile the whole plane. Many people know that identical regular hexagons can tile the plane properly as well. In fact, even the bees know and use this fact! Is there any other convex domain that can tile the Euclidean plane? Of course, there is a long list of them! To find the list and to show the completeness of the list is a unique drama in mathematics which has lasted for more than one century, and the completeness of the list has been mistakenly announced more than once! Up to now, the list consists of triangles, quadrilaterals, fifteen types of pentagons, and three types of hexagons. In 2017, Michaël Rao announced a computer proof for the completeness of the list. Meanwhile, Qi Yang and Chuanming Zong made a series of unexpected discoveries in multiple tilings in the Euclidean plane. For example, besides parallelograms and centrally symmetric hexagons, there is no other convex domain that can form any two-, three-, or fourfold translative tiling in the plane. However, there are two types of octagons and one type of decagon that can form nontrivial fivefold translative tilings. In fact, parallelograms, centrally symmetric hexagons, and these three types of polygons are the only convex polygons that can form fivefold translative tilings. This paper reviews the dramatic progress.

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DOI: https://doi.org/10.1090/noti2075

Introduction

Tiling the plane is an ancient subject in our civilization. It has been considered in the arts by craftsmen since antiquity. According to Gardner [4], the ancient Greeks knew that, among the regular polygons, only the triangle, the square, and the hexagon can tile the plane. Aristotle apparently knew this fact, since he made a similar claim in the space: *Among the five Platonic solids, only the tetrahedron and the cube can tile the space*. Unfortunately, he was wrong: *Identical regular tetrahedra cannot tile the whole space*!

The first recorded scientific investigation into tilings was made by Kepler. Assume that $\mathscr{T}$ is a tiling of the Euclidean plane $\mathbb{E}^2$ by regular convex polygons. If the polygons are identical (congruent), the answer was already known to the ancient Greeks. When different polygons are allowed, the situation becomes more complicated and more interesting. In particular, an edge-to-edge tiling $\mathscr{T}$ by regular polygons is said to be of type $(n_1, n_2, \ldots, n_r)$ if each vertex $\mathbf{v}$ of $\mathscr{T}$ is surrounded by an n_1-gon, an n_2-gon, and so on in a cyclic order, where edge-to-edge means that every pair of neighbors shares an entire common edge. Usually, they are known as *Archimedean tilings*. In 1619, Kepler enumerated all such tilings as (3, 3, 3, 3, 3, 3), (3, 3, 3, 3, 6), (3, 3, 3, 4, 4), (3, 3, 4, 3, 4), (3, 4, 6, 4), (3, 6, 3, 6), (3, 12, 12), (4, 4, 4, 4), (4, 6, 12), (4, 8, 8), and (6, 6, 6). Beautiful illustrations of the Archimedean tilings can be found in many references.

If $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are n linearly independent vectors in the n-dimensional Euclidean space $\mathbb{E}^n$, then the set

$$\Lambda = \left\{ \sum z_i \mathbf{a}_i \ : \ z_i \in \mathbb{Z} \right\}$$

is an *n-dimensional lattice*. Clearly, lattices are the most natural periodic discrete sets in the plane and space. Therefore, many pioneering scientists like Kepler, Huygens, Haüy, and Seeber took lattice packings and lattice tilings as the scientific foundation for crystals. In 1885, the famous crystallographer Fedorov [3] discovered that *A convex domain can form a lattice tiling of $\mathbb{E}^2$ if and only if it is a parallelogram or a centrally symmetric hexagon; a convex body can form a lattice tiling in $\mathbb{E}^3$ if and only if it is a parallelotope, a hexagonal prism, a rhombic dodecahedron, an elongated octahedron, or a truncated octahedron.*

Usually, tilings allow very general settings without restriction on the shapes of the tiles and the number of the different shapes. However, to avoid complexity and confusion, in this paper we deal only with the tilings by identical convex polygon tiles. In other words, all the tiles are congruent to one convex polygon. In particular, we call it a *translative tiling* if all the tiles are translates of one another and call it a *lattice tiling* if it is a translative tiling and all the translative vectors together form a lattice.

In 1900, Hilbert [8] proposed a list of mathematical problems in his ICM lecture in Paris. As a generalized inverse of Fedorov's discovery, he wrote in the second part of his 18th problem that a *fundamental region of each group of motions, together with the congruent regions arising from the group, evidently fills up space completely. The question arises: whether polyhedra also exist which do not appear as fundamental regions of groups of motions, by means of which nevertheless by a suitable juxtaposition of congruent copies a complete filling up of all space is possible.* Here Hilbert did not restrict to convex ones.

Hilbert proposed his problem in the space; perhaps he believed that there is no such domain in the plane. When Reinhardt started his doctoral thesis at Frankfurt am Main in the 1910s, Bieberbach (see [14]) suggested that he *determine all the convex domains which can tile the whole plane* and to verify in this way that Hilbert's problem indeed has a positive answer in the plane. This is the origin of the following natural problem:

Bieberbach's Problem. To determine all the two-dimensional convex tiles.

In 1917 Reinhardt was an assistant of Hilbert's at Göttingen and likely discussed this problem with him. It is worth mentioning that in 1911, Bieberbach himself solved the first part of Hilbert's 18th problem: *Is there in n-dimensional Euclidean space also only a finite number of essentially different kinds of groups of motions with a fundamental region*?

Reinhardt's List

In 1918, Reinhardt received his doctoral degree under the supervision of Bieberbach at Frankfurt am Main with a thesis "On Partitioning the Plane into Polygons" (Über die Zerlegung der Ebene in Polygone). This is the first approach to characterizing all the convex domains that can tile the whole plane. First, he studied the tiling networks (the vertices, edges, and faces of the tilings) and obtained an expression for the mean of the number of vertices over faces. As a corollary of the formula, he obtained the following result.

Theorem 1 (Reinhardt [14]). *A convex m-gon can tile the whole plane $\mathbb{E}^2$ by identical copies only if*

$$m \leq 6.$$

In fact, as Reinhardt and several other authors pointed out (see [4, 10, 12, 14]), this theorem can be easily deduced by *Euler's formula*

$$v - e + f = 1, \tag{1}$$

where v, e, and f stand for the numbers of vertices, edges, and faces of a polygonal division of a finite polygon.

Let $\mathcal{T} = \{T_1, T_2, T_3, \ldots\}$ be a tiling of $\mathbb{E}^2$ such that all tiles T_i are congruent to a convex m-gon P_m, and let H_ℓ be a regular hexagon of edge length ℓ centered at the origin of $\mathbb{E}^2$. Assume that H_ℓ contains $f(\ell)$ tiles $T_1, T_2, \ldots, T_{f(\ell)}$ of $\mathcal{T}$ and the boundary of H_ℓ intersects $g(\ell)$ tiles $T'_1, T'_2, \ldots, T'_{g(\ell)}$ of $\mathcal{T}$, and let u_i denote the number of vertices of the tiling network on the boundary of T_i. Clearly we have $u_i \geq m$ and

$$\lim_{\ell \to \infty} \frac{g(\ell)}{f(\ell)} = 0. \tag{2}$$

Applying Euler's formula to $\mathcal{T} \cap H_\ell$, when ℓ is sufficiently large we get

$$f \leq f(\ell) + g(\ell) \lesssim f(\ell), \tag{3}$$

$$e \geq \frac{1}{2} \sum_{i=1}^{f(\ell)} u_i \geq \frac{m}{2} \cdot f(\ell), \tag{4}$$

$$v \lesssim \frac{1}{3} \sum_{i=1}^{f(\ell)} u_i \lesssim \frac{m}{3} \cdot f(\ell), \tag{5}$$

$$v - e + f \lesssim \left(1 - \frac{m}{6}\right) f(\ell), \tag{6}$$

and therefore

$$m \leq 6, \tag{7}$$

where $g(\ell) \lesssim cf(\ell)$ means

$$\lim_{\ell \to \infty} \frac{g(\ell)}{f(\ell)} \leq c.$$

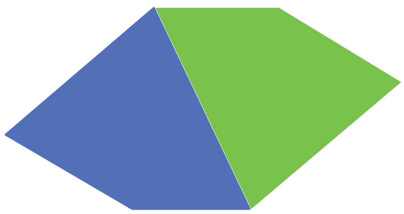

Figure 1. Two quadrilaterals make a centrally symmetric hexagon.

Apparently, two identical triangles can make a parallelogram and two identical quadrilaterals can make a centrally symmetric hexagon (see Figure 1). Thus, by Fedorov's theorem, identical triangles or quadrilaterals can always tile the plane nicely. However, it is easy to see that identical regular pentagons or some particular hexagons cannot tile the plane. Then, Bieberbach's problem can be reformulated as:

What kind of convex pentagons or hexagons can tile the plane?

Let P_n denote a convex n-gon with vertices $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ in an anti-clock order, let G_i denote the edge with ends $\mathbf{v}_{i-1}$ and $\mathbf{v}_i$, where $\mathbf{v}_0 = \mathbf{v}_n$, let ℓ_i denote the length of G_i, and let α_i denote the inner angle of P_n at $\mathbf{v}_i$.

Reinhardt's thesis obtained the following solution to the hexagon case of Bieberbach's problem.

Theorem 2 (Reinhardt [14]). *A convex hexagon P_6 can tile the whole plane $\mathbb{E}^2$ by identical copies if and only if it satisfies one of the three groups of conditions:*

(1) $\alpha_1 + \alpha_2 + \alpha_3 = 2\pi$, *and* $\ell_1 = \ell_4$.
(2) $\alpha_1 + \alpha_2 + \alpha_4 = 2\pi$, $\ell_1 = \ell_4$, *and* $\ell_3 = \ell_5$.
(3) $\alpha_1 = \alpha_3 = \alpha_5 = \frac{2}{3}\pi$, $\ell_1 = \ell_2$, $\ell_3 = \ell_4$, *and* $\ell_5 = \ell_6$.

The "if" part of this theorem is relatively simple. It is illustrated by Figure 2. However, the "only if" part is much more complicated. Reinhardt deduced the only if part by considering six cases with respect to how many edges of the considered hexagon are equal. His proof was very sketchy and difficult to understand and check. It seems that he considered only the edge-to-edge tilings.

Fortunately, this theorem has been verified by several other authors. For example, without knowledge of Reinhardt's thesis, in 1963 Bollobás made the following surprising observation, which guarantees the sufficiency of Reinhardt's consideration.

Lemma 1 (Bollobás [2]). *If $\mathcal{T}$ is a tiling of the plane by identical convex hexagons and ℓ is any given positive number, there*

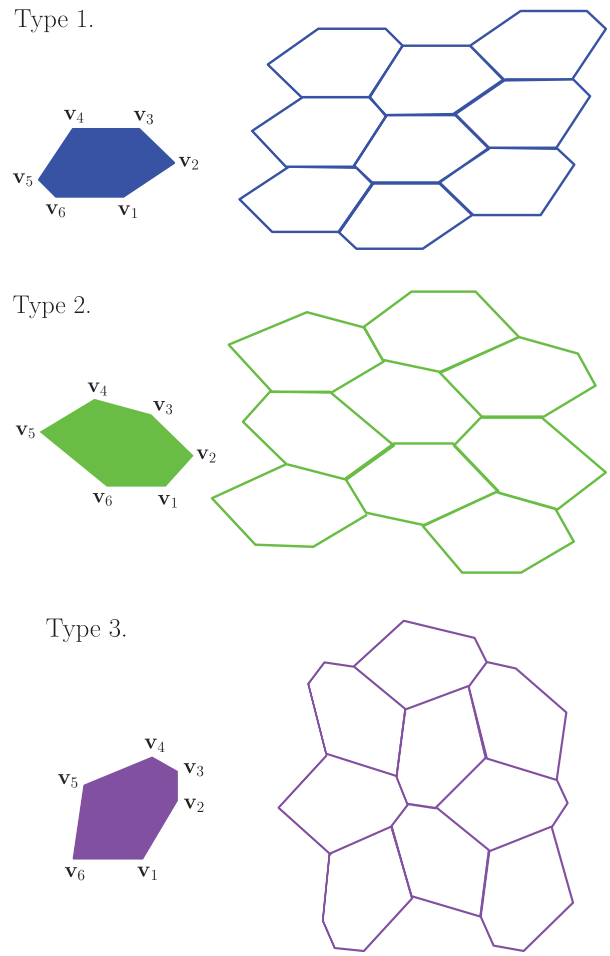

Figure 2. Reinhardt's hexagonal tiles and their local tilings.

is a square of edge length ℓ in which the tiling is edge-to-edge and every vertex is surrounded by three hexagons.

Let S be a big square of edge length ℓ centered at the origin of the plane and consider the network of $N = \mathcal{T} \cap S$. Let n_1 denote the number of vertices in N that appear also in the relative interior of some edges, let n_2 denote the number of vertices in N at which three hexagons join properly at their vertices, and let n_3 denote the number of all other vertices of N. Lemma 1 can be proved by studying the quotients

$$\frac{n_2}{n_1 + n_3}$$

for S and its subsquares for sufficiently large ℓ. It is interesting to notice that there are hexagon tilings in which $n_3 \neq 0$.

For the pentagon tilings, by considering five cases with respect to how many edges are equal, Reinhardt obtained the following result.

Theorem 3 (Reinhardt [14]). *A convex pentagon P_5 can tile the whole plane $\mathbb{E}^2$ by identical copies if it satisfies one of the five groups of conditions:*

(1) $\alpha_1 + \alpha_2 + \alpha_3 = 2\pi$.
(2) $\alpha_1 + \alpha_2 + \alpha_4 = 2\pi$, *and* $\ell_1 = \ell_4$.
(3) $\alpha_1 = \alpha_3 = \alpha_4 = \frac{2}{3}\pi$, $\ell_1 = \ell_2$, *and* $\ell_4 = \ell_3 + \ell_5$.
(4) $\alpha_1 = \alpha_3 = \frac{1}{2}\pi$, $\ell_1 = \ell_2$, *and* $\ell_3 = \ell_4$.
(5) $\alpha_1 = \frac{1}{3}\pi$, $\alpha_3 = \frac{2}{3}\pi$, $\ell_1 = \ell_2$, *and* $\ell_3 = \ell_4$.

Figuring out the list is nontrivial. However, as shown in Figure 3, it is easy to check that all the pentagons listed in Theorem 3 indeed can tile the plane. Reinhardt himself did not claim the completeness of the pentagon tile list. However, according to Gardner [4] *it is quite clear that Reinhardt and everyone else in the field thought that the Reinhardt pentagon list was probably complete.*

As observed by Reinhardt [14], all triangles, quadrilaterals, the three types of hexagons listed in Theorem 2, and the five classes of pentagons listed in Theorem 3 are indeed fundamental domains of some groups of motions. Hilbert and Bieberbach would have been happy to know this.

In 1928, Reinhardt discovered a (nonconvex) three-dimensional polytope that can form a tiling in the space but is not the fundamental domain of any group of motions! This is the first counterexample to the second part of Hilbert's 18th problem.

Inspired by Reinhardt's discovery, in 1935 Heesch [7] obtained a two-dimensional nonconvex counterexample to Hilbert's problem. In other words, there exists a nonconvex polygon that can tile the whole plane; however, it is not the fundamental region of any group of motions.

Thirty years later, Heesch and Kienzle presented a rather detailed treatment of plane tilings in a book entitled *Flächenschluß: System der Formen lückenlos aneinanderschliessender Flachteile*, including nonconvex tiles. No new convex tile was discovered. It was claimed that their treatment was complete.

An End, or a New Start

In 1968, fifty years after Reinhardt's pioneering thesis, Kershner surprisingly discovered three new classes of pentagons that can pave the whole plane without gaps or overlapping.

Theorem 4 (Kershner [10]). *A convex pentagon P_5 can tile the whole plane $\mathbb{E}^2$ by identical copies if it satisfies one of the three groups of conditions:*

(6) $\alpha_1+\alpha_2+\alpha_4 = 2\pi$, $\alpha_1 = 2\alpha_3$, $\ell_1 = \ell_2 = \ell_5$, *and* $\ell_3 = \ell_4$.
(7) $2\alpha_2 + \alpha_3 = 2\alpha_4 + \alpha_1 = 2\pi$, *and* $\ell_1 = \ell_2 = \ell_3 = \ell_4$.
(8) $2\alpha_1 + \alpha_2 = 2\alpha_4 + \alpha_3 = 2\pi$, *and* $\ell_1 = \ell_2 = \ell_3 = \ell_4$.

According to Kershner, having been *intrigued by this problem for some 35 years, he finally discovered a method of classifying the possibilities for pentagons in a more convenient way than Reinhardt's to yield an approach that was humanly possible to carry to completion*. Unfortunately, Kershner's paper contains no hint of his method. Of course, the three classes of new pentagon tiles were indeed surprising, though verifications are simple (see Figure 5).

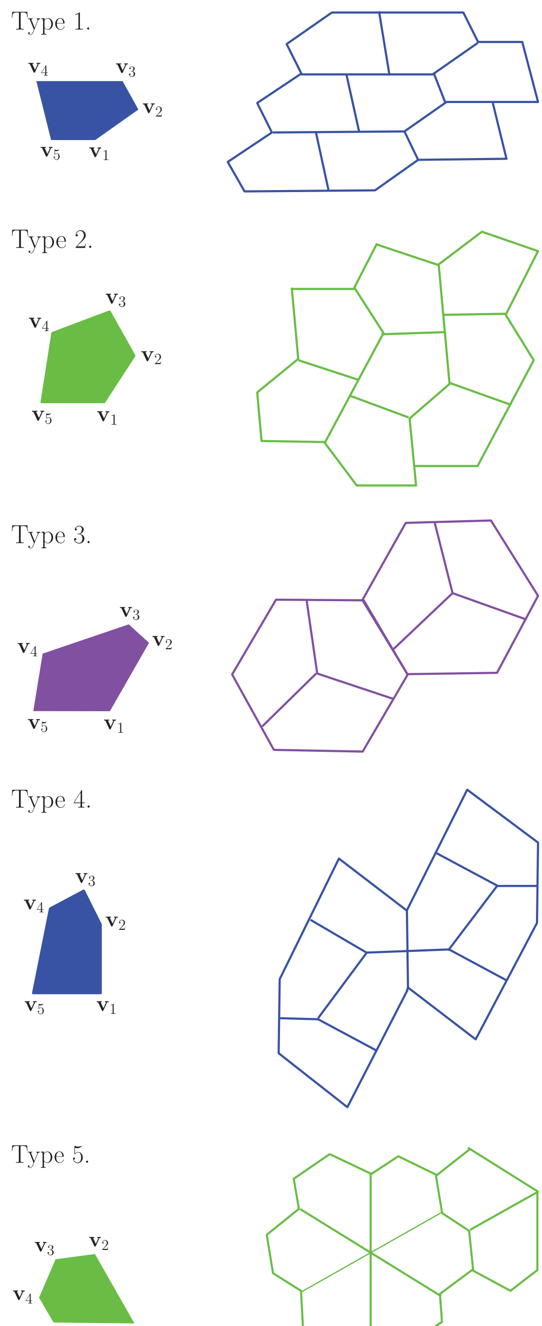

Figure 3. Reinhardt's pentagonal tiles and their local tilings.

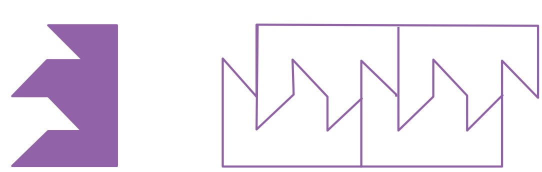

Figure 4. Heesch's counterexample to Hilbert's problem.

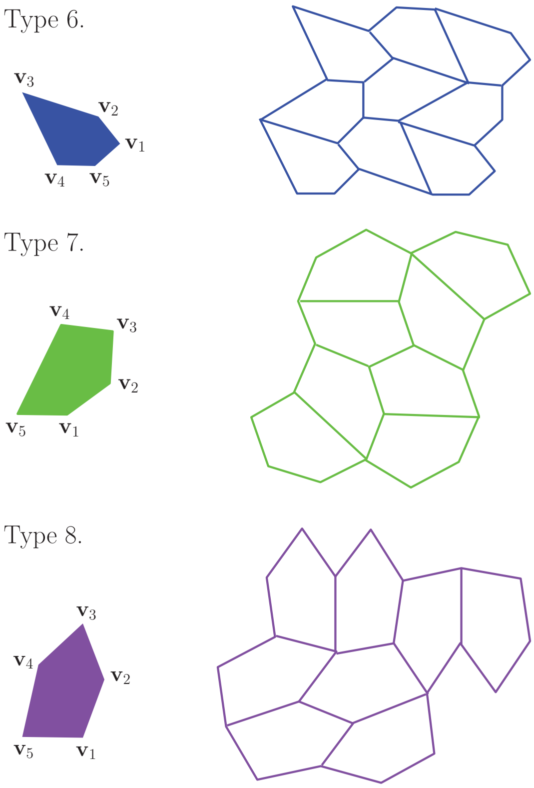

Figure 5. Kershner's pentagonal tiles and their local tilings.

Remark 1. Kershner's discovery was unexpected. Even more surprising was that all the pentagons of Types 6–8 are counterexamples to the second part of Hilbert's 18th problem! In other words, they can tile the whole plane; nevertheless they are not the fundamental regions of any group of motions. Hilbert, Bieberbach, Reinhardt, Heesch, and others would have been surprised by Kershner's elegant examples! Kershner himself did not mention this fact in his papers. Perhaps he overlooked it. This fact has been mentioned in many books and survey papers (see [6]). Inductively, n-dimensional counterexamples to Hilbert's problem can be constructed as cylinders over $(n-1)$-dimensional ones. For example, if D is a domain of Type 6 and H is the cylinder of height one over D, then H is a counterexample to the second part of Hilbert's 18th problem in $\mathbb{E}^3$.

In 1975, Reinhardt and Kershner's discoveries were introduced and popularized by Martin Gardner, a famous scientific writer, in the "Mathematical Games" column of the *Scientific American* magazine. Since then, the tiling problem has stimulated many amateurs who went on to make significant contributions to this problem.

Soon after Gardner's popular paper, based on the known Archimedean tiling $(4, 8, 8)$ by octagons and squares together, as shown in Figure 6, a computer scientist, Richard James III, discovered a class of new tiles.

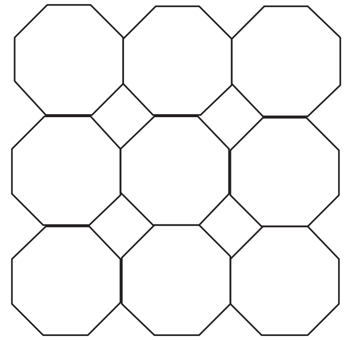

Figure 6. The Archimedean tiling of $(4, 8, 8)$ type.

Theorem 5 (James [9]). *A convex pentagon P_5 can tile the whole plane $\mathbb{E}^2$ by identical copies if it satisfies the following group of conditions:*

$$(9)\quad \alpha_5 = \frac{\pi}{2},\ \alpha_1 + \alpha_4 = \pi,\ 2\alpha_2 - \alpha_4 = 2\alpha_3 + \alpha_4 = \pi,\ \text{and}\ \ell_1 = \ell_2 + \ell_4 = \ell_5.$$

This result can be easily verified by argument based on Figure 7. In principle, Lemma 1 guarantees that every hexagon tiling is edge-to-edge. However, James's discovery shows that this is no longer true in some pentagon tilings. Theorem 5 also served to point out that Kershner had taken edge-to-edge as a hidden assumption in his consideration.

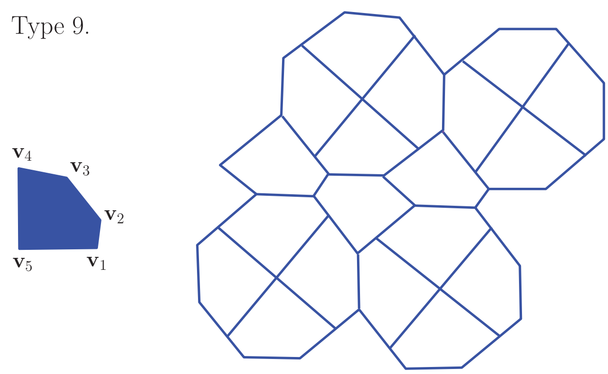

Figure 7. James's pentagonal tile and its local tiling.

Meanwhile, mathematical amateur Marjorie Rice made some truly astonishing discoveries that made the news. She was a true amateur. According to Schattschneider [15], Rice had no mathematical training beyond "the bare minimum they required...in high school over 35 years ago." Even so she was able to consider the problem with a systematic method based on the possible local structures of the pentagon tilings at a given vertex. By dealing with more than sixty cases, she discovered four types of new pentagon tiles!

Theorem 6 (Rice [15]). *A convex pentagon P_5 can tile the whole plane $\mathbb{E}^2$ by identical copies if it satisfies one of the four groups of conditions:*

$$(10)\quad \alpha_2 + 2\alpha_5 = 2\pi,\ \alpha_3 + 2\alpha_4 = 2\pi,\ \text{and}\ \ell_1 = \ell_2 = \ell_3 = \ell_4.$$

(11) $\alpha_1 = \frac{\pi}{2}$, $\alpha_3 + \alpha_5 = \pi$, $2\alpha_2 + \alpha_3 = 2\pi$, *and* $2\ell_1 + \ell_3 = \ell_4 = \ell_5$.

(12) $\alpha_1 = \frac{\pi}{2}$, $\alpha_3+\alpha_5 = \pi$, $2\alpha_2+\alpha_3 = 2\pi$, *and* $2\ell_1 = \ell_3+\ell_5 = \ell_4$.

(13) $\alpha_1 = \alpha_3 = \frac{\pi}{2}$, $2\alpha_2 + \alpha_4 = 2\alpha_5 + \alpha_4 = 2\pi$, $\ell_3 = \ell_4$, *and* $2\ell_3 = \ell_5$.

It is routine to verify this theorem based on Figure 8. Nevertheless, it is rather surprising to notice that the tilings produced by the pentagons of Type 10 are edge-to-edge, a fact that was missed by both Reinhardt and Kershner. It is even more surprising that all the pentagons of Types 9–13 are counterexamples to Hilbert's problem as well (see [6]). In other words, they can tile the whole plane; however, they are not the fundamental domains of any group of motions.

Marjorie Rice died on July 2, 2017, at the age of 94. A lobby floor of the Mathematical Association of America in Washington is paved with one of Rice's pentagon tiles in her honor. On July 11, 2017, *Quanta Magazine* published an article in her memory.

Rice's method was systematic, in the sense that it was based on a geometric principle. In any case, the method was not strong enough to guarantee the completeness of the list. In 1985, Rolf Stein reported another one.

Theorem 7 (Stein [16]). *A convex pentagon P_5 can tile the whole plane $\mathbb{E}^2$ by identical copies if it satisfies the following conditions:*

(14) $\alpha_1 = \frac{\pi}{2}$, $2\alpha_2 + \alpha_3 = 2\pi$, $\alpha_3 + \alpha_5 = \pi$, *and* $2\ell_1 = 2\ell_3 = \ell_4 = \ell_5$.

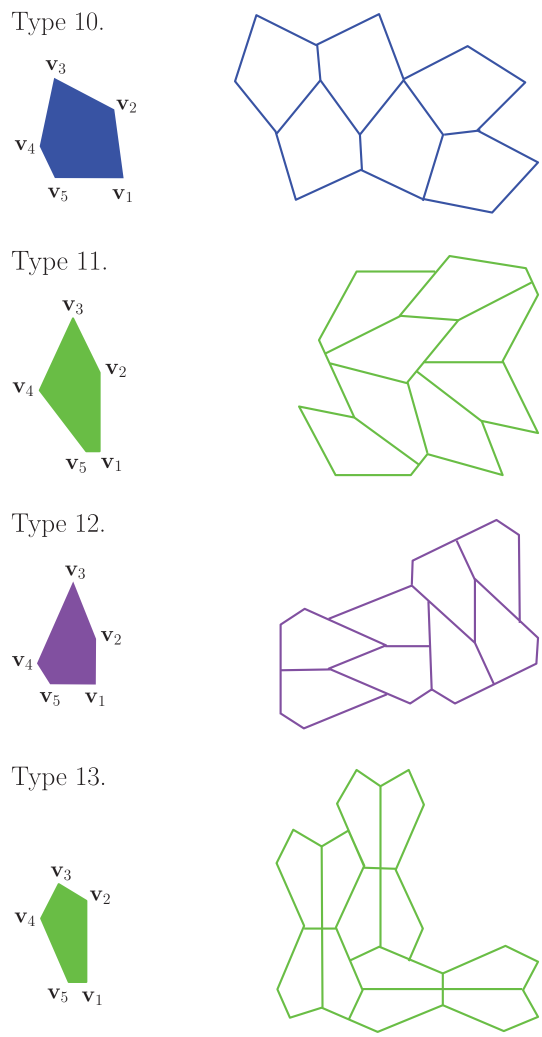

Figure 8. Rice's pentagonal tiles and their local tilings.

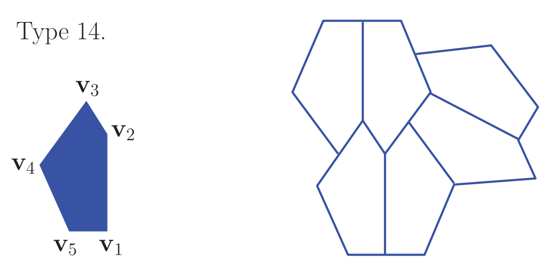

Figure 9. Stein's pentagonal tile and its local tiling.

Fifteen, and Only Fifteen

Let $\mathcal{T}$ denote a tiling of $\mathbb{E}^2$ with congruent tiles. A *symmetry* of $\mathcal{T}$ is an isometry of $\mathbb{E}^2$ that maps the tiles of $\mathcal{T}$ onto tiles of $\mathcal{T}$, and the *symmetry group* $\mathcal{G}$ of $\mathcal{T}$ is the collection of all such symmetries associated with isometry multiplications. Two tiles, T_1 and T_2 of $\mathcal{T}$, are said to be equivalent if there is a symmetry $\sigma \in \mathcal{G}$ such that $\sigma(T_1) = T_2$. If all the tiles of $\mathcal{T}$ are equivalent to one tile T, the tiling $\mathcal{T}$ is said to be *transitive* (or *isohedral*) and T is called a *transitive tile*. Then, the second part of Hilbert's 18th problem can be reformulated as:

Is every polytope that can tile the whole space a transitive tile?

A tiling $\mathcal{T}$ of $\mathbb{E}^2$ by identical convex pentagons is called an *n-block transitive tiling* if it has a block B consisting of n (minimum) connected tiles such that $\mathcal{T}$ is a transitive tiling of B. If a convex pentagon T can form an n-block transitive tiling but not an m-block transitive tiling for any $m < n$, then we call it an *n-block transitive tile*. Clearly, all the tiles of Types 1–5 are one-block transitive. In other words, they are transitive tiles. According to [12, 15], all the tiles of Types 5–14 except Type 9 are two-block transitive, and the tiles of Type 9 are three-block transitive.

From the intuitive point of view, it is reasonable to believe that periodic structure is inevitable in pentagon tilings and the period cannot be too large. Based on this belief, Mann, McLoud-Mann, and Von Derau [12] developed an algorithm for enumerating all the n-block transitive pentagon tiles. When they checked the three-block case, surprisingly, they discovered a new type of pentagon tiles.

Theorem 8 (Mann, McLoud-Mann, and Von Derau [12]). *A convex pentagon P_5 can tile the whole plane $\mathbb{E}^2$ by identical copies if it satisfies the following conditions:*

$$\alpha_1 = \frac{\pi}{3},\ \alpha_2 = \frac{3\pi}{3},\ \alpha_3 = \frac{7\pi}{12},\ \alpha_4 = \frac{\pi}{2},\ \alpha_5 = \frac{5\pi}{6},\ \text{and}\ \ell_1 = 2\ell_2 = 2\ell_4 = 2\ell_5. \tag{15}$$

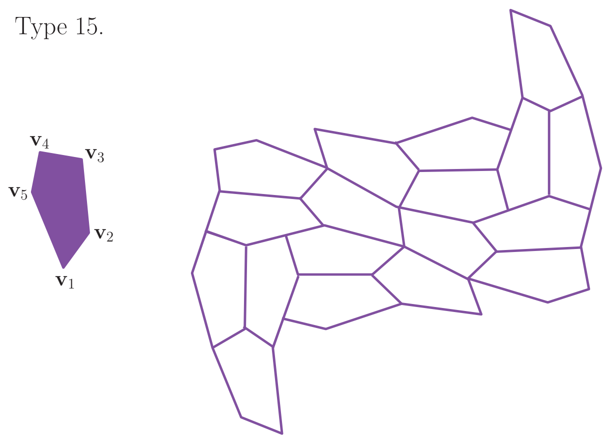

Figure 10. Mann, McLoud-Mann, and Von Derau's pentagonal tile and its local tiling.

Remark 2. It was shown by Mann, McLoud-Mann, and Von Derau [12] that there is no other n-block transitive pentagon tile with $n \le 4$. The completeness of the list emerges again.

Since Hales's computer proof for the Kepler conjecture, more and more geometers have turned to computers for help when their mathematical problems can be reduced into a large number of cases. Characterizing all the pentagon tiles seems to be a perfect candidate for such purpose.

In 2017, one century after Bieberbach proposed the characterization problem, Michaël Rao announced a computer proof for the completeness of the known pentagon tile list. Rao's approach is based on a graph expression. First he proved that if a pentagon tiles the plane, then it can form a tiling such that every vertex type has positive density. Clearly, this is a weak version of the periodic tiling. Second, it was shown that there are only a finite number of possible vertex types in the modified pentagon tiling. In fact, he reduced them to 371 types. Then, by testing the 371 cases, Rao announced the following theorem.

Theorem 9 (Rao [13]). *A convex pentagon P_5 can tile the whole plane $\mathbb{E}^2$ by identical copies if and only if it belongs to one of the fifteen types listed in Theorems* 3–8.

Computer proofs are still not as acceptable as transparent logical proofs within the mathematical community. However, we have to admit that the complexity of the mathematical problems ranges from zero to infinity, and there indeed exist problems that have no transparent logical proofs.

In 1980, Grünbaum and Shephard [6] made the following comment when they wrote about the tiling problems: *Current fashions in mathematics applaud abstraction for its own sake, regarding it as the highest intellectual activity—whether or not it is, in any sense, useful or related to other endeavors. Mathematicians frequently regard it as demeaning to work on problems related to "elementary geometry" in Euclidean space of two or three dimensions. In fact, we believe that many are unable, both by inclination and training, to make meaningful contributions to this more "concrete" type of mathematics; yet it is precisely these and similar considerations that include the results and techniques needed by workers in other disciplines.* Clearly, the proof history of Bieberbach's problem indeed confirms their comment.

Multiple Tilings

Intuitively speaking, tiling the plane is to pave the whole plane flat with identical tiles. As one can see from previous sections, only a few types of polygons are qualified for the job. However, if multiple layers are permitted, we will have many more choices for the shape of the tile.

Let K denote an n-dimensional convex body with interior $\mathrm{int}(K)$ and boundary $\partial(K)$. In particular, let D denote a two-dimensional convex domain.

Assume that $\mathcal{F} = \{K_1, K_2, K_3, \ldots\}$ is a family of convex bodies in $\mathbb{E}^n$ and k is a positive integer. We call $\mathcal{F}$ a *k-fold tiling* of $\mathbb{E}^n$ if every point $\mathbf{x} \in \mathbb{E}^n$ belongs to at least k of these convex bodies and every point $\mathbf{x} \in \mathbb{E}^n$ belongs to at most k of the $\mathrm{int}(K_i)$. In other words, a k-fold tiling of $\mathbb{E}^n$ is both a *k-fold packing* and a *k-fold covering* in $\mathbb{E}^n$. In particular, we call a k-fold tiling of $\mathbb{E}^n$ a *k-fold congruent tiling*, a *k-fold translative tiling*, or a *k-fold lattice tiling* if all K_i are congruent to K_1, all K_i are translates of K_1, or all K_i are translates of K_1 and the translative vectors form a lattice in $\mathbb{E}^n$, respectively. In these particular cases, we call K_1 a *k-fold congruent tile*, a *k-fold translative tile*, or a *k-fold lattice tile*, respectively. Clearly, a k-fold translative tiling of the plane $\mathbb{E}^2$ is a nice pavement with identical copies. In other words, it covers every point of the plane with the same multiplicity, excepting the boundary points of the tiles.

For a fixed convex body K, we define $\tau^{\bullet}(K)$ to be the smallest integer k such that K can form a k-fold congruent tiling in $\mathbb{E}^n$, $\tau(K)$ to be the smallest integer k such that K can form a k-fold translative tiling in $\mathbb{E}^n$, and $\tau^*(K)$ to be the smallest integer k such that K can form a k-fold lattice tiling in $\mathbb{E}^n$. For convenience, if K cannot form any multiple congruent tiling, translative tiling, or lattice tiling, we will define $\tau^{\bullet}(K) = \infty$, $\tau(K) = \infty$, or $\tau^*(K) = \infty$, respectively. Clearly, for every convex body K we have

$$\tau^{\bullet}(K) \le \tau(K) \le \tau^*(K). \tag{8}$$

By looking at the separating hyperplanes between tangent neighbors, it is obvious that a convex body can form a multiple tiling only if it is a polytope.

If σ is a nonsingular affine linear transformation from $\mathbb{E}^n$ to $\mathbb{E}^n$, then $\mathcal{F} = \{K_1, K_2, K_3, \ldots\}$ forms a k-fold tiling of $\mathbb{E}^n$ if and only if $\mathcal{F}' = \{\sigma(K_1), \sigma(K_2), \sigma(K_3), \ldots\}$ forms a k-fold tiling of $\mathbb{E}^n$. Consequently, for any n-dimensional convex body K and any nonsingular affine linear transformation σ we have both

$$\tau(\sigma(K)) = \tau(K) \tag{9}$$

and

$$\tau^*(\sigma(K)) = \tau^*(K). \tag{10}$$

Unfortunately, $\tau^{\bullet}(K)$ is not an invariant for the linear transformation group.

Clearly, onefold tilings are the usual tilings. In the plane, we have

$$\tau(D) = \tau^*(D) = 1 \tag{11}$$

if and only if D is a parallelogram or a centrally symmetric hexagon, and

$$\tau^{\bullet}(D) = 1 \tag{12}$$

if and only if D is a triangle, a quadrilateral, a pentagon belonging to one of the fifteen types listed in Theorems 3–8, or a hexagon belonging to one of the three types listed in Theorem 2.

Taking a usual tiling and stacking it on top of itself k times forms a k-fold tiling. Similarly, by stacking j copies of a k-fold tiling on top of each other, we get a jk-fold tiling. However, we are interested in the nontrivial multiple tilings.

Since 1936, multiple tilings have been studied by P. Furtwängler, G. Hajós, R. M. Robinson, U. Bolle, N. Gravin, M. N. Kolountzakis, S. Robins, D. Shiryaev, and many others. Nevertheless, many natural problems are still open. In the forthcoming sections we will introduce some fascinating new results about multiple tilings in the plane.

Multiple Lattice Tilings

In 1994, Bolle studied the two-dimensional lattice multiple tilings. Let D be a convex domain, let Λ be a lattice, and assume that $D + \Lambda$ is a k-fold lattice tiling of $\mathbb{E}^2$. It is easy to see that D must be a polygon. Let E be an edge of D, let L be the straight line containing E, let H_1 and H_2 denote the two closed half-planes with L as their boundary, and for a general point $\mathbf{p} \in \mathrm{int}(E)$ define

$$n_i(\mathbf{p}) = \sharp\{\mathbf{g} : \ \mathbf{g} \in \Lambda,\ D + \mathbf{g} \in H_i,\ \mathbf{p} \in \partial(D) + \mathbf{g}\}. \tag{13}$$

By studying $n_1(\mathbf{p})$ and $n_2(\mathbf{p})$ for general $\mathbf{p} \in \mathrm{int}(E)$, Bolle proved the following criterion:

Lemma 2 (Bolle [1]). *A convex polygon D is a k-fold lattice tile for a lattice Λ and some positive integer k if and only if the following conditions are satisfied:*

(1) *It is centrally symmetric.*
(2) *When D is centered at the origin, the relative interior of each edge G of D contains a point of $\frac{1}{2}\Lambda$.*
(3) *If the midpoint of G is not in $\frac{1}{2}\Lambda$, then G is a lattice vector of Λ.*

Based on Bolle's criterion, Gravin, Robins, and Shiryaev [5] discovered the following example.

Example 1. Let Λ denote the two-dimensional integer lattice $\mathbb{Z}^2$ and let P_8 denote the octagon with vertices $\mathbf{v}_1 = (\frac{1}{2}, -\frac{3}{2})$, $\mathbf{v}_2 = (\frac{3}{2}, -\frac{1}{2})$, $\mathbf{v}_3 = (\frac{3}{2}, \frac{1}{2})$, $\mathbf{v}_4 = (\frac{1}{2}, \frac{3}{2})$, $\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$, $\mathbf{v}_7 = -\mathbf{v}_3$, and $\mathbf{v}_8 = -\mathbf{v}_4$, as shown in Figure 11. Then $P_8 + \Lambda$ is a sevenfold lattice tiling of $\mathbb{E}^2$.

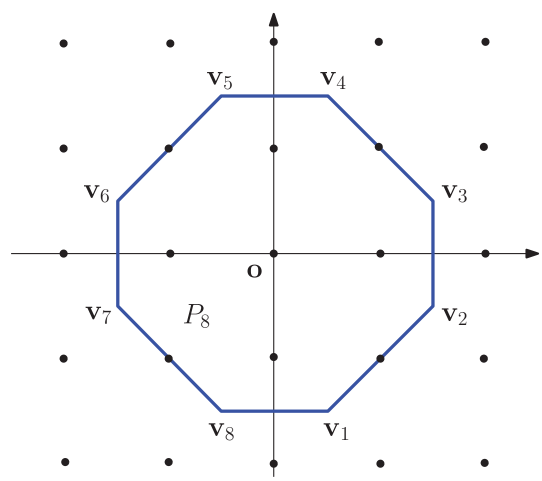

Figure 11. Gravin, Robins, and Shiryaev's octagonal sevenfold lattice tile.

Let $\mathcal{D}$ denote the family of all two-dimensional convex domains and let $\mathcal{P}_{2m}$ denote the family of all centrally symmetric convex $2m$-gons. Since the octagon of Example 1 is the simplest centrally symmetric polygon (except parallelograms and hexagons) satisfying the criterion of Lemma 2, one may conjecture that

$$\min_{D \in \mathcal{D} \setminus \{\mathcal{P}_4 \cup \mathcal{P}_6\}} \tau^*(D) \ge 7.$$

However, based on the known results on multiple lattice packings by V. C. Dumir, R. J. Hans-Gill, and G. Fejes Tóth (see Zong [19]), in 2017 Yang and Zong discovered the following unexpected result.

Theorem 10 (Yang and Zong [17]). *If D is a two-dimensional convex domain that is neither a parallelogram nor a centrally symmetric hexagon, then we have*

$$\tau^*(D) \ge 5,$$

where the equality holds at some particular decagons.

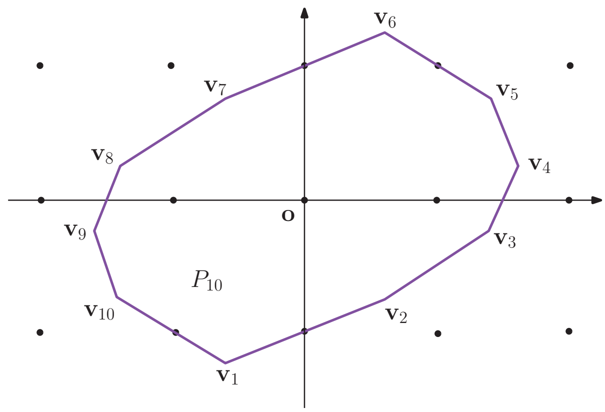

Figure 12. Yang and Zong's decagonal fivefold lattice tile.

Let Λ be the integer lattice $\mathbb{Z}^2$ and let P_{10} denote a decagon whose edge midpoints are $\mathbf{u}_1 = (0,-1)$, $\mathbf{u}_2 = (1,-\frac{1}{2})$, $\mathbf{u}_3 = (\frac{3}{2},0)$, $\mathbf{u}_4 = (\frac{3}{2},\frac{1}{2})$, $\mathbf{u}_5 = (1,1)$, $\mathbf{u}_6 = -\mathbf{u}_1$, $\mathbf{u}_7 = -\mathbf{u}_2$, $\mathbf{u}_8 = -\mathbf{u}_3$, $\mathbf{u}_9 = -\mathbf{u}_4$, and $\mathbf{u}_{10} = -\mathbf{u}_5$, as shown in Figure 12. By Lemma 2, it can be easily verified that $P_{10} + \Lambda$ is indeed a fivefold lattice tiling of $\mathbb{E}^2$.

Even more unexpected, by studying lattice polygons, all the fivefold lattice tiles can be nicely characterized. There are two classes of octagons and one class of decagons besides the parallelograms and the centrally symmetric hexagons.

Theorem 11 (Zong [20]). *A convex domain D can form a fivefold lattice tiling of the Euclidean plane if and only if D is a parallelogram or centrally symmetric hexagon or, up to affine linear transformation, D is a centrally symmetric octagon with vertices $\mathbf{v}_1 = (-\alpha,-\frac{3}{2})$, $\mathbf{v}_2 = (1-\alpha,-\frac{3}{2})$, $\mathbf{v}_3 = (1+\alpha,-\frac{1}{2})$, $\mathbf{v}_4 = (1-\alpha,\frac{1}{2})$, $\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$, $\mathbf{v}_7 = -\mathbf{v}_3$, and $\mathbf{v}_8 = -\mathbf{v}_4$, where $0 < \alpha < \frac{1}{4}$, or with vertices $\mathbf{v}_1 = (\beta,-2)$, $\mathbf{v}_2 = (1+\beta,-2)$, $\mathbf{v}_3 = (1-\beta,0)$, $\mathbf{v}_4 = (\beta,1)$, $\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$, $\mathbf{v}_7 = -\mathbf{v}_3$, $\mathbf{v}_8 = -\mathbf{v}_4$, where $\frac{1}{4} < \beta < \frac{1}{3}$, or a centrally symmetric decagon whose edge midpoints are $\mathbf{u}_1 = (0,-1)$, $\mathbf{u}_2 = (1,-\frac{1}{2})$, $\mathbf{u}_3 = (\frac{3}{2},0)$, $\mathbf{u}_4 = (\frac{3}{2},\frac{1}{2})$, $\mathbf{u}_5 = (1,1)$, $\mathbf{u}_6 = -\mathbf{u}_1$, $\mathbf{u}_7 = -\mathbf{u}_2$, $\mathbf{u}_8 = -\mathbf{u}_3$, $\mathbf{u}_9 = -\mathbf{u}_4$, and $\mathbf{u}_{10} = -\mathbf{u}_5$.*

Let P_{2m} be a centrally symmetric convex $2m$-gon centered at the origin $\mathbf{o}$ of $\mathbb{E}^2$. It is reasonable to believe that $\tau^*(P_{2m})$ is big when m is sufficiently large. In fact, by studying the local structure of a multiple tiling (see next section), Yang and Zong [18] proved that

$$\tau^*(P_{2m}) \geq \begin{cases} m-1 & \text{if } m \text{ is even,} \\ m-2 & \text{if } m \text{ is odd.} \end{cases} \tag{14}$$

Furthermore, by detailed geometric analysis based on Lemma 2, Lemma 4, and Pick's theorem, Zong [20] proved that

$$\tau^*(P_{14}) \geq 6, \tag{15}$$

$$\tau^*(P_{12}) \geq 6, \tag{16}$$

$$\tau^*(P_{10}) \geq 5, \tag{17}$$

where equality in (17) holds if and only if (after a suitable affine linear transformation) P_{10} is a centrally symmetric decagon whose edge midpoints are $\mathbf{u}_1 = (0,-1)$, $\mathbf{u}_2 = (1,-\frac{1}{2})$, $\mathbf{u}_3 = (\frac{3}{2},0)$, $\mathbf{u}_4 = (\frac{3}{2},\frac{1}{2})$, $\mathbf{u}_5 = (1,1)$, $\mathbf{u}_6 = -\mathbf{u}_1$, $\mathbf{u}_7 = -\mathbf{u}_2$, $\mathbf{u}_8 = -\mathbf{u}_3$, $\mathbf{u}_9 = -\mathbf{u}_4$, and $\mathbf{u}_{10} = -\mathbf{u}_5$ (as shown by Figure 12), and

$$\tau^*(P_8) \geq 5, \tag{18}$$

where the equality holds if and only if (after a suitable affine linear transformation) P_8 is either a centrally symmetric octagon $P_8(\alpha)$ (see Figure 13, top) with vertices $\mathbf{v}_1 = (-\alpha,-\frac{3}{2})$, $\mathbf{v}_2 = (1-\alpha,-\frac{3}{2})$, $\mathbf{v}_3 = (1+\alpha,-\frac{1}{2})$, $\mathbf{v}_4 = (1-\alpha,\frac{1}{2})$, $\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$, $\mathbf{v}_7 = -\mathbf{v}_3$, and $\mathbf{v}_8 = -\mathbf{v}_4$, where $0 < \alpha < \frac{1}{4}$, or a centrally symmetric octagon $P_8(\beta)$ (see Figure 13, bottom) with vertices $\mathbf{v}_1 = (\beta,-2)$, $\mathbf{v}_2 = (1+\beta,-2)$, $\mathbf{v}_3 = (1-\beta,0)$, $\mathbf{v}_4 = (\beta,1)$, $\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$, $\mathbf{v}_7 = -\mathbf{v}_3$, $\mathbf{v}_8 = -\mathbf{v}_4$, where $\frac{1}{4} < \beta < \frac{1}{3}$. Let Λ be the integer lattice $\mathbb{Z}^2$; it can be verified that both $P_8(\alpha) + \Lambda$ and $P_8(\beta) + \Lambda$ are indeed fivefold lattice tilings of $\mathbb{E}^2$.

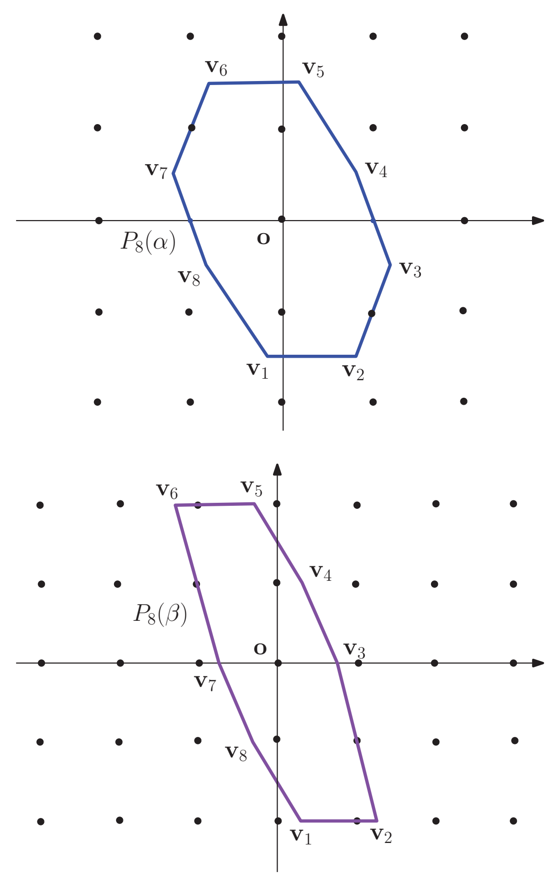

Figure 13. Zong's octagonal fivefold lattice tiles.

Clearly, Theorem 11 follows from (14)–(18). The proofs of these inequalities are complicated, in particular (17) and (18). Their proofs rely on carefully designed area estimations by dealing with many cases. Nevertheless, unlike Rao's proof for Theorem 9, computer checking is not necessary here.

To describe the structure of the decagon in Theorem 11 more explicitly we have the following theorem.

Theorem 12 (Zong [20]). *Let W denote the quadrilateral with vertices $\mathbf{w}_1 = (-\frac{1}{2}, 1)$, $\mathbf{w}_2 = (-\frac{1}{2}, \frac{3}{4})$, $\mathbf{w}_3 = (-\frac{2}{3}, \frac{2}{3})$, and $\mathbf{w}_4 = (-\frac{3}{4}, \frac{3}{4})$. A centrally symmetric convex decagon can take $\mathbf{u}_1 = (0, -1)$, $\mathbf{u}_2 = (1, -\frac{1}{2})$, $\mathbf{u}_3 = (\frac{3}{2}, 0)$, $\mathbf{u}_4 = (\frac{3}{2}, \frac{1}{2})$, $\mathbf{u}_5 = (1, 1)$, $\mathbf{u}_6 = -\mathbf{u}_1$, $\mathbf{u}_7 = -\mathbf{u}_2$, $\mathbf{u}_8 = -\mathbf{u}_3$, $\mathbf{u}_9 = -\mathbf{u}_4$, and $\mathbf{u}_{10} = -\mathbf{u}_5$ as the middle points of its edges if and only if one of its vertices is an interior point of W.*

Similarly, the sixfold lattice tiles can be characterized as follows:

Theorem 13 (Zong [20]). *A convex domain D can form a sixfold lattice tiling of the Euclidean plane if and only if D is a parallelogram or centrally symmetric hexagon or, up to affine linear transformation, D is a centrally symmetric octagon with vertices $\mathbf{v}_1 = (\alpha - 1, 2)$, $\mathbf{v}_2 = (\alpha, 2)$, $\mathbf{v}_3 = (1 - \alpha, 0)$, $\mathbf{v}_4 = (1 + \alpha, -1)$, $\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$, $\mathbf{v}_7 = -\mathbf{v}_3$, and $\mathbf{v}_8 = -\mathbf{v}_4$, where $0 < \alpha < \frac{1}{6}$, a centrally symmetric decagon whose edge midpoints are $\mathbf{u}_1 = (-1, \frac{1}{2})$, $\mathbf{u}_2 = (\frac{1}{2}, 1)$, $\mathbf{u}_3 = (\frac{3}{2}, 1)$, $\mathbf{u}_4 = (2, \frac{1}{2})$, $\mathbf{u}_5 = (2, 0)$, $\mathbf{u}_6 = -\mathbf{u}_1$, $\mathbf{u}_7 = -\mathbf{u}_2$, $\mathbf{u}_8 = -\mathbf{u}_3$, $\mathbf{u}_9 = -\mathbf{u}_4$, and $\mathbf{u}_{10} = -\mathbf{u}_5$, or a centrally symmetric decagon whose edge midpoints are $\mathbf{u}_1 = (-\frac{1}{2}, 1)$, $\mathbf{u}_2 = (\frac{1}{2}, 1)$, $\mathbf{u}_3 = (\frac{3}{2}, \frac{1}{2})$, $\mathbf{u}_4 = (2, 0)$, $\mathbf{u}_5 = (\frac{3}{2}, -\frac{1}{2})$, $\mathbf{u}_6 = -\mathbf{u}_1$, $\mathbf{u}_7 = -\mathbf{u}_2$, $\mathbf{u}_8 = -\mathbf{u}_3$, $\mathbf{u}_9 = -\mathbf{u}_4$, and $\mathbf{u}_{10} = -\mathbf{u}_5$.*

Theorem 14 (Zong [20]). *Let Q denote the quadrilateral with vertices $\mathbf{q}_1 = (0, 1)$, $\mathbf{q}_2 = (0, \frac{5}{6})$, $\mathbf{q}_3 = (-\frac{1}{4}, \frac{3}{4})$, and $\mathbf{q}_4 = (-\frac{1}{3}, \frac{5}{6})$. A centrally symmetric convex decagon P_{10} can take $\mathbf{u}_1 = (-1, \frac{1}{2})$, $\mathbf{u}_2 = (\frac{1}{2}, 1)$, $\mathbf{u}_3 = (\frac{3}{2}, 1)$, $\mathbf{u}_4 = (2, \frac{1}{2})$, $\mathbf{u}_5 = (2, 0)$, $\mathbf{u}_6 = -\mathbf{u}_1$, $\mathbf{u}_7 = -\mathbf{u}_2$, $\mathbf{u}_8 = -\mathbf{u}_3$, $\mathbf{u}_9 = -\mathbf{u}_4$, and $\mathbf{u}_{10} = -\mathbf{u}_5$ as the middle points of its edges if and only if one of its vertices is an interior point of Q.*

Let Q^* denote the quadrilateral with vertices $\mathbf{q}_1 = (0, \frac{5}{4})$, $\mathbf{q}_2 = (\frac{1}{6}, \frac{7}{6})$, $\mathbf{q}_3 = (0, 1)$, and $\mathbf{q}_4 = (-\frac{1}{6}, \frac{7}{6})$. A centrally symmetric convex decagon P_{10}^* can take $\mathbf{u}_1 = (\frac{1}{2}, -1)$, $\mathbf{u}_2 = (\frac{3}{2}, -\frac{1}{2})$, $\mathbf{u}_3 = (2, 0)$, $\mathbf{u}_4 = (\frac{3}{2}, \frac{1}{2})$, $\mathbf{u}_5 = (\frac{1}{2}, 1)$, $\mathbf{u}_6 = -\mathbf{u}_1$, $\mathbf{u}_7 = -\mathbf{u}_2$, $\mathbf{u}_8 = -\mathbf{u}_3$, $\mathbf{u}_9 = -\mathbf{u}_4$, and $\mathbf{u}_{10} = -\mathbf{u}_5$ as the middle points of their edges if and only if one of its vertices is an interior point of Q^*.

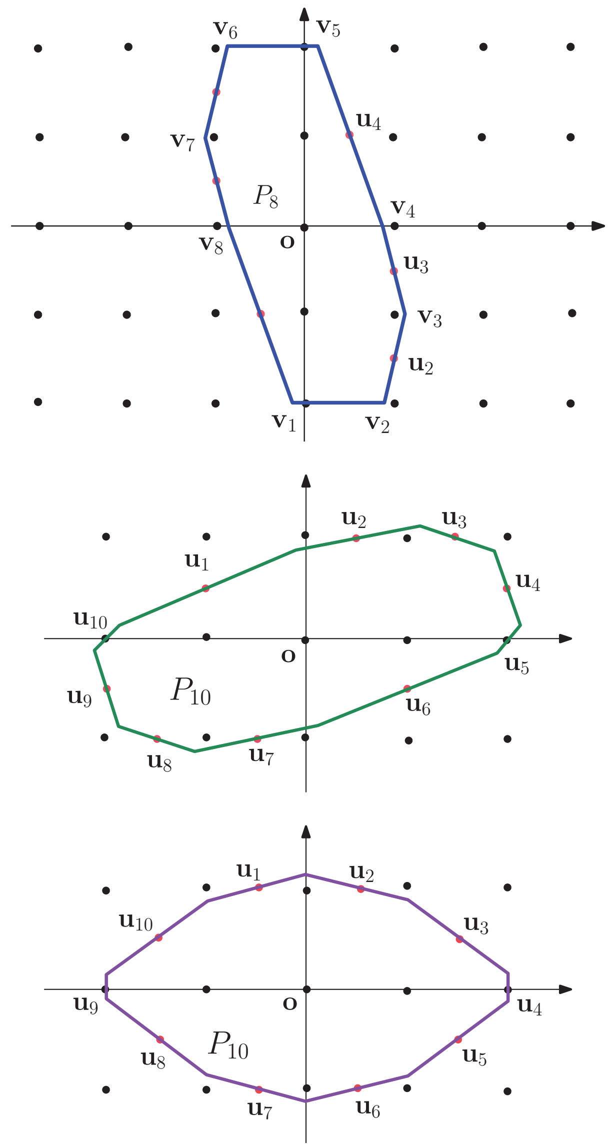

Figure 14. Zong's sixfold lattice tiles, one class of octagons and two classes of decagons.

Multiple Translative Tilings

In 2012, Gravin, Robins, and Shiryaev [5] proved that an n-dimensional convex body can form a multiple translative tiling of the space only if it is a centrally symmetric polytope with centrally symmetric facets. Therefore, to study multiple translative tilings in the plane, we need to deal only with the centrally symmetric polygons.

Let P_{2m} denote a centrally symmetric convex $2m$-gon centered at the origin, with vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{2m}$ enumerated in the clock-order, and write $V = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{2m}\}$. Assume that $P_{2m} + X$ is a $\tau(P_{2m})$-fold translative tiling in $\mathbb{E}^2$, where $X = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots\}$ is a discrete multiset with $\mathbf{x}_1 = \mathbf{o}$. By studying the local structure of $P_{2m} + X$ at the vertices $\mathbf{v} \in V + X$, Yang and Zong [18] discovered some fascinating results.

Theorem 15 (Yang and Zong [18]). *If D is a two-dimensional convex domain that is neither a parallelogram nor a centrally symmetric hexagon, then we have*

$$\tau(D) \geq 5,$$

where the equality holds if D is some particular centrally symmetric octagon or some particular centrally symmetric decagon.

Remark 3. It is known that

$$\tau(D) \leq \tau^*(D)$$

holds for every convex domain D. Therefore, Theorem 15 implies Theorem 10.

At this point, it is natural to ask for a characterization of all fivefold translative tiles and in particular to determine if these are just the known fivefold lattice tiles.

Theorem 16 (Yang and Zong [18]). *A convex domain D can form a fivefold translative tiling of the Euclidean plane if and only if D is a parallelogram or centrally symmetric hexagon or, up to affine linear transformation, D is a centrally symmetric octagon with vertices* $\mathbf{v}_1 = \left(\frac{3}{2} - \frac{5\alpha}{4}, -2\right)$, $\mathbf{v}_2 = \left(-\frac{1}{2} - \frac{5\alpha}{4}, -2\right)$, $\mathbf{v}_3 = \left(\frac{\alpha}{4} - \frac{3}{2}, 0\right)$, $\mathbf{v}_4 = \left(\frac{\alpha}{4} - \frac{3}{2}, 1\right)$, $\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$, $\mathbf{v}_7 = -\mathbf{v}_3$, *and* $\mathbf{v}_8 = -\mathbf{v}_4$, *where* $0 < \alpha < \frac{2}{3}$, *or with vertices* $\mathbf{v}_1 = (2-\beta, -3)$, $\mathbf{v}_2 = (-\beta, -3)$, $\mathbf{v}_3 = (-2, -1)$, $\mathbf{v}_4 = (-2, 1)$, $\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$, $\mathbf{v}_7 = -\mathbf{v}_3$, *and* $\mathbf{v}_8 = -\mathbf{v}_4$, *where* $0 < \beta \leq 1$, *or a centrally symmetric decagon whose edge midpoints are* $\mathbf{u}_1 = (0, -1)$, $\mathbf{u}_2 = (1, -\frac{1}{2})$, $\mathbf{u}_3 = (\frac{3}{2}, 0)$, $\mathbf{u}_4 = (\frac{3}{2}, \frac{1}{2})$, $\mathbf{u}_5 = (1, 1)$, $\mathbf{u}_6 = -\mathbf{u}_1$, $\mathbf{u}_7 = -\mathbf{u}_2$, $\mathbf{u}_8 = -\mathbf{u}_3$, $\mathbf{u}_9 = -\mathbf{u}_4$, *and* $\mathbf{u}_{10} = -\mathbf{u}_5$.

The proofs for Theorems 15 and 16 are extremely complicated. They consist of a series of lemmas showing that

$$\tau(P_{2m}) \geq \begin{cases} m-1 & \text{if } m \text{ is even,} \\ m-2 & \text{if } m \text{ is odd,} \end{cases} \tag{19}$$

$$\tau(P_{14}) \geq 6, \tag{20}$$

$$\tau(P_{12}) \geq 6, \tag{21}$$

$$\tau(P_{10}) \geq 5, \tag{22}$$

where in (22) equality holds if and only if P_{10} is a centrally symmetric decagon that can form a fivefold lattice tiling of $\mathbb{E}^2$, and

$$\tau(P_8) \geq 5, \tag{23}$$

where the equality holds if and only if (after a suitable affine linear transformation) P_8 is either a centrally symmetric octagon $P_8'(\alpha)$ (see Figure 15, top) with vertices $\mathbf{v}_1 = \left(\frac{3}{2} - \frac{5\alpha}{4}, -2\right)$, $\mathbf{v}_2 = \left(-\frac{1}{2} - \frac{5\alpha}{4}, -2\right)$, $\mathbf{v}_3 = \left(\frac{\alpha}{4} - \frac{3}{2}, 0\right)$, $\mathbf{v}_4 = \left(\frac{\alpha}{4} - \frac{3}{2}, 1\right)$, $\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$, $\mathbf{v}_7 = -\mathbf{v}_3$, and $\mathbf{v}_8 = -\mathbf{v}_4$, where $0 < \alpha < \frac{2}{3}$, or a centrally symmetric octagon $P_8'(\beta)$ (see Figure 15, bottom) with vertices $\mathbf{v}_1 = (2-\beta, -3)$, $\mathbf{v}_2 = (-\beta, -3)$, $\mathbf{v}_3 = (-2, -1)$, $\mathbf{v}_4 = (-2, 1)$,

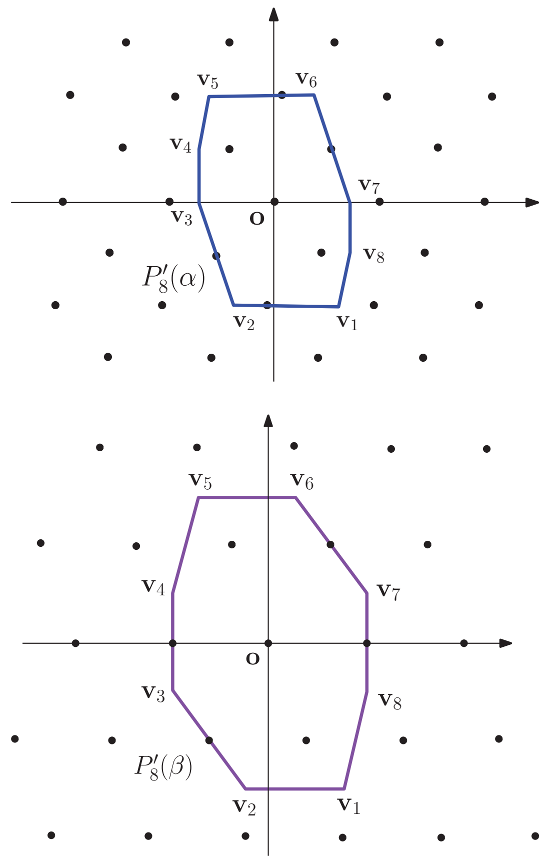

Figure 15. Yang and Zong's octagonal fivefold translative tiles.

$\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$, $\mathbf{v}_7 = -\mathbf{v}_3$, and $\mathbf{v}_8 = -\mathbf{v}_4$, where $0 < \beta \leq 1$.

Clearly, Theorems 15 and 16 follow by (19)–(23) and (17).

Though the statements (19)–(23) are more or less identical to (14)–(18), their proofs are very different. While the lattice case was based on lattice polygon checking, the translative case is based on combinatorial analysis.

In fact, the two classes $P_8(\alpha)$ and $P_8'(\beta)$ shown in Figures 13 and 15, respectively, are equivalent under suitable linear transformations, as well as the two classes $P_8(\beta)$ and $P_8'(\alpha)$. Therefore, we have the following theorem.

Theorem 17 (Yang and Zong [18]). *A convex domain can form a fivefold translative tiling of the Euclidean plane if and only if it can form a fivefold lattice tiling in* $\mathbb{E}^2$.

Open Problems

To end this paper, let us list three open problems about multiple tilings that are closely related to the known results.

Problem 1. Is there a two-dimensional convex domain D satisfying $\tau(D) \neq \tau^*(D)$?

In 2000, Kolountzakis [11] proved that if a convex polygon that is not a parallelogram can form a multiple translative tiling of the plane, then the translative set must be a finite union of translated lattices. To improve this result and to answer a question of Gravin, Robins, and Shiryaev [5], B. Liu and Q. Yang independently proved that if a convex domain D can form a multiple translative tiling of the plane, then it also can form a multiple lattice tiling of the plane. However, we do not know if $\tau(D) = \tau^*(D)$ holds for every convex domain D.

Problem 2. Is there an integer $k \geq 6$ such that $\tau(D) \neq k$ (or $\tau^*(D) \neq k$) holds for all the two-dimensional convex domains D?

As noticed by Yang and Zong [17], based on the two-dimensional examples, for any $n \geq 3$ one can construct n-dimensional centrally symmetric polytopes P satisfying

$$2 \leq \tau^*(P) \leq 5$$

and

$$2 \leq \tau(P) \leq 5.$$

Then, we have the following natural problem.

Problem 3. Assume that $k = 2, 3$, or 4, and $n \geq 3$. Is there an n-dimensional polytope P satisfying $\tau(P) = k$ (or $\tau^*(P) = k$)?

Besides these results and open problems for $\tau^*(P)$ and $\tau(P)$, analogous problems for $\tau^\bullet(P)$ are interesting and worth studying as well.

ACKNOWLEDGMENTS. The author is grateful to the referees and the handling editor for their helpful suggestions, which essentially improved the quality of the paper. This work was supported by the National Natural Science Foundation of China (NSFC11921001) and the National Key Research and Development Program of China (2018YFA0704701).

References

[1] U. Bolle, *On multiple tiles in E^2*, Intuitive geometry (Szeged, 1991), Colloq. Math. Soc. János Bolyai, vol. 63, North-Holland, Amsterdam, 1994, pp. 39–43. MR1383609

[2] B. Bollobás, *Filling the plane with congruent convex hexagons without overlapping*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **6** (1963), 117–123. MR166683

[3] E. S. Fedorov, *Načala učeniya o figurah* (Russian), Izdat. Akad. Nauk SSSR, Moscow, 1953. MR0062061

[4] M. Gardner, *On tessellating the plane with convex polygon tiles*, Scientific American (July 1975), 112–117.

[5] N. Gravin, S. Robins, and D. Shiryaev, *Translational tilings by a polytope, with multiplicity*, Combinatorica **32** (2012), no. 6, 629–649, DOI 10.1007/s00493-012-2860-3. MR3063154

[6] B. Grünbaum and G. C. Shephard, *Tilings with congruent tiles*, Bull. Amer. Math. Soc. (N.S.) **3** (1980), no. 3, 951–973, DOI 10.1090/S0273-0979-1980-14827-2. MR585178

[7] H. Heesch, *Aufbau der Ebene aus kongrenten Bereichen*, Nachr. Ges. Wiss. Göttingen **1** (1935), 115–117.

[8] D. Hilbert, *Mathematical problems*, Bull. Amer. Math. Soc. **8** (1902), no. 10, 437–479, DOI 10.1090/S0002-9904-1902-00923-3. MR1557926

[9] R. James, *New pentagonal tiling reported by Martin Gardner*, Scientific American (December 1975), 117–118.

[10] R. B. Kershner, *On paving the plane*, Amer. Math. Monthly **75** (1968), 839–844, DOI 10.2307/2314332. MR236822

[11] M. N. Kolountzakis, *On the structure of multiple translational tilings by polygonal regions*, Discrete Comput. Geom. **23** (2000), no. 4, 537–553, DOI 10.1007/s004540010014. MR1753701

[12] C. Mann, J. McLoud-Mann, and D. Von Derau, *Convex pentagons that admit i-block transitive tilings*, Geom. Dedicata **194** (2018), 141–167, DOI 10.1007/s10711-017-0270-9. MR3798065

[13] M. Rao, *Exhaustive search of convex pentagons which tile the plane*, arXiv:1708.00274, 2017.

[14] K. Reinhardt, *Über die Zerlegung der Ebene in Polygone*, Dissertation, Universität Frankfurt am Main, 1918.

[15] D. Schattschneider, *Tiling the plane with congruent pentagons*, Math. Mag. **51** (1978), no. 1, 29–44, DOI 10.2307/2689644. MR493766

[16] R. Stein, *A new pentagon tiler*, Math. Magazine **58** (1985), 308.

[17] Q. Yang and C. Zong, *Multiple lattice tilings in Euclidean spaces*, Canad. Math. Bull. **62** (2019), no. 4, 923–929, DOI 10.4153/s0008439518000103. MR4028498

[18] Q. Yang and C. Zong, *Characterization of the two-dimensional five-fold translative tiles*, arXiv:1712.09732, 2017.

[19] C. Zong, *Packing, covering and tiling in two-dimensional spaces*, Expo. Math. **32** (2014), no. 4, 297–364, DOI 10.1016/j.exmath.2013.12.002. MR3279483

[20] C. Zong, *Characterization of the two-dimensional five- and six-fold lattice tiles*, arXiv:1712.01122, 2017, arXiv:1904.06911, 2019.

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