Finding Solitons

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1. Introduction

The concept of soliton provides a useful way to find or discover elements in a given set that are somehow distinguished. Heuristically, only three ingredients are needed to define a soliton:

- A set Γ endowed with some kind of tangent space or space of directions $T_\gamma\Gamma$ at each $\gamma \in \Gamma$.
- An equivalence relation $\simeq$ on Γ collecting in each equivalence class $[\gamma]$ all the elements that cannot be distinguished from γ in relation to the question to be studied.
- An optimal or preferred direction at each point, $q(\gamma) \in T_\gamma\Gamma$, viewed as a "direction of improvement" in some sense.

In that case, $\gamma \in \Gamma$ is called a *soliton* if

$$q(\gamma) \in T_\gamma[\gamma]; \qquad (1)$$

that is, γ is in a way nice enough that it cannot be improved toward $q(\gamma)$ (see Figure 1). The relation $\simeq$ is typically defined by the action of a group H, and if q is H-equivariant, then γ is a soliton if and only if the whole class $[\gamma]$ consists of solitons.

By assuming enough differentiability in the situation, we may consider the evolution differential equation defined by q on Γ,

$$\frac{\partial}{\partial t}\gamma(t) = q(\gamma(t)), \qquad \gamma(0) = \gamma.$$

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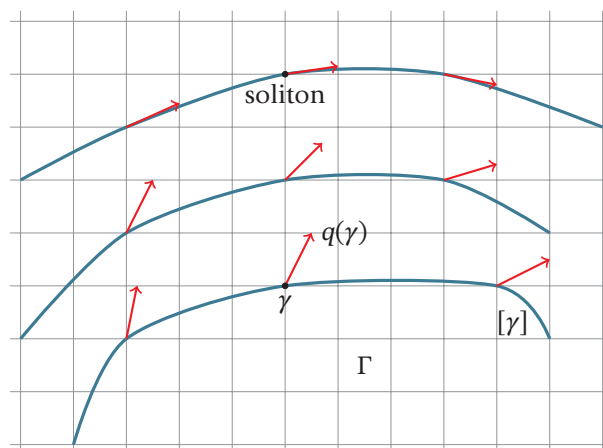
Figure 1. Solitons are not actually improved toward the "direction of improvement."

The existence of solutions is not guaranteed. Solitons are not in general fixed points of this evolution flow (i.e., zeroes of q). However, γ is a soliton if and only if $\gamma(t) \in [\gamma]$ for all t, called a *self-similar* solution (see Figure 1 and the opener image on the left). In other words, solitons are not improved by the flow, and so their existence is not that welcomed if one is hoping to use the flow to find a fixed point in Γ. Indeed, an element may be attracted or stopped in its way to a fixed point by a soliton.

On the other hand, the existence of solitons is great news for the search for canonical or distinguished elements in Γ beyond the zeroes of q.

We note that if $q(\gamma) = -\operatorname{grad}(F)|_\gamma$ for some functional $F : \Gamma \to \mathbb{R}$ that is constant on equivalence classes (or H-invariant), then γ is a soliton if and only if $q(\gamma) = 0$; that is, solitons are precisely the critical points of F or the fixed points of the corresponding flow. Most interesting phenomena occur when this is not the case.

The evocative word *soliton* first appeared in PDE theory in the context of the Korteweg–de Vries equation to name certain solutions resembling solitary water waves that evolve only by translation without losing their shape. More generally, in the study of geometric flows, solitons refer to geometric structures that evolve along symmetries of the flow (i.e., self-similar solutions). The use of the word *soliton* was initiated by Hamilton in the 1980s in the context of Ricci flow to name *Ricci solitons* (see [8]) and nowadays is spread over the fields of differential geometry and geometric analysis.

In this article we discuss and find solitons in many different contexts, including matrices, polynomials, plane curves, Lie group representations (moment maps), and the variety of Lie algebras, as well as in the context of geometric structures (Riemannian, Hermitian, almost-Kähler, and G_2) and their homogeneous versions on Lie groups.[1]

2. Matrices

Consider $\Gamma = \mathfrak{gl}_n$, the vector space of all real $n \times n$ matrices. As is well known, a matrix is semisimple (i.e., diagonalizable over $\mathbb{C}$) if and only if it is conjugate to a normal matrix (i.e., $[A, A^t] = 0$). The subset of normal matrices is invariant under scaling and the action of the orthogonal group $O(n)$ by conjugation, and there is exactly one $O(n)$-orbit of normal matrices in each semisimple conjugacy class. With the aim of finding distinguished matrices other than normal matrices, we consider orthogonal conjugation and scaling as the equivalence relation, that is,

$$[A] := H \cdot A = \{chAh^{-1} : c \in \mathbb{R}^*, \ h \in O(n)\}, \quad H := \mathbb{R}^* O(n).$$

It is easy to see that the tangent space at a matrix A of its conjugacy class $GL_n \cdot A$ is given by $T_A GL_n \cdot A = [A, \mathfrak{gl}_n]$, where GL_n is the group of all invertible real $n \times n$ matrices, so the simplest preferred direction that will make normal matrices solitons is

$$q(A) := [A, [A, A^t]].$$

According to (1), a matrix A is a soliton if and only if

$$[A, [A, A^t]] = cA + [A, B] \in T_A[A], \qquad c \in \mathbb{R}, \quad B \in \mathfrak{so}(n),$$

where $\mathfrak{so}(n)$ denotes the space of skew-symmetric matrices. Since $[A, B] \perp A, A^t$ (relative to the usual inner product $\operatorname{tr} XY^t$), we obtain that A is a soliton if and only if either A is normal ($c = 0$ implies $-|[A, A^t]|^2 = \langle [A, [A, A^t]], A \rangle = 0$) or A is nilpotent ($c \neq 0$ implies $\operatorname{tr} A^k = 0$ for any k) and satisfies the following matrix equation:

$$[A, [A, A^t]] = -\frac{|[A, A^t]|^2}{|A|^2} A.$$

Remark 2.1. Arguing as above, one obtains that the even simpler possibility $q(A) = [A, A^t]$ gives that only normal matrices are solitons.

Besides normal matrices, it is straightforward to check that the following nilpotent matrices are also solitons:

$$\begin{bmatrix} 0 & \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & & \\ \sqrt{2} & 0 & \\ 0 & \sqrt{2} & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & & & \\ \sqrt{3} & 0 & & \\ 0 & 2 & 0 & \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & & & \\ 2 & 0 & & \\ 0 & \sqrt{6} & 0 & \\ 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}, \quad \cdots \quad, \begin{bmatrix} 0 & & & \\ a_1 & 0 & & \\ & \ddots & \ddots & \\ & & a_{k-1} & 0 \end{bmatrix}, \tag{2}$$

where $a_i := \sqrt{i(k-i)}$ (rather than the expected matrices with $a_i = 1$ for all i). Any nilpotent matrix is therefore conjugate to a soliton by using the Jordan canonical form.

An easy computation gives that actually $q(A) := -\frac{1}{4} \operatorname{grad}(E)|_A$, where $E : \mathfrak{gl}_n \to \mathbb{R}$ is the functional $E(B) := |[B, B^t]|^2$ measuring how far a matrix is from being normal. Note that E is not constant on $[A]$ due to scaling, but it is easy to see that A is a soliton if and only if it is a critical point of the normalized functional $\overline{E}(B) := E(B)/|B|^4$.

An interesting characterization of normal matrices, perhaps less known, is that they have minimal norm among their conjugacy classes. Also, it is not hard to show that if $A = S + N$, where S is semisimple, N nilpotent, and $[S, N] = 0$, then $S \in \overline{GL_n \cdot A}$. In particular, if a conjugacy class $GL_n \cdot A$ is closed, then A is necessarily semisimple (also note that $0 \in \overline{GL_n \cdot N}$ for any nilpotent matrix N). The following nice properties of soliton matrices follow from well-known results in geometric invariant theory (GIT for short) and the fact that the moment map (to be defined later) for the GL_n-action on $\mathfrak{gl}_n$ by conjugation is precisely $m(A) = [A, A^t]$:

- For any nilpotent matrix A, there is exactly one $\mathbb{R}^* O(n)$-orbit of solitons in its conjugacy class $GL_n \cdot A$.
- Any soliton A is a minimum of $\overline{E}$ restricted to $GL_n \cdot A$; that is, a nilpotent soliton is in a sense the matrix closest to being normal in its conjugacy class.
- A conjugacy class $GL_n \cdot A$ is closed if and only if A is semisimple, if and only if $GL_n \cdot A$ contains a matrix of minimal norm (or normal matrix).
- The negative gradient flow solution $A(t)$ of the functional $\overline{E}$ starting at A stays in the conjugacy class $GL_n \cdot A$, and if $A = S + N$ as above, then $A(t)$ converges as $t \to \infty$ to either a normal matrix in the conjugacy class of S or, in the case $S = 0$, to a soliton in the conjugacy class of N.

3. Polynomials

The space Γ is now given by $P_{n,d}$, the vector space of all homogeneous polynomials of degree d in n variables with coefficients in $\mathbb{R}$ (e.g., quadratic ($d = 2$) and binary ($n = 2$)

forms). There is a natural left GL_n-action on $P_{n,d}$ given by $h \cdot f := f \circ h^{-1}$ and the inner product for which the basis of monomials

$$\left\{ x^D := x_1^{d_1} \cdots x_n^{d_n} : d_1 + \cdots + d_n = d \right\}, \quad D := (d_1, \dots, d_n),$$

is orthogonal and $\left| x^D \right|^2 := d_1! \cdots d_n!/d!$ is $O(n)$-invariant. The role of normal matrices in the previous section is played here by the renowned *harmonic* polynomials, i.e., $\Delta f = 0$, where Δ is the *Laplace operator* defined by

$$\Delta : P_{n,d} \longrightarrow P_{n,d-2}, \qquad \Delta := \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

Δ is $O(n)$-equivariant, and so the subspace $\mathcal{H}_{n,d} \subset P_{n,d}$ of all harmonic polynomials is invariant under scalings and orthogonal maps. Moreover, $\mathcal{H}_{n,d}$ is known to be $O(n)$-*irreducible* (i.e., the only $O(n)$-invariant subspaces are $\{0\}$ and $\mathcal{H}_{n,d}$).

It is therefore natural to consider the equivalence defined by $H = \mathbb{R}^* O(n) \subset GL_n$ and as a preferred direction at $f \in P_{n,d}$,

$$q(f) := -r^2 \Delta f \in P_{n,d} = T_f P_{n,d},$$

where

$$r^2 := x_1^2 + \cdots + x_n^2 \in P_{n,2}.$$

Note that harmonic polynomials are the fixed points of the corresponding flow. A polynomial f is a soliton (see (1)) if and only if

$$r^2 \Delta f = cf + \theta(A)f \in T_f(H \cdot f), \qquad c \in \mathbb{R}, \quad A \in \mathfrak{so}(n),$$

where $\theta(A)f := \frac{d}{dt}\big|_0 e^{tA} \cdot f$. Are there solitons other than harmonic polynomials?

We first note that

$$P_{n,d} = \mathcal{H}_{n,d} \oplus r^2 \mathcal{H}_{n,d-2} \oplus r^4 \mathcal{H}_{n,d-4} \oplus \cdots \tag{3}$$

is a decomposition of $P_{n,d}$ in irreducible $O(n)$-invariant subspaces[2] (note that r^k is fixed by $O(n)$ for any $k \in \mathbb{N}$). This is suggesting candidates for solitons. For instance, a straightforward computation gives that if $f = r^{2k}g$ with $g \in \mathcal{H}_{n,d-2k}$, then

$$r^2 \Delta f = \lambda_k f, \qquad \text{where } \lambda_k := 2k(2d - 2k + n - 2), \tag{4}$$

and thus f is a soliton.[3] Moreover, by writing any polynomial according to (3) and using that $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$ and $\theta(\mathfrak{so}(n))f \perp f$ for any $f \in P_{n,d}$, it is easy to see that any soliton is actually of this form. Thus the subset of solitons is precisely the union of the $O(n)$-irreducible subspaces in decomposition (3).

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[2]*Since $r^2 P_{n,d-2}$ is the orthogonal complement of $\mathcal{H}_{n,d}$ in $P_{n,d}$, which is straightforward to check, and hence $\Delta P_{n,d} = P_{n,d-2}$.*

[3]*This is strongly related to the fact that $d(d + n - 2)$ is precisely the dth eigenvalue of the Laplace–Beltrami operator on the sphere S^{n-1} with eigenspace $\mathcal{H}_{n,d}|_{S^{n-1}}$.*
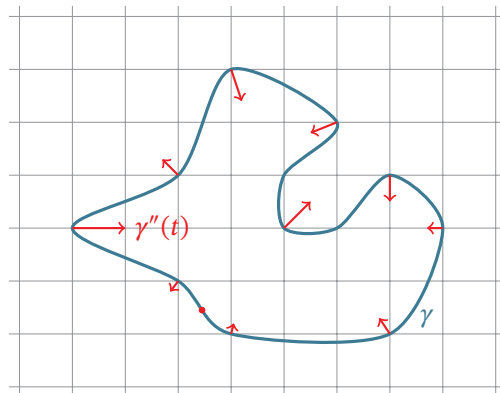

Figure 2. Curvature of a plane curve.

Concerning evolution, given $f = f_j + f_{j+1} + \cdots + f_k$, $f_j, f_k \neq 0$, $j < k$, relative to decomposition (3) (i.e., $f_i \in r^{2i} \mathcal{H}_{n,d-2i}$), the solution to the corresponding flow

$$\frac{d}{dt}f(t) = -r^2 \Delta f(t), \qquad f(0) = f,$$

is given by $f(t) = e^{-t\lambda_j} f_j + \cdots + e^{-t\lambda_k} f_k$. This implies that

$$\lim_{t \to \infty} \frac{1}{|f(t)|} f(t) = \frac{1}{|f_j|} f_j,$$

and so each polynomial in the open and dense subset of $P_{n,d}$ defined by $f_0 \neq 0$ flows to some harmonic polynomial. On the contrary, any polynomial f with $f_0 = 0$ will be stopped in its way to $\mathcal{H}_{n,d}$ by a soliton.

4. Plane Curves

Plane curves naturally flow according to their curvature, and this may be considered as the gene of all geometric flows.

Let Γ be the space of all regular plane curves,

$$\Gamma := \{ \gamma : \mathbb{R} \longrightarrow \mathbb{R}^2 : \gamma \text{ is differentiable} \}.$$

Two curves are considered equivalent if their traces coincide up to rotations, translations, and scaling. We note that an element of $T_\gamma \Gamma$ consists of a vector field along the curve γ or, in other words, a smooth family of vectors, one at each point of the trace of the curve (see Figure 2). After assuming that γ is parametrized by arc length (i.e., $|\gamma'| \equiv 1$), the most natural preferred or optimal direction is its "curvature,"

$$q(\gamma) := \gamma''.$$

Being a measure of how sensitive your constant velocity car γ' is to passing through the point $\gamma(s)$, $\gamma''(s)$ certainly provides a good perception of how curved the trace of γ is at that point (see Figure 2).

The evolution equation defined by this preferred direction is called the *curve shortening flow* (CSF for short); the following are just a few of its several wonderful properties:

- Each of the following kinds of curves is invariant under the flow: embedded, closed, simple, and convex.
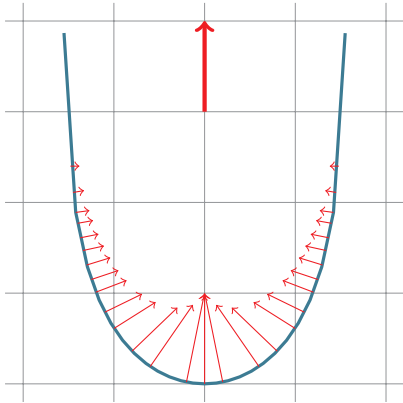
Figure 3. The unique translating soliton is the Grim Reaper found by Calabi, given by the graph of the function $y = -\log(\cos x)$. Its curvature produces the precise gentle breeze needed to move it up without losing its shape.

- For closed curves, the CSF is precisely the negative gradient flow of the length; that is, $q(\gamma) = \gamma''$ is the optimal direction to shorten a closed curve, explaining the name of the flow.
- Grayson proved that under CSF, any simple curve becomes convex (cf. Figure 2 and the opener image on the right), and Gage–Hamilton showed that once it is convex, it converges toward a round point (i.e., asymptotically becoming a circle), collapsing in finite time.

We note that according to the equivalence relation on Γ considered above, a curve is a soliton if and only if it evolves under CSF without losing its shape, i.e., by only a combination of rotations, translations, and scalings (possibly expanding or shrinking). It is therefore easy to convince ourselves that a circle γ is a soliton; indeed, $q(\gamma)$ is in the appropriate sense tangent to the subset of all circles with the same center as γ. It is not so easy, however, to figure out what would be another example of soliton, other than straight lines, which are the trivial solitons with $q = 0$.

The complete classification of CSF-solitons was obtained by Halldorsson in [7]. We give in Figures 3, 4, 5, and 6 examples of all the behaviors that appear. Note that in particular solitons that translate and are scaled at the same time do not exist.
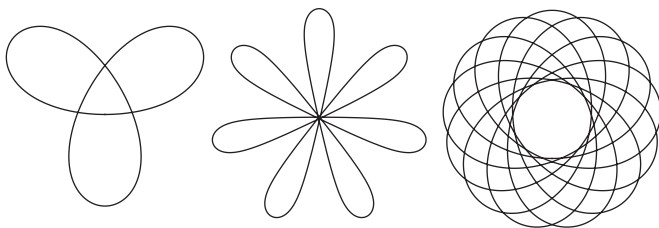

Figure 4. Three shrinking solitons from the infinite discrete family obtained in the classification by Abresch–Langer. Their evolution consists of the very same flower, just reducing its size.
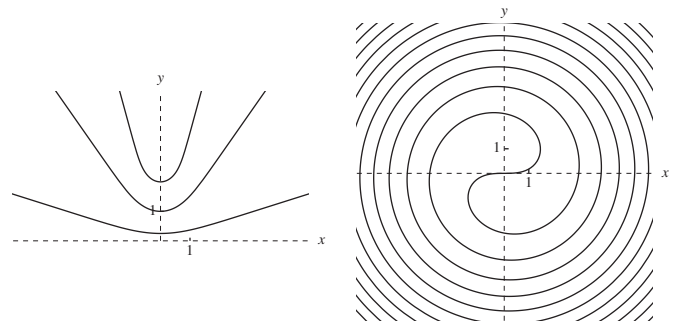

Figure 5. Three expanding solitons (left) and one rotating soliton (right).
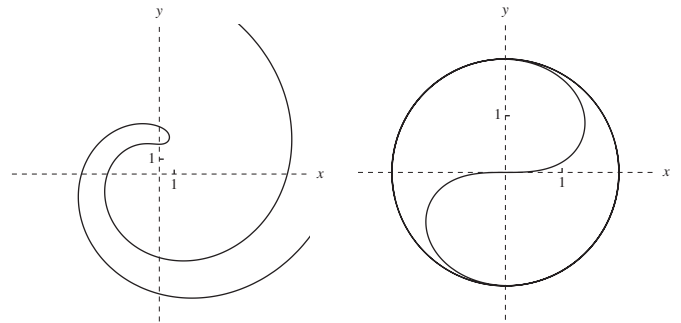

Figure 6. Rotating and expanding soliton (left) and rotating and shrinking soliton (right).

5. Lie Group Representations

As a massive generalization of the matrices example given above, we may consider any linear action of a *Lie group*[4] G on a real vector space V. Thus $\Gamma = V$, and a natural question arises: What would be a distinguished vector $v \in V$ analogous to a normal matrix? If we endow V with an inner product, then natural candidates are *minimal vectors*; i.e., $|v| \le |h \cdot v|$ for any $h \in G$ (recall the characterization of normal matrices as minimal vectors in their conjugacy classes). Note that any closed G-orbit contains a minimal vector. The next question, more intriguing, is, What should the other solitons be, playing the role of nilpotent soliton matrices (see (2)) in this much more general context?

Motivated by the following equation satisfied in the case of matrices,

$$\langle [A, A^t], B \rangle = \frac{1}{2} \left. \frac{d}{dt} \right|_{t=0} \left| e^{tB} A e^{-tB} \right|^2,$$

we fix inner products on the Lie algebra $\mathfrak{g}$ of G and on V and consider for each $v \in V$ the element $\mathrm{m}(v) \in \mathfrak{g}$ implicitly defined by

$$\langle \mathrm{m}(v), X \rangle = \frac{1}{2} \left. \frac{d}{dt} \right|_{t=0} \left| \exp tX \cdot v \right|^2 = \langle \theta(X)v, v \rangle \quad \forall X \in \mathfrak{g}, \tag{5}$$

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[4]*A group that is also, compatibly, a differentiable manifold.*

where $\theta : \mathfrak{g} \to \mathfrak{gl}(V)$ is the corresponding Lie algebra *representation* (i.e., $\theta(X)v := \frac{d}{dt}\big|_0 \exp tX \cdot v \in T_v(G \cdot v)$). Thus $m(v)$ encodes the behavior of the norm of vectors inside the orbit $G \cdot v$ in a neighborhood of v. Moreover, by (5), $-\theta(m(v))v \in T_v(G \cdot v)$ is the direction of fastest norm decreasing tangent to the orbit $G \cdot v$ at v, in the sense that

$$\frac{d}{dt}\Big|_0 |\exp(-t\, m(v)) \cdot v|^2 = -2\langle \theta(m(v))v, v\rangle = -2|m(v)|^2$$

$$\leq 2\langle m(v), X\rangle = \langle 2\theta(X)v, v\rangle$$

$$= \frac{d}{dt}\Big|_0 |\exp tX \cdot v|^2,$$

for any $X \in \mathfrak{g}$ such that $|X| = |m(v)|$, where equality holds if and only if $X = -m(v)$ (note that any minimal vector v satisfies $m(v) = 0$).

We assume from now on that the following conditions on the G-action on V hold. If

$$\mathfrak{k} := \{X \in \mathfrak{g} : \theta(X)^t = -\theta(X)\},$$

$$\mathfrak{p} := \{X \in \mathfrak{g} : \theta(X)^t = \theta(X)\},$$

then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $G = K \exp \mathfrak{p}$, where $K := \{h \in G : v \mapsto h \cdot v \text{ is orthogonal}\}$. It follows that $\mathfrak{g}$ is *reductive* (i.e., semisimple modulo an abelian factor), $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition (i.e., $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$), K is a maximal compact subgroup of G with Lie algebra $\mathfrak{k}$, and the function $K \times \mathfrak{p} \to G$, $(h, X) \mapsto h \exp X$ is a diffeomorphism. By (5), $m(v) \in \mathfrak{p}$ for any $v \in V$, and the function $m : V \to \mathfrak{p}$, called in GIT the *moment map*[5] (or G-gradient map) for the action, is K-equivariant.

Similarly to matrices, we consider $\Gamma = V$, $[v] = K \cdot v$, and as the preferred direction,

$$q(v) := -\operatorname{grad}(E)|_v, \quad \text{where} \quad E : V \setminus \{0\} \to \mathbb{R},$$

$$E(v) := \frac{|m(v)|^2}{|v|^2},$$

is the K-invariant functional measuring how far v is from being a minimal vector. Therefore, solitons are precisely the critical points of E (i.e., $q(v) = 0$), and $\frac{d}{dt}v(t) = q(v(t))$ is the negative gradient flow of the functional E. A straightforward computation gives that

$$\operatorname{grad}(E)|_v = \frac{4}{|v|^2}\left(\theta(m(v))v - |m(v)|^2 v\right);$$

hence v is a soliton if and only if

$$\theta(m(v))v \in \mathbb{R}v. \tag{6}$$

This suggests that, as in the case of matrices, there may be solitons other than minimal vectors, probably having non-closed G-orbits (see examples below). The following are nice and important results from real GIT (see [3, 9]):

[5]*The suggestive name comes from the fact that if we complexify everything, then* m *is precisely the moment map for the Hamiltonian action of K on the complex projective space $P(V_{\mathbb{C}})$.*

- A G-orbit is closed if and only if it contains a minimal vector. The closure of any G-orbit contains exactly one K-orbit of minimal vectors.
- The subset of solitons of a given G-orbit is either empty or consists of exactly one K-orbit (up to scaling).
- Every soliton v is a minimum of the functional E restricted to $G \cdot v$. Solitons are therefore the vectors closest to being a minimal vector in their G-orbit in a sense.
- The negative gradient flow solution of E starting at any $v \in V$ stays in $G \cdot v$ and converges as $t \to \infty$ to a soliton $w \in \overline{G \cdot v}$. Moreover, there is exactly one K-orbit (up to scaling) of solitons $z \in \overline{G \cdot v}$ such that $m(z) \in K \cdot m(w)$, which is the limit set towards which the whole orbit $G \cdot v$ is flowing.

In what follows, we analyze the existence of solitons on some particular examples of representations.

Ternary cubics. Consider $G = \mathrm{SL}_3$ acting on $V = P_{3,3}$, the vector space of all homogeneous polynomials of degree 3 on 3 variables with real coefficients. It follows that $\mathfrak{g} = \mathfrak{sl}_3$, $K = \mathrm{SO}(3)$, $\mathfrak{k} = \mathfrak{so}(3)$, and $\mathfrak{p} = \mathrm{sym}_0(3)$, the space of traceless symmetric 3×3 matrices. It is easy to compute that the moment map $m : P_{3,3} \to \mathrm{sym}_0(3)$ is given by

$$m(f) = I - \frac{1}{|f|^2}\left[\left\langle x_j \frac{\partial f}{\partial x_i}, f\right\rangle\right].$$

Thus $m(x^D) = \mathrm{Diag}(1-d_1, 1-d_2, 1-d_3)$ for any monomial x^D, $D = (d_1, d_2, d_3)$, and so any monomial is a soliton by (6) with critical value $E(x^D) = -1 + \sum d_i^2$ (this holds on any $P_{n,d}$). It also easily follows that $E(p) = 0$ for $p := x_1 x_2 x_3$; that is, p is a minimal vector and its SL_3-orbit is therefore closed. On the other hand, the polynomials

$$g = x_1^2 x_3 + x_1 x_2^2 \quad \text{and} \quad f = x_1^2 x_3 + \left(\frac{5}{27}\right)^{\frac{1}{2}} x_2^3$$

are both solitons with critical values $E(g) = \frac{1}{2}$ and $E(f) = \frac{155}{49} - 3 < \frac{1}{2}$, respectively, which in particular implies that $f \notin \overline{\mathrm{SL}_3 \cdot g}$.

Algebras. We consider the vector space

$$V = \mathcal{A} := \{\mu : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n : \mu \text{ is bilinear}\},$$

parametrizing the set of all n-dimensional algebras over $\mathbb{R}$. Note that isomorphism classes of algebras are precisely G-orbits, where $G = \mathrm{GL}_n$, relative to the standard GL_n-action on V given by $h \cdot \mu := h\mu(h^{-1}\cdot, h^{-1}\cdot)$. The corresponding representation,

$$\theta(A)\mu = A\mu - \mu(A\cdot, \cdot) - \mu(\cdot, A\cdot) \quad \forall A \in \mathfrak{gl}_n, \quad \mu \in \mathcal{A}, \tag{7}$$

measures how far A is from being a derivation of the algebra μ. Thus $\mathfrak{k} = \mathfrak{so}(n)$, $\mathfrak{p} = \mathrm{sym}(n)$, $K = \mathrm{O}(n)$, and it

is straightforward to see that the moment map $\mathrm{m} : \mathcal{A} \to \mathrm{sym}(n)$ is given by

$$\langle \mathrm{m}(\mu)X, X \rangle = -\tfrac{1}{2} \sum \langle \mu(e_i, e_j), X \rangle^2 \tag{8}$$
$$+ \tfrac{1}{8} \sum \langle \mu(X, e_i), e_j \rangle^2$$
$$+ \tfrac{1}{8} \sum \langle \mu(e_i, X), e_j \rangle^2 \quad \forall X \in \mathbb{R}^n.$$

Since $\operatorname{tr} \mathrm{m}(\mu) = -|\mu|^2$ by (5) and (7), the only minimal vector is the trivial algebra $\mu = 0$, which is also the only closed GL_n-orbit (note that actually $0 \in \overline{\mathrm{GL}_n \cdot \mu}$ for any $\mu \in \mathcal{A}$). However, there are closed SL_n-orbits and SL_n-minimal vectors.

According to (6), an algebra product $\mu \in \mathcal{A}$ is a soliton if and only if the following nice compatibility condition between μ and the fixed inner product $\langle \cdot, \cdot \rangle$ holds:

$$\mathrm{m}(\mu) = cI + D, \qquad c \in \mathbb{R}, \quad D \in \mathrm{Der}(\mu). \tag{9}$$

Which algebras are isomorphic to a soliton? How special are they?

Lie algebras. We now list only a few of several known results on solitons in the case of Lie algebras (see [12]). Note that the set of all n-dimensional Lie algebras is parametrized by the GL_n-invariant algebraic subset

$$\mathcal{L} \subset \mathcal{A} \tag{10}$$

of all algebras that are in addition skew-symmetric and satisfy the Jacobi condition,[6] called the *variety of Lie algebras*.

- $\mathrm{SL}_n \cdot \mu$ is closed if and only if μ is semisimple. Moreover, μ is an SL_n-minimal vector if and only if the Killing form B_μ is either a negative multiple of $\langle \cdot, \cdot \rangle$ and μ is compact semisimple, or B_μ has exactly two opposite eigenvalues (relative to $\langle \cdot, \cdot \rangle$) and the eigenspace decomposition is a Cartan decomposition.
- There is a soliton in the isomorphism class of each of the fifty nilpotent Lie algebras of dimension ≤ 6 (see [20]). In dimension 7, there are infinitely many nilpotent Lie algebras that are not isomorphic to a soliton, and a complete classification was obtained in [6].
- The only known general obstruction in the nilpotent case is that any soliton μ has to admit an $\mathbb{N}$-gradation. Everything seems to indicate that a full structural characterization of nilpotent solitons may be hopeless.
- A Lie algebra μ is a soliton if and only if its nilradical $\mathfrak{n}$ is a soliton and the orthogonal complement $\mathfrak{r}$ of $\mathfrak{n}$ is a reductive Lie algebra such that $\mathrm{ad}_\mu X|_\mathfrak{n}^t \in \mathrm{Der}(\mathfrak{n})$ for any $X \in \mathfrak{r}$. In that case, $\mathrm{ad}_\mu X|_\mathfrak{n}$ is a normal operator for any X in the center of $\mathfrak{r}$ and the subspaces $\mathfrak{k} := \{X : \mathrm{ad}_\mu X|_\mathfrak{n}^t = -\mathrm{ad}_\mu X|_\mathfrak{n}\}$ and $\mathfrak{p} := \{X : \mathrm{ad}_\mu X|_\mathfrak{n}^t = \mathrm{ad}_\mu X|_\mathfrak{n}\}$ give rise to a Cartan decomposition $[\mathfrak{r}, \mathfrak{r}] = \mathfrak{k} \oplus \mathfrak{p}$ of the semisimple Lie subalgebra $[\mathfrak{r}, \mathfrak{r}]$.

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[6] $\mu(\mu(e_i, e_j), e_k) + \mu(\mu(e_j, e_k), e_i) + \mu(\mu(e_k, e_i), e_j) = 0$ *for all i, j, k.*

The study of soliton Lie algebras was strongly motivated by their relationship with left-invariant Ricci solitons and Einstein metrics on Lie groups (see [12]). The author is not aware of any study of solitons in other classes of algebras, such as associative or Jordan algebras.

6. Geometric Structures

Let M be a differentiable manifold. We consider the space Γ of all geometric structures on M of a given type, e.g., Riemannian metrics, almost-Hermitian structures, G_2-structures, etc. As usual, Γ is identified with a subset of the vector space $\mathcal{T}^{r,s}M$ of all tensor fields of some type (r, s), or tuples of tensors, and the equivalence relation is scaling and pulling back by diffeomorphisms. Thus the equivalence class of $\gamma \in \Gamma$ is determined by the natural action of the group $H := \mathrm{Diff}(M) \times \mathbb{R}^*$ on tensor fields:

$$[\gamma] := H \cdot \gamma = \{ch^*\gamma : c \in \mathbb{R}^*, \ h \in \mathrm{Diff}(M)\}.$$

The preferred direction

$$\gamma \mapsto q(\gamma) \in T_\gamma \Gamma \subset \mathcal{T}^{r,s}M$$

is typically given by a curvature tensor associated to some affine connection associated with γ, or the gradient of a natural geometric functional, or the Hodge–Laplacian on differential forms, etc. Thus q is in most cases diffeomorphism equivariant, i.e., $q(h^*\gamma) = h^*q(\gamma)$ for any $h \in \mathrm{Diff}(M)$, which implies that if γ is a soliton, then any $h^*\gamma$ is also a soliton.

Γ is many times open in a vector subspace $\mathcal{T} \subset \mathcal{T}^{r,s}M$, in which case one has that $T_\gamma \Gamma = \mathcal{T}$, and so any tensor field in $\mathcal{T}$ can be the "direction of improvement" $q(\gamma)$ at the structure $\gamma \in \Gamma$.

Once the space Γ and the preferred direction q have been specified, it follows from (1) that $\gamma \in \Gamma$ is a soliton if and only if

$$q(\gamma) \in T_\gamma(H \cdot \gamma), \qquad H = \mathrm{Diff}(M) \times \mathbb{R}^*,$$

which is equivalent to

$$q(\gamma) = c\gamma + \mathcal{L}_X \gamma, \qquad c \in \mathbb{R}, \quad X \in \mathfrak{X}(M), \tag{11}$$

where $\mathcal{L}_X$ denotes the Lie derivative with respect to the vector field X of M. Indeed, recall that if X is defined by a one-parameter family $f(t) \in \mathrm{Diff}(M)$ with $f(0) = \mathrm{id}$, then $\frac{d}{dt}\big|_0 f(t)^*\gamma = \mathcal{L}_X \gamma$.

Example 6.1 (Ricci solitons). Consider $\Gamma = \mathcal{M}$, the space of all Riemannian metrics on M. Thus $\mathcal{M}$ is open in $\mathcal{T} = \mathcal{S}^2 M \subset \mathcal{T}^{2,0}M$, the vector space of all symmetric 2-tensors on M. A natural preferred direction is $q(g) := -2\,\mathrm{Ric}_g$, where Ric_g is the Ricci tensor of the metric $g \in \mathcal{M}$, giving rise to the well-known Ricci solitons (see [5, Chapter 1]). Note that the corresponding evolution equation is precisely the famous Ricci flow $\frac{\partial}{\partial t}g(t) = -2\,\mathrm{Ric}_{g(t)}$ introduced in the 1980s by Hamilton and used as a primary

tool by Perelman to prove the Poincaré and geometrization conjectures.

In the case we want to consider a space Γ of geometric structures satisfying some extra properties, e.g., an integrability-like condition such as Hermitian, almost-Kähler, or closedness/coclosedness for G_2-structures, we have to reduce accordingly the group $H \subset \mathrm{Diff}(M) \times \mathbb{R}^*$ determining the equivalence between structures. The possibilities for preferred directions also decrease, and the vector field X in the definition of soliton (11) must be tangent to H as $q(\gamma) \in T_\gamma(H \cdot \gamma)$. Note that q is assumed to be only H-equivariant in this situation rather than diffeomorphism equivariant. On the other hand, Γ is no longer open in $\mathcal{T}$.

Example 6.2. If a complex manifold (M, J) is fixed and Hermitian metrics or any other kind of geometric structures on (M, J) are to be considered, then $H = \mathrm{Aut}(M, J) \times \mathbb{R}^*$, where $\mathrm{Aut}(M, J)$ is the group of bi-holomorphic diffeomorphisms, q is assumed to be only $\mathrm{Aut}(M, J)$-equivariant, and X has to be a holomorphic field. Analogously, in the symplectic case, $H = \mathrm{Aut}(M, \omega)$, the group of symplectomorphisms of a fixed symplectic manifold (M, ω).

Concerning the associated evolution equation,

$$\frac{\partial}{\partial t}\gamma(t) = q(\gamma(t)), \qquad \gamma(0) = \gamma, \qquad (12)$$

one easily obtains that γ is a soliton if and only if

$$\gamma(t) = c(t)f(t)^*\gamma, \qquad c(t) \in \mathbb{R}, \quad f(t) \in \mathrm{Diff}(M); \quad (13)$$

that is, $\gamma(t)$ is a self-similar solution. It is worth pointing out at this point that the natural preferred direction q chosen may or may not produce a flow, as the existence of solutions to the PDE (12) is not always guaranteed. So possibly, a study of solitons can be worked out without any reference to a flow. An example of this situation is Ricci solitons in pseudo-Riemannian geometry (see [4]). On the other hand, even though the flow is not defined on the whole space Γ, there may be special subclasses $\Gamma' \subset \Gamma$ on which solutions to (12) do exist, e.g., homogeneous structures (see below).

Assuming that the scaling behavior of the preferred direction q is given by $q(c\gamma) = c^\alpha\gamma$ for any $c \in \mathbb{R}^*$, $\gamma \in \Gamma$, for some fixed $\alpha < 1$, it is easy to check that the scaling in (13) is given by

$$c(t) = ((1 - \alpha)ct + 1)^{\frac{1}{1-\alpha}},$$

where c is the constant appearing in the soliton equation (11). The soliton γ is therefore called *expanding, steady*, or *shrinking* depending on whether $c > 0$, $c = 0$, or $c < 0$. The maximal time intervals of the self-similar solutions are

respectively given by

$$(-T_\alpha, \infty), \quad (-\infty, \infty), \quad (-\infty, T_\alpha),$$

$$\text{where} \quad T_\alpha := \frac{1}{(1-\alpha)|c|} > 0,$$

often called *immortal, eternal*, and *ancient* solutions, respectively. For instance, $\alpha = 0$ if q is the Ricci tensor or form of any connection associated to a metric or to an almost-Hermitian structure, and $\alpha = \frac{1}{3}$ for most of the known flows for G_2-structures.

In what follows, we give an overview of different kinds of solitons in complex, symplectic, and G_2 geometries.

Chern–Ricci solitons. For a given complex manifold (M, J), consider the space Γ of all Hermitian metrics on M (or J-invariant, i.e., $g(J\cdot, J\cdot) = g$). Thus $\mathcal{T} \subset \mathcal{T}^{2,0}M$ is the vector space of holomorphic (or J-invariant) symmetric 2-tensors, and the group providing the equivalence relation is the subgroup $H \subset \mathrm{Diff}(M)$ of bi-holomorphic diffeomorphisms (i.e., $h^*J = J$). The Chern connection ∇^C, being the only *Hermitian connection* (i.e., $\nabla^C g = 0$, $\nabla^C J = 0$) with an anti-J-invariant torsion, provides us with the natural preferred direction $q(g) := -2\,\mathrm{Ric}_g^C \in \mathcal{T}$, where Ric_g^C is the corresponding Chern–Ricci tensor of the Hermitian metric g. Note that g is a soliton, called the *Chern–Ricci soliton*, if and only if $\mathrm{Ric}_g^C = cg + \mathcal{L}_X g$, for some $c \in \mathbb{R}$ and holomorphic vector field X on M. The corresponding Chern–Ricci flow was introduced by Gill and has been studied by Tosatti–Weinkove among others. Examples of homogeneous Chern–Ricci solitons were given in [16].

Remark 6.3. Another possible preferred direction for Hermitian metrics is to just take the J-invariant part of the Ricci tensor, given by

$$q(g) := -\mathrm{Ric}_g^{1,1} = -\frac{1}{2}\left(\mathrm{Ric}_g + \mathrm{Ric}_g(J\cdot, J\cdot)\right).$$

This choice does not give rise to any geometric flow on the set of all Hermitian metrics. However, as recently shown by Lafuente–Pujia–Vezzoni, it coincides with the *Hermitian curvature flow* (HCF for short) introduced by Streets–Tian among left-invariant Hermitian metrics on complex unimodular Lie groups.

Pluriclosed solitons. Consider now on (M, J) the space Γ of all Hermitian metrics on M that satisfy the *pluriclosed* condition $\partial\bar{\partial}\omega = 0$, where $\omega = g(J\cdot, \cdot) \in \Omega^2 M$ (also called SKT metrics). A natural preferred direction here is given by $q(g) := -\left(\mathrm{Ric}_g^B\right)^{1,1}$, where Ric_g^B denotes the Bismut–Ricci tensor of g associated to the Bismut connection.[7] Pluriclosed solitons (i.e., $\mathrm{Ric}_g^B = cg + \mathcal{L}_X g$) and the corresponding pluriclosed flow, which coincides with HCF

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[7]*The only Hermitian connection whose torsion satisfies that $(X, Y, Z) \mapsto g(T^B(X, Y), Z)$ is a 3-form.*

on SKT metrics, have been studied by several authors (see [1]).

Anticomplexified Ricci solitons. Let (M, ω) be a symplectic manifold and let Γ denote the space of all compatible metrics (i.e., $J^2 = -I$ if $\omega = g(J\cdot, \cdot)$). For each $g \in \Gamma$, the pair (ω, g) is called an *almost-Kähler structure* (in other words, an almost-Hermitian structure (ω, g, J) such that $d\omega = 0$). It follows that $\mathcal{T}$ is the vector space of anti-J-invariant symmetric 2-tensors (i.e., $q(J\cdot, J\cdot) = -q$) and $H \subset \text{Diff}(M)$ is the subgroup of symplectomorphisms (i.e., $h^*\omega = \omega$). As a preferred direction, the simplest option is

$$q(g) := \text{Ric}_g^{2,0+0,2} = \tfrac{1}{2}\left(\text{Ric}_g - \text{Ric}_g(J\cdot, J\cdot)\right),$$

the anti-J-invariant part of the Ricci tensor Ric_g. This flow was introduced by Le–Wang, and examples of homogeneous solitons were given by Fernández–Culma (see [13]).

Symplectic curvature flow solitons. We now consider the larger and trickier space Γ of all almost-Hermitian structures (i.e., a 3-tuple (ω, g, J) such that $g = \omega(J\cdot, \cdot)$) on a manifold M. It is easy to see that in this case

$$\mathcal{T} = \left\{ (\overline{\omega}, \overline{g}) \in \Omega^2 M \times \mathcal{S}^2 M : \overline{g}^{1,1} = \overline{\omega}^{1,1}(\cdot, J\cdot) \right\}, \quad (14)$$

and we take the full $H = \text{Diff}(M) \times \mathbb{R}_{>0}$, so the equivalence class of an almost-Hermitian structure $\gamma = (\omega, g)$ is given by $[(\omega, g)] = \{(ch^*\omega, ch^*g) : c \in \mathbb{R}^*, \ h \in \text{Diff}(M)\}$.

As above, we consider the Chern–Ricci tensor $\text{Ric}_{(\omega,g)}^C$ and the corresponding Chern–Ricci form $p(\omega, g) := \text{Ric}_{(\omega,g)}^C(J\cdot, \cdot)$, which is a very natural preferred direction at ω to choose (indeed, $dp = 0$ if $d\omega = 0$, so p is tangent to the set of almost-Kähler structures). According to (14), the J-invariant part of the preferred direction at g must be given by $p(\omega, g)^{1,1}(\cdot, J\cdot)$; hence it only remains to set the anti-J-invariant part, for which we can just choose the simplest one considered above. In this way, we arrive at the following natural preferred direction:

$$q(\omega, g) := \left(p(\omega, g),\ p(\omega, g)^{1,1}(\cdot, J\cdot) + \text{Ric}_g^{2,0+0,2} \right) \in \mathcal{T}.$$

Thus (ω, g) is a soliton if and only if there exist $c \in \mathbb{R}$ and $X \in \mathfrak{X}(M)$ such that

$$\begin{cases} p(\omega, g) = c\omega + \mathcal{L}_X \omega, \\ p^{1,1}(\cdot, J\cdot) + \text{Ric}^{2,0+0,2} = cg + \mathcal{L}_X g. \end{cases}$$

The flow is called *symplectic curvature flow* (SCF for short) and was introduced by Streets–Tian; see [17] for a study of homogeneous SCF-solitons. The fact that the three structures ω, g, and J are actually evolving is particularly challenging.

Remark 6.4. The classification of complex surfaces is a major problem motivating the study of all the above flows among different subclasses of structures. They all coincide with the Kähler–Ricci flow among *Kähler* structures (i.e., $\nabla J = 0$ for the Levi–Civita connection ∇ of g).

Laplacian solitons. A G_2-*structure* on a 7-dimensional differentiable manifold M is a differential 3-form φ that is positive (or definite), in the sense that φ (uniquely) determines a Riemannian metric g on M together with an orientation. The space Γ of all G_2-structures on M is an open subset of $\mathcal{T} = \Omega^3 M \subset \mathcal{T}^{3,0}$, and the equivalence is determined by $H = \text{Diff}(M) \times \mathbb{R}^*$. A very natural preferred direction is $q(\varphi) = \Delta_\varphi \varphi$, where $\Delta_\varphi = *d*d - d*d*$ denotes the Hodge Laplace operator on forms and $*$ the Hodge star operator attached to g and the orientation. Indeed, if M is compact and $\Delta_\varphi \varphi = 0$, then g is Ricci flat and has holonomy group contained in the exceptional compact simple Lie group G_2.

The corresponding *Laplacian flow* $\frac{\partial}{\partial t}\varphi(t) = \Delta_{\varphi(t)}\varphi(t)$ was introduced back in 1992 by Bryant as a tool to try to deform closed G_2-structures toward holonomy G_2 and has recently been deeply studied by Lotay–Wei (see [18]). We refer to [14] for an account of the existence and structure of Laplacian solitons. Interestingly, the shrinking Laplacian solitons found on certain solvable Lie groups are the only Laplacian flow solutions with a finite-time singularity known so far.

7. Geometric Structures on Lie Groups

The role of (locally) homogeneous manifolds in Ricci flow theory has been very important (see [2]). More recently, Lie groups have also played an even stronger role in the study of geometric flows in complex, symplectic, and exceptional holonomy geometries, due mainly to the lack of explicit examples (see [13]).

We continue in this section the study of solitons in differential geometry initiated above. However, our fixed manifold is here a *Lie group*[8] G, and we assume that Γ consists of *left-invariant* geometric structures, i.e., $L_a^* \gamma = \gamma$ for any $a \in G$, where L_a is the diffeomorphism of G defined by $L_a(b) = ab$ for all $\gamma \in \Gamma$. Thus each γ is determined by its value at the identity $e \in G$ and so is identified with the tensor γ_e (or a tuple of tensors) on the Lie algebra $\mathfrak{g} = T_e G$ of G. In this way,

$$\Gamma \subset T^{r,s}\mathfrak{g},$$

the finite-dimensional vector space of all left-invariant tensor fields of type (r, s) on G. Note that Γ is usually contained in a single GL($\mathfrak{g}$)-orbit, which is many times open in some suitable vector subspace $T \subset T^{r,s}\mathfrak{g}$ (e.g., inner products, nondegenerate 2-forms, almost-Hermitian structures, SU(3)-structures, positive 3-forms, etc.).

Accordingly, we consider the equivalence between left-invariant structures to be defined by scalings and the particular diffeomorphisms of G that are also group morphisms, i.e., by the group

$$H := \text{Aut}(G) \times \mathbb{R}^*,$$

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[8]*A differentiable manifold that is also, compatibly, a group.*

which is identified with $\mathrm{Aut}(\mathfrak{g}) \times \mathbb{R}^*$, where $\mathrm{Aut}(\mathfrak{g})$ is the automorphism group of $\mathfrak{g}$, as G is assumed to be simply connected from now on.

Any preferred direction q from the general case that is diffeomorphism equivariant produces a left-invariant one. Indeed, one obtains a preferred direction determined by a function

$$\gamma \mapsto q(\gamma) \in T_\gamma \Gamma \subset T^{r,s}\mathfrak{g}.$$

We therefore have that $\gamma \in \Gamma$ is a soliton, called a *semi-algebraic soliton* in the literature, if and only if

$$q(\gamma) = c\gamma + \mathcal{L}_{X_D}\gamma, \qquad c \in \mathbb{R}, \quad D \in \mathrm{Der}(\mathfrak{g}), \qquad (15)$$

where $X_D \in \mathfrak{X}(G)$ is defined at each point $a \in G$ by $X_D(a) = \frac{d}{dt}\big|_{t=0} f(t)(a)$ and $f(t) \in \mathrm{Aut}(G)$ is determined by $df(t)|_e = e^{tD} \in \mathrm{Aut}(\mathfrak{g})$ (since $X_D(e) = 0$, these fields are never left invariant). Note that (G, γ) is also a soliton from the general point of view considered in (11). It is easy to check that, algebraically, the Lie derivative is simply given by

$$\mathcal{L}_{X_D}\gamma = -\theta(D)\gamma, \qquad (16)$$

where θ denotes the usual $\mathfrak{gl}(\mathfrak{g})$-representation on tensors. As in the general case, additional conditions on the derivation D may apply if Γ satisfies extra properties (see Example 6.2).

Remark 7.1. These vector fields X_D on Lie groups attached to a derivation D may be viewed as a generalization of *linear vector fields* on $\mathbb{R}^n$ (i.e., $X_v = Av$, $A \in \mathfrak{gl}_n$) and have been strongly used in control theory since a pioneering article by Ayala–Tirao.

It is important to point out that in the Lie group case considered in this section, the corresponding geometric flow

$$\frac{d}{dt}\gamma(t) = q(\gamma(t)), \qquad \gamma(0) = \gamma, \qquad (17)$$

is actually an ODE rather than a PDE. In particular, short time existence and uniqueness of solutions are always guaranteed. It is also known that $|q(\gamma(t))|$ must blow up at any finite-time singularity (see [13]). Note that γ is a semialgebraic soliton if and only if the solution $\gamma(t)$ to (17) is given by $\gamma(t) = c(t)f(t)^*\gamma$ for some $c(t) \in \mathbb{R}$ and $f(t) \in \mathrm{Aut}(G)$.

Due perhaps to its neat definition as a combination of geometric and algebraic aspects of (G, γ) (cf. (15) and (16)), the concept of semialgebraic soliton has a long and fruitful history in the Ricci flow case (see [10–12]) and has also been a quite useful tool to address the existence problem of soliton structures for all the geometric flows in complex, symplectic, and G_2 geometries given above.

Remark 7.2. Given a semialgebraic soliton (G, γ), if G has a cocompact discrete subgroup Λ, then the solution $\gamma(t)$ also solves (17) on the compact manifold G/Λ. However, in general, the locally homogeneous manifold $(G/\Lambda, \gamma)$ is no longer a soliton, since the field X_D does not descend to G/Λ. The solution $(G/\Lambda, \gamma(t))$ is very peculiar though: it is "locally self-similar" in the sense that $\gamma(t)$ is locally equivalent to γ up to scaling for all t.

The moving-bracket approach. The following viewpoint is suggested by the fact that all the geometric information on a Lie group endowed with a left-invariant geometric structure, say (G, γ), is encoded in just the tensor $\gamma \in T^{r,s}\mathfrak{g}$ and the Lie bracket μ of $\mathfrak{g}$. We consider the variety of Lie algebras $\mathcal{L} \subset \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$ as in (10) (i.e., the algebraic subset of all Lie brackets on the vector space $\mathfrak{g}$) and fix a suitable tensor γ on $\mathfrak{g}$. Each $\mu \in \mathcal{L}$ is therefore identified with (G_μ, γ), the simply connected Lie group G_μ with Lie algebra $(\mathfrak{g}, \mu)$ endowed with the left-invariant geometric structure on G_μ defined by the fixed γ:

$$\mu \longleftrightarrow (G_\mu, \gamma). \qquad (18)$$

The natural $\mathrm{GL}(\mathfrak{g})$-actions on tensors provide the following key equivalence between geometric structures:

$$(G_{h\cdot\mu}, \gamma) \underset{\simeq}{\to} (G_\mu, h^*\gamma) \quad \forall h \in \mathrm{GL}(\mathfrak{g}), \qquad (19)$$

given by the Lie group isomorphism $G_{h\cdot\mu} \to G_\mu$ with derivative h^{-1}. Since $\Gamma \subset \mathrm{GL}(\mathfrak{g}) \cdot \gamma$, it follows from (18) and (19) that the isomorphism class $\mathrm{GL}(\mathfrak{g}) \cdot \mu$ contains all geometric structures of the same type of γ (up to equivalence) on the Lie group G_μ for each $\mu \in \mathcal{L}$. Thus one has inside $\mathcal{L}$, all together, all Lie groups of a given dimension endowed with left-invariant geometric structures of a given type.

Remark 7.3. The usual convergence of a sequence of brackets produces convergence of the corresponding geometric structures in well-known senses such as pointed (or Cheeger–Gromov) and smooth up to pull-back by diffeomorphisms, under suitable conditions (see [13]). In particular, a degeneration (i.e., $\lambda \in \overline{\mathrm{GL}(\mathfrak{g}) \cdot \mu} \setminus \mathrm{GL}(\mathfrak{g}) \cdot \mu$) gives rise to the convergence of a sequence of geometric structures on a given Lie group toward a structure on a different Lie group, which may be nonhomeomorphic.

The moving-bracket approach has actually been used for decades in homogeneous geometry (see [13, Section 5] and [15]). In most applications, concepts and results from GIT, including moment maps and their convexity properties, closed orbits, stability, categorical quotients, and Kirwan stratification, have been exploited in one way or another.

A particularly fruitful interplay occurs in the Riemannian case, which relies on the fact that if μ is nilpotent, then the Ricci operator of $(G_\mu, \langle \cdot, \cdot \rangle)$ is precisely the moment map $\mathrm{m}(\mu)$ defined in (8) (up to scaling). This implies that, remarkably, soliton nilpotent Lie algebras (see (9)) and semialgebraic Ricci solitons (see (15) and (16)) on nilpotent Lie groups (called *nilsolitons*) are the same thing. In particular, the uniqueness up to isometry and scaling of nilsolitons on a given nilpotent Lie group follow from the

uniqueness of critical points of $E(\mu) = |m(\mu)|^2$ on a given nilpotent $GL(\mathfrak{g})$-orbit up to the action of $O(\mathfrak{g})$ and scaling.

The bracket flow. Provided by equivalence (19), a main tool to study the geometric flow (17) is a dynamical system defined on the variety of Lie algebras $\mathcal{L}$ called the *bracket flow*, which is equivalent in a precise sense to the geometric flow (17). It is defined by

$$\frac{d}{dt}\mu(t) = \theta(Q_{\mu(t)})\mu(t), \quad \mu(0) = \mu, \tag{20}$$

where $Q_\mu \in \mathfrak{gl}(\mathfrak{g})$ is a suitable (unique) operator such that $\theta(Q_\mu)\gamma = q(G_\mu, \gamma)$.[9] Since $\frac{d}{dt}\mu(t) \in T_{\mu(t)}(GL(\mathfrak{g}) \cdot \mu(t))$, the solution $\mu(t) \in GL(\mathfrak{g}) \cdot \mu$ for all t, and so each $\mu(t)$ represents a structure on G_μ. However, $\mu(t)$ may converge to a Lie bracket $\lambda \in \overline{GL(\mathfrak{g}) \cdot \mu}$, i.e., toward a structure on a different Lie group G_λ (cf. Remark 7.3). For instance, this occurs already in dimension 3 for the Ricci flow.

The bracket flow is useful to better visualize the possible pointed limits of solutions under diverse rescalings, as well as to address regularity issues. Immortal, ancient, and self-similar solutions naturally arise from the qualitative analysis of the bracket flow (see [1, 2, 13, 17]).

Algebraic solitons. In light of the equivalence between the flows (17) and (20), a natural question arises: How do solitons evolve according to the bracket flow? It is natural to expect an evolution of a very special kind.

If μ is a fixed point (up to scaling) of the bracket flow (20), i.e., $\mu(t) = c(t)\mu$ for $c(t) \in \mathbb{R}$, then $\theta(Q_\mu)\mu = -c\mu$ for some $c \in \mathbb{R}$ by evaluating (20) at $t = 0$, and one obtains from (7) that

$$Q_\mu = cI + D, \qquad c \in \mathbb{R}, \quad D \in \text{Der}(\mu). \tag{21}$$

In that case, (G_μ, γ) is called an *algebraic soliton*. Note that these are semialgebraic solitons, since by (16), $q(G_\mu, \gamma) = \bar{c}\gamma - \mathcal{L}_{X_D}\gamma$ for some $\bar{c} \in \mathbb{R}$. They are distinguished though; indeed, it is proved in [15] that in terms of the operator Q_μ, (G_μ, γ) is a semialgebraic soliton if and only if $Q_\mu = cI + p(D)$ for some $c \in \mathbb{R}$ and $D \in \text{Der}(\mu)$, where $p : \mathfrak{gl}(\mathfrak{g}) \to \mathfrak{q}$ is the projection with respect to the decomposition[10]

$$\mathfrak{gl}(\mathfrak{g}) = \mathfrak{k} \oplus \mathfrak{q}, \qquad \mathfrak{k} := \{A \in \mathfrak{gl}(\mathfrak{g}) : \theta(A)\gamma = 0\}.$$

Thus algebraic solitons are the solitons for which $Q_\mu = cI + p(D)$ holds for a special derivation D such that $p(D) = D$ (see (21)).

Any Ricci soliton on a Lie group is isometric to an algebraic soliton (see [10]). On the other hand, examples of Laplacian and pluriclosed semialgebraic solitons that are not isometric to any algebraic soliton were found in [19] and [1], respectively.

As expected, algebraic solitons are distinguished from many other points of view. Some of the results supporting this follow:

- Consider the Ricci pinching functional

$$F(g) := \frac{\text{scal}_g^2}{|\text{Ric}_g|^2},$$

measuring in a sense how far a homogeneous metric g is from being Einstein (indeed, $F(g) \leq \dim M$, and equality holds if and only if (M, g) is Einstein). As shown by Lauret–Will, algebraic Ricci solitons are precisely the global maxima for F restricted to the set of all left-invariant metrics on any unimodular Lie group, as well as on any solvable Lie group with codimension one nilradical.

- Böhm–Lafuente proved that the dimension of the isometry group of an algebraic Ricci soliton on a solvable Lie group S (called *solvsolitons*) is maximal among all left-invariant metrics on S. A stronger symmetry maximality condition was shown to hold in the case when S is in addition unimodular by Jablonski: the isometry group of a solvsoliton contains all possible isometry groups of left-invariant metrics on S up to conjugation by a diffeomorphism.

- A closed G_2-structure φ is called *extremally Ricci-pinched* (ERP for short) if

$$d\tau = \frac{1}{6}|\tau|^2\varphi + \frac{1}{6} * (\tau \wedge \tau),$$

where $\tau = -*d*\varphi$ is the torsion 2-form of φ. They are characterized in the compact case as the structures at which equality holds in the following Ricci curvature estimate for closed G_2-structures discovered by Bryant:

$$\int_M \text{scal}^2 *1 \leq 3 \int_M |\text{Ric}|^2 * 1. \tag{22}$$

It was proved by Lauret–Nicolini that any left-invariant ERP G_2-structure on a Lie group is necessarily a steady algebraic Laplacian soliton and its attached metric is an expanding algebraic Ricci soliton.

Concerning bracket flow evolution of a semialgebraic soliton that is not algebraic, we know that $\mu(t)/|\mu(t)|$ is either periodic or not periodic and the following chaotic behavior occurs: for each t_0 there exists a sequence $t_k \to \pm\infty$ such that $\mu(t_k)/|\mu(t_k)|$ converges to $\mu(t_0)/|\mu(t_0)|$ (see [15]). The existence of a soliton of this last kind is an open problem.

Remark 7.4. More generally, the whole picture developed in this section essentially works for G-invariant geometric structures on a homogeneous space G/K, though a more technical exposition would be necessary (see [13, 15]).

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[9]*The existence of such an operator relies on the fact that $GL(\mathfrak{g}) \cdot \gamma$ is open in T.*
[10]*For instance, $p(D) = \frac{1}{2}(D + D^t)$ if γ is a metric or a closed G_2-structure.*

8. Concluding Remarks

As discussed in the Introduction, solitons play the role of "best" elements in a given set in the case when the most natural ones are not available. A main aim of this article was to show how fruitful this has been in the study of geometric structures on manifolds, with particular strength on Lie groups.

On each solvable Lie group, there is at most one solvsoliton up to isometry and scaling. This allows us to endow several Lie groups that do not admit Einstein metrics (e.g., nilpotent or unimodular solvable Lie groups) with a canonical Riemannian metric. Analogously, Chern–Ricci, pluriclosed, and HCF (resp., SCF) algebraic solitons provide distinguished Hermitian (resp., almost-Kähler) structures for Lie groups on which Kähler metrics do not exist. Laplacian algebraic solitons play the same role in the homogeneous case, where holonomy G_2 is out of reach since Ricci flat implies flat.

The moving-bracket approach allows the rich interplay between soliton geometric structures on Lie groups and soliton Lie algebras, paving the way to many beautiful applications of GIT to differential geometry.

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