WHAT IS...

a Minuscule Representation?

Richard M. Green

A minuscule representation is a particular type of highest weight representation. Highest weight representations play a key role in the representation theory of several classes of algebraic objects occurring in Lie theory, including Lie algebras, Lie groups, algebraic groups, Chevalley groups, and quantized enveloping algebras.

A highest weight representation is called *minuscule* if the associated Weyl group acts transitively on the weights. The highest weight of a minuscule representation is known as a *minuscule weight*, which is a translation of Bourbaki's term *poids minuscule* [1, VIII, §7.3]. The French word "minuscule" can mean "lower case" instead of "tiny". Although minuscule representations can have large dimensions, their structure is in some sense more basic than that of representations in general, so they can be thought of as "lower case representations".

For our purposes, we will regard minuscule representations as representations of finite-dimensional simple Lie algebras over the complex numbers. The representation theory of these algebras is described in detail in the books by Erdmann and Wildon [3], Carter [2], and Kac [5]. The aim of this article is to explain how minuscule representations can be used to construct almost all of these simple Lie algebras, by using combinatorial methods and with relatively little theory. The framework we discuss here is described in the *ADE* case in Wildberger's paper [7], but that paper does not include all the proofs. For a detailed development of the ideas in this article, including proofs and comprehensive references, the reader is referred to the author's book [4].

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A significant advantage of minuscule representations is that they can be easier to construct than the Lie algebras themselves. When a simple Lie algebra over $\mathbb{C}$ has a minuscule representation, it is possible to construct the algebra in terms of its action on the corresponding minuscule module. The good news is that minuscule representations exist in almost all cases, and one way to construct them is by using certain linear operators associated to a combinatorial structure called a "heap".

Heaps and linear operators. A *heap* is a certain labeled partially ordered set (or *poset* for short) where the labels themselves are the vertices of a certain graph. Figure 1 shows the heap of the unique minuscule representation of the simple Lie algebra of type E_7. In this case, the underlying graph is the Dynkin diagram of type E_7, which is shown in Figure 2. Because of its shape, this particular heap has been variously nicknamed "the swallow" or "the bat".

Rather than give the axiomatic definition of heaps, we will work with reference to this particular example. An important property of a heap is that the set of all elements of the heap that have a particular label (for example, the label "4") forms a *chain*, meaning a totally ordered subset. Another important property along these lines is that if we choose a pair of labels that are adjacent in the graph, for example "3" and "7", then the set of all elements of the heap having labels in the set $\{3, 7\}$ also forms a chain.

The heaps associated to minuscule representations are required to satisfy some additional combinatorial rules. We will not go into full details here, but an example of one such rule that applies to this case is that the interval between any two consecutive elements with label p is required to contain precisely two elements having labels that are adjacent (in the underlying graph) to p. For example, we can check by inspecting the heap in Figure 1 that between any two consecutive occurrences of the label 4 there

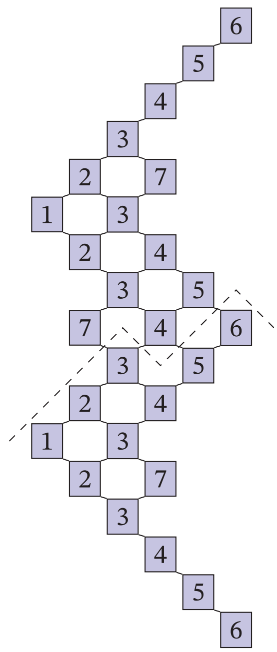

Figure 1. The heap of the unique minuscule representation in type E_7, together with one of the ideals of the heap.

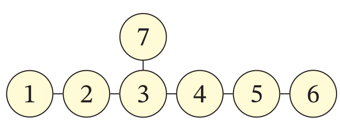

Figure 2. The Dynkin diagram of type E_7.

are either two elements with label 3, or an element with label 3 and an element with label 5.

An *ideal* of a partially ordered set E is a subset I of E that is downward-closed; in other words, whenever y is an element of I and $x \in E$ satisfies $x \leq y$, then we must also have $x \in I$. The set of ideals of a partially ordered set E is denoted by $J(E)$, which in turn becomes a partially ordered set under the "subset" relation. (Note that $\emptyset$ and E are both elements of $J(E)$.) Another example of an ideal of the heap in Figure 1 consists of all the elements below the given dashed line. This particular heap has 56 ideals in total, which is the same as the dimension of the representation it is encoding.

The underlying posets of the heaps of minuscule representations can be characterized concisely by the following theorem. Recall that if E_1 and E_2 are posets, then $E_1 \times E_2$ becomes a poset by stipulating that $(x_1, y_1) \leq (x_2, y_2)$ if and only if both $x_1 \leq x_2$ and $y_1 \leq y_2$. Let $[k]$ denote a chain with k elements, and let J^r denote the result of applying the J operator r times.

Theorem. *The only posets that occur as the underlying poset of the heap of a minuscule representation are the following:*

$$\begin{array}{ll} [s] \times [t] & (s, t \geq 1), \\ J([2] \times [t]) & (t \geq 1), \\ J^k([2] \times [2]) & (k \geq 0), \\ J^2([2] \times [3]) & \textit{and} \\ J^3([2] \times [3]). & \end{array}$$

Proof. This is proved in [4, Theorem 11.2.8 (iv)]. □

Despite its somewhat intricate appearance, the partially ordered set in Figure 1 can be simply described as $J^3([2] \times [3])$.

The posets in the theorem above appear as the answer to various combinatorial problems; for example, they are the only known examples of connected Gaussian posets, in the sense of Stanley [6, Exercise 4.25].

The reason that heaps such as the one in Figure 1 are useful in representation theory is that one can use their combinatorial properties to define linear operators on certain vector spaces.

If E is the heap of a minuscule representation, we define V_E to be the $\mathbb{C}$-vector space with basis $\{v_I : I \in J(E)\}$. Whenever p is the label of an element of the heap E, we can define linear operators X_p and Y_p on V_E by their effect on basis elements as follows:

$$X_p(v_I) = \begin{cases} v_K & \text{if } K \backslash I \text{ is a singleton with label } p, \\ 0 & \text{otherwise;} \end{cases}$$

$$Y_p(v_I) = \begin{cases} v_L & \text{if } I \backslash L \text{ is a singleton with label } p, \\ 0 & \text{otherwise.} \end{cases}$$

(The ideals K and L given above are necessarily unique if they exist.)

Example. Consider the ideal I of the heap E that is given by the dashed line in Figure 1. Since I has a maximal element x with label 6, it follows that $Y_6(v_I) = v_L$, where L is the ideal given by $I \backslash \{x\}$. On the other hand, $E \backslash I$ has a minimal element y with label 4, which means that $X_4(v_I) = v_K$, where $K = I \cup \{y\}$. Similar arguments show that $X_7(v_I) \neq 0$ and $Y_3(v_I) \neq 0$. However, $X_5(v_I)$ is equal to zero, because $E \backslash I$ has no minimal element with a label of 5.

There are also operators H_p and S_p on V_E. These can be defined in terms of the operators X_p and Y_p as follows:

$$H_p(v_I) = \begin{cases} -v_I & \text{if } X_p(v_I) \neq 0, \\ v_I & \text{if } Y_p(v_I) \neq 0, \\ 0 & \text{otherwise;} \end{cases}$$

$$S_p(v_I) = \begin{cases} X_p(v_I) & \text{if } X_p(v_I) \neq 0, \\ Y_p(v_I) & \text{if } Y_p(v_I) \neq 0, \\ v_I & \text{otherwise.} \end{cases}$$

It is not immediately clear that the various cases above are exclusive, but this can be proved from the combinatorial properties of the heap in question.

Representations of Lie algebras over $\mathbb{C}$. The operators X_p, Y_p, and H_p defined in the previous section give a representation of a particular simple Lie algebra. Recall that a *representation* of a Lie algebra $\mathfrak{g}$ is a homomorphism of Lie algebras $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}_n(\mathbb{C})$, where $\mathfrak{gl}_n(\mathbb{C})$ is regarded as a Lie algebra with the bracket $[A, B] := AB - BA$. A *simple* Lie algebra $\mathfrak{g}$ is one for which $[\mathfrak{g}, \mathfrak{g}]$ is nonzero and $\mathfrak{g}$ has no ideals other than itself and the zero ideal.

The following theorem uses the linear operators defined above to construct all minuscule representations of simple Lie algebras over $\mathbb{C}$. This procedure also constructs the simple Lie algebras themselves and their Weyl groups.

Theorem. *For every minuscule representation of a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, there exists a heap E with set of labels Γ, satisfying the following properties.*

(i) *The subalgebra of $\mathfrak{gl}(V_E)$ generated by the operators $\{X_p, Y_p, H_p : p \in \Gamma\}$ is isomorphic to the Lie algebra $\mathfrak{g}$. This construction endows V_E with the structure of a $\mathfrak{g}$-module affording the minuscule representation in question.*

(ii) *The operators $\{S_p : p \in \Gamma\}$ give permutations of $J(E)$ of order 2. The group generated by these permutations is isomorphic to the Weyl group, W, of $\mathfrak{g}$.*

The generators X_i, Y_i, and H_i for $\mathfrak{g}$ that are given in the theorem agree with the usual *Serre generators* e_i, f_i, and h_i for $\mathfrak{g}$. Similarly, the generators S_i for W agree with the usual *Coxeter generators* s_i for W.

Weights. Recall that a representation is called minuscule if the Weyl group acts transitively on the weights. The weights of a minuscule representation are particularly easy to work with, because they can be identified with the ideals I of the associated heap E.

An important property of the basis elements v_I is that they are simultaneous eigenvectors for the operators H_p. The *weight* associated to v_I can be identified as the formal linear combination $\sum_{p \in \Gamma} c_p \omega_p$, where c_p is the eigenvalue of H_p acting on v_I.

Example. Consider the ideal I of the heap E given in Figure 1. A direct check using the definitions shows that H_4 and H_7 act on v_I with the eigenvalue -1, H_3 and H_6 act on v_I with the eigenvalue 1, and the other H_i act on v_I with the eigenvalue zero. It follows that the weight of v_I is $\lambda = \omega_3 - \omega_4 + \omega_6 - \omega_7$.

In fact, the operators $\{H_p : p \in \Gamma\}$ are a basis for the Cartan subalgebra, and $\{\omega_p : p \in \Gamma\}$ is the associated dual basis.

The weights of a highest weight representation can be made into a partially ordered set in a natural way, and this corresponds to the natural order on ideals given by the subset relation. It follows from this that the ideals E and $\emptyset$ are the highest and lowest weights, respectively, of the corresponding minuscule representation. Since any ideal of E may be transformed to the empty ideal by successive removal of maximal elements, it follows that the action of the Weyl group on the ideals of E (and therefore on the weights of the representation) is transitive.

By inspection of the heap shown in the diagram, we see that the minuscule representation in type E_7 has highest weight ω_6 and lowest weight $-\omega_6$. Because this particular heap has symmetry about a horizontal axis, it follows that the weights of the corresponding representation are closed under negation.

Every minuscule module V_E whose weights are closed under negation can be endowed with a bilinear form $p : V_E \otimes_{\mathbb{C}} V_E \longrightarrow \mathbb{C}$ given by its effect on basis elements as follows:

$$p(v_I, v_K) = \begin{cases} (-1)^{|I|} & \text{if } \mathrm{wt}(v_I) = -\mathrm{wt}(v_K), \\ 0 & \text{otherwise.} \end{cases}$$

Theorem. *Let E be the heap of a minuscule module V_E that has the property that the weights are closed under negation, and let p be the bilinear form defined above.*

(i) *The form p is nondegenerate, and it is invariant in the sense that for every $x \in \mathfrak{g}$ and every $v \in V_E$, we have $p(x.v, v) + p(v, x.v) = 0$.*

(ii) *If the heap E has an even number of elements, then p is orthogonal, meaning that we always have $p(v_1, v_2) = p(v_2, v_1)$.*

(ii) *If the heap E has an odd number of elements, then p is symplectic, meaning that we always have $p(v_1, v_2) = -p(v_2, v_1)$.*

Proof. This is proved in [4, Theorem 5.6.4]. This result is true in a more general context than minuscule representations, as proved by Bourbaki [1, VIII, Chapter 7, Proposition 12]. □

When do minuscule representations exist? The finite-dimensional simple Lie algebras over $\mathbb{C}$ were classified by W. Killing and E. Cartan in the late nineteenth century. It turns out that there are four infinite families, known as A_n, B_n, C_n, and D_n, together with five exceptional cases, known as E_6, E_7, E_8, F_4, and G_2. The number of (isomorphism classes of) minuscule representations of each simple Lie algebra is given in Table 1.

In more detail, the minuscule representations are as follows. In type A_n, there is an $n + 1$-dimensional minuscule representaiton arising from the realization of the Lie algebra as $\mathfrak{sl}_{n+1}(\mathbb{C})$. The other minuscule representations come from the exterior powers of this representation; they have dimensions $\binom{n+1}{k}$ for $1 \leq k \leq n$. The minuscule representation in type B_n is the spin representation of dimension 2^n, and the minuscule representation in type C_n is the natural representation of dimension $2n$. In type D_n, the three minuscule representations are the natural representation of dimension $2n$, and the two half spin representations of dimension 2^{n-1}. Finally, there are two minuscule

Type	Number of minuscule representations
A_n	n
B_n	1
C_n	1
D_n	3
E_6	2
E_7	1
E_8	0
F_2	0
G_2	0

Table 1

representations in type E_6, both of dimension 27, and one minuscule representation in type E_7, of dimension 56.

The correspondence between posets and minuscule representations works as follows. The posets $[s] \times [t]$ give rise to the minuscule representations in types A_n. The heaps for type C_n are chains, which corresponds to the special case $t = 1$. The posets $J([2] \times [t])$ give the spin and half spin representations in types B_n and D_n. The posets $J^k([2] \times [2])$ give the natural representations in type D_n. The poset $J^2([2] \times [3])$ gives the two 27-dimensional representations in type E_6, and the poset $J^3([2]\times[3])$ gives the 56-dimensional representation in type E_7.

Remark. If we define N to be the number of minuscule representations of the simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, then there are several interesting Lie-theoretic interpretations of the number $N + 1$. For example, $N + 1$ is equal to (a) the determinant of the Cartan matrix of $\mathfrak{g}$; (b) the index of the root lattice of $\mathfrak{g}$ in the weight lattice of $\mathfrak{g}$; and (c) the ratio $|\overline{G}|/|G|$, where G and $\overline{G}$ are the automorphism groups of the Dynkin diagram of $\mathfrak{g}$ and of the corresponding untwisted affine Dynkin diagram, respectively.

Connections with geometry. It turns out that there is a natural way to identify weights of a minuscule representation with points in Euclidean space to form a highly symmetric polytope on which the Weyl group W acts by orthogonal transformations [4, §8.4]. In particular, one can obtain the simplex, the hypercube, and the hyperoctahedron from the weights of suitable minuscule representations in type A_n, B_n, and C_n, respectively. Given an arbitrary minuscule representation, it can be shown that if (λ_1, λ_2) and (μ_1, μ_2) are two pairs of weights, then there exists an element $w \in W$ satisfying $(w.\lambda_1, w.\lambda_2) = (\mu_1, \mu_2)$ if and only if the Euclidean distances $d(\lambda_1, \lambda_2)$ and $d(\mu_1, \mu_2)$ are equal. This observation provides a more detailed view of the transitive action of the Weyl group on the weights of a minuscule representation; in particular, this can be used to calculate the rank of the Weyl group as a permutation group [4, Theorem 8.2.22].

The 56-dimensional minuscule representation in type E_7 has a geometric interpretation in terms of the del Pezzo surface that arises from blowing up seven points in general position in the complex projective plane. (Here, "general position" means that no three points are collinear, and no six lie on a conic.) This gives rise to 56 curves of self-intersection -1 that can be identified with the weights of the representation in a way that is naturally compatible with the action of the Weyl group. By pairing up these 56 curves in a canonical way, we obtain the action of the Weyl group of type E_7 on the 28 bitangents to a plane quartic curve. This action has many interesting combinatorial properties, which are the subject of [4, §9].

The 27-dimensional minuscule representations in type E_6 are another very interesting case that arises from blowing up six points in general position. Here, the 27 weights correspond to a famous configuration of 27 lines on a cubic surface [4, §10.1]. The Euclidean distances between the weights then determine whether a pair of these lines is incident or skew.

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