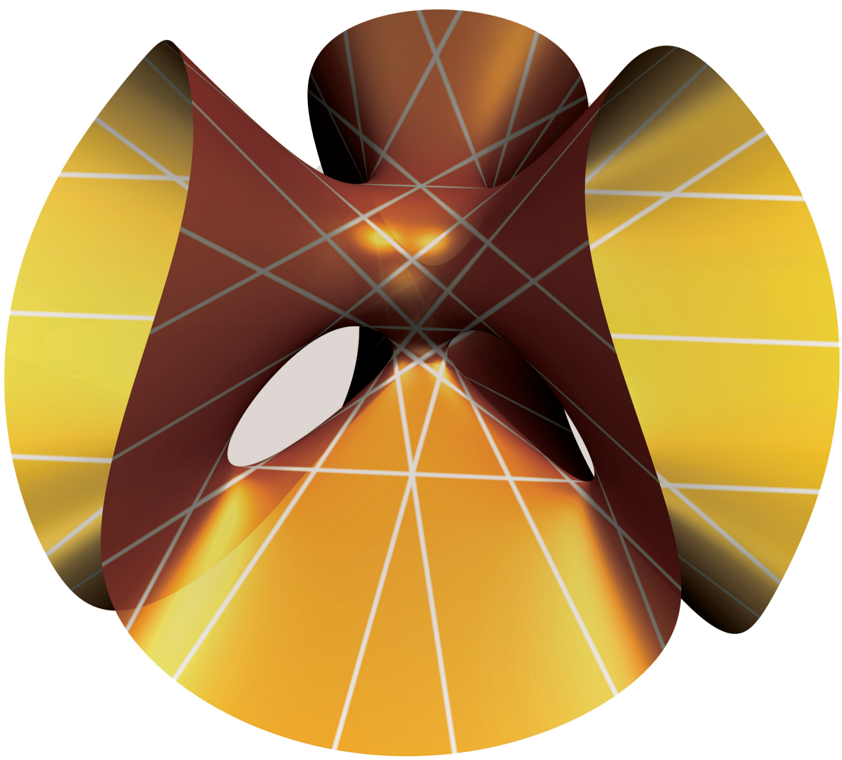
The Prym Map: From Coverings to Abelian Varieties



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Introduction

Abelian varieties are very important objects that lie at the crossroads of several areas in mathematics: algebraic geometry, complex analysis, and number theory. So they can be studied from different points of view. An *abelian variety* over a field K is an abelian group which admits an embedding in a projective space $\mathbb{P}(K)$, so it is also a projective variety. Although it can be defined over an arbitrary field K, in this note we will only consider the case of *complex abelian varieties*, that is $K = \mathbb{C}$.

Abelian varieties made their appearance at the beginning of the nineteenth century, when Niels Henrik Abel and Carl Gustav Jacob Jacobi studied *hyperelliptic integrals*. A hyperelliptic integral is an integral of the form

$$\int_\gamma \frac{p(z)}{\sqrt{q(z)}} dz,$$

where $p(z)$ is a polynomial, $q(z) = (z - a_1) \cdots (z - a_n)$ is a polynomial of degree n with pairwise distinct roots a_i, and γ is a path in $\mathbb{C}$. At the time of Abel and Jacobi it was known that for $n = 1, 2$ this integral could be solved by means

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of trigonometric and logarithmic functions. For $n = 3, 4$ one can solve it using elliptic functions, but for $n \geq 5$ no way of integrating this function is known in general. Considered as a function on $\mathbb{C}$, the differential $\frac{1}{\sqrt{q(z)}}dz$ is not singled-valued, but it can be seen as a holomorphic differential on the compact Riemann surface (which is also a smooth algebraic curve over $\mathbb{C}$) associated to $\sqrt{q(z)}$, that is, the double covering C of $\mathbb{P}^1$ ramified at the points $a_1, \dots, a_n$. The difficulty to integrate $\frac{1}{\sqrt{q(z)}}dz$ comes from the topological structure of C. Abel and Jacobi had the idea to integrate all the holomorphic differentials on C at the same time by considering a basis of the form $\omega_i := z^{i-1}\frac{1}{\sqrt{q(z)}}dz$, $i = 1, \dots, [\frac{n-1}{2}]$. The dimension $g := [\frac{n-1}{2}]$ of the vector space of holomorphic differentials (denoted by $H^0(C, \omega_C)$) is called the *genus* of C. Topologically, the genus is the number of "holes" of the Riemann surface C (see Figure 1). Let $p_0 \in C$ be a fixed point. Around this point we can define the map

$$p \mapsto \left(\int_{p_0}^{p} w_1, \dots, \int_{p_0}^{p} w_g\right), \tag{1}$$

but it cannot be extended to the whole C because these integrals depend on the path between p_0 and p. The way out is to consider the image modulo the values of integrals along closed paths to obtain a well-defined map. Let $H_1(C, \mathbb{Z}) \simeq \mathbb{Z}^{2g}$ be the group of closed paths in C (which does not depend on the starting point) modulo homology. This group can be seen as a full rank lattice inside $H^0(C, \omega_C)^*$, via the injective map

$$\gamma \mapsto \left\{\omega \mapsto \int_\gamma \omega\right\}$$

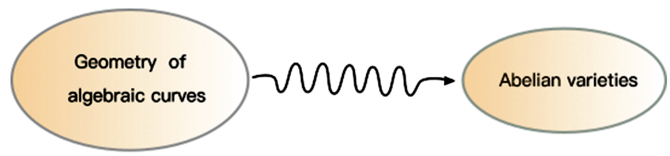
assigning to a path γ the functional which integrates the holomorphic differentials along γ. Thus, the map (1) extends to a holomorphic map on C,

$$\alpha : p \mapsto \left(\int_{p_0}^{p} w_1, \dots, \int_{p_0}^{p} w_g\right) \mod H_1(C, \mathbb{Z}),$$

called the *Abel–Jacobi map*, and the complex torus $JC := H^0(C, \omega_C)^*/H_1(C, \mathbb{Z})$ is the *Jacobian variety* of C. Consequently, the problem of integrating holomorphic differentials on C is equivalent to describing the Jacobian variety of C and the *Abel–Jacobi map* $\alpha : C \to JC$. Historically, the Jacobian of a compact Riemann surface C is the first example of an abelian variety.

The main idea we want to convey in this article is that one can construct and study abelian varieties using curves and its surrounding geometry.

Abelian varieties and Jacobians. One can show that a *complex abelian variety* A is the quotient of a complex vector space V by a full rank lattice Γ in V together with an ample



line bundle L on the torus V/Γ, called a *polarization*. The sections of $L^{\otimes k}$, for some $k > 0$, provide an embedding of A into a projective space[1] $\mathbb{P}H^0(A, L^{\otimes k})^* \simeq \mathbb{P}^N$:

$$x \mapsto [s_0(x) : s_1(x) : \cdots : s_N(x)],$$

where $s_0, s_1, \dots, s_N$ form a basis of the space of sections $H^0(A, L^{\otimes k})$. A polarization has several incarnations: (1) an ample line bundle[2] L; (2) a nondegenerated alternating bilinear form $E : V \times V \to \mathbb{R}$ with $E(\Gamma, \Gamma) \subset \mathbb{Z}$; (3) a nondegenerated Hermitian form H on V with $\operatorname{Im} H(\Gamma, \Gamma) \subset \mathbb{Z}$; and (4) a Weil divisor $\Theta \subset X$ with the property that the set $\{a \in A \mid \Theta - a \sim \Theta\}$ is finite. One has $L = \mathcal{O}_A(\Theta)$, the associated line bundle to the divisor Θ; conversely, giving a polarization L one recovers Θ as the zero locus of a section of L. There is a basis of V with respect to which the alternating form E has as matrix $\left(\begin{smallmatrix} 0 & D \\ -D & 0 \end{smallmatrix}\right)$, where D is a diagonal matrix with positive integer entries $d_1, \dots, d_g$ satisfying $d_i | d_{i+1}$ for $i = 1, \dots, g-1$. The vector $(d_1, \dots, d_g)$ is called the *type of the polarization* of L, and when it is of the form $(1, \dots, 1)$ the polarization is *principal*. In this case the dimension of the space of sections of L is one, so we have canonically associated to the abelian variety a unique geometric object, its *theta divisor*. This is one of the reasons why principal polarizations are the most studied ones.

As we have seen, the Jacobian of an algebraic curve C (or compact Riemann surface) is the complex torus

$$JC = H^0(C, \omega_C)^*/H_1(C, \mathbb{Z}).$$

The intersection product on $H_1(C, \mathbb{Z})$ induces an alternating form E on $V := H^0(C, \omega_C)^*$. More precisely, if we choose a basis over $\mathbb{Z}$, $\gamma_1, \dots, \gamma_{2g}$ of $H_1(C, \mathbb{Z})$ as in Figure 1, the intersection product has as matrix $\left(\begin{smallmatrix} 0 & \mathbf{1}_g \\ -\mathbf{1}_g & 0 \end{smallmatrix}\right)$. As $H_1(C, \mathbb{Z})$ is a full rank lattice in V, the $\{\gamma_i\}$ form also a basis of V as $\mathbb{R}$-vector space. One verifies then that with respect to this basis, the intersection matrix gives an alternating form E on V defining a principal polarization Θ.

A one-dimensional abelian variety is also an algebraic curve of genus one, that is, an elliptic curve. The Jacobian of a genus one curve is then isomorphic to the curve itself.

Algebraic geometers typically gather their objects of study in families to investigate a *general* property or to single out interesting elements. Ideally, the set of all the objects is itself an algebraic variety where one can apply known tools. This leads to the notion of *moduli space*, which is the variety parametrizing the objects. Fortunately,

[1] *This is essentially the definition of ampleness of a line bundle.*

[2] *Or more precisely its first Chern class.*

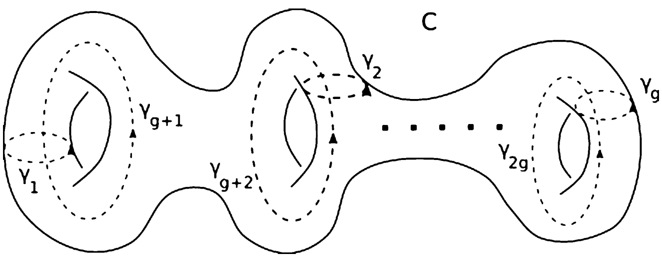


Figure 1. Curve of genus g.

there exists a nice parameter space for all principally polarized abelian varieties (ppav) of fixed dimension g (up to isomorphism). This moduli space can be constructed as the quotient of the *Siegel upper half plane*

$$\mathfrak{h}_g := \{\tau \in M_{g\times g}(\mathbb{C}) \mid \tau^t = \tau,\ \mathrm{Im}\,\tau > 0\}$$

by the action of the symplectic group

$$Sp_{2g}(\mathbb{Z}) = \left\{ M \in GL_{2g}(\mathbb{Z}) \ :\ M\begin{pmatrix} 0 & \mathbf{1_g} \\ -\mathbf{1_g} & 0 \end{pmatrix}{}^tM = \begin{pmatrix} 0 & \mathbf{1_g} \\ -\mathbf{1_g} & 0 \end{pmatrix} \right\}$$

given by

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp_{2g}(\mathbb{Z}), \quad M \cdot \tau = (a + b\tau)(c + d\tau)^{-1}.$$

Thus, every point in the quotient $\mathfrak{h}_g/Sp_g(\mathbb{Z})$ represents an isomorphism class of a principally polarized abelian variety of dimension g: for each $\tau \in \mathfrak{h}_g$ set $A_\tau = \mathbb{C}^g/\tau\mathbb{Z}^g \oplus \mathbb{Z}^g$, then

$$A_\tau \simeq A_{\tau'} \text{ as ppav} \Leftrightarrow \exists\, M \in Sp_{2g}(\mathbb{Z}) \quad \text{s.t.} \quad \tau' = M \cdot \tau.$$

In what follows, we denote by $\mathcal{A}_g$ the moduli space of principally polarized abelian varieties of dimension g. Observe that the dimension of this space is the same as the dimension of the space of symmetric matrices of size g; thus $\dim A_g = \frac{g(g+1)}{2}$.

The space parametrizing isomorphism classes of smooth projective curves has no such simple description as the moduli of abelian varieties, but it is nevertheless a well-behaved algebraic variety. Let $\mathcal{M}_g$ be the moduli space of smooth projective curves of genus g. It is an irreducible algebraic variety of dimension $3g - 3$. By associating to each smooth curve $[C] \in \mathcal{M}_g$ its Jacobian, we get the *Torelli map:*

$$\mathfrak{t} : \mathcal{M}_g \to \mathcal{A}_g, \qquad [C] \mapsto (JC, \Theta).$$

The celebrated *Torelli theorem* states that the map $\mathfrak{t}$ is an embedding. In particular, one can recover a smooth algebraic curve C from the pair (JC, Θ). Comparing the dimensions of both spaces, one deduces that the general principally polarized abelian variety of dimension < 3 is the Jacobian of some curve.

Which other abelian varieties can be obtained from curves?

Since much more is known about the moduli space of curves than the moduli of abelian varieties, the answer to this question will help us understand better abelian varieties and their moduli spaces.

Coverings of curves and Prym varieties. Consider a finite covering $f : \widetilde{C} \to C$ of degree d between two smooth projective curves, and let g and $\tilde{g}$ denote the genera of C and $\widetilde{C}$, respectively. By the Riemann–Hurwitz formula these genera are related by

$$\tilde{g} = d(g-1) + \frac{\deg R}{2} + 1, \tag{2}$$

where R denotes the ramification divisor of f, that is, the set of points in $\widetilde{C}$ (counted with multiplicities) where the map is not locally a homeomorphism. The map f induces a map between the Jacobians of the curves, the *norm map*. As a group, the Jacobian JC is generated by the points of the curve C, and in fact JC parametrizes classes of linear equivalence of divisors of degree zero. With this in mind one can simply define the norm map as the pushforward of divisors from $\widetilde{C}$ to C:

$$\mathrm{Nm}_f : J\widetilde{C} \to JC, \qquad \left[\sum_i n_i p_i\right] \mapsto \left[\sum_i n_i f(p_i)\right],$$

where the sum is finite, $\sum n_i = 0$ with $n_i \in \mathbb{Z}$, and the bracket denotes the class of linear equivalence. The kernel of Nm_f is not necessarily connected, but since Nm_f is a homomorphism of groups the connected component containing the zero is naturally a subgroup of $J\widetilde{C}$. This subgroup is the *Prym variety of* f denoted by

$$P(f) := (\mathrm{Ker}\,\mathrm{Nm}_f)^0 \subset J\widetilde{C}. \tag{3}$$

Moreover, the restriction of the principal polarization Θ on $J\widetilde{C}$ to $P(f)$ defines a polarization Ξ; hence $(P(f), \Xi)$ is an abelian subvariety of the Jacobian $J\widetilde{C}$ of dimension

$$\dim P(f) = \dim J\widetilde{C} - \dim JC = \tilde{g} - g.$$

The Prym variety can be regarded as the complementary variety of the image of $f^* : JC \to J\widetilde{C}$ inside $J\widetilde{C}$. It was shown by Mumford in [Mum74] that the restricted polarization Ξ is principal when f is an étale (unramifed) double covering, or when it is a double covering ramified in exactly two points. These are basically the only cases when one can obtain a principal polarization on $P(f)$. Assume now that f is an étale double covering. According to (2) the dimension of the corresponding Prym variety is $\dim P(f) = 2(g-1) - g = g - 1$. Thus, this construction provides us a way to associate to each étale double covering $f : \widetilde{C} \to C$ over a smooth curve C of genus g a principally polarized abelian variety; this is the Prym map. In order to make the definition precise we need to introduce the moduli space

$$\mathcal{R}_g := \{[C,\eta] \mid [C] \in \mathcal{M}_g,\ \eta \in \mathrm{Pic}^0(C) \setminus \{\mathcal{O}_C\},\ \eta^{\otimes 2} \simeq \mathcal{O}_C\}/\cong$$

parametrizing all the (nontrivial) étale double coverings over curves of genus g up to isomorphism. Given a pair $[C,\eta] \in \mathcal{R}_g$ the isomorphism $\eta^{\otimes 2} \simeq \mathcal{O}_C$ endows $\mathcal{O}_C \oplus \eta$ with a ring structure (actually with a structure of $\mathcal{O}_C$-algebra). Thus, the corresponding double covering is given by taking the spectrum $\widetilde{C} := \mathrm{Spec}(\mathcal{O}_C \oplus \eta)$, and the map f is just the natural projection $\mathrm{Spec}(\mathcal{O}_C \oplus \eta) \to C = \mathrm{Spec}\, \mathcal{O}_C$, induced by the inclusion $\mathcal{O} \hookrightarrow \mathcal{O}_C \oplus \eta$. There are finitely many "square roots" of $\mathcal{O}_C$, that is, line bundles η with $\eta^{\otimes 2} \simeq \mathcal{O}_C$. More precisely, the forgetful map

$$\mathcal{R}_g \to \mathcal{M}_g, \qquad [C,\eta] \mapsto \eta$$

is finite of degree $2^{2g} - 1$, and hence $\dim \mathcal{R}_g = \dim M_g = 3g - 3$. The *Prym map* is then defined as

$$Pr_g : \mathcal{R}_g \to \mathcal{A}_{g-1} \qquad [C,\eta] \mapsto (P(f), \Xi).$$

By comparing the dimensions on both sides, one sees that $\dim \mathcal{R}_g \geq \dim \mathcal{A}_{g-1} = \frac{g(g-1)}{2}$ for $g \leq 6$, so it makes sense to ask if for low values of g the Prym map is dominant, i.e., if we can realize a (general) principally polarized abelian variety of dimension ≤ 6 as the Prym variety of some covering. The following theorem, representing work of several mathematicians, gives a positive answer to this question and summarizes the situation for the classical Prym map.

Theorem 4. *(a) The Prym map is dominant if $g \leq 6$.*
(b) The Prym map is generically injective if $g \geq 7$.
(c) The Prym map is never injective.

Let $\mathcal{P}_{g-1}$ denote the image of Pr_g. Wirtinger showed ([Wir95]) that the closure $\overline{\mathcal{P}}_{g-1}$ is an irreducible subvariety in $\mathcal{A}_{g-1}$ of dimension $3g-3$, so $\overline{\mathcal{P}}_{g-1} = \mathcal{A}_{g-1}$ for $g \leq 6$, which implies part (a). Moreover, he also proved that the Jacobian locus in $\mathcal{A}_{g-1}$ (i.e., the image of the Torelli map $\mathfrak{t}$) is contained in $\overline{\mathcal{P}}_{g-1}$. In this sense, Pryms are a generalization of Jacobians. Part (b) was first proved by R. Friedman and R. Smith ([FS82]), and for $g \geq 8$ by V. Kanev ([Kan82]) by using degeneration methods. More geometric proofs were given by G. Welters ([Wel87]) and later by O. Debarre ([Deb89]), in the spirit of the proof of Torelli's theorem. The fact that the Prym map is noninjective was first observed by Beauville ([Bea77]), who produced, using *Recillas' trigonal construction*, examples of nonisomorphic coverings $(\widetilde{C}_1, C), (\widetilde{C}_2, C_2)$ in $\mathcal{R}_g$ for $g \leq 10$, whose Prym varieties are isomorphic as principally polarized abelian varieties. Later, Donagi's tetragonal construction ([Don81]) (of which Recillas's construction is a degeneration) provided examples for the noninjectivity of the Prym map in any genus. We will discuss this construction in the next section.

When $g < 6$ the fibers of the Prym map are positive dimensional and they are geometrically well understood.

The Prym Map $Pr_6 : \mathcal{R}_6 \to \mathcal{A}_5$

The case of étale double coverings over a genus 6 curve deserves special attention. We have $\dim \mathcal{R}_6 = \dim \mathcal{A}_5 = 15$, and the map Pr_6 being dominant implies that it is also generically finite. Can one determine the degree or even describe its generic fibre? There is not only a positive answer to this question but also a beautiful one. The degree of $Pr_6 : \mathcal{R}_6 \to \mathcal{A}_5$ is 27, which is also the number of lines on every smooth cubic surface. This is not a coincidence, because the monodromy group of the Prym map equals the Weyl group $W(E_6)$, which governs the incidence structure of the lines in a smooth cubic surface ([Don92]). On the other hand, Pr_6 fails to be finite over the locus of all the Jacobians of curves of genus 5, as well as over the locus of intermediate Jacobians of cubic threefolds.[3] The fibres and the blowup of the Prym map along these loci are explicitly described in [Don92]. There is even a procedure to pass from one element to another on a general fibre of Pr_6: the *tetragonal construction*. First, notice that a general curve $[C] \in \mathcal{M}_6$ carries exactly five g^1_4's, that is, line bundles of degree 4 on C whose space of sections is two-dimensional. Thus, a g^1_4 on C is equivalent to having a 4:1 map $C \to \mathbb{P}^1$. Now, given a double covering $[f : \widetilde{C} \to C] \in \mathcal{R}_6$ and a g^1_4 on C, one can construct two other coverings in $\mathcal{R}_6$ as follows. Let $C^{(4)}$ (respectively, $\widetilde{C}^{(4)}$) be the fourth symmetric product of C (respectively, $\widetilde{C}$), parametrizing divisors of degree 4 on the corresponding curves. Define the curve $\widetilde{X}$ by the cartesian diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r, hook] \arrow[d, "16:1"', "f^{(4)}_{|\widetilde{X}}"] & \widetilde{C}^{(4)} \arrow[d, "2^4:1"', "f^{(4)}"] \\ \mathbb{P}^1 = g^1_4 \arrow[r, hook] & C^{(4)} \end{tikzcd}$$

where $f^{(4)}$ is the natural map $p_1 + \cdots + p_4 \mapsto f(p_1) + \cdots + f(p_4)$. The involution σ on $\widetilde{C}$ induces an involution $\tilde{\sigma} : p_1 + \cdots + p_4 \mapsto \sigma(p_1) + \cdots + \sigma(p_4)$ on $\widetilde{X}$. It turns out that $\widetilde{X}$ consists of two disjoint nonsingular connected components $\widetilde{X}_0, \widetilde{X}_1$ and the involution $\tilde{\sigma}$ acts without fixed points on each of these components. Moreover, the restriction of the map $f^{(4)}$ to $\widetilde{X}_0$ and $\widetilde{X}_1$ defines 8:1 maps fitting in the following diagram:

$$\begin{tikzcd} & \widetilde{X}_0 \arrow[dl, "f_0"'] \arrow[ddr, "8:1"] & \sqcup & \widetilde{X}_1 \arrow[dr, "f_1"] \arrow[ddl, "8:1"'] & \\ X_0 \arrow[drr, "4:1"'] & & & & X_1 \arrow[dll, "4:1"] \\ & & \mathbb{P}^1 & & \end{tikzcd}$$

[3] *The intermediate of a cubic threefold Y is the complex torus $H^{1,2}(Y)/H_3(Y,\mathbb{Z})$.*

where $X_i := \widetilde{X}_i/\tilde{\sigma}$ for $i = 0, 1$. Therefore, from an element $[\widetilde{C} \xrightarrow{f} C] \in \mathcal{R}_6$ together with a pencil g^1_4 on C we have constructed two other elements in $\mathcal{R}_6$, namely, $[\widetilde{X}_0 \xrightarrow{f_0} X_0]$ and $[\widetilde{X}_1 \xrightarrow{f_1} X_1]$, together with g^1_4's on X_0 and X_1. These three pairs are in general not isomorphic, but the associated Pryms are isomorphic ([Don81]). If one applies the construction to $[\widetilde{X}_0 \xrightarrow{f_0} X_0]$ using the obtained g^1_4, one gets back the original two coverings, but if one uses another pencil on X_0, one gets two other nonisomorphic coverings. By repeating this procedure on the coverings using different pencils on their base curves, we obtain eventually all the elements on the fibre. This is also the structure on the lines of a smooth cubic surface. Two coverings in the fibre are related by the tetragonal construction, as two lines on a cubic surface are incident. The triad $\{(\widetilde{C}, C), (\widetilde{X}_0, X_0), (\widetilde{X}_1, X_1)\}$ corresponds to a triangle in the surface. The fact that each line in the cubic surface is a "side" of five triangles is reflected in the construction by the five g^1_4's that a curve C possesses. This particular case shows that the fibres of the Pym map display rich and beautiful geometry.

Nonprincipally polarized abelian varieties. One can of course use more general coverings to construct abelian varieties, for instance, coverings of degree higher than 2 or ramified ones. In these cases one obtains abelian varieties which are no longer principally polarized. In this section we will discuss cyclic coverings ramified at $2r$ points with $r \geq 0$.

By definition, an étale cyclic covering $\widetilde{C} \to C$ of degree d admits an automorphism σ of order d on $\widetilde{C}$ such that $C = \widetilde{C}/\langle\sigma\rangle$. For simplicity, we can assume that d is a prime number (otherwise one can factorize the map through cyclic coverings of smaller degree). If the cyclic covering is branched over a divisor $B \subset C$, the ramification index at each ramified point is d; i.e., all the branches come together in that point. In particular, if $\deg B = dm$, for some integer $m > 0$, the ramification divisor is of degree

$$\deg R = 2r = (d-1)dm.$$

Similarly to the étale double cover case, giving a ramified cyclic covering is equivalent to giving a triple (C, B, η) with η a line bundle on C (of degree m) satisfying $\eta^{\otimes d} \simeq \mathcal{O}_C(B)$. More precisely, take a section s of $\mathcal{O}_C(B)$ vanishing exactly along B (if $B = \emptyset$, take s the constant section 1). Denote by $Tot(\eta)$ the total space of η and let $p : Tot(\eta) \to C$ be the projection. If t is the tautological section of the line bundle $p^*\eta \to Tot(\eta)$, then the locus where the section $p^*s - t^d$ vanishes defines the curve $\widetilde{C}$ inside $Tot(\eta)$.

Recall that points in the Jacobian represent line bundles of degree zero on the corresponding curve. Hence, every automorphism of a curve induces an automorphism of its Jacobian by taking the pullback of the bundle under the automorphism. If we denote by the same letter the automorphism on $J\widetilde{C}$ induced by σ, it is more convenient to define the Prym variety of the covering f as

$$P(f) := \mathrm{Im}(1-\sigma) \subset J\widetilde{C},$$

which is an abelian subvariety of $J\widetilde{C}$ of dimension

$$\dim P(f) = \tilde{g} - g = (d-1)(g-1) + r =: p,$$

with polarization Ξ given by the restriction of the principal polarization $\widetilde{\Theta}$ on $J\widetilde{C}$. It is a polarization of type $D = (1, \dots, 1, d, \dots, d)$, where 1 occurs $p - (g-1)$ times and d occurs $g-1$ times if $r = 0$, and 1 occurs $p - g$ times and d occurs g times if $r > 0$. This definition coincides with (3), since in that case $\mathrm{Im}(1-\sigma) = \ker(1+\sigma)^0$. Let

$$\begin{aligned}\mathcal{R}_g(d,r) := \{&(C, B, \eta) \mid \\ &\eta \in \mathrm{Pic}^m(C),\ B \text{ reduced divisor in } |\eta^{\otimes d}|\}/\cong\end{aligned}$$

denote the moduli space parametrizing cyclic coverings of degree d over a curve of genus g totally ramified over dm points, and let $\mathcal{A}^D_p$ be the moduli space of abelian varieties of dimension p and polarization type D. In this case the Prym map is given by

$$Pr_g(d,r) : \mathcal{R}_g(d,r) \to \mathcal{A}^D_p,\ [\widetilde{C} \xrightarrow{f} C] \mapsto (P(f), \Xi).$$

In order to compute the dimension of its image one needs to know when this map is generically finite. This is the case when the differential map of $Pr_g(d,r)$ is injective at a generic point of $\mathcal{R}_g(d,r)$ or equivalently when the codifferential map $d^*Pr_g(d,r)$ is surjective. One of the advantages of considering cyclic covering is that the tangent space at $0 \in P(f)$ to the Prym variety can be identified with the direct sum of spaces of sections

$$T_0P \simeq \bigoplus_{i=1}^{d-1} H^0(C, \omega_C \otimes \eta^i)^*,$$

where each summand is an eigenspace for the action of σ on $H^0(\widetilde{C}, \omega_{\widetilde{C}})^*$. Notice that the forgetful map $[C, B, \eta] \mapsto [C, B]$ is finite over the moduli space $\mathcal{M}_{g,dm}$ of dm-pointed smooth curves of genus g. Therefore the cotangent space to a generic point $[C, \eta, B] \in \mathcal{R}_g(d,r)$ can be identified to the cotangent space to $\mathcal{M}_{g,dm}$ at $[C, B]$. By identifying the cotangent spaces

$$T^*_{(P,\Xi)}\mathcal{A}^D_p \simeq \mathrm{Sym}^2(T_0P)^*,$$
$$T^*_{[C,\eta,B]}\mathcal{R}_g(d,r) \simeq H^0(C, \omega_C(B)),$$

we obtain that the codifferential of $P_g(d,r)$ at a generic point $[C, B, \eta]$ is given by the multiplication of sections

$$d^*Pr_g(d,r) : \mathrm{Sym}^2(T_0P)^* \to H^0(C, \omega_C^2 \otimes \mathcal{O}(B)).$$

In the cases when this map is surjective at the generic point $[(C, B, \eta)]$ we get that the Prym map $Pr_g(d,r)$ is generically finite onto its image. Recall that dm is the degree of the branch locus of the covering.

Theorem 1 ([LO11]). *If*

- $g \geq 2$ *and* $dm \geq 6$ *for* d *even or* $dm \geq 7$ *for* d *odd,*
- $g \geq 3$ *and* $(d, m) \in \{(4, 1), (5, 1), (2, 2), (3, 2)\}$,
- $g \geq 5$ *and* $(d, m) \in \{(2, 1), (3, 1)\}$,

then the Prym map $Pr_g(d, r)$ *is generically finite.*

In the case of ramified double coverings we know that the Prym map is generically injective as soon as the dimensions of the moduli spaces in the source and target allow it (see [MP12], [MN14], [NO19]), except when $g = 3, r = 2$, where $\deg Pr_3(2, 2) = 3$. There are actually at least two different ways to interpret this degree ([BCV95], [NR95]). As we have seen in the previous section, it is particularly appealing to study the geometry of the fibres of the Prym map. Let $\mathcal{B}_D$ be the component of the moduli space $\mathcal{A}_p^D$ of elements (P, Ξ) such that the polarization Ξ is compatible with σ, i.e., $\sigma^*\Xi \equiv \Xi$. In particular, $\mathcal{B}_D = \mathcal{A}_p^D$ when $d = 2$. One checks that $\operatorname{Im} Pr_g(d, r) \subset \mathcal{B}_D$. By computing the dimension of $\mathcal{B}_D$ one can prove that only in the following cases is $Pr_g(d, r)$ generically finite and dominant over $\mathcal{B}_D$ ([LO18]):

(g, d, r)	$\deg Pr_g(d, r)$	$\dim P$	Polarization
(6,2,0)	27	5	(2,2,2,2,2)
(3,2,2)	3	4	(1,2,2,2)
(1,2,3)	1	3	(1,1,2)
(4,3,0)	16	6	(1,1,1,3,3,3)
(2,7,0)	10	6	(1,1,1,1,1,7)
(2,3,3)	?	5	(1,1,1,3,3)

The degree of the Prym map $Pr_2(3, 3)$ seems to be unknown. We would like to emphasize that the fibre of the different Prym maps carries a peculiar structure, so the way of computing the degree has been ad hoc for each case.

Beyond coverings: Prym–Tyurin varieties. In the previous construction the Prym variety appears as a subvariety of some Jacobian. It is known that to every abelian subvariety of a Jacobian corresponds an endomorphism, called the *norm-endomorphism*. Thus, in order to extend the definition of Prym variety, one needs to look at Jacobians admitting *nontrivial* endomorphisms, i.e., endomorphisms which are not of the form $a \mapsto n \cdot a$, with $n \in \mathbb{Z}$ (every abelian variety contains $\mathbb{Z}$ as a subgroup of endomorphisms). As we mentioned before, every automorphism of the curve induces an automorphism of the Jacobian, and more generally, every endomorphism of JC comes from a *correspondence* on the curve. A correspondence on C is a divisor $D \subset C \times C$, which induces a map from the curve to the group of divisors on C of fixed degree:

$$C \to \operatorname{Div}^n(C), \qquad x \mapsto D|_{\{x\} \times C}.$$

Since the curve C generates JC as a group, by the universal property of the Jacobian we can extend this map to an endomorphism of JC. A nontrivial endomorphism γ_D on JC defines a proper subvariety $P := \operatorname{Im}(1 - \gamma_D)$ of the Jacobian.

We say that a principally polarized abelian variety (P, Ξ) is a *Prym–Tyurin variety*[4] for the curve C of exponent e if $P \overset{\iota}{\hookrightarrow} JC$ is a subvariety and $\iota^*\Theta \equiv e\Xi$. One checks that in this case the endomorphism Norm $N_P \in \operatorname{End} JC$ associated to P satisfies the equation $N_P(e - N_P) = 0$. For example, Jacobians are Prym–Tyurin varieties of exponent one, and classical Prym varieties are Prym–Tyurin of exponent two. Welters ([BL04, Corollary 12.2.4]) proved that *every* principally polarized abelian variety of dimension g is in fact a Prym–Tyurin variety for some curve of a very large genus, with exponent $3^{g-1}(g-1)!$. It is an open problem to find, for a given g, the smallest integer c such that every principally polarized abelian variety of dimension g is a Prym–Tyurin variety of exponent $e \leq c$. The issue is that, as with automorphisms, the general curve does not carry a nontrivial correspondence, which leaves us with little room to look for Prym–Tyurin varieties. However, many examples of families of Prym–Tyurin varieties have been constructed by employing different tools like representation theory or the geometry of lines in some Fano varieties.

The following *Prym–Tyurin map*, worked out in [ADF+] after the original construction of V. Kanev ([Kan89]), gives us a parametrization of $\mathcal{A}_6$ through special coverings; it illustrates how to use geometric objects to produce Prym–Tyurin varieties. Consider a smooth cubic threefold V in $\mathbb{P}^4$ and choose a pencil of hyperplanes $\{H_\lambda\}_{\lambda \in \mathbb{P}^1}$ in $\mathbb{P}^4$, that is, a family of hyperplanes parametrized by $\mathbb{P}^1$. With the right choice of the pencil, the intersection $V_\lambda := H_\lambda \cap V$ is a smooth cubic surface for the *general element* of the family. Actually, one can compute that there are exactly 24 critical values of λ for which this intersection is a singular cubic surface. On these singular surfaces some of the lines come together, so they contain 21 lines instead of 27. We define the curve of all the lines in the family:

$$C := \{(\lambda, \ell) \mid \ell \text{ a line in } V_\lambda\},$$

which is a smooth curve in the Grassmannian $\mathbb{G}(1, 4)$ of lines in $\mathbb{P}^4$. The projection on the first factor defines a finite morphism $h : C \to \mathbb{P}^1$ of degree 27, since a smooth cubic surface contains 27 lines. Moreover, over each of the 24 critical points the map h has 6 simple ramification points, corresponding to the places where 2 lines collide giving rise to a singular cubic surface. By the Riemann–Hurwitz formula it is easy to compute that C has genus 46. Now, we will use the incidence of the lines to define a correspondence on C:

$$D := \{((\lambda, \ell), (\lambda, \ell')) \in C \times C \mid \ell \cap \ell' \neq \emptyset \text{ in } V_\lambda\}$$

[4] *Also called generalized Prym variety.*

and we denote by $D(\ell)$ the divisor of degree 10 in C corresponding to the lines which are incident to ℓ. The geometry of the lines in a cubic surface imposes a relation

$$D(D(\ell)) + 4D(\ell) - 5\ell = 5\, g^1_{27}$$

as an equality of divisor classes on C, where the right-hand side is the divisor consisting of five times the fibre of ℓ. This equality implies that the divisor D induces an endomorphism γ_D on the Jacobian JC, satisfying the quadratic equation

$$\gamma_D^2 + 4\gamma_D - 5 = (\gamma_D + 5)(\gamma_D - 1) = 0.$$

It is not difficult to check that the correspondence has no fixed points (the diagonal in $C \times C$ does not intersect D), and it is symmetric, that is, $(\lambda, \ell), (\lambda, \ell')) \in D \Leftrightarrow ((\lambda, \ell), (\lambda, \ell')) \in D$. Then, by a criterion due to Kanev, one can conclude that the abelian variety defined as

$$PT(C, D) := \mathrm{Im}(\gamma_D - 1) \subset JC$$

carries a principal polarization Ξ, more precisely $\Theta_{|PT(C,D)} \equiv 6\Xi$. Therefore $PT(C, D)$ is a Prym–Tyurin variety of exponent 6 inside JC, and one computes that $\dim PT(C, D) = 6$. This is a fortunate observation since the attempt to dominate the moduli space $\mathcal{A}_g$ using Prym varieties failed for $g \geq 6$. In fact, a breakthrough theorem states that $\mathcal{A}_g$ is of general type for $g \geq 7$ ([Mum83]), which in particular means that these moduli spaces are as far as possible from being parametrized by projective spaces. On the other hand, $\mathcal{A}_g$ is unirational[5] for $g \leq 5$. But how about $\mathcal{A}_6$?

By counting the parameters involved in the construction one can verify that there are not enough Prym–Tyurin varieties constructed as above to dominate $\mathcal{A}_6$. The space parametrizing all cubic threefolds is of dimension 10, and the dimension of the pencils of hyperplanes in $\mathbb{P}^4$ is 6, so this geometrical construction does not cover the moduli space $\mathcal{A}_6$ which is of dimension 21. The key point is that we do not need cubic threefolds to build up a curve C with a correspondence of degree 10. What is essential is the action of the Weyl group $W(E_6)$ on the lines of a smooth cubic surface. We define the following candidate for a parameter space of $\mathcal{A}_6$:

$$\begin{aligned}\mathcal{H}_{E_6} = \{f : C \xrightarrow{27:1} \mathbb{P}^1 \mid Mon(f) = W(E_6),\ f \text{ is}\\ \text{ramified over 24 points with ramification}\\ \text{pattern determined by root reflexion}\}.\end{aligned}$$

This is a special Hurwitz scheme that incorporates the incidence correspondence of lines on a cubic surface in the monodromy of the covering; that is, for each $[C \xrightarrow{f} \mathbb{P}^1] \in \mathcal{H}_{E_6}$ we have a correspondence on C of degree 10 that behaves just like in the family of lines constructed above. If γ denotes the induced endomorphism on JC we define the associated Prym–Tyurin variety

$$(PT(C, \gamma), \Xi)$$

as the image of $\gamma - 1$ inside JC, and since the correspondence has the desired properties, the induced polarization is indeed principal, so it defines an element in $\mathcal{A}_6$. One observes that $\mathcal{H}_{E_6}$ depends on

$$\#\{\text{branch points}\} - \dim \mathrm{Aut}(\mathbb{P}^1) = 24 - 3 = 21$$

parameters. The following theorem confirms the intuition that $\mathcal{H}_{E_6}$ is a parameter space for $\mathcal{A}_6$ ([ADF$^+$]):

Theorem 2. *The Prym–Tyurin map*

$$PT : \mathcal{H}_{E_6} \to \mathcal{A}_6, \qquad [C \xrightarrow{f} \mathbb{P}^1] \mapsto (PT(C, \gamma), \Xi)$$

is generically finite. In particular, the general principally polarized abelian variety of dimension 6 *is a Prym–Tyurin variety of exponent* 6 *for a curve of genus* 46.

There are of course other spaces parametrizing $\mathcal{A}_6$ but this one involves curves, generalizing somehow the classical Prym map for $g = 4, 5$. A great deal is known about the geometry of Hurwitz schemes, so hopefully this construction will be useful to study 6-dimensional abelian varieties and the birational nature of $\mathcal{A}_6$. Notice the big leap on the complexity of the parameter space of $\mathcal{A}_g$: for $g = 5$ the group acting on the coverings is $\mathbb{Z}_2$, whereas for $g = 6$ the Jacobians of the coverings in $\mathcal{H}_{E_6}$ admit an action of $W(E_6)$ which is of order 51,840. It is also shown in [ADF$^+$] that a compactification $\overline{\mathcal{H}}_{E_6}$ of this Hurwitz space is of general type, which leaves us with the open question of whether $\mathcal{A}_6$ is of general type or not.

[5] *That is, there is a dominant rational map* $\mathbb{P}^N \dashrightarrow \mathcal{A}_g$.

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