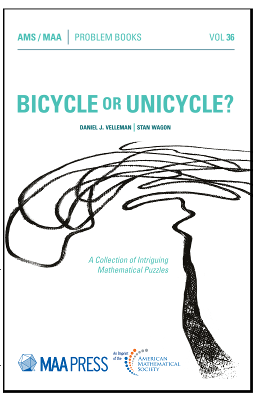
BOOK REVIEW

Bicycle or Unicycle: A Collection of Intriguing Mathematical Puzzles

Reviewed by Jason Rosenhouse



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Bicycle or Unicycle: A Collection of Intriguing Mathematical Puzzles
By Daniel J. Velleman and Stan Wagon

At Thanksgiving last year, my fourteen-year-old nephew showed me an interesting puzzle:

Puzzle 1. *You are playing solitaire in the first quadrant of the Cartesian plane, the lower corner of which is shown in Figure 1. You begin with a single checker on square a1. On each turn, a legal move consists of removing one checker from the board and then placing two new checkers in the cells immediately above and to the right of the original checker. If either of those two cells is occupied, then the move is illegal, and a different checker must be selected for removal. A possible sequence for the first two moves is shown in Figure 1. You keep making moves in this manner for as long as you wish. Prove that no matter how you move your checkers, it is impossible to contrive a situation in which the lower left 3×3 square (between rows 1–3, and between columns a–c) is entirely empty. That is, there will always be at least one checker in one of those nine cells.*

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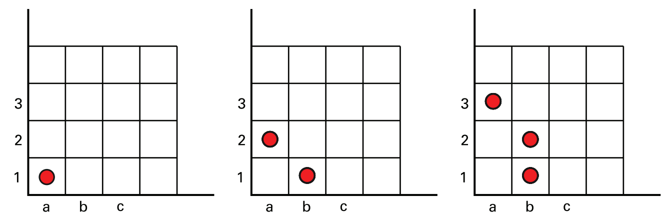

Figure 1. A possible sequence for the first two moves in our solitaire game. The only possible first move is to remove a1 and to place new checkers on a2 and b1. For the second move we chose to remove a2, and to place new checkers on a3 and b2.

My nephew participates in a math circle, meaning he comes to family gatherings armed to the teeth with puzzles like this. I was unable to solve it expeditiously, and his obvious delight in having stumped his mathematician uncle bordered on unseemly.

The solution is to assign a fraction to each cell, like this:

⋮	⋮	⋮	⋮	
1/8	1/16	1/32	1/64	...
1/4	1/8	1/16	1/32	...
1/2	1/4	1/8	1/16	...
1	1/2	1/4	1/8	...

Since each move entails removing one checker and replacing it with two checkers of half the value, the total sum of all the checkers is always one. A straightforward calculation shows that the cells outside the lower 3×3 square sum to less than one. It follows that there must always be at least one checker in the lower 3×3 square to make the sum of all the checkers work out properly. Very nice!

This sort of thing makes you want to philosophize about the difference between a problem and a puzzle. Problems are the sort of thing we inflict on students to force them to practice basic skills, while puzzles induce a smile and provide occasions for family bonding, at least among the mathematically inclined family members.

The line between a problem and a puzzle is often just one of presentation. I could imagine a calculus textbook asking the student to sum the entire array above, but then it would cease to be a puzzle. It would just be a tedious exercise in applying the formula for the sum of a geometric series. It is the snappy presentation in the form of a solitaire game, coupled with the flash of insight needed to see that infinite series are relevant, that transforms this into a fun and engaging puzzle.

Discussing this with my nephew reminded me of a puzzle I heard when I was in middle school. My older brother was taking algebra, and his teacher presented him with this:

Puzzle 2. *There are some horses and chickens in a barn, fifty animals in all. Horses have four legs while chickens have two. If there are* 130 *legs in the barn, then how many horses and how many chickens are there?*

This is not terribly difficult, of course. The stronger students will quickly think to write down the system (using the obvious notation)

$$\begin{aligned} H + C &= 50 \\ 4H + 2C &= 130, \end{aligned}$$

and will then apply one of the standard methods for solving such things. As a textbook problem, we might suspect this is meant simply as an exercise in solving linear systems, with the horses and chickens thrown in to make it into a "word problem."

This is not what made the impression on me. Rather, the memorable part was when my brother pointed out that this can be solved easily without writing down any equations at all. You just tell the horses to stand on their hind legs. Now there are fifty animals each with two legs on the ground, accounting for one hundred legs. That means there are thirty legs in the air. Since every horse has two legs in the air, we find that there are fifteen horses, and therefore thirty-five chickens.

I thought about these questions while reading Daniel Velleman's and Stan Wagon's latest collection of mathematical puzzles, which I enjoyed immensely. Fair warning, though: These puzzles are *hard*! I was looking for fodder to spring on my nephew at our next family gathering, but I fear he is a little too early in his mathematical education for most of what is here (though I can certainly imagine giving him a copy as a Hanukkah present a few years down the line). I will confess that for some of the puzzles I found it too much trouble even to work through the solutions, let alone to try to solve them myself. But those were in the minority, and most of the puzzles are both fun to solve and thought-provoking for the mathematical issues they raise.

There are 105 puzzles in all, with separate chapters for geometry, number theory, combinatorics, probability, calculus, algorithms and strategy, and miscellaneous puzzles. The title problem, too complex to state here, gets its own chapter. If you enjoyed Peter Winkler's puzzle collections [6, 7], then you will enjoy Velleman and Wagon as well. On the other hand, if Martin Gardner [3] is more to your liking, then you might find many of these puzzles a bit too intense. In the remainder of this review, I will spotlight a few of the puzzles that I found especially interesting.

Velleman and Wagon followed tradition in grouping their puzzles by mathematical discipline, but as I read I often found myself formulating my own groupings. For example, some of the puzzles I mentally placed under the heading, "Gosh! *Somebody* noticed that."

Here is one such puzzle, referred to by the authors as "Pascal's Determinant":

Puzzle 3. *Suppose that Pascal's triangle is written as follows:*

$$\begin{matrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & \cdots \\ 1 & 3 & 6 & 10 & 15 & \cdots \\ 1 & 4 & 10 & 20 & 35 & \cdots \\ 1 & 5 & 15 & 35 & 70 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{matrix}$$

The first row and column consist entirely of 1*s, and every other number is the sum of the number to its left and the number above. For each positive number* n*, let* P_n *denote the matrix consisting of the first* n *rows and columns of this array. What is the determinant of* P_n*?*

There is a trick to this: The matrix P_n can be factored into the product of a lower triangular matrix and an upper triangular matrix. More specifically, consider these two arrays:

$$\begin{matrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & \cdots \\ 1 & 4 & 6 & 4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{matrix} \qquad\qquad \begin{matrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 2 & 3 & 4 & \cdots \\ 0 & 0 & 1 & 3 & 6 & \cdots \\ 0 & 0 & 0 & 1 & 4 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{matrix}$$

If we let L_n and U_n denote, respectively, the first n rows and columns of the left and right arrays, then it turns out that $P_n = L_n U_n$. This immediately implies that $\det P_n = 1$ for all n (since the determinant is multiplicative, and since the determinants of triangular matrices are given by the products of their diagonal entries). Velleman and Wagon provide a very nontrivial combinatorial proof for this factorization. (They also refer readers to a paper by Edelman and Strang [2] that provides three additional proofs of the factorization.)

Do you see what I mean? *Somebody* noticed that!

My favorite puzzle in this category comes from the chapter on geometry:

Puzzle 4. *Bisect the four sides of a convex quadrilateral and connect each midpoint to an opposite corner, choosing the first one in counterclockwise order as in Figure 2. Prove that the central quadrilateral has the same area as the union of the four corner triangles.*

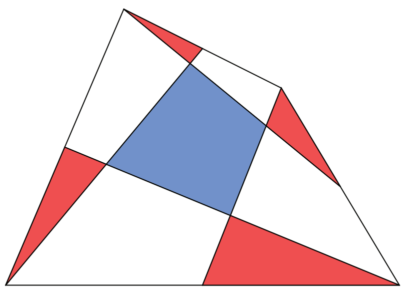

Figure 2. The construction described in Puzzle 4.

The solution is a marvelous "proof without words," as shown in Figure 3.

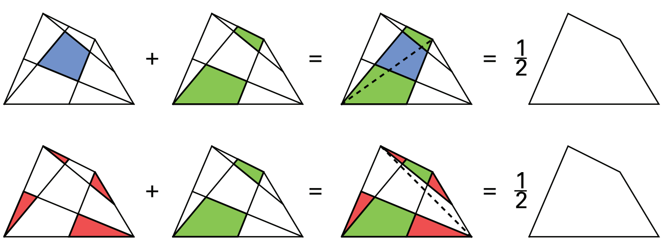

Figure 3. The two dashed lines divide the original quadrilateral into two triangles. Within each triangle, the colored and white areas are themselves triangles, each having the same base and height.

One more time: *Somebody* noticed that! (Velleman and Wagon cite Ash, Ash, and Ash [1] and Mabry [4] for further information on this puzzle.)

Another category is: "Puzzles that make you say, 'Really?'" Sometimes the task that is put before you seems so impossible, or the result that is proved seems so surprising, that you just have to stare for a while before coming to terms with it. Here is an especially enjoyable example:

Puzzle 5. *A round cake has icing on the top but not the bottom. Cut out a piece in the usual shape (a sector of a circle with vertex at the center), remove it, turn it upside down, and replace it in the cake to restore roundness. Do the same with the next piece; i.e. take a piece with the same vertex angle, but rotated counterclockwise from the first one so that one boundary edge coincides with a boundary edge of the first piece. Remove it, turn it upside-down, and replace it. Keep doing this in a counterclockwise direction. If θ is the central angle of the pieces, let $f(\theta)$ be the smallest number of steps needed so that all icing returns to the top of the cake, with $f(\theta)$ set to ∞ if this never happens. For example, $f(90°) = 8$.*

- *What is $f(181°)$?*
- *What is $f(1\text{ radian})$?*

The answers are wonderfully counterintuitive, and I would urge you to have a go at this one yourself. Wagon has produced an online resource that will allow readers to simulate the process described in the puzzle for any starting angle θ [5].

The number theory chapter is especially rife with surprising results:

Puzzle 6. *Let a, b, c be positive integers with no factor in common to all three such that*

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c}.$$

Prove that $a + b, a - c$, and $b - c$ are all perfect squares.

Puzzle 7. *Prove that a positive integer n is prime if and only if there is a unique pair of positive integers j and k such that*

$$\frac{1}{j} - \frac{1}{k} = \frac{1}{n}.$$

The solutions to these are too complex to include here, but the results themselves strike me as fascinating and surprising. Egyptian fractions, meaning sums of fractions all of which have numerator one, just seem to be a bottomless pit of interesting problems!

Another favorite category is "Quickies." Some puzzles require an elaborate setup and a fair amount of thought just to understand what is being asked. The quickies, by contrast, have a stark simplicity that makes them all the more enjoyable. Any good puzzle has a snappy, clever solution, but with the quickies there is something ingenious just in asking the question in the first place.

Velleman and Wagon offer several such examples, among which I especially liked these two (again, with apologies for not including the complex solutions):

Puzzle 8. *Can the rational numbers in the interval $[0, 1]$ be enumerated as a sequence $q_1, q_2, \ldots$ in such a way that $\sum_{n=1}^{\infty} q_n/n$ converges?*

Puzzle 9. *Consider a standard, circular 12-hour clock where the three hands have equal length, and all point vertically upward at midnight.*

- *Is there a time when the ends of the three hands form an equilateral triangle?*
- *Same question, but assume there are 11 hours in the full circle as opposed to 12 (but still with 60 minutes to an hour and 60 seconds to a minute.)*

Puzzles of this sort are enjoyable to read even if you do not attempt to solve them. Under this category, I would also mention this one:

Puzzle 10. *Suppose the parabola $y = x^2$, with focus at $(0, 1/4)$, rolls without slipping along the x-axis. What is the locus of the focus?*

It is not that I think this is an especially ingenious puzzle. I mention it only because the phrase "locus of the focus" made me laugh, since it reminded me of Danny Kaye in *The Court Jester*. (Google "chalice from the palace" if you do not know the reference.)

There are many anthologies of mathematical brainteasers available, but most of them are rather basic for those of us in the business. Books of high-level mathematical puzzles are harder to come by. Velleman and Wagon have produced an admirable addition to this literature. I suspect most of the puzzles will be new to most readers, and their presentations of better-known puzzles invariably bring some new angle to the discussion. Anyone with a taste for mathematical puzzles, which I assume includes the entire readership of the *Notices*, will benefit from reading this book.

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Credits

Figure 1 and author photo are courtesy of Jason Rosenhouse. Figures 2 and 3 are courtesy of Daniel J. Velleman and Stan Wagon.