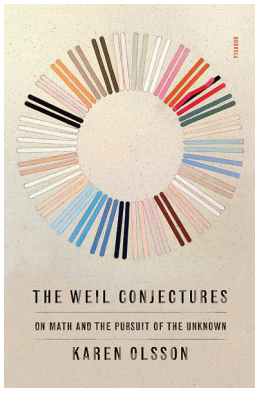
BOOK REVIEW

The Weil Conjectures

Reviewed by Brian Hayes



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The Weil Conjectures
On Math and the Pursuit of the Unknown
by Karen Olsson

In the winter of 1940 Simone Weil was urging her brother André to write an expository account of his mathematical research, for her own benefit and for the world at large. He had plenty of time, she pointed out. He was confined to a French prison, awaiting trial on charges of failing to report for military service. André replied:

> Telling nonspecialists of my research or of any other mathematical research, it seems to me, is like explaining a symphony to a deaf person. It could be attempted, you could talk of images and themes, of sad harmonies or triumphant dissonances, but in the end what would you have? A kind of poem, good or bad, unrelated to the thing it pretends to describe.

André's dismissive response strikes me as surly and arrogant, and yet there's surely truth in it. The latest ideas from the frontiers of research are seldom fit for armchair consumption. It's the nature of a frontier that you have to do some bushwhacking to get there. On the other hand, if the discoveries of a research community are so abstruse that they can never be understood outside a small coterie of initiates, what's the point of discovering them? Somehow, the explorers of new territory have to send the occasional dispatch back to civilization, to let us know what they've found.

André did eventually write the account that Simone had asked for, but it was highly technical, "a treatise in the form of a letter," perhaps meant more to intimidate than to inform. (An English translation was published in the *Notices* in 2005 [1].) Simone wrote back, "I understood nothing." She continued to worry that mathematics was becoming too remote from ordinary life. With a fierce sense of social responsibility, she wanted her brother's work to serve human needs, or at least to reveal something about the world we all live in.

Karen Olsson, who tells this story in *The Weil Conjectures: On Math and the Pursuit of the Unknown*, seems sympathetic to both sides of the dispute. Like André, she believes that mathematics should speak for itself. She wants to understand it in mechanistic detail; she won't settle for gauzy metaphors or analogies. But she also shares Simone's concerns. Why should I care about these forbidding abstractions? she asks herself. What do they have to do with my life as a writer, a parent, a citizen? No clear answers are

Figure 1. André and Simone, 1922.

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forthcoming, and yet still she yearns to learn, as if echoing Hilbert's "We must know! We will know!"

Before going further I should make clear that *The Weil Conjectures* is not a textbook or a scholarly monograph. It is not addressed to an audience of mathematicians. But it raises questions about relations between mathematics and society that may well be of interest to the mathematical community. This issue is commonly discussed in terms of *outreach*—the challenge of communicating research-level mathematics to the public. In Olsson's case it also becomes a question of *inreach*: how can we help someone who feels a powerful attraction to mathematical ideas but cannot negotiate the rugged terrain of prerequisite knowledge?

The heart of Olsson's book is a personal essay, in which she describes her own intense and turbulent encounters with the world of mathematics. That narrative is braided into the stories of the Weil siblings—whose lives were also marked by intensity and turbulence.

André Weil (1906–1998) was a child wonder who grew into an enfant terrible, swaggering across the landscape of French mathematics while still in his twenties. He was a founder and ringleader of the Bourbaki collective. During his jail time in 1940 he proved a variant of the Riemann hypothesis for curves over finite fields. The conjectures referred to in Olsson's title extend this result from curves to varieties (the higher-dimensional analogs of curves) and forge an unexpected link between two distant realms—number theory and topology. By now the conjectures are all theorems (proved by Alexander Grothendieck and Pierre Deligne, among others), yet they are still widely known as the Weil conjectures.

Simone Weil (1909–1943), equally brilliant and precocious, was her older brother's first pupil and devoted childhood companion. Olsson sets the scene: "Simone and André memorize long sections of verse by Corneille and Racine, and they recite pieces of them in turn, staring bug-eyed at each other. It's a contest: although they smirk as they call out the lines, every time one of them misses a word or mangles a phrase, the other delivers a hard slap to the face."

Brother and sister both went on to study at the École Normale Supérieure, but Simone drifted into philosophy rather than mathematics. Later she moved on to social and political activism, and then mystical theology. She could not endure a comfortable life while others suffered, so she sought out work in factories and mines—jobs for which she was utterly unsuited. She volunteered on the Republican side in the Spanish Civil War, but her career at the front ended when she stepped into a cooking pot and scalded her leg.

André's reluctance to take up arms a few years later was not based on pacifist convictions; rather, he felt that his *dharma*—his fate and his duty—was to make mathematics, not war. To get out of jail he ultimately agreed to report for military service, but a few weeks later the French forces surrendered.

The Weil family escaped Europe at the last possible moment before the Nazis closed the exits. In the New World, André had several vagabond years before finding a home at the University of Chicago; later he moved to the Institute for Advanced Study. In 1942 Simone insisted on returning to Europe, with a plan to parachute into battle and nurse the wounded. She died before that fantasy could be fulfilled, succumbing to tuberculosis and self-induced malnutrition at age 34.

The Weils are fascinating, larger-than-life figures, but their stories have been told elsewhere [2, 3]. In the rest of this review I want to focus on the more personal part of Olsson's narrative. As an adolescent on the lookout for role models, she was drawn to the writings and the life story of Simone, but she also developed an early enthusiasm for mathematics, which eventually led her to an interest in André's work.

"There's a certain kind of young kid for whom the word *algebra* has a magical shimmer," she writes, "portending the enigmas of grades not yet reached, all the unimaginable revelations of junior high and high school." She couldn't wait to get to the x's and y's. And the romance did not end when those first secrets were revealed. In her sophomore year at Harvard, in the 1990s, she weighed her options and interests, and chose a mathematics major. She remembers late-night walks across a snowy campus. "I experienced then … a kind of pleasure that (for me) came only after having thought hard about math, the mental equivalent of having gone for a long run. A gentle euphoria."

The pleasures were not all solitary. "We were a small band of students giddily, exhaustedly trekking through an abstract moonscape, helping one another across patches of ice or fighting over which directions to head next. The egos, the insecurities, the unabashed nerdiness! I miss it still …. Then there was the fact that I had a serious boyfriend for the first time …. Part of loving math, for me, was loving a person who also loved math."

But there's more to the story of how Olsson wound up a math major, then *didn't* wind up a mathematician. There was an attack of impostor syndrome (though she never uses that phrase):

> From those exceptional kids I detected (or at least imagined) some mix of scorn and pity for someone like me, smart enough to get by, but just the ordinary type of smart. Much as mathematics came with a democratic ideology, according to which it was a realm of rarefied delights open to anyone who wished to work her way along its paths, there also seemed to be an unstated but obvious hierarchy. If math to me was a dark place where I went groping around on my hands and knees, here were these other

> people with (it seemed to me) killer night vision who could see everything at once, go prancing from one topic to the next."

Besides, her real ambition was to be a novelist. Upon graduating, she tucked away her mathematics degree and went off in another direction. She became a newspaper reporter and editor in Austin, Texas, and eventually she published a couple of novels (*Waterloo* and *All the Houses*, both set in a political milieu). It was not until 20 years later that the mathematical itch came back, provoked in part by her young son's awakening interest. ("Give me an algebra problem, he begs.")

For reasons that aren't made clear, the Weil conjectures became a focal point of Olsson's mathematical revival. "Though I didn't go far enough in math to really understand the Weil conjectures, nevertheless I wonder, to what extent could I appreciate more about them? A bee in my bonnet, a dubious goal: maybe I could try to apprehend something of their flavor, I think, but at the same time I don't know what that would mean."

She turns to YouTube, where she finds a series of lectures in abstract algebra recorded in 2003 by Benedict Gross [4]. It's a course she took at Harvard a decade earlier, though with a different instructor. She still has the textbook. (Based on her description of the cover, it's Michael Artin's *Algebra*.) As she works her way through the lectures, her comments emphasize blackboard mannerisms, historical digressions, and classroom interruptions. Gross presents a theorem "like a magician announcing his next trick," she says. She tells us less about the mathematics itself. And she doesn't report any progress in penetrating the mysteries of the Weil conjectures.

Later she discovers that a classmate from her Harvard years is now a professor at her hometown university. After weeks of hesitation she sends him a carefully composed email, asking if they might chat sometime about the Weil conjectures. Weeks pass; there's no reply.

It's back to YouTube, but that doesn't go well either.

> In the middle of watching the twenty-first online algebra lecture, I hit a wall .... While Professor Gross was elaborating on the Sylow theorems, as he was saying that "any two p-Sylow subgroups H and H' are conjugate," I became instantly tetchy, I couldn't take it any longer. Who cares? I am a midlife mother of two, I thought morosely, and this is the most pointless thing I could possibly be doing.

In the neighborhood supermarket, she spots the professor who never answered her email, and chases him through the aisles until she corners him with her shopping cart in the tortilla section. He apologizes for not responding. They speak about getting together sometime to talk math, but they don't set a date.

As Olsson relates these disheartening developments in her own mathematical journey, she is also wrapping up the narrative thread on the life of André Weil. She passes along some poignant stories of André's last years told by his daughter Sylvie [5], and she admires an elegy written by Goro Shimura [6]. "Is that what I am writing, I wonder, some sort of elegy for math, or for my own entanglement with math? At times it feels that way, but I don't think that's what this is. As it turns out, one stilted encounter in the supermarket is enough to send me back to the algebra lectures, which for no good reason I still want to finish. And so it's on to ring theory, which is of course nothing I need to remember, nothing I need to know."

Much as I enjoyed tagging along with Olsson on her mathematical ramble, I am mildly disappointed with the way the journey ends. I had looked forward to finally reading her account of what the Weil conjectures are all about, and what they mean to her—however fuzzy and fragmentary the appraisal might have to be. Earlier in the book she demonstrates an impressive talent for mathematical exposition. For example, she gives a deft and sure-footed explanation of a fixed point of a continuous function, in terms intelligible to readers who don't know what a function is, or why continuity would matter. I strongly suspect Olsson knows more than she lets on. Nevertheless, when it comes to the big challenge, which I had thought would be the climax to the story, she ducks.

I am also annoyed at her failure to exploit all of the resources available to her. Tuning in to the Gross lectures was surely a good idea, but what about that textbook she mentions? Apparently she never opens it; the book comes off the shelf only as a plaything for her six-year-old. Giving up on the search for a mentor after a single failed attempt also seems pretty feckless. I say this as someone who has often needed help of the same kind, and the mathematical community has always responded generously. If Olsson had persisted, I'm sure she would have found someone eager to offer guidance.

As far as I can tell, Olsson also never consulted the primary literature. If she tried to read the brief paper [7] in which Weil introduced the conjectures, she does not tell us about it. Nor does she mention any of the various commentaries, tutorials, and review articles [8–12] that endeavor to explain the conjectures and the subsequent proofs. None of these publications are easy reading. They assume familiarity with a whole catalog of ideas from abstract algebra, number theory, and topology: finite fields, varieties, rings, zeta functions, fixed-point theorems. That's a heavy cargo of conceptual baggage to tote around. Nevertheless, mastering some of this material seems like a necessary step if you want to write about the subject. There is no royal road to algebraic geometry.

Although Olsson's story can be exasperating at times, on the whole I find her quest inspiring. It's not every day you meet a journalist and novelist who longs for a deeper

understanding of varieties over finite fields and their zeta functions. She is not doing it for grades or for glory, but simply because something about mathematics calls out to her. I hope she will continue, and eventually find fulfillment rather than frustration.

For the mathematics community, Olsson's experience raises the question of how research-level mathematics can be made intelligible to those outside the field. Is it possible? Is it worth the effort? We already have André Weil's answer. He seemed quite comfortable with the idea of mathematics as an elite guild, open only to those of exceptional talent; the rest of the world is deaf to the symphony. Perhaps he was right, but if so the situation is rather sad. Beautiful music is played in an empty concert hall, with no one but the composer and the orchestra able to appreciate it. And there's a practical concern: in general it's the audience that provides material support to the musicians.

Weil's inward-looking view is certainly not universal. For many others, mathematics is something worth sharing—a thing of beauty, a useful tool for understanding the world we live in, a window onto an unexpected universe. They work to engage the public through teaching, lectures, expository writing, mentoring. Olsson's story offers a bit of cheerful news to these evangelists: it's proof that someone out there is listening, keen to hear the message. But it also underlines how much hard work is needed to open a line of communication between research mathematicians and the general public.

The task is not merely translation or interpretation—making the vocabulary of mathematics comprehensible. The crucial challenge is motivation: conveying a sense of why a mathematical idea is worth the trouble of understanding. After giving a brief explanation of Fermat's Little Theorem, Olsson remarks:

> Even Fermat's relatively simple theorem starts to grow hair when I try to lay it out in ordinary language, I realize, and it's hard to articulate why it's interesting without invoking more math. At the end of the day, why should a nonmathematician care that André Weil figured out how to count solutions to polynomial equations in finite number fields? In one sense, I myself don't care. I don't understand it well enough to care. But there's a kind of … not quite gratification but the prospect of it, a door that cracks open just a sliver when I learn about these constructed realms and the relations within and among them ….

Can we open that door just a little wider?

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Credits

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