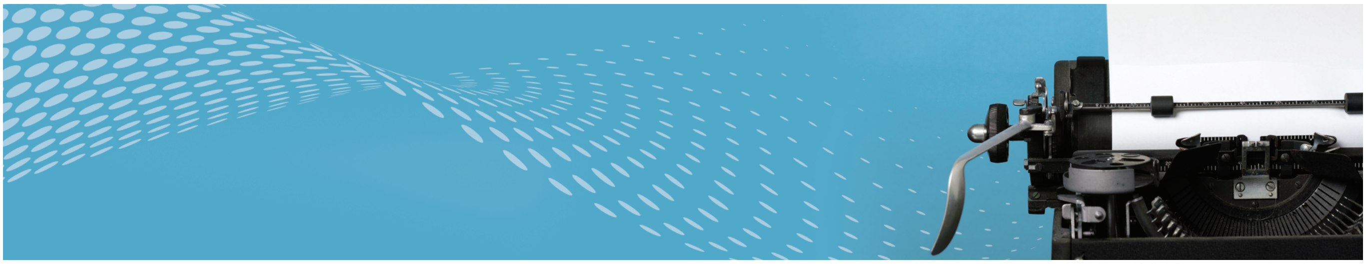

Six Surfaces (Almost) Surely

Danny Calegari

In the television show "Star Trek," whenever Spock and Kirk and an anonymous ensign marked for death beam down to a strange alien planet, chances are they'll encounter a landscape oddly reminiscent of Southern California, populated by a race of English-speaking bipedal humanoids whose otherness is certified by one or two facial prosthetics and the unusual color of their polyester jumpsuits.

This uniformity, given the diversity of biological lifeforms and habitats on even our own planet, is an artefact of the limited budget and special effects capabilities of Paramount studios in the 1960s. But even in mathematical environments unimaginably rich in possibility, the dice can be loaded in surprisingly frugal ways.

One of my favorite examples is an astonishing theorem of Étienne Ghys, on the topology of "typical" leaves of foliations. A foliation is a manifold clothed in a stripy fabric: the space is filled up with submanifolds (the "leaves" of the foliation) of lower dimension, which locally are arranged in products, but globally can exhibit all kinds of interesting behavior.

Any reasonable nowhere zero vector field integrates to give a 1-dimensional foliation by integral curves. Likewise, a holomorphic vector field on a complex manifold integrates to a foliation by leaves of real dimension two. Though there are some restrictions in higher dimensions, John Cantwell and Larry Conlon proved that *every* 2-dimensional manifold—i.e., every surface—arises as a leaf of some foliation of some compact manifold.

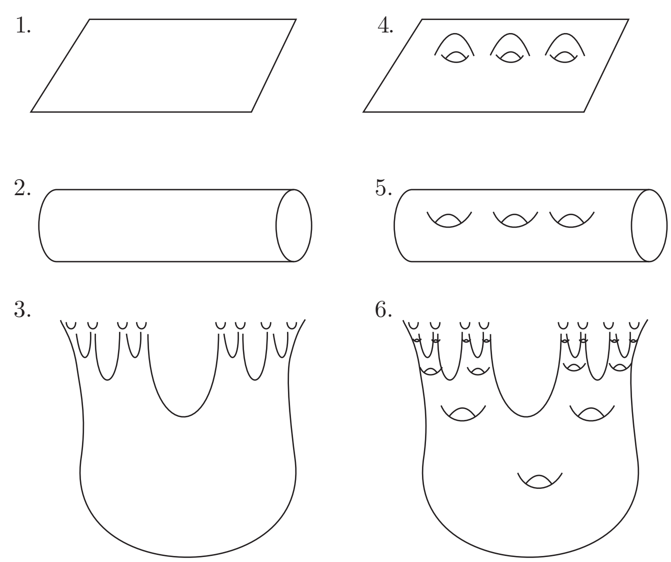

Figure 1. The usual suspects.

Compact (oriented) surfaces are classified by genus—the number of handles—which can be any nonnegative integer. Noncompact (oriented) surfaces are more numerous. There are uncountably many of them. They are classified by two invariants. First there is the space of *ends*. The "ends" of a topological space are—roughly speaking—the "connected components at infinity." Technically, the path-connected components (i.e., π_0) of the complements

Danny Calegari is a professor of mathematics at the University of Chicago. His email address is dannyc@math.uchicago.edu.

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of bigger and bigger compact subsets of a space form an inverse system, and the ends are the points in the inverse limit of this system. The space of ends of a surface is always compact and totally disconnected, so that it embeds in a Cantor set. Every possible homeomorphism type of a (nonempty) compact subset of a Cantor set can arise as the space of ends of some noncompact surface. Second, one considers genus (i.e., number of "handles"). If the genus is finite, it can be any nonnegative integer. If it is infinite, one must consider how the handles interact with the ends—some ends have a neighborhood which is planar, the others (a closed subset of them) are accumulated by handles. This subset of nonplanar ends may again be any closed subset of the space of all ends, and the pair of spaces (ends, nonplanar ends) up to homeomorphism is a complete invariant.

For example: there is exactly one oriented surface (up to homeomorphism) whose space of ends is homeomorphic to the ordinal $\omega^\omega + 1$, and for which the only ends accumulated by handles are those whose ordinal is divisible by ω^{666}. But out of this crazy proliferation of possibilities, Ghys's theorem says that a *typical noncompact leaf* is almost surely homeomorphic to one of only *six* (!) possibilities, illustrated in Figure 1.

Three of the surfaces are planar: the plane (one end), the cylinder (two ends), and the sphere minus a Cantor set (sometimes called the *Cantor tree* surface); the points in the missing Cantor set are precisely the ends of this surface. The other three surfaces are obtained from these three by plumbing with infinitely many "handles," accumulating to every end. Colloquial names for the latter three are the *Loch Ness monster*, the *ladder* surface, and the *blooming Cantor tree* surface.

To understand Ghys's theorem we must first understand what is meant by the expression "a typical leaf." The word "typical" refers to a certain kind of probability measure on the foliated manifold called a *harmonic measure*, and a "typical point" is a point chosen at random with respect to this measure. Finally, the leaf containing this typical point is a "typical leaf." It's important to get the order right: Dennis Sullivan famously observed that "there is no measurable way to pick a point on a leaf."

On a compact Riemannian manifold, any initial heat distribution becomes equidistributed over time under the heat flow. Another way of saying this is that the uniform (volume) measure is the unique probability measure invariant under heat flow. Equivalently, it is preserved (on average) by *Brownian motion*, the random process obtained as a parabolically scaled limit of random walk. On a (compact, Riemannian) foliated manifold we want to look for probability measures that are invariant under a restricted sort of heat flow, where heat is only allowed to diffuse along the leaves. Any such probability measure is called a *harmonic measure* for the foliation. Lucy Garnett proved that every foliation of a compact manifold admits at least one harmonic measure. There might be more than one; for example, the uniform volume measure concentrated on any compact leaf is a harmonic measure. With respect to the local product structure of a foliation, any harmonic measure disintegrates leafwise into the leafwise volume measure times a leafwise harmonic function. If some leaves are noncompact, they might admit many nonconstant positive harmonic functions, and in general harmonic measures can be quite interesting.

Ghys's theorem arises from the fact that we can think about leafwise Brownian motion in two quite distinct ways: as a process on a compact manifold, or as a process on a (typically) noncompact leaf. Leafwise Brownian motion preserves harmonic measure. If we think of it as a measure-preserving random process on a compact manifold, then Poincaré recurrence means that for every Borel set B of positive harmonic measure, leafwise Brownian motion starting at (almost) any $x \in B$ will return to B infinitely often, and in fact the expected time of first return is finite. On the other hand, if we think of Brownian motion as a random process on any particular noncompact leaf, it will turn out that the expected time of return to any compact subset of the leaf is *infinite*.

The simplest case to think about is an end with the geometry of a half-infinite cylinder. Brownian motion in this end is roughly like simple random walk on the positive integers. As is well known, a simple random walk started at any fixed positive integer n will return to 0 almost surely. However, the expected time before it returns is infinite! Let's see why. Let $R(n)$ be the expected time for a random walk beginning at n to return to 0. Evidently $R(n)$ is monotone increasing; we will show in fact that $R(1)$ (and hence $R(n)$ for all positive n) is infinite.

From the definition, $R(0) = 0$ and

$$R(n) = 1 + 1/2(R(n-1) + R(n+1))$$

for every positive n. Summing the latter formula for n ranging from 1 to some big N and rearranging, we see that $R(N) = R(N+1) - R(1) + 2N$. If $R(1)$ were finite, then for sufficiently big N we would have $R(N) > R(N+1)$, which is absurd. So $R(1)$ (and therefore also $R(n)$ for all positive n) is infinite.

By comparing leafwise Brownian motion in M and in a leaf λ, we deduce that for any Borel set $B \subset M$ of positive harmonic measure and for almost every $x \in B$, every end of the leaf $\lambda(x)$ containing x intersects B. How might we apply this, focusing for concreteness on the case of a compact manifold foliated by surfaces? We want to show that a leaf with more than two ends has an entire Cantor set of ends; and furthermore, that a leaf of positive genus has every end accumulated by handles. A surface S has more than two ends if and only if it contains a compact subset P whose complement has at least three unbounded components. Let's let $B(n)$ be the union of all such compact subsets of (leafwise) diameter at most n over all leaves, and let $B = \bigcup_n B(n)$. It turns out that each $B(n)$ is Borel. To see this, let $B(n, N)$ denote the set of leafwise compact subsets P of diameter at most n whose complement in a ball of radius $N + n$ has at least three components of diameter at least N. The property of being some P in $B(n, N)$ depends only on the geometry of a ball of radius $N + n$ in some leaf, and therefore the set $B(n, N)$ is *closed*. Since $B(n) = \bigcap_N B(n, N)$, it follows that $B(n)$ (and also B) is Borel.

A leaf disjoint from B has at most two ends. Any other leaf must intersect some $B(n)$, and therefore (almost surely), every end intersects $B(n)$. This means that every end contains a subset P of diameter at most n with at least three unbounded components. This implies that no end is isolated, so that the space of ends is therefore perfect, and thus homeomorphic to a Cantor set.

In a similar way, a surface S has positive genus if and only if it contains a handle—i.e., a subset H homeomorphic to a once-punctured torus. Let $C(n)$ denote the union of all leafwise handles of diameter at most n, and let $C = \bigcup_n C(n)$. Then each $C(n)$ is closed, and therefore Borel. A leaf disjoint from C is planar. Any other leaf must intersect some $C(n)$, and therefore (almost surely) every end intersects $C(n)$, so that every end is accumulated by handles. From this Ghys's theorem follows immediately.

AUTHOR'S NOTE. Ghys's theorem is proved in "Topologie des feuilles génériques," Ann. Math. 141 (1995), 387–422.

Danny Calegari

Credits

Figure 1 was created by the American Mathematical Society. Photo of Danny Calegari is courtesy of Danny Calegari.