Eigenfunction Concentration via Geodesic Beams

Yaiza Canzani and Jeffrey Galkowski

A vast array of physical phenomena, ranging from the propagation of waves to the location of quantum particles, is dictated by the behavior of Laplace eigenfunctions, i.e., solutions to the Helmholtz equation

$$-\Delta\phi_\lambda = \lambda\phi_\lambda.$$

In quantum mechanics, $|\phi_\lambda(x)|^2$ describes the probability density of finding a free quantum particle of energy λ at the point x. It is then natural to ask

how large can $\phi_\lambda(x)$ be?

Starting in the 1950s, Avakumovich, Levitan, and Hörmander proved that, for a smooth compact Riemannian manifold M, there is a constant C such that

$$\max_{x\in M} |\phi_\lambda(x)| \leq C\lambda^{(n-1)/4}, \qquad (*)$$

where n is the dimension of M. This bound is sharp, with the zonal harmonics on the round sphere saturating it (these are eigenfunctions that have sharp peaks near the north and south poles). However, for most manifolds, the eigenfunctions are expected to be much smaller than the bound in $(*)$. More importantly, for a fixed manifold and most points $x \in M$, the value $\phi_\lambda(x)$ is not expected to saturate the bound.

In this talk we will discuss how the growth of $\phi_\lambda(x)$ responds to the long-time behavior of the geodesics that run through x.

To study this problem, we developed a framework in which the eigenfunction ϕ_λ is decomposed as a sum of what we call *geodesic beams* near the point x. In broad terms, a geodesic beam is a piece of the eigenfunction that has been localized to a geodesic that runs through x. This localization is accomplished using semiclassical analysis and is done in such a way that each geodesic beam is (locally) an approximate solution to the Helmholtz equation.

In this talk, we present the geodesic beam techniques and explain how to use them to obtain *quantitative improvements* on standard estimates such as $(*)$. For example, we will see that

$$|\phi_\lambda(x)| \leq C\lambda^{(n-1)/4}/\sqrt{\log\lambda},$$

whenever the point x is not maximally self-conjugate at $1/\log\lambda$ scales.

Remarkably, this framework allows for the treatment of several other problems related to eigenfunction concentration, including L^p norms, averages over submanifolds, and both pointwise and integrated Weyl laws. One consequence of this method is that if M is *any* nontrivial product manifold with Laplace eigenvalues $0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$, then the eigenvalue counting function, $N(\lambda) = \#\{j : \lambda_j \leq \lambda\}$, satisfies the Weyl law

$$N(\lambda) = \frac{1}{(2\pi)^n}\mathrm{vol}_{\mathbb{R}^n}(B)\mathrm{vol}(M)\lambda^{\frac{n}{2}} + O\Big(\frac{\lambda^{\frac{n-1}{2}}}{\log\lambda}\Big),$$

as $\lambda \to \infty$, where B is the unit ball in $\mathbb{R}^n$.


Yaiza Canzani


Jeffrey Galkowski

Credits

Photo of Yaiza Canzani is courtesy of Nicolás Fraiman.
Photo of Jeffrey Galkowski is courtesy of Marit Orav.

Yaiza Canzani is an assistant professor in the department of mathematics at UNC-Chapel Hill. Her email address is canzani@email.unc.edu.

Jeffrey Galkowski is an associate professor in the department of mathematics at University College London. His email address is j.galkowski@ucl.ac.uk.

DOI: https://doi.org/10.1090/noti2247