Heat Flow and Sets of Finite Perimeter

Fabrice Baudoin

A basic motivating question is: *what is a good mathematical structure on a space that will allow us to define an intuitively reasonable and useful notion of perimeter for "good" sets?* The question is, of course, a little vague but thinking about it from different viewpoints is certainly fruitful and yields an interesting mathematical adventure.

We will argue that Dirichlet spaces provide a convenient framework in which we can define sets of finite perimeter and prove theorems generalizing classical results from Euclidean space, like the classical isoperimetric inequality, in an elegant way.

The isoperimetric inequality in the plane. Dido was, according to ancient Greek and Roman sources, the founder and first Queen of Carthage (in modern-day Tunisia). The legend says that when Dido arrived in 814 BCE on the coast of Tunisia, she asked for a piece of land. Her request would be satisfied provided that the land could be encompassed by an oxhide. This land became Carthage and Dido became the queen. But: *what is the shape of the land chosen by Queen Dido?* Mathematically, we can first simplify and rephrase this problem as: in the Euclidean plane, what is the curve which encloses the maximum area A for a given perimeter P?

This is the isoperimetric problem in the plane. If we restrict ourselves to smooth closed curves and understand the perimeter of a set as the length of its boundary, then the problem can be solved using basic calculus: the method of Lagrange multipliers. As is intuitively clear, the curve which encloses the maximum area A for a given perimeter P is the circle and one has the isoperimetric inequality

$$A \leq \frac{1}{4\pi} P^2.$$

So the city limits of Carthage formed a (half) circle. We can still visit the ruins of Carthage today, and a small museum there explains the isoperimetric story of Queen Dido.

Modern footprints. As mathematicians, we like to formulate problems at different levels of generality and in a form that can be generalized to different settings. To formulate the isoperimetric problem in the n-dimensional Euclidean space R^n, one needs a notion of volume of a set and a notion of perimeter of a set. The rigorous and deep understanding of these two notions has motivated many of the spectacular developments in geometric measure theory throughout the 20th and beginning of the 21st centuries. Our modern understanding of the notion of volume is largely based on the seminal works (1901–1902) by H. Lebesgue. Nowadays, one understands volume in the category of measure spaces. The notion of perimeter is more elusive and can be understood from different viewpoints, yielding different generalizations. The Italian school in geometric measure theory played a major role in advancing the theory.

The first idea for how to define the perimeter of a Borel set $E \subset R^n$ is due to R. Caccioppoli (1904–1959) and is based on a generalization of the notion of rectifiability for a curve. Essentially, the perimeter of a set is the limit of the surface areas of polyhedra approximating E. This is a perfectly reasonable take on what quantity the perimeter of a set should represent. However, this definition is somehow rigid. It makes strong use of the Euclidean structure of the space, using the notion of a polyhedron and its area. Also, this definition is difficult to reconcile with the variational interpretation of perimeter that is given by the calculus of variations (the Gauss-Green formula). A somehow dual, more flexible and easier to work with, definition of the perimeter of a set is also due to R. Caccioppoli. It coincides with the previous one and is given by

$$P(E) = \sup \int_E \mathbf{div}(\phi) dx,$$

where the supremum is taken over the set of smooth and compactly supported vector fields in R^n such that $\|\phi\|_\infty \leq 1$. We note that this definition only requires a differential structure and a Riemannian metric to define the divergence of vector fields; it can therefore be extended to any Riemannian manifold.

More recently, following works by E. De Giorgi (1928–1996) and M. Ledoux, a characterization of sets with finite perimeter in R^n that uses very little structure was discovered. If E is a Borel set in R^n, then E has finite perimeter if and only if

$$\liminf_{t\to 0} \frac{\sqrt{\pi}}{2\sqrt{t}} \int_{R^n}\int_{R^n} |1_E(x) - 1_E(y)| p_t(x,y) dx\, dy < +\infty,$$

where $p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{\|y-x\|^2}{4t}}$ is the Euclidean heat kernel. Moreover, in that case

$$P(E) = \lim_{t\to 0} \frac{\sqrt{\pi}}{2\sqrt{t}} \int_{R^n}\int_{R^n} |1_E(x) - 1_E(y)| p_t(x,y) dx\, dy.$$

This characterization of sets of finite perimeter, which is due to M. Ledoux for balls and in 2007 to M. Miranda Jr., D. Pallara, F. Paronetto, and M. Preunkert in the general

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DOI: https://doi.org/10.1090/noti2257

case, only requires a heat semigroup and is therefore appealing since heat semigroups might be defined in the large category of Dirichlet spaces. Thus, at least formally, sets of finite perimeter may be perfectly well defined in any Dirichlet space and we do not need a gradient or a distance. This is the point of view we will take and will explain the associated theory. A key point is understanding the normalizing factor $\frac{1}{\sqrt{t}}$ which, in Euclidean space, reflects its "smooth" structure, but which has to be modified to understand sets of finite perimeter in nonsmooth spaces like fractals.

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Credits

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