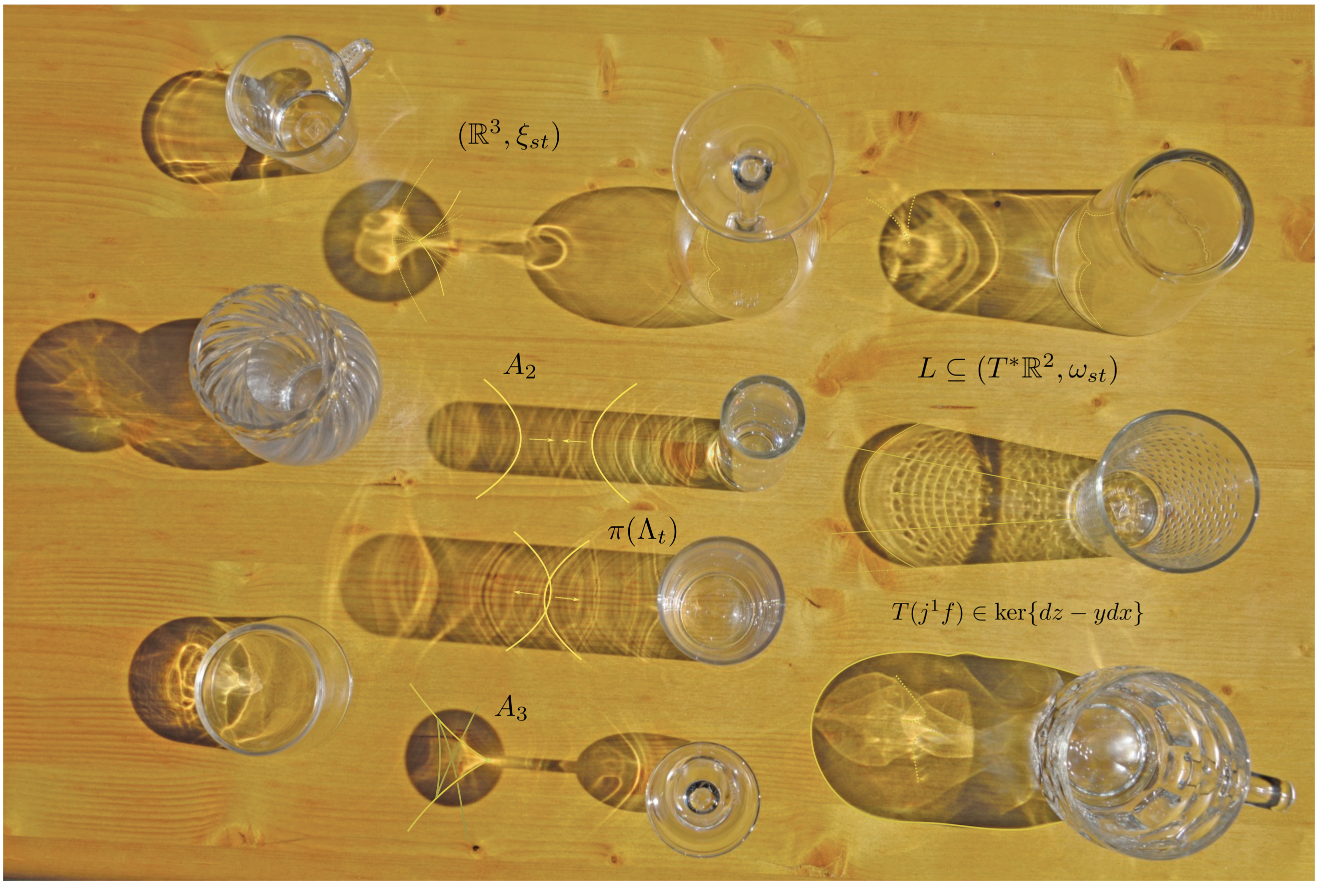
Contact and Symplectic Topology: Mastering the Art of Front Cooking



Roger Casals

A Bird's Eye View

Classification results are relevant in many areas of mathematics. The initial data is often a set of objects and an isomorphism relation—we invite the reader to think of their favorite classification problem. A few examples are:

(i) knots $K \subseteq \mathbb{R}^3$, classified up to smooth isotopy, or smooth manifolds up to diffeomorphism,
(ii) metric spaces, e.g., Riemannian manifolds, understood up to isometries or quasi-isometries,
(iii) discrete groups, up to (virtual) isomorphism, or Lie groups and their Lie algebras,
(iv) the classification of integer quadratic forms, classified up to classes or genera—or the solutions to a Diophantine equation up to descent,
(v) the (space of) functions solving a partial differential equation, solutions possibly considered up to compactly supported smooth functions or gauge equivalence.

Roger Casals is a professor of mathematics at the University of California Davis. His email address is `casals@ucdavis.edu`.

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Classification problems aim to either prove two given objects X_1 and X_2 are isomorphic, $X_1 \cong X_2$, or that they are *not* isomorphic. In the former case, an isomorphism must be constructed, either explicitly or by abstract means. In the latter, one must argue such an equality cannot exist; this is frequently achieved by constructing and computing an *invariant* $I(X)$, associated to each object X.

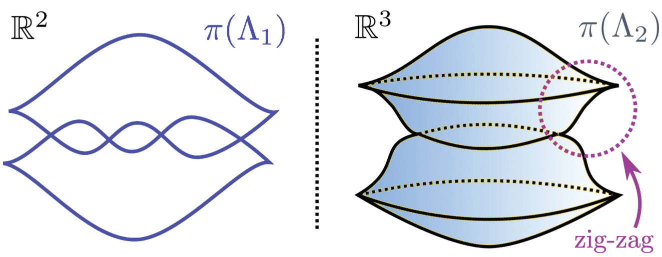

Figure 1. Legendrian wavefront $\pi(\Lambda_1) \subseteq \mathbb{R}^2$ for a (2,4)-torus link $\Lambda_1 \subseteq \mathbb{R}^3$ (left). Legendrian wavefront $\pi(\Lambda_2) \subseteq \mathbb{R}^3$ for a 2-sphere $\Lambda_2 \subseteq \mathbb{R}^5$ (right).

This invariant I could be anything: a number, a polynomial, an Abelian group, a category,[1] a topological space, or a kind of potato—as long as it satisfies that $I(X_1) \cong I(X_2)$ if $X_1 \cong X_2$, it can be used to argue that $X_1 \ncong X_2$ by showing that $I(X_1) \ncong I(X_2)$.

Examples. Instances of invariants include the dimension of a vector space, the index of Fredholm operator, the class number of a number field, the genus of a surface, curvatures in Riemannian manifolds, the Jones polynomial of a knot, the Poincaré polynomial of a space, the homology groups of a space, or the category of coherent sheaves on an algebraic variety.

In general, one still has to show $I(X_1) \ncong I(X_2)$, which is also a classification problem—the trick is to choose invariants I which are easily distinguished: most mathematicians would be comfortable telling numbers, polynomials, and Abelian groups apart, and such invariants are most frequently used (categories, spaces, and potatoes might have a more niche crowd).

The field of **contact and symplectic topology** [1, 6] has seen developments on both sides of the aisle: new invariants I distinguishing objects, and new techniques to construct isomorphisms.

Developments: Two New Directions

This article will introduce some of the objects studied in contact and symplectic topology and discuss some of the results that have been proved.[2] We focus on two particular developments:

(1) *The discovery of a flexible class of contact structures and symplectic structures.*

(2) *The study of Legendrian and Lagrangian submanifolds using microlocal sheaf theory.*

In these results, *Legendrian wavefronts*, which are certain singular hypersurfaces $\pi(\Lambda) \subseteq \mathbb{R}^n$, have played a crucial role; see Figure 1. These will be discussed in depth in the upcoming section. The title of this article is a reference to *Mastering the Art of French Cooking*. For us, it is the study of (wave)front diagrams, with all its flavors and different recipes, that allows us to prove new results and enjoy some of the wonders of our mathematical kitchen.

Example. For a Legendrian knot $\Lambda \subseteq (\mathbb{R}^3, \xi_{st})$, the front $\pi(\Lambda) \subseteq \mathbb{R}^2$ is akin to a planar knot diagram, except it allows for cusps and *all* planar crossings are overcrossings;[3] see Figure 1 (left) and Figure 4. For a Legendrian surface $\Lambda \subseteq (\mathbb{R}^5, \xi_{st})$, the front $\pi(\Lambda) \subseteq \mathbb{R}^3$ is a singular surface in $\mathbb{R}^3$, as in Figure 1 (right) or the blue flying saucer in Figure 2. Note that we are drawing surfaces in 5-dimensions by drawing fronts in 3-space.

Given a Legendrian wavefront $\pi(\Lambda)$, we will build a symplectic manifold $W(\Lambda)$ and two contact manifolds $Y_{\pm 1}(\Lambda)$, and also construct invariants. Figure 2 schematically shows these possibilities.

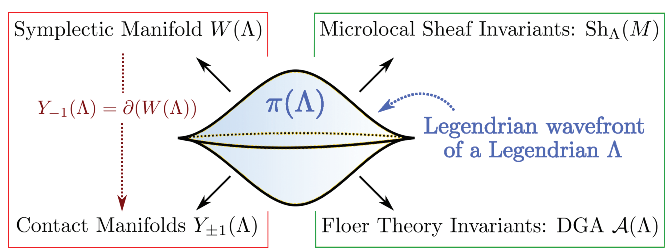

Figure 2. A Legendrian wavefront $\pi(\Lambda)$, from which we can build geometric manifolds (left) or compute algebraic invariants (right). M is the smooth manifold where the wavefront lives, typically $M = \mathbb{R}^n$.

Part (1) above belongs to the realm of h-principles [12]. The theory of h-principles is *constructive*, a tenet being that "if certain algebraic obstructions vanish, then a geometric construction is possible." (In broad terms, h-principles are theorems where "algebra implies geometry.") In our context, h-principles are results allowing one to conclude that, for certain classes of objects and (actually computable) invariants I, the equality $I(X_1) \cong I(X_2)$ implies that $X_1 \cong X_2$. By definition, X_1, X_2 are *flexible* with respect to an invariant I when $I(X_1) \cong I(X_2)$ implies $X_1 \cong X_2$. Thus, should these flexible objects exist, the invariant I gives a complete classification! For a given invariant I, it is interesting to study

[1] *E.g., "computing a category" might mean finding a set of generators, or presenting it as the differential graded derived (dg-derived) category of modules over an algebra we understand.*

[2] *The four main protagonists (contact, symplectic, Legendrian, and Lagrangian) will be defined momentarily and the reader is welcome to skip to the next section if needed.*

[3] *Northwest-Southeast strand above Southwest-Northeast.*

whether a nonempty class of flexible objects exists and, if so, decide whether a given object is flexible.

In contact and symplectic topology, these flexible classes are defined by the existence of a *local zig-zag* in Legendrian wavefronts;[4] such a zig-zag is shown in Figure 1 (right). A wavefront with a zig-zag is said to be stabilized. The following result [2, 5, 6] summarizes[5] three flexible developments.

Theorem 1. *In higher dimensions, i.e.,* $\dim(\Lambda) \geq 2$ *for Legendrians and* $\dim(W) \geq 6$ *for symplectic manifolds, the following hold:*

(i) *Let $\Lambda_1, \Lambda_2 \subseteq (Y, \xi)$ be smoothly isotopic Legendrian submanifolds, each admitting a stabilized wavefront. Then Λ_1 is Legendrian isotopic to Λ_2.*

(ii) *Let W_1, W_2 be diffeomorphic symplectic manifolds, each admitting a stabilized front handlebody. Then W_1 is symplectic isomorphic to W_2.*

(iii) *Let Y_1, Y_2 be diffeomorphic contact manifolds, each obtained by $(+1)$-surgery on a stabilized wavefront. Then Y_1 is contact isomorphic to Y_2.*

In fact, we also know several equivalent characterizations [5] of the hypothesis in Theorem 1(iii), e.g., in terms of contact open books and the existence of overtwisted disks.

Each of the three statements in Theorem 1 is of the following form: we have two objects X_1, X_2 which we want to compare. Then if both X_1, X_2 verify a certain *topological* property (being diffeomorphic and "stabilized"), then $X_1 \cong X_2$. The notions featured in Theorem 1 will be explained shortly. The message is that, in contact and symplectic topology, we now have verifiable **local** properties (of a wavefront) which **globally** characterize certain isomorphism classes of geometric objects, i.e., if you see a zig-zag somewhere in your wavefront diagram $\pi(\Lambda)$, this characterizes completely the symplectic and contact topology of Λ, $W(\Lambda)$, and $Y_{\pm 1}(\Lambda)$ given the underlying smooth data. (This is rather exclusive to contact and symplectic topology: e.g., for a knot diagram or a Riemannian manifold, no local property near *one* point will typically be able to globally characterize the knot or the Riemannian metric.)

In contrast, part (2) above has provided new invariants to help the classification and study of contact and symplectic structures.[6] This relies on new and fruitful connections to cluster algebras [3, 17], differential equations, and homological mirror symmetry. A class of invariants that are shaping the field is given by studying *categories* of constructible sheaves, with appropriate constraints [13–15].

Let M be a smooth manifold. These sheaves are intuitively given by assigning a finite-dimensional vector space at each point of M. The dimensions of the vector spaces might potentially jump as we vary the point. Legendrian wavefronts $\pi(\Lambda) \subseteq M$ serve as constraint: we only allow these dimensional jumps to happen when we cross the wavefront. This is morally what being *constructible* with respect to a wavefront means; the set of such sheaves is denoted by $\mathrm{Sh}_\Lambda(M)$. In precise terms, let $T^\infty M$ be a unit cotangent bundle. Then, we have the following.

Theorem 2 ([13, 14, 18])**.** *The category $\mathrm{Sh}_\Lambda(M)$ of constructible sheaves microlocally supported at a Legendrian $\Lambda \subseteq T^\infty M$ is a Legendrian invariant $I(\Lambda)$. Furthermore, it can often be computed combinatorially by using a wavefront for Λ.*

The computability is a strong virtue of such categories $\mathrm{Sh}_\Lambda(M)$. The result above is remarkably efficient at distinguishing Legendrian submanifolds $\Lambda_1, \Lambda_2 \subseteq T^\infty M$ which are smoothly isotopic but not Legendrian isotopic. Many interesting results in the field have been discovered or reproved thanks to the use of microlocal sheaf theory [13–15]. Two new examples [3, 17] are given in the following theorem.

Theorem 3. (i) *The moduli space of Lagrangian fillings of positive Legendrian links $\Lambda \subseteq (\mathbb{R}^3, \xi_{st})$ admits a (partial) cluster structure.*

(ii) *The Legendrian (max-tb) representatives of (n, m) torus links, $n \geq 3, m \geq 6$, admit infinitely many distinct Lagrangian fillings. In fact, most max-tb Legendrian representatives of a positive braid admit infinitely many distinct Lagrangian fillings.*

The Hamiltonian isotopy class of an embedded exact Lagrangian filling specifies a cluster chart in the moduli space of Theorem 3(i), and it is shown that these cluster charts can be used to distinguish exact Lagrangian fillings as in Theorem 3(ii).[7]

Contact and Symplectic Topology

Let us first introduce the four main characters in the field: symplectic and contact structures, and Legendrian and Lagrangian submanifolds. In this article, the symplectic and Lagrangian dimensions will be $2n$ and n, and the contact and Legendrian dimensions will be $2n-1$ and $n-1$, respectively.

Definition 1. A symplectic structure (X, ω) on a real $2n$-dimensional smooth manifold X is the choice of a nondegenerate closed 2-form ω on X.

[4] *The importance of zig-zags in h-principles, and in general wrinkled maps, goes back to Eliashberg-Mishachev (1997).*

[5] *For notational simplicity, we write smooth isotopy for a smooth isotopy which includes the corresponding formal data required by the h-principle; the same for diffeomorphisms.*

[6] *Floer-theoretic invariants date back to the 1980s, the dg-algebras $\mathcal{A}$ being developed in the 2000s. The resurgence of microlocal sheaf theory belongs to the last decade 2010–20.*

[7] *This relates to the wall-crossing phenomena present in Morse theory and mirror symmetry.*

In practice, the integral $\int_S \omega \in \mathbb{R}$ over a 2-dimensional surface S generalizes the notion of 2-dimensional area to any surface $S \subseteq X$. In contrast to a Riemannian metric, the 2-form ω is antisymmetric and perceives orientations. Thus, nonpositive (symplectic) areas are allowed. Note that the three infinite families of simple Lie algebras are $\mathfrak{sl}_n$, corresponding to volume-preserving geometry, $\mathfrak{so}_n$, giving (pseudo)Riemannian geometry, and $\mathfrak{sp}_n$, which yields symplectic geometry.

Example. Consider the $2n$-dimensional ball

$$\mathbb{D}^{2n} := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \|x\|^2 + \|y\|^2 \le 1\}.$$

Then $\omega_{\mathrm{st}} := dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$ is a symplectic form. Given an embedded surface $S \subseteq \mathbb{R}^{2n}$, the real number $\int_S \omega_{\mathrm{st}} \in \mathbb{R}$ is obtained as follows: for each $i \in [1,n]$, project S to the 2-plane $\langle x_i, y_i \rangle \subseteq \mathbb{R}^{2n}$ and compute the signed area A_i of this image. Then $\int_S \omega_{\mathrm{st}} \in \mathbb{R}$ is the sum $A_1 + \cdots + A_n$. See Figure 3 and note that some, even all, of the A_i might be zero, e.g., the 2-torus $L := \{x_1^2 + y_1^2 = 1, x_2^2 + y_2^2 = 1, x_j = y_j = 0, 2 < j \le n\} \subseteq \mathbb{R}^{2n}$ has all $A_i = 0$.

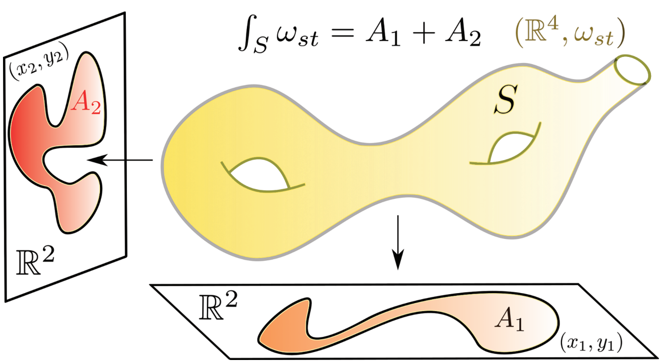

Figure 3. A surface $S \subseteq \mathbb{R}^4$ and the number $\int_S \omega_{\mathrm{st}}$ given by the symplectic structure.

Symplectic structures are abundant: complex projective manifolds, or any manifold $X = \{p_1 = \cdots = p_n = 0\} \subseteq \mathbb{C}^N$ cut out by complex polynomials, are symplectic. The phase space of any physical system on a configuration space M, i.e., the cotangent bundle (T^*M, λ_{can}) and its reductions, also admits symplectic structures. These come from the 2-from $d\lambda_{can}$, where λ_{can} is the canonical Liouville 1-form on T^*M. Most interesting aspects of symplectic topology are *global*: near a point $p \in (X, \omega)$, the symplectic structure is always given by the above example $(\mathbb{D}^{2n}, \omega_{\mathrm{st}})$ [1].

Definition 2. A contact structure on a real $(2n-1)$-dimensional smooth manifold Y is a (locally) generic[8] and smooth choice of hyperplane $\xi_y \subseteq T_yY$ at each tangent space.

[8]*The precise condition is "maximally nonintegrable"; by the Frobenius Integrability Theorem, this is equivalent to the algebraic condition* $\xi = \ker(\alpha)$ *and* $\alpha \wedge (d\alpha)^{n-1} \neq 0$, *where* α *is a 1-form in* Y.

Example. Consider the sphere $\mathbb{S}^{2n-1} = \partial \mathbb{D}^{2n}$. The kernel $\xi_{\mathrm{st}} := \ker(\lambda_{\mathrm{st}})$ is a contact structure, where λ_{st} is the restriction of the 1-form $\lambda_{\mathrm{st}} := \frac{1}{2}\sum_{i=1}^n (x_i dy_i - y_i dx_i)$ to the boundary $\partial \mathbb{D}^{2n}$. Removing a point from the sphere yields a contact structure $(\mathbb{R}^{2n-1}, \xi_{\mathrm{st}})$.

Contact structures are also bountiful. In many cases, the intersection $Y = X \cap \mathbb{S}^{2N-1}$ of a manifold $X = \{p_1 = \cdots = p_n = 0\} \subseteq \mathbb{C}^N = \mathbb{R}^N \oplus i\mathbb{R}^N$ with the unit sphere $\mathbb{S}^{2N-1} \subseteq \mathbb{C}^N$ has a canonical contact structure $\xi = TY \cap i(TY)$. Similarly, the energy level sets $T^\infty M$ of phase spaces and their reductions generically have contact structures coming from $\ker(\lambda_{can})$. Contact topology is also *global* in nature: near a point $p \in (Y, \xi)$, the contact structure is given by the example $(\mathbb{R}^{2n-1}, \xi_{\mathrm{st}})$ above [9].

Remark. É. Cartan classified all distributions $D \subseteq TY$ that locally have a *unique* normal form. There are four classes: (i) nonvanishing vector fields, which lead to smooth dynamics, (ii) Engel structures, which are 4-dimensional, (iii) even–contact structures, and (iv) contact structures.

A salient relation between a symplectic manifold (X, ω) and a contact manifold (Y, ξ) arises when $Y = \partial X$ is the boundary of X and $\omega = d\lambda$ is the differential of a 1-form. Then the hyperplanes $\xi_y = \{v \in T_yY : \lambda_y(v) = 0\}$ are the linear subspaces $\ker(\lambda)$, and these often form a contact structure $(\partial X, \xi)$ if λ is appropriately chosen. Note that the two examples $(\mathbb{S}^{2n-1}, \xi_{\mathrm{st}}) = \partial(\mathbb{D}^{2n}, \omega_{\mathrm{st}})$ above fit into this framework as $d\lambda_{\mathrm{st}} = \omega_{\mathrm{st}}$. In general, it is fruitful to think of contact manifolds $(Y, \xi) = (\partial X, \ker \lambda)$ as boundaries of (exact) symplectic manifolds $(X, \omega) = (X, d\lambda)$.[9]

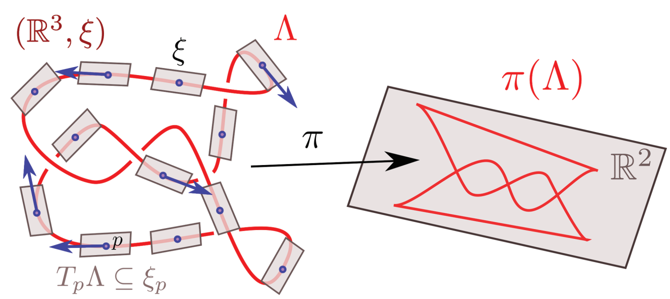

Figure 4. A Legendrian trefoil knot $\Lambda \subseteq \mathbb{R}^3$ and its wavefront $\pi(\Lambda) \subseteq \mathbb{R}^2$. *Legendrian* means that the tangent vectors to Λ belong to the contact 2-plane ξ.

A *Lagrangian* submanifold $L \subseteq (X, \omega)$ is any submanifold such that the restriction $\omega|_L = 0$ is zero and

[9]*This apparently innocent pairing is strengthened by a tenet in the field: "the contact boundary* $(\partial X, \ker \lambda)$ *knows about the symplectic interior* $(X, d\lambda)$*." For instance, suppose that* $(X, d\lambda)$ *symplectically coincides with* $(\mathbb{R}^{2n}, \omega_{\mathrm{st}})$ *away from a compact set. Then* X *must be diffeomorphic to* $\mathbb{R}^{2n}$, *and even symplectic isomorphic to* $(\mathbb{R}^4, \omega_{\mathrm{st}})$ *for* $n = 2$.

$\dim(X) = 2\dim(L)$. On the contact side, an *isotropic* submanifold $\Lambda \subseteq (Y, \xi)$ is a submanifold of Y such that at each point $p \in \Lambda$, the tangent space $T_p\Lambda$ is contained in the hyperplane ξ_p. An isotropic submanifold $\Lambda \subseteq (Y, \xi)$ is said to be *Legendrian* if $\dim(Y) = 2\dim(\Lambda) + 1$.

In fact, $\omega|_L = 0$ implies the dimensional inequality $2\dim(L) \leq \dim(X)$ and, similarly, $T\Lambda \subseteq \xi$ implies $2\dim(\Lambda) + 1 \leq \dim(Y)$. Thus, Lagrangian and Legendrian submanifolds have the maximal possible dimension given their defining (isotropic) constraints. Figure 4 (left) depicts a Legendrian knot $\Lambda \subseteq (\mathbb{R}^3, \ker \lambda)$.

Example. Lagrangian submanifolds $L \subseteq (\mathbb{R}^{2n}, \omega_{\mathrm{st}})$ generalize the graphs $gr(df) := \{y = df(x)\} \subseteq \mathbb{R}^{2n}$ of the derivative $df : \mathbb{R}^n \to \mathbb{R}^n$ of an $\mathbb{R}$-valued function $f = f(x) \in C^\infty(\mathbb{R}^n)$, which are indeed Lagrangian for ω_{st}. In fact, *any* Lagrangian is locally of this form. Similarly, Legendrian submanifolds $\Lambda \subseteq (\mathbb{R}^{2n-1}, \xi_{\mathrm{st}})$, with $\xi_{\mathrm{st}} = \ker\{dz - y_1 dx_1 - \cdots - y_{n-1}dx_{n-1}\}$, generalize the 1-jet graphs $j^1(f) := \{(x, y, z) : y = df(x), z = f(x)\} \subseteq \mathbb{R}^{2n-1}$ of $f \in C^\infty(\mathbb{R}^{n-1})$, containing the information of the function f and its derivatives df. Any such $j^1(f)$ is Legendrian for ξ_{st} and, conversely, *any* Legendrian is locally of this form.

The relation between L and Λ parallels that of X and ∂X: in many cases, the boundary $\partial L = L \cap \partial X$ of a properly embedded Lagrangian submanifold $L \subseteq (X, d\lambda)$ is a Legendrian submanifold $\Lambda = \partial L \subseteq (\partial X, d\lambda)$.[10]

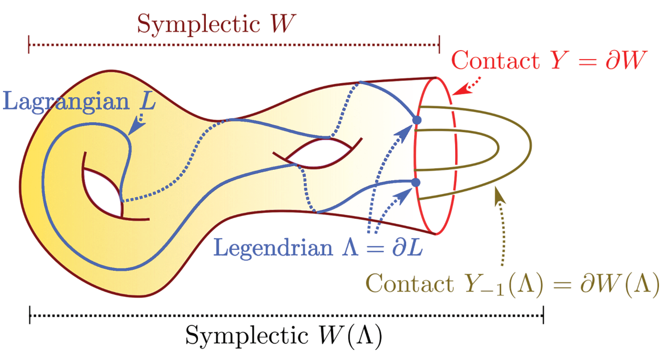

Figure 5. A symplectic manifold W^{2n} with contact boundary $Y^{2n-1} = \partial W$, and a Lagrangian $L^n \subseteq W$ with Legendrian boundary $\Lambda^{n-1} = \partial L$.

Pictorial conclusion. Figure 5 depicts a rather general situation where the four main characters appear at once. Researchers in the field look at many different questions; including the classification of all symplectic (or contact) structures on a smooth manifold, as well as the classification of Lagrangian (or Legendrian) submanifolds. Theorems 1, 2, and 3 above help us classify *some* such structures.

Studying Legendrians: Wavefronts

Symplectic and contact structures are rather hard to visualize: one is a 2-form and the other a hyperplane distribution. We will be able to translate problems on contact and symplectic structures into Legendrians, which are often more easily visualized and manipulated. This leads to some key questions: e.g., given two Legendrian submanifolds Λ_1, Λ_2, how do we show $\Lambda_1 \cong \Lambda_2$ or prove that $\Lambda_1 \ncong \Lambda_2$?

At its core, the answer to this question is by **drawing them**: by picturing a Legendrian Λ we will be able to both pin down properties of the drawing that might characterize the Legendrian isotopy class of Λ and compute invariants $I(\Lambda)$ that distinguish Legendrians. Let us focus on $(Y, \xi) = (\mathbb{R}^{2n-1}, \xi_{\mathrm{st}})$, as this is the local model. The projection $\pi : \mathbb{R}^{2n-1} \longrightarrow \mathbb{R}^n$, $\pi(x, y, z) = (x, z)$, has the following property: the image $\pi(\Lambda) \subseteq \mathbb{R}^n$ of any Legendrian recovers the Legendrian $\Lambda \subseteq (\mathbb{R}^{2n-1}, \xi_{\mathrm{st}})$ by setting the y_i-coordinates to be the x_i-slope of the tangent plane of $\pi(\Lambda)$, i.e., $y_i := \partial_{x_i} z(x_1, \dots, x_{n-1})$. Indeed, this is just the analytic incarnation of the condition $T\Lambda \subseteq \xi_{\mathrm{st}}$.

The image $\pi(\Lambda) \subseteq \mathbb{R}^n$ is known as the wavefront of Λ. Figures 1, 2, and 4 depict examples of wavefronts. The wavefront $\pi(\Lambda)$ is typically a singular hypersurface, despite Λ being embedded: the singularities acquired by $\pi(\Lambda)$ come from the singularities of the restricted projection $\pi|_\Lambda$. Hence, we can study Legendrians in $(\mathbb{R}^{2n-1}, \xi_{\mathrm{st}})$ by studying *certain* singular hypersurfaces in $\mathbb{R}^n$. Building on the theory of singularities [1], there is a diagrammatic calculus for wavefronts [1, 4, 11]. In the same way that one may tackle knots through their diagrams, and manipulate smooth 4-manifolds with Kirby calculus, contact and symplectic topologists can manipulate Legendrians and Lagrangians using wavefront diagrams.

Two Applications

Sample problems that we can now address with contact and symplectic techniques include the following two problems.

Problem 1 (Affine varieties). Let us consider the two symplectic manifolds $W_1 = (\mathbb{C}^3, \omega_{\mathrm{st}})$ and

$$W_2 = \{(x, y, z, w) \in \mathbb{C}^4 : x + x^2 y + z^2 + w^3 = 0\},$$

where W_2 inherits the symplectic structure from $(\mathbb{C}^4, \omega_{\mathrm{st}})$. It can be shown that as smooth manifolds $W_2 \cong \mathbb{R}^6$, hence $W_1 \cong W_2$ smoothly. In contrast, $W_1 \ncong W_2$ as affine algebraic varieties. (This W_2 is an *algebraically exotic* structure on $\mathbb{C}^3$, known as the Koras-Russell cubic. Being algebraic isomorphic implies being symplectic isomorphic, but the

[10] *The previous tenet also holds: "the Legendrian boundary $\Lambda = \partial L \subseteq (\partial X, \ker \lambda)$ knows about the Lagrangian interior $L \subseteq (X, d\lambda)$." E.g., if an embedded (exact) Lagrangian surface $L \subseteq (\mathbb{D}^4, \omega_{\mathrm{st}})$ bounds a Legendrian knot $\Lambda = \partial L \subseteq \partial(\mathbb{D}^4, \omega_{\mathrm{st}})$, then L must have the minimal 4-genus of the knot $\Lambda \subseteq \mathbb{S}^3$.*

converse may fail.) Now we can study the question: are W_1 and W_2 isomorphic as symplectic manifolds? That is, does there exist a diffeomorphism $f : W_1 \to W_2$ such that $f^*(\omega_2) = \omega_1$?

Here is a variation on Problem 1: consider $\mathcal{C}_m = \{(x, y, z, w) \in \mathbb{C}^4 : x^m y + zw = 1\}$. As smooth manifolds $\mathcal{C}_m \cong \mathbb{S}^3 \times \mathbb{R}^3$ but, for instance, $\mathcal{C}_1 \not\cong \mathcal{C}_2$ are algebraically distinct. Are they symplectically isomorphic? Both versions of this problem will be solved using Legendrian handlebodies, Legendrian wavefronts, and Theorems 1 and 2.

Problem 2 (Propagation of singularities)**.** Consider a planar wavefront $\pi \subseteq \mathbb{R}^2$ in the shape of an ellipse moving inwards. Imagine an elliptical source of light, or a elliptical water wave[11] propagating inwards, as depicted in Figure 6 (upper-left). As wavefronts evolve in time, they develop singularities and it is interesting to understand which singularities are created and how they propagate.[12] For instance, the *Four Cusps Theorem* states that, generically, *any* sequence of wavefronts that starts and ends as in Figure 6 *must* have a wavefront in the middle with at least *four cusps*.

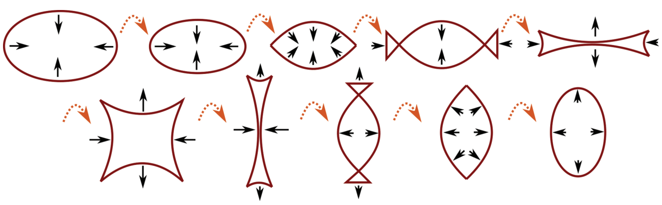

Figure 6. An eversion of an elliptic wavefront. Fixing the first and last fronts, we can ask which and how many singularities *must* occur for any sequence of intermediate fronts.

In general, given two (possibly singular) wavefronts $\pi_1, \pi_2 \subseteq M$ in a smooth manifold, one can ask *which* (and how many) singularities must appear or persist for *any* (generic) sequence of fronts starting at π_1 and ending at π_2. The invariant $\mathrm{Sh}_\Lambda(M)$, which studies sheaves constrained by a wavefront $\pi(\Lambda)$, will allow us to answer some such questions.

Legendrian Handlebodies

Consider the situation in Figure 5, focusing on a Legendrian sphere $\Lambda \subseteq (\partial W, \xi)$ in the contact boundary.

From Legendrians to contact and symplectic. We can use this Legendrian $\Lambda \subseteq \partial W$ to construct a new symplectic manifold $W(\Lambda) := W \cup_\Lambda T^*\mathbb{D}^n$, where $T^*\mathbb{D}^n$ denotes the (unit disk) cotangent bundle of $\mathbb{D}^n$. This $W(\Lambda)$ is obtained by attaching the symplectic piece $T^*\mathbb{D}^n = \mathbb{D}^n \times_{\mathbb{D}^n}$, called a handle, to (a neighborhood of) the Legendrian Λ along its boundary $\partial\mathbb{D}^n \times_{\{0\}}$.[13] For the appropriate choice of framing (cf. [6, 20]), A. Weinstein showed that $W(\Lambda)$ admits a symplectic structure [20] and its boundary $Y_{-1}(\Lambda) := \partial(W(\Lambda))$ is a contact manifold [9].

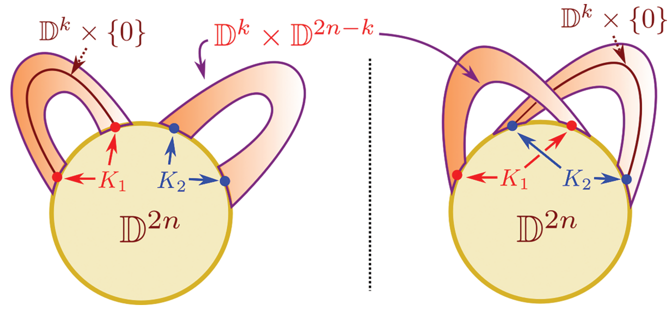

Figure 7. The handle attachment construction of symplectic manifolds: the cores $\mathbb{D}^k \times_{\{0\}}$ of the handles $T^*\mathbb{D}^k \times_{T^*\mathbb{D}^{n-k}}$ are attached along (open neighborhoods of) the isotropic spheres $K_1, K_2 \cong \mathbb{S}^{k-1} \subseteq \partial\mathbb{D}^{2n}$. Differing spheres K_1, K_2 may lead to different contact and symplectic structures.

Given a k-dimensional isotropic sphere $S \subseteq (\partial W, \xi)$, we can similarly construct a new symplectic manifold $W(S)$ by attaching the handle $T^*\mathbb{D}^k \times_{T^*\mathbb{D}^{n-k}}$ along its boundary $\partial\mathbb{D}^k \times_{\{0\}}$ to S. The contact boundary of $W(S)$ is denoted $Y_{-1}(S)$. In either case, the pieces we attach are simple, as they are standard symplectic disks $(\mathbb{D}^{2n}, \omega_{\mathrm{st}})$. The data that most enriches this construction is the choice of *where* the piece is attached, which we specify by choosing an isotropic submanifold. Intuitively, the wealth of Legendrian submanifolds which are *smoothly isomorphic* (i.e., smoothly isotopic), but *distinct as Legendrians*, accounts for the additional richness of contact and symplectic topology, in comparison to differential topology.

Remark. The above construction is a *symplectic* incarnation of handlebody decompositions for smooth manifolds; see Figure 8. The advantage of this technique—exemplified by 4-dimensional Kirby calculus—is that it translates problems about smooth manifolds into (a generalization of) knot theory, where pictorial and combinatorial techniques can be successfully used to manipulate diagrams.

A key distinction is that isotropic submanifolds of $\dim(S) \leq n-2$ are significantly different than Legendrian submanifolds, i.e., those with $\dim(S) = n-1$.

Theorem 4. *Let $S_1, S_2 \subseteq (Y, \xi)$ be isotropic, $\dim(S_i) < n-1$, and S_1 smoothly isotopic*[14] *to S_2. Then S_1 is isotopic to S_2, i.e., there exists a 1-parameter family S_t of isotropic submanifolds,*

[11] *These are rare in the ocean, but common in round kiddie pools, fish farms, or a cup of tea.*

[12] *These problems already appear in classical geometric optics; this is the theory of caustics and wavefronts, key to understanding oscillatory integrals.*

[13] *The second factor of $\mathbb{D}^n \times_{\mathbb{D}^n}$ is written in a smaller font to emphasize that its only purpose is to thicken $\mathbb{D}^n$ to the necessary dimension 2n, and make the total space symplectic.*

[14] *For experts, it should read "formal isotropic isotopic" [6], as we use "smoothly isotopic" for "formally isotopic."*

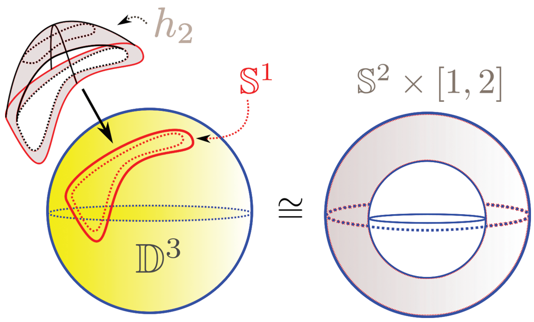

Figure 8. The space $\mathbb{S}^2 \times [1,2]$ constructed by attaching a 3-dimensional 2-handle h_2 (an igloo) to a yellow solid 3-disk. The attachment is specified by the red circle in $\partial\mathbb{D}^3$.

$t \in [1,2]$, starting at S_1 and ending at S_2. In contrast, there exist pairs of Legendrian submanifolds $\Lambda_1, \Lambda_2 \subseteq (Y,\xi)$ which are smoothly isotopic but **not** *Legendrian isotopic.*

The message of Theorem 4 is that when we encode contact and symplectic structures in terms of isotropic submanifolds, only those of largest dimension, i.e., Legendrian submanifolds, contain any information beyond smooth topology. In the statement, Λ_1, Λ_2 are *Legendrian* isotopic if there exists a 1-parameter family Λ_t of Legendrian submanifolds, $t \in [1,2]$, starting at Λ_1 and ending at Λ_2. Let us formalize the above construction.

Definition 3. (i) Let (W,ω) be a symplectic manifold. A Legendrian handlebody is a decomposition of the form $W_0 = \mathbb{D}^{2n} \subseteq W_1 \subseteq \cdots \subseteq W_{s-1} \subseteq W_s = W$, where $W_{i+1} = W_i(\Lambda_i)$ and $\Lambda_i \subseteq (\partial W_i, \xi_i)$ is a Legendrian (or isotropic) sphere [6, 20].

(ii) Let (Y,ξ) be a contact manifold. A surgery diagram is a sequence $(Y^0, Y^1, \ldots, Y^{s-1}, Y^s)$, $Y^0 = \mathbb{S}^{2n-1}, Y^s = Y$, where $Y^{i+1} = Y^i_{\pm 1}(\Lambda_i)$ and $\Lambda_i \subseteq (Y^i, \xi_i)$ is a Legendrian sphere [9, 16] (or an isotropic sphere if doing $Y^i_{-1}(\Lambda_i)$).

The contact manifold $Y_{+1}(\Lambda)$ is the *lower* boundary of the (symplectic) manifold $(Y \times [0,1]) \cup_\Lambda T^*\mathbb{D}^n$. This latter manifold is obtained by attaching a $T^*\mathbb{D}^n$ symplectic handle, with boundary the Legendrian sphere Λ, to the lower boundary of the symplectic neighborhood $Y \times [0,1]$.

From contact and symplectic to Legendrian. An advantage of these decompositions is generality.

Theorem 5 ([10]). *Any closed symplectic manifold (X,ω) admits a codimension-2 symplectic submanifold $(C, \omega|_C) \subseteq (X,\omega)$ such that $X \setminus C$ admits a Legendrian handlebody decomposition.*

Thus, proceeding inductively, we assume $(C, \omega|_C)$ is understood and study *any* symplectic manifold (X,ω) by studying the *Legendrian* submanifolds that describe $X \setminus C$. Similarly, many interesting contact manifolds (Y,ξ) can be realized as the boundary of $(\mathbb{D}^{2n}, d\lambda_{st})$ after having either attached or deleted symplectic handles along Legendrian spheres in $(\partial\mathbb{D}^{2n-1}, \ker\lambda_{st})$ [9, 16]. Legendrian submanifolds can then be studied using diagrammatic calculus for their wavefronts.

Flexibility: Recent Developments

Let us address Theorem 1, which pinpoints geometric properties for X_1, X_2 that suffice to conclude isomorphisms $X_1 \cong X_2$. Legendrians being the central character, we start with the following definition.

Definition 4. A Legendrian $\Lambda \subseteq (Y,\xi)$ is said to be stabilized if there exist a point $p \in \Lambda$ and an open neighborhood $\mathcal{O}p(p) = (\mathbb{R}^{2n-1}, \xi_{st})$, such that its wavefront $\pi(\Lambda) \subseteq \mathbb{R}^n$ looks like $Z \times \mathbb{S}^{n-2}$, where Z is the planar zig-zag in Figure 9(i). In the lowest dimension $n = 2$, a Legendrian link with one zig-zag Z, instead of two disjoint zig-zags $Z \times \mathbb{S}^0$, is also considered stabilized.

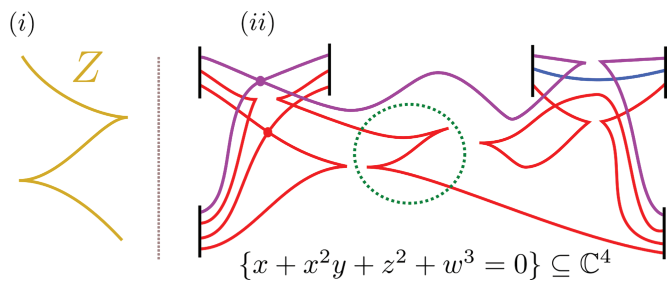

Figure 9. The zig-zag (left). A slice of a Legendrian wavefront handlebody for the Koras-Russell cubic W_2, with its zig-zag emphasized in the dotted green circle (right).

Any smooth submanifold $N^{n-1} \subseteq (Y,\xi)$ can be approximated by a stabilized Legendrian. In contrast, the Legendrian 2-sphere whose wavefront is drawn in Figure 2 is not stabilized. A reason stabilized Legendrians are useful is the following result, which establishes them as flexible objects with respect to the property of being stabilized (and smoothly isotopic).

Theorem 6. *Let $\Lambda_1, \Lambda_2 \subseteq (Y,\xi)$ be smoothly isotopic and each of them (sufficiently) stabilized. Then there exists a Legendrian isotopy from Λ_1 to Λ_2.*

For $n = 2$, this is a theorem of D. Fuchs and S. Tabachnikov (1997) and the *sufficiently* hypothesis is required; that is, there must exist sufficiently many points $p \in Y$ with a zig-zag near them. For higher dimensions $n \geq 3$, this is the result of E. Murphy's thesis (2012), and the adverb *sufficiently* is not needed: one zig-zag is enough. The stabilization procedure used in Theorem 6 was introduced by Ekholm-Etnyre-Sullivan (2005).

The strength of this result is that, in general, the existence of a smooth isotopy is *not* enough to conclude a Legendrian isotopy. Nevertheless, *if* we draw Λ_1 and Λ_2 *and* we see zig-zags in their fronts, *then* we can conclude that there is a Legendrian isotopy.

An analogue of Definition 4 for contact and symplectic manifolds uses the **Legendrian handlebodies** and **surgery diagrams** introduced above.

Definition 5. A symplectic structure $(W, d\lambda)$ is said to be flexible if it admits a Legendrian handlebody whose Legendrians are all stabilized.[15]

A contact structure is *overtwisted* if it admits a contact surgery diagram where a handle attached along a stabilized Legendrian has been deleted, i.e., $Y = Y_{+1}(\Lambda)$ for Λ a stabilized Legendrian.

It is proven in [5] that this definition of overtwisted contact structures coincides with the classical definition of Y. Eliashberg (1989). In line with Theorem 6, we know uniqueness for these *flexible* objects, i.e., their diffeomorphism type and being *stabilized* completely characterizes their contact and symplectic geometry.

Theorem 7. *Let $(W_1, d\lambda_1)$ and $(W_2, d\lambda_2)$ be two flexible symplectic structures and W_1 diffeomorphic[16] to W_2. Then W_1 is symplectomorphic to W_2. Similarly, let (Y_1, ξ_1) and (Y_2, ξ_2) be overtwisted contact structures and Y_1 diffeomorphic to Y_2. Then (Y_1, ξ_1) is contact isomorphic to (Y_2, ξ_2).*

Fix a smooth manifold. Theorem 7 states that, should either of these structures exist—flexible symplectic or overtwisted contact—then they are unique. We also emphasize that the general existence of contact structures has also been proved.

Theorem 8 ([2])**.** *Let Y^{2n-1} be a smooth manifold whose tangent bundle TY is an almost-complex vector bundle. Then Y admits a (overtwisted) contact structure.*

Solution to Problem 1. The symplectic manifold $W_1 = \mathbb{C}^3$ is tautologically flexible, as it admits the empty handlebody $W_1 = \mathbb{D}^6(\emptyset)$. To study W_2, we use the Legendrian handlebody $W_2 = \mathbb{D}^6(\Lambda)$, where Λ is the Legendrian surface whose (sliced) wavefront is drawn in Figure 9. Since this Λ is stabilized *and* W_2 is smoothly identical to W_1, Theorem 7 implies that W_1 is symplectomorphic to W_2. In fact, this same argument shows that W_1 is Stein deformation equivalent to W_2 [6].

Partial solution to Variation 1. The wavefronts of the Legendrian handlebodies for $\mathcal{C}_m$ are drawn in Figure 10. Since they have zig-zags, this proves that $\mathcal{C}_m$ are flexible for $m \geq 2$ and thus—by Theorem 7—$\mathcal{C}_l$ is symplectomorphic (and even Stein equivalent) to $\mathcal{C}_m$ for any $l, m \geq 2$.

[15] *Each Legendrian component has to admit a zig-zag in the complement of the other Legendrians.*

[16] *Recall that we use the term diffeomorphic to also mean that the formal obstructions distinguishing W_1, W_2 vanish.*

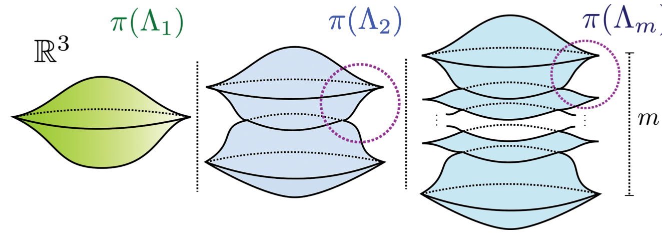

Figure 10. The wavefronts $\pi(\Lambda_m)$ for the symplectic manifolds $\mathcal{C}_m = \mathbb{D}^6(\Lambda_m)$.

Note that the front for $\mathcal{C}_1 = \mathbb{D}^6(\Lambda_1)$ does not readily exhibit a zig-zag: to completely resolve Variation 1, we either manipulate Λ_1 until a zig-zag appears[17] or, if we believe Λ_1 is not equal to the other Λ_m, use an invariant such that $I(\Lambda_1) \neq I(\Lambda_m)$, $m \geq 2$. This matter is resolved in the next section. (It turns out that the latter is correct, so an invariant is needed.)

Rigidity: New Invariants and Results

Let us consider a Legendrian $\Lambda \subseteq (\mathbb{R}^{2n-1}, \xi_{st})$: we would like to come up with an invariant $I(\Lambda)$, i.e., $I(\Lambda_1) = I(\Lambda_2)$ if Λ_1, Λ_2 are Legendrian isotopic.[18] In addition, we want this invariant $I(\Lambda)$ to be effective, in that it is often able to tell apart distinct pairs of Legendrians $\Lambda_1, \Lambda_2 \subseteq (\mathbb{R}^{2n-1}, \xi_{st})$ which are smoothly equal. Thanks to studying the wavefront projection $\pi(\Lambda) \subseteq \mathbb{R}^n$, such a new invariant has been constructed and studied recently.

Theorem 9 ([13, 14, 18])**.** *The category $\mathrm{Sh}_\Lambda(\mathbb{R}^n)$ of constructible sheaves microlocally supported at the Legendrian Λ is a Legendrian invariant. In addition, it is effective and can be computed in many cases.*

Let us informally explain why we end up considering such invariant $I(\Lambda) := \mathrm{Sh}_\Lambda(\mathbb{R}^n)$:

(1) The wavefront $\pi(\Lambda)$ gives a **stratification** $\mathcal{S}(\pi(\Lambda))$ of $\mathbb{R}^n$. The singularities of $\pi(\Lambda)$ are responsible for the richness of these stratifications;[19] we can then apply techniques to study stratified spaces, such as constructible sheaves.[20]

(2) A Legendrian Λ is recovered from its wavefront $\pi(\Lambda)$ by taking the first *partial derivatives*: we capture the information about the first derivatives with the notion of microlocal support $\mu\mathrm{supp}(F) \subseteq T^\infty\mathbb{R}^n$ of a sheaf F.

Intuitively, $\mu\mathrm{supp}(F) \subseteq T^*\mathbb{R}^n$ encodes points $p \in \mathbb{R}^n$ and (unit) directions $v \in T_p\mathbb{R}^n$ such that there are

[17] *This would prove Λ_1 is stabilized.*

[18] *The next paragraphs can be generalized to $\mathrm{Sh}_\Lambda(M)$ for a smooth manifold M and $\Lambda \subseteq T^\infty M$ a Legendrian submanifold of the unit cotangent bundle $T^\infty M$.*

[19] *If $\pi(\Lambda)$ were smooth, there would only be two strata: $\pi(\Lambda)$ and $\mathbb{R}^n \setminus \pi(\Lambda)$. Intuitively, the invariant in this case is given by the derived category of local systems on $\mathbb{R}^n \setminus \pi(\Lambda)$.*

[20] *The classical stratified theory developed notions such as stratified Morse theory, perversity, and intersection homology.*

sections of F which cannot be extended as we move from p in the direction v. This is a sheaf-theoretic analogue, due to Kashiwara-Schapira (1982), of Hörmander's wavefront set of a function (1971). The category $Sh_{\Lambda}(\mathbb{R}^n)$ is then defined as the (dg-derived) category of sheaves F on $\mathbb{R}^n$ which are both constructible with respect to $\mathcal{S}(\pi(\Lambda))$ and satisfy $\mu\text{supp}(F) \subseteq \Lambda$. In many instances, this category can be computed combinatorially, by assigning a local system to each stratum of $\mathcal{S}(\pi(\Lambda))$ and maps between these local systems.

Solution to Variation 1. Consider the wavefronts $\pi(\Lambda_1)$ and $\pi(\Lambda_m)$, $m \geq 2$, in Figure 10. The category $I(\Lambda_\eta) := Sh_{\Lambda_\eta}(\mathbb{R}^3)$ has at least one object for $\eta = 1$ and none for $\eta = m$. By Theorem 9, Λ_1 is not Legendrian isotopic to Λ_m. This Legendrian wavefront description of the Stein manifolds $\mathcal{C}_n$ also implies that $\mathcal{C}_1$ is not symplectomorphic to $\mathcal{C}_m$, $m \geq 2$.

Solution to Problem 2. The category $Sh_{\Lambda}(\mathbb{R}^n)$ also knows about certain geometric properties for *all* possible wavefronts $\pi(\Lambda)$ of a Legendrian. This is beautifully used by S. Guillermou[21] to prove some of Arnol'd's Cusps Conjectures [13].

Theorem 10. (*4 Cusps*) *A generic family of wavefronts starting and ending with the same wavefronts as in Figure 6 must have an intermediate wavefront with at least four cusps.*

(*3 Cusps*) *Any generic wavefront deformation of a cotangent fiber in $\mathbb{RP}^2$ must have at least three cusps.*

The proof of the 3 Cusps Conjecture [13] studies the category $Sh_{\Lambda_0}(\mathbb{S}^2)$ for the initial point wavefront Λ_i, which has countably many "simple" objects. Then it is argued that the category $Sh_{\Lambda_f}(\mathbb{S}^2)$ associated to *any* generic wavefront deformation Λ_f with only one cusp must have uncountably many such objects. Since $Sh_{\Lambda_f}(\mathbb{S}^2)$ should be an invariant under deformation of Λ_i, and the number of cusps is odd, there must be at least three cusps.

Take-home nugget. We can describe contact (Y, ξ) and symplectic (W, ω) manifolds with Legendrian submanifolds Λ via **Legendrian handlebodies** and surgeries. Legendrians can themselves be studied using a diagrammatic calculus for their **wavefronts**: they are completely classified *if they have a zig-zag in their wavefront*, and else we have computable invariants $I(\Lambda)$ which both distinguish them and tell us about their geometric properties. These Legendrian invariants $I(\Lambda)$ can in turn be used to provide invariants of contact and symplectic manifolds (Y, ξ) and (W, ω) and help their classification.

Sins of omission. Floer theory and the study of (pseudo)holomorphic curves are also pillars of modern contact and symplectic topology. They have been successfully applied to obtain groundbreaking results for the last three decades, preceding the developments presented here. There is a (currently) conjectural relation between Floer-theoretical and sheaf-theoretical invariants [8].[22]

Finally, a few examples of other recent developments in the field are: results towards "2-or-∞ many Reeb orbits," by M. Hutchings et al., the resolution of the Simplicity Conjecture [7] in C^0-symplectic geometry,[23] and the use of stability conditions in the study of symplectic mapping class groups and auto-equivalences of Fukaya categories, after I. Smith et al. Equally interesting, the construction of infinitely many Lagrangian tori by D. Auroux and R. Vianna, the development of arboreal Lagrangian skeleta [19] after D. Nadler, L. Starkston, et al., and the study of Liouville sectors and their invariants [8].

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Roger Casals

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