Lucien Szpiro (1941–2020)

Renée Elkik, Dorian Goldfeld, Mireille Martin-Deschamps, Christian Peskine, Thomas Tucker, and Shou-Wu Zhang

Lucien Szpiro was born in France, in 1941, during the Second World War. He was one of France's "hidden children," protected during the Nazi occupation by being sent to live with a French non-Jewish family in the countryside. His father was in the French resistance, and was captured and killed in Auschwitz. After the war, Lucien was reunited with his mother. She remarried and had two more sons. The family lived in the small village of Livry Gargan, outside of Paris, before moving to Paris in the 1950s.

When Lucien was in high school, he often got into trouble at school. He was not a big fan of following rules. He preferred making the class laugh at his jokes to being quiet and obedient. He didn't do his homework. He regularly confronted right-wing youth gangs waiting for a fight at the exit of the high school. Outside the classroom he had many interests: cinema, chess, music, politics. He often visited the famous Cinémathéque Française, which French New Wave directors Truffaut and Goddard had frequented only a few years earlier. But in school, mathematics was the one subject that strongly interested him. His mathematics teachers took an interest in him and became his mentors and supporters, and he was designated to participate in the highly competitive, countrywide Concours Géneral in math.

After high school, Lucien attended Université Paris-Sud, where he earned a PhD under Pierre Samuel. From 1965 to 1969, he was an assistant professor at Sorbonne Université. For the next 30 years, Lucien held positions in CNRS, beginning as an attaché at Paris Université Paris Diderot and ultimately becoming Distinguished Professor (Directeur de Recherche de Classe Exceptionnelle) at Université Paris-Sud. In 1999, he took a position as Distinguished Professor at City University of New York Graduate Center, where he lived with his wife Beth Pessen.

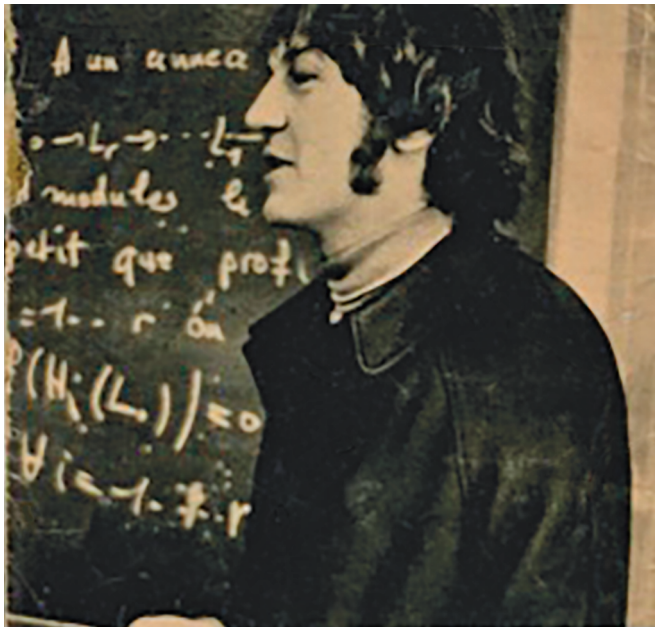

Figure 1. Lucien Szpiro, 1960s.

What follows are the remembrances, both mathematical and personal, of some of the mathematicians who knew him best.

Christian Peskine

I met Lucien Szpiro in Paris during the academic year 1964/65. Pierre Samuel, our future advisor, had recently

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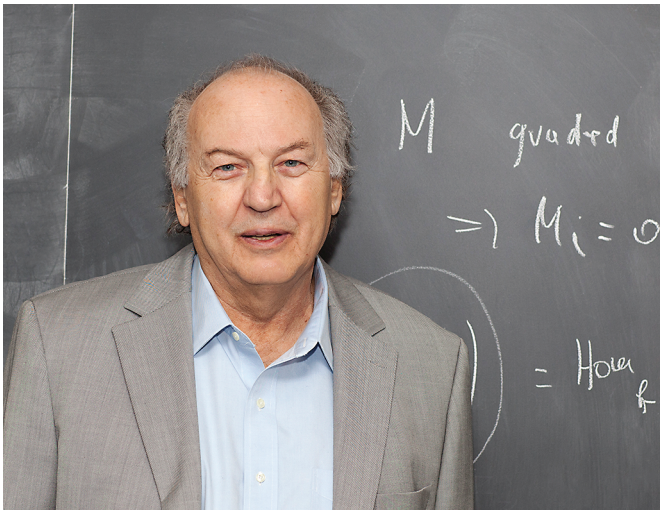


Figure 2. Lucien Szpiro, February 2012.

arrived in Paris and his new master's level course (the name was DEA at that time) attracted a lot of young students of all backgrounds, not only as a consequence of Samuel's recent results concerning multiplicities of intersection and Hilbert polynomials, but also because his years near Oscar Zariski influenced his personality and the way he taught. Compared to many other courses at this level, all taking place in the "Institut Henri Poincaré" (IHP), the atmosphere was pleasant and open. Furthermore, the course had a very new homological flavour, an exciting attraction for some of us (a bizarre abstraction for others). Pierre Samuel was a popular, generous, and nice person. Consequently he found himself very soon with (too) many PhD students.

It is difficult to describe the working conditions for PhD students in Paris at the time. There was no research library open to students, except for students at the École Normale Supérieure. For the rest of us, the mathematical life in Paris took place inside IHP. We had to apply in advance to the IHP library to consult a book. The consultation would take place in the early morning on Saturday (to prove that we were serious...). In order to work without constraints, the best option was to buy several Bourbaki and all EGA volumes. If I remember well, Samuel had half an office there for half a day every other week or something equivalent. In fact, many doctoral students would meet their advisor for an hour once or twice a year.

Lucien understood early and very clearly the need to be strategic and efficient to adapt to this very difficult environment. He contacted a few of us (in particular, Marguerite Mangeny, Daniel Ferrand, and myself) to organize a working group, a learning seminar, which would meet every week to discuss and study a subject. The main sources were Bourbaki, EGA, SGA, Cartan-Eilenberg, and Zariski-Samuel. Each of us would use as many hours as necessary to explain a theme to the others.

We met a new difficulty: finding a work space in the IHP for our meetings. I remember following Lucien from one office to another to apply for a room, for a book, for everything. I liked him immediately for being such a fighter. He understood the rules better than I did and I understood that he was used to fighting for his rights. Amusingly enough, he knew that after having used a seminar room two weeks in a row at the same day and same time, we acquired a sort of "priority" to use this room at the same time the following weeks. There was no such thing as a "graduate school"; the administration served the faculty and the whole thing seemed to be a private club to me. What a closed world and how lucky I was to discover it with this new "fighting" friend.

For the academic year 1965/66, I think that Lucien had obtained a job of assistant at the University of Paris. We were both working on our master's project (called "thèse de 3ème cycle"). Lucien was working on "valuations et fonctions d'ordre." Sometime during the year 1966 we began to collaborate. At some point we made a bizarre and naive agreement: we would have a common research project for the years to come. We did not realize the amount of trust carried by this agreement, for both of us, but time proved it to be a smart approach. During the approximately seven following years of common mathematical work, we may have had some disagreements now and then, but we always trusted each other.

A stroke of luck came for our project, hence for us, at the beginning of 1967. Samuel had invited Maurice Auslander to Paris for the spring semester. He gave a series of four lectures early in the semester. They attracted many young foreign mathematicians (Dan Laksov, Armand Brumer,...). With the help of Marguerite Mangeney we started writing detailed notes of the talks that we presented at some point to Maurice, who was very generous with his time and with his delicious sense of humor. There were many exciting questions related to these talks and the recent papers they referred to (in particular "On the ubiquity of Gorenstein rings" by Hyman Bass). Lucien and I understood quickly that meeting Maurice and discussing math regularly with him was a unique piece of luck. Interacting with him, exchanging mathematical ideas, and joking at the same time with him was not only possible but marvelously easy. At the end of the semester, it was decided that Maurice would try to find a way to invite us for the academic year 1968/69 at Brandeis University. Once back in Brandeis, he made the necessary arrangements shortly thereafter.

The year 1967/68 was special. Lucien and I were on the one hand constructing our project on intersection theory through weekly discussions, and on the other hand

participating actively in our learning group. We began attending the IHES algebraic geometry seminar. The extraordinary stature of A. Grothendieck was very impressive. During one of the rare personal discussions we had with him, in his office if I remember well, he gave us an enormous pile of notes (several hundred pages...) and suggested that we should put all this in clear form as a serious start in our mathematical life. A magnificent and generous offer which left us rather excited. While we began thinking about this new possibility, the turmoil of the spring of '68 started. My impression is that Lucien really took pleasure in this excitement and participated in many ways. Did he believe that this was the beginning of something really important? It is impossible for me to say for sure. He wore the uniform (long hair,...) but I think that he was too much of an individualist to be fully in. We did go on working together thoroughly and regularly, and he was as serious about it as before. At the same time the fun and the noise in the streets of Paris would keep everybody busy for a good part of the day. Furthermore, since Lucien was teaching, he had to participate in hours of discussions about courses and exams and in important decisions about how to maintain some sort of academic life during this incredible period. All this came to an end, and we started organizing the American project which would deeply change Lucien's life (and mine).

In August 1968, the discovery of Brandeis' Math department was a transformative experience for Lucien and I. Our colleagues, both junior and senior, were there full time, open to answer to any (possibly stupid) mathematical question. All of them were very nice to the two young Frenchmen. The common room discussions were always fun. All offices were alike (something new for us), with many open doors. Without delay we went to work in the best possible conditions. Lucien taught a master's level course during the first semester and calculus during the second. He did adapt very fast and enjoyed everything about life in Boston (he would take the train to Waltham almost every day). Among our many junior colleagues, Lucien developed a special relationship with Mike Shub. They would often be together. They had the same taste for derision, and the same way of showing that they were different. I remember Charles Pugh as another member of this friendly academic group.

We attended the algebraic geometry seminar at Harvard; we had our own seminar at Brandeis, where Pierre Dolbeaut participated now and then; and we went on with our project. By the end of the year we began appreciating the power of Frobenius and changed our main intersection conjecture to an intersection theorem in positive characteristic. We gave up on the Grothendieck project (this was a difficult choice). Shortly thereafter we transported our intersection theorem to characteristic 0 (not in full generality but for geometric rings) by a natural reduction method. The second semester was partly used to write "Notes sur un air de Bass" ("Notes on a Bass tune"), a Brandeis preprint and our first written version of this work. Throughout the year Maurice Auslander supported us with his comments and advice. His influence was both mathematical and personal. He was very generous with his time, particularly considering the fact that his mathematical interests were moving in a completely different direction. Lucien and I agreed that this year with Maurice profoundly influenced the rest of our lives, and particularly the years to come, back in Paris.

The two next years in Paris were used to finish our common thesis. We had to produce more mathematical results. The recent paper of Robin Hartshorne about cohomological dimension was full of interesting questions and results. Answers and comments concerning some of these questions became a chapter in our thesis.

The learning seminar developed with new members. J. L. Colliot Thélène appeared and later Mireille Martin-Deschamps and Renée Elkik. Lucien was absolutely gifted for bringing people together in a common working project. The atmosphere was as nice and open as ever. Each seminar was followed by a lunch at an especially inexpensive restaurant, the Volcan, near Place de la Contrescarpe in the latin quarter. We learned a lot, unofficially and with great pleasure.

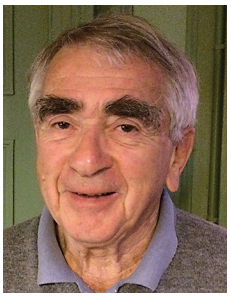
Lucien moved from the university to the Centre National de la Recherche Scientifique (CNRS), changing his working conditions for the better. We even got a very large office (shared with our friend Marguerite Mangeney) in the new building in Jussieu, with a view on Place Jussieu. At the same time, Pierre Samuel moved to the new University of Orsay where we registered as doctoral students. Rather soon it became clear to Samuel and to his colleagues that Lucien and I intended to write a common thesis. To begin with, the answer from the "administration" was negative. It was suggested that since there was enough mathematics for two, an easy solution was to cut the text into two separate theses. Our proposition became odd pages for one and even for the other, to be chosen by drawing lots. Once more we were saved by Maurice Auslander who made it clear—from far away—that this was a common work and that it was pure nonsense to attribute any part of it to just one of us. At the same time, our text was accepted for publication in the "cahiers bleus de l'IHES" and everything settled down nicely. We presented our thesis, first and second thesis for each of us, the same day, in front of two distinct juries. Maurice Auslander and Pierre Samuel, who were participating in both, had a long working day.

The following academic year, 1971/72, was our last year of complete collaboration. After an exchange with Michel Raynaud we got interested in "liaison" (linkage). Homological methods in classical projective geometry were blossoming at this time. After revisiting the linkage theorem of Gaeta we wrote what we believed to be a complete description of this question. The paper was quickly written and accepted for publication in *Inventiones*. The same year, Lucien gave a remarkable Bourbaki talk (June '72) on "special divisors," based on the works of Kempf, Kleiman, and Laksov.

We began to understand that our tastes were diverging. Lucien was more and more attracted by arithmetic. He would regularly describe (with a smile) a project to prove Fermat's theorem (often by using Frobenius). I had decided to classify space curves. He wanted to be in Paris, I wanted to leave Paris. This was the end of a collaboration that both of us had enjoyed deeply, the end of our years of training. I left Paris for many years. As a friendly sign, Lucien gave a series of lectures on space curves at the Tata Institute. Later, he would visit me in Strasbourg and Oslo and I would participate several times in his Oberwolfach workshops.

During the eighties, I heard a lot about the working group which slowly became "le séminaire Szpiro," and particularly about the positive influence it had on several younger mathematicians. In 1985, Lucien had an indirect, but very friendly, role in bringing me back to Paris. For a few years, he moved from Orsay to the same lab as me in Paris but then went back to Orsay, where he belonged mathematically.

From our discussions I understood very early in the nineties that he really loved working in New York where he was a frequent visitor at Columbia. He felt at home in New York where he decided to settle with Beth as soon as possible. Not surprisingly he became much more relaxed after he did so. Sometime near the end of the twentieth century we began to call each other by our Christian names. Our friendly regular dinners in Paris, when he was visiting, were a great pleasure for our two couples. What fun! I liked him.



Christian Peskine

Figure 3. Szpiro with his motorcycle, 1970s.

Mireille Martin-Deschamps

I first met Lucien Szpiro in 1969.

In 1969–70, the Orsay mathematics department offered a special program in algebraic geometry. A small group of researchers who were former students of Grothendieck—Michel Demazure, Jean Giraud, Michel Raynaud, and Jean-Louis Verdier—offered courses to introduce students to this new vision of algebraic geometry. I followed this high-level program, which was difficult and demanding. The courses were supplemented by an impressive seminar, in which I generally did not understand much. The atmosphere was a bit rough for beginners.

Lucien Szpiro, whom I met at the seminar, was somewhat atypical with his libertarian and sixty-eight'ish look. In addition to being an excellent mathematician, he was a person with a great sense of humor, he regarded the world in a curious and very critical way, but with great kindness.

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Figure 4. Hiking in the Alps with his son, circa 2000.

He invited me to participate in a very informal and confidential working group that met at the IHP (Institut Henri Poincaré). In this group, there were mathematicians of his generation—already established, yet only a few years our elders—and beginners like me, to whom he extended his hand. I found that this group had a stimulating atmosphere, which was at the same time warm and friendly. For years, we met on Friday mornings at the bistro rue des Feuillantines, before going to the IHP for a work session followed by lunch in a small Greek restaurant, Le Volcan. We took turns giving lectures on a theme chosen for the year. The rule was that everyone should understand everything. Hence a lot of discussion and criticism, often fierce, but in a benevolent and very egalitarian atmosphere. It was a place where you did not feel judged, a welcoming place.

Welcoming in particular for the female mathematicians, as the research environment in algebraic geometry was—and has undoubtedly remained—very masculine. Szpiro had surely understood the discomfort experienced by the young women researchers. He never remarked on it, but the fact that our working group was gender balanced was no accident. I found the scientific and psychological support that allowed me to build myself as a mathematician. Jean-François Boutot, Renée Elkik, Marguerite Flexor, Daniel Ferrand, Natacha Ménégaux were the other regular participants. Others came less regularly, such as Yves Colombé, Laurent Gruson, Christian Peskine, and Ragni Piene (when she was in Paris). Jean-Louis Colliot-Thélène and Laurent Moret-Bailly joined us a few years later. From 1975 on, every other year our group moved to Oberwolfach for a week of commutative algebra.

Over the years we gained self-confidence and the working group became a more official seminar. In the early 1980s Lucien Szpiro was called to the mathematics department of the ENS (École Normale Supérieure, rue d'Ulm) and offered me an office there. At the same time the seminar moved to the ENS and its proceedings were published (*Astérisque*, no. 86 in 1981 and no. 127 in 1985).

Then our mathematical interests diverged. However, the bonds created at that time never broke, even if some slackened a little. We met again in New York for Lucien's 70th birthday party and conference, to which he had invited us.

Many times in my professional life I needed his help, and I could always count on him. He was a generous person and a faithful friend. We shared very good mathematical moments, but also festive, convivial, and warm moments. These are precious memories.

Mireille Martin-Deschamps

Renée Elkik

I met Lucien Szpiro first in the fall of 1972 at Orsay when he gave a course on his thesis with Christian Peskine. It has always been very pleasant and interesting to talk with him, about any subject, including of course, mathematics; Lucien Szpiro himself liked the informal character of these easy-going conversations. Some people feel at ease only with a final mathematical product. In contrast, Lucien Szpiro rejoiced in explaining his thought process, sometimes repeatedly, until a final product coalesced. From the mid-seventies and for about fifteen years he organized a seminar at the ENS, rue d'Ulm. In the beginning, we were about ten young mathematicians. In order to reserve time to learn, to think, to understand, and to ask naive questions, Lucien Szpiro wished to stay away from the Parisian pressure. In the first years, no advertising was done, so one might have called it a secret seminar, albeit a very open secret, indeed. The seminar gradually became well known, formally announced, and more people attended.

Although his early work focused on commutative algebra, he turned towards geometry, with a special focus on characteristic p, and towards arithmetic geometry. Today, he is especially known for his work in Arakelov

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geometry and his influence on the wide growth of this subject through the school he created about it. There is a clear path with Lucien Szpiro's imprint from the beginnings of its development in the early eighties.

During the winter of 1975–76, he gave a course in Bombay entitled "Lectures on equations defining space curves," which was written up by an audience member, Muhan Kumar. The second chapter reflects Szpiro's concerns at this period and is extended by the report on his lecture at the International Symposium in Tokyo (1977): "Le théorème de la régularité de l'adjointe de Gorenstein à Kodaira."

I will say two words on this.

Let X be a smooth projective surface over a field k, $C \hookrightarrow X$ an integral divisor, $f : \tilde{X} \to X$ a birational transform such that the proper transform of C, say $\tilde{C}$, is smooth, and $\mathcal{J}$ the annihilator of $f_* O_{\tilde{C}}/O_C$ in O_X.

The adjoint curves are sections of $\omega_X(C)$ lying in $H^0(X, \mathcal{J}\omega_X(C))$ and their regularity is governed by the dimension of the image of this vector space in $H^0(C, \omega_C)$. In his Kyoto article, Szpiro shows that the regularity is equivalent to $H^1(\tilde{X}, O_{\tilde{X}}(-\tilde{C})) = 0$.

Recall that Kodaira's Vanishing Theorem was strengthened by Ramanujan in 1972, but both statements are only valid in characteristic 0. What about characteristic p? The first counterexample in characteristic p had only recently been discovered by Raynaud in 1977. More counterexamples would be found by Szpiro and Raynaud later.

At the end of his Kyoto article, Szpiro outlines the following vanishing lemma.

Lemma 1. *Let X be a normal projective surface over a field of characteristic $p > 0$, let E be a locally free module of finite rank over X, and let F denote the Frobenius morphism. If $H^0(X, F^*E \otimes \Omega^1_X) = H^1(X, F^*E) = 0$, then $H^1(X, E) = 0$.*

As everybody knows, Szpiro was especially fond of arguments toggling between characteristic 0 and characteristic p, and especially the passage from characteristic p to characteristic 0. From the lemma, he deduces a proof of Kodaira's vanishing in characteristic 0! This is the kind of proof he was fond of.

A detailed proof of the lemma was announced in an article entitled "L'annulation du Achun par la face Nord"; the title was a joke, but the article will never be written. There is a good reason for this: shortly thereafter, Szpiro found a fertile application of it; this is the next story.

During the summer of 1978, "Les journées de Géométrie Algébrique" took place in Rennes, and Szpiro gave a lecture entitled "Sur le théorème de rigidité de Parsin et Arakelov." In it, he first gave a complete proof of the previous lemma and used it to establish the following rigidity statement.

Theorem 2. *Let X be a smooth projective surface, and let C be a smooth projective curve, both defined over a field of characteristic $p > 0$. Let $X \to C$ be a semistable nonisotrivial fibration with fibers of genus $g \geq 2$. Then, $H^1(X, \omega^{-1}_{X/C}) = 0$.*

This result, plus an upper bound on $\langle \omega_{X/C}, \omega_{X/C} \rangle$, allowed him to answer, for a function field of characteristic p, the following question asked by Shafarevich in 1962 (for a number field); the case of a function field of characteristic 0 had been answered by Parsin and Arakelov.

Question 3. Let C be a smooth projective curve of genus $g \geq 2$, over a number field k, $S \subset C(k)$ a finite set. Is the set of semistable nonisotrivial fibrations over C, with fibers of genus g and smooth outside S, finite?

Thanks to Szpiro's result, one can obtain a new proof of Mordell's conjecture in characteristic p, following a method already used by Parsin in characteristic 0.

An overview of these questions and complete proofs was given in the seminar which took place at the ENS in 1979–80 and is published in *Astérisque* (number 86), "Seminaire sur les pinceaux de courbes de genre au moins 2."

At this point, it became clear that a natural continuation of research concerns the situation over number fields. In 1980, J. P. Serre gave a course at the College de France entitled "Autour du théorème de Mordell Weil." Lucien told me "this means that Serre believes Mordell's conjecture is ripe."

In 1982, Lucien stated a conjecture, now known as Szpiro's conjecture, relating the conductor and the discriminant of an elliptic curve over a number field.

I don't know when he read Arakelov's paper on the intersection theory of arithmetic surfaces, which had been published in 1974; but in the early eighties, he was one of the first (with Parsin) to be convinced that this subject was promising and must be deepened. The first game he liked to play was to translate geometry statements on the integers of a number field and their proofs into the language of Arakelov; next, he would try to compute on a fibration over O_K as if over a projective curve. It would, of course, be audacious to assert that he already predicted around 1982–83 the outstanding development of Arakelov's geometry, but he was enthusiastic and spread this approach in Paris and Columbia. In 1983, Faltings proved Tate's conjecture and consequently Mordell's conjecture. Even if this original proof did not use Arakelov' theory, Szpiro was an interlocutor for Faltings and the seminar held at rue d'Ulm in 1983–84 entitled "Séminaire sur les pinceaux arithmétiques : la conjecture de Mordell" makes a large place for it, provides enrichment to Faltings' proof, and marks the beginning of the wide interest Arakelov's theory generated. In the following years and until recently, Lucien

Szpiro attracted young researchers and supervised a consequently large number of theses. Several of his students are well-known mathematicians today.

Today we are sad because he was so unique, but it was a beautiful adventure.

Renée Elkik

Dorian Goldfeld

I first met Lucien Szpiro when he gave a colloquium talk at Harvard in the early 1980s. He was dressed in a black leather jacket and was tentatively slowly pacing the stage, carefully eyeing the audience. After a while he started his lecture, which I found mind-blowing because of the way he presented an approach for possibly proving Fermat's last theorem and other diophantine conjectures that seemed completely out of reach at the time. After his lecture I introduced myself and we went for a very long walk where I asked a lot of questions and got a lot of answers, which I only vaguely understood at the time. Lucien opened my eyes to a whole new mathematical world for which I am forever grateful.

My own deep interest and delvings into Lucien's mathematical program really began in the mid 1980s when the English mathematician David Masser [4] (University of Basel, Switzerland) and the French mathematician Joseph Oesterlé [5] (University of Paris VI), inspired by work of Szpiro, introduced the *abc* conjecture. For every positive integer $n \in \mathbb{Z}$ let $\mathrm{Rad}(n)$ denote the radical (square-free part) of n which is just the product of the distinct primes dividing n. Here is the original *abc* conjecture of Oesterlé and Masser.

Conjecture 4. *For every $\varepsilon > 0$ there are only finitely many pairwise coprime triples a, b, c of positive integers satisfying*

- $a + b = c$ *with* $a < b$,
- $c > \mathrm{Rad}(abc)^{1+\varepsilon}$.

We now quote from a public lecture of Brian Conrad given at Stony Brook University in September 2013.

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Figure 5. Hiking in the Alps with a donkey, 1985.

"Masser and Oesterlé were led to the abc conjecture because Oesterlé was interested in a new conjecture of Szpiro about elliptic curves (smooth cubic curves, such as $y^2 = x^3 + 8$) which has applications to number-theoretic properties of elliptic curves.

Masser heard Oesterlé's lecture on Szpiro's conjecture and wanted to formulate it without using elliptic curves. Eventually it turned out that the abc Conjecture and Szpiro's Conjecture are equivalent."

In my expository article [2] on the *abc* conjecture, it is shown via an elementary argument that the *abc* conjecture implies that there are only finitely many solutions in integers (x, y, z, n) with $\gcd(x, y, z) = 1$ and $n > 5$ to the equation

$$x^n + y^n = z^n. \tag{1}$$

It is of course possible to introduce stronger versions of the *abc* conjecture that imply Fermat's last theorem which states that (1) has no solutions with $n > 2$. The *abc* conjecture has received a huge amount of international publicity in the last eight years due to Shinichi Mochizuki's surprising release of four preprints in 2012 on his web page (www.kurims.kyoto-u.ac.jp/~motizuki/top-english.html) which claim a proof of the *abc* conjecture.

J. Oesterlé and A. Nitaj proved that the *abc* conjecture implies the famous Szpiro's conjecture [9] which was first stated in 1981.

Conjecture 5 (Szpiro's conjecture). *Let E be an elliptic curve over $\mathbb{Q}$ with minimal discriminant Δ and conductor N. Then for every $\varepsilon > 0$ there exists a constant $c(\varepsilon) > 0$ such that*

$$|\Delta| < c(\varepsilon) N^{6+\varepsilon}.$$

Thinking of a nonzero integer D as the discriminant of an elliptic curve over $\mathbb{Q}$, it can be shown that the following conjecture is equivalent to the *abc* conjecture.

Conjecture 6 (Generalized Szpiro's conjecture). *Let $\varepsilon > 0$ and $M > 0$. Then there exists a constant $c(\varepsilon, M) > 0$ such that*

	Number Field	Function Field
Discriminant	$D = a^3 - 27b^2$	$D = \sum_{\substack{P=\text{zero of } a^3-27b^2 \\ n_P=\text{multiplicity of } P}} n_P$
Conductor	$N = \prod_{p \mid D} p$	$N = q^{\#\{P \mid n_P \neq 0\}}$
One Cycles	Mordell-Weil group	Néron-Severi group
Two Cycles	Tate-Shafarevich group	Brauer group
Intersections	Néron Tate height and Arakelov intersection	Intersection Theory on the Surface
Model	Minimal Model over O_K	Relative Minimal Elliptic Surface over a curve C

Table 1.

for all

$$\{(x,y) \in \mathbb{Z}^2 \mid D = 4x^3 - 27x^2 \text{ with } D \neq 0\},$$

where the greatest prime factor of x and y is bounded by M, we have

$$\max\left(|x|^3, y^2, |D|\right) \ < \ c(\varepsilon, M) \cdot \mathrm{Rad}(D)^{6+\varepsilon}.$$

I moved to Columbia University in 1985. Lucien had a visiting position at Columbia at that time, so we were able to meet regularly both socially and mathematically in the following years. During this period, I attended all of his courses at Columbia, which were extremely popular with the graduate students. He also attended some of my graduate courses. We became quite close and my wife, older daughter, and I visited him in Paris at his beautiful flat near the Luxembourg Gardens, where Lucien and I occasionally played tennis on the public clay courts. He was a superb chef with a remarkable wine cellar and he introduced me to the best (lower cost) restaurants in both Paris and New York.

During this period we continued our discussions on possible approaches to the *abc* conjecture and thanks to his influence and inspiration, I obtained some new equivalences between the *abc* conjecture and periods of elliptic curves. Finally, in the early 1990s Lucien and I started working on a joint project which I shall now describe.

We became interested in the problem of determining whether Szpiro's conjecture could in any way be related to the famous conjecture that the Tate-Shafarevich group of an elliptic curve over $\mathbb{Q}$ is finite. Lucien had proved earlier in [9] that his Conjecture 5 holds in the case of a function field k of one variable over a finite field. Let Ш denote the Tate-Shafarevich group of an elliptic curve E of conductor N over the function field k. We were able to show (see [3]) that if the Birch–Swinnerton-Dyer conjecture holds for E over k, then for every $\varepsilon > 0$ and $N \to \infty$ we have

$$|\text{Ш}| = \mathcal{O}_\varepsilon\left(N^{\frac{1}{2}+\varepsilon}\right), \tag{2}$$

where the constant implied by the $\mathcal{O}_\varepsilon$-symbol depends at most on ε.

In [3] we considered a semistable elliptic curve $E : y^2 = 4x^3 - ax - b$ defined over a global field K which is either an algebraic number field or a function field of one variable defined over a finite field of q elements. Set $f(x,y) = y^2 - 4x^3 - ax - b$ with $a, b \in O_K$, where O_K denotes the ring of integers of K. Then $O_K[x,y]/f$ has Krull dimension two, so E may be viewed as a surface of $\mathrm{Spec}\,(O_K)$. We introduced the dictionary in Table 1 relating invariants of E for a number field or function field.

This leads us to the following conjecture.

Conjecture 7. *Let E be an elliptic curve over a fixed algebraic number field of discriminant D and conductor $N \to \infty$. Let Ш denote the Tate-Shafarevich group of E. Then for every fixed $\varepsilon > 0$*

$$|\text{Ш}| = \mathcal{O}_\varepsilon\left(N^{\frac{1}{2}+\varepsilon}\right).$$

In [3] we were able to prove the equivalence of Conjecture 7 and Conjecture 5 of Szpiro for elliptic curves E of fixed rank over $\mathbb{Q}$ assuming the Birch–Swinnerton-Dyer conjecture. Actually, the assumption of the Birch–Swinnerton-Dyer conjecture is not needed in showing that

$$\text{Conjecture 7} \implies \text{Szpiro's conjecture},$$

since we may pass (by quadratic base change) to the case of rank zero where the Birch–Swinnerton-Dyer conjecture is proved by Kolyvagin.

Returning to the function field case, it is still not known that the Tate-Shafarevich group is finite. However, in our joint paper [3] we prove (in the function field case) that Ш is finite if and only if

$$|\text{Ш}_\ell| = 1$$

(where Ш_ℓ denotes the ℓ torsion in Ш for a suitable prime ℓ). This provides an effective algorithm for determining if Ш is finite in the case of function fields. As is well known, such an algorithm is not presently available for elliptic curves over number fields.

In 1999, Lucien became a Distinguished Professor at the Graduate Center of the City University of New York (CUNY). He built up the number theory group there by bringing in Victor Kolyvagin in 2002 as Mina Rees Chair in Mathematics, and Alex Gamburd as Presidential Professor in 2011, establishing the CUNY Grad Center as a leading international center for number theory. In 2003 Victor Kolyvagin, Peter Sarnak (who was at the Courant Institute at the time), Lucien, and I established the joint Columbia-CUNY-NYU number theory seminar which rotates weekly among the three universities. In the last 17 years, this seminar has had a weekly attendance of 30–60 participants from all over the greater New York-New Jersey metropolitan region. As part of the joint seminar Lucien and I, together with several other mathematicians, organized three mini-conferences: *Algebraic Dynamics* in 2004, *Workshop on p-adic methods in automorphic forms* in 2005, *Ergodic theory and Diophantine problems* in 2006.

Lucien was very devoted to the joint seminar and attended every talk and seminar dinner until the last two years when his health declined and he couldn't make it 100% of the time. He helped transform New York City into a vibrant number theory center making students, postdocs, and professors at the various universities and colleges feel as if they were part of an extended family. At the end of every academic year he and his wife Beth hosted a big party at their condo near Central Park with superb food, wine, live music, and sometimes mathemagic.

Lucien Szpiro was a wonderful man, inspiring colleague, and close personal friend. I learned a lot of mathematics from him and relished his joy of life. He will be dearly missed.

Dorian Goldfeld

Shou-Wu Zhang

I first met Lucien Szpiro when he visited Columbia in the spring of 1987. When I walked into the classroom on the first day of his class, I was a first-year graduate student whose dream was to work in arithmetic geometry. On that day, I had no idea that Lucien would be the master arithmetic geometer who would make my dream a reality.

After so many years, I still remember my very first impression of him: a special blend of fun and rigor. Lucien's lecture was remarkably fun: he wrote on the blackboard very slowly, always missing the letter "h" in words like "what," "which," and "where"; a satisfied verbal "bon" here and a "bien" there; and a wonderful joie de vivre that emanated through excited hand gestures after he had written a pleasing result on the board. His lecture was also technically inspiring: he began his first lecture with a proof of his Riemann–Roch theorem for "arithmetic curves," and then proceeded to use it to reprove all of the classical results in algebraic number theory, such as the finiteness of class numbers, the Dirichlet unit theorem, and the simply connectedness of $\operatorname{Spec}\mathbb{Z}$. Lucien had a chef-like ability to bring out all of the flavors of algebraic number theory, algebraic curves, and arithmetic geometry in one lecture with so much passion.

After Lucien's semester at Columbia ended, I wrote a short letter to him in the fall of 1987 asking him for a problem to work on. He replied to me with just half a page with the prompt:

> *Show the "separateness" of sections of powers of canonical bundle on arithmetic surfaces.*

With this half-page letter, I jumped at the opportunity and privilege to become his first remote student, across the Atlantic at Columbia while he was at the CNRS in Paris. Before I explain the thesis problem in the letter, I will explain two of Lucien's programs about *small points and* ω^2.

Effective Mordell conjecture. The first program is related to the "effective Mordell conjecture." The original Mordell conjecture proved by Faltings in 1983 says that for a smooth and proper curve C over number field K of genus $g \geq 2$, the set of rational points $C(K)$ is finite. The proof of Faltings is not effective in the sense that it does not offer an effective bound for the heights of these solutions. This is clearly a fundamental question in Diophantine geometry left after Faltings' work and thus a big topic in mathematics. To understand Lucien's program on the effective Mordell conjecture, let us review his work on the Shafarevich and Mordell conjectures over function fields in positive

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characteristic in his papers [10], extending previous work of Parshin and Arakelov in characteristic 0. Let $f : X \longrightarrow B$ be a proper and flat morphism from a smooth surface to a curve over an algebraically closed field k. Assume each fiber is semistable, connected, and is of genus g. We also assume that the induced morphism $B \longrightarrow M_g$ to the coarse moduli space of genus g curves is separable when k has positive characteristic. The main discovery in this paper is the existence of a small point $P : C \longrightarrow X$ finite over B in the sense that

$$\frac{\deg P^*\omega_{X/B}}{[C : B]} \leq 2g(B) - 2 + |S|,$$

where S is the subset of points of B over which f is singular. The key to proving such an inequality is to combine his (at the time) recently proved vanishing theorem $H^1(X, L^{-1}) = 0$ for an ample line bundle L and the nonvanishing $\kappa \neq 0$ of the Kodaira–Spencer class

$$\kappa \in \mathrm{Hom}(f_*(\omega_{X/B}^{\otimes 2}), \omega_B(S)).$$

One consequence of this inequality is the following bound on the self-intersection of $\omega_{X/B}$:

$$\omega_{X/B}^2 \leq 8g(g-1) \cdot (2g(B) - 2 + |S|).$$

To apply this inequality to get an effective Mordell conjecture, one uses Parshin's construction: for any curve C over the function field $K = k(B)$, there is a family $Y \longrightarrow C$ of curves. Now for each point $P \in C(K)$, we get a curve Y_P over K. Then we can apply the inequality to the integral model of Y_P to get a bound for the height of P in terms of B and S. Notice that when $g = 1$, the same Kodaira–Spencer class gives the so-called Szpiro inequality:

$$\deg \Delta_{E/B} \leq 6(2g(B) + 2 + |S|)$$

for an elliptic curve over K with semistable reduction.

As a general principle in Diophantine geometry, if a statement is true over function fields, then some variation of it should be true over number fields. This was certainly a main motivation for Lucien to promote Arakelov theory for the proof of the effective Mordell and Shafarevich conjectures. Now we start with a smooth and geometrically connected curve X_K of genus $g \geq 2$ over a number field K. Assume that X_K has a regular semistable model $X \longrightarrow B = \mathrm{Spec}\mathcal{O}_K$ over the ring of integers of $\mathcal{O}_K$ and let S be the set of places of bad reduction, including archimedean places. Arakelov developed an intersection theory on X. Some properties, such as the Hodge index theorem and the semipositivity of the relative dualizing sheaf $\omega_{X/B}$, have been proved by Faltings. The analogue of the above theorem is the following conjecture of Lucien's: *for each g, there is a constant $A(g)$ such that for any curve X/K as above, there is a point $P \in X(\bar{K})$ such that*

$$h_{\omega_{X/B}}(P) \leq A(g) \cdot (\log D_K + \log N_S),$$

where D_K is the absolute discriminant of K, and N_S is the norm of S. By the Hodge index theorem, this bound will give an upper bound for ω^2 and thus an effective Shafarevich and Mordell conjecture. Despite its massive importance in Diophantine geometry, we still have no clue on how to prove this conjecture as there is no Kodaira–Spencer class available in the number field case.

Bogomolov conjecture. The second of Lucien's programs is related to the Bogomolov conjecture about the topology on $X(\bar{K})$ with $g(X) \geq 2$ and the norm $\|x\|^2 := h_{\mathrm{NT}}(x - \frac{\omega}{2g-2})$ defined as the Néron–Tate height on the Jacobian: *There is an $\epsilon > 0$ such that the set of points $x \in X(\bar{K})$ with $\|x\|^2 < \epsilon$ is finite.* Using the Hodge index theorem, Lucien had shown that when X/K has good reduction everywhere and $\omega_{X/B}^2 > 0$, then the above conjecture holds. He also mentioned how to prove the converse: if $\omega_{X/B}^2 = 0$, then one can construct points with small heights by sections of powers $\omega(\epsilon F)$ provided these sections have no fixed horizontal part. Lucien proposed two conjectures about $\omega_{X/B}$: the positivity $\omega_{X/B}^2 > 0$ for all X/B and the separateness of sections of $\bigcup_{n\geq 1} \Gamma(\omega_{X/B}^{\otimes n})$.

Our story on Lucien's second program. The problem that he wrote in his 1987 half-page letter gave the following even more general notion of separateness:

> *Given a Hermitian line bundle L on X which is numerically positive, show that the set of small sections of positive powers of L has no fixed horizontal part.*

This was a tantalizing problem, but I had no idea of how to start. This problem was too difficult for me to attack.

In the spring of 1988, Lucien returned to Columbia and immediately asked me to meet with him in his office for one hour per week. At our first meeting, I was a bit nervous and I told him that I had no progress on the problem that he had assigned to me. Then he gave me a lesson about his recent work on the formula for the arithmetic degree of the dualizing sheaf on a regular and semistable elliptic curve over a ring of integers O_K over a number field:

$$12 \deg_O \omega_{E/\mathcal{O}_K} = \log \mathrm{N}(\Delta_{E/\mathcal{O}_K}).$$

Thus $\log \mathrm{N}(\Delta_{E/\mathcal{O}_K})$, the finite part of Faltings height $h_F(E)$, has a separate global arithmetic meaning. He proves this formula by using a projection formula for computing the self-intersection O_E^2 of the unit section via isogeny $[2] : E \longrightarrow E$:

$$([2]^* O_E)^2 = 4\, O_E^2.$$

The proof is short and elegant; it reflected his deep understanding of Green's functions at archimedean places as analogues on the polygons associated to bad reductions of E/O_K. He raised a question about the maximal values of these Green's functions. Fortunately, I was able to solve this problem by applying a Fourier transform to the

Faltings formula for Green's functions. Lucien was very pleased with my minor result and even mentioned it in his article for the Grothendieck Festschrift; his enthusiasm and happiness with this gave me a boost of confidence that would help me throughout graduate school.

During the remainder of the semester, I would excitedly show Lucien several "proofs" of the effective Mordell conjecture on one day, only to have to point out the gaps and mistakes in those "proofs" the next. He was always very patient when listening to all of my nonsense and never once complained. Lucien always knew how to let his students make and learn from their own mistakes with immense patience. I could not be more grateful for his mentorship.

At the end of the semester, he made arrangements for me to visit him in Paris in the coming year. Given my already-poor English and nonexistent French, this was something that had the potential to really test his patience.

In February 1989, my wife Min and I arrived at Charles de Gaulle Airport, where Lucien personally received us. Lucien not only brought us to the IHES at Bures-sur-Yvette and our accommodations at the Résidence de l'Ormaille, which was already more than we could have expected from him, but also helped us with paperwork at the office building, helped us register for meals at the IHES, and immediately introduced me to R. Elkik at Paris 11 in Orsay who helped me enter their graduate system for library and cafeteria usage. Lucien went so far out of his way to make us feel at home in this country that was so foreign to us.

After a few days of settling in, I took a trip to the Institut de Henri Poincaré in Paris where Lucien was working as a CNRS fellow. After he arranged for our living stipend at the CNRS office, I showed him two of my new results. One was about ω^2 for a modular curve $X(N)$, and another was a small improvement of a new result of Silverman–Hindry.

Lucien was very happy with these results, saying that they were enough for a PhD thesis. No longer concerned with being able to graduate, I became free to focus on the problem from Lucien's 1987 letter that I had been hopelessly stuck on.

I started working on my original thesis problem in a more systematic way. I had some idea of how to prove the arithmetic Nakai–Moishezon just for $\mathbb{P}^1$ for some easy cases. In the general case for an arithmetic surface, I managed to translate "ampleness" for line bundles on varieties over a discrete valuation ring into a statement in Kähler geometry. After talking to several differential geometers visiting IHES, I realized that it was not known whether my desired statement was true in Kähler geometry.

After my wonderful stay in Paris with Lucien, I used a Sloan Research Fellowship to become a visiting student at Princeton under the supervision of G. Faltings. At Princeton, I wrote to Professor S. T. Yau asking about my statement in Kähler geometry. He answered my letter with the PhD thesis of G. Tian, which contained a proof of my desired statement. So by the end of 1989, I completely solved the original thesis problem raised by Lucien with some applications including the Bogomolov conjecture for curves in multiplicative groups. Lucien accepted it as my PhD thesis after R. Elkik checked the details.

Lucien could not visit the US in 1990, so I spent another year at Columbia before having my PhD defense in May 1991. Lucien held a big party at his apartment after the defense, with more than a dozen bottles of French wine and Chinese food prepared by my wife—a wonderful combination, as Lucien knew how to match French wine with Chinese food! Soon after my thesis defense, I discovered a new intersection theory on curves called *admissible pairings* giving the inequality

$$\omega^2_{X/B} \geq \omega^2_a \geq 0,$$

where the first equality holds exactly when X/B has good reduction everywhere and the second equality holds exactly when the Bogomolov conjecture fails.

This is precisely a generalization of Szpiro's theorem and conjecture in the smooth case, developed following Lucien's earlier insights.

When I was a postdoc at the IAS and then a junior faculty member at Princeton from 1992–1995, I kept visiting Lucien in New York, Paris, and Bures-sur-Yvette, where I continued to learn new mathematics from him and his students. For example, motivated by his lecture at Columbia about dynamical systems on projective spaces, I proposed an intersection theory for adelically metrized line bundles and used it to prove some special cases of the Bogomolov conjecture. Together with Lucien and E. Ullmo, we discovered a new tool in Arakelov theory: the equidistribution of small points. This equidistribution and adelic intersection theory eventually led E. Ullmo and I to prove the full Bogomolov conjecture with great generality in the summer of 1996. This brought a satisfying conclusion to Lucien's program about small points and the Bogomolov conjecture.

In the summer of 1996, I moved back to Columbia as a faculty member; Lucien moved to CUNY just a few years later. We met regularly at our joint number theory seminar and with occasional dinners at each other's homes. I started to work more on the Gross–Zagier formula rather than Arakelov theory, but Lucien's work still regularly inspired me. As Lucien moved into algebraic dynamics, I supervised three doctoral students who worked on algebraic dynamics: Xinyi Yuan, Xander Faber, and Alon Levy. Later, I wrote a paper relating ω^2_a with heights of Gross–Schoen cycles. This led Z. Cinkir to prove an effective version of the Bogomolov conjecture for curves over function fields in characteristic 0. More recently, the full

Bogomolov conjecture over function fields in characteristic 0, where equidistributions are not available, was proved by S. Cantat, Z. Gao, P. Habegger, and J. Xie using a new tool: Betti foliations.

All of these results were exciting developments that extended Lucien's original program on the Bogomolov conjecture and his more recent work on dynamical systems.

Throughout my interactions with Lucien, he taught me a lot of mathematics; he taught me how to learn mathematics on my own; and he taught me a bit of his joie de vivre. In between the mathematics, Lucien taught this young graduate student from the Chinese countryside a bit about tennis, wine, and the finer things in life. Tennis was not actually so difficult to pick up given its similarities to badminton, a popular sport in China. On the other hand, Lucien had to exercise real patience to introduce me to wine. Wine is very different from Chinese liquor, so Lucien gave me a thorough education progressing from light white wines to bold red wines.

At one point, I spent two weeks in Paris living in the attic of Lucien's apartment which he called "only suitable for people under 35." This small attic room had a bed, a bathroom, and dozens of French novels on the wall. Lucien had wanted to teach me about cinema and literature, but we unfortunately never got around to that. Perhaps I picked up an integral part of my French mathematics education here though, as he invited me to have breakfast with him every day and taught me to use a French press, and coffee has since become a necessity in my life.

Looking back on my time with Lucien, I realize I learned so much from him, both from his mathematical programs and from his personal mentorship. I am very fortunate to have contributed to Lucien's programs on the one hand and to have learned techniques from him which have been so useful in other areas on the other. I am grateful to him for his generosity in sharing his insights with me and for all that he has done to help me as a person. In mathematics as well as in life, tennis, wine, food, and humor, Lucien's influence will live on—but I will dearly miss him.

Shou-Wu Zhang

Thomas Tucker

During Lucien's time at the Graduate Center, he focused a great deal of his energy on creating a warm, welcoming research environment for graduate students and postdocs. He cofounded the joint Columbia-CUNY-NYU number theory seminar, which became an important meeting place and social activity for everyone in number theory at the Graduate Center. He made sure that his postdocs and graduate students were heavily involved with the seminar; the seminars even included a special pretalk aimed exclusively at graduate students. It wasn't unusual to see 50 or 60 people at some of the talks, and it became a great way for younger mathematicians in the area to network and make new friends and potential coauthors. Every spring, Lucien and his wife Beth held a party at his home for everyone in the CUNY number theory community. These became beloved, well-attended affairs that often featured musical entertainment and, on at least one occasion, a magic show. Lucien also made a point of introducing his students and postdocs to many of the great cultural and culinary things that New York has to offer. He took me and others to jazz shows, movies, and restaurants (his French background always came through in his judicious choices here). Lucien loved New York: the cultural melting pot; the energy; the music. Lucien had seven graduate students and five postdocs during his time at the Graduate Center, and he invested an enormous amount of time, both mathematically and personally, in every one of them.

Dynamical Mahler measures. Lucien's research came to focus on the new emerging field of arithmetic dynamics. His equidistribution result with Ullmo and Zhang for points of small height on abelian varieties helped lead to Ullmo's and Zhang's proof of the Bogomolov conjecture. In this case, the underlying measure was simply the Haar measure, and Lucien wondered if it was possible to do similar things with the Mané-Lyubich measure for rational functions (a measure associated to a rational function that can in some sense be thought of as a general dynamical analog of the Haar measure on a topological group). When one takes the rational function $f(x) = x^2$, the Mané-Lyubich measure is simply the Haar measure on the unit circle. In the 1930s, Mahler defined the Mahler measure $M(F)$ of a polynomial F to be the integral of $\log|F|$ against this measure, i.e., as $M(F) = \int_0^1 \log|F(e^{2\pi i\theta})|\,d\theta$. The Weil height of an algebraic number, which is usually defined using absolute values on number fields, may also be defined by $h(z) = \frac{1}{[\mathbb{Q}(z):\mathbb{Q}]}M(F_z)$, where F_z is the monic

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polynomial of minimal degree with coefficients in $\mathbb{Q}$ having z as a root (for example, if $z = a/b \in \mathbb{Q}$, then $F_z = x - a/b$ and $h(z) = \log\max(|a|, |b|)$). Lucien asked if there was a dynamical version of Mahler's formula. To describe this, we will need a little bit of notation. For a rational function φ of degree $d > 1$, Call and Silverman, following ideas of Tate, introduced a canonical height function for φ defined as $h_\varphi(z) = \lim_{n\to\infty} \frac{h(\varphi^n(z))}{d^n}$ for $z \in \bar{\mathbb{Q}}$. This has many of the same properties as the Néron-Tate heights on abelian varieties; for example, $h_\varphi(\varphi(z)) = dh_\varphi(z)$ for all $z \in \bar{\mathbb{Q}}$ and $h_\varphi(z) = 0$ if and only if z is preperiodic for φ. Lucien asked if one could obtain $h_\varphi(z)$ from the "dynamical Mahler measure" $\frac{1}{[\mathbb{Q}(z):\mathbb{Q}]} \int \log|F_z|\, d\mu_\varphi$, where F_z is the minimal monic for z over $\mathbb{Q}$ and $d\mu_\varphi$ is the Mané-Lyubich measure associated to φ. He was able to prove that indeed $h_\varphi(z)$ is equal to the "dynamical Mahler measure" $\frac{1}{[\mathbb{Q}(z):\mathbb{Q}]} \int \log|F_z|\, d\mu_\varphi$ plus some additional terms coming from places of bad reduction. Later these terms at the places of bad reduction were interpreted as integrals at the places of bad reduction on Berkovich spaces by Favre and Rivera-Letelier.

The Mané-Lyubich measure can be calculated by averaging over preperiodic points. More precisely, if φ is a rational function over $\mathbb{C}$ of degree greater than 1, f is a continuous function on the Riemann sphere, and $\mathcal{S}_{m,n}$ denotes the set of points of φ having preperiodic m and period n, then

$$\lim_{m+n\to\infty} \frac{1}{\#\mathcal{S}_{m,n}} \sum_{z\in\mathcal{S}_{m,n}} f(z) = \int f\, d\mu_\varphi.$$

Note however that this works only for continuous functions f and not necessarily for functions like $\log|F|$, for F a polynomial, which have poles. Lucien and his coauthors were able to show that when φ and F have coefficients in $\bar{\mathbb{Q}}$, then one can indeed calculate the dynamical Mahler measure of F by averaging over preperiodic points [6]. This is very useful for many applications because it is often much easier to calculate a limit of sums of values of $\log|F|$ than to attempt to integrate $\log|F|$ against a complicated measure. In particular, this result has found use in the computation of Lyapunov exponents.

A symmetric relation between rational maps. Many of Lucien's insights came from very simple, easy-to-state ideas. For example, Lucien wondered if it was possible to find a symmetric relation between points of small canonical height for two different rational functions φ and ψ; more specifically, he asked if there was a symmetry between the values of h_φ on preperiodic points of ψ and h_ψ on preperiodic points of φ. In its most naïve form, where one asks for some sort of equality at the level of individual points, there is not much of a pattern, but Lucien persisted in pursuing some kind of symmetric relation. Ultimately, he and his coauthors were able to show that when one passes to a limit, a relation of the exact form that he proposed does indeed exist [8]. The result is most easily expressed using a pairing between rational functions first introduced by Arakelov and later refined by Zhang and others. Crucially, this pairing is symmetric so $\langle\varphi,\psi\rangle = \langle\psi,\varphi\rangle$ for any rational functions φ, ψ.

Theorem 8. *Let $\{x_n\}$ be a sequence of distinct points in $\mathbb{P}^1(\bar{\mathbb{Q}})$ such that $h_\psi(x_n) \to 0$, and let $\{y_n\}$ be a sequence of distinct points in $\mathbb{P}^1(\bar{\mathbb{Q}})$ such that $h_\varphi(y_n) \to 0$. Then $h_\varphi(x_n) \to \langle\varphi,\psi\rangle$ and $h_\psi(y_n) \to \langle\varphi,\psi\rangle$.*

The proof is essentially an application of equidistribution results of Baker, Chambert-Loir, Favre, Rivera-Letelier, and Rumely, along with results of Arakelov and Zhang establishing basic properties of the pairing. Lucien's great insight here was that there should be some kind of underlying symmetry. Theorem 8 has found wide usage recently in arithmetic dynamics, most notably perhaps in recent groundbreaking work of DeMarco, Krieger, and Ye proving a conjecture of Bogomolov, Fu, and Tschinkel on uniform bounds on the number of common torsion points between two elliptic curves.

Dynamical heights over function fields. Another simple question of Lucien's is as follows. Suppose that we take a rational function φ of degree greater than 1 defined over a function field L. One can still define a Weil height $h(\cdot)$ and create a canonical height $h_\varphi(z) = \lim_{n\to\infty} \frac{h(\varphi^n(z))}{d^n}$. However, since one no longer has the Northcott property for the Weil height (which says that there are finitely many points of bounded height and bounded degree), it is no longer clear that $h_\varphi(z) = 0$ means that z must be preperiodic. Indeed, when φ is defined over the base field of L this will not be the case in characteristic 0. Lucien proposed that if φ is not isotrivial (that is, cannot be defined over the base field of L), then $h_\varphi(z) = 0$ means that z must indeed be preperiodic. This led to a great deal of interesting activity in the area, with Benedetto soon proving that this is true for polynomials and Baker shortly after proving it for rational functions. Later, Chatzidakis and Hrushovski translated an idea of Lucien's about bounded families into the language of model theory and produced a remarkable result about points with canonical height zero in higher dimensions. One of Lucien's crucial insights here was the idea that isotriviality should be equivalent to having good reduction at all primes; Benedetto and Baker proved this for morphisms of the projective line, and Lucien and his coauthors later proved it for morphisms of projective space of any dimension [7].

Extending polarized maps. Another result of Lucien's came from an interesting question arising in work of Fakhruddin, having to do with polarized morphisms. If X is a projective variety and $\varphi : X \longrightarrow X$ is a morphism, φ is said to be *polarized* if there is an ample line bundle L on X such that $\varphi^* L \cong L^{\otimes d}$ for some $d > 1$. The idea is that d is the analog of the degree of self-map on projective space. Polarized maps have many good properties, such as the existence of well-behaved canonical height functions. Fakhruddin proved that if $\varphi : X \longrightarrow X$ is a polarized map, then there is an embedding $i : X \longrightarrow \mathbb{P}^m$ (for some m) along with a map $\psi : \mathbb{P}^m \longrightarrow \mathbb{P}^m$ extending φ, i.e., $\psi \circ i = i \circ \varphi$. There is no particular association between the line bundle L and the embedding i, however. When L is very ample it gives rise to an embedding $j : X \longrightarrow \mathbb{P}^n$ (where $n+1$ is the dimension of the space of global sections of L). Lucien wondered if it was true that there was some $\psi : \mathbb{P}^n \longrightarrow \mathbb{P}^n$ such that $\psi \circ j = j \circ \varphi$. It turns out that there may not be, but Lucien and his coauthor [1] were able to show that for some iterate φ^ℓ of φ, there is some $\psi : \mathbb{P}^n \longrightarrow \mathbb{P}^n$ with the property $\psi \circ j = j \circ \varphi^\ell$. The proof is quite short and elegant.

Dynamical Shafarevich-Faltings. The dynamical work that Lucien was perhaps most proud of was his work on a dynamical analog of the Shafarevich-Faltings theorem, which states that for a finite set of places S of a number field K, there are at most finitely many abelian varieties of bounded dimension having good reduction at all places outside of S. Shafarevich proved this for elliptic curves, using a relatively simple diophantine argument, while Faltings' proof in higher dimensions is much more difficult and played a crucial role in Faltings' celebrated proof of the Mordell conjecture. On an abelian variety, a prime $\mathfrak{p}$ of good reduction is one at which the variety remains nonsingular when one reduces modulo $\mathfrak{p}$. For example, an elliptic curve $y^2 = f(x)$ has good reduction at all primes $\mathfrak{p}$ where f does not have multiple roots modulo $\mathfrak{p}$. Good reduction can also be interpreted in terms of multiplication maps on abelian varieties—namely, are they still well-behaved modulo $\mathfrak{p}$—and it was this notion that Lucien sought to generalize to a dynamical context. The usual notion of good reduction for a rational function φ over a number field K is that φ has good reduction at $\mathfrak{p}$ if φ descends to a well-defined map on $\mathbb{P}^1$ modulo $\mathfrak{p}$. But for this notion there is clearly no analogue of the Shafarevich-Faltings theorem; for example $x^2 + m$ has good reduction at all primes when m is an integer, so no reasonable finiteness statement is possible. Lucien concocted the clever idea of an alternate notion of good reduction, what he called "critically good reduction." A rational function φ is said to have critically good reduction at a prime $\mathfrak{p}$ if none of the critical points (that is, the points z such that $\varphi'(z) = 0$) meet at $\mathfrak{p}$ and also none of the critical values (the image of the critical points under φ) meet at $\mathfrak{p}$. This is related to the notion of good reduction on an elliptic curve $y^2 = f(x)$, because when one considers projection onto the x-coordinate, the ramification points are simply the roots of f (plus the point at infinity) so one has good reduction for the elliptic curve at $\mathfrak{p}$ exactly when the ramification points of this projection map do not meet at $\mathfrak{p}$. Using this notion one does indeed obtain a finiteness theorem with a notion of equivalence that is slightly different than conjugacy. We will say that two rational functions $\varphi_1, \varphi_2 : \mathbb{P}^1_K \longrightarrow \mathbb{P}^1_K$ are *equivalent* if there are automorphisms σ, τ defined over K such that $\sigma\varphi_1\tau = \varphi_2$. With this terminology, Lucien and his coauthor proved the following [14].

Theorem 9. *Let S be a finite set of places of a number field K, and let d be an integer greater than 1. Then there are at most finitely many equivalence classes of rational functions on $\mathbb{P}^1_K$ of degree d having at least three critical points that have critically good reduction outside of S.*

(The exact statement of the theorem is slightly more technical since it involves fixing a model for the projective line over the ring of integers over the number field.) Later, Lucien realized that with a slightly different notion of good reduction, what he called "critically excellent reduction," one could prove the same finiteness result up to conjugacy rather than equivalence. This work has yet to be published.

Lucien's greatest ambition was to prove the Szpiro conjecture (equivalent to the Masser-Oesterlé *abc* conjecture), which states that the discriminant of an elliptic curve can be bounded in terms of its conductor. With his work on dynamical analogs of the Shafarevich-Faltings theorem, what he hoped to do was come up with different but related notions of conductors and discriminants where one might hope to more easily prove something along those lines. In particular, Lucien hoped to be able to relate the "critical conductor" of a rational function, the primes at which the critical values or critical points meet, with some kind of critical discriminant, which would take into account some kind of intersection multiplicities between the critical values or critical points at these primes. This, he hoped, would lead to a proof of his original conjecture. It is a very ambitious idea and typical of Lucien's approach to mathematics. Lucien was always focused on the biggest problems in number theory and was more than willing to take approaches which others might see as longshots if he thought there was any chance that they might work.

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References

[1] Anupam Bhatnagar and Lucien Szpiro, *Very ample polarized self maps extend to projective space*, J. Algebra **351** (2012), 251–253, DOI 10.1016/j.jalgebra.2011.11.010. MR2862209

[2] Dorian Goldfeld, *Beyond the last theorem*, The Sciences, New York Academy of Sciences, March/April 1996, pp. 34–40.

[3] Dorian Goldfeld and Lucien Szpiro, *Bounds for the order of the Tate-Shafarevich group*, Compositio Math. **97** (1995), no. 1-2, 71–87. Special issue in honour of Frans Oort. MR1355118

[4] D. W. Masser, *Open problems*, Proceedings of the Symposium on Analytic Number Theory, W. W. L. Chen (ed.), Imperial College, London, 1985.

[5] Joseph Oesterlé, *Nouvelles approches du "théorème" de Fermat* (French), Astérisque **161-162** (1988), Exp. No. 694, 4, 165–186 (1989). Séminaire Bourbaki, Vol. 1987/88. MR992208

[6] Jorge Pineiro, Lucien Szpiro, and Thomas J. Tucker, *Mahler measure for dynamical systems on $\mathbb{P}^1$ and intersection theory on a singular arithmetic surface*, Geometric methods in algebra and number theory, Progr. Math., vol. 235, Birkhäuser Boston, Boston, MA, 2005, pp. 219–250, DOI 10.1007/0-8176-4417-2_10. MR2166086

[7] Clayton Petsche, Lucien Szpiro, and Michael Tepper, *Isotriviality is equivalent to potential good reduction for endomorphisms of $\mathbb{P}^N$ over function fields*, J. Algebra **322** (2009), no. 9, 3345–3365, DOI 10.1016/j.jalgebra.2008.11.027. MR2567424

[8] Clayton Petsche, Lucien Szpiro, and Thomas J. Tucker, *A dynamical pairing between two rational maps*, Trans. Amer. Math. Soc. **364** (2012), no. 4, 1687–1710, DOI 10.1090/S0002-9947-2011-05350-X. MR2869188

[9] Lucien Szpiro, *Seminaire us les pinceaux des courbes de genre au moins deux*, Astérisque **86** (1981), no. 3, 44–78.

[10] Lucien Szpiro, *Propriétés numériques du faisceau dualisant relatif* (French), Astérisque **86** (1981), 44–78. Seminar on Pencils of Curves of Genus at Least Two. MR3618571

[11] Lucien Szpiro, *Degrés, intersections, hauteurs* (French), Astérisque **127** (1985), 11–28. Seminar on arithmetic bundles: the Mordell conjecture (Paris, 1983/84). MR801917

[12] Lucien Szpiro, *Un peu d'effectivité* (French), Astérisque **127** (1985), 275–287. Seminar on arithmetic bundles: the Mordell conjecture (Paris, 1983/84). MR801928

[13] Lucien Szpiro, *Sur les propriétés numériques du dualisant relatif d'une surface arithmétique* (French), The Grothendieck Festschrift, Vol. III, Progr. Math., vol. 88, Birkhäuser Boston, Boston, MA, 1990, pp. 229–246, DOI 10.1007/978-0-8176-4576-2_9. MR1106917

[14] Lucien Szpiro and Thomas J. Tucker, *A Shafarevich-Faltings theorem for rational functions*, Pure Appl. Math. Q. **4** (2008), no. 3, Special Issue: In honor of Fedor Bogomolov, 715–728, DOI 10.4310/PAMQ.2008.v4.n3.a4. MR2435841

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