SHORT STORIES

The Gauss–Lucas Theorem

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The Gauss–Lucas theorem says that for any complex polynomial P, the roots of the derivative P' lie in the convex hull of the roots of P. In other words, the roots of P' lie inside the smallest convex subset of the complex plane containing all the roots of P. This theorem is not hard to prove, but is there an intuitive explanation? In fact there is, using physics—or more precisely, electrostatics in 2-dimensional space [2].

Here is the basic idea. For simplicity, suppose P has no repeated roots. Put a particle of charge 1 at each root. These particles create a vector field called the electric field, $\vec{E}$. The roots of P' are precisely the points where this electric field vanishes! In fact, each particle creates its own electric field pointing radially outward, and we sum these fields to get $\vec{E}$. Because of this, $\vec{E}$ can only vanish inside the convex hull of the roots of P.

Figure 1, drawn by Greg Egan, shows an example. Here we see five charged particles marked in red, located at the roots of some quintic polynomial P. The curves are the level curves of a function ϕ, called the **potential**, such that $\vec{E} = -\vec{\nabla}\phi$. The potential has three critical points, marked in blue. The electric field vanishes only at these points. Note that in some sense these points lie "between" the charged particles. This is in accord with the Gauss–Lucas theorem, and also physical intuition: since each particle creates its own electric field pointing radially outwards, these fields can only cancel at locations between particles.

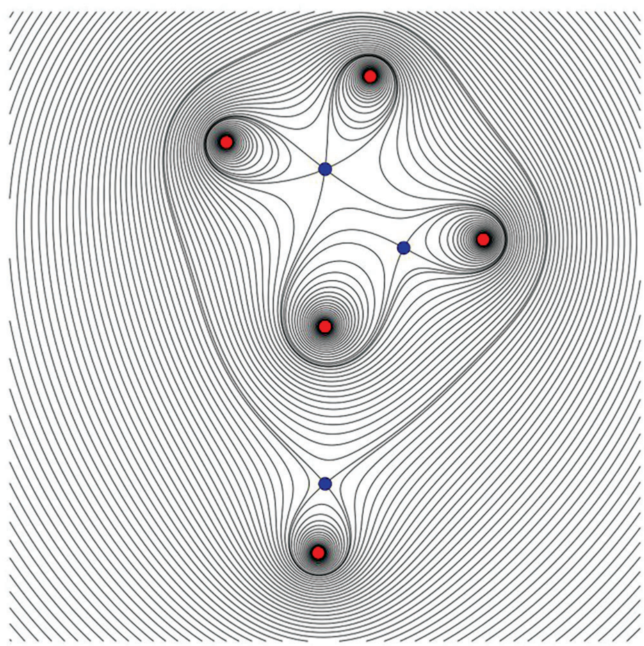

Figure 1.

Note also that our polynomial P has five roots, but its derivative has only three. But there is no paradox here: in this example, the uppermost blue dot is a root of multiplicity two for P'. Thus, P' has degree four.

To prove the Gauss–Lucas theorem using these ideas, we need some mathematics coming from physics. The equations of electrostatics make sense in any dimension. In $\mathbb{R}^n$ they say the potential $\phi : \mathbb{R}^n \to \mathbb{R}$ obeys $\nabla^2\phi = -\rho$, where $\rho : \mathbb{R}^n \to \mathbb{R}$ is the charge density. The case $n = 2$ is special because we can identify $\mathbb{R}^2$ with the complex plane. In

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this case, when ρ is a Dirac delta at the origin, there is a solution

$$\phi(z) = -\frac{1}{2\pi} \ln |z|.$$

This gives an electric field $\vec{E}$ pointing radially outward, with magnitude $1/2\pi|z|$. This is the electric field produced by a point particle of charge 1 at the origin. Since the equations of electrostatics are linear and translation-invariant, it is then easy to solve them for any collection of charged particles. If we have particles of charge 1 at points $a_1, \dots, a_k \in \mathbb{C}$, the potential is

$$\phi(z) = -\frac{1}{2\pi} \sum_{i=1}^{k} \ln |z - a_i|.$$

Now suppose P is a polynomial without repeated roots, having roots at the points $a_1, \dots, a_k \in \mathbb{C}$. Putting particles of charge 1 at these points, we get the above potential ϕ. Thus, $|P|$ is some nonzero constant times

$$\prod_{i=1}^{k} |z - a_i| = \exp\left(\sum_{i=1}^{k} \ln |z - a_i|\right) = \exp(-2\pi\phi).$$

The critical points of P are the same as those of $|P|$, except for critical points where P vanishes—but these are forbidden, since we are assuming P has no repeated roots. So, the critical points of P are precisely the critical points of $\exp(-2\pi\phi)$. These, in turn, are the same as the critical points of ϕ. But since $\phi = -\nabla\vec{E}$, these are the points where the electric field vanishes. In short, P' vanishes precisely where $\vec{E}$ vanishes.

Now let us use this to prove the Gauss–Lucas theorem. Say $z \in \mathbb{C}$ lies outside the convex hull of the roots $a_1, \dots, a_k$. Think of all these points as points $\vec{a}_i$ and $\vec{z}$ in $\mathbb{R}^2$. By the separating hyperplane theorem, we can draw a line with $\vec{z}$ on one side and all the points $\vec{a}_i$ on the other side. Let $\vec{v}$ be a vector perpendicular to this line and pointing toward the $\vec{z}$ side. The electric field $\vec{E}$ at $\vec{z}$ is a sum of vectors pointing from the charges $\vec{a}_i$ to $\vec{z}$, since

$$\vec{E}(\vec{z}) = -\vec{\nabla}\phi(\vec{z}) = \frac{1}{2\pi} \sum_{i=1}^{k} \frac{\vec{z} - \vec{a}_i}{|\vec{z} - \vec{a}_i|^2}.$$

But $(\vec{z} - \vec{a}_i) \cdot \vec{v} > 0$, so $\vec{E}(\vec{z})$ cannot vanish.

This proves the Gauss–Lucas theorem for polynomials without repeated roots. But we only used the assumption that all the a_i were distinct at one place in the argument: to avoid situations where both P and P' vanish at the same point. These situations pose no problem: a root of P' that is also a root of P is *obviously* in the convex hull of the roots of P.

For more thoughts along these lines, we recommend the book by Marden [2]. For example, in the late 1800s, Siebeck, Lucas, and Bôcher independently showed that if P is a cubic whose vertices form a triangle, the roots of P' lie at foci of an ellipse inscribed in this triangle—and in fact, this ellipse touches the midpoints of the triangle's sides! It is called the **Steiner inellipse**, and it is also the ellipse of largest area that fits inside the triangle. Bôcher asked if this result could be generalized to polynomials of higher degree. In 1919, Linfield showed that indeed it can, using a general concept of "foci" of a real plane algebraic curve [1]. It would be nice if there were a way to understand some of these more advanced results using electrostatics.

References

[1] Ben-Zion Linfield, *On the relation of the roots and poles of a rational function to the roots of its derivative*, Bull. Amer. Math. Soc. **27** (1920), no. 1, 17–21, DOI 10.1090/S0002-9904-1920-03350-1. MR1560349

[2] Morris Marden, *Geometry of polynomials*, 2nd ed., Mathematical Surveys, No. 3, American Mathematical Society, Providence, R.I., 1966. MR0225972

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Credits

Figure 1 is courtesy of Greg Egan.

Photo of John C. Baez is courtesy of Shaun Martin.