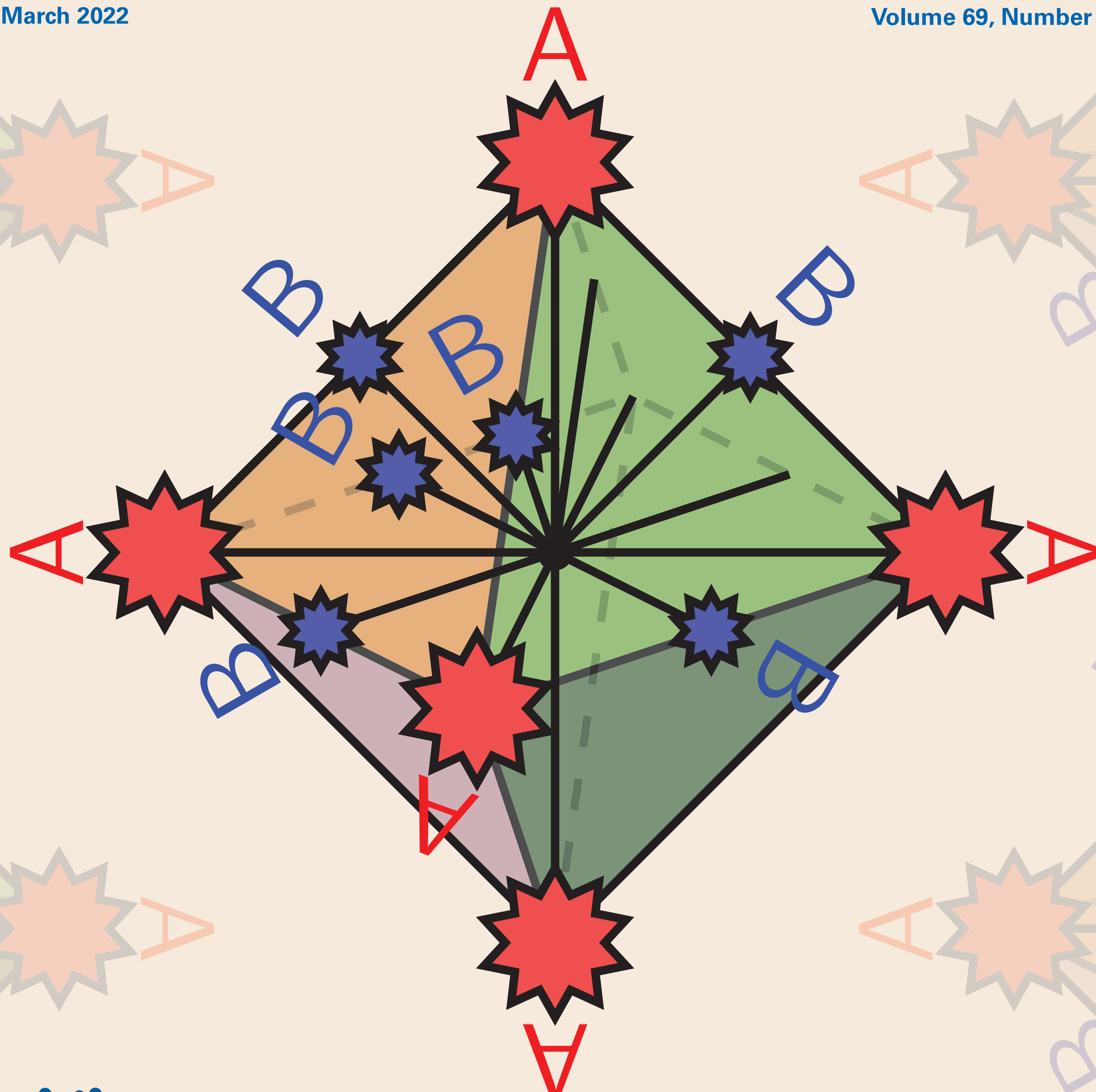
ISSN 0002-9920 (print)
ISSN 1088-9477 (online)

Notices

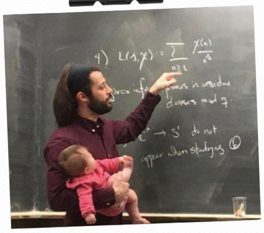
of the American Mathematical Society

March 2022 Volume 69, Number 3



AMERICAN MATHEMATICAL SOCIETY
Advancing research. Creating connections.

The cover design is based on imagery from "Gems from the Work of Georgia Benkart," page 375.



Bryn Mawr

Photo by: AMS Graduate Student Chapter at Bryn Mawr

University of Louisville

Photo by: AMS Graduate Student Chapter at University of Louisville

University of Houston

Photos by: AMS Graduate Student Chapter at University of Houston

The Graduate Research Seminar, held biweekly, is an opportunity for graduate students to present, share, and get feedback from peers. Topics can include research, expository work, coursework, and information from conferences attended.

At AMS Chapter meetings throughout the fall semester, professors gave talks on research or topics in mathematics they find intriguing and students practiced presenting research in front of their peers.

In the spring, the chapter held their recruiting and fundraising event.

Author Ben Orlin gave a presentation of ideas from his next book: "Four Mathematical Games That I Can't Stop Thinking About".

University of Nebraska–Lincoln

Photos by: AMS Graduate Student Chapter at University of Nebraska–Lincoln

University of New Orleans

Photos by: AMS Graduate Student Chapter at University of New Orleans

The always popular annual Pi-Day Pie Baking Contest gave the graduate students and faculty an opportunity to socialize and eat some great pie.

"Pi Talks!" Seminars presented by math faculty and graduate students were held throughout the school year. A profitable and tasty bake sale was held to raise additional funds to support the activities of the UNO chapter of the AMS.

AMS Graduate

Student Chapter

Luncheon

Photos by: Kate Awtrey, Atlanta Convention Photography

2020 JOINT MATHEMATICS MEETINGS

Notices of the American Mathematical Society

March 2022
Women's History Month

Cover Credit: The image used in the cover design appears in "Gems from the Work of Georgia Benkart," p. 375, and is courtesy of Tom Halverson.

FEATURES

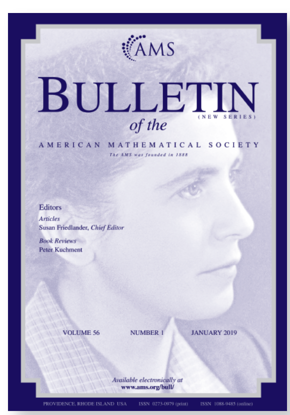
FROM THE AMS SECRETARY

INVITATIONS FROM THE AMS

Member Benefit Reminder

If you wish to receive a print copy of either *Notices of the AMS* or *Bulletin of the AMS* in addition to the electronic version, please select that option in your online profile. We encourage you to also take a moment to ensure that your mailing address and other information is up to date.

1. Log in to your member profile at www.ams.org/account.
2. In the Membership column, click "Edit" next to *Notices* or *Bulletin*, which will open a new webpage.
3. Next to *Notices* and/or *Bulletin*, remove the checkmark on "Opt Out of Paper."



AMERICAN MATHEMATICAL SOCIETY

Advancing research. Creating connections.

For more information about AMS Membership visit **www.ams.org/membership**

Notices
of the American Mathematical Society

SUBSCRIPTION INFORMATION
Subscription prices for Volume 69 (2022) are US$720 list; US$576 institutional member; US$432 individual member; US$648 corporate member. (The subscription price for members is included in the annual dues.) A late charge of 10% of the subscription price will be imposed upon orders received from non-members after January 1 of the subscription year. Add for postage: Domestic—US$6.50; International—US$24. Surface delivery outside the US and India—US$27; in India—US$40; expedited delivery to destinations in North America—US$35; elsewhere—US$120. Subscriptions and orders for AMS publications should be addressed to the American Mathematical Society, PO Box 845904, Boston, MA 02284-5904 USA. All orders must be prepaid.

ADVERTISING
Notices publishes situations wanted and classified advertising, and display advertising for publishers and academic or scientific organizations. Advertising requests, materials, and/or questions should be sent to:
classads@ams.org (classified ads)
notices-ads@ams.org (display ads)

PERMISSIONS
All requests to reprint *Notices* articles should be sent to:
reprint-permission@ams.org.

SUBMISSIONS
The editor-in-chief should be contacted about articles for consideration after potential authors have reviewed the "For Authors" page at **www.ams.org/noticesauthors**.

The managing editor should be contacted for additions to our news sections and for any questions or corrections. Contact the managing editor at: **notices@ams.org**.

Letters to the editor should be sent to: **notices-letters@ams.org**

To make suggestions for additions to other sections, and for full contact information, see **www.ams.org/noticescontact**.

Supported by the AMS membership, most of this publication, including the opportunity to post comments, is freely available electronically through the AMS website, the Society's resource for delivering electronic products and services. Use the URL **www.ams.org/notices** to access the *Notices* on the website. The online version of the *Notices* is the version of record, so it may occasionally differ slightly from the print version.

The print version is a privilege of Membership. Graduate students at member institutions can opt to receive the print magazine by updating their individual member profiles at **www.ams.org/member-directory**. For questions regarding updating your profile, please call 800-321-4267.

For back issues see **www.ams.org/backvols**. Note: Single issues of the *Notices* are not available after one calendar year.

The American Mathematical Society is committed to promoting and facilitating equity, diversity and inclusion throughout the mathematical sciences. For its own long-term prosperity as well as that of the public at large, our discipline must connect with and appropriately incorporate all sectors of society. We reaffirm the pledge in the AMS Mission Statement to "advance the status of the profession of mathematics, encouraging and facilitating full participation of all individuals," and urge all members to conduct their professional activities with this goal in mind. (*as adopted by the April 2019 Council*)

[Notices of the American Mathematical Society (ISSN 0002-9920) is published monthly except bimonthly in June/July by the American Mathematical Society at 201 Charles Street, Providence, RI 02904-2213 USA, GST No. 12189 2046 RT****. Periodicals postage paid at Providence, RI, and additional mailing offices. POSTMASTER: Send address change notices to Notices of the American Mathematical Society, PO Box 6248, Providence, RI 02904-6248 USA.] Publication here of the Society's street address and the other bracketed information is a technical requirement of the US Postal Service.

A WORD FROM...

Catherine A. Roberts, AMS Executive Director

The opinions expressed here are not necessarily those of the Notices *or the AMS.*

Photo is courtesy of Mike Nyman Photography.

Supporting a Welcoming Environment at the AMS

With February's Black History Month and March's Women's History Month, now seems like an opportune time to describe efforts to ensure that everyone participating in AMS committee meetings and AMS mathematics conferences experiences a welcoming environment. The AMS is committed to advancing mathematics research and creating connections. Our Mission Statement says that our publications, meetings, advocacy, and other programs encourage and facilitate the full participation of ***all*** individuals. And yet, stories have been shared with the AMS of instances where members of our mathematics community did not feel welcome. This is a concern we take seriously and are working to address.

The AMS Policy on a Welcoming Environment[1] was adopted by the January 2015 AMS Council and revised in January 2019. It applies to all AMS activities, including committee meetings and conferences.

The Policy reads as follows:

> The AMS strives to ensure that participants in its activities enjoy a welcoming environment. In all its activities, the AMS seeks to foster an atmosphere that encourages the free expression and exchange of ideas. The AMS supports equality of opportunity and treatment for all participants, regardless of gender, gender identity or expression, race, color, national or ethnic origin, religion or religious belief, age, marital status, sexual orientation, disabilities, veteran status, or immigration status.
>
> Harassment is a form of misconduct that undermines the integrity of AMS activities and mission.
>
> The AMS will make every effort to maintain an environment that is free of harassment, even though it does not control the behavior of third parties. A commitment to a welcoming environment is expected of all attendees at AMS activities, including mathematicians, students, guests, staff, contractors and exhibitors, and participants in scientific sessions and social events. To this end, the AMS will include a statement concerning its expectations towards maintaining a welcoming environment in registration materials for all its meetings, and has put in place a mechanism for reporting violations. Violations may be reported confidentially and anonymously to 855-282-5703 or at `www.mathsociety.ethicspoint.com`. The reporting mechanism ensures the respect of privacy while alerting the AMS to the situation.

The goals contained in the AMS Mission Statement and the Welcoming Environment Policy, while laudable, unfortunately remain aspirational. Take note, for example, of interviews conducted for and described in the March 2021 report[2] "Towards a Fully Inclusive Mathematics Profession," written by the AMS Council's *Task Force on Understanding and Documenting the Historical Role of the AMS in Racial Discrimination*. Chapter 5 of the report discusses the report's finding that "Black mathematicians suffer from a lack of professional respect, even today." The chapter says Black mathematicians report commonly experiencing microaggressions at AMS meetings. In subsequent discussions of this report with multiple stakeholders in our community, it has been suggested that the AMS increase

---

Catherine A. Roberts is the executive director of the American Mathematical Society. Her email address is `croberts@ams.org`.

[1] `https://www.ams.org/welcoming-environment-policy`

[2] `https://www.ams.org/understanding-ams-history`

efforts to support our Welcoming Environment Policy, particularly at conferences. First, I will describe how we currently support the Welcoming Environment Policy, and then I will describe a new addition to our efforts.

Since 2015, the AMS has subscribed to the external service called EthicsPoint to help us address reported conduct violations experienced by members of our community. This service is used by several professional societies. The EthicsPoint system allows the reporter to correspond with the AMS anonymously and confidentially. Cases involving AMS staff are handled by our director of human resources, while cases involving members of our community are overseen by the executive director. Each case is unique. Consultation with legal counsel and the AMS president can occur, but we are not conducting formal investigations. The goal is to resolve each case in a respectful and effective way. A brief report with very basic details of EthicsPoint cases from the prior year is provided to the Board of Trustees each spring by the executive director. To date the AMS has addressed a total of thirteen cases, all involving bias or harassment. These cases provide insight into the types of issues being encountered by our community. Our goal is to work with complainants to ensure their satisfaction that their concern has received adequate attention, as well as to improve the AMS's approaches to supporting a welcoming environment. For one case, for example, the AMS arranged for personal security for a conference attendee. In another case, a person with multiple complaints against them was banned from future AMS conferences.

At AMS committee meetings, the Welcoming Environment Policy is included in each agenda. Time is spent at the start of each committee meeting describing the essence of the policy, addressing any questions, and explaining how someone could proceed if they feel the policy is not being followed. In addition to speaking directly to the committee chair, AMS secretary, or AMS executive director, a concerned person could choose to submit a concern through EthicsPoint.

At AMS conferences, the Welcoming Environment Policy is included in the conference program, along with information about how to contact that conference's ombudsperson. A trained AMS staff member is designated as the on-site ombudsperson for each conference and is accessible to anyone who would like to discuss a matter of concern. Anyone at an AMS conference can help direct an inquiry to the ombudsperson. The EthicsPoint portal is also available to anyone who would like to make an anonymous report.

A new endeavor called MathSafe offers an additional way to support the mathematics profession at meetings and conferences. MathSafe will be piloted at AMS conferences and will be reviewed and adapted as needed to support our community effectively. The intent is for MathSafe to further support our AMS conferences. This program is actively being considered for adoption by other mathematics professional societies to support their code of conduct policies at their meetings and conferences.

Briefly, MathSafe introduces a mechanism for trained volunteers to support the Welcoming Environment Policy during conferences. Volunteers wear a MathSafe button to indicate their willingness to assist anyone who feels they have witnessed or experienced an unwelcoming environment. It is important to emphasize that volunteers are *not* actively enforcing codes of conduct, nor will they be settling any arguments. By wearing a MathSafe button, they are signaling their willingness to listen and help. These volunteers can easily connect those who experience harassment, bullying, or other unwelcome behavior to access existing support services and formal reporting channels. For a further description of MathSafe, including FAQs (frequently asked questions), please visit the website at `http://www.mathsafe.org`.

Where did the idea for MathSafe come from? The National Academies of Sciences, Engineering, and Medicine[3] released a consensus report[4] in 2018 entitled "Sexual Harassment of Women: Climate, Culture, and Consequences in Academic Sciences, Engineering, and Medicine." This report documented serious issues with harassment of women, much of it occurring at professional conferences. Subsequently, the Societies Consortium on Harassment in STEMM[5] was formed. The mathematics discipline is represented in this consortium by the AMS, SIAM,[6] MAA,[7] ASA,[8] and AWM.[9] They join over 130 professional societies representing disciplines in science, technology, engineering, and medicine. At a Societies

[3] https://www.nationalacademies.org/

[4] https://www.nationalacademies.org/our-work/sexual-harassment-in-academia

[5] https://societiesconsortium.com/

[6] https://siam.org/

[7] https://www.maa.org/

[8] https://www.amstat.org/

[9] https://awm-math.org/

Consortium convening, members were introduced to Safe AGU,[10] a program of the American Geophysical Union.[11] This was held up as a model for consideration by other disciplines. The executive directors of these five mathematical societies recognized the potential to improve the climate for mathematics conference attendees. This was discussed at, for example, the Joint Policy Board for Mathematics (JPBM[12]). We decided to work together to develop something similar to Safe AGU that could be shared for possible adoption at a range of mathematics conferences.

In January 2020, the presidents of the four member societies of the JPBM (AMS, SIAM, ASA, MAA) sent a letter to the Association for LGBTQ+ Mathematicians (Spectra[13]) in response to Spectra leadership's expression of concern about climate and safety regarding the 2022 International Congress of Mathematicians (ICM[14]) being held in Russia. A copy of this response was sent to the Local Organizing Committee (LOC[15]) of the ICM. The letter included the following paragraph:

> *We are pleased to share that the AMS has offered to help ensure a welcoming environment with a program that this society will be launching at JMM2022. The program, modeled on that of another professional society, provides trained and easily identified on-site staff and volunteers who can effectively address issues related to climate and to the welcoming environment policies. For ICM, the AMS will work in partnership with the LOC to offer such a program.*

The program this letter refers to had no name at the time, but was inspired by ongoing discussions to introduce something similar to Safe AGU at mathematics conferences. Consequently, the AMS took the lead in developing what is now known as MathSafe. While we had hoped to pilot this at the Joint Mathematics Meetings, we now plan to do so at future AMS conferences. Since the pilot launch is delayed, the 2022 ICM does not plan to offer it as of now.

Over 130 people signed up for the first MathSafe volunteer training held on November 4, 2021. A second training for leaders involved in running the program on-site occurred the following week. This first round of training was delivered by the same consultant who trains volunteers for the Safe AGU program. These trainings included representatives from the ICM LOC, as well as some JMM partner societies (AWM, SIAM, COMAP,[16] ASL,[17] NAM[18]). From the AMS, all staff who attend AMS conferences are trained, although it is each employee's personal choice whether or not to volunteer. Several people from AMS governance, including some from the AMS Committee on Meetings and Conferences and the AMS Committee on Human Rights of Mathematicians, attended the trainings.

Once MathSafe is piloted, it will be reviewed by the AMS Committee on Meetings and Conferences. We will adapt this program to meet our needs going forward. We anticipate offering MathSafe at future AMS conferences and we expect that other professional societies may also adopt MathSafe. We imagine that program refinement will be ongoing, to ensure that it is supporting our Welcoming Environment Policy in the ways we expect. If you are curious and want to attend a future MathSafe training, please visit the website to sign up.

Catherine A Roberts

Catherine A. Roberts
AMS Executive Director

---

[10] https://www.agu.org/Learn-About-AGU/About-AGU/Ethics/SafeAGU

[11] https://www.agu.org/

[12] https://en.wikipedia.org/wiki/Joint_Policy_Board_for_Mathematics

[13] http://lgbtmath.org/

[14] https://www.mathunion.org/icm/icm-2022

[15] https://icm2022.org/organization

[16] https://www.comap.com/

[17] http://aslonline.org/

[18] https://www.nam-math.org/

Grace Wahba and the Wisconsin Spline School

Chong Gu and Yuedong Wang

Chong Gu is a professor of statistics at Purdue University. His email address is `chong@purdue.edu`.

Yuedong Wang is a professor of statistics at the University of California, Santa Barbara. His email address is `yuedong@pstat.ucsb.edu`.

The opening photo is from a conference in honor of Wahba, 2014. Front from left: Scott Gamlen, Linda Gamlen*, Grace Wahba, Sydney Goldsmith*. Middle from left: Anna Liu (UMass Amherst), Zehua Chen (NUS), Yuedong Wang (UCSB), Jyh-Jen Shiau (National Chiao Tung U, Taiwan), Ronaldo Dias (State U of Campinas, Brazil), Yoonkyung Lee (Ohio State), Helen Zhang (U Arizona), Wing Hung Wong (Stanford), Wensheng Guo (UPenn), Chenlei Leng (Warwick). Back from left: Doug Nychka (Colorado School of Mine), obscured, Zhigeng Geng (Apple), Yufeng Liu (UNC), Jim Wendelberger (Los Alamos Lab, U New Mexico), David Callan*, Dong Xiang (Amazon), Bin Dai (Tower Research Capital), Shilin Ding (Facebook), Ming Yuan (Columbia), Héctor Corrada Bravo (Genetech), Robert Krafty (Emory). Names with * are Grace's family members. All of the rest are Grace's academic descendants followed by current affiliations.*

Communicated by Notices *Associate Editor Richard Levine.*

DOI: https://doi.org/10.1090/noti2438

Grace Wahba (née Goldsmith, born August 3, 1934) joined the Department of Statistics at the University of Wisconsin–Madison in 1967, and remained on its faculty until her retirement in 2019. During her luminous career at Madison of more than half a century, Grace graduated 39 doctoral students and has over 400 academic descendants to date. Over the years, Grace has worked on countless interesting problems, covering a broad spectrum from theory to applications. Above all, Grace is best known as the mother of the Wisconsin spline school, the primary driving force behind a rich family of data smoothing methods developed by herself, collaborators, and generations of students. Some of Grace's life stories can be found in a recent entertaining piece by Nychka, Ma, and Bates [10]. Here, we try to complement that piece with some technical discussions concerning the Wisconsin spline school.

As Grace's former students, our first set of reading assignments consisted of the defining document of reproducing kernel Hilbert spaces by Aronszajn [1] and three early papers by Kimeldorf and Wahba [4–6]. As we shall demonstrate below, some of those early results had far-reaching impacts on developments decades later. According to [10], many of the ideas of Kimeldorf and Wahba were inspired by discussions during tea parties at the Mathematics Research Center at UW Madison in the late 1960s/early 1970s, with participants including Issac Schoenberg, Carl de Boor, and Larry Schumaker.

In the sections to follow, we shall outline the smoothing spline approach to data smoothing, addressing numerous aspects and noting similarities and differences compared to related techniques. We highlight Grace's many original contributions, but otherwise focus on the flow of presentation; for more accurate attributions of credit, one may consult the forward of [16] and the bibliographic notes in [3].

1. Smoothing Splines

Given pairs (x_i, y_i), $i = 1, \dots, n$, $x_i \in [a, b]$, one may obtain a smoothing spline via the minimization of

$$\frac{1}{n}\sum_{i=1}^{n}\left(y_i - \eta(x_i)\right)^2 + \lambda\int_a^b \left(\eta''(x)\right)^2 dx. \qquad (1)$$

The minimizer of (1) is a piecewise cubic polynomial, three times differentiable, with the third derivative jumping at the distinctive x_i points.

In the mathematics or numerical analysis literature, a spline typically refers to a piecewise polynomial, one is concerned with function interpolation or approximation, and the (x_i, y_i) pairs are exact samples satisfying $y_i = \eta(x_i)$.

With stochastic data, one does not have exact samples of the function and needs statistical models. A regression model behind (1) has $y_i = \eta(x_i) + \epsilon_i$, $\epsilon_i \sim N(0, \sigma^2)$ independent, and the minimizer of (1) provides an estimate of the regression function $\eta(x)$.

The first term in (1) pushes for a close fit of η to the data and the second term penalizes the roughness of η, with the smoothing parameter λ controlling the tradeoff between the two conflicting goals. As $\lambda \to \infty$, one approaches a simple linear regression line $\eta(x) = \beta_0 + \beta_1 x$; $\lambda = 0_+$ yields the minimum curvature interpolant.

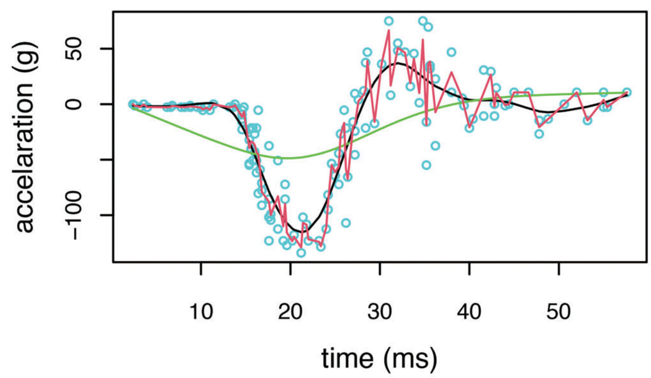


Figure 1. Cubic smoothing splines.

Figure 1 illustrates the cubic smoothing spline of (1) applied to some data from a simulated motorcycle accident, found in R package `MASS` as data frame `mcycle`, where x is time and y is head acceleration. The data are in circles and the three lines correspond to $\log_{10} n\tilde{\lambda} = -8.5, -4.08, -1.5$, from rough to smooth (red, black, and green, respectively), with $\log_{10} n\tilde{\lambda} = -4.08$ (black color) selected by cross validation via (15) using `ssanova0` in R package `gss`; $\lambda/\tilde{\lambda} = 60^3$ accounts for the mapping of $[0, 60]$ onto $[0, 1]$ in the software implementation.

Regression analysis, a primary tool of supervised learning, is widely used in applications. Traditional parametric regression was developed when data were scarce, restricting $\eta(x)$ to low-dimensional function spaces with a few free parameters; this maintains sufficient samples per parameter to effectively control "variance" in estimation, but may incur serious model bias. As the sample size n increases, "variance" is less of a concern and one looks to reduce model bias, yielding numerous non-parametric regression techniques that permit "flexible" forms of $\eta(x)$. The smoothing spline of (1) presents a tidy approach to non-parametric regression, tuning λ to control the effective dimension of model space.

Many non-parametric regression methods exist, and all perform equally well in one dimension. The real challenge is in the modeling/estimation of functions on multidimensional domains, and generalizations of (1) lead to unparalleled operational convenience and a rich collection of modeling tools.

1.1. Penalized likelihood method.

The penalized likelihood method results from an abstraction of (1). To estimate a function of interest η on a generic domain $\mathcal{X}$ using stochastic data, one may minimize

$$L(\eta|\text{data}) + \lambda J(\eta), \qquad (2)$$

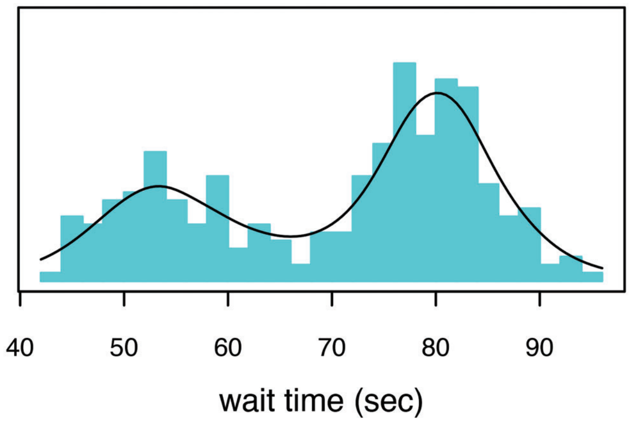


Figure 2. Density estimation.

where $L(\eta|\text{data})$ can be taken as the minus log likelihood of the data and $J(\eta)$ is a quadratic functional quantifying the roughness of η.

In a step towards an abstract $J(\eta)$, Kimeldorf and Wahba [4–6] considered $J(\eta) = \int_a^b (L\eta)^2(x)dx$ on $\mathcal{X} = [a,b]$ for some general differential operator L, with $J(\eta) = \int_a^b \left(\eta^{(m)}(x)\right)^2 dx$ being special cases yielding polynomial smoothing splines. More examples of $\mathcal{X}$ and $J(\eta)$ will follow shortly.

A smoothing spline is defined as the solution to a variational problem of the form given in (2). Depending on the configuration, it may or may not reduce to a piecewise polynomial.

The first term in (1) is proportional to the minus log likelihood of the Gaussian regression model stated above. Two more examples of $L(\eta|\text{data})$ follow.

Example 1 (Logistic regression). Consider $y_i \sim \text{Bin}(m_i, p_i)$, where $\log\{p_i/(1-p_i)\} = \eta(x_i)$. One may use $L(\eta) = -\sum_i \{y_i\eta(x_i) - m_i \log\left(1+e^{\eta(x_i)}\right)\}$ for the estimation of the logit function $\eta(x)$. □

Example 2 (Density estimation). Let x_i be independent samples from a probability density $p(x)$ on domain $\mathcal{X}$. To estimate $p(x) = e^{\eta(x)}/\int_{\mathcal{X}} e^{\eta(x)}$, one may use

$$L(\eta) = -\sum_i \{\eta(x_i) - \log \int_{\mathcal{X}} e^{\eta(x)}\}.$$

□

Example 1 is a special case of non-Gaussian regression. A variant of Example 2 was studied by Silverman [11].

Shown in Figure 2 is an example of one-dimensional density estimation using $J(\eta) = \int_a^b \left(\eta''(x)\right)^2 dx$; the data are 272 waiting times between eruptions of the Old Faithful geyser, found in the `R` data frame `faithful`. Plotted are a cross-validated density estimate (see §3) using `ssden` in `R` package `gss`, along with a histogram of the data. This is an instance of unsupervised learning.

The penalized likelihood of (2) is in fact performing constrained maximum likelihood estimation,

$$\min L(\eta|\text{data}) \quad \text{s.t. } J(\eta) \le \rho, \tag{3}$$

using the Lagrange method, with λ being the Lagrange multiplier; see [3, Theorem 2.12], where $\lambda \propto \rho^{-1}$ is also quantified.

The null space of $J(\eta)$, $\mathcal{N}_J = \{\eta : J(\eta) = 0\}$, specifies a parametric model for η. With a $\rho > 0$, data-adaptive through the selection of λ, one allows for η to depart from the parametric model.

1.2. Reproducing kernel Hilbert spaces.

The minimization of (2) is implicitly in the space $\{\eta : J(\eta) < \infty\}$ or a subspace therein, and function evaluations typically appear in $L(\eta|\text{data})$. To facilitate analysis and computation, one needs a metric and a geometry in the function space, and needs the evaluation functional to be continuous.

A reproducing kernel Hilbert space is a Hilbert space $\mathcal{H}$ of functions on a domain $\mathcal{X}$ in which the evaluation functional $[x]\eta = \eta(x)$ is continuous, $\forall x \in \mathcal{X}$, $\forall \eta \in \mathcal{H}$. A comprehensive theory of reproducing kernel Hilbert spaces can be found in [1].

By Riesz representation, there exists a reproducing kernel, a non-negative definite bivariate function $R(x,y)$ dual to the inner product $\langle\cdot,\cdot\rangle$ in $\mathcal{H}$, which satisfies $\langle R(x,\cdot), \eta(\cdot)\rangle = \eta(x)$, $\forall x \in \mathcal{X}$, $\forall \eta \in \mathcal{H}$.

A reproducing kernel Hilbert space can also be generated from its reproducing kernel $R(x,y)$, for which *any* non-negative definite function qualifies, as the "column space" $\text{span}\{R(x,\cdot), x \in \mathcal{X}\}$. The corresponding inner product may or may not have an explicit expression, however.

For use in (2), one takes $\mathcal{H} = \{\eta : J(\eta) < \infty\}$ equipped with $\langle\cdot,\cdot\rangle = J(\cdot,\cdot) + \tilde{J}(\cdot,\cdot)$, where $J(\cdot,\cdot)$ is the semi-inner product associated with the quadratic functional $J(\eta)$ and $\tilde{J}(\cdot,\cdot)$ is an inner product in the null space $\mathcal{N}_J$. One has a tensor-sum decomposition $\mathcal{H} = \mathcal{N}_J \oplus \mathcal{H}_J$ with $J(\cdot,\cdot)$ being a full inner product in $\mathcal{H}_J = \{\eta : J(\eta) < \infty, \tilde{J}(\eta) = 0\}$.

Example 3 (Cubic spline). Consider, on $\mathcal{X} = [0,1]$, $J(\eta) = \int_0^1 \left(\eta''(x)\right)^2 dx$. For an inner product in $\mathcal{N}_J$, take $\tilde{J}(\eta,\eta) = \left(\int_0^1 \eta(x)dx\right)^2 + \left(\int_0^1 \eta'(x)dx\right)^2$. It follows that $\int_0^1 \eta(x)dx = \int_0^1 \eta'(x)dx = 0$, $\forall \eta \in \mathcal{H}_J$.

The reproducing kernel in $\mathcal{H}_J$ is known to be $R_J(x,y) = k_2(x)k_2(y) - k_4(x-y)$, where $k_\nu = B_\nu/\nu!$ are scaled Bernoulli polynomials. One may further decompose $\mathcal{N}_J = \{1\} \oplus \{k_1(x)\}$ for $k_1(x) = x - 0.5$, with the respective reproducing kernels given by $R_{00}(x,y) = 1$ and $R_{01}(x,y) = k_1(x)k_1(y)$. □

Facts concerning tensor-sum decompositions of reproducing kernel Hilbert spaces can be found in [3, Theorem 2.5], and technical details of Example 3 are in Craven and Wahba [2].

The theory of reproducing kernel Hilbert spaces provides an abstract mathematical framework encompassing a great variety of problems. The abstract setting allows many important issues, such as the computation and the asymptotic convergence of the minimizers of (2), be treated in a unified fashion. As summarized in Grace's 1990 monograph [16], much of her work up to that date, from approximation theory to spline smoothing, fit under the general framework.

1.3. Tensor product splines.

A statistical model should be interpretable, which distinguishes it from mere function approximation or some black-box predictor/classifier. Two main challenges in the non-parametric modeling of multivariate data are weak interpretability and the curse of dimensionality, which might be alleviated via a hierarchical structure of the functional ANOVA decomposition.

Functional ANOVA decomposition. Consider a bivariate function $\eta(x) = \eta(x_{\langle 1\rangle}, x_{\langle 2\rangle})$ on $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$; subscripts in brackets denote coordinates of a point on a multidimensional domain while ordinary subscripts are reserved for multiple points. One may write

$$\begin{aligned}\eta(x) =&(I - A_1 + A_1)(I - A_2 + A_2)\eta \\ =&A_1A_2\eta + (I - A_1)A_2\eta \\ &+ A_1(I - A_2)\eta + (I - A_1)(I - A_2)\eta \\ =&\eta_\emptyset + \eta_1(x_{\langle 1\rangle}) + \eta_2(x_{\langle 2\rangle}) + \eta_{12}(x_{\langle 1\rangle}, x_{\langle 2\rangle}), \end{aligned} \tag{4}$$

where I is the identity operator, A_1, A_2 are averaging operators acting on arguments $x_{\langle 1\rangle}$, $x_{\langle 2\rangle}$, respectively, that satisfy $A1 = 1$, η_1, η_2 are main effects, and η_{12} is the interaction, satisfying $A_1\eta_1 = A_1\eta_{12} = 0$, $A_2\eta_2 = A_2\eta_{12} = 0$. Similar constructions in more than two dimensions are straightforward.

Examples of averaging operators include $A\eta = \int_a^b \eta(x)dx/(b-a)$ on $[a,b]$, $A\eta = \sum_{i=1}^m \eta(x_i)/m$ on any domain; averaging operators on different axes are independent of each other.

For $\mathcal{X}_1 \times \mathcal{X}_2$ discrete, $\eta(x_{\langle 1\rangle}, x_{\langle 2\rangle})$ is a matrix of treatment means usually denoted by μ_{ij} in a standard two-way ANOVA model, with (4) in the form

$$\begin{aligned}\mu_{ij} =&\mu_{\cdot\cdot} + (\mu_{i\cdot} - \mu_{\cdot\cdot}) \\ &+ (\mu_{\cdot j} - \mu_{\cdot\cdot}) + (\mu_{ij} - \mu_{i\cdot} - \mu_{\cdot j} + \mu_{\cdot\cdot}) \\ =&\mu + \alpha_i + \beta_j + (\alpha\beta)_{ij},\end{aligned}$$

where $\mu_{i\cdot} = \sum_j c_j\mu_{ij}$ for $\sum_j c_j = 1$, $\mu_{\cdot j} = \sum_i d_i\mu_{ij}$ for $\sum_i d_i = 1$, and $\mu_{\cdot\cdot} = \sum_{i,j} c_jd_i\mu_{ij}$.

Selective term elimination in functional ANOVA decompositions helps to combat the curse of dimensionality in estimation and facilitates the interpretation of the analysis. For example, the so-called additive models, those containing only main effects, are easier to estimate and interpret than ones involving interactions. As with classical ANOVA models on discrete domains, the inclusion of higher-order interactions are to be avoided in practice.

For η a log density, absence of selected interactions may imply (conditional) independence structures among random variables. Taking random variables (X, Y, Z) on domain $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$, say, a log density of the form $\eta = \eta_\emptyset + \eta_x + \eta_y + \eta_z + \eta_{xy} + \eta_{xz}$ implies the conditional independence of Y and Z given X, or $Y \perp Z | X$, where the notation for ANOVA terms of $\eta(x,y,z)$ parallels that in (4).

Tensor product spaces. For the estimation of η on product domains via (2), functional ANOVA decompositions can be built in through the construction of tensor product reproducing kernel Hilbert spaces.

Theorem 1. *Consider $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$ and $x, y \in \mathcal{X}$. For $R_{\langle 1\rangle}(x_{\langle 1\rangle}, y_{\langle 1\rangle})$ non-negative definite on $\mathcal{X}_1$ and $R_{\langle 2\rangle}(x_{\langle 2\rangle}, y_{\langle 2\rangle})$ non-negative definite on $\mathcal{X}_2$, $R(x,y) = R_{\langle 1\rangle}(x_{\langle 1\rangle}, y_{\langle 1\rangle})R_{\langle 2\rangle}(x_{\langle 2\rangle}, y_{\langle 2\rangle})$ is non-negative definite on $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$.*

To construct a reproducing kernel Hilbert space, it suffices to specify a reproducing kernel. The following example illustrates a construction on $\mathcal{X} = [0,1]^2$ using the results of Example 3.

Example 4 (Tensor-product cubic spline). The space $\mathcal{H} = \left\{\eta : \int_0^1 \left(\eta''(x)\right)^2 dx < \infty\right\}$ is decomposed as $\{1\} \oplus \{k_1(x)\} \oplus \mathcal{H}_J$ in Example 3. We rewrite the decomposition as $\mathcal{H}_{00} \oplus \mathcal{H}_{01} \oplus \mathcal{H}_1$, and denote the respective reproducing kernels as R_{00}, R_{01}, and R_1. A one-way ANOVA decomposition is built in, with $\eta \in \mathcal{H}_{01} \oplus \mathcal{H}_1$ satisfying $A\eta = \int_0^1 \eta(x)dx = 0$.

Using these reproducing kernels on the two axes and taking products as in Theorem 1, one has nine reproducing kernels on $[0,1]^2$, each yielding a tensor product space, as laid out below:

$$\begin{matrix}\mathcal{H}_{00,00} & \mathcal{H}_{00,01} & \mathcal{H}_{00,1} \\ \mathcal{H}_{01,00} & \mathcal{H}_{01,01} & \mathcal{H}_{01,1} \\ \mathcal{H}_{1,00} & \mathcal{H}_{1,01} & \mathcal{H}_{1,1}\end{matrix}$$

The ANOVA decomposition of (4) is built in, with $\eta_\emptyset \in \mathcal{H}_{00,00}$, $\eta_1 \in \mathcal{H}_{01,00} \oplus \mathcal{H}_{1,00}$, etc.

Pasting these spaces together via tensor-sum, one has what is needed for use in (2). The four spaces on the upper-left corner are of one dimension each, and can be lumped into $\mathcal{N}_J$. The remaining five spaces can be put together as $\mathcal{H}_J$ with a reproducing kernel

$$\begin{aligned}R_J =&\theta_{00,1}R_{00,1} + \theta_{1,00}R_{1,00} \\ &+ \theta_{01,1}R_{01,1} + \theta_{1,01}R_{1,01} + \theta_{1,1}R_{1,1},\end{aligned}$$

where the θ's are extra smoothing parameters adjusting the relative contribution of each component to the overall roughness measure.

To enforce an additive model, one removes $\mathcal{H}_{01,01}$ from $\mathcal{N}_J$ and sets $\theta_{1,01} = \theta_{01,1} = \theta_{1,1} = 0$. □

The construction described in Example 4 is at an abstract level, requiring little specifics of cubic splines on

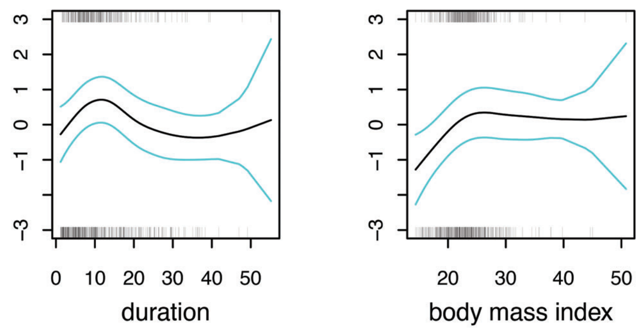


Figure 3. Additive logistic regression.

$[0,1]$. All one needs are reproducing kernels on marginal domains. Functional ANOVA decomposition follows trivially from one-way ANOVA decompositions on marginal domains.

Denote by $\mathcal{H}_\beta$ the component spaces of $\mathcal{H}_J$ in Example 4 and write R_β, J_β as the respective reproducing kernels and the associated square norms. With $R_J = \sum_\beta \theta_\beta R_\beta$, the corresponding square norm in $\mathcal{H}_J$ is given by $J(\eta) = \sum_\beta \theta_\beta^{-1} J_\beta(\eta)$, to be used in (2).

As an illustration, we fit a logistic regression model of Example 1 to the `wesdr` data found in R package `gss`, concerning the progression of diabetic retinopathy. One has $x = (x_{\langle 1\rangle}, x_{\langle 2\rangle}, x_{\langle 3\rangle}) \in R^3$ and a binary $y \sim \text{Bin}(1, p(x))$, and the full model would include three main effects, three two-way interactions, and a three-way interaction. After some exploration, the interaction terms are found to be negligible so an additive model appears adequate, and one of the main effects is actually linear. We finally fit a model

$$\eta(x) = \beta_0 + \beta_1 x_{\langle 1\rangle} + \eta_2(x_{\langle 2\rangle}) + \eta_3(x_{\langle 3\rangle}),$$

where $\eta = \log p/(1-p)$; this is actually a partial spline model discussed at the end of this section. Plotted in Figure 3 are estimated η_2 and η_3 along with their Bayesian confidence intervals (see §4). The rugs on the ceiling and the floor mark the observed $y = 1$ and $y = 0$, respectively, though $p(x)$ does not depend on $x_{\langle 2\rangle}$ or $x_{\langle 3\rangle}$ alone; the confidence intervals do get wider where data are sparse. The analysis was done using the `gssanova` facilities in package `gss`. Further details can be found in [3, §5.5.3].

Grace's signature was all over the tensor product spline technique, from the inception of the idea to the ensuing rigorous developments, involving several of her students including your authors; see, e.g., [16, Chap. 10] and [18].

1.4. More splines. Real intervals are the most encountered domains in practice, which can be mapped onto $[0,1]$, and the cubic spline on $[0,1]$ with $J(\eta) = \int_0^1 \left(\eta''(x)\right)^2 dx$ is the commonly used configuration for data smoothing in the setting. On domains other than real intervals or to accommodate various special needs, alternatives to cubic splines are sometimes called for.

We now present a variety of configurations tuned to various situations, which may be used directly in (2) on the respective designated domains, or be used as building blocks to construct tensor product splines.

Periodic splines. To accommodate recurring patterns such as circadian rhythms or seasonal effects, one may consider only periodic functions on a real interval. Mapping the interval onto $[0,1]$ and setting $J(\eta) = \int_0^1 \left(\eta''(x)\right)^2 dx$ for η periodic, one has $\mathcal{H} = \{1\} \oplus \mathcal{H}_J$, with component space reproducing kernels $R_0(x,y) = 1$ and $R_J(x,y) = -k_4(x-y)$; see Craven and Wahba [2].

L-splines. On $[0,1]$, one may configure an L-spline by setting $J(\eta) = \int_0^1 (L\eta)^2(x)h(x)dx$, where L is a general differential operator and $h(x) > 0$ is a weight function. With $L = D^m$ and $h(x) = 1$, it reduces to a polynomial spline; $m = 2$ yields the cubic spline.

When the null space $\mathcal{N}_L = \{\eta : L\eta = 0\}$ forms a more desirable parametric model than lower-order (say linear) polynomials, the corresponding L-spline is preferred over a polynomial spline.

Example 5 (Exponential spline). For $\theta > 0$, set $L = D(D-\theta)$ and $h(x) = e^{-3\theta x}$. The null space $\mathcal{N}_L = \{\eta : \eta(x) = \beta_0 + \beta_1 e^{\theta x}\}$ makes a reasonable model for a growth curve.

Transforming x by $\tilde{x} = (e^{\theta x} - 1)/\theta$, it can be shown that $\int_0^1 (L\eta)^2(x)h(x)dx = \int_0^a (d^2\eta/d\tilde{x}^2)^2 d\tilde{x}$, where $a = (e^\theta - 1)/\theta$, yielding a cubic spline in $\tilde{x}$. □

As noted earlier, L-splines were studied by Kimeldorf and Wahba [4–6], showcasing an abstract $J(\eta)$.

Thin-plate splines. On $\mathcal{X} = R^d$, for $2m > d$, one may use

$$J(\eta) = \sum_{\alpha_1+\cdots+\alpha_d=m} \frac{m!}{\alpha_1!\cdots\alpha_d!} \int\cdots\int \left(\frac{\partial^m \eta}{\partial x_{\langle 1\rangle}^{\alpha_1}\cdots\partial x_{\langle d\rangle}^{\alpha_d}}\right)^2 dx_{\langle 1\rangle}\cdots dx_{\langle d\rangle},$$

which is invariant to coordinate rotation and shift. For $d = 1$, $m = 2$, this reduces to a cubic spline. For $d = 2$, $m = 2$, one has

$$J(\eta) = \int\int \left(\left(\eta_{1,1}^{(2)}\right)^2 + 2\left(\eta_{1,2}^{(2)}\right)^2 + \left(\eta_{2,2}^{(2)}\right)^2\right) dx_{\langle 1\rangle} dx_{\langle 2\rangle},$$

where $\eta_{1,2}^{(2)} = \partial^2\eta/\partial x_{\langle 1\rangle}\partial x_{\langle 2\rangle}$, etc. The reproducing kernels are rather involved to specify for $d > 1$; technical details and references can be found in [3, §4.3] and [16, §2.4].

Thin-plate splines provide natural analytical tools for spatial smoothing and geographic mapping, where directional decompositions such as latitude/longitude effects may not make sense. When used as marginal domains in tensor product splines, the mathematically multidimensional R^d acts as an inseparable entity, contributing one logical dimension.

Some early derivations and meteorology applications of thin-plate splines can be found in Wahba and Wendelberger [19].

Spherical splines. To estimate functions on small geographic areas, one may use thin-plate splines on R^2, but surface curvature cannot be ignored on larger geographic regions or for global mapping. The spherical splines of Wahba [13] were developed just for this purpose.

On the unit sphere $\mathcal{S}$, consider the Laplace-Beltrami operator

$$\Delta = \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right)$$

and set $J(\eta) = \int_0^{2\pi}\int_0^{\pi}(\Delta\eta)^2 \sin\theta\, d\theta\, d\phi$, where $\theta \in [0,\pi]$ is latitude and $\phi \in [0, 2\pi]$ is longitude. Such a $J(\eta)$ is invariant under coordinate rotation. The null space is $\mathcal{N}_J = \{1\}$, and the reproducing kernel associated with $J(\eta)$ is available as an infinite sum involving spherical harmonics.

Technical details rely heavily on the mathematics of spherical harmonics, as outlined in [3, §4.4] and [16, §2.2]. Similar to thin-plate splines, the unit sphere $\mathcal{S}$ contributes one logical dimension in tensor product splines.

Discrete splines. Consider a discrete domain $\{1, \dots, K\}$, on which a function is a vector, and a bivariate function a matrix.

When the domain is nominal, or the labeling of the elements is arbitrary, a natural choice is

$$J(\eta) = \sum_{x=1}^{K}\big(\eta(x) - \bar{\eta}\big)^2,$$

where $\bar{\eta} = K^{-1}\sum_{x=1}^{K}\eta(x)$.

When the domain is ordinal, one may use $J(\eta) = \sum_{x=2}^{K}\big(\eta(x) - \eta(x-1)\big)^2$.

Write $J(\eta) = \eta^T J\,\eta$, where $\eta = \big(\eta(1), \dots, \eta(K)\big)^T$ and the matrix J is non-negative definite. $J(\eta)$ is a full square norm in $\mathcal{H}_J = \{\eta : \eta = J\mathbf{c}\}$, the column space of J, with reproducing kernel $R_J = J^+$, the Moore-Penrose inverse of J. See [3, §2.2].

Standing alone, penalized estimation on discrete domains is known as shrinkage having its own literature. The purpose of this discussion is to configure building blocks for use in tensor product splines when some of the variables are discrete.

Partial splines. As evident by now, estimation via (2) along with the functional ANOVA structure provides a rich family of non-parametric statistical models. When knowledge is sufficient to justify parametric forms for part of the model, one has semiparametric models [20, Chap. 8]. A special case of such models can be written as $y_i = \mathbf{z}_i^T\beta + \eta(x_i) + \epsilon_i$ in a Gaussian regression setting, where $\mathbf{z}_i^T\beta$ comprises partial terms, and one may estimate β and $\eta(x)$ jointly via the minimization of

$$\sum_i \big(y_i - \mathbf{z}_i^T\beta - \eta(x_i)\big)^2 + \lambda J(\eta).$$

The partial terms can be readily accommodated in (2) via trivial manipulations of the $L(\eta)$ term. Caution must be exercised in practice, however, as the partial terms $\mathbf{z}_i^T\beta$ may not be identifiable from $\eta(x_i)$ using the available data.

Some theoretical analysis of partial splines can be found in [16, Chap. 6].

2. Representation

The function space $\mathcal{H}$ is of infinite dimension in general. To numerically calculate the minimizer of (2) in $\mathcal{H}$, one needs adequate explicit expression for $\eta \in \mathcal{H}$.

2.1. Finite-dimensional solution. On domain $\mathcal{X}$, consider the minimization of

$$\frac{1}{n}\sum_{i=1}^{n}\big(y_i - \eta(x_i)\big)^2 + \lambda J(\eta) \tag{5}$$

over $\eta \in \mathcal{H} = \mathcal{N}_J \oplus \mathcal{H}_J$, where $\mathcal{N}_J \subseteq \{\eta : J(\eta) = 0\}$ is of finite dimension and $J(\eta) = J(\eta, \eta)$ is the square norm in $\mathcal{H}_J$ dual to reproducing kernel $R_J(x, y)$; $J\big(R_J(x, \cdot), \eta(\cdot)\big) = \eta(x)$, $\forall\eta \in \mathcal{H}_J$.

Theorem 2. *The minimizer of* (5) *has a finite-dimensional representation*

$$\eta(x) = \sum_{\nu=1}^{m} d_\nu\phi_\nu(x) + \sum_{i=1}^{n} c_i R_J(x_i, x), \tag{6}$$

where $\{\phi_\nu\}_{\nu=1}^m$ *is a basis of* $\mathcal{N}_J$.

This cornerstone result is due to Kimeldorf and Wahba [6], presented in an L-spline setting, the most abstract form known at the time.

The proof of the theorem is via a simple, clever geometric argument. Functions in $\mathcal{H}$ can be expressed in a form

$$\eta(x) = \sum_{\nu=1}^{m} d_\nu\phi_\nu(x) + \sum_{i=1}^{n} c_i R_J(x_i, x) + \rho(x), \tag{7}$$

where $\rho(x) \in \mathcal{H}_J \ominus \operatorname{span}\{R_J(x_i, \cdot), i = 1, \dots, n\}$. Plugging (7) into (5), one has

$$(\mathbf{y} - S\mathbf{d} - Q\mathbf{c})^T(\mathbf{y} - S\mathbf{d} - Q\mathbf{c}) + n\lambda\,\mathbf{c}^T Q\mathbf{c} + n\lambda J(\rho), \tag{8}$$

where $\mathbf{y} = (y_1, \dots, y_n)^T$, S is $n \times m$ with the (i, ν)th entry $\phi_\nu(x_i)$, and Q is $n \times n$ with the (i, j)th entry $R_J(x_i, x_j) = J\big(R_J(x_i, \cdot), R_J(x_j, \cdot)\big)$; note that $\rho(x_i) = J\big(R_J(x_i, \cdot), \rho(\cdot)\big) = 0$.

Clearly, $\rho(x) = 0$ for the minimizer of (5), yielding (6). The result holds in general when $L(\eta)$ in (2) depends on η only through function evaluations $\eta(x_i)$.

The abstract setting allows generic algorithms to be developed and implemented, and the numerical calculation requires only a basis of $\mathcal{N}_J$ and the reproducing kernel R_J of $\mathcal{H}_J$; specifically, an explicit form of $J(\eta)$ is *not* needed.

For tensor product splines, $R_J = \sum_\beta \theta_\beta R_\beta$, and the solution is in the form

$$\eta(x) = \sum_{\nu=1}^{m} d_\nu \phi_\nu(x) + \sum_{i=1}^{n} c_i \Big(\sum_\beta \theta_\beta R_\beta(x_i, x) \Big). \qquad (9)$$

ANOVA terms are readily available in partial sums of $d_\nu \phi_\nu(x)$ and $\theta_\beta \sum_{i=1}^n c_i R_\beta(x_i, x)$; note the same c_i's used in all penalized terms.

2.2. **Asymptotic convergence.** Denote by η_λ the minimizer of (5) in $\mathcal{H}$, and let $f(x)$ be the limiting density of $\{x_i\}_{i=1}^n$ on $\mathcal{X}$. Under conditions, one can show that as $n \to \infty$, $\lambda \to 0$,

$$(V + \lambda J)(\eta_\lambda - \eta) = O_p(\lambda^p + n^{-1}\lambda^{-1/r}), \qquad (10)$$

where $V(\eta) = \int_\mathcal{X} \eta^2(x) f(x) dx$, $p \in [1,2]$ depending on how smooth the true η is, and r codes the regulating power of $J(\eta)$. For the cubic splines on $[0,1]$, $r = 4$, $p = 1$ if $\eta(x)$ "barely" satisfies $\int_0^1 (\eta''(x))^2 dx < \infty$, and $p = 2$ if $\int_0^1 (\eta^{(4)}(x))^2 dx < \infty$.

$V(\eta_\lambda - \eta) \approx n^{-1} \sum_{i=1}^n (\eta_\lambda(x_i) - \eta(x_i))^2$, the mean square error at data points. The optimal convergence rate $O_p(n^{-pr/(pr+1)})$ is achieved at $\lambda \asymp n^{-r/(pr+1)}$. For $p = 2$, $r = 4$, the rate would be $O_p(n^{-8/9})$, not too far from the $O_p(n^{-1})$ rate for parametric models. The minimizer η_λ of (5) can indeed be a good estimate of η, assuming λ can be selected properly.

Smoothing via $J(\eta) \le \rho \propto \lambda^{-1}$ is like applying a low-pass filter, and the effective "active" subspace should be of dimension in the ballpark of $O(\lambda^{-1/r})$. The solution expression in (6) asserts that this "active" subspace is entirely contained in $\mathcal{N}_J \oplus \mathrm{span}\{R_J(x_i, \cdot), i = 1, \dots, n\}$, but there likely remain further redundancies among $R_J(x_i, \cdot)$'s.

In fact, the same convergence rate as in (10) also holds for the minimizer η_λ^* of (5) in a space

$$\mathcal{H}^* = \mathcal{N}_J \oplus \mathrm{span}\{R_J(z_j, \cdot), j = 1, \dots, q\}, \qquad (11)$$

where $q\lambda^{2/r} \to \infty$ and $\{z_j\}_{j=1}^q$ is a random subset of $\{x_i\}_{i=1}^n$. Note the role $f(x)$ plays in the definition of $V(\eta)$, and a random selection of $\{z_j\}$ mimics $f(x)$. It suffices to use $q \asymp \lambda^{-2/r} n^\epsilon$, $\forall \epsilon > 0$, and for $\lambda \asymp n^{-r/(pr+1)}$, this leads to $q \asymp n^{2/(pr+1)+\epsilon}$.

The q $R_J(z_j, \cdot)$'s, $q \asymp \lambda^{-2/r} n^\epsilon$, provide sufficient coverage of the "active" subspace of dimension $O(\lambda^{-1/r})$. In principle, one might be able to achieve the same effect with a smaller set of bases via delicate selection in specific settings, but the worry-free random selection in (11) is justified in an abstract setting. Computation using q bases is of order $O(nq^2)$.

As an estimate of η, η_λ^* is as efficient asymptotically performance-wise, thus is called an efficient approximation of η_λ.

Similar results hold for other configurations of $L(\eta)$ in (2), but the definition of $V(\eta)$ naturally varies with the settings. See [3, Chap. 9] for detailed asymptotic analysis in a variety of stochastic settings, using techniques originated in [11].

When $L(\eta)$ depends on η through more than a finite number of function evaluations, such as with the density estimation of Example 2, the minimizer η_λ in $\mathcal{H}$ may not be computable, but the efficient approximation η_λ^* makes the method practically applicable.

Functions in $\mathcal{H}^*$ of (11) have an expression

$$\eta(x) = \sum_{\nu=1}^{m} d_\nu \phi_\nu(x) + \sum_{j=1}^{q} c_j R_J(z_j, x),$$

with (6) as a special case at $q = n$.

3. Smoothing Parameter Selection

The convergence rates confirm that the method is capable of delivering good estimates, and the finite-dimensional representation makes computation possible. Varying the λ in front of $J(\eta)$ and the possible θ's hidden in $J(\eta)$ as with tensor product splines, one has available a family of estimates, from which we hope to pick well-performing members.

The practical success of (2) hinges on the proper selection of smoothing parameters. In what follows, we shall use λ to denote both the λ in front of $J(\eta)$ and the possible θ's hidden therein. The minimizer of (2) to be calculated, likely in $\mathcal{H}^*$, is denoted by η_λ.

3.1. **Performance measures.** The purpose of λ selection is to pick well-performing estimates, but one first needs to define what good performance means. Performance measures naturally vary with stochastic settings.

Gaussian regression. For Gaussian regression via (5), a natural performance measure is the mean square error over $\{x_i\}$,

$$L(\lambda) = \frac{1}{n} \sum_{i=1}^{n} \big(\eta_\lambda(x_i) - \eta(x_i)\big)^2. \qquad (12)$$

$L(\lambda)$ in (12) is a statistical loss, with minimum at say λ_o, the data-specific optimal choice. The optimal λ_o is beyond reach in practice, as $\eta(x)$ is unknown except in simulations.

Density estimation. For another stochastic setting, consider the estimation of probability density $p(x) = e^{\eta(x)} / \int_\mathcal{X} e^{\eta(x)} dx$ on $\mathcal{X}$ via the minimization of

$$-\frac{1}{n} \sum_{i=1}^{n} \eta(x_i) + \log \int_\mathcal{X} e^{\eta(x)} dx + \frac{\lambda}{2} J(\eta).$$

A natural performance measure is the Kullback-Leibler divergence,

$$L(\lambda) = \mu_\eta(\eta - \eta_\lambda) - \log \int_\mathcal{X} e^{\eta(x)} dx + \log \int_\mathcal{X} e^{\eta_\lambda(x)} dx, \quad (13)$$

where $\mu_g(h) = \int_\mathcal{X} h(x) e^{g(x)} dx / \int_\mathcal{X} e^{g(x)} dx$.

3.2. Cross validation. The statistical loss $L(\lambda)$ involves the unknown η and the candidate estimate η_λ. In many stochastic settings, one may decompose

$$L(\lambda) = A(\eta_\lambda) + B(\eta, \eta_\lambda) + C(\eta),$$

where $A(\eta_\lambda)$ can be computed, $C(\eta)$ can be dropped as it does not involve λ, and $B(\eta, \eta_\lambda)$ is to be estimated via cross validation, yielding some $V(\lambda) = A(\eta_\lambda) + \hat{B}(\eta, \eta_\lambda)$ for use as the selection tool.

In general, one is looking for some computable $V(\lambda)$ that roughly parallels $L(\lambda)$, so the minimizer of $V(\lambda)$ may have a good chance to deliver near optimal performance.

Density estimation. An example of the structure described above is the Kullback-Leibler divergence in (13), where $A(\eta_\lambda) = \log \int_{\mathcal{X}} e^{\eta_\lambda(x)} dx$ and $B(\eta, \eta_\lambda) = -\mu_\eta(\eta_\lambda)$. One may estimate $\mu_\eta(\eta_\lambda)$ using a cross-validated sample mean, $n^{-1} \sum_{i=1}^n \eta_\lambda^{[i]}(x_i)$, with $\eta_\lambda^{[i]}$ minimizing

$$-\frac{1}{n-1} \sum_{j \neq i} \eta(x_j) + \log \int_{\mathcal{X}} e^{\eta(x)} dx + \frac{\lambda}{2} J(\eta).$$

Gaussian regression. For Gaussian regression via (5), an ordinary cross validation score is given by

$$V_0(\lambda) = \frac{1}{n} \sum_{i=1}^n (y_i - \hat{y}_{i(i)})^2, \qquad (14)$$

where $\hat{y}_{i(i)} = \eta_\lambda^{[i]}(x_i)$ for $\eta_\lambda^{[i]}$ minimizing

$$\frac{1}{n} \sum_{j \neq i} \big(y_j - \eta(x_j)\big)^2 + \lambda J(\eta).$$

Write $\hat{y}_i = \eta_\lambda(x_i)$ and $\hat{\mathbf{y}} = A(\lambda)\mathbf{y}$, where $A(\lambda)$ is known as the smoothing matrix. It can be shown that $y_i - \hat{y}_{i(i)} = (y_i - \hat{y}_i)/(1 - a_{i,i})$, where $a_{i,i}$ is the (i,i)th entry of $A(\lambda)$, so

$$V_0(\lambda) = \frac{1}{n} \sum_{i=1}^n \frac{(y_i - \hat{y}_i)^2}{(1 - a_{i,i})^2}.$$

An invariance argument suggests replacing $a_{i,i}$'s by their average value $\mathrm{tr}A(\lambda)/n$, yielding the renowned generalized cross validation (GCV) score of Craven and Wahba [2],

$$V(\lambda) = \frac{n^{-1}\mathbf{y}^T\big(I - A(\lambda)\big)^2\mathbf{y}}{\big\{n^{-1}\mathrm{tr}\big(I - A(\lambda)\big)\big\}^2}. \qquad (15)$$

In standard multiple regression, the score in (14) is the familiar PRESS statistic for model selection, where $J(\eta)$ is irrelevant and the continuous scale λ is replaced by a discrete set of candidate models.

3.3. Optimality of cross validation. We first reiterate some simple logic. There are two functions of λ involved here, the performance measure $L(\lambda)$, and the selection score $V(\lambda)$. The minimizer λ_o of $L(\lambda)$ defines the data-specific optimal choice, the best one can hope to do given the data. The minimizer λ_v of $V(\lambda)$ is the practical selection.

Figure 4. Manny Parzen's 60th birthday party, 1989. From left: Don Ylvisaker, Grace Wahba, Joe Newton, Marcello Pagano, Randy Eubank, Manny Parzen, Will Alexander, Marvin Zelen, Scott Grimshaw.

Once again, λ_v is *not* optimal by definition, but may deliver near optimal performance. $V(\lambda)$ is a selection tool, not a performance measure.

To explore the effectiveness of a cross validation score, one may conduct simulation studies, where one knows η so can compute $L(\lambda)$ and identify λ_o. If the ratio $L(\lambda_o)/L(\lambda_v)$ is frequently scattered near one, then $V(\lambda)$ is likely a winner. Note that the proximity of λ_o and λ_v is meaningless for the purpose, as the bottom of $L(\lambda)$ may be flat or may be steep.

For technical analysis, suggestive results for various cross validation scores in regression settings exist in the form of $V(\lambda) - L(\lambda) - K = o_p\big(L(\lambda)\big)$, where K is some random quantity not involving λ [3, §§6.2.3, 6.3.3]; such results fall far short of justifying the use of $V(\lambda)$, however, as the behaviors of the random λ_o and λ_v are not accounted for.

The only rigorous theoretical justification known to date was by Ker-Chau Li [8], who showed that the use of the GCV score $V(\lambda)$ in (15) is optimal for the minimization of $L(\lambda)$ in (12), in the sense that $L(\lambda_v)/L(\lambda_o) = 1 + o_p(1)$ as $n \to \infty$.

4. Bayes Model

Consider $y_i = \eta(x_i) + \epsilon_i$, $\epsilon_i \sim N(0, \sigma^2)$ independent, $x_i \in \{1, \dots, K\}$, and assume $\eta = \mu\mathbf{1} + \alpha$, where $\mu\mathbf{1}$ is fixed effect and $\alpha \sim N\big(\mathbf{0}, b(I - \mathbf{1}\mathbf{1}^T/K)\big)$ is random effect. The posterior mean of η is given by the minimizer of

$$\frac{1}{\sigma^2} \sum_{i=1}^n \big(y_i - \eta(x_i)\big)^2 + \frac{1}{b} \sum_{x=1}^K \big(\eta(x) - \bar{\eta}\big)^2; \qquad (16)$$

this is discrete spline with $J(\eta) = \sum_{x=1}^K \big(\eta(x) - \bar{\eta}\big)^2$ and $\lambda = \sigma^2/nb$.

In general, let $\eta(x) = \eta_0(x) + \eta_1(x)$, where $\eta_0(x)$ is diffuse in $\mathcal{N}_J$ (fixed effect) and $\eta_1(x)$ is a mean zero

Figure 5. First meeting in the Old Hospital of the Thursday Group, 2004. Back from left: Joung Youn Kim, Yongho Jeon, Fan Lu, John Carew, Hyonho Chun, Xianhong Xie, Weiliang Shi. Front from left: Grace Wahba, Yi Lin.

Gaussian process on $\mathcal{X}$ with covariance function $E[\eta_1(x)\eta_1(y)] = bR_J(x,y)$. Observing $y_i = \eta(x_i) + \epsilon_i$, $\epsilon_i \sim N(0,\sigma^2)$ independent, the posterior mean of $\eta(x)$ is given by $\eta_\lambda(x)$, with η_λ minimizing (5) for $\lambda = \sigma^2/nb$. See Wahba [12].

For tensor product splines with $\mathcal{N}_J = \bigoplus_\nu\{\phi_\nu\}$, $J(\eta) = \sum_\beta \theta_\beta^{-1} J_\beta(\eta)$, and $R_J = \sum_\beta \theta_\beta R_\beta$, one has $\eta = \sum_\nu \psi_\nu + \sum_\beta \eta_\beta$, with ψ_ν diffuse in $\{\phi_\nu\}$, η_β's being mean zero Gaussian processes with covariance functions $b\theta_\beta R_\beta$, independent of each other. The terms $d_\nu\phi_\nu(x)$ and $\theta_\beta \sum_{i=1}^n c_i R_\beta(x_i, x)$ in (9) then give the posterior means of $\psi_\nu(x)$ and $\eta_\beta(x)$, respectively. See [3, Theorem 3.8].

The correspondence between Bayesian estimation with Gaussian process priors and smoothing with quadratic roughness penalties was first observed by Kimeldorf and Wahba [4, 5]. An instance of the former is the kriging method widely used in geostatistics, where the covariance function of the Gaussian process is known as the variogram.

Despite the common mathematical structure, the underlying statistical models for spline smoothing and kriging are fundamentally different. With spline smoothing, the true η is a smooth function and a comprehensible $J(\eta)$ provides intuitions about what the method does, whereas in kriging the true η is a realization of stochastic process and specific models are defined by the variogram.

4.1. Bayesian confidence intervals. Under the Bayes model, $\eta_\lambda(x)$ is the posterior mean of $\eta(x)$, and one may also calculate the posterior standard deviation $s(x)$, leading to say a 95% Bayesian confidence interval $\eta_\lambda(x) \pm 1.96s(x)$.

Despite the derivation under the Bayes model, such intervals possess a certain average coverage property in the spline smoothing setting, as shown by Wahba [14] via heuristic arguments and empirical simulations; the coverage percentage over the data points, $\#\{\eta(x_i) \in \eta_\lambda(x_i) \pm 1.96s(x_i)\}/n$, averages to nearly the nominal 95% over replicates.

For tensor product splines, posterior covariances can be calculated for the $\psi_\nu(x)$'s and $\eta_\beta(x)$'s, yielding Bayesian confidence intervals for the ANOVA terms. The term-wise intervals do not possess the average coverage property, but do demonstrate favorable behavior such as being tighter where data are dense.

The posterior distributions are with $b = \sigma^2/n\lambda$ for fixed λ. In practice, one selects λ by cross validation and also needs an estimate of σ^2.

When $L(\eta)$ in (2) is convex but not quadratic, such as with the logistic regression of Example 1, the respective minimizer $\eta_\lambda(x)$ is closer to a posterior mode than a posterior mean; precise Bayesian derivation can be subtle. Substituting $L(\eta)$ by its quadratic approximation at η_λ, things reduce to a weighted version of (5), based on which approximate Bayesian confidence intervals can be constructed.

In the Bayesian statistics literature, a Gaussian approximation of the posterior is known as Laplace approximation.

4.2. Restricted maximum likelihood. Under the Bayes model, $\lambda = \sigma^2/nb$ is a model parameter via σ^2 and b in the data-generating mechanism, not just an external tuning parameter associated with an estimation method. One may write

$$\mathbf{y} = S\mathbf{d} + \epsilon^*, \quad \epsilon^* \sim N(\mathbf{0}, \sigma^2 I + bQ),$$

where $\mathbf{y}$, S, $\mathbf{d}$, and Q are as in (8).

Let F be $n\times(n-m)$ orthogonal satisfying $F^TS = O$. One may obtain the maximum likelihood estimates of (σ^2, b) using $F^T\mathbf{y} \sim N(\mathbf{0}, \sigma^2 I + bF^TQF)$, where the nuisance parameter $\mathbf{d}$ is eliminated. This is known as the restricted maximum likelihood (REML) in the mixed-effect model literature.

In the spline smoothing setting, this approach to the selection of λ does not target any statistical loss as cross validation does, but provides a stable empirical performer nevertheless. It makes a good fallback option when effective cross validation scores are not available, say when $\sigma^2 I$ above is replaced by some non-trivial covariance matrices.

When used in (5), Wahba [15] showed that the REML selection λ_m cannot exceed the order of $n^{-r/(r+1)}$, achieving a convergence rate no better than

$$O_p(n^{-pr/(r+1)} + n^{-r/(r+1)}) = O_p(n^{-r/(r+1)})$$

by (10); the best possible rate is $O_p(n^{-pr/(pr+1)})$, however, achievable by the optimally performing GCV score in (15), at $\lambda_v \asymp n^{-r/(pr+1)}$. When $p > 1$, or the true η is "super smooth" in Wahba's terms, the REML selection is suboptimal.

4.3. **Efficient approximation.** For the minimizer η^*_λ of (5) in $\mathcal{H}^*$ of (11), $\eta^*_\lambda(x)$ is the posterior mean of $\eta(x) = \eta_0(x) + \eta_1(x)$, with η_0 diffuse in $\mathcal{N}_J$ and η_1 being a mean zero Gaussian process with covariance function $E\big[\eta_1(x)\eta_1(y)\big] = bR_J(x, \mathbf{z}^T)Q^+R_J(\mathbf{z}, y)$, where Q is $q{\times}q$ with the (j,k)th entry $R_J(z_j, z_k)$; see [3, §3.5.2].

Substituting $R_J(x,\mathbf{z}^T)Q^+R_J(\mathbf{z},y)$ for $R_J(x,y)$ resembles the Nyström matrix approximation, with which a nonnegative definite $K = \left(\begin{smallmatrix} A & B^T \\ B & C\end{smallmatrix}\right)$ is to be approximated by DA^+D^T for $D = \left(\begin{smallmatrix} A \\ B\end{smallmatrix}\right)$.

5. Machine Learning

The 1990s witnessed the start of some explosive developments in modern machine learning techniques, and the timely publication of Grace's 1990 monograph [16] played a pivotal role in introducing the powerful reproducing kernel Hilbert space machinery to the broader research community. The kernel learning approach bears close resemblance to spline smoothing and kriging. Some classification methods such as the support vector machines can be cast as solutions to regularization problems similar to (2).

Consider pairs of training samples (x_i, y_i), where $x_i \in \mathcal{X}$ are in the feature space and y_i are binary class labels. The task is to develop rules to classify future items into one of the two classes based on the features. The performance of classifiers is typically assessed by misclassification rate.

Assuming $y \sim \mathrm{Bin}\big(1, p(x)\big)$ with $p(x)$ known, the Bayes rule, which minimizes misclassification rate, would classify an item as $y = 1$ when $p(x) > 0.5$, or $\eta(x) = \log p(x)/(1-p(x)) > 0$.

It is tempting to employ the logistic regression of Example 1 to estimate the logit $\eta(x)$, then approximate the Bayes rule using $\eta_\lambda(x)$ in its place. This is called soft classification by Wahba [17]. Techniques developed for logistic regression aim to minimize a statistical loss

$$n^{-1}\sum_{i=1}^{n}(p_\lambda - p)(x_i)(\eta_\lambda - \eta)(x_i),$$

however, which is not directly related to misclassification rate. Also, logistic regression is technically challenged when the true $p(x)$ are mostly near 0 or 1, that is, when the two labels are well separated in the feature space, which however should be a more favorable environment to train classifiers.

Targeting the Bayes rule directly, one may set $y_i \in \{-1, 1\}$ and minimize

$$\frac{1}{n}\sum_{i=1}^{n}\big(1 - y_i\eta(x_i)\big)_+ + \lambda J(\eta) \tag{17}$$

over $\eta \in \{1\} \oplus \mathcal{H}_J$, where $(x)_+ = \max(x, 0)$, and classify a future item by $y = \mathrm{sign}\{\eta(x)\}$; this yields a support vector machine. See [17]. Note that the $\eta(x)$ in (17) is *not* the logit, but defining a classifier in $\mathrm{sign}\{\eta(x)\}$.

When the two types of misclassification incur unequal costs, or when $\sum_i I_{[y_i=1]}/\sum_i I_{[y_i=-1]}$ is far from the natural ratio due to retrospective sampling, the Bayes rule minimizing the misclassification cost is of the form $\mathrm{sign}\{p(x) - p_0\}$ for some $p_0 \neq 0.5$. To target such a rule in the form of $\mathrm{sign}\{\eta(x)\}$, one may modify the terms in (17) as $L(y_i)\big(1 - y_i\eta(x_i)\big)_+$ for $L(1) = p_0 = 1 - L(-1)$. See [9].

The solution expression in (6) still holds for the minimizer η_λ of (17), though the performance of $\mathrm{sign}\{\eta_\lambda(x)\}$ may not be as sensitive to the selection of λ. Technical underpinnings, implementation details, and empirical performances can be found in the references cited above, all with Grace among the authors. Extensions of the techniques to multiclass classification are to be found in [7].

Figure 6. Receiving honorary doctorate from the University of Chicago, 2007. Back from left: Peter McCullagh, Michael Wichura, Stephen Stigler, Per Mykland, Mathias Drton, Greg Lawler, Dan Nicolai, Yali Amit, Ron Thisted. Front from left: Grace Wahba, Mary Sara McPeek, Linda Collins.

6. Epilogue

As noted earlier, Grace has worked on a great variety of interesting problems over the years; there was even a Wahba's problem named after her, for her work done while a graduate student at Stanford working part time for IBM [10]. A thorough presentation of the entire body of Grace's life work would be way above our pay grade.

Our attempt in this piece is to provide an overview of a family of spline smoothing techniques that owe their flourishing developments mainly to Grace's ingenuity and persistence. These techniques form a comprehensive, coherent system, anchored in reproducing kernel Hilbert spaces.

Earlier in her career, Grace had also published on approximation theory in the mathematics literature. In recent years, Grace has explored methodologies such as the use of L_1-type penalties for variable selection and the embedding of general discrepancy measures in normed spaces. While a theoretician by training, Grace had a genuine passion for practical applications, and much of her work was motivated by scientific problems in meteorology and the life sciences.

Grace is a member of the National Academy of Sciences since 2000, and received an honorary degree from the University of Chicago in 2007. She is a fellow of the American Academy of Arts and Sciences, the American Association for the Advancement of Science, the American Statistical Association, and the Institute of Mathematical Statistics. Grace was also recognized by the Committee of Presidents of Statistical Societies, receiving the Elizabeth Scott Award in 1996 and the R. A. Fisher Lectureship in 2014. To honor Grace's monumental contributions to statistics and science, a Grace Wahba Award and Lecture was recently established by the Institute of Mathematical Statistics.

References

[1] N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68** (1950), 337–404, DOI 10.2307/1990404. MR51437

[2] P. Craven and G. Wahba, *Smoothing noisy data with spline functions. Estimating the correct degree of smoothing by the method of generalized cross-validation*, Numer. Math. **31** (1978/79), no. 4, 377–403, DOI 10.1007/BF01404567. MR516581

[3] C. Gu, *Smoothing spline ANOVA models*, 2nd ed., Springer Series in Statistics, vol. 297, Springer, New York, 2013, DOI 10.1007/978-1-4614-5369-7. MR3025869

[4] G. Kimeldorf and G. Wahba, *A correspondence between Bayesian estimation on stochastic processes and smoothing by splines*, Ann. Math. Statist. **41** (1970), 495–502, DOI 10.1214/aoms/1177697089. MR254999

[5] G. Kimeldorf and G. Wahba, *Spline functions and stochastic processes*, Sankhyā Ser. A **32** (1970), 173–180. MR303594

[6] G. Kimeldorf and G. Wahba, *Some results on Tchebycheffian spline functions*, J. Math. Anal. Appl. **33** (1971), 82–95, DOI 10.1016/0022-247X(71)90184-3. MR290013

[7] Y. Lee, Y. Lin, and G. Wahba, *Multicategory support vector machines: theory and application to the classification of microarray data and satellite radiance data*, J. Amer. Statist. Assoc. **99** (2004), no. 465, 67–81, DOI 10.1198/016214504000000098. MR2054287

[8] K.-C. Li, *Asymptotic optimality of C_L and generalized cross-validation in ridge regression with application to spline smoothing*, Ann. Statist. **14** (1986), no. 3, 1101–1112, DOI 10.1214/aos/1176350052. MR856808

[9] Y. Lin, G. Wahba, H. Zhang, and Y. Lee, *Properties and adaptive tuning of support vector machines*, Mach. Learn. **48** (2002), 115–136.

[10] D. Nychka, P. Ma, and D. Bates, *A conversation with Grace Wahba*, Statist. Sci. **35** (2020), no. 2, 308–320, DOI 10.1214/19-STS734. MR4106607

[11] B. W. Silverman, *On the estimation of a probability density function by the maximum penalized likelihood method*, Ann. Statist. **10** (1982), no. 3, 795–810. MR663433

[12] G. Wahba, *Improper priors, spline smoothing and the problem of guarding against model errors in regression*, J. Roy. Statist. Soc. Ser. B **40** (1978), no. 3, 364–372. MR522220

[13] G. Wahba, *Spline interpolation and smoothing on the sphere*, SIAM J. Sci. Statist. Comput. **2** (1981), no. 1, 5–16, DOI 10.1137/0902002. MR618629

[14] G. Wahba, *Bayesian "confidence intervals" for the cross-validated smoothing spline*, J. Roy. Statist. Soc. Ser. B **45** (1983), no. 1, 133–150. MR701084

[15] G. Wahba, *A comparison of GCV and GML for choosing the smoothing parameter in the generalized spline smoothing problem*, Ann. Statist. **13** (1985), no. 4, 1378–1402, DOI 10.1214/aos/1176349743. MR811498

[16] G. Wahba, *Spline models for observational data*, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 59, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1990, DOI 10.1137/1.9781611970128. MR1045442

[17] G. Wahba, *Soft and hard classification by reproducing kernel Hilbert space methods*, Proc. Natl. Acad. Sci. USA **99** (2002), no. 26, 16524–16530, DOI 10.1073/pnas.242574899. MR1947755

[18] G. Wahba, Y. Wang, C. Gu, R. Klein, and B. Klein, *Smoothing spline ANOVA for exponential families, with application to the Wisconsin Epidemiological Study of Diabetic Retinopathy*, Ann. Statist. **23** (1995), no. 6, 1865–1895, DOI 10.1214/aos/1034713638. MR1389856

[19] G. Wahba and J. Wendelberger, *Some new mathematical methods for variational objective analysis using splines and cross validation*, Monthly Weather Rev. **108** (1980), 1122–1145.

[20] Y. Wang, *Smoothing splines: Methods and applications*, Monographs on Statistics and Applied Probability, vol. 121, CRC Press, Boca Raton, FL, 2011, DOI 10.1201/b10954. MR2814838

Chong Gu

Yuedong Wang

Credits

Opening photo is courtesy of Chong Gu.
Figures 1–3 are courtesy of the authors.
Figures 4 and 5 are courtesy of Grace Wahba.
Figure 6 is courtesy of Stephen M. Stigler.
Author photos are courtesy of the authors.

Calibrating Computational Complexity via Definability: The Work of Julia F. Knight

Karen Lange

Figure 1. Knight receiving an Honorary Professorship with the Siberian Branch of the Russian Academy of Sciences.

Julia F. Knight, until recently the Charles L. Huisking Professor of Mathematics at the University of Notre Dame, continues to make a significant impact on computability theory and model theory, two subfields of logic, through her research, mentorship, and collaborative approach.

Karen Lange is the Theresa Mall Mullarkey Associate Professor at Wellesley College. Her email address is `karen.lange@wellesley.edu`.

Communicated by Notices *Associate Editor Antonio Montalbán.*

For permission to reprint this article, please contact: `reprint-permission@ams.org`.

DOI: https://doi.org/10.1090/noti2436

Computability theory provides a framework for calibrating the (idealized) computational content of a mathematical object, whereas model theory is about definability—what can be expressed about a structure in terms of formal syntax. In a nutshell, Knight's work explores how the complexity of syntactic definitions is strongly connected to the computational content of structures. Knight trained as a model theorist at the University of California, Berkeley, under the supervision of Robert Vaught, a key figure in the field's development. After completing her doctorate in 1972 and spending a few years at Penn State, Knight arrived at the University of Notre Dame in 1977, where she has remained throughout her career.

Although she has done some purely model-theoretic work, she was drawn early on into questions about the computational content of structures satisfying the axioms of first-order Peano arithmetic. This work marked the start of her path into computability theory, and her role in shaping and promoting computable structure theory, a subfield of computability that focuses on understanding the computational content encoded by mathematical structures.

The primary goal of this article is to describe Knight's wide-ranging research contributions to logic without assuming background in the field. Her work includes now standard conceptual frameworks that involve an appealing mix of computability-theoretic and model-theoretic ideas and their application to natural examples, some of which are described in her classic text [2] with Chris Ash. In recognition of her research contributions, she was part of the inaugural class of Fellows of the American Mathematical Society in 2012, and she delivered the Association for Symbolic Logic's Gödel Lecture in 2014. She also holds an Honorary Professorship with the Siberian Branch of the Russian Academy of Sciences.

1. Background

1.1. Structures, formulas, & arithmetic. A *structure* $\mathcal{A}$ consists of an underlying set of elements, known as the *domain*, and whatever functions or relations on the domain that are of interest—the set of symbols L used to denote these functions and relations is called the *language* of $\mathcal{A}$.

One structure studied extensively in logic is the *standard model of arithmetic* $\mathbb{N}$, whose domain is ω, the set of natural numbers.[1] We'll fix the language of $\mathbb{N}$ as $L_{\mathbb{N}} = \{S, +, \cdot, <, 0\}$ so $\mathbb{N}$ comes equipped with the functions giving the successor operation $x \to x+1$, addition, multiplication, the usual order relation, and a constant for 0.

Once a language L is fixed, one can define a variety of classes of formal statements using symbols in L, logical connectives, variables, and quantifiers. There are natural rules for interpreting the truth or meaning of these statements within a particular structure. In finitary (elementary) first-order logic, formulas are finite in length and variables are intended to range over only the domain of the structure.

For example, for all natural numbers m, consider the first-order formula

$$\theta_m = (\exists x)(\overbrace{SS\cdots S}^{m}0 < x).$$

When interpreted in $\mathbb{N}$, formula θ_m asserts "there is an element in the domain greater than m." Formulas, like θ_m, in which all variables are bound by a quantifier are called *sentences*. A sentence is either true or false in a structure, according to natural interpretation rules; certainly $\mathbb{N}$ *models* θ_m, in that θ_m is true in $\mathbb{N}$. Given a collection of sentences $\mathcal{C}$, a structure $\mathcal{A}$ is a *model* of $\mathcal{C}$ if each of the sentences in $\mathcal{C}$ is true in $\mathcal{A}$. The collection of sentences that are true in $\mathcal{A}$ is called the *theory* of $\mathcal{A}$, so a structure is a model of its own theory. The theory of $\mathbb{N}$, which contains θ_m for all m, is known as *true arithmetic* (TA).

Nonsentences, such as the subformula of θ_m,

$$\theta'_m(x) = (\overbrace{SS\cdots S}^{m}0 < x),$$

are valuable for describing subsets of (and relations on) a fixed structure $\mathcal{A}$. If $\delta(x)$ is a formula with one unbound variable x, then $\delta(x)$ *defines* the following subset of $\mathcal{A}$:

$$\delta(\mathcal{A}) \coloneqq \{a \in \text{domain of } \mathcal{A} \mid \delta(a) \text{ is true in } \mathcal{A}\}.$$

Since $\theta'_m(\mathbb{N}) = \{n \in \omega \mid n > m\} \neq \emptyset$ for all natural numbers m, the Compactness Theorem for finitary first-order logic guarantees that there are *nonstandard* models of true arithmetic, ones in which there are elements larger than any natural number. In fact, the standard model $\mathbb{N}$ can be characterized as the only model of TA, up to isomorphism, without an element x satisfying the collection of formulas $\{x > \overbrace{SS\cdots S}^{m}0 \mid m \in \omega\}$. Several of Knight's early papers (e.g., [13]) explored "omitting" collections of formulas in this sense.

The existence of nonstandard models of TA indicates a limitation in the expressiveness of finitary first-order logic. There is no single finitary first-order formula in $L_{\mathbb{N}}$ stating that some elements are "infinite." But such a formula exists in the infinitary (but still first-order) logic $\mathcal{L}_{\omega_1,\omega}$, which allows formulas with countable conjunctions and disjunctions, namely the (infinitary) formula

$$(\exists x) \bigwedge_{m\in\omega} (\overbrace{SS\cdots S}^{m}0 < x).$$

In fact, for each countable structure, there is a single sentence in $\mathcal{L}_{\omega_1,\omega}$ that distinguishes the given structure from all others up to isomorphism.

Theorem 1 (Scott Isomorphism Theorem). *If $\mathcal{A}$ is a countable structure in a countable language, then there is a sentence in $\mathcal{L}_{\omega_1,\omega}$ that is true in just the countable structures isomorphic to $\mathcal{A}$.*

The sentence guaranteed by Theorem 1 for a structure $\mathcal{A}$ is called the *Scott sentence* of $\mathcal{A}$.

Peano arithmetic represents mathematicians' best attempt at giving a useful set of axioms describing the standard model of arithmetic. The axioms of Peano arithmetic are the usual rules for the arithmetic operations and order, along with axioms of the form

$$[\psi(0) \,\&\, (\forall x)(\psi(x) \to \psi(Sx))] \to (\forall x)\psi(x), \qquad (1)$$

which state that induction holds for any finitary first-order formula $\psi(x)$. The theory of Peano arithmetic, which we denote by PA, is the set of all logical consequences of the axioms. As an example, the statement "all nonzero elements have predecessors" is in PA since PA contains the induction axiom in (1) for the formula

$$x = 0 \vee (\exists y)(Sy = x)$$

and can carry out the inductive proof.

Gödel's Incompleteness Theorems imply that PA (and any reasonable axiomatization of TA) fails to prove all the statements in TA, and that PA, in fact, can't even prove its own consistency. Further, the inner workings of these results demonstrate that there is no algorithm to determine whether a sentence is in TA, or even in PA. (There is, however, an algorithm for deciding whether a sentence is an axiom of PA.) The relationship between computational complexity and arithmetic goes much deeper and turns out to depend on definability in $\mathbb{N}$.

1.2. Computability. Alan Turing provided the conceptual framework for determining the computational complexity of sets of natural numbers. At the most basic level, a set A of natural numbers is *computable* if there is an algorithm[2] that computes the characteristic function χ_A of A,

[1] *The symbol ω comes from viewing the set of natural numbers as the first infinite ordinal.*

[2] *By algorithm we really mean a Turing machine, but an informal understanding suffices here by Church's thesis.*

and A is *computably enumerable (c.e.)* if there is an algorithm for listing out the elements of A (not necessarily in order). Note that any computable set A is computably enumerable since we could use χ_A to algorithmically enumerate the elements of A in order.

We can discuss the computability of anything that can be encoded by sets of natural numbers. For example, even though neither TA nor PA is computable, PA is computably enumerable. Indeed, we can algorithmically list all possible deductions from the computable set of axioms for PA, by interleaving deductions of varying proof lengths. As we'll see, TA is far from being computably enumerable.

Using the same ideas, we can compare the relative complexity of two subsets of natural numbers A and B. The set A is *B-computable* (or *computable relative to B*) if there's an algorithm that, given access to the values of χ_B, can compute χ_A on any input. In this case, we write $A \leq_T B$, and the definition of B-computably enumerable is analogous. We describe the computational complexity of a set A by its *Turing degree*, the equivalence class of sets that are Turing equivalent to A under $\leq_T$. In other words, the Turing degree of A is the collection of sets that both compute A and that are A-computable.

Just as PA can be computably enumerated, we can effectively enumerate all possible algorithms (written in some specified format) to obtain a computable list of all partial B-computable functions on the natural numbers. We'll denote this list by $(\Phi_e^B)_{e\in\omega}$. Here $\Phi_e^B(n)$ is defined only if the eth algorithm reaches an output, i.e., *halts*, on input n after finitely many steps when given access to χ_B. One can use this list to diagonalize outside the class of B-computable sets.

Theorem 2. *The* halting set relative to B, *defined as*

$$B' := \{e \in \omega \mid (\exists s)\Phi_e^B(e) \text{ halts by stage } s\},$$

is a B-computably enumerable set that is not B-computable.

Observe that we can B-computably check whether the eth algorithm halts by stage s on input e, when given access to χ_B. This property can be used to B-computably enumerate the elements of B', making B' c.e. relative to B. Furthermore, note that the definition of B' is an existential one in terms of a B-computable property. The set B' is often called the *jump* of B because the complexity of B' "jumps above" that of B, i.e., $B' >_T B$. Hence, the set $\emptyset'$ is a natural example of a c.e. set that is not computable. We can continue to take jumps to obtain

$$\emptyset <_T \emptyset' <_T \emptyset'' = \emptyset^{(2)} <_T \emptyset''' = \emptyset^{(3)} <_T \cdots.$$

These sets end up serving as an important measuring stick for computational power in the *arithmetical hierarchy*.

1.2.1. *The arithmetical hierarchy.* Even with its expressive limitations, finitary first-order logic can define many subsets of $\mathbb{N}$. Matiyasevich's negative resolution of Hilbert's Tenth Problem implies that all computably enumerable sets are definable in $\mathbb{N}$ using relatively simple formulas. Hilbert's Tenth Problem asks whether there's an algorithm for determining whether a given *Diophantine equation*, one of the form $p(\bar{x},\bar{y}) = 0$ where $p(\bar{x},\bar{y}) \in \mathbb{Z}[\bar{x},\bar{y}]$, has a solution in the integers. Building on the work of Robinson, Davis, and Putnam, Matiyasevich provided a characterization of the solution sets of Diophantine equations.

Theorem 3 (MRDP Theorem). *Let S be a subset of $\mathbb{Z}^j$. The set S is computably enumerable if and only if*

$$S = \{\bar{n} \in \mathbb{Z}^j \mid (\exists \bar{y} \in \mathbb{Z}^k)[p(\bar{n},\bar{y}) = 0\}$$

for some $p(\bar{x},\bar{y}) \in \mathbb{Z}[\bar{x},\bar{y}]$.

Theorem 3 together with the existence of noncomputable c.e. sets like $\emptyset'$ ensure that no algorithm of the kind Hilbert hoped for exists. Moreover, since the integers can be represented simply in terms of natural numbers, Theorem 3 implies that all c.e. sets are definable in $\mathbb{N}$ by finitary formulas whose only quantifiers are existential. Such formulas are known as Σ_1^0 formulas.

This relationship between computational complexity and definability extends to all *arithmetical sets*, those that are definable in $\mathbb{N}$ by a formula in finitary first-order logic. The *arithmetical hierarchy* classifies these definable subsets by the syntactic complexity of the formulas defining them. A formula (in prenex normal form) is Σ_{n+1}^0 if it starts with an existential quantifier followed by at most n alternations between blocks of existential and universal quantifiers, and then a quantifier-free formula.[3] The Π_{n+1}^0 formulas are similarly defined, with the initial quantifier being universal. Finally, a set of natural numbers is described as Σ_n^0 (respectively, Π_n^0) if it is defined in $\mathbb{N}$ by a formula at that level and Δ_n^0 if it is both Σ_n^0 and Π_n^0. Sometimes one wants to describe a property in terms of some parameter set B of natural numbers. One can make the same definitions all relative to the given B, so we can talk about $\Sigma_n^0(B)$ or $\Pi_n^0(B)$ subsets of natural numbers.

The arithmetical hierarchy is tightly aligned with the computational complexity of subsets of $\mathbb{N}$, when measured against the sequence $(\emptyset^{(n)})_{n\in\omega}$. Post's Theorem explains this relationship[4] and shows that the hierarchy does not collapse.

Theorem 4 (Post). *Let A be a set of natural numbers.*

1. *A is Δ_{n+1}^0 if and only if $A \leq_T \emptyset^{(n)}$.*
2. *A is Σ_{n+1}^0 if and only if A is $\emptyset^{(n)}$-computably enumerable. Hence, A is definable by a Σ_1^0 formula in $\mathbb{N}$ if and only if A is c.e.*

[3] *The superscript 0 indicates the formulas are first order.*

[4] *The formulation here of the arithmetical hierarchy relies on Theorem 3, a much later result than Post's.*

3. $\emptyset^{(n)}$ *is* Σ^0_n*-complete for all* $n > 0$ *in that*
   (a) $\emptyset^{(n)}$ *is* Σ^0_n*, and*
   (b) *for every* Σ^0_n *set* B*, there is a one-to-one computable function* f *such that* $x \in B$ *if and only if* $f(x) \in \emptyset^{(n)}$.

2. Complexity of Structures

While Knight was at Penn State, Mark Nadel recruited her to join him at the University of Notre Dame in 1977. Like Knight, Nadel was a model theorist studying models of arithmetic from that field's perspective, but their joint efforts began to require some methods from computability. At the same time, others had become interested in the computational content of models of PA and TA, including Carl Jockusch, a computability theorist at Urbana-Champaign, and David Marker and his advisor Angus Macintyre, two model theorists then at Yale. Working on the computational content of models of arithmetic solidified Knight's path into computability theory.

Here we've begun blurring the idea of a structure with a specific encoding of one. To encode a (countable) structure $\mathcal{A}$, one labels its domain with constants $(c_i)_{i\in\omega}$ and considers the collection $D(\mathcal{A})$ of quantifier-free facts about $\mathcal{A}$ in terms of these constants. The collection $D(\mathcal{A})$ is known as the *atomic diagram* of $\mathcal{A}$. Observe that the collection $D(\mathcal{A})$ here depends on the labeling of the domain of $\mathcal{A}$. Each possible encoding of $D(\mathcal{A})$ is called a *presentation* or *copy* of $\mathcal{A}$, whose computational content can be measured. From here on out, we will identify a structure $\mathcal{A}$ with a particular encoding of $D(\mathcal{A})$. So, the computational content of $\mathcal{A}$ is measured by the Turing degree of its particular encoding, the equivalence class of all sets equivalent to this encoding under $\leq_T$.

2.1. Degrees of models of arithmetic. Though Tennenbaum had earlier shown that there are no computable nonstandard models of PA (and hence TA), Jockusch and Soare had demonstrated in 1972 that "low" nonstandard models of PA exist. A *low* set X is one that is almost computable in the sense that its jump X' is as small as possible, i.e., $X' \leq_T \emptyset'$.

The situation for TA is quite different; this theory and its nonstandard models are quite complex. By its construction, TA can uniformly compute whether a given formula holds in $\mathbb{N}$. Since Post's Theorem states that $\emptyset^{(n)}$ is defined by a Σ^0_n formula, any set X computing TA can uniformly compute $\emptyset^{(n)}$ for all n.

As for models of TA, a result of Fefferman (which Knight and Nadel later generalized) shows that if $\mathcal{A}$ is a nonstandard model of TA, then $\deg(\mathcal{A}) >_T \emptyset^{(n)}$ for all n. Knight conjectured that Fefferman's result was the only limitation on the complexity of nonstandard models of TA, i.e., that, for each $X >_T \emptyset^{(n)}$ for all n, there would be a model of TA having the same degree as X. In 1984, Knight, together with Lachlan and Soare, proved this conjecture to be false. (See [12] for references in this section.)

2.2. Degrees of presentations. The above work motivated Knight and others to study different presentations of a given countable structure. The degrees of these presentations can vary widely, making the following definition natural.

Definition 1. The *degree spectrum* of a countable structure $\mathcal{A}$ is

$$DgSp(\mathcal{A}) = \{degree(\mathcal{B}) \mid \mathcal{B} \cong \mathcal{A}\}.$$

In 1982, Marker proved that the degree spectrum of a model $\mathcal{A}$ of PA is upwards closed, in that, given a degree $\mathbf{d}$ Turing above a presentation of $\mathcal{A}$, there is another copy of $\mathcal{A}$ of exactly degree $\mathbf{d}$. Macintyre asked what other kinds of structures, beyond models of arithmetic, have this property. Knight's answer [14], obtained while visiting Yale in the fall of 1982, is a classic result of computable structure theory.

Theorem 5 (Knight). *Let* $\mathcal{A}$ *be a countable structure. The following are equivalent:*

1. *The degree spectrum* $DgSp(\mathcal{A})$ *is upward closed.*
2. $\mathcal{A}$ *is* automorphically nontrivial, *in that there is no finite subset* S *of the domain of* $\mathcal{A}$ *such that all permutations fixing* S *are in fact automorphisms.*

Her proof shows how automorphically nontrivial structures are rich enough to allow (and are necessary) for the coding of additional computational information. Knight continued to study questions involving the degree spectra of structures, e.g., what sets are computable in all copies of a given structure, while visiting Jockusch at Urbana-Champaign for her sabbatical in 1984–85 and throughout her career.

3. Complexity Between Copies

Knight met Chris Ash, a computability theorist at Monash University who would become a close collaborator, during her 1984–85 sabbatical. Their highly regarded book [2] describes their shared interests (see this text for all references in this section). The book was not intended to be jointly authored, but Knight completed the project after Ash's untimely death in 1995. The book addresses three kinds of problems regarding the complexity between and within particular copies of a fixed structure. So as to give some concrete motivation,[5] we'll first state their most basic incarnations.

Problem 1. Given a computable structure $\mathcal{A}$ and a relation R on $\mathcal{A}$, what syntactic conditions ensure that the image of R is c.e. under all isomorphisms from $\mathcal{A}$ to another computable copy?

[5] *This exposition is inspired by Rod Downey's excellent Math Review of* [2].

A relation on a structure $\mathcal{A}$ is called *intrinsically c.e.* if it has the property in Problem 1. Let $(\omega, <)$ be the standard linear order on the natural numbers. The relation "$x < y$ but y is not the successor of x" in $(\omega, <)$ is intrinsically c.e.; given a computable copy of $(\omega, <)$, one can computably enumerate the pairs of domain elements (x, y) whenever another element z is placed between x and y. Observe that this relation is defined by the Σ^0_1 formula $(\exists z)(x < z \;\&\; z < y)$, suggesting the relevance of definability. On the other hand, the successor relation itself turns out *not* to be intrinsically c.e., even though it's computable in the standard model $(\omega, <)$.

Problem 2. Given a computable structure $\mathcal{A}$, what syntactic conditions ensure that every computable copy $\mathcal{B}$ of $\mathcal{A}$ is *computably* isomorphic to $\mathcal{A}$?

Structures $\mathcal{A}$ with the property described in Problem 2 are called *computably categorical*. For example, the usual linear order on the rationals $(\mathbb{Q}, <)$ is computably categorical; the construction of an isomorphism between two countable dense linear orders via the usual back and forth argument is an algorithmic process. But, $(\omega, <)$ is not computably categorical, since that would imply this structure's successor relation would be intrinsically c.e., which, as we stated above, is false.

Problem 3. Given a computable structure $\mathcal{A}$, what syntactic conditions ensure that, for every computable copy $\mathcal{B}$ of $\mathcal{A}$, *every* isomorphism from $\mathcal{B}$ to $\mathcal{A}$ is computable?

Though $(\omega, <)$ is not computably categorical (and hence does not have the desired property in Problem 3), all isomorphisms between its computable copies *are* Δ^0_2. Indeed, any isomorphism between copies of $(\omega, <)$ must match up the least elements, and then their successors, and then those elements' successors, etc. Combining this process with the fact that the least element and successor relations are both Δ^0_2-definable gives the desired result.

In earlier work, others explored Problems 1, 2, and 3 as stated, in general and in specific kinds of structures, e.g., Goncharov and independently Remmel proved that the computably categorical linear orders are those with only finitely many successor pairs. However, the answers to the original problems are typically quite complicated. As we'll see, the problems have more elegant solutions when the copies range over *all* copies of a fixed structure, rather than only those of a given complexity. For example, a computable structure $\mathcal{A}$ is called *relatively computably categorical* if for every copy $\mathcal{B}$ of $\mathcal{A}$, there is a $\mathcal{B}$-computable isomorphism from $\mathcal{B}$ to $\mathcal{A}$. Changing the problems in this way is one approach to relativizing. Another approach is to relativize the other objects under consideration (e.g., replace "computable isomorphism" with "Δ^0_2-isomorphism" as done in the example after Problem 3). Knight and Ash, often with others, focused on relativizing these problems according to both approaches, to the *hyperarithmetical hierarchy*, an extension of the arithmetical hierarchy.

3.1. Hyperarithmetical hierarchy.

The measuring stick of the arithmetical hierarchy is the sequence $(\emptyset^{(n)})_{n\in\omega}$, in the sense that a set of natural numbers is arithmetical if it is computable from $\emptyset^{(n)}$ for some n. We can extend this measuring stick by continuing to take jumps along the infinite ordinals, as long as the ordinals (and hence this process) remain sufficiently effective:

$$\emptyset <_T \emptyset' <_T \emptyset^{(2)} <_T \cdots <_T \emptyset^{(\omega)} <_T \emptyset^{(\omega+1)} <_T \cdots.$$

Such ordinals are those that are order-isomorphic to a computable well-ordering. These *computable ordinals* form a countable initial segment of all ordinals; the notation ω_1^{CK} denotes the least noncomputable ordinal.

A subset of $\mathbb{N}$ is *hyperarithmetical* if it is computable from $\emptyset^{(\alpha)}$ for some computable ordinal α. True arithmetic provides a natural example; the degree of TA is that of $\emptyset^{(\omega)}$.

3.1.1. *Computable infinitary formulas.*

Just like the arithmetical sets, the hyperarithmetical sets can be described in terms of definability. Here we use *computable infinitary formulas*, formulas from $L_{\omega_1,\omega}$ whose countable conjunctions and disjunctions are computably enumerable. As in the finitary first-order setting, formula complexity is measured by alternation of quantifiers. The quantifier-free finitary first-order formulas are deemed both Σ^c_0 and Π^c_0. Then, for any computable ordinal α, a formula is Σ^c_α if it is essentially a (possibly infinite) c.e. disjunction of formulas $(\exists\bar{u})\psi(\bar{u})$, where the $\psi(\bar{x})$ are Π^c_β formulas for varying $\beta < \alpha$. Analogously, a formula is Π^c_α if it is a c.e. conjunction of formulas $(\forall\bar{u})\psi(\bar{u})$, where the $\psi(\bar{x})$ are Σ^c_β formulas for different $\beta < \alpha$.

The next theorem, which follows from results related to Theorem 8, provides evidence that the computable infinitary formulas are the "right" logic for the hyperarithmetical setting.

Theorem 6. *A set is hyperarithmetical if and only if it is definable in $\mathbb{N}$ by a computable infinitary formula. Moreover, if a set A is hyperarithmetical, then for any computable $\alpha \geq \omega$,*

- *A is definable by a Σ^c_α and a Π^c_α formula if and only if A is $\emptyset^{(\alpha)}$-computable;*
- *A is definable by a Σ^c_α formula if and only if A is $\emptyset^{(\alpha)}$-computably enumerable.*

We remark that the Σ^c_n and Π^c_n formulas do not define anything in $\mathbb{N}$ beyond what their finitary analogues could. As before, we declare a set of natural numbers Σ^0_α if it is definable by a Σ^c_α formula; Π^0_α or Δ^0_α subsets are defined similarly.

3.1.2. *Connection to analytic hierarchy.*

The hyperarithmetical sets can also be defined in terms of the first level of the *analytical hierarchy*, a classification of subsets of $\mathbb{N}$ in terms of their definability in second-order arithmetic, a logic that allows variables ranging over not just natural numbers but also *sets* of natural numbers. Here quantifiers ranging over sets (and their alternations) are what drive complexity. A formula of this kind is Σ^1_1 (resp., Π^1_1) if it has only existential (resp., universal) set quantifiers.[6] The hyperarithmetical sets coincide with the Δ^1_1 sets, those that are definable in $\mathbb{N}$ by both Σ^1_1 and Π^1_1 formulas.

Computable infinitary formulas are also the "right" logic for understanding computable structures (and hyperarithmetical ones too). Most crucially, for any computable ordinal α, a Σ^c_α formula defines a Σ^0_α subset of the domain of a computable structure (and an analogous result holds for Π^c_α ones). They also have a compactness theorem akin to the one for finitary first-order logic, that can even provide computable models.

Theorem 7 (Barwise-Kreisel Compactness). *If Γ is a Π^1_1 set of computable infinitary sentences and every hyperarithmetical subset of Γ has a (computable) model, then Γ has a (computable) model.*

One can apply Barwise-Kreisel Compactness to show that computable infinitary formulas are capable of distinguishing between nonisomorphic computable structures.

Corollary 1. *If $\mathcal{A}$ and $\mathcal{B}$ are computable (or hyperarithmetical structures) satisfying the same computable infinitary sentences, then $\mathcal{A} \cong \mathcal{B}$.*

3.2. **Results on the relativizations.** By their nature, Problems 1, 2, and 3 in the introduction to §3 require understanding how computational power is connected to definability. In fact, the resolution of the full relativization of Problem 1 is purely about definability.

Theorem 8 (Ash, Knight, Manasse, & Slaman, independently Chisholm). *Let R be a relation on a computable structure $\mathcal{A}$. The following are equivalent:*

1. *R is definable in $\mathcal{A}$ by a Σ^c_α formula referencing a finite tuple of parameters from $\mathcal{A}$.*
2. *For any $\mathcal{B} \cong \mathcal{A}$, the image of R in $\mathcal{B}$ is $\Sigma^0_\alpha(\mathcal{B})$.*

A relation R on a computable structure $\mathcal{A}$ satisfying statement 2 of Theorem 8 is called *relatively intrinsically Σ^0_α on $\mathcal{A}$*. Recall that the adverb "relatively" here indicates that $\mathcal{B}$ is ranging over *all* copies of $\mathcal{A}$, not just the computable ones. This aspect of the definition reflects the first approach to relativizing described at the end of the introduction to §3. Considering images of relations that are $\Sigma^0_\alpha(\mathcal{B})$, rather than just those that are $\Sigma^0_1(\mathcal{B})$, reflects the second approach mentioned. Because there are no restrictions on the complexity of $\mathcal{B}$, this "relative" definition requires only that the image of R in $\mathcal{B}$ be $\Sigma^0_\alpha(\mathcal{B})$, rather than plainly Σ^0_α.

[6]*The subscript "1" indicates that there are element variables and set variables.*

The syntactic conditions in Theorem 8 were first identified in the computable setting by Ash and Anil Nerode and then Ash's student Ewan Barker. Though those results came first, Theorem 8 can be stated and proven more elegantly, suggesting that combining "relative" approaches leads to stronger results. Progress on Problems 2 and 3 evolved similarly.

The solutions to Problems 2 and 3 (whether relativized or not) involve the ability to effectively enumerate formulas that define certain parts of the structure. Goncharov showed the solution to Problem 2 involves a special kind of *Scott family*, an important tool in the proof of the Scott Isomorphism Theorem.

Definition 2. A *Scott family for a structure $\mathcal{A}$* is a collection of formulas $\mathcal{F}$ in $\mathcal{L}_{\omega_1,\omega}$ (in terms of some fixed parameters) so that

- each tuple in $\mathcal{A}$ satisfies some formula in $\mathcal{F}$, and
- if two tuples in $\mathcal{A}$ satisfy the same formula in $\mathcal{F}$, then there's an automorphism of $\mathcal{A}$ taking one to the other.

Theorem 9 (Ash, Knight, Manasse, & Slaman, independently Chisholm). *For a computable structure $\mathcal{A}$, the following are equivalent:*

1. *$\mathcal{A}$ has a c.e. Scott family consisting of Σ^c_α formulas in terms of a fixed tuple of parameters.*
2. *For any $\mathcal{B} \cong \mathcal{A}$, there is a $\Delta^0_\alpha(\mathcal{B})$-computable isomorphism from $\mathcal{A}$ to $\mathcal{B}$, i.e., $\mathcal{A}$ is relatively Δ^0_α-categorical.*

Showing that the given computational feature implies the desired definability is the difficult direction in both Theorems 8 and 9. The proofs of these portions use forcing, a powerful technique of mathematical logic. Forcing here involves building a "generic" copy $\mathcal{B}$ of the given structure $\mathcal{A}$, whose construction "forces" decisions about statements describing the computational properties of $\mathcal{B}$. Since $\mathcal{B}$ is, in fact, a copy of $\mathcal{A}$ and by assumption $\mathcal{B}$ has the specified computational feature, this fact must be forced at some point in the construction, allowing one to extract the desired definability statement.

3.3. **Priority arguments.** While Theorems 8 and 9 rely on forcing, the solutions of the original three problems use *finite injury priority constructions*, a fundamental proof technique in computability theory. Friedberg and independently Muchnick first used this method to build a c.e. set B strictly between $\emptyset$ and $\emptyset'$. A priority construction is a stage-by-stage process for making an object according to a list of "requirements" that together ensure desired properties. Typically the requirements make use of the list of all partial B-computable functions $(\Phi^B_e)_{e\in\omega}$. For example, a single requirement R_e in the Friedberg-Muchnick Theorem aims to ensure that the eth partial computable function $\Phi^{\emptyset}_e$ does not compute B. Satisfying requirement R_e for all e guarantees that $B >_T \emptyset$.

The strategies for satisfying distinct requirements may conflict. In fact, satisfying one requirement may "injure" or undo the satisfaction of another. Hence, an ordering is put on the requirements to make clear at any stage which requirement has priority to act. In a finite injury construction, like that of the Friedberg-Muchnick Theorem and those for the original three problems above, each requirement is injured only finitely often. Though these constructions are computable, knowing when requirements are permanently satisfied is $\emptyset'$-computable.

Shoenfield and Sacks independently invented *infinite injury constructions*, to obtain theorems like Sacks' result that the degrees of c.e. sets are dense. Determining how requirements are satisfied in these constructions requires increasing amounts of power. For example, Harrington used his "workers" method to obtain a nonstandard model of arithmetic that was arithmetical but whose theory was not. Determining how requirements are met in this construction requires $\emptyset^{(\omega)}$. Knight and others tried with some success to employ Harrington's approach in other settings, but priority constructions need increasingly intricate mechanisms as their requirements involve more complicated information. Coming up with, understanding, and verifying these constructions was becoming unmanageable.

To ameliorate this situation, Ash developed a black box approach. His "metatheorem" guarantees that if certain effectiveness conditions are satisfied, then an associated infinite injury priority construction will succeed. Knight joined forces with Ash and others to refine this method, develop new variations, e.g., [15], and explore applications, including ones related to the three problems.

4. Complexity of Classes of Structures

Though well aware of his work before, Knight met Sergei Goncharov (Novosibirsk State University) at an Association for Symbolic Logic conference in Leeds in the summer of 1997 before joining him at a conference in Kazan. There Goncharov gave a talk describing a variety of effective classification results for specific classes of computable structures (e.g., computable linear orders) and calling for similar results for other natural classes. At the talk, Richard Shore asked Goncharov what would make him give up on finding an effective classification of a particular class. Shore's question led Knight and Goncharov to explore three known approaches to effective classification and to identify a satisfying answer in [9]. This work generated a new wave of interest in both the theory of effective classification and its application to natural classes, from graphs, to fields, to groups.

Let $\mathcal{K}$ be a class of structures that is closed under isomorphism (e.g., fields of characteristic zero). Goncharov and Knight aimed to understand *characterizations* and *classifications* of $\mathcal{K}^c$, the set of computable presentations of elements in $\mathcal{K}$. For them, an effective characterization of $\mathcal{K}$ is a way to understand the full diversity of structures in $\mathcal{K}^c$ (possibly with repetition), and an effective classification of $\mathcal{K}$ is a method that also distinguishes between isomorphism types within $\mathcal{K}^c$. (See [4] for a more comprehensive view of these approaches and for the references in this section.)

4.1. Three approaches.

4.1.1. *Enumerations.*

Classification in mathematics often takes the form of a list of all distinct possibilities, up to a fixed notion of equivalence, e.g., the classification of finite simple groups. This idea motivates the first approach.

Definition 3. An *enumeration* of $\mathcal{K}^c$ is a sequence $(\mathcal{C}_n)_{n\in\omega}$ of presentations in $\mathcal{K}^c$ in which every $\mathcal{A} \in \mathcal{K}^c$ is isomorphic to some $\mathcal{C}_n$.

A *Friedberg enumeration* of $\mathcal{K}^c$ is an enumeration of $\mathcal{K}^c$ in which the isomorphism types of the $\mathcal{C}_n$ are distinct.

Thus, an enumeration characterizes the computable structures in $\mathcal{K}$, and a Friedberg enumeration classifies them. The moniker "Friedberg" has its origin in Friedberg's classic result providing a computable enumeration of all c.e. sets without repetition. Goncharov and Knight viewed an enumeration, whether Friedberg or not, as being effective if the enumeration is hyperarithmetical.

4.1.2. *Computable infinitary descriptions & Scott rank.*

The second approach is syntactic, relating to Scott sentences and Scott families. In this approach, an effective characterization of $\mathcal{K}$ is a computable infinitary definition of $\mathcal{K}^c$, if any exists. Describing effective classification in this approach requires more effort.

In a standard proof of the Scott Isomorphism Theorem, one first shows that there is a Scott family for the given structure $\mathcal{A}$ and an ordinal α in which the definitions of all orbits of tuples in $\mathcal{A}$ are Π^0_α. The least such ordinal α for which such a Scott family exists can be used to define the *Scott rank* of $\mathcal{A}$, a measure of syntactic complexity of certain automorphisms. The Scott ranks of computable structures have nice features.

Theorem 10 (Nadel)**.** *Let $\mathcal{A}$ be a computable structure.*

1. *The Scott rank of $\mathcal{A}$ is at most $\omega_1^{CK} + 1$.*
2. *$\mathcal{A}$ has computable Scott rank (so less than ω_1^{CK}) if and only if $\mathcal{A}$ has a computable infinitary Scott sentence.*

Turning to a class $\mathcal{K}$ of structures, Goncharov and Knight deemed $\mathcal{K}$ effectively classifiable if there is a computable ordinal bound on the Scott ranks of the elements of $\mathcal{K}^c$. Such a bound implies that there is a computable ordinal α such that structures in $\mathcal{K}^c$ all have Π^c_α Scott sentences, which distinguish these structures up to isomorphism. Hence, this bound on Scott rank is a proxy for effective classification.

4.1.3. *Index sets & the isomorphism problem.* The third approach measures the computational content of sets involving the "names" of presentations in $\mathcal{K}^c$.

Definition 4. The *index set of* $\mathcal{K}$ is the set $I(\mathcal{K})$ of indices e of functions $\Phi_e^{\emptyset}$ that encode a presentation in $\mathcal{K}^c$.

The *isomorphism problem of* $\mathcal{K}$ is the set $E(\mathcal{K})$ of ordered pairs (e, e') so that $e, e' \in I(\mathcal{K})$ and the structures computed by $\Phi_e^{\emptyset}$ and $\Phi_{e'}^{\emptyset}$ are isomorphic.

Here Goncharov and Knight viewed $\mathcal{K}$ as having an effective characterization if $I(\mathcal{K})$ is hyperarithmetical and an effective classification if $E(\mathcal{K})$ is.

4.1.4. *Relationships between approaches.* At the level of determining whether a class $\mathcal{K}$ is effectively classifiable, all three approaches turn out to be equivalent, under the assumption that $\mathcal{K}$ can be characterized by a computable infinitary sentence. The complexity of such a syntactic description of $\mathcal{K}$ gives an upper bound on the computational content of $I(\mathcal{K})$—e.g., if $\mathcal{K}^c$ is described by a Π_3^c statement, then $I(\mathcal{K})$ is Π_3^0. Even better, Goncharov and Knight showed that $\mathcal{K}$ has a computable infinitary characterization if and only if $I(\mathcal{K})$ is hyperarithmetical. Building on Goncharov and Knight's work, Calvert and Knight stated the following result.

Theorem 11. *If $I(\mathcal{K})$ is hyperarithmetical, then the following are equivalent:*

1. *$\mathcal{K}$ has a hyperarithmetical Friedberg enumeration.*
2. *There is a computable ordinal α such that any two members of $\mathcal{K}^c$ satisfying the same Π_α^c sentences are isomorphic.*
3. *$E(\mathcal{K})$ is hyperarithmetical.*

Theorem 11 provides a robust test for determining whether a class is effectively classifiable. Applying Theorem 11, one can show natural classes like vector spaces over a fixed infinite computable field, algebraically closed fields of fixed characteristic, and archimedean real closed fields are effectively classifiable whereas classes like undirected graphs, fields of fixed characteristic, real closed fields, and linear orders are not.

4.2. **Finer-grained complexity.** A hallmark of Knight's work is the marriage of theory-building and application, and Knight and her collaborators have used all three approaches to obtain finer-grained information about specific effectively classifiable classes. Exact complexity calculations make use of the structural properties of the class and often employ results of classical (noneffective) mathematics. In this section, we highlight some of this work, beginning with Knight and her collaborators work on free groups in [5]. All references in §4.2.1 can be found there.

4.2.1. *Free groups.* In 1945, Tarski asked whether all free groups with at least two, but not infinitely many, generators are *elementarily equivalent*, i.e., satisfy the same finitary first-order statements in the language of groups. In 2006, Sela and independently Kharlampovich and Myasnikov proved the answer is positive—finitary first-order logic is unable to distinguish between finitely generated free groups of rank at least two. Since all countable free groups have computable presentations, they are distinguished from one another by computable infinitary formulas by Corollary 1. Hence, it's natural to search for computable infinitary descriptions of these groups. Any such description of these groups puts an upper bound on the computational content of their index sets. A description of a group G is optimal if $I(G)$ is m-complete at the level of the description.

Definition 5. Let G be a countable group. The index set $I(G)$ is *m-complete* at a given complexity level if

1. $I(G)$ has the given complexity, and
2. for every set X of the given complexity, there is a uniformly computable sequence of groups $(G_n)_{n\in\omega}$ such that $n \in X$ if and only if $G_n \cong G$.

Let F_n denote the free group of rank n, i.e., one having a basis of size n, and let F_∞ be the free group of countably infinite rank. Recall that a basis for a free group is a generating set for the group in terms of which there is no nontrivial expression of the identity.

Knight and her collaborators, including some of her then current and former students, found optimal descriptions for F_n, F_∞, and related classes of groups in [5]. They showed that F_∞ has a Π_4^c description, which McCoy and Wallbaum, a subset of the authors, showed was optimal. Simply determining whether a presentation of a group is free is quite complicated, so the full team also looked for descriptions that distinguish one specific free group from other free groups. The group F_∞ has an optimal Π_3^c description of this latter kind.

When applied to the class of all groups, however, the Π_3^c description of F_∞ no longer characterizes F_∞, since it may describe groups that are *locally free* but not free. A group is *locally free* if every finitely generated subgroup is a free group. An example of a locally free but not free group that satisfies the Π_3^c description is the abelian group generated by $\{b_n \mid n \in \omega\}$ such that $b_{n+1}^2 = b_n$. To obtain the Π_4^c description that picks out exactly F_∞ amongst all groups, the authors needed to describe tuples of elements that can be part of a basis. They relied on *Nielsen transformations* on collections of words, which are analogous in flavor to row reductions on matrices. Just as there is reduced echelon form for matrices, there is an *N-reduced form* on words. Given a tuple of words, N-reduced form can be used to determine whether the tuple could be part of a basis or not, if applied to a set of basis elements. Moreover, the processes involved in making this determination are computable.

Interestingly, the optimal descriptions of the groups and classes are not always the "natural" ones. Knight and her collaborators note that an obvious description of F_n states that there is an n-tuple, having no nontrivial relationships, so that every other group element can be written as a word on this tuple. This description is Σ^c_3. When they went to show this description was optimal, they became stuck, motivating them to search for (and find) a simpler but less expected description. Thus, computable structure theory helps to pinpoint wherein lies the true complexity within a problem.

4.2.2. *Friedberg enumerations.* Though the three approaches agree on whether a class has an effective classification or not, the exact complexities may differ between approaches. For example, computable Friedberg enumerations exist for vector spaces over a fixed infinite computable field and algebraically closed fields of fixed characteristic, but Calvert showed that the isomorphism problem $E(\mathcal{K})$ for these classes is m-complete Π^0_3, i.e., is Π^0_3 and computably encodes any Π^0_3 set. The complexities are relatively low since (countable) vector spaces and algebraically closed fields have simple invariants in terms of dimension and transcendence degree.

Equivalence structures, ones whose domain ω is equipped with an equivalence relation, also have relatively simple invariants in terms of the number of equivalence classes of a given size, but they are slightly more syntactically difficult to describe. Indeed, the computable bound on the Scott rank of equivalence structures is higher than that of those on vector spaces and algebraically closed fields, and the isomorphism problem $E(\mathcal{K})$ is a level more complex at m-complete Π^0_4.

Determining whether the class of equivalence structures has a computable Friedberg enumeration proved challenging. Goncharov and Knight were able to show that the class of all equivalence structures with infinitely many infinite classes has a computable Friedberg enumeration, but conjectured that the full class did not. Others proved related but not definitive existence and nonexistence results. However, Goncharov and Knight's conjecture was false—Downey, Melnikov, and Ng provided a Friedberg enumeration of all equivalence structures in 2017. (See [7] for the attributions in §4.2.2.)

4.2.3. *Results when $\mathcal{K}$ is nonclassifiable.* If a class $\mathcal{K}$ is not effectively classifiable, Theorems 10 and 11 imply that the bound on the Scott ranks of elements of $\mathcal{K}^c$ must be "high," either ω_1^{CK} or $\omega_1^{CK}+1$. Knight and her collaborators have a large body of work on computable structures of high Scott rank. Prior to their work, it was well known that, for any computable ordinal α, a computable structure of Scott rank α exists. Researchers also knew of computable examples of rank $\omega_1^{CK}+1$—the most famous being the Harrison order. Even though ω_1^{CK} has no computable presentation by definition, Harrison proved that there is a computable linear order whose order type consists of an initial segment of type ω_1^{CK} followed by densely many copies of ω_1^{CK}. This order has Scott rank $\omega_1^{CK}+1$. That left the question of whether there exist computable structures of Scott rank ω_1^{CK}. In 1981 Makkai proved that an arithmetical structure of rank ω_1^{CK} exists. Knight and Jessica Millar showed that this structure can be made computable.

Theorem 12 (Knight & Millar). *There is a computable structure of Scott rank ω_1^{CK}.*

With Calvert, Goncharov, and Millar, Knight showed that there are computable structures of Scott rank ω_1^{CK} in natural classes such as undirected graphs, trees, fields of any characteristic, and linear orders. Other classes cannot sustain such examples, e.g., abelian p-groups by a result of Barwise. The computable infinitary theories of these examples are all $\aleph_0$-*categorical* in that they have exactly one countable model up to isomorphism. Millar and Sacks produced a nonhyperarithmetical example of Scott rank ω_1^{CK} whose theory is not $\aleph_0$-*categorical*, but asked whether the example could be made computable. In 2018, Harrison-Trainor, Igusa, and Knight produced such an example (see [10] for the references in this section).

Knight, together with a large team, also studied the class of high rank structures using the tools described earlier. They showed that the index set of the class of high rank structures is as complicated as possible, specifically Σ^1_1-complete [3].

4.3. **Another approach to classification.** In the early 2000s, Alexander Kechris suggested to Knight that an approach to classification from descriptive set theory might have a natural effective analogue. In descriptive set theory, as above, structures are identified with their atomic diagrams, which in turn are identified with their encodings as elements of Cantor space 2^ω, the set of infinite sequences of 0's and 1's. The class of all structures in a fixed countable language forms a *Polish space*, a separable and complete metric space. For a class $\mathcal{K}$ of structures closed under isomorphism, one can show $\mathcal{K}$ is Borel if and only if $\mathcal{K}$ is definable by a sentence in $L_{\omega_1,\omega}$.

A coarse measure of classification complexity in this setting is whether the isomorphism relation on $\mathcal{K}$ is Borel. The following approach gives more information.

Definition 6 (Friedman & Stanley). Let $\mathcal{K}$ and $\mathcal{K}'$ be classes of structures that are closed under isomorphism. A *Borel embedding* of $\mathcal{K}$ into $\mathcal{K}'$ is a Borel function F from $\mathcal{K}$ to $\mathcal{K}'$ such that

$$\mathcal{A} \cong \mathcal{A}' \iff F(\mathcal{A}) \cong F(\mathcal{A}').$$

A class is *Borel complete* if it is Borel and every class is Borel embeddable in it.

Examples of Borel complete classes include linear orders, trees, and undirected graphs. But Borel reducibility does not allow for distinctions between classes having countably many isomorphism types.

Following Kechris' suggestion, Knight and her then students Calvert, Cummings, and Miller introduced *computable* and *Turing computable embeddings*, as two ways to distinguish between classes with countably many isomorphism types. Here these embeddings directly translate statements about a structure in $\mathcal{K}$ into statements about a structure in $\mathcal{K}'$. In [16], Knight and a different group of students proved a "Pull-back Theorem" that guarantees that syntactic invariants for the codomain class can be transformed into syntactic invariants in the domain class in both approaches (see this citation for all prior references in §4.3).

Knight and others [8] also explored an analogous kind of computable embedding introduced by Friedman and Fokina that maps elements of $I(\mathcal{K})$ to elements of $I(\mathcal{K}')$. Knight and her various collaborators have applied these frameworks to a wide variety of classes and compared their behavior to each other and to related approaches.

5. Further Contributions

Though we've described many important threads of Knight's work, there are others. We briefly mention a few more.

5.1. Computable model theory. Knight would continue to explore ideas connected to those in §2.1. If a theory T (such as TA) has a computable model, the set of Σ_n^0 statements in T is Σ_n^0 uniformly in n. Andrews, Lempp, and Schweber called such theories *Solovay theories* (see their paper [1] for all references in this section). Earlier, Knight and Solovay proved that a set computes a copy of some model of every Solovay theory if and only if it computes a nonstandard model of TA. However, when the Solovay theory T has exactly one countable model up to isomorphism, Knight (building on work of Lerman and Schmerl) obtained a better result. In this case, she showed that the only countable model of T has a $\emptyset'$-computable copy.

More recently, Knight and Andrews studied Solovay theories that are *strongly minimal*, an important class of theories in model-theoretic stability. A *minimal structure* is one in which each subset defined by a finitary first-order formula is finite or cofinite. In other words, the only definable subsets of the domain are those that are definable using equality and inequality alone. A theory is *strongly minimal* if all its models are minimal. Many natural theories are strongly minimal, e.g., those of vector spaces over $\mathbb{Q}$ and algebraically closed fields of a fixed characteristic.

Knight and Andrews proved that all countable models of a strongly minimal Solovay theory have a $\emptyset^{(3)}$-computable copy and asked whether $\emptyset^{(3)}$ could be replaced by $\emptyset^{(2)}$. Andrews, Lempp, and Schweber recently proved the answer is no, by characterizing the sets X that compute a copy of every countable model of a strongly minimal Solovay theory. These are precisely the *high* sets over $\emptyset^{(2)}$, i.e., sets X satisfying $X \geq_T \emptyset''$ and $X' \geq_T \emptyset^{(4)}$.

5.2. Recursive saturation, integer parts, & Hahn fields. Recursive saturation is a property of abundance, in that structures with this property have elements that satisfy any reasonable, describable property. Knight extensively studied this important model-theoretic notion and its relationship to models of arithmetic.

Definition 7. A structure is *recursively saturated* if any computable collection of finitary first-order formulas that is consistent with the structure is modeled by some elements in the structure.

As one example of her work in this area, Knight, D'Aquino, and Starchenko studied integer parts of real closed fields [6]. (See this citation for references in §5.2.) A real closed field is an ordered field that has the same theory as $\mathbb{R}$, i.e., one in which square roots of positive elements and roots of odd degree polynomials exist. An *integer part* I of a real closed field R is a discrete ordered ring in which every element of R is within distance of 1 of exactly one element of I. So, an integer part I sits inside R just as $\mathbb{Z}$ does in $\mathbb{R}$. Mourgues and Ressayre proved every real closed field has at least one integer part. While archimedean real closed fields have $\mathbb{Z}$ as their unique integer part, nonarchimedean ones may have many.

Shepherdson showed that a discrete ordered ring I is an integer part of a real closed field if and only if I is a model of a subset of PA called $IOpen$, in which the induction axioms are limited to quantifier-free formulas. The following theorem illuminates what happens when an integer part I is a model of full PA.

Theorem 13 (D'Aquino, Knight, Starchenko).

1. *If a real closed field R has an integer part I that is a nonstandard model of PA, then R is recursively saturated. Hence, in particular, the real closure of I is recursively saturated.*
2. *If R is a countable recursively saturated real closed field, then R has an integer part I such that R is the real closure of I and I is a nonstandard model of PA.*

To prove a given real closed field R has an integer part, Mourgues and Ressayre showed that R can be specially embedded in a *Hahn field*, a field of generalized power series in which the series have ordinal lengths. Understanding this process led Knight and her collaborators to study the complexity of root-taking in Hahn fields and the simpler setting of fields of Puiseux series, when these fields are algebraically closed and of characteristic zero. They proved that Newton's method for finding roots over a field of Puiseux series is fully computable, and uniformly so for nonconstant polynomials (see [11] for all citations in this

Figure 2. Julia and her husband Bill with collaborators and friends in Bulgaria.

paragraph). In the Hahn field setting, Knight and Lange found sharp bounds on the ordinal lengths of roots in terms of the lengths of the polynomial's coefficients. These length bounds seem to play an important role in understanding the exact complexity of the root-taking process in Hahn fields, an area of ongoing work.

5.3. *o*-minimality.
Knight also played a role in the development of *o-minimality*, an important generalization of strong minimality formalized by Anand Pillay and Charles Steinhorn. An *o-minimal structure* is one with an order in which each subset of the domain defined by a finitary first-order formula is a finite union of points and intervals, i.e., those that are definable using only equality and the order alone. In analogy with the strongly minimal setting, a theory is *o-minimal* if all its models are *o*-minimal. The canonical example of an *o*-minimal theory is the theory of $\mathbb{R}$ as an ordered field (which, in contrast with TA, is computable). The study of *o*-minimality has allowed for the generalization of results about real algebraic geometry and semialgebraic sets. In particular, Knight, Pillay, and Steinhorn [17] proved a fundamental cell decomposition theorem for *o*-minimal theories.

6. Epilogue

Mathematicians can shape their discipline not only through their intellectual contributions but also their approach to the mathematics community. Knight collaborates extensively with others at all career stages, and she generously acknowledges others' ideas (even when rescuing a germ of a good idea from an ill-conceived proposal).

She supports collaboration and communication in other ways as well. Early in her career, she cofounded the Midwest Model Theory seminar with John Baldwin at Illinois-Chicago. This seminar continues to run today and inspired analogous seminar series in computability in the midwest and northeast. A lack of communication between the East and West led to substantial research duplication in computability during and after the Cold War. Knight has been a major proponent of East/West collaboration, facilitating funding and travel in both directions for logicians. Her efforts along with those of others, particularly Steffen Lempp who started an NSF East/West collaboration grant in the late 1990s that Knight now administers, have helped to integrate these research communities and propel the field forward. She continues to steward the logic community as the immediate past president of the Association for Symbolic Logic.

Knight is an exemplary mentor and advocate for early-career mathematicians (including but not limited to her eighteen doctoral students), greeting new faces at conferences, securing funding, and making valuable introductions. These efforts are not limited to logicians either. She served as Director of Graduate Studies at Notre Dame for many years and was awarded the university's 2007 James A. Burns, C.S.C., Graduate School Award for distinction in graduate education.

ACKNOWLEDGMENTS. The author is very grateful for Knight's ongoing mentorship, collaboration, and friendship, and her assistance in putting together this article. She also thanks the anonymous referees for their thoughtful suggestions.

References

[1] Uri Andrews, Steffen Lempp, and Noah Schweber, *Building models of strongly minimal theories*, Adv. Math. **386** (2021), Paper No. 107802, 25, DOI 10.1016/j.aim.2021.107802. MR4266747

[2] C. J. Ash and J. Knight, *Computable structures and the hyperarithmetical hierarchy*, Studies in Logic and the Foundations of Mathematics, vol. 144, North-Holland Publishing Co., Amsterdam, 2000. MR1767842

[3] W. Calvert, E. Fokina, S. S. Goncharov, J. F. Knight, O. Kudinov, A. S. Morozov, and V. Puzarenko, *Index sets for classes of high rank structures*, J. Symbolic Logic **72** (2007), no. 4, 1418–1432, DOI 10.2178/jsl/1203350796. MR2371215

[4] Wesley Calvert and Julia F. Knight, *Classification from a computable viewpoint*, Bull. Symbolic Logic **12** (2006), no. 2, 191–218. MR2223921

[5] J. Carson, V. Harizanov, J. Knight, K. Lange, C. McCoy, A. Morozov, S. Quinn, C. Safranski, and J. Wallbaum, *Describing free groups*, Trans. Amer. Math. Soc. **364** (2012), no. 11, 5715–5728, DOI 10.1090/S0002-9947-2012-05456-0. MR2946928

[6] P. D'Aquino, J. F. Knight, and S. Starchenko, *Real closed fields and models of Peano arithmetic*, J. Symbolic Logic **75** (2010), no. 1, 1–11, DOI 10.2178/jsl/1264433906. MR2605879

[7] Rodney G. Downey, Alexander G. Melnikov, and Keng Meng Ng, *A Friedberg enumeration of equivalence structures*, J. Math. Log. **17** (2017), no. 2, 1750008, 28, DOI 10.1142/S0219061317500088. MR3730564

[8] Ekaterina B. Fokina, Sy-David Friedman, Valentina Harizanov, Julia F. Knight, Charles McCoy, and Antonio Montalbán, *Isomorphism relations on computable structures*, J. Symbolic Logic **77** (2012), no. 1, 122–132, DOI 10.2178/jsl/1327068695. MR2951633

[9] S. S. Goncharov and Dzh. Naĭt, *Computable structure and antistructure theorems* (Russian, with Russian summary), Algebra Logika **41** (2002), no. 6, 639–681, 757, DOI 10.1023/A:1021758312697; English transl., Algebra Logic **41** (2002), no. 6, 351–373. MR1967769

[10] Matthew Harrison-Trainor, Gregory Igusa, and Julia F. Knight, *Some new computable structures of high rank*, Proc. Amer. Math. Soc. **146** (2018), no. 7, 3097–3109, DOI 10.1090/proc/13967. MR3787370

[11] J. Knight, K. Lange, and D. R. Solomon, *Roots of polynomials in fields of generalized power series*, Proceedings for Aspects of Computation, World Scientific, to appear.

[12] Julia Knight, Alistair H. Lachlan, and Robert I. Soare, *Two theorems on degrees of models of true arithmetic*, J. Symbolic Logic **49** (1984), no. 2, 425–436, DOI 10.2307/2274174. MR745370

[13] Julia F. Knight, *Omitting types in set theory and arithmetic*, J. Symbolic Logic **41** (1976), no. 1, 25–32, DOI 10.2307/2272941. MR406793

[14] Julia F. Knight, *Degrees coded in jumps of orderings*, J. Symbolic Logic **51** (1986), no. 4, 1034–1042. MR865929

[15] Julia F. Knight, *Requirement systems*, J. Symbolic Logic **60** (1995), no. 1, 222–245, DOI 10.2307/2275519. MR1324511

[16] Julia F. Knight, Sara Miller, and M. Vanden Boom, *Turing computable embeddings*, J. Symbolic Logic **72** (2007), no. 3, 901–918, DOI 10.2178/jsl/1191333847. MR2354906

[17] Julia F. Knight, Anand Pillay, and Charles Steinhorn, *Definable sets in ordered structures. II*, Trans. Amer. Math. Soc. **295** (1986), no. 2, 593–605, DOI 10.2307/2000053. MR833698

Karen Lange

Credits

Figure 1 is courtesy of Wesley Calvert.
Figure 2 is courtesy of Charlie McCoy.
Photo of Karen Lange is courtesy of Darlene Howland.

Stochastic Networks and Reflecting Brownian Motion: The Mathematics of Ruth Williams

Ioana Dumitriu, Todd Kemp, and Kavita Ramanan

Ioana Dumitriu is a professor of mathematics at the University of California, San Diego. Her email address is idumitriu@ucsd.edu.

Todd Kemp is a professor of mathematics at the University of California, San Diego. His email address is tkemp@ucsd.edu.

Kavita Ramanan is the Roland George Dwight Richardson University Professor of Applied Mathematics at Brown University. Her email address is kavita_ramanan@brown.edu.

Opening photo is Ruth Williams in 2016.

Communicated by Notices *Associate Editor Scott Sheffield.*

For permission to reprint this article, please contact: reprint-permission@ams.org.

DOI: https://doi.org/10.1090/noti2437

Introduction

When Ruth Williams enrolled at the University of Melbourne, Australia, as an undergraduate pursuing an honors BSc in mathematics, she launched a stellar mathematical career that spans five decades and is still going strong. After completing a second (research masters) degree in mathematics from Melbourne, she crossed the Pacific to begin her PhD studies at Stanford University. There were three women in her PhD cohort; by a twist of fate, all three took an early reading course from Sam Karlin, and all three pursued dissertations related to probability theory.

Figure 1. Some members of the Bendigo Computer Club, circa 1970. Photo provided by Ruth Williams (center).

Williams took a number of probability courses from Kai Lai Chung at Stanford. He posed an open problem about stopped Feynman–Kac functionals and the reduced Schrödinger equation which he had solved in one dimension; she realized that, using methods from PDE, she could solve the higher-dimensional open question, which led to her first single-authored publication as a PhD student. Chung became her advisor. While she did not pursue further research in the direction of Schrödinger equations, she found stochastic processes particularly appealing: their study involves rigorous analysis, and they arise naturally in a wide range of applications. While taking a course from a young professor in the Stanford Business School, Michael Harrison, she learned about reflecting Brownian motion (RBM): a diffusion process constrained to stay inside a region by "reflecting" at the boundary, with many associated challenging open problems (at the time). During the remainder of her time at Stanford, and the following year (1983–84) as a Postdoctoral Visiting Member at the Courant Institute working with S. R. Srinivasa Varadhan, she worked on foundational theory for Brownian motion with oblique reflection in a wedge. This set the stage for the nature of much of her future work—development of rigorous theory motivated by applications.

She was recruited to UC San Diego by its historically very strong group in stochastic processes, anchored by Ron Getoor and Michael Sharpe.[1] She began her University of California career in a lively fashion: on her first campus visit, she was serendipitously "interviewed" by Paul Erdős (who spent a substantial amount of time in San Diego during this period).

Nearly four decades later, Ruth Williams is still at UC San Diego, where she is now a Distinguished Professor and holds the Charles Lee Powell Chair in Mathematics I. She is one of the most celebrated active probabilists in the world. Early recognition of her work came in the form of an Alfred P. Sloan Fellowship (1988–1992) and an NSF Presidential Young Investigator award (1987–1994), followed by an NSF Faculty Award for Women (1991–1997). Her fundamental contributions to 20th century probability theory—in particular stochastic processes—were honored with an invited talk at the 1998 International Congress of Mathematicians in Berlin, and with the prestigious Guggenheim Fellowship (2001–2002). She has had continuous NSF support since 1984.

Her accomplishments are so widely recognized that she has received highest honors from five professional associations: in addition to being an Inaugural Fellow of the AMS (2012), she is a Fellow of the Institute of Mathematical Statistics (1992), the American Association for the Advancement of Science (1995), the Institute for Operations Research and Management Sciences (2008), and the Society for Industrial and Applied Mathematics (2020). In 2016 she was awarded, jointly with Martin Reiman,[2] the highly prestigious John von Neumann Theory Prize from INFORMS, for seminal research contributions to the theory and applications of stochastic networks and their heavy traffic approximations.

Williams was elected to the American Academy of Arts and Sciences (2009), the National Academy of Sciences (2012), and was elected to be a Corresponding Member of the Australian Academy of Science (2018). She has been awarded honorary doctorates by the University of Melbourne and by La Trobe University, both in Melbourne, Australia.

Her research is interdisciplinary, involving the development of fundamental mathematical theory in order to provide insight into real-world phenomena from a variety of fields, ranging from communication networks to (more recently) systems biology. She has published more than eighty papers and two cornerstone books. Her 2006 textbook *Introduction to the Mathematics of Finance*, published by the AMS, is widely used in graduate courses in mathematical finance. Her 1983 book *Stochastic Integration* with Kai Lai Chung was, when it first appeared, the most comprehensive and comprehensible treatment of the subject, and it remains a highly regarded and widely used source today (the second author used the 2nd edition, published in 1990, as recently as 2019 as a primary source to teach a

[1] *UC San Diego seems to have made some excellent hires in 1983: Jim Agler and S. T. Yau also joined the faculty that year.*

[2] *Reiman was also a Stanford PhD student, supervised by Michael Harrison several years before Williams.*

popular advanced graduate course on stochastic differential equations). In addition to her transformative research, Ruth Williams is also widely known for the outstanding quality of her expository work: she has written several survey papers [Wil95, Wil16] that serve as introductions to the field and describe important open problems, and which have stimulated further research, including some early works by the third author. Williams is also a dynamic and highly skilled public speaker. In addition to her invited ICM address, she has given a long series of prestigious invited lectures such as a Plenary AMS Invited Address in 1994, the Markov Lecture of the Applied Probability Society in 2007, the Doob Lecture at the 2011 meeting on Stochastic Processes and their Applications, and the Le Cam Lecture at the IMS annual meeting in Vilnius, Lithuania in 2018.

Given the breadth and depth of Ruth Williams' work, along with the fact that she continues to be very active, it would be impossible to provide an exhaustive overview in this (or indeed any) article. Here, we will present the main themes of her prodigious career and highlight some of her most influential contributions. In the broadest possible terms, Ruth Williams has made foundational contributions to the understanding of *reflecting diffusions* and, using these as tools, has dramatically advanced the scientific understanding of a wide array of *stochastic networks* experiencing *heavy traffic*, by approximating them using rigorous probabilistic scaling limits.

Scaling limits. Probability theory has enjoyed over a century of remarkable success in analyzing and predicting the behavior of very complex systems. One reason is the central idea of scaling limits: encoding main parameters of a random system and scaling them together (in different proportions) to identify more tractable limit objects that can be analyzed more directly, and which then serve as useful approximations of the original system. This paradigm shows up in the first major theorems that are the capstone of any introductory probability course at the graduate or undergraduate level: the Strong Law of Large Numbers and the Central Limit Theorem. In their simplest form, these state that the *empirical average* of n independent identically distributed (i.i.d.) random variables $\{X_i\}_{i\in\mathbb{N}}$ converges, with probability one, to their common deterministic mean m (whenever the latter is well-defined) as $n \to \infty$, while in the case when X_i has finite variance, the *centered and rescaled sum* (divided by $\sqrt{n}$ instead of n) converges to an object that is still random: a normal random variable. In other words, taken together, these classical limit theorems show that empirical averages concentrate about their deterministic common mean, and have Gaussian fluctuations on the scale $n^{-1/2}$.

There are two versions of these core theorems that give different perspectives, which are relevant to the present story. Instead of averaging random variables X_i directly, consider their *empirical distribution* $\nu_n = \frac{1}{n}\sum_{i=1}^{n}\delta_{X_i}$, which is a (random) probability measure-valued statistic that places equal mass at each point X_i. If the common law of the random variables is μ, then the Strong Law of Large Numbers tells us that ν_n converges *weakly in distribution* to μ almost surely, meaning that for each real-valued test function $f : \mathbb{R} \to \mathbb{R}$, $\int f\,d\nu_n$ converges to $\int f\,d\mu$ with probability one. The Central Limit Theorem then states that the fluctuations $\int f\,d\nu_n - \int f\,d\mu$ are of order $n^{-1/2}$, and $n^{1/2}[\int f\,d\nu_n - \int f\,d\mu]$ converges to a centered normal random variable (with variance $\int f^2\,d\mu - (\int f\,d\mu)^2$). (The original statements of these limit theorems mentioned above correspond to the special case $f(x) = x$.)

The above limit theorems concern real- or measure-valued random variables; we can consider analogous scaling limits for random elements of more exotic state spaces, such as paths (with some regularity, like continuity or at least right continuity with finite left limits) in some metric space. Such continuous-time stochastic processes model the evolution of dynamical systems that can have random influences. In this context, there are *functional* laws of large numbers and central limit theorems, with the latter also being referred to as *invariance principles*.

Let $\{X_i\}_{i\in\mathbb{N}}$ be i.i.d. *standardized* random variables (having mean zero and variance one), and denote $S(n) = X_1 + \cdots + X_n$. We can connect the dots (linear interpolation from $S(n-1)$ to $S(n)$) to create a piecewise affine random path $(S(t))_{t\geq 0}$. In this formulation, the two functional limit theorems can be phrased in terms of *rescaling space and time* in different proportions. For the Strong Law of Large Numbers, the statement is simply that, for any $t > 0$, $\lim_{r\to\infty} S(rt)/r = 0$ with probability one. (Had we not centered the random variables, the limit here would be the deterministic drift process $t\cdot m$ where m is the common mean of the random variables X_i.) The Central Limit Theorem in this context, known as Donsker's invariance principle, uses the different scaling $S^{(r)}(t) = S(rt)/\sqrt{r}$; here, the stochastic processes $S^{(r)}$ converge (weakly in distribution) as $r \to \infty$, to Brownian motion $B = (B(t))_{t\geq 0}$, the central Gaussian object in stochastic processes.

Brownian motion has quadratic scaling: for any $r > 0$, the new process $B^{(r)}(t) = B(rt)/\sqrt{r}$ is also a Brownian motion (it has the same law on path space). For this reason, one could just as well do the scaling $S(r^2t)/r$ in Donsker's theorem, going further out in time. There is a subtle but important difference when taken in concert with the law of large numbers, however: the pair $(S(rt)/r, S(rt)/\sqrt{r})$ scales differently from the pair $(S(rt)/r, S(r^2t)/r)$. In this example, it doesn't matter. In more complex examples the choice of whether to contract the space scaling or accelerate the time scaling can, in some circumstances, yield different results;

moreover, the latter can sometimes be more useful. This was a key insight, originally due to Michael Harrison, that played an important role in some of Ruth Williams' scaling limit theorems described below.

For more general stochastic processes (not arising from i.i.d. data), the law of large numbers kind of limit where space and time are scaled at the same rate is called a *fluid* or *hydrodynamic* limit, while scaling time with the square of the spatial rate is often referred to as a *diffusion* limit. A major theme of Ruth Williams' research program throughout her illustrious career has revolved around fluid and diffusion limits of a class of stochastic processes (discrete, continuous, or even measure-valued) that model *multiclass queueing networks* and more general *stochastic processing networks*. In order to provide some context for her work, we start by describing multiclass queueing networks, heavy traffic limits, and reflecting Brownian motions (RBMs).

Multiclass queueing networks. Queueing systems arise as models in a variety of applications, including computer systems, communication networks, transportation, service systems, and complex manufacturing systems. More recently, they have also been used by Ruth Williams in systems biology, for example as models of enzymatic processing. A *multiclass queueing network* consists of a fixed set of nodes (or stations), at which there are entities or jobs, which could represent customers or packets of data to be processed, and a server (or a pool of statistically homogeneous servers) capable of processing those jobs. The jobs at each node may belong to one of a finite number of types or classes depending on their arrival characteristics, service requirements, and routing needs, all of which may be random. A node is sometimes also referred to as a queue, which comprises the server(s), the jobs being processed, and the jobs awaiting processing at that node. If there is more than one class of job at a node, the node is called a multiclass queue; otherwise, it is called a single-class queue. Similarly, if there are multiple servers at a node, then it is referred to as a many-server queue; if not, it is said to be a single-server queue. When a job has finished service at a node, it either departs the system or changes class via a routing mechanism, which may be probabilistic. Networks in which jobs eventually leave the system are referred to as open networks. Networks can also differ in terms of the "service discipline" or protocol used by a server to process entities at its node. For example, under a head-of-the-line (or HL) protocol, entities of the same class that are awaiting service at a node are processed in the order in which they arrived to that node.

Quantities of interest in such systems include conditions for stability of the network dynamics, statistics of queue lengths of different classes of jobs at different nodes, the workload at each node (which is the amount of server effort required to serve all of the jobs at that node),

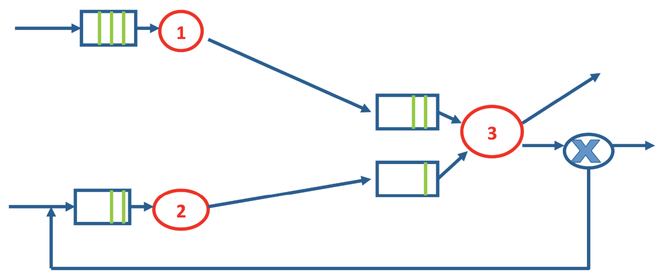


Figure 2. An example of a multiclass queueing network.

probabilities of critical rare events, and steady state or equilibrium distributions of these quantities. Starting with the Danish engineer A. K. Erlang in 1917, early work in queueing theory focused on exact closed form expressions for various statistics related to single-class queues. The first general results for networks were obtained by Jackson for open networks of single-server single-class HL queues with Markovian routing, where exogenous arrivals to each node are described by independent Poisson processes, jobs have independent exponentially distributed service times, and the service rate at each node is a function of the queue size. In particular, in 1963, Jackson showed that the equilibrium distribution of such networks has an explicit product form, which implies that in equilibrium, the numbers of jobs in distinct queues are independent. This was later generalized by several authors including Baskett et al. (1975) and Kelly (1979), who identified special classes of multiclass queueing networks that also have product form stationary distributions.

Heavy traffic limits and RBMs. However, beyond these special cases, typically it is not possible to compute performance measures of even HL multiclass queueing networks with general arrival processes and service distributions exactly. A particular regime of interest from an operations point of view is the so-called *heavy traffic regime*, where networks are congested or near capacity in the sense that the rate at which work is input to the system is approximately balanced by the capacity of the system to process that work. At such near-equilibrium regimes, performance can be strongly influenced by stochastic variability. Although early work of Kingman, Borovkov, and Prohorov in the early 1960s established approximations for steady-state distributions or finite-dimensional distributions of single-class queues, Iglehart and Whitt (1970) were the first to consider a functional heavy traffic approximation for a HL single-class (multi-server) queue, showing that a suitably rescaled job count process converges in distribution to a diffusion limit that is a so-called *reflecting Brownian motion* (RBM).

Standard Brownian motion takes both positive and negative values almost surely, and so it is not a good limit model for any random quantity that is by definition positive (like a queue length or workload process). Instead, reflecting Brownian motion is a process whose increments coincide with that of Brownian motion on intervals when the process is positive, but is then modified when it hits zero. In fact, as shown by Skorokhod in 1961, one-dimensional reflecting Brownian motion Z can be represented as

$$Z(t) = B(t) + L(t), \tag{1}$$

where B is a standard Brownian motion and $L(t) \coloneqq \sup_{s\in[0,t]} \max(-B(s), 0)$ is (proportional to) the so-called Brownian *local time*, which characterizes the amount of time Brownian motion spends near zero.

The construction in (1) yields the reflecting Brownian motion Z as a continuous function of the driving Brownian motion B. In the case where B is a standard one-dimensional Brownian motion, by a theorem of Lévy, the process Z has the same distribution as the "reflected" or absolute value process $|B|$; this is where the terminology "reflecting Brownian motion" comes from. This equivalence no longer holds true for Brownian motion with a drift, and in many higher-dimensional contexts, although the name is still used. As will be evident from the more precise definition given below, it is more accurate to think of a RBM or more general reflecting stochastic process as a process whose increments behave like those of the original process on the time intervals when the reflecting process lies in the interior of the state space, but is then suitably constrained to live within (the closure of) a domain (which is the non-negative reals in the one-dimensional case).

Skorokhod's idea was extended by Harrison and Reiman to study heavily loaded networks of single-class queues. In this case, each coordinate of the limit process represents the queue length at a node, and so the limit process must lie in the positive orthant. In 1981, Harrison and Reiman developed a multi-dimensional analog of the Skorokhod map in the positive orthant, and subsequently, Reiman exploited its continuity properties to show that the heavy traffic limit of open single-class HL queueing networks (with generally distributed interarrival and service times with finite moment conditions) is a reflecting Brownian motion in the orthant. Furthermore, their definition guaranteed that the process is a semimartingale, which means that it admits a decomposition as the sum of a (local) martingale and an adapted process that is (locally) of bounded variation. The semimartingale property is useful because it allows an easy application of stochastic calculus to study the evolution of sufficiently regular functionals of the process. The Skorokhod map is useful in that it is pathwise and, when continuous, it defines what is known as a strong solution to the corresponding stochastic differential equation with reflection (which means that the solution is measurable with respect to the filtration generated by the driving Brownian motion). However, it turns out that the Skorokhod map may fail to be well-defined or continuous for data associated with multiclass queueing networks and more general stochastic processing networks. An alternative is to consider distributional, rather than pathwise, limits and to characterize RBMs using the so-called submartingale problem introduced by Stroock and Varadhan in the 1970s to study (weak solutions to) stochastic differential equations with reflection in smooth domains with smooth boundary conditions.

Figure 3. Ruth Williams, Michael Harrison, and Jim Dai, at a conference in honor of Michael Harrison, 2009.

Ruth Williams' Contributions

Ruth Williams' mathematical career has centered on developing methodologies for the analysis of stochastic processing networks, proving hydrodynamic and heavy traffic limit theorems that yield fluid and diffusion approximations, and analyzing these approximations. Ruth Williams' most influential early work [VW85, Wil87, RW88, TW93, DW94] focused on developing the foundations of RBM in the orthant with discontinuities in the oblique reflection field at the boundary interfaces. At the time, there was limited theory for such non-smooth, non-symmetric situations, where novel behavior such as hitting corners can occur (in contrast to Brownian motion which hits individual points with probability zero in dimensions greater than one). In applications to queueing networks, the oblique reflection directions arise from routing in the network, and the orthant state space represents the fact that queue lengths are always non-negative with intersections of faces corresponding to several queues being empty simultaneously. This work on RBMs is beautifully summarized in the survey paper [Wil95], in which Williams succinctly defines RBMs in such domains, and discusses existence and uniqueness in law and characterizations of stationary distributions.

After establishing the foundations for these RBMs and the appearance in the early 1990s of surprising examples showing that the stability and heavy traffic behavior of multiclass queueing networks are more intricate than that of single-class queueing networks, Ruth Williams turned to establishing invariance principles and heavy traffic limit theorems for multiclass queueing networks. An excellent short survey is in [Wil98a] (the paper accompanying her 1998 ICM talk), which describes the general modular framework she developed (with Maury Bramson) for establishing sufficient conditions for heavy traffic limit theorems for HL multiclass queueing networks.

Subsequently, at the turn of the century, she started analyzing more general stochastic processing networks, including those with resource sharing, such as in processor sharing and bandwidth sharing networks [GPW02, KW04, KKLW09, MPW19, PW16, FW21]. Often in resource sharing, service is shared amongst all entities and one needs to keep track of more information than queuelengths to describe the dynamics; also, these are non-head-of-the-line (non-HL) networks. This presents new mathematical challenges, which Williams overcame by introducing measure-valued stochastic processes to represent the dynamics of these networks and by developing new techniques for proving hydrodynamic and heavy traffic limit theorems for them.

Over the last fifteen years, catalyzed by participation in a meeting at the Institute for Mathematics and its Applications (IMA), Williams has also expanded her research to include applications in systems biology [MHTW10, LW19, AHLW19].

In what follows, we describe some of her research contributions in greater detail.

(i) Reflecting Brownian motion (RBM). We start by defining reflecting Brownian motions in domains with piecewise smooth boundaries [KW07]. Let $\{G_i\}_{i\in\ell}$ be a finite collection of open subsets of $\mathbb{R}^d$, each with continuously differentiable boundary, and let $G = \bigcap_{i\in\ell} G_i$. Fix vector fields $\gamma^i : \mathbb{R}^d \to \mathbb{R}^d$, $i \in \ell$. A *semimartingale reflecting Brownian motion* (SRBM) Z is a stochastic process on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ taking values only in $\overline{G}$, which has a decomposition of the form

$$Z(t) = X(t) + \sum_{i\in\ell} \int_{(0,t]} \gamma^i(Z(s))\, dY_i(s),$$

where X is a Brownian motion in $\mathbb{R}^d$ with respect to the filtration $(\mathscr{F}_t)_{t\geq 0}$ (with initial distribution supported in $\overline{G}$, and some fixed drift and covariance), and each Y_i is a continuous, adapted, non-decreasing process that only increases at times s when $Z(s) \in \partial G \cap \partial G_i$ (i.e., Z lies on the corresponding part of the boundary of the domain). In the one-dimensional setting where $G = (0, \infty)$, the process $Y = Y_1$ is the Brownian local time L and the vector field is simply $\gamma^1(x) = 1$ pointing into the region. In general, there is no reason to assume that the vector field γ^i is a normal vector field on ∂G_i. In particular, the geometry of the directions of reflection that arise in heavy traffic limit theorems for queueing networks is dictated by the routing structure in the network, and generally leads to *obliquely* reflecting Brownian motions. Such reflecting Brownian motions are related to elliptic PDE with oblique derivative boundary conditions in much the same way that Brownian motion is related to the Laplace equation.

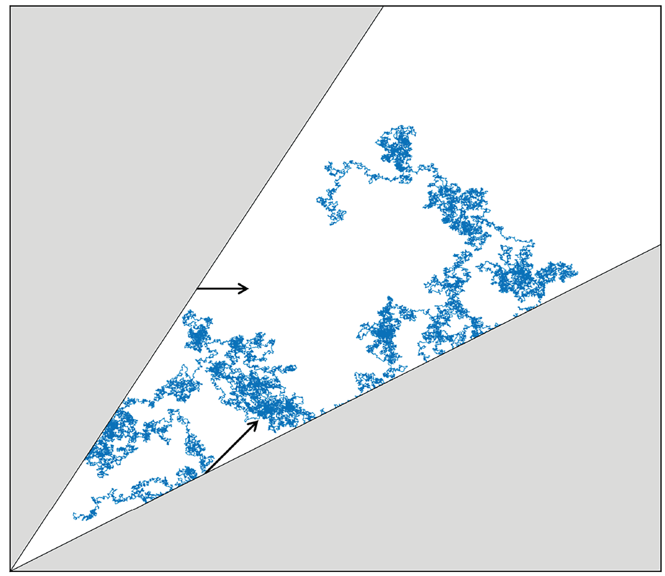


Figure 4. A simulation of a two-dimensional RBM (reflecting Brownian motion) in a wedge, with oblique reflection field. This simulation was provided by Prof. Xinyun Chen at the School of Data Science (SDS) in the Chinese University of Hong Kong, Shenzhen.

It is far from obvious that SRBMs should exist, and indeed some natural conditions on the domain G and the vector fields γ^i are required. The vector fields must be sufficiently regular and must, in a general sense, "point inward" on the boundary to have any chance of pushing the Brownian motion back into G when it tries to escape. More precisely, at each point $x \in \partial G$, some convex combination of the vectors $\gamma^i(x)$ for $i \in \ell$ such that $x \in \partial G \cap \partial G_i$ should point inward into G. In addition, the vector fields should not stray "too far" from the unit normal field, in a broad sense—this is to guarantee that the process does not oscillate too wildly near boundary intersections to be reflected in a meaningful way.

A special case of broad interest is when G is a polyhedral domain in the positive orthant, and the vector fields are constant on each face. In that case, the definition becomes somewhat simpler: $Z = X + RY$ where $Y = [Y_1, \ldots, Y_d]^\top$ as above and R is a $d \times d$ matrix, called the *reflection matrix* (or, more accurately, constraint matrix), whose columns are the (constant) vector fields. When G is the entire positive orthant, Ruth Williams and her student Lisa Taylor [TW93] identified sufficient conditions on the vector fields for the (weak) existence and uniqueness in law of SRBMs, and showed that the conditions were also necessary in a paper with Martin Reiman [RW88]. The technical conditions on the vector fields can be stated succinctly in an algebraic form that R is a *completely-$\mathcal{S}$ matrix*, which means that for every principal submatrix $\tilde{R}$ of R, there is a vector $\tilde{y}$ in the positive orthant for which $\tilde{R}\tilde{y}$ lies in the positive orthant. This is a subtle result. Firstly, it should be noted that this condition is only necessary for a *semimartingale* RBM to exist; one can still have a well-posed RBM that is not a semimartingale when the completely-$\mathcal{S}$ condition fails and such non-semimartingale RBMs can also arise as heavy traffic limits of multiclass queueing networks. Moreover, the RBM constructed here is what is known as a weak solution to the stochastic differential equation with reflection. A longstanding open question that is still unresolved is whether strong solutions also exist under this condition.

Constraining Brownian motion to stay in a region by pushing it in the allowed "reflection" or constraint directions at the boundary can be thought of as a stochastic control problem, with highly singular controls. Proving that such processes exist and are unique in law is highly nontrivial. In the general piecewise smooth boundary case covered in [KW07], the proof of existence was tied together with the other side of the story: an invariance principle describing when an SRBM (or rather an *extended* SRBM, consisting of the triple (X, Y, Z)) arises as the diffusion scaling limit of a system $(X^{(r)}, Y^{(r)}, Z^{(r)})$ that only satisfies the boundary control approximately. (Again, the main technical hurdle is controlling oscillations at the boundary; achieving this even locally turns out to be enough to guarantee the requisite tightness for the diffusion limit to emerge.) Kang and Williams then proved the existence of such general SRBMs by exhibiting approximate extended systems and constructing the SRBM as their diffusion scaling limit.

(ii) Stationary distributions of RBMs. Williams simultaneously also initiated the study of the stationary distributions of SRBMs, which are Markov processes; this was natural given the importance of stationary measures for stochastic networks and Reiman's (1982) result on RBMs in the orthant arising as heavy traffic limits of open HL single-class networks. With Paul Dupuis [DW94], she obtained a general sufficient condition for the positive recurrence (or ergodicity) of SRBMs in the orthant, which reduced the problem to studying the long-time behavior of a deterministic constrained dynamical system (the "fluid" model) in the orthant. Next, in view of the fact that one-dimensional RBM is well-known to have a stationary distribution of exponential form and Jackson's result on product-form stationary distributions for a special class of open single-class networks, she set out to identify when SRBMs also exhibit analogous product-form or exponential stationary distributions.

In [HW87], Michael Harrison and Ruth Williams first studied this question for obliquely reflecting SRBMs on bounded domains with smooth boundaries governed by a smooth, possibly oblique, but non-tangential "reflection" vector field γ. Existence and uniqueness in law of such SRBMs follows from classical results of Stroock and Varadhan, and drawing on the classical connection between stationary distributions of reflected processes and elliptic PDE with (oblique) derivative boundary conditions, they showed that the RBM has an explicit stationary density with an exponential product form if and only if γ satisfies the following *skew-symmetry condition*: for all $x, \tilde{x} \in \partial G$,

$$\langle n(x), \gamma(\tilde{x}) - n(\tilde{x})\rangle + \langle \gamma(x) - n(x), n(\tilde{x})\rangle = 0,$$

where n denotes the inward normal vector field on the boundary ∂G, and $\gamma - n$ is the tangential part of γ.

For the case of RBMs (with covariance equal to the identity matrix) in a polyhedral domain in $\mathbb{R}^d$ with normal vector field n^i and a constant reflection vector field γ^i on the ith face, they also studied the formal analogue of the analytical PDE characterization of the stationary density that arises in the smooth case, dubbed it the *basic adjoint relation (BAR)*, and showed that the solution p of the BAR has the form

$$p(x) = \prod_{i=1}^{d} \exp(c_i x_i), \quad i = 1, \dots, d$$

for suitable real-valued constants $c_i, i = 1, \dots, d$, if and only if for any two distinct faces of G, labelled i and j,

$$\langle n^i, \gamma^j - n^j\rangle + \langle n^j, \gamma^i - n^i\rangle = 0.$$

In the particular case where the domain is the orthant and the associated reflection matrix R is normalized to have 1's along its diagonal, this reduces to saying that the matrix $R - I$ is skew-symmetric; here I is the $d \times d$ identity matrix. In all cases, the explicit dependence of the vector $c = (c_1, \dots, c_d)$ on the drift of the RBM was identified. In a separate paper [Wil87], Williams justified that the solution of the BAR is indeed the stationary density of the corresponding RBM. This was done by approximating the SRBMs in polyhedral domains with piecewise constant reflection vector fields by RBMs in certain approximating smooth domains with smooth vector fields, and showing that under the skew-symmetry condition, the RBM does not reach the non-smooth parts of the boundary. The latter property is of independent interest and in general fails when the skew-symmetry condition does not hold. These general results for RBMs have also been used in other applications, such as Atlas models in finance, which describe the evolution of equity markets in terms of rank-based stochastic differential equations.

(iii) Heavy traffic limits and multiplicative state space collapse. Although it was well-known that not all multiclass queueing networks could be approximated by SRBMs, Williams and Maury Bramson [Wil98a] laid out a modular approach to identifying classes of networks for which such an approximation is possible. Specifically, they identified general sufficient conditions under which such a limit theorem holds: first, the reflection matrix describing the SRBM must satisfy the completely-$\mathcal{S}$ condition and second, one must check whether a certain multiplicative state space collapse condition holds. As mentioned above, the completely-$\mathcal{S}$ condition guarantees well-posedness of the associated SRBM, and also enables proof of an invariance principle [Wil98b] that shows convergence of approximating processes to the SRBM under general conditions. The second condition is to establish what is known as *multiplicative state space collapse*, which is a generalization of the notion of state space collapse first considered by Reiman in 1984 and later used by Peterson in his 1991 work on heavy traffic limits for feedforward networks. Loosely speaking, state space collapse holds if, in diffusion scale, the job count process can be approximately recovered from the (typically lower-dimensional) workload process and the precision of this approximation becomes exact in the heavy traffic limit. The multiplicative version introduced by Bramson involves a normalization by the amount of work in the system. It is often easier to verify and can frequently be shown to imply state space collapse. Bramson and Williams also verified the sufficient conditions for several classes of networks including first-in-first-out (FIFO) networks of so-called Kelly type and networks with a HL proportional processor sharing service discipline. Taken together, these results represent a culmination of one and a half decades of focused effort by Ruth Williams to develop the requisite mathematical theory to identify and rigorously justify heavy traffic approximations of several families of multiclass queueing networks.

(iv) Resource sharing in stochastic processing networks. Having brought some measure of order to the understanding of HL multiclass queueing networks, at the turn of the century Ruth Williams started studying resource sharing problems in more general stochastic processing networks. With her student Steven Bell, Williams considered dynamic scheduling (or control) for parallel server systems with HL scheduling policies, and then later shifted her focus to the study of the non-HL Processor Sharing (PS) scheduling policy, and more general bandwidth sharing networks. The PS protocol, in which each server at any time divides its processing capacity equally amongst all jobs present in the queue at that time, seeks to provide an egalitarian allocation of a scarce resource among competing users and is an idealization of the round-robin protocol in time sharing computer systems.

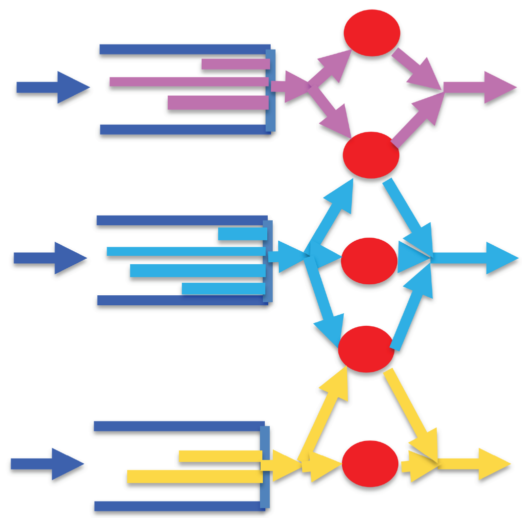


Figure 5. An example of a bandwidth sharing stochastic processing network. The split arrows indicate simultaneous resource possession.

There is a large body of literature on PS queues. However, with only rare exceptions, most of the literature imposes the stringent parametric assumptions of Poisson arrivals and/or exponential service requirements. Under these assumptions, the queue process is a Markov process in the sense that its instantaneous evolution at any time depends only on its current state (and not on the history), which greatly simplifies the analysis. Unfortunately, these assumptions are typically not satisfied in real-world applications.

In [GPW02], another paper written with her postdoctoral fellow Amber Puha, and the PhD thesis of her student Christian Gromoll, fluid and diffusion approximations were developed for PS queues with arrivals that form what is known as a renewal process, and jobs that have independent and identically distributed general service requirements. Since, under these more general distributions, the queue process on its own need no longer be Markovian, they introduced a measure-valued state representation that at any time t has a mass at the residual (remaining processing) service time of each job, from which one can recover traditional performance measures such as queue length and workload. They then established fluid and diffusion limit theorems for this measure-valued process.

These three papers together garnered the authors a "Best Publication Award" from the INFORMS Applied Probability Society in 2007. The citation stated that these papers "solve outstanding difficult problems, which advance the state of the art of Applied Probability." Ruth Williams' commitment to mentorship is evident from the fact that she told the last author of the present article at the time that the best thing about the award was that it was given jointly with her mentees.

Figure 6. INFORMS Best Publication Award prize ceremony, 2007. From left to right: Amber Puha, Christian Gromoll, Ruth Williams, Jim Dai.

These papers also served as the starting point for the study of bandwidth sharing communication networks. In 2000, Massoulié and Roberts introduced a connection-level model of Internet congestion control that represents the randomly varying flows in a network where bandwidth is shared fairly between file transfers, with fairness modulated by a parameter α. With Poisson arrivals and exponentially distributed file sizes, this model can be phrased as a multi-dimensional Markov chain in which the transition rates are solutions of concave optimization problems. Conditions for stability (positive recurrence) of this Markov chain were established early on, but characterizing the heavy traffic behavior was more challenging, because these are stochastic processing networks with simultaneous resource possession in which processing of files uses capacity from multiple resources simultaneously.

In 2001-02, while visiting Stanford on her Guggenheim fellowship, Ruth Williams initiated a collaboration on this problem with Frank Kelly, who was also visiting Stanford that year. Their goal was to obtain heavy traffic diffusion approximations for α-fair bandwidth sharing models by extending to this more complicated setting the approach developed earlier by Ruth Williams and Maury Bramson for multiclass queueing networks. First, in the work [KW04] with Kelly, Ruth Williams established long-time convergence of critical fluid model solutions to the set of invariant states. Then in [KKLW09], with Kelly and Ruth Williams' PhD students Weining Kang and Nam Lee, she used the asymptotic behavior of the fluid model to establish a dimension reduction called *multiplicative state space collapse*. Furthermore, in the case $\alpha = 1$, which corresponds to the natural case of proportional fair sharing of bandwidth, the multiplicative state space collapse property was combined with an invariance principle Ruth Williams established with W. Kang in [KW07] and her previous results on well-posedness of reflected diffusions in polyhedral domains, to show that the heavy traffic limit is a reflected diffusion in a polyhedral cone. In this case, it can also be deduced from previous work of Ruth Williams with Michael Harrison [HW87] that the stationary distributions of the heavy traffic limit are explicit and of product-form.

It should be emphasized that these limit theorems do not merely yield mathematical statements, but actually shed insight into the qualitative phenomenon of entrainment in these networks, whereby congestion at some resources may prevent other resources from working at their full capacity. Ruth Williams continues to work on this problem, with the ultimate goal to generalize these results to cover a more realistic version of this model that has generally distributed file sizes. In this case, the dynamics are represented by measure-valued processes, where understanding long-time behavior is much more complicated. Building on related works with Justin Mulvany and Amber Puha for the processor sharing model [MPW19, PW16], Ruth Williams and her PhD student Yingjia Fu have made recent progress on this subject. Specifically, in [FW21], Fu and Williams construct Lyapunov functions based on f-divergence (a generalization of relative entropy) to understand the long-time behavior of critical (measure-valued) fluid models in the presence of general file size distributions.

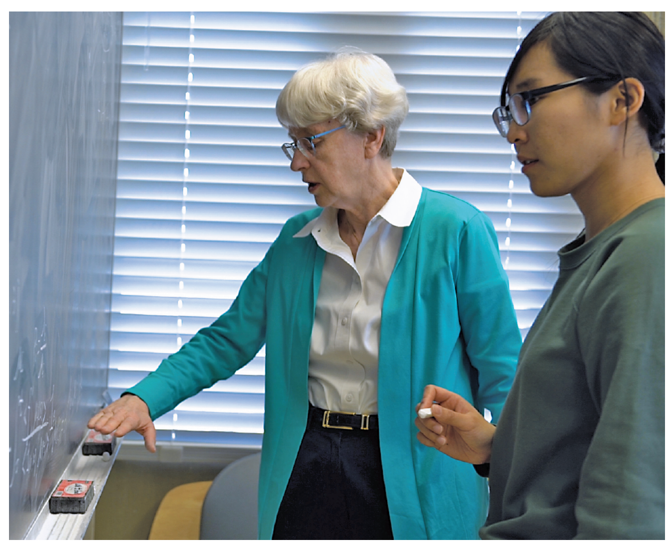


Figure 7. Ruth Williams working with her student Yingjia Fu, 2019.

(v) Constrained Langevin approximations for biochemical reaction networks. Key processes in chemical and biological systems are described by complex networks of chemical reactions, which are frequently not amenable to exact analysis. Classically, the evolution of molecular concentrations is often modelled by coupled systems of nonlinear differential equations, which can be justified via a functional law of large numbers, in the limit as the number of molecules of all species goes to infinity. However, in systems biology the concentrations of some constituent molecules can be low, and thus deterministic models are inadequate. A common stochastic model of chemical kinetics treats the system as a continuous time Markov chain that tracks the number of molecules of each chemical species, and quantities of interest are then approximated by Monte Carlo estimates using simulations of the sample paths. However, since each reaction is accounted for in this model, these simulations can become computationally prohibitive even for a modest number of species. When the number of molecules is moderately large (though still not sufficiently large to ignore stochastic fluctuations), this model is often replaced by solutions of associated stochastic differential equations (SDE), referred to as diffusion approximations, which can be simulated more efficiently. Two commonly used diffusion approximations are the so-called linear noise approximation, obtained by linearizing fluctuations around the deterministic approximation, and the chemical Langevin equation. However, both approximations have serious drawbacks. The linear approximation fails to capture fluctuations due to nonlinearities in the reaction rates and, unlike the Markov chain models, its solution can become negative, which is not physically meaningful. On the other hand, the Langevin equation is better at capturing nonlinearities and serves as a good approximation as long as it is valid, but since its coefficients involve square roots of the concentrations of the species, it is typically ill-posed beyond the first time any coordinate of the solution reaches zero.

Several alternative models to deal with this negativity issue were proposed, including other Langevin-type models as well as hybrid methods that tried to combine the accuracy and robustness of the Markov chain models with the computational efficiency of diffusion approximations. Ruth Williams realized that some of the fixes unnecessarily perturb the *global* dynamics to deal with what is inherently a *local* issue (near the boundary of the orthant); she instead proposed a *constrained Langevin approximation*, which is an obliquely reflected diffusion in the orthant satisfying the non-negativity constraints of the component processes [AHLW19, LW19]. She presented preliminary results on this work as part of her Kolmogorov lecture at the World Congress in Probability and Statistics in July 2016. As demonstrated there, this approximation agrees with the chemical Langevin approximation until the first time any component goes negative, but is well-defined for all time and performs better than the existing approximations. Subsequently, Ruth Williams and Saul Leite rigorously showed that this reflected diffusion process arises as the weak limit of a sequence of jump-diffusion Markov processes that mimic the Langevin system in the interior and behave like a scaled version of the Markov chain on the boundary [LW19], which in particular required generalizing previous results on well-posedness of reflecting diffusions.

Continuing Legacy

Through her extraordinary continuing career, Ruth Williams has left a large imprint on probability theory and on mathematics in general. Her influence has been felt not only through her groundbreaking research, but through her direct involvement in the community. She has advised eleven PhD students (all of whom graduated from UC San Diego) and she is currently advising three more. She has supervised many postdoctoral fellows, masters students, and undergraduates (at UC San Diego). The research work that she did with her advisees and mentees has earned many accolades, some of which were highlighted above, and others are too numerous to mention.

Another constant in Ruth Williams' career has been her unwavering commitment to supporting and promoting women and underrepresented minorities. From organizing and speaking at women-centered and AWM-sponsored mathematical conferences, to extensive mentorship of junior colleagues and involvement in university-wide postdoctoral initiatives at the University of California, she has always been a strong advocate for the advancement of underrepresented groups in mathematics and science. In recognition of her dedication to this cause, INFORMS presented her with the prestigious Award for the Advancement of Women in Operations Research and Management Sciences (2017).

In conjunction with her many research accomplishments and accolades, Williams has provided a truly astonishing array of service to her department, to UC San Diego, and to the international mathematics and scientific communities. A complete list would go on for pages; we mention only a few highlights here. She has devoted decades to editorial boards of highly respected journals such as *Annals of Applied Probability, Electronic Journal of Probability* and *Electronic Communications in Probability*, and *Mathematics of Operations Research*. She has served on the Council (2003–2006) and as President (2011–2012) of the IMS (Institute of Mathematical Statistics). As IMS President, she spearheaded the effort to become an Associate Member of ICIAM (the International Council for Industrial and Applied Mathematics) to foster stronger ties to the applied mathematics community. In order to be more welcoming to junior researchers, she also arranged for tutorials to be added to the annual SSP (Seminar on Stochastic

Processes); in particular, she created a subcommittee of the IMS New Researchers Committee to suggest speakers for the SSP tutorials.

She was on the Bernoulli Society Council (2001–2004) and on the Board of Governors for the Institute for Mathematics and its Applications (2003–2006). She served as Chair of the Joint Program Committee for the 7th World Congress in Probability and Statistics (2008). She helped found the Steering Committee of the *Stochastic Networks* conference series initiated by Peter Glynn, Thomas Kurtz, and Peter Ney. She was a member of the Governing Board for the Australian Mathematics Research Institute, MATRIX (2015–2020). She currently serves on the Governing Council and the Executive Committee of the National Academy of Sciences. She has profoundly broken the stereotype partitioning mathematicians into those who are talented at research and mentorship and those who are devoted to service; Ruth Williams is a paragon of the mentor-scholar-academic.

Figure 8. Group photo from a conference in honor of Ruth Williams at the IMA, 2016. `https://www.ima.umn.edu/2014-2015/SW6.25-27.15`.

In San Diego, Ruth Williams met and married Bill Helton: a fellow UC San Diego mathematician who, like her, straddles the divide between pure and applied mathematics. These days, they enjoy spending their leisure time outdoors, gardening or hiking. He has been her constant companion and, in recent years, occasional collaborator. As it happens, her initial forays into systems biology applications included her first paper coauthored with Gheorghe Craciun and Bill Helton, on homotopy methods for counting equilibria in dynamic models of chemical reaction networks.

Williams is as active as ever, finding new ways to use mathematics to explain the world around us. Her current major research interests include stochastic models in systems biology, and entropy methods in the analysis of stochastic processing networks. On the first front, she has collaborated with the biodynamics lab at UC San Diego, led by Jeff Hasty and Lev Tsimring, on enzymatic processing networks. In connection with this area, she has worked with a PhD student, David Lipshutz, on (stochastic) differential delay equations relating to delayed protein degradation. Stochastic modeling of genetic circuits holds the promise of new understanding in cellular and molecular biology, a rapidly expanding quantitative field. She is currently collaborating with Domitilla Del Vecchio and Ron Weiss at MIT on stochastic modeling of epigenetic cell memory.

On the second front, Williams' current work using entropy-like notions has been very fruitful in analyzing fluid limits of certain non-HL systems: bandwidth sharing networks. These constitute just one of a huge number of non-HL real world networks, and there are many reasons to believe the Lyapunov approach can help understand these. This has the potential to make a huge impact on the field, since the relationship between bandwidth sharing models and more general non-HL stochastic processing networks is analogous to the relationship between bananas and non-banana fruits. Ruth Williams will no doubt leave a lasting mark on these problems—as she is fond of saying, "I eat problems for breakfast."

References

[AHLW19] David F. Anderson, Desmond J. Higham, Saul C. Leite, and Ruth J. Williams, *On constrained Langevin equations and (bio)chemical reaction networks*, Multiscale Model. Simul. **17** (2019), no. 1, 1–30, DOI 10.1137/18M1190999. MR3895328

[DW94] Paul Dupuis and Ruth J. Williams, *Lyapunov functions for semimartingale reflecting Brownian motions*, Ann. Probab. **22** (1994), no. 2, 680–702. MR1288127

[FW21] Y. Fu and R. J. Williams, *Asymptotic behavior of a critical fluid model for bandwidth sharing with general file size distributions*, 2021. To appear.

[GPW02] H. Christian Gromoll, Amber L. Puha, and Ruth J. Williams, *The fluid limit of a heavily loaded processor sharing queue*, Ann. Appl. Probab. **12** (2002), no. 3, 797–859, DOI 10.1214/aoap/1031863171. MR1925442

[HW87] J. M. Harrison and R. J. Williams, *Brownian models of open queueing networks with homogeneous customer populations*, Stochastics **22** (1987), no. 2, 77–115, DOI 10.1080/17442508708833469. MR912049

[KKLW09] W. N. Kang, F. P. Kelly, N. H. Lee, and R. J. Williams, *State space collapse and diffusion approximation for a network operating under a fair bandwidth sharing policy*, Ann. Appl. Probab. **19** (2009), no. 5, 1719–1780, DOI 10.1214/08-AAP591. MR2569806

[KW04] F. P. Kelly and R. J. Williams, *Fluid model for a network operating under a fair bandwidth-sharing policy*, Ann. Appl. Probab. **14** (2004), no. 3, 1055–1083, DOI 10.1214/105051604000000224. MR2071416

[KW07] W. Kang and R. J. Williams, *An invariance principle for semimartingale reflecting Brownian motions in domains with piecewise smooth boundaries*, Ann. Appl. Probab. **17** (2007), no. 2, 741–779, DOI 10.1214/105051606000000899. MR2308342

[LW19] Saul C. Leite and Ruth J. Williams, *A constrained Langevin approximation for chemical reaction networks*, Ann. Appl. Probab. **29** (2019), no. 3, 1541–1608, DOI 10.1214/18-AAP1421. MR3914551

[MHTW10] W. H. Mather, J. Hasty, L. S. Tsimring, and R. J. Williams, *Correlation resonance generated by coupled enzymatic processing*, Biophys. J. **99** (2010), 3172–81.

[MPW19] Justin A. Mulvany, Amber L. Puha, and Ruth J. Williams, *Asymptotic behavior of a critical fluid model for a multiclass processor sharing queue via relative entropy*, Queueing Syst. **93** (2019), no. 3-4, 351–397, DOI 10.1007/s11134-019-09629-8. MR4032930

[PW16] Amber L. Puha and Ruth J. Williams, *Asymptotic behavior of a critical fluid model for a processor sharing queue via relative entropy*, Stoch. Syst. **6** (2016), no. 2, 251–300, DOI 10.1214/15-SSY198. MR3633537

[RW88] M. I. Reiman and R. J. Williams, *A boundary property of semimartingale reflecting Brownian motions*, Probab. Theory Related Fields **77** (1988), no. 1, 87–97. MR921820

[TW93] L. M. Taylor and R. J. Williams, *Existence and uniqueness of semimartingale reflecting Brownian motions in an orthant*, Probab. Theory Related Fields **96** (1993), no. 3, 283–317, DOI 10.1007/BF01292674. MR1231926

[VW85] S. R. S. Varadhan and R. J. Williams, *Brownian motion in a wedge with oblique reflection*, Comm. Pure Appl. Math. **38** (1985), no. 4, 405–443, DOI 10.1002/cpa.3160380405. MR792398

[Wil16] R. J. Williams, *Stochastic processing networks*, Annual Review of Statistics and Its Application **3** (2016), 323–345.

[Wil87] R. J. Williams, *Reflected Brownian motion with skew symmetric data in a polyhedral domain*, Probab. Theory Related Fields **75** (1987), no. 4, 459–485, DOI 10.1007/BF00320328. MR894900

[Wil95] R. J. Williams, *Semimartingale reflecting Brownian motions in the orthant*, Stochastic networks, IMA Vol. Math. Appl., vol. 71, Springer, New York, 1995, pp. 125–137. MR1381009

[Wil98a] R. J. Williams, *Diffusion approximations for open multiclass queueing networks: sufficient conditions involving state space collapse*, Queueing Systems Theory Appl. **30** (1998), no. 1-2, 27–88, DOI 10.1023/A:1019108819713. MR1663759

[Wil98b] R. J. Williams, *An invariance principle for semimartingale reflecting Brownian motions in an orthant*, Queueing Systems Theory Appl. **30** (1998), no. 1-2, 5–25, DOI 10.1023/A:1019156702875. MR1663755

Ioana Dumitriu

Todd Kemp

Kavita Ramanan

Credits

Opening photo is courtesy of Erik Jepsen/UC San Diego Publications.

Figures 1, 2, 3, and 5 are courtesy of Ruth Williams.

Figure 4 is courtesy of Xinyun Chen.

Figure 6 is courtesy of Marty Reiman.

Figure 7 is courtesy of UC San Diego Publications.

Figure 8 is courtesy of the Institute for Mathematics and its Applications.

Photo of Ioana Dumitriu is courtesy of Alex Matthews/UCSD.

Photo of Todd Kemp is courtesy of UC San Diego Publications.

Photo of Kavita Ramanan is courtesy of Paul Rochford at Brown University.

Gems from the Work of Georgia Benkart

Tom Halverson and Arun Ram

Georgia Benkart completed her PhD in 1974 at Yale University, where she was the 30th of Nathan Jacobson's 34 PhD students. From there she joined the faculty at the University of Wisconsin–Madison, where she is now Professor Emerita. Since her retirement from teaching she has provided tremendous service to the mathematical community, notably as President of the AWM and as an Associate Secretary of the AMS for more than a decade.

In this article we highlight a few selected gems from her extensive contribution to our field, organized in a roughly chronological sequence of vignettes and images (which can be read or viewed in any order). Our hope is that we can capture and transmit a snapshot of Georgia's rich mathematics, beautiful style, and wonderful mathematical personality.

Tom Halverson is the Armstrong Professor of Mathematics at Macalester College. His email address is `halverson@macalester.edu`.

Arun Ram is a professor of mathematics at the University of Melbourne. His email address is `aram@unimelb.edu.au`.

Communicated by Notices *Associate Editor Steven Sam.*

For permission to reprint this article, please contact: `reprint-permission@ams.org`.

DOI: https://doi.org/10.1090/noti2447

Classifying simple Lie algebras. In algebra in 1974, the air was thick with the classification of finite simple groups, with new finite simple groups being discovered in a frenzy, and the question always in the air:

"Have we found them all?"

At that time there was another such classification effort beginning: a search for all of the finite-dimensional simple Lie algebras.

In characteristic 0 the problem had been completed by Cartan and Killing around 1894, resulting in the list of Dynkin diagrams (Figure 1), which are in bijection with the finite-dimensional simple Lie algebras. Over an algebraically closed field of characteristic $p > 7$, four additional series occur:

- the Witt Lie algebras $W(m, \underline{n})$,
- the special Lie algebras $S(m, \underline{n})^{(1)}$,
- the Hamiltonian Lie algebras $H(2m, \underline{n})^{(2)}$,
- the contact Lie algebras $K(2m+1, \underline{n})^{(1)}$.

The monograph by Benkart, Gregory, and Premet [BGP09] provides complete details on these algebras. They are known as the generalized Cartan-type Lie algebras, because they are derived from Cartan's four infinite families (Witt, special, Hamiltonian, contact) of *infinite-dimensional* complex Lie algebras. Cartan's work set the stage for Kostrikin–Šafarevič [KŠ66], who identified the above four unifying families of simple Lie algebras living in the Witt algebras. Earlier work of George Seligman [Sel67] (also at Yale) emphasized the role and the importance of the Lie algebras of Cartan type. George was one of Jacobson's first students and Georgia was one of his last.

In 1966, Kostrikin and Šafarevič conjectured that the Cartan-type Lie algebras and the Lie algebras coming from characteristic 0 were *all* of the finite-dimensional simple Lie algebras (over an algebraically closed field) in characteristic p. The original formulation was for "restricted" Lie algebras, and the general statement for finite-dimensional simple Lie algebras is the "Generalized Kostrikin–Šafarevič conjecture."

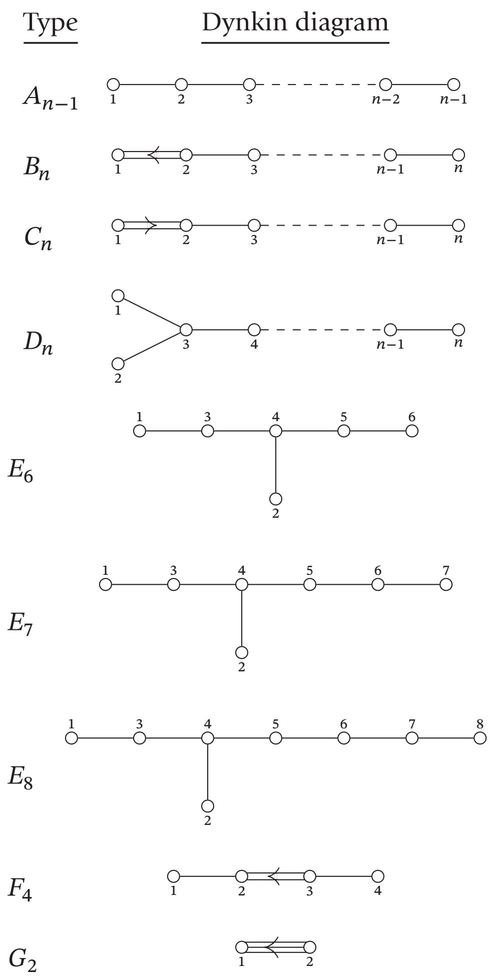


Figure 1. The classification of finite-dimensional simple Lie algebras in characteristic 0 is by the above Dynkin diagrams. In characteristic $p > 3$ there are five additional series of algebras:
(1) the Witt Lie algebras $W(m, \underline{n})$,
(2) the special Lie algebras $S(m, \underline{n})^{(1)}$,
(3) the Hamiltonian Lie algebras $H(2m, \underline{n})^{(2)}$, and
(4) the contact Lie algebras $K(2m+1, \underline{n})^{(1)}$;
and when $p = 5$ there is one more additional series:
(5) the Melikyan Lie algebras $M(2, \underline{n})$.

The study and proof of the Kostrikin–Šafarevič conjecture inspired work by many people around the world. Georgia brought to Wisconsin the mindset of the Jacobson school, emphasizing a module-theoretic approach to the classification of algebraic systems. She joined a thriving algebra community that included Marty Isaacs, Marshall Osborn, Donald Passman, and Louis Solomon. Also in the thick of the action around the Kostrikin–Šafarevič conjecture were Richard Block, Robert Wilson (who had been a student of George Seligman at Yale), and Victor Kac, who seemed to be everywhere, classifying all things Lie.

Benkart and Osborn [BO84] classified the finite-dimensional simple Lie algebras of characteristic $p > 7$ with a one-dimensional Cartan subalgebra, showing that they are either $\mathfrak{sl}(2)$ or Albert–Zassenhaus Lie algebras (the algebras $W(1, \underline{n})$ and a family of Hamiltonian Lie algebras). Their paper [BO90] studied the subalgebra $L^{(\alpha)} = L_0 \oplus L_\alpha \oplus L_{2\alpha} \oplus \cdots \oplus L_{(p-1)\alpha}$ of a finite-dimensional simple Lie algebra L determined by a root α. Modulo the radical, these one-sections $L^{(\alpha)}$ are isomorphic to either $\mathfrak{sl}(2)$, $W(1, \underline{1})$, or to a subalgebra of $H(2, \underline{1})$ containing $H(2, \underline{1})^{(2)}$.

The results of Benkart and Osborn, along with their proof techniques, were ultimately absorbed into the general classification process. In the 1990s, Alexander Premet and Helmut Strade pulled it all together, methodically completing every step to a full classification.

Of course, as with any huge project, there were many other important contributors in addition to those named here. In the middle of it all, in 1980, Melikyan found a new finite-dimensional simple Lie algebra in characteristic 5, of dimension 125. That certainly put a wrench into things, and increased the worry that, in those small p cases, there might exist even more fascinating and untamed algebras that nobody had seen before. Fortunately, now the whole project is finished for $p > 3$ and is comprehensively exposited in the 1100 pages of the three volumes of Helmut Strade's books, *Simple Lie algebras over fields of positive characteristic Vols. I*, *II*, and *III* [Str17a, Str17b, Str13].

Quoting from the Math Review of Vol. III:

> Kac's recognition theorem is one major result whose proof is not included in the book. All details for an arbitrary $p > 3$ can be found in a paper of G.M. Benkart, T.B. Gregory and Premet [BGP09].

The Recognition Theorem was a *hugely* important step on the long road to completion of the classification. To quote from the introduction of [BGP09]: "The Recognition Theorem is used several times throughout the classification; its first application results in a complete list of the simple Lie algebras of absolute toral rank two, and its last application yields a crucial characterization of the Melikyan Lie algebras, thereby completing the classification." Finally those mysterious Melikyan algebras (they had multiplied in the interim and become a whole family) were under control in the sense that the freedom that causes them to appear had been pinpointed, and it had been checked carefully that this freedom doesn't cause other sporadic examples of this nature. The monograph of Benkart, Gregory, and Premet is a wonderful work to read: thorough, efficient, elementary, with precise definitions; it contains a clear big-picture point of view. It is absolutely beautifully written.

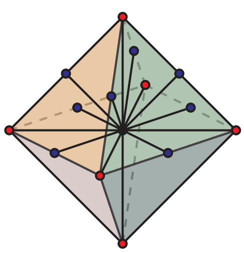
Infinite dimensions and magic squares. The structure of a *finite-dimensional* Lie algebra $\mathfrak{g}$ corresponding to one of the Dynkin diagrams in Figure 1 is governed by its root

Figure 2. Georgia Benkart on February 22, 1979 during a visit to Indiana University. An image from the Paul R. Halmos Photograph Collection.

system Δ, and $\mathfrak{g}$ decomposes into a direct sum of the form,

$$\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha), \quad \text{with} \quad \dim(\mathfrak{g}_\alpha) = 1 \tag{1}$$

and $\dim(\mathfrak{h})$ equal to the number of vertices in the Dynkin diagram. Furthermore, the root system Δ has a geometric description connecting it to the world of polytopes (see Figure 3).



The blue and red vectors form the root system Δ of the finite-dimensional Lie algebra $\mathfrak{g} = \mathfrak{sp}_6$.

Figure 3. The root system Δ for a Lie algebra $\mathfrak{g}$ corresponding to the Dynkin diagram C_3. The root system Δ consists of the vectors from the center to the vertices and from the center to the midpoints of the edges of the octahedron. See equation (1).

The second half of the 20th century produced a huge expansion into the universe of *infinite-dimensional* Lie algebras. The finite-dimensional Cartan-type Lie algebras in Figure 1 are the characteristic p versions of infinite-dimensional characteristic 0 Lie algebras that arose from Cartan's study of "pseudogroups." The study of Feynman path integrals and the development of string theory also produced new examples of infinite-dimensional Lie algebras with interesting structure.

The underlying structure of the infinite-dimensional Lie algebra L comes from a finite-dimensional $\mathfrak{g}$ sitting inside L. This property was formalized in the early 1990s by Berman and Moody when they defined Δ-graded Lie algebras. A Δ-*graded Lie algebra* L contains a subalgebra $\mathfrak{g}$ corresponding to a Dynkin diagram, and the whole Lie algebra L decomposes into root spaces L_μ indexed by the root system Δ of $\mathfrak{g}$,

$$\mathfrak{g} \subseteq L \qquad \text{and} \qquad L = \bigoplus_{\alpha\in\Delta\cup\{0\}} L_\alpha.$$

Berman and Moody classified the Δ-graded Lie algebras for which the Dynkin diagram does not have double or triple edges by viewing them as Lie algebras analogous to $\mathfrak{sl}_n(R)$, where R is an (associative) algebra. Favorite examples are the polynomial rings $R = \mathbb{C}[t_1, \dots, t_n]$ and the Laurent polynomial rings $R = \mathbb{C}[t_1^{\pm1}, \dots, t_n^{\pm1}]$, but R can be much more general.

Berman and Moody's classification leads one to wonder what happens when the Dynkin diagram has multiple edges. Efim Zelmanov had started to study these cases, and in the course of his work gave a few lectures in the seminar at the University of Wisconsin. One morning Georgia came in and indicated that she thought that some of the ideas from her thesis might apply to this question. It didn't take long before Georgia and Efim hunkered down and quickly polished off all the other cases and completed the amazing theorem that

$$\begin{array}{l}\textit{all } \Delta\textit{-graded Lie algebras have the form} \\ L = (\mathfrak{g}\otimes A) \oplus (W \otimes B) \oplus D,\end{array} \tag{2}$$

where $\mathfrak{g}$ is a finite-dimensional Lie algebra with root system Δ, W is a small $\mathfrak{g}$-module, and D is a subalgebra of derivations that acts on the algebra $\mathfrak{a} = A \oplus B$. Hence the infinite-dimensional Lie algebra L is something like that in Figure 4, where A and B are visualized as appendages to the root system Δ. The Benkart–Zelmanov paper [BZ96] explaining how this works has become a classic.

Georgia didn't stop there. There are two basic steps in the classification of Δ-graded Lie algebras:

First: One has to show that the only possible forms that a Δ-graded Lie algebra can take are $L = (\mathfrak{g}\otimes A) \oplus (W \otimes B) \oplus D$.

Second: After narrowing down the possibilities, one has to show that they all occur in reality and do, in fact, produce Δ-graded Lie algebras.

This second step is obtained by powerful constructions which go by various names (see Tables 1 and 2): "Freudenthal's magic square," the "Tits–Kantor–Koecher construction," "generalized octonions." These constructions were originally conceived to build the Lie algebras corresponding to the Dynkin diagrams E_6, E_7, E_8, F_4, and G_2. They were vastly generalized by Benkart–Zelmanov to construct Δ-graded Lie algebras and by Benkart–Elduque and Elduque to extend to exceptional Lie superalgebras and Lie algebras and Lie superalgebras in characteristic p.

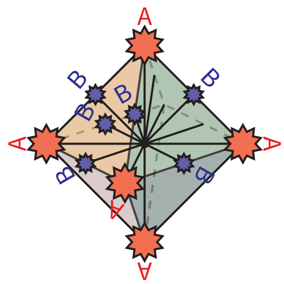


Figure 4. The infinite-dimensional Δ-graded Lie algebra corresponding to the root system Δ for the Dynkin diagram C_3**.** The octahedron provides the structure of the root system Δ of the finite-dimensional Lie algebra $\mathfrak{g}$. The Δ-graded Lie algebra $L = (\mathfrak{g} \otimes A) \oplus (W \otimes B) \oplus D$ is built by fitting $\mathfrak{g}$-modules A and B into sockets on the mother board $\mathfrak{g}$ labeled by the elements of Δ. See (2).

$\mathcal{T}(C,J)$	$H_3(\mathbb{F})$	$H_3(\mathbb{F}^2)$	$H_3(M_2(\mathbb{F}))$	$H_3(C(\mathbb{F}))$
$\mathbb{F}$	A_1	A_2	C_3	F_4
$\mathbb{F}^2$	A_2	$A_2 \oplus A_2$	A_5	E_6
$M_2(\mathbb{F})$	C_3	A_3	D_6	E_7
$C(\mathbb{F})$	F_4	E_6	E_7	E_8

Table 1. The Tits–Kantor–Koecher construction. In this table $\mathbb{F}^2 = \mathbb{F} \times \mathbb{F}$, $M_2(\mathbb{F})$ denotes the algebra of 2×2 matrices with entries from a field $\mathbb{F}$, and $H_3(C')$ is the Jordan algebra of 3×3 Hermitian matrices over the unital composition algebra C'.

$\mathfrak{g}(C,C')$	1	2	4	8
1	A_1	A_2	C_3	F_4
2	A_2	$A_2 \oplus A_2$	A_5	E_6
4	C_3	A_3	D_6	E_7
8	F_4	E_6	E_7	E_8

Table 2. Freudenthal's magic square or the symmetric (Vinberg) construction. The rows are indexed by $\dim(C)$, and the columns are indexed by $\dim(C')$.

The construction has two forms: the first method is to take a Jordan algebra J and a composition algebra C and twist them together to get a Lie algebra $L = \mathcal{T}(C,J)$. The other version of the construction (introduced by Vinberg) builds the Lie algebra L from two composition algebras C and C'. In this version, the symmetry of Freudenthal's magic square is embedded into the construction.

The wonderful article of Elduque in the Tits 80th birthday volume [Eld11] provides an accessible survey of the various constructions of Freudenthal's magic square, along with recent advances in the theory involving Georgia and her coauthors and a nice entrée into open questions and current research in this vein. The original paper of Benkart–Zelmanov [BZ96] classified Δ-graded Lie algebras for the cases where the Dynkin diagram of Δ is B_r, C_r, F_4, and G_2. The AMS Memoir of Allison, Benkart, and Gao [ABG02] provides an amazing resource for understanding all parts of the classification of Δ-graded Lie algebras, the analysis of their derivations, central extensions and invariant forms, and their constructions, including the Tits–Kantor–Koecher constructions.

Elemental Lie algebras. Imagine that it is the early 1800s and you are Dalton, or Gay-Lussac, or Avogadro, trying to figure out how atoms combine to make molecules. There are two fundamental problems to solve:

(a) What are the individual elements?
(b) How do they combine to make molecules?

Now imagine that it is the turn of the 21st century and you are Georgia Benkart trying to figure out how Lie algebras are built. There are two fundamental problems:

(a) What are the littlest Lie algebras?
(b) How do they combine to make larger Lie algebras?

A motivating phenomenon is that all finite-dimensional simple Lie algebras $\mathfrak{g}$ (in characteristic 0) and all Kac–Moody Lie algebras are constructed from the little Lie algebras $\mathfrak{sl}_2$ glued together appropriately.

Letting $[a,b] = ab - ba$,

$$\mathfrak{sl}_2 = \left\{ \begin{pmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{pmatrix} \right\} = \mathrm{span}\{x, y, h\},$$

where

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and

$$[x,y] = h, \quad [h,x] = 2x, \quad [h,y] = -2y.$$

Another little Lie algebra is the three-dimensional Heisenberg Lie algebra

$$\mathcal{H} = \left\{ \begin{pmatrix} 0 & a_1 & a_3 \\ 0 & 0 & a_2 \\ 0 & 0 & 0 \end{pmatrix} \right\} = \mathrm{span}\{x, y, h\},$$

where

$$x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$h = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$[x,y] = h, \quad [h,x] = 0, \quad [h,y] = 0.$$

These algebras are strikingly similar in presentation, but different in application. If one has these examples in mind, then it is not very surprising that Georgia has sequences of papers engaged in the study of families of "elemental" algebras over a field $\mathbb{F}$:

(A) the *parametric family*

$$\begin{aligned} &A_h, \text{ generated by } x \text{ and } y \text{ with} \\ &\quad [x,y] = h, \quad \text{where } h \in \mathbb{F}[x], \text{ and} \end{aligned} \tag{3}$$

(B) the *down-up algebras* [BR98], depending on parameters $\alpha, \beta, \gamma \in \mathbb{F}$:

$$\begin{aligned} &A(\alpha,\beta,\gamma), \text{ generated by } u \text{ and } d \text{ with} \\ &\quad d^2u = \alpha dud + \beta ud^2 + \gamma d \quad \text{and} \\ &\quad du^2 = \alpha udu + \beta u^2 d + \gamma u. \end{aligned} \tag{4}$$

If $\gamma = 0$, $\beta = -1$, and $\alpha = 2$, then $d^2u - 2dud + u^2d = 0$, and we recover the Heisenberg algebra. This is because in $\mathcal{H}$, the relation $0 = [h,x]$ expands to $0 = hx - xh = [x,y]x - x[x,y] = (xy-yx)x - x(xy-yx) = xyx - yx^2 - x^2y + xyx = -(x^2y - 2xyx + yx^2)$.

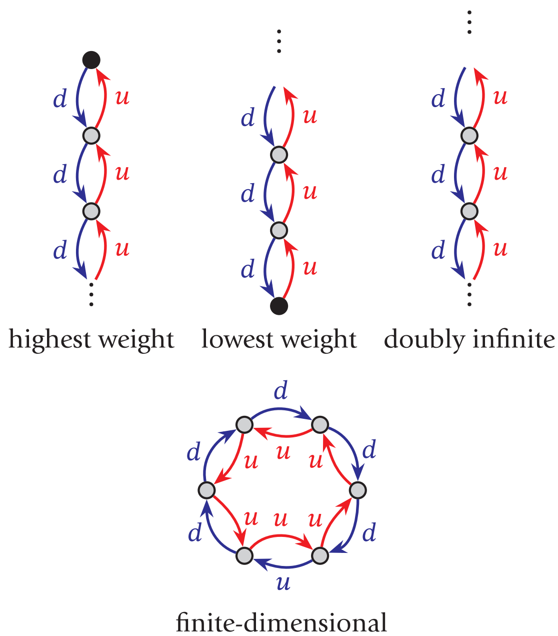


Figure 5. Irreducible modules for the down-up algebras $A(\alpha,\beta,\gamma)$. Up to constants depending on the parameters α, β, γ, the u operators act according the red edges and the d operators act according the blue edges. The black vertex represents the highest weight and the lowest weight, respectively. See (4).

These algebras A_h and $A(\alpha,\beta,\gamma)$ capture the core underlying structures that join together to make larger Lie algebras and their quantum groups. Georgia and her collaborators have done thorough studies of the properties of these "little quantum groups" by determining all of the following: automorphisms, inner automorphisms, centers, derivations, inner derivations, their Hochschild cohomology HH^1, prime ideals, primitive ideals, Duflo correspondences between primitive ideals and annihilators of simple modules, highest weight modules, lowest weight modules, finite-dimensional modules, Whittaker modules, and also some tensor product rules for simple modules in case that wasn't enough already.

Just to highlight a tiny portion of these results, Georgia and her collaborators determine precisely all the possible "shapes" of irreducible modules of down-up algebras $A(\alpha,\beta,\gamma)$. These are shown pictorially in Figure 5.

Because the algebras A_h and $A(\alpha,\beta,\gamma)$ are so "elemental" (generalizing the structures from $\mathfrak{sl}_2$ and three-dimensional Hesenberg algebras), one has confidence that they will be useful to mathematicians of the future in the same way that intimate knowledge of Mendeleev's periodic table is indispensible for any post-19th century chemist. The elemental Lie algebras are the atoms from which larger Lie algebras and quantum groups that arise in nature (i.e., many other parts of mathematics and physics) are built.

Talking the talk: A Tale of Two Groups. In a Dickensian plenary address at the 1994 Joint Math Meetings, Georgia told the story of Schur–Weyl duality as a "Tale of Two Groups." See [Ben96]. The protagonist is a group G acting on tensor powers of a defining representation, and the antagonist is the algebra of endomorphisms $\mathrm{End}_G(V^{\otimes n})$ that commute with G. See Figure 6.

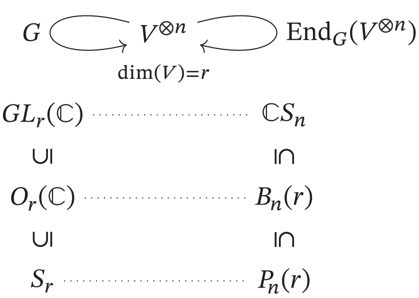
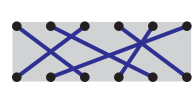
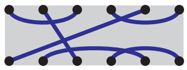


Figure 6. Schur–Weyl duality between the general linear group $GL_r(\mathbb{C})$ and the symmetric group S_n, between the orthogonal group $O_r(\mathbb{C})$ and the Brauer algebra $B_n(r)$, and between the symmetric group S_n and the partition algebra $P_n(r)$.

In his groundbreaking thesis at the turn of the 20th century, Schur used these methods to construct the irreducible polynomial representations of the general linear group $G = GL_r(\mathbb{C})$. He showed that $\mathrm{End}_G(V^{\otimes n})$ is generated by $\mathbb{C}S_n$, the algebra of permutations, displayed here as a permutation diagram,

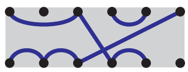
 acting on $V^{\otimes n}$ by tensor place permutation.

In the 1930s Brauer showed that if $G = O_n(\mathbb{C})$, then $\mathrm{End}_G(V^{\otimes n})$ is generated by the algebra of Brauer diagrams, which correspond to arbitrary matchings of $2n$ vertices,



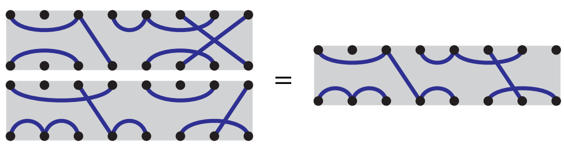
acting on $V^{\otimes n}$ by permutation and contraction onto subspaces.

In about 1990, Paul Martin and Vaughan Jones showed that if $G = S_r$, the symmetric group, then the centralizer $\mathrm{End}_G(V^{\otimes n})$ is generated by set partition diagrams,



acting on $V^{\otimes n}$ by permutation, contraction, and fragmentation.

Set partition diagrams multiply with one another via concatenation:

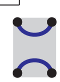


Schur–Weyl duality allows information to flow back and forth between the group G and its centralizer $\mathrm{End}_G(V^{\otimes n})$. In an AMS Memoir [BBL90] Georgia and her coauthors, Dan Britten and Frank Lemire, study finite-dimensional representations of $GL_r(\mathbb{C})$, $SL_r(\mathbb{C})$, $O_r(\mathbb{C})$, and $Sp_{2r}(\mathbb{C})$. They identify submodules for these G inside the tensor space $V^{\otimes n}$ and use the combinatorics of the centralizer, for example,

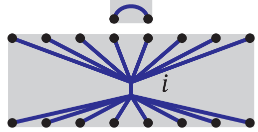
Young tableaux

1	1	2	3	3
2	4	4	5	
3	5	5	7	
4	7	7		
5	9			

contraction mappings

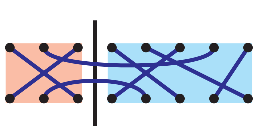


and Young symmetrizers



to understand stability properties for irreducible G-modules as r grows. Georgia [Ben90] and Sheila Sundaram [Sun90] each give elegant descriptions of these combinatorial methods in representation theory.

In 1989, in a collaboration [BCH+94] with *five* graduate students at the University of Wisconsin, Georgia defined the *walled-Brauer algebra* by determining the centralizer of the $GL_n(\mathbb{C})$ on $V^{\otimes n} \otimes (V^*)^{\otimes m}$, where V^* is the dual module to V. This time, the diagrams come with a left part and a right part separated by a wall, with the constraint that horizontal edges must cross the wall and top-to-bottom edges must not cross the wall,



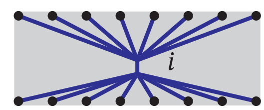
acting on $V^{\otimes n} \otimes (V^*)^{\otimes m}$.

This collaboration, with Georgia leading a group of five junior mathematicians at once, was unusual at the time. Now, this is more common and one finds, among Georgia's recent papers, several team collaborations that include early-career researchers who have been stimulated by Georgia's leadership. Not only is Georgia a natural and inspiring mentor for these teams, but she initiated them long before there were organizations like Banff (see Figure 11) and MSRI helping so effectively to make it happen.

Walking the walk: The Representation Theory Way. In 2014, Georgia delivered the Noether Lecture at the International Congress of Mathematicians in Seoul, Korea entitled, "Walking on Graphs the Representation Theory Way." The motivating idea is that one can build

every irreducible G-module V_i
from *a single* well-chosen G-module V,

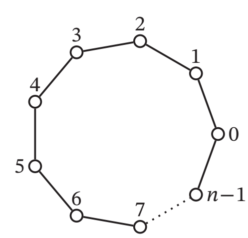
by applying idempotents p_i of $\mathrm{End}_G(V^{\otimes n})$ to $V^{\otimes n}$. The idempotent

$p_i =$ 

is a projection onto the irreducible G summand V_i. A powerful way to study this is by building a graph that keeps track of what happens when one tensors by V. This *representation graph*, or *McKay quiver*, has vertices V_i and r edges $V_i \to V_j$ if V_j appears r times in $V_i \otimes V$. For example, if $G = \{1, g, g^2, \ldots, g^{n-1}\}$ is the cyclic group of order n, and V is the two-dimensional representation of G corresponding to the matrix

$$g = \begin{pmatrix} \omega^{-1} & 0 \\ 0 & \omega \end{pmatrix}, \quad \omega = e^{2\pi i/n},$$

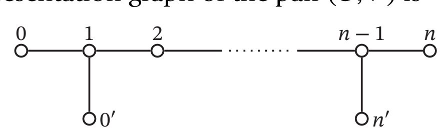
then the representation graph of the pair (G, V) is

$\hat{A}_{n-1}$  .

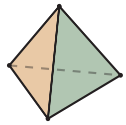
If $G = \{g, h \mid g^{2n} = 1, h^2 = g^n, gh = hg^{-1}\}$ is the binary dihedral group of order $4n$ and V is the two-dimensional representation given by the matrices

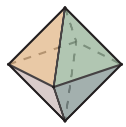
$$g = \begin{pmatrix} \zeta^{-1} & 0 \\ 0 & \zeta \end{pmatrix}, \ \zeta = e^{\pi i/n}, \quad \text{and} \quad h = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

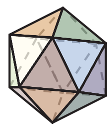
then the representation graph of the pair (G, V) is

$\hat{D}_n$ 

Finally, if G is one of the three polyhedral groups,

$\mathbb{T} = \{$symmetries of  $\}$, of order 24,

$\mathbb{O} = \{$symmetries of  $\}$, of order 48,

$\mathbb{I} = \{$symmetries of  $\}$, of order 120,

and V is the two-dimensional representation of G, then the representation graphs of the pairs (G, V) are the graphs in Figure 7. The observation that the graphs $\hat{A}_{n-1}$, $\hat{D}_n$, $\hat{E}_6$, $\hat{E}_7$, $\hat{E}_8$ are exactly the "simply-laced affine Dynkin diagrams" is the amazing McKay correspondence. These same graphs also describe (see [Kac90] and [Bri71]) the internal structure of the Lie algebras of loop groups as well as the structure of the subregular nilpotent orbits for reductive algebraic groups!

In these examples, if we now let $Z_n = \mathrm{End}_G(V^{\otimes n})$, then the two commuting actions of G and Z_n on $V^{\otimes n}$

$$G \curvearrowright V^{\otimes n} \curvearrowleft Z_n = \mathrm{End}_G(V^{\otimes n})$$

give a decomposition of $V^{\otimes k}$ into irreducible (G, Z_n)-bimodules,

$$V^{\otimes n} = \bigoplus_i V_i \otimes Z_n^i.$$

The walks on the representation graph $\Gamma(G, V)$ encode multiplicities and dimensions:

$$\begin{aligned} &\#\{\text{walks of length } n \text{ from } 0 \text{ to } i \text{ on } \Gamma(G,V)\} \\ &\quad = \textit{multiplicity} \text{ of the } G\text{-module } V_i \text{ in } V^{\otimes n} \\ &\quad = \textit{dimension} \text{ of the } Z_n\text{-module } Z_n^i, \end{aligned}$$

and $\dim(Z_n)$ is the number of walks that come back home (to the node labeled 0) after $2n$ steps:

$$\begin{aligned} &\dim(Z_n) = \textstyle\sum_i \dim(Z_n^i)^2 \\ &= \#\{\text{walks of length } 2n \text{ from } 0 \text{ to } 0 \text{ on } \Gamma(G,V)\}. \end{aligned}$$

A particularly elegant way to enumerate walks on the representation graph $\Gamma(G, V)$ is to expand them into paths on the corresponding *Bratteli diagram* $\mathcal{B}(G, V)$, which is an infinite lattice organized so that the nodes on level n are those that can be reached by an n-step walk starting at the root on $\Gamma(G, V)$ (see Figure 8).

With several collaborators, Georgia has used walks on these representation graphs to answer many questions in combinatorial representation theory. To name just a few: they describe the projection operators in McKay and Motzkin centralizer algebras; they characterize the kernel of the partition algebra on tensor space; they describe walks on hypercubes; and they are used to perform chip firing on Dynkin diagrams and McKay quivers.

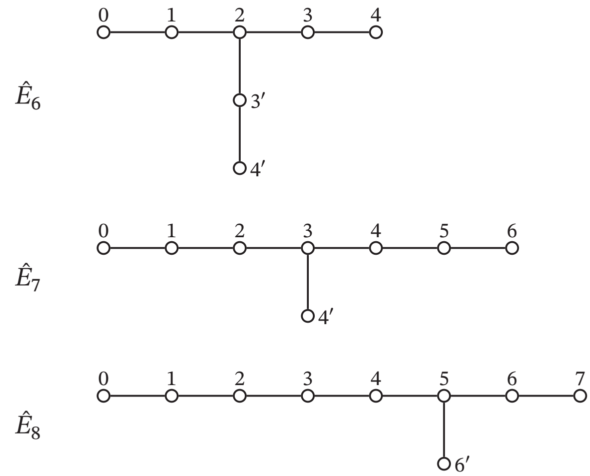


Figure 7. The representation graphs of the binary tetrahedral, octahedral, and icosahedral groups are the simply-laced affine Dynkin diagrams of type $\hat{E}_6, \hat{E}_7$, and $\hat{E}_8$.

Fusion rules! Georgia's most recent talks and collaborations have centered around fusion rules. Fusion matrices encode the rules that determine the decomposition of the tensor product of two modules into a direct sum of simple modules. In the case of the McKay correspondence, the fusion matrices are the adjacency matrices of the representation graphs in Figure 7, and in conformal field theory in physics, integrable models are described by the fusion rules for their charges.

In a group project [BBK+21] that began at the workshop in Leeds for Women in Noncommutative Algebra and Representation Theory (WINART3), Georgia and her collaborators compute fusion matrices for certain classes of finite-dimensional Hopf algebras. They express the eigenvalues and eigenvectors of these matrices in terms of Chebyshev polynomials, furthering the case that Chebyshev polynomials are as dense in representation theory as they are in numerical analysis. A key step is to relate the eigenvectors to characters, and an overarching question in this work is to find a good notion of a character table for a Hopf algebra.

In another exciting collaboration, Georgia worked with Persi Diaconis, Martin Liebeck, and Pham Huu Tiep (see [BDLT20]) at MSRI to use fusion matrices to analyze families of Markov chains. They studied walks in a similar manner to the case pictured in Figure 8 above, except now using

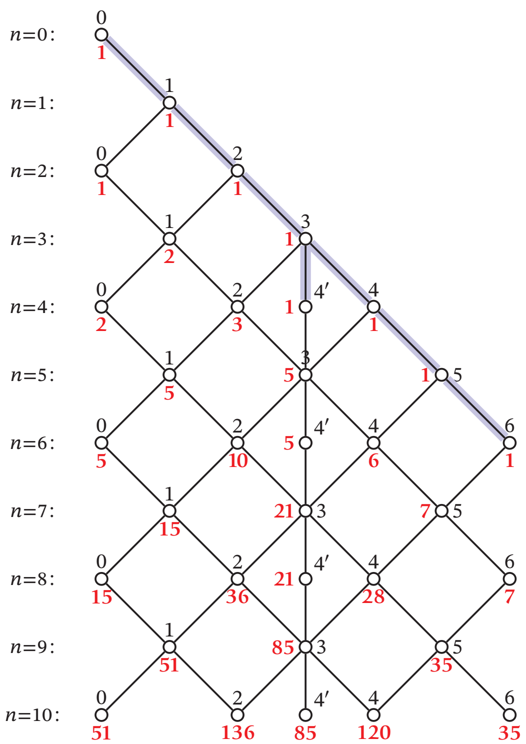


Figure 8. The Bratteli diagram for $\tilde{E}_7$. Surprising and beautiful things happen in this diagram. The Dynkin diagram $\hat{E}_7$ is embedded at the top of the Bratteli diagram (shaded in blue). The dimension of the irreducible Z_n-modules are the red labels, which satisfy a Pascal's triangle-like addition rule. The dimension $\dim(Z_n)$ is the number of paths ending at 0 on level $2n$, i.e., the numbers $1, 1, 2, 5, 15, 51, \ldots$. Thus the red number at node 0 on level $2n$ is the sum of the squares of the red numbers on level n. For example, $\dim(Z_5) = 5^2 + 5^2 + 1^2 = 51$.

groups and quantum groups like

$$SL_2(\mathbb{F}_p), \quad SL_3(\mathbb{F}_p), \qquad SL_2(\mathbb{F}_{2^n}), \quad SL_2(\mathbb{F}_{p^2}),$$

$$\text{and} \quad U_\xi(\mathfrak{sl}_2),$$

instead of the octahedral group used in Figure 8.

The game is similar to walking on graphs with representations and the McKay correspondence. You start with an empty mixing bowl, choose a small representation, put it in the bowl, and hand it to the next cook. The second cook chooses a small representation to tensor with, and mixes it into the bowl (i.e., calculates the tensor product with what is already there) and hands it on to the next cook in line. This process continues …, and there's one person at the restaurant (Persi Diaconis) who always wants to know when the food is going to arrive, i.e., how long it takes for all this mixing and cooking to get to the stationary state.

There are several finicky issues that have to be dealt with:

(a) In characteristic p, the tensor products don't always decompose as direct sums.

(b) Tensoring by the natural module V doesn't always produce all representations.

They fix the issue in (a) by using the Grothendieck ring (Brauer characters) in some cases and by using indecomposable representations instead of irreducible representations in others. They fix the issue in (b) by tensoring by $V \oplus \text{triv}$ or by tensoring with $V \oplus V^{(p)}$, where $V^{(p)}$ is the Frobenius twist of V by the pth power field automorphism, and by restricting attention only to the representations of a normal subgroup called the Frobenius kernel.

A few selected answers for the walks and their convergence rates are as follows:

(a) For $G = SL_2(\mathbb{F}_{p^2})$, when mixing (tensoring) by the two-dimensional natural representation V at each step, the walk takes p^4 steps to equilibriate.

(b) For $G = SL_2(\mathbb{F}_{2^n})$, when walking (tensoring) by the two-dimensional natural representation V at each step, the mixing takes 2^{2n} steps to converge to stationarity.

See Figure 9 and Figure 10.

The bottom line. Georgia Benkart is a clear and creative writer and speaker, who finds great joy in peppering her talks with inventive, mostly deadpan, and always amusing mathematical puns.

The first article Georgia coauthored as an undergraduate appeared in the *Pi Mu Epsilon Journal*. She was crushed when the publication appeared: they had listed her name as George Benkart. This rather inauspicious beginning to publishing papers was followed by graduate school at Yale University and, in 1974, a postdoc at the University of Wisconsin–Madison. By 1983, she had risen to full professor at Wisconsin–Madison. In 1991, of the 340 tenured or tenure-track faculty at the ten top-ranked schools in mathematics, 12 were women (see Table 2 in [BLW21]). Two decades prior to 1991, it was likely a quarter to a half that number.

Georgia's research on Lie theory, representation theory, combinatorics, and noncommutative algebra has resulted in over 130 journal publications and research monographs. The more than 350 invited talks she has given during her career include plenary lectures at the Joint Mathematics Meetings on three different occasions and at the annual meetings of the Canadian Mathematical Society and the Mathematical Association of America. In 2014, she was chosen to give both the AWM Noether Lecture at the Joint Mathematics Meetings and the International Mathematical Union's Emmy Noether Lecture at the International Congress of Mathematicians in Seoul.

Georgia was President of the Association for Women in Mathematics from 2009 to 2011 and one of the five US delegates to the 2014 International Mathematical Union General Assembly. She has served as an Associate Secretary of

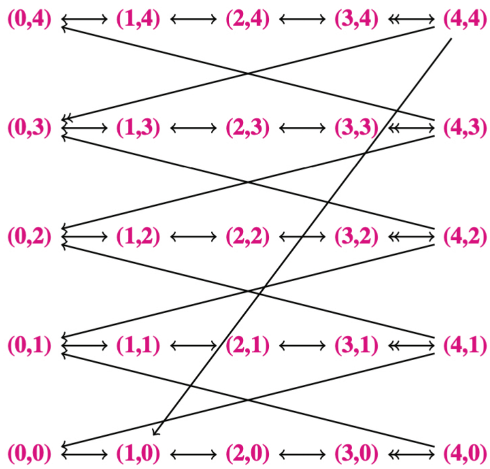


Figure 9. The representation graph for $G = SL_2(\mathbb{F}_{p^2})$ and the two-dimensional natural representation $V = (1,0)$. The double headed arrows indicate that the representation appears twice in the tensor product decomposition. The walk has a drift to the left and a drift downward. Heuristically, the walk moves back and forth at a fixed horizontal level. Once it hits the right-hand wall, it usually bounces back, but with small probability (order $\frac{1}{p}$), it jumps up or down by one level. The walk takes order p^4 steps to totally equilibriate.

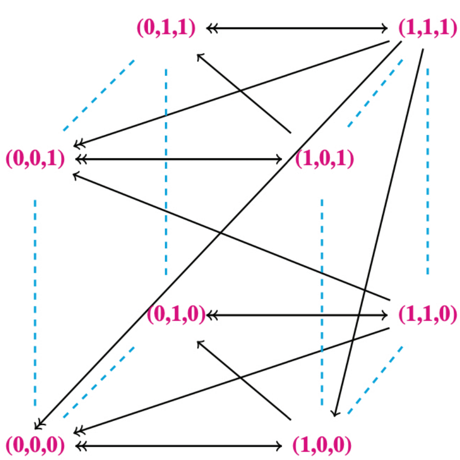


Figure 10. The representation graph for $G = SL_2(\mathbb{F}_{2^3})$ and the two-dimensional natural representation $V = (1,1,0)$. The double headed arrows indicate that the representation appears twice in the tensor product decomposition. For $G = SL_2(\mathbb{F}_{2^3})$, this walk takes order $2^{2\cdot 3}$ steps to reach stationarity.

the AMS, as a member of the AMS Council, as a member of the US National Committee for Mathematics of the National Academies, and on several editorial boards, including *Journal of Algebra*, *Algebra and Number Theory*, and *AMS Mathematical Surveys and Monographs*.

Figure 11. 2011 "Algebraic Combinatorixx (11w5025)" workshop, taken at Banff International Research Station in Banff, Alberta.

However, we feel that Georgia's contribution to our discipline goes well beyond this astonishing catalog of research papers, monographs, lectures, and service roles. She has left an indelible mark on a generation of mathematicians through supportive collaborations with more than 90 coauthors, many of whom are (or, more accurately, were) early-career researchers. And there are even more mathematicians who were not her coauthors but for whom Georgia's mentoring, advice, and support made it possible for them to achieve much more than they ever expected of themselves. Georgia, always humbly and perfectly, serves as a role model and mentor to all.

In the acknowledgments at the opening of her PhD thesis Georgia thanked the many people who supported her by saying,

> Many people have contributed to my mathematical education. I owe them all my sincerest thanks.
>
> I would like to express my special appreciation to my advisor, Professor Nathan Jacobson, and to Professor George Seligman who first suggested inner ideals as a possible avenue of research. Among the other individuals who helped with the preparation of this dissertation are Professors Wallace Martindale and James Lepowsky, Darrell Haile, Carl Bumiller, Nicholas Bourbaki, Jr., and Mary Ellen DelVecchio. I am also profoundly grateful for the financial support awarded me through National Science Foundation graduate fellowships and National Science Foundation grant GP-33591.

Now in 2022 it is our turn to sincerely thank Georgia for teaching us so much beautiful mathematics and helping us to begin and sustain careers in research mathematics. Most meaningful to us all is her kindness, decency, and humanity.

References

[ABG02] Bruce Allison, Georgia Benkart, and Yun Gao, *Lie algebras graded by the root systems BC_r, $r \geq 2$*, Mem. Amer. Math. Soc. **158** (2002), no. 751, x+158, DOI 10.1090/memo/0751. MR1902499

[BBK+21] Georgia Benkart, Rekha Biswal, Ellen Kirkman, Van C. Nguyen, and Jieru Zhu, *McKay matrices for finite-dimensional Hopf algebras*, Canadian Journal of Mathematics (2021), 1–46.

[BBL90] G. M. Benkart, D. J. Britten, and F. W. Lemire, *Stability in modules for classical Lie algebras—a constructive approach*, Mem. Amer. Math. Soc. **85** (1990), no. 430, vi+165, DOI 10.1090/memo/0430. MR1010997

[BCH+94] Georgia Benkart, Manish Chakrabarti, Thomas Halverson, Robert Leduc, Chanyoung Lee, and Jeffrey Stroomer, *Tensor product representations of general linear groups and their connections with Brauer algebras*, J. Algebra **166** (1994), no. 3, 529–567, DOI 10.1006/jabr.1994.1166. MR1280591

[BDLT20] Georgia Benkart, Persi Diaconis, Martin W. Liebeck, and Pham Huu Tiep, *Tensor product Markov chains*, J. Algebra **561** (2020), 17–83, DOI 10.1016/j.jalgebra.2019.10.038. MR4135538

[Ben90] Georgia Benkart, *Partitions, tableaux, and stability in the representation theory of classical Lie algebras*, Lie theory, differential equations and representation theory (Montreal, PQ, 1989), Univ. Montréal, Montreal, QC, 1990, pp. 47–76. MR1121952

[Ben96] Georgia Benkart, *Commuting actions—a tale of two groups*, Lie algebras and their representations (Seoul, 1995), Contemp. Math., vol. 194, Amer. Math. Soc., Providence, RI, 1996, pp. 1–46, DOI 10.1090/conm/194/02387. MR1395593

[BGP09] Georgia Benkart, Thomas Gregory, and Alexander Premet, *The recognition theorem for graded Lie algebras in prime characteristic*, Mem. Amer. Math. Soc. **197** (2009), no. 920, xii+145, DOI 10.1090/memo/0920. MR2488391

[BLW21] Georgia Benkart, Kristin Lauter, and Sylvia Wiegand, *AWM at 50 and beyond*, Notices Amer. Math. Soc. **68** (2021), no. 3, 387–397, DOI 10.1090/noti2239. MR4218176

[BO84] Georgia M. Benkart and J. Marshall Osborn, *Rank one Lie algebras*, Ann. of Math. (2) **119** (1984), no. 3, 437–463. MR744860

[BO90] Georgia Benkart and J. Marshall Osborn, *Simple Lie algebras of characteristic p with dependent roots*, Trans. Amer. Math. Soc. **318** (1990), no. 2, 783–807. MR955488

[BR98] Georgia Benkart and Tom Roby, *Down-up algebras*, J. Algebra **209** (1998), no. 1, 305–344, DOI 10.1006/jabr.1998.7511. MR1652138

[Bri71] E. Brieskorn, *Singular elements of semi-simple algebraic groups*, Actes du Congrès International des Mathématiciens (Nice, 1970), Gauthier-Villars, Paris, 1971, pp. 279–284. MR0437798

[BZ96] Georgia Benkart and Efim Zelmanov, *Lie algebras graded by finite root systems and intersection matrix algebras*, Invent. Math. **126** (1996), no. 1, 1–45, DOI 10.1007/s002220050087. MR1408554

[Eld11] Alberto Elduque, *Tits construction of the exceptional simple Lie algebras*, Pure Appl. Math. Q. **7** (2011), no. 3, Special Issue: In honor of Jacques Tits, 559–586, DOI 10.4310/PAMQ.2011.v7.n3.a4. MR2848587

[Kac90] Victor G. Kac, *Infinite-dimensional Lie algebras*, 3rd ed., Cambridge University Press, Cambridge, 1990, DOI 10.1017/CBO9780511626234. MR1104219

[KŠ66] A. I. Kostrikin and I. R. Šafarevič, *Cartan's pseudogroups and the p-algebras of Lie* (Russian), Dokl. Akad. Nauk SSSR **168** (1966), 740–742. MR0199235

[Sel67] G. B. Seligman, *Modular Lie algebras*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 40, Springer-Verlag New York, Inc., New York, 1967. MR0245627

[Str13] Helmut Strade, *Simple Lie algebras over fields of positive characteristic. III: Completion of the classification*, De Gruyter Expositions in Mathematics, vol. 57, Walter de Gruyter GmbH & Co. KG, Berlin, 2013. MR3025870

[Str17a] Helmut Strade, *Simple Lie algebras over fields of positive characteristic. Vol. 1: Structure theory*, De Gruyter Expositions in Mathematics, vol. 38, De Gruyter, Berlin, 2017. Second edition [of MR2059133]. MR3642321

[Str17b] Helmut Strade, *Simple Lie algebras over fields of positive characteristic. Vol. II: Classifying the absolute toral rank two case*, De Gruyter Expositions in Mathematics, vol. 42, De Gruyter, Berlin, 2017. Second edition [of MR2573283]. MR3642323

[Sun90] Sheila Sundaram, *Tableaux in the representation theory of the classical Lie groups*, Invariant theory and tableaux (Minneapolis, MN, 1988), IMA Vol. Math. Appl., vol. 19, Springer, New York, 1990, pp. 191–225. MR1035496

Tom Halverson

Arun Ram

Credits

Opening photo is courtesy of Yvonne Nagel.

Figures 1 and 3–8 are courtesy of Tom Halverson.

Figure 2 is courtesy of the University of Texas.

Figures 9 and 10 are courtesy of Georgia Benkart. Graphics appeared previously in Georgia Benkart, Persi Diaconis, Martin W. Liebeck, and Pham Huu Tiep, "Tensor product Markov chains," *Journal of Algebra*, Volume 561, 2020.

Figure 11 is courtesy of the Banff International Research Station.

Photo of Tom Halverson is courtesy of David J. Turner / Macalester College.

Photo of Arun Ram is courtesy of Yvonne Nagel.

Nina Nikolaevna Uraltseva

Darya Apushkinskaya, Arshak Petrosyan, and Henrik Shahgholian

Nina Nikolaevna Uraltseva was born on May 24, 1934, in Leningrad, USSR (currently St. Petersburg, Russia), to parents Nikolai Fedorovich Uraltsev (an engineer) and Lidiya Ivanovna Zmanovskaya (a school physics teacher). Nina Uraltseva was attracted to both mathematics and physics from the early stages of her life.[1] She was a student at the now famous school no. 239, then a school for girls, which later became specialized in mathematics and physics and produced many notable alumni. Together with her friends, Nina Uraltseva initiated a mathematical study group at her school, under the supervision of Mikhail Birman, then a student at the Faculty of Mathematics and Mechanics of Leningrad State University (LSU). In the higher grades of the school, she was actively involved in the Mathematical Circle at the Palace of Young Pioneers, guided by Ilya Bakelman, and became a two-time winner of the citywide mathematical olympiad.

Nina Uraltseva graduated from school in 1951 (with the highest distinction—a gold medal) and started her study at the Faculty of Physics of LSU. She was an active participant in an (undergraduate) student work group founded by Olga Aleksandrovna Ladyzhenskaya, that gave her the opportunity to further deepen her study into the analysis of partial differential equations (PDEs). In 1956, she graduated from the university and the same year married Gennady Lvovich Bir (a fellow student at the Faculty of Physics). The young couple were soon blessed with a son (and only child) Kolya.[2]

During her graduate years, Uraltseva continued to be supervised by Olga Ladyzhenskaya. This mentorship transformed into a lifelong productive collaboration and warm friendship until 2004, when Olga Ladyzhenskaya passed away.

Darya Apushkinskaya is a professor at the S. M. Nikol'skii Mathematical Institute, Peoples' Friendship University of Russia (RUDN University), and the Department of Mathematics and Computer Science, St. Petersburg State University. Her email address is `apushkinskaya@gmail.com`.

Arshak Petrosyan is a professor of mathematics at Purdue University. His email address is `arshak@purdue.edu`.

Henrik Shahgholian is a professor of mathematics at KTH Royal Institute of Technology. His email address is `henriksh@kth.se`.

A version of this paper with more complete references can be found at `https://arxiv.org/abs/2109.00658`.

The opening photo is Nina Uraltseva with Mount Ararat in the background, Khor Virap, Armenia, 2004.

[1] *Uraltseva's prematurely deceased younger brother (Igor Uraltsev) was a famous physicist, a specialist in epsilon spectroscopy in semiconductors. The Spin Optics Laboratory at St. Petersburg State University is named after him.*

Communicated by Notices *Associate Editor Daniela De Silva.*

For permission to reprint this article, please contact: `reprint-permission@ams.org`.

DOI: https://doi.org/10.1090/noti2440

[2] *Tragically, Kolya (Nikolai Uraltsev) passed away from a heart attack in 2013 (in Siegen, Germany). He was a renowned nuclear physicist, author of 120 papers published in the world's top scientific journals, most of them very well known internationally (with approximately 6000 references), and two of them are in the category of renowned. Kolya's son, Gennady Uraltsev, is currently a postdoctoral fellow at the University of Virginia, working in harmonic analysis.*

Figure 1. Nina Uraltseva in a schoolgirl uniform, Leningrad, 1951.

Nina Uraltseva defended her Candidate of Science[3] thesis entitled "Regularity of solutions to multidimensional quasilinear equations and variational problems" in 1960. Four years later, she became a Doctor of Science[4] with a thesis "Boundary-value problems for quasilinear elliptic equations and systems of second order." Since 1959, she has been a member of the Chair of Mathematical Physics at the Faculty of Mathematics and Mechanics of LSU (currently St. Petersburg State University), where she became a Full Professor in 1968 and served as the head of the chair since 1974.

For her fundamental contributions to the theory of partial differential equations in the 1960s, Nina Uraltseva (jointly with Olga Ladyzhenskaya) was awarded the Chebyshev Prize of the Academy of Sciences of the USSR (1966) as well as one of the highest honors of the USSR, the USSR State Prize (1969).

Throughout her career, Nina Uraltseva has been an invited speaker at many meetings and conferences, including the International Congress of Mathematicians in 1970 and 1986. In 2005, she was chosen as the Lecturer of the European Mathematical Society.

Nina Uraltseva's mathematical achievements are highly regarded throughout the world, and have been acknowledged by various awards, such as the titles of Honorary Scientist of the Russian Federation in 2000, Honorary Professor of St. Petersburg State University in 2003, and Honorary Doctor of KTH Royal Institute of Technology, Stockholm, Sweden, in 2006. In the same year, in recognition of her academic record, she received the Alexander von Humboldt Research Award. In 2017, the Government of St. Petersburg recognized her recent research by its Chebyshev Award.

Nina Uraltseva's interests are not limited to scientific activities only. In her youth, she used to be a very good basketball player and an active member of the university basketball team. She enjoyed hiking in the mountains, canoeing, and driving a car. In the 1980s, Nina took part in five archaeological expeditions in the north of Russia (the Kola Peninsula and the Kotlas area) and excavated Paleolithic ceramics. She is also a passionate lover of classical music and a regular visitor at philharmonic concerts.

Mathematical Contributions

Nina Uraltseva has made lasting contributions to mathematics with her pioneering work in various directions in analysis and PDEs and the development of elegant and sophisticated analytical techniques. She is most renowned for her early work on linear and quasilinear equations of elliptic and parabolic type in collaboration with Olga Ladyzhenskaya, which is the category of classics, but her contributions to the other areas such as degenerate and geometric equations, variational inequalities, and free boundaries are equally deep and significant. Below, we summarize Nina Uraltseva's work with some details on selected results.

1. Linear and Quasilinear Equations

1.1. Hilbert's 19th and 20th problems. The first three decades of Nina Uraltseva's mathematical career were devoted to the theory of linear and quasilinear PDEs of elliptic and parabolic type. Her first round of works in the 1960s, mostly in collaboration with Olga Ladyzhenskaya, was related to Hilbert's 19th and 20th problems on the existence and regularity of the minimizers of the energy integrals

$$I(u) = \int_\Omega F(x, u, \nabla u)dx,$$

where $F(x, u, p)$ is a smooth function of its arguments and Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$. In her Candidate of Science thesis, based on work [17],[5] Nina Uraltseva has shown that under the assumption that F is $C^{2,\alpha}$ and satisfies the uniform ellipticity condition

$$F_{p_i p_j}\xi_i\xi_j \geq m|\xi|^2, \quad m > 0,$$

[3] *Equivalent of PhD in many countries.*

[4] *Equivalent of Habilitation in many European countries.*

[5] *In those years, it was quite unusual to base the Candidate of Science thesis on just a single paper and some of the committee members voiced their concerns. However, Olga Ladyzhenskaya objected decisively that it depends on the quality of the paper.*

Figure 2. Left to right: Nina Uraltseva, Olga Ladyzhenskaya, and Vladimir Smirnov in a seminar on mathematical physics, Leningrad, 1968.

the minimizers u are $C^{2,\alpha}$ locally in Ω (i.e., on compact subdomains of Ω), provided they are Lipschitz. (It has to be mentioned here that the Lipschitz regularity of the minimizers was known from the earlier works of Ladyzhenskaya under natural growth conditions on F and its partial derivatives.) Uraltseva has also shown that $C^{2,\alpha}$ regularity extends up to the boundary $\partial\Omega$ under the natural requirement that both $\partial\Omega$ and $u|_{\partial\Omega}$ are $C^{2,\alpha}$. This generalized the results of Morrey in dimension $n = 2$ to higher dimensions.

Uraltseva's proof was based on a deep extension of the ideas of De Giorgi for the solutions of uniformly elliptic equations in divergence form with bounded measurable coefficients, which were applicable only to the integrands of the form $F(\nabla u)$. In particular, one of the essential steps was to establish that $v = \pm u_{x_i}$, $i = 1, \dots, n$, which are assumed to be bounded, satisfy the energy inequalities

$$\int_{A_{k,\rho}} |\nabla v|^2 \zeta^2 \le C \int_{A_{k,\rho}} (v-k)^2 |\nabla \zeta|^2 + C|A_{k,\rho}| \qquad (1)$$

for all $|k| \le M$, where $A_{k,\rho}$ is the intersection of $\{v > k\}$ with the ball $B_\rho(x^0) \Subset \Omega$, ζ is a cutoff function, and M is a bound for $\max|\nabla u|$.

Using similar ideas, Uraltseva was able to deduce the existence and regularity of solutions for the class of quasilinear uniformly elliptic equations in divergence form,

$$\partial_{x_i}(a_i(x, u, \nabla u)) + a(x, u, \nabla u) = 0, \qquad (2)$$

under natural growth conditions on $a_i(x, u, p)$, a and some of their partial derivatives, which were mainly needed for proving the bounds on $\max|\nabla u|$. These results were further refined in the joint works with Olga Ladyzhenskaya in 1961. In [18], Uraltseva extended these results to problems with Neumann-type boundary conditions as well as to certain quasilinear diagonal systems (important, e.g., for the applications in harmonic maps).

Quasilinear uniformly elliptic equations in nondivergence form,

$$a_{ij}(x, u, \nabla u)\, u_{x_i x_j} + a(x, u, \nabla u) = 0, \qquad (3)$$

were trickier to treat, but already in her thesis Uraltseva found a key: quadratic growth of $a(x, u, p)$ in the p-variable,

$$|a(x, u, p)| \le \mu(1 + |p|)^2,$$

along with the corresponding conditions on the partial derivatives of a and a_{ij} in their variables. In [19], Uraltseva proved $C^{1,\alpha}$ a priori bounds for solutions of (3), as well as for diagonal systems of similar type.

The results in the elliptic case were further extended to the parabolic case (including systems) in a series of works of Ladyzhenskaya and Uraltseva [9].

This extensive research, that went far beyond the original scope of Hilbert's 19th and 20th problems, was summarized in two monographs, *Linear and Quasilinear Equations of Elliptic Type* (1964) (substantially enhanced in the 2nd edition in 1973) and *Linear and Quasilinear Equations of Parabolic Type* (1967), written in collaboration with Vsevolod Solonnikov; see Figure 3. The monographs became instant classics and were translated to English [8, 12] and other languages and have been extensively used for generations of mathematicians working in elliptic and parabolic PDEs and remain so to this date.

1.2. **Equations with unbounded coefficients.** In a series of papers in 1979–1985, summarized in her talk at the International Congress of Mathematicians in Berkeley, CA, 1986 and a survey paper with Ladyzhenskaya [11], Uraltseva and collaborators have studied uniformly elliptic quasilinear equations of nondivergence type (3) and their parabolic counterparts, when a and the first derivatives of a_{ij} are possibly unbounded. The typical conditions read

$$|a(x, u, p)| \le \mu|p|^2 + b(x)|p| + \Phi(x),$$

where μ is a constant and $b, \Phi \in L^q(\Omega)$, $q > n$. Uraltseva and collaborators were able to establish the existence and up to the boundary $C^{1,\alpha}$ regularity of $W^{2,n}$ strong solutions of the problem, vanishing on $\partial\Omega$ (provided the latter is sufficiently regular). The proofs were based on the extension of methods of Ladyzhenskaya and Uraltseva already in their books [8, 12], as well as those of Krylov and Safonov using the Aleksandrov-Bakelman-Pucci (ABP) estimate, in the elliptic case, and a parabolic version of the ABP estimate due to Nazarov and Uraltseva (1985), in the parabolic case.

Most recent results of Nina Uraltseva in this direction are in the joint work with Alexander Nazarov [13] on the linear equations in divergence form,

$$\partial_{x_i}(a_{ij}(x)u_{x_j}) + b_i(x)u_{x_i} = 0 \quad \text{in } \Omega,$$

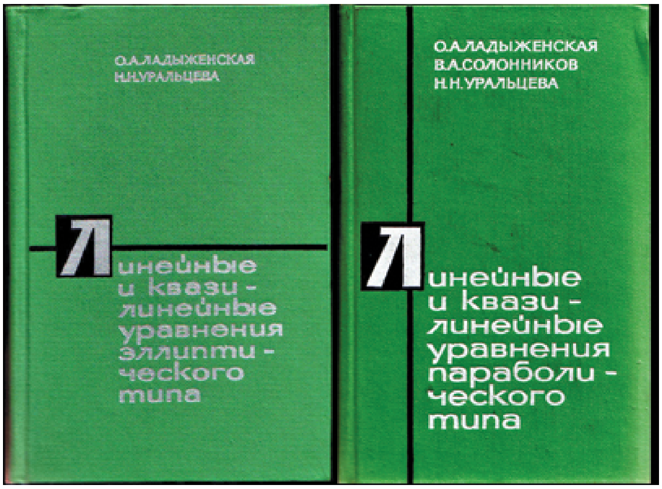


Figure 3. The famous books: the iconic green Russian editions of the elliptic (2nd ed., 1973) and parabolic (1967) versions of Uraltseva's books with Ladyzhenskaya and Solonnikov.

and their parabolic counterparts. Their goal was to find conditions on the lower-order coefficients $\mathbf{b} = (b_1, \dots, b_n)$ that guarantee the validity of classical results such as the strong maximum principle, Harnack's inequality, and Liouville's theorem. It was shown by Trudinger (1973) that such results hold when $\mathbf{b} \in L^q$, $q > n$. Motivated by applications in fluid dynamics, in one of their theorems Nazarov and Uraltseva showed that under the additional assumption

$$\operatorname{div} \mathbf{b} = 0, \tag{4}$$

the condition on $\mathbf{b}$ can be relaxed to being in the Morrey space

$$\sup_{B_r(x^0)\subset\Omega} r^{q-n} \int_{B_r(x^0)} |\mathbf{b}|^q < \infty$$

for some $n/2 < q \leq n$. In the borderline case $q = n$, the Morrey space above is locally the same as L^n. Remarkably, in that case the divergence-free condition (4) on $\mathbf{b}$ can be dropped when $n \geq 3$, i.e., $\mathbf{b} \in L^n_{\mathrm{loc}}$ alone is sufficient to have the classical theorems; moreover, this result is optimal. In dimension $n = 2$, to drop (4) one needs the stronger condition $\mathbf{b} \ln^{1/2}(1 + |\mathbf{b}|) \in L^2_{\mathrm{loc}}$.

2. Nonuniformly Elliptic and Parabolic Equations

2.1. Degenerate equations. Nina Uraltseva has also made a pioneering work on the regularity theory for degenerate quasilinear equations. A particular result in this direction is her 1968 proof [20] of the $C^{1,\alpha}$ regularity of p-harmonic functions, $p > 2$, which are the weak solutions of the p-Laplace equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \quad \text{in } \Omega, \tag{5}$$

or, equivalently, are the minimizers of the energy functional

$$\int_\Omega |\nabla u|^p dx.$$

The difficulty here lies in the fact that the p-Laplace equation (5) degenerates at the points where the gradient vanishes and that the solutions are not generally twice differentiable in the Sobolev sense. As stated in her paper, this problem was posed to Nina Uraltseva by Yurii Reshetnyak in relation with the study of quasiconformal mappings in higher dimensions.

Uraltseva has obtained the $C^{1,\alpha}$ regularity of p-harmonic functions as an application of the Hölder regularity of the solutions of the degenerate quasilinear diagonal systems

$$\partial_{x_i}(a_{ij}(x, \mathbf{u})\,\mathbf{u}_{x_j}) = \mathbf{0},$$

with scalar coefficients a_{ij} satisfying the degenerate ellipticity condition

$$\nu(|\mathbf{u}|)|\xi|^2 \leq a_{ij}(x, \mathbf{u})\xi_i\xi_j \leq \mu\nu(|\mathbf{u}|)|\xi|^2,$$

with $\mu \geq 1$ and a nonnegative increasing function $\nu(\tau)$ satisfying $\nu(\lambda\tau) \leq \lambda^s\nu(\tau)$ for $\lambda \geq 1$ and $s > 0$.

Unfortunately, despite the utmost importance of this result, Nina Uraltseva's proof remained unknown outside of the Soviet Union. In 1977, nine years later, it was independently reproved by Karen Uhlenbeck. Other proofs were given by Craig Evans (1982), John Lewis (1983), who extended the range of exponents to $1 < p \leq 2$, and Di Benedetto (1983) and Tolksdorf (1984), who both extended it to the case of general degenerate quasilinear equations in divergence form.

Another work in this area that has gained the status of classic is the paper of Nina Uraltseva and Anarkul Urdaletova [25], where they proved uniform gradient estimates for bounded solutions of anisotropic degenerate equations,

$$\partial_{x_i}(a_i(x, u_{x_i})) + a(x, u, \nabla u) = 0 \quad \text{in } \Omega,$$

under ellipticity, growth, and monotonicity conditions on the coefficients. Their results were applicable to the minimizers of the energy functional

$$\int_\Omega \sum_{i=1}^n |u_{x_i}|^{m_i} + f(x, u),$$

with the exponents $m_1, \dots, m_n$ satisfying $m_i > 3$, $2m_i > m_0$, $i = 1, \dots, n$, $m_0 = \max\{m_i\}$, under the monotonicity condition $f_u(x, u) \geq 0$. This was the very first paper to prove regularity results for degenerate quasilinear equations with nonstandard growth, which appeared first in the 1980s, motivated by applications in elasticity and material science, and continue to be the subject of extensive research today. Major contributions in this direction have been made by Paolo Marcellini and many others.

2.2. Geometric equations. In [10], Ladyzhenskaya and Uraltseva developed a method of local a priori estimates for nonuniformly elliptic and parabolic equations, including the equations of minimal surface type,

$$\operatorname{div} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = a(x,u,\nabla u) \quad \text{in } \Omega.$$

A particular case with $a(x,u,\nabla u) = \kappa u$, $\kappa > 0$, together with the Neumann-type condition $\partial_\nu u/\sqrt{1+|\nabla u|^2} = \varkappa$ on $\partial\Omega$, $|\varkappa| < 1$, is known as the capillarity problem. The boundary estimates, as well as the existence of classical solutions for such problems, were proved by Uraltseva in [21]. Remarkably, the results in this paper required only the smoothness of the domain Ω, but not its convexity.

In the 1990s, in a series of joint works with Vladimir Oliker (see [14] and the references therein), Nina Uraltseva studied the evolution of surfaces $S(t)$ given as graphs $u = u(x,t)$ over a bounded domain $\Omega \subset \mathbb{R}^n$ with the speed depending on the mean curvature of $S(t)$ under the condition that the boundary of the surface $S(t)$ is fixed. More precisely, they considered a parabolic PDE of the type

$$u_t = \sqrt{1+|\nabla u|^2}\, \operatorname{div} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \quad \text{in } \Omega\times(0,\infty)$$

with the boundary condition $u(x,t) = \phi(x)$ on $\partial\Omega\times(0,\infty)$ and initial condition $u(x,0) = u_0(x)$. Even in the stationary case, when this problem is the Dirichlet problem for the mean curvature equation, the existence of up to the boundary classical solutions requires a geometric condition on the domain Ω, namely, the nonnegativity of the mean curvature of $\partial\Omega$. For such domains, Huisken (1989) has shown the existence of the classical solutions of the evolution problem and proved that the surfaces $S(t)$ converge to a classical minimal surface S as $t \to \infty$. Oliker and Uraltseva have studied this problem with no geometric conditions on the domain Ω. For this purpose, they introduced a notion of a generalized solution to the parabolic problem (as a limit of regularized problems). They have proved its existence and convergence $u(\cdot,t) \to \Phi$ as $t \to \infty$ to a generalized solution Φ of the stationary problem, in the sense that Φ minimizes the area functional

$$\int_\Omega \sqrt{1+|\nabla u|^2} + \int_{\partial\Omega} |u-\phi|$$

among all competitors in $W^{1,1}(\Omega)$. Such minimizer Φ is unique, but may differ from the Dirichlet data ϕ on the "bad" part of the boundary where the mean curvature is negative. The study of the behavior of the minimizer near the "contact points" on the boundary where $\Phi|_{\partial\Omega}$ "detaches" from ϕ later served as one of Uraltseva's motivations for studying the touch between free and fixed boundaries; see Section 4.1.

3. Variational Inequalities

Another area in which Nina Uraltseva has made significant contributions is variational inequalities, including variational problems with convex constraints that often exhibit a priori unknown sets known as free boundaries. An important example is the Signorini problem from elasticity, which describes equilibrium configurations of an elastic body resting on a rigid frictionless surface.

In a series of papers in the 1970s, as well as in the period 1986–1996, together with Arina Arkhipova, Nina Uraltseva studied elliptic and parabolic variational inequalities with unilateral and bilateral boundary constraints, known as the boundary obstacle problems, which can be viewed as scalar versions of the Signorini problem. Ultimately, these results played a fundamental role in Schumann's proof (1989) of the $C^{1,\alpha}$ regularity for the solution of the Signorini problem in the vectorial case.

Below, we give a more detailed description of some of her most impactful results in this direction.

3.1. Problems with unilateral constraints. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with a smooth boundary and S a relatively open nonempty subset of $\partial\Omega$. Suppose we are also given two functions $\psi, g \in W^{1,2}(\Omega)$ satisfying $g \geq \psi$ on S (in the sense of traces). Consider then a closed convex subset $\mathfrak{K} \subset W^{1,2}(\Omega)$ defined by

$$\mathfrak{K} := \{v \in W^{1,2}(\Omega) : v \geq \psi \text{ on } S,\ v = g \text{ on } \partial\Omega \setminus S\}.$$

In other words, $\mathfrak{K}$ consists of functions that need to stay above ψ, called a boundary (or thin) obstacle, on S and equal to g on $\partial\Omega \setminus S$. Then, one wants to find $u \in \mathfrak{K}$ that minimizes the generalized Dirichlet energy

$$J(v) = \int_\Omega a_{ij}(x) v_{x_j} v_{x_i} + 2f(x)u,$$

where $a_{ij}(x)$ are uniformly elliptic coefficients and f is a certain function. Equivalently, the minimizer u satisfies the variational inequality

$$u \in \mathfrak{K}, \quad \int_\Omega a_{ij}(x) u_{x_j} (v-u)_{x_i} + f(x)(v-u) \geq 0 \quad \text{for any } v \in \mathfrak{K}.$$

In turn, it is equivalent to the boundary value problem

$$\begin{aligned} \partial_{x_i}(a_{ij}(x)u_{x_j}) &= f(x) \quad \text{in } \Omega,\\ u &= g \quad \text{on } \partial\Omega \setminus S,\\ u \geq \psi,\ \partial_\nu^A u \geq 0,\ (u-\psi)\partial_\nu^A u &= 0 \quad \text{on } S, \end{aligned}$$

to be understood in the appropriate weak sense, where $\partial_\nu^A u := a_{ij}(x)\nu_j u_{x_j}$ is the conormal derivative of u on $\partial\Omega$, with $\nu = (\nu_1, \dots, \nu_n)$ being the outward unit normal. The conditions on S are known as the Signorini complementarity conditions and are remarkable because they imply that

$$\text{either } u = \psi \text{ or } \partial_\nu^A u = 0 \text{ on } S,$$

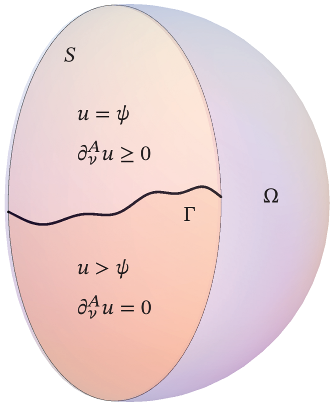


Figure 4. Boundary obstacle problem.

yet the exact sets where the first or the second equality holds are unknown. The interface Γ between these sets in S is called the free boundary (see Figure 4). The study of the free boundary is one of the main objectives in such problems (see Section 4 for Uraltseva's contributions in that direction), yet the regularity of the solutions u is a challenging problem by itself and is often an important step towards the study of the free boundary.

One of the theorems of Nina Uraltseva [22] states that when

$$\begin{aligned} &a_{ij} \in W^{1,q}(\Omega), \\ &\psi \in W^{1,2}(\Omega) \cap W^{2,q}_{\mathrm{loc}}(\Omega \cup S), \\ &f \in L^q(\Omega) \end{aligned}$$

for some $q > n$, then

$$u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega \cup S),$$

with a universal exponent $\alpha \in (0,1)$. Prior to this result, a similar conclusion was known only under higher regularity assumptions on the coefficients and the obstacle in the works of Caffarelli (1979) and Kinderlehrer (1981). The lower regularity assumptions in Uraltseva's result, particularly on the obstacle ψ, were instrumental in Schumann's proof (1989) of the corresponding result in the vectorial case. The parabolic counterpart of Uraltseva's theorem, with similar assumptions on the coefficients and the obstacle, was established later in a joint work of Arkhipova and Uraltseva [5].

The idea of Uraltseva's proof is based on an interplay between De Giorgi-type energy inequalities and the Signorini complementarity condition. Locally, near $x^0 \in S$, one can assume that $S = \{x_n = 0\}$ and $\psi = 0$. First, working with the regularized problem, one can establish that for any partial derivative $v = \pm u_{x_i}$, $i = 1, \dots, n$, there holds an energy inequality (similar to (1) in the unconstrained case)

$$\int_{A_{k,\rho}} |\nabla v|^2 \zeta^2 \leq C \int_{A_{k,\rho}} (v-k)^2 |\nabla \zeta|^2 + C_0 |A_{k,\rho}|^{1-2/q}$$

for any $k > 0$, $0 < \rho < \rho_0$, and a cutoff function ζ in $B_\rho(x^0)$, where $A_{k,\rho} = \{v > k\} \cap B_\rho(x^0) \cap \Omega$. Next, one observes that as a consequence of the Signorini complementarity conditions, one has

$$u_{x_i} u_{x_n} = 0 \quad \text{on } \{x_n = 0\} \cap B_\rho(x_0)$$

for all $i = 1, \dots, n-1$ and hence either the normal derivative $v = u_{x_n}$ or all tangential derivatives $v = u_{x_i}$, $i = 1, \dots, n-1$, vanish at least on half of $\{x_n = 0\} \cap B_\rho(x^0)$ (by measure). This allows one to apply Poincare's inequality in one of the steps and obtain a geometric improvement of the Dirichlet energy for v going from radius ρ to $\rho/2$. By iteration, this gives that either

$$\sum_{i=1}^{n-1} \int_{\Omega \cap B_\rho(x^0)} |\nabla u_{x_i}|^2 \leq C\rho^{n-2+2\alpha} \quad \text{or} \tag{6}$$

$$\int_{\Omega \cap B_\rho(x^0)} |\nabla u_{x_n}|^2 \leq C\rho^{n-2+2\alpha} \tag{7}$$

holds, with C depending on the distance from x^0 to $\partial\Omega \setminus S$. However, using the PDE satisfied by u, it is easy to see that (6) implies (7), and hence (7) always holds. From there, the $C^{1,\alpha}$ regularity of u follows by standard results for the solutions of the Neumann problem.

3.2. Diagonal systems.

The results described above were extended by Arkhipova and Uraltseva [7] to the problem with two obstacles $\psi_- \leq \psi_+$ on S, that corresponds to the constraint set

$$\begin{aligned} \mathfrak{K} = \{v \in W^{1,2}(\Omega) : \psi_- \leq u \leq \psi_+ \text{ on } S, \\ u = g \text{ on } \partial\Omega \setminus S\}. \end{aligned}$$

While substantial difficulties arise near the set where $\psi_- = \psi_+$, the results are as strong as in the case of a single obstacle. In their further work, Arkhipova and Uraltseva studied related problems for quasilinear elliptic systems with diagonal principal part. To describe their results, let $V = W^{1,2}(\Omega; \mathbb{R}^N) \cap L^\infty(\Omega; \mathbb{R}^N)$ and

$$\mathfrak{K} = \{\mathbf{u} \in V : \mathbf{u}(x) \in K(x) \text{ for every } x \in \partial\Omega\},$$

where $K(x)$ are given convex subsets of $\mathbb{R}^N$ for every $x \in \partial\Omega$. Then consider the variational inequality of the type

$$\begin{aligned} \mathbf{u} \in \mathfrak{K}, \quad &\int_\Omega \left(a_{ij}(x,\mathbf{u})\mathbf{u}_{x_j} + \mathbf{b}_i(x,\mathbf{u})\right)(\mathbf{v}-\mathbf{u})_{x_i} \\ &+ \mathbf{f}(x,\mathbf{u},\nabla\mathbf{u})(\mathbf{v}-\mathbf{u}) \geq \mathbf{0} \\ &\qquad\qquad \text{for any } \mathbf{v} \in \mathfrak{K}, \end{aligned}$$

where a_{ij} are scalar uniformly elliptic coefficients, $\mathbf{b}_i$ and $\mathbf{f}$ are N-dimensional vector functions, and $\mathbf{f}(x,\mathbf{u},\mathbf{p})$ grows at most quadratically in $\mathbf{p}$. We note that the problem with two obstacles $\psi_- \leq \psi_+$ on $\partial\Omega$ fits into this framework with

$N = 1$ and $K(x) = [\psi_-(x), \psi_+(x)]$. Assume now that the convex sets $K(x)$ are of the form

$$K(x) = T(x)K_0 + \mathbf{g}(x),$$

where K_0 is a convex set in $\mathbb{R}^N$ with a nonempty interior and a smooth (C^2) boundary, $T(x)$ is an orthogonal $N \times N$ matrix, and $\mathbf{g}(x)$ is an N-dimensional vector. A theorem of Arkhipova and Uraltseva [6] then states that when the entries of T and $\mathbf{g}$ are extended to $W^{2,q}$ functions in Ω, $q > n$, $a_{ij}(\cdot, \mathbf{u})$ and $\mathbf{b}_i(\cdot, \mathbf{u})$ are in $W^{1,q}(\Omega)$, uniformly in $\mathbf{u}$ and have at most linear growth in $\mathbf{u}$, and $\mathbf{f}$ has at most quadratic growth in $\mathbf{p}$, then

$$\mathbf{u} \in C^{1,\alpha}_{\mathrm{loc}}(\Omega \cup S),$$

provided $\mathbf{u}$ is Hölder continuous in $\Omega \cup S$. The Hölder continuity assumption on $\mathbf{u}$ can be replaced by a bound on the oscillation in Ω and a local uniqueness of the solutions, which is also necessary for the continuity of the solutions of the nonlinear systems of the type

$$\partial_{x_i}(a_{ij}(x,\mathbf{u})\mathbf{u}_{x_j} + \mathbf{b}_i(x,\mathbf{u})) + \mathbf{f}(x,\mathbf{u},\nabla\mathbf{u}) = \mathbf{0} \quad \text{in } \Omega.$$

For a more complete overview of Uraltseva's results on variational inequalities, we refer to her own survey paper [23].

4. Free Boundary Problems

In the last 25 years, Uraltseva's work has dealt with regularity issues arising in free boundary problems. She has developed powerful techniques, which have led to proving the optimal regularity results for solutions and for free boundaries. She has systematically studied how the free boundaries approach the fixed boundaries, and has developed tools to study free boundary problems for weakly coupled systems, as well as two-phase problems. The graduate textbook *Regularity of Free Boundaries in Obstacle-Type Problems* [15], written in collaboration with two of us, contains these and related results.

Some of Uraltseva's major contributions (results, approaches) in free boundary problems are addressed below in more detail.

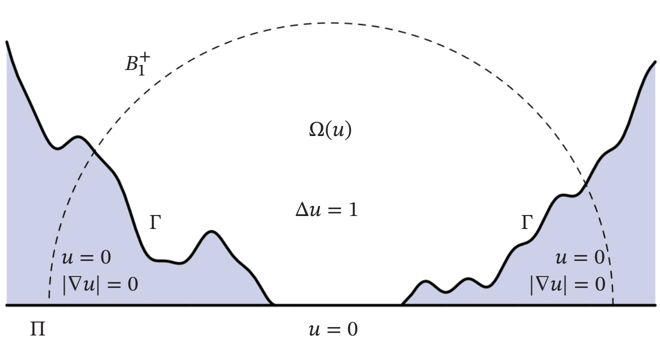


Figure 5. Touch between the free boundary $\Gamma = \partial\Omega(u)$ and the fixed boundary Π in problem (8).

4.1. Touch between free and fixed boundary.

In [3] (joint with one of us) and her follow-up paper [24], Uraltseva studied the obstacle problem close to a Dirichlet data, for smooth boundaries, where she proves that the free boundary touches the fixed boundary tangentially. The idea seemed to be inspired by related works with Oliker (see Section 2.2) and the Dam-problem in filtration.

During the potential theory program at Institute Mittag-Leffler (1999–2000) she started working on free boundary problems that originated in potential theory. Specifically, the harmonic continuation problem in potential theory, that was strongly tied to the obstacle problem, but with the lack of having a sign for the solution function. The simplest way to formulate this problem is as follows:

$$\begin{aligned} &\Delta u = \chi_{\Omega(u)} \quad \text{in } B_1^+, \\ &\qquad\qquad \text{with } \Omega(u) := \{u = |\nabla u| = 0\}^c, \\ &u = 0 \quad \text{on } \Pi \cap B_1, \end{aligned} \tag{8}$$

where $B_1^+ = \{|x| < 1,\ x_1 > 0\}$ and $\Pi = \{x_1 = 0\}$; see Figure 5. The question of interest was the behavior of the free boundary $\Gamma = \partial\Omega(u)$ close to the fixed boundary Π.

In [2], and several follow-up papers in the parabolic regime, she shows that the free boundary Γ is a graph of a C^1-function close to points on Π, where $\Gamma \cap B_1^+$ touches Π, or comes too close to Π.

To prove this, and the related parabolic results, there was a need for developing new tools and approaches. This was possible partly due to the availability of monotonicity formulas, such as that of Alt, Caffarelli, and Friedman (1984). One version of the latter asserts that for continuous subharmonic functions h_1, h_2 in $B_R(x^0)$, satisfying $h_1 h_2 = 0$ and $h_1(x^0) = h_2(x^0) = 0$, we have $\varphi(r) \nearrow$ for $0 < r < R$, where

$$\varphi(r) = \phi(r, h_1, x^0)\, \phi(r, h_2, x^0) \tag{9}$$

with

$$\phi(r, h_i, x^0) := \frac{1}{r^2} \int_{B(x^0,r)} \frac{|\nabla h_i|^2 dx}{|x - x^0|^{n-2}}.$$

One can use the monotonicity of the function $\varphi(r)$ to prove several important properties for u and the free boundary. Indeed, one first extends u to be zero in $B_1^- = \{|x| < 1,\ x_1 < 0\}$ and applies the monotonicity formula (9) to $h_1 = (\partial_e u)^+$ and $h_2 = (\partial_e u)^-$, where e is any vector tangent to the plane $\{x_1 = 0\}$. Using the fact that at least one of the sets $\{\pm\partial_e u > 0\}$ has positive volume density at x^0, we shall have

$$c_0 |\nabla \partial_e u(x^0)|^4 = \lim_{r \to 0} \varphi(r) \le \varphi(1) \le C_0.$$

Combining this with equation (8) we obtain the bound for $u_{x_1 x_1}(x^0)$. From here, the uniform $C^{1,1}$ regularity for u in $B_{1/2}^+$ follows.

The $C^{1,1}$ regularity is instrumental for any analysis of the properties of the free boundary. Indeed, to study the free

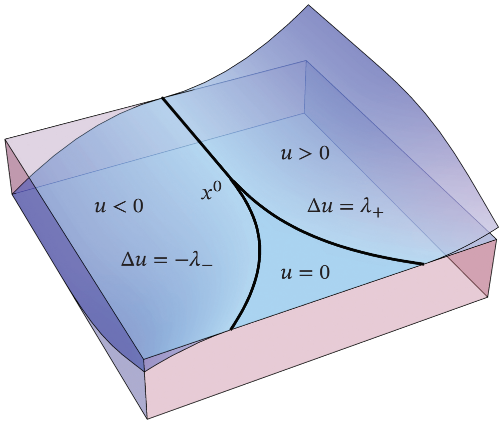


Figure 6. Two-phase problem: branch point x^0.

boundary at points where it touches the fixed boundary, one needs to rescale the solution quadratically, $u_r(x) = u(rx+x^0)/r^2$, which keeps the equation invariant. Indeed, this scaling and "blow-up"[6] brings one to a global setting of equation (8) in $\mathbb{R}^n_+$, where solutions can be classified (in a rotated system) as one of the following:

(i) $u(x) = \frac{1}{2}x_1^2 + ax_1x_2 + \alpha x_1 \quad (a > 0,\ \alpha \in \mathbb{R})$,

(ii) $u(x) = \frac{1}{2}((x_1 - a)^+)^2 \quad (a > 0)$.

The proof of the classification of global solutions uses an array of geometric tools and the monotonicity function $\varphi(r)$, implying that if $\{u = 0\} \cap \{x_1 > 0\} \neq \emptyset$, then $\partial_e u \equiv 0$ for any direction e tangential to Π. The case when this set is empty is easily handled by Liouville's theorem.

Once this classification is done, one can argue by indirect methods that the free boundary $\partial\{u > 0\} \cap \{x_1 > 0\}$ approaches the fixed one, at touching points, tangentially, and that it is a C^1-graph locally, which is optimal in the sense that (in general) it cannot be $C^{1,\mathrm{Dini}}$.

4.2. **Two-phase obstacle type problems.** If one considers extension of equation (8) into B_1, by an odd reflection, then one obtains a specific example of a general problem that is referred to as the two-phase obstacle problem, and is formulated as

$$\Delta u = \lambda_+ \chi_{\{u>0\}} - \lambda_- \chi_{\{u<0\}} \quad \text{in } B_1(0),$$

where $\lambda_\pm$ are positive bounded Lipschitz functions. Figure 6 illustrates this problem.

In [16], Nina Uraltseva (with coauthors) proves that at any branch point $x^0 \in \partial\{u > 0\} \cap \partial\{u < 0\}$ with $u(x^0) = |\nabla u(x^0)| = 0$, the free boundaries $\partial\{u > 0\} \cap B_{r_0}(0)$ and $\partial\{u < 0\} \cap B_{r_0}(0)$ are C^1-surfaces, that touch each other tangentially at x^0.

The proof of this and several similar results (also in the parabolic setting) relies heavily on the monotonicity function φ mentioned above as well as on the balanced energy functional

$$\Phi_{x_0}(r) := r^{-n-2} \int_{B_r(x_0)} \left(|\nabla u|^2 + \lambda_+ u^+ + \lambda_- u^-\right) - 2r^{-n-3} \int_{\partial B_r(x_0)} u^2, \quad (10)$$

which is strictly monotone in r, unless u is homogeneous. Using these two monotonicity functionals in combination with geometric tools brings us to the fact that any global solution u_0 to the two-phase problem is one-dimensional and, in a rotated and translated system of coordinates,

$$u_0 = \frac{\lambda_+}{2}(x_1^+)^2 - \frac{\lambda_-}{2}(x_1^-)^2.$$

From here one uses a revised form of the so-called directional monotonicity argument of Luis Caffarelli, that in this setting boils down to the fact that close to branch points x^0 one can show that in a suitable cone of directions $\mathcal{C}$ one has $\partial_e u \geq 0$ in $B_r(x^0)$ for $e \in \mathcal{C}$ and r universal. This in particular implies that the free boundaries $\partial\{\pm u > 0\}$ are Lipschitz graphs locally close to branch points.

The approaches here generated further application of the techniques to problems with hysteresis; see, e.g., [4].

4.3. **Free boundaries for weakly coupled systems.** In her work with coauthors [1], Uraltseva considers the following vectorial energy minimizing functional:

$$E(\mathbf{u}) := \int_{B_1} \left(|\nabla \mathbf{u}|^2 + 2|\mathbf{u}|\right) dx.$$

Here B_1 is the unit ball in $\mathbb{R}^n$ $(n \geq 1)$, and we minimize over the Sobolev space $\mathbf{g} + W_0^{1,2}(B_1; \mathbb{R}^N)$ for some smooth boundary values $\mathbf{g} = (g_1, \dots, g_N)$. The minimizer(s) are vector-valued functions $\mathbf{u} = (u_1, \dots, u_N)$, with components u_i satisfying

$$\Delta u_i = \frac{u_i}{|\mathbf{u}|}, \quad i = 1, \dots, N.$$

Since the set $\{|\mathbf{u}| > 0\}$ competes with the Dirichlet energy, by taking the boundary values small we may obtain $\{\mathbf{u} = \mathbf{0}\} \neq \emptyset$, which is in contrast to standard variational problems. The set $\partial\{|\mathbf{u}| > 0\}$ is called the free boundary. One observes that when $N = 1$ (scalar case) then we fall back to the two-phase problem.

Simple examples of solutions to this problem are:

(i) $u_i = \alpha_i P(x)$, with $P(x) \geq 0$, $\Delta P(x) = 1$, and $\sum_{i=1}^N \alpha_i^2 = 1$,

(ii) $u_i = \frac{\alpha_i}{2}(x_1^+)^2 + \frac{\beta_i}{2}(x_1^-)^2 \quad$ (2-phase), $\sum_{i=1}^N \alpha_i^2 = 1, \quad \sum_{i=1}^N \beta_i^2 = 1$,

(iii) $u_i = \frac{\alpha_i}{2}(x_1^+)^2 \quad$ (1-phase), $\sum_{i=1}^N \alpha_i^2 = 1$.

[6] *Blow-up refers to* $\lim_{r\to 0} u_r(x)$, *whenever it exists.*

Figure 7. Nina Uraltseva in 2013.

Using the vectorial version of the monotonicity formula (10), one can show that $\mathbf{u}$ has a quadratic growth away from the free boundary.

The regularity of the free boundary follows through the homogeneity improvement approach with the so-called epiperimetric inequality, which is used to show that the functional

$$\mathcal{M}(\mathbf{v}) := \int_{B_1(0)} (|\nabla \mathbf{v}|^2 + 2|\mathbf{v}|) - \int_{\partial B_1(0)} |\mathbf{v}|^2$$

satisfies

$$|\mathcal{M}(\mathbf{u}_{r_1}) - \mathcal{M}(\mathbf{u}_{r_2})| \leq c|r_2 - r_1|^\alpha, \quad \alpha > 0,$$

where $\mathbf{u}_r = \mathbf{u}(x^0 + rx)/r^2$ and x^0 is such that $\mathbf{u}_r$ is close to the rotated version of a half-space solution of type $\mathbf{h} = \frac{1}{2}(x_1^+)^2\mathbf{e}$.

This, in particular, gives uniqueness of the blow-ups, and can be used to show that (in a rotated system of coordinates) there exist $\beta' > 0$, $r_0 > 0$, and $C < \infty$ such that

$$\int_{\partial B_1(0)} |\mathbf{u}_r - \mathbf{h}| \leq Cr^{\beta'} \qquad \text{for every } x^0 \in \mathcal{R}_{\mathbf{u}} \text{ and every } r \leq r_0,$$

where $\mathcal{R}_{\mathbf{u}}$ is the set of free boundary points whose blow-ups are half-spaces. This implies that $\mathcal{R}_{\mathbf{u}}$ is locally in $B_{1/2}$ a $C^{1,\beta}$-surface.

Nina's Impact

Nina Uraltseva has over 100 publications[7] and over 8000 citations in MathSciNet. Her famous book *Linear and Quasilinear Equations of Parabolic Type* [8] (joint with Ladyzhenskaya and Solonnikov) has over 4600 citations, and the elliptic version of this book [12] (joint with Ladyzhenskaya) has over 1600 citations in MathSciNet. This naturally gives a picture of a mathematician with tremendous impact on the field of partial differential equations. Needless to say that, even though there are many new books on the topic of PDEs, these books stay equally important and extremely valuable to many PhD students and early-career analysts.

Nina Uraltseva has, over the years, contributed to the mathematical community by serving on many important committees; e.g., chairing the PDE Panel of the International Congress of Mathematicians in Berlin, Germany, 1998, and the Prize Committee of the European Congress of Mathematics in Stockholm, Sweden, 2004. She also served as an expert for research foundations such as the European Research Council and the Russian Foundation for Basic Research.

She has been an editor for several journals,[8] and has been a frequent visitor of many universities all over the world and presented talks at various international conferences and schools. In her role as a world leading expert in analysis of PDEs she has captured the attention of many female students in all areas of mathematics, and attracted them to further pursue research and start a career in mathematics. Her motivational talks at many conferences, especially meetings related to "connection to women," have been an important factor in attracting several females to mathematics.

The instructional aspect of her work and her dedication to educating PhD students,[9] as well as unselfishly being available to students and colleagues for discussions and brain-storming of their problems, make her one of the most prominent and devoted persons in the mathematical community.

Nina Uraltseva has dedicated her life to mathematics, and in her scientific journey through the years she has made many friends all over the world. Her kind personality and utmost politeness on one side and her unbiased style and open mindedness towards diverse mathematical problems have made her extremely popular among colleagues and students, not only as a mathematician but also as a human being.

[7] *See:* `https://www.scilag.net/profile/nina-uraltseva`.

[8] *Editor in Chief for* Proceedings of St. Petersburg Math. Society *and* Journal of Problems in Mathematical Analysis; *member of the editorial committee for* Algebra and Analysis *(translated in* St. Petersburg Mathematical Journal*)*, Vestnik of St. Petersburg State University, Lithuanian Mathematical Journal.

[9] *Uraltseva has supervised 13 PhD students, four of which have habilitated.*

References

[1] John Andersson, Henrik Shahgholian, Nina N. Uraltseva, and Georg S. Weiss, *Equilibrium points of a singular cooperative system with free boundary*, Adv. Math. **280** (2015), 743–771, DOI 10.1016/j.aim.2015.04.014. MR3350233

[2] D. E. Apushkinskaya, H. Shahgholian, and N. N. Uraltseva, *Boundary estimates for solutions of a parabolic free boundary problem* (English, with English and Russian summaries), Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **271** (2000), no. Kraev. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funkts. 31, 39–55, 313, DOI 10.1023/A:1023357416587; English transl., J. Math. Sci. (N.Y.) **115** (2003), no. 6, 2720–2730. MR1810607

[3] D. E. Apushkinskaya and N. N. Ural′tseva, *On the behavior of the free boundary near the boundary of the domain* (Russian, with English and Russian summaries), Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **221** (1995), no. Kraev. Zadachi Mat. Fiz. i Smezh. Voprosy Teor. Funktsiĭ. 26, 5–19, 253, DOI 10.1007/BF02355579; English transl., J. Math. Sci. (New York) **87** (1997), no. 2, 3267–3276. MR1359745

[4] Darya E. Apushkinskaya and Nina N. Uraltseva, *On regularity properties of solutions to the hysteresis-type problem*, Interfaces Free Bound. **17** (2015), no. 1, 93–115, DOI 10.4171/IFB/335. MR3352792

[5] A. Arkhipova and N. Uraltseva, *Sharp estimates for solutions of a parabolic Signorini problem*, Math. Nachr. **177** (1996), 11–29, DOI 10.1002/mana.19961770103. MR1374941

[6] A. A. Arkhipova and N. N. Ural′tseva, *Regularity of the solutions of variational inequalities with convex constraints on the boundary of the domain for nonlinear operators with a diagonal principal part* (Russian, with English summary), Vestnik Leningrad. Univ. Mat. Mekh. Astronom. **vyp. 3** (1987), 13–19, 127. MR928154

[7] A. A. Arkhipova and N. N. Ural′tseva, *Regularity of the solution of a problem with a two-sided limit on a boundary for elliptic and parabolic equations* (Russian), Trudy Mat. Inst. Steklov. **179** (1988), 5–22, 241. Translated in Proc. Steklov Inst. Math. **1989**, no. 2, 1–19; Boundary value problems of mathematical physics, 13 (Russian). MR964910

[8] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural′ceva, *Linear and quasilinear equations of parabolic type* (Russian), Translations of Mathematical Monographs, Vol. 23, American Mathematical Society, Providence, R.I., 1968. Translated from the Russian by S. Smith. MR0241822

[9] O. A. Ladyženskaja and N. N. Ural′ceva, *A boundary-value problem for linear and quasi-linear parabolic equations. I, II, III*, Iaz. Akad. Nauk SSSR Ser. Mat. **26** (1962), 5-52; ibid. **26** (1962), 753- 780; ibid. **27** (1962), 161–240. MR0181837

[10] O. A. Ladyzhenskaya and N. N. Ural′tseva, *Local estimates for gradients of solutions of non-uniformly elliptic and parabolic equations*, Comm. Pure Appl. Math. **23** (1970), 677–703, DOI 10.1002/cpa.3160230409. MR265745

[11] O. A. Ladyzhenskaya and N. N. Ural′tseva, *A survey of results on the solvability of boundary value problems for uniformly elliptic and parabolic second-order quasilinear equations having unbounded singularities* (Russian), Uspekhi Mat. Nauk **41** (1986), no. 5(251), 59–83, 262. MR878325

[12] Olga A. Ladyzhenskaya and Nina N. Ural′tseva, *Linear and quasilinear elliptic equations*, Academic Press, New York-London, 1968. Translated from the Russian by Scripta Technica, Inc; Translation editor: Leon Ehrenpreis. MR0244627

[13] A. I. Nazarov and N. N. Ural′tseva, *The Harnack inequality and related properties of solutions of elliptic and parabolic equations with divergence-free lower-order coefficients*, Algebra i Analiz **23** (2011), no. 1, 136–168. MR2760150

[14] Vladimir I. Oliker and Nina N. Uraltseva, *Evolution of nonparametric surfaces with speed depending on curvature. III. Some remarks on mean curvature and anisotropic flows*, Degenerate diffusions (Minneapolis, MN, 1991), IMA Vol. Math. Appl., vol. 47, Springer, New York, 1993, pp. 141–156, DOI 10.1007/978-1-4612-0885-3_10. MR1246345

[15] Arshak Petrosyan, Henrik Shahgholian, and Nina Uraltseva, *Regularity of free boundaries in obstacle-type problems*, Graduate Studies in Mathematics, vol. 136, American Mathematical Society, Providence, RI, 2012, DOI 10.1090/gsm/136. MR2962060

[16] Henrik Shahgholian, Nina Uraltseva, and Georg S. Weiss, *The two-phase membrane problem—regularity of the free boundaries in higher dimensions*, Int. Math. Res. Not. IMRN **8** (2007), Art. ID rnm026, 16, DOI 10.1093/imrn/rnm026. MR2340105

[17] N. N. Ural′ceva, *Regularity of solutions of multidimensional elliptic equations and variational problems*, Soviet Math. Dokl. **1** (1960), 161–164. MR0126742

[18] N. N. Ural′ceva, *Boundary-value problems for quasi-linear elliptic equations and systems with principal part of divergence type*, Dokl. Akad. Nauk SSSR **147** (1962), 313–316. MR0142886

[19] N. N. Ural′ceva, *General second-order quasi-linear equations and certain classes of systems of equations of elliptic type*, Dokl. Akad. Nauk SSSR **146** (1962), 778–781. MR0140817

[20] N. N. Ural′ceva, *Degenerate quasilinear elliptic systems*, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **7** (1968), 184–222. MR0244628

[21] N. N. Ural′ceva, *The solvability of the capillarity problem. II*, Vestnik Leningrad. Univ. **1**, Mat. Meh. Astronom. vyp. **1** (1975), 143–149, 191. Collection of articles dedicated to the memory of Academician V. I. Smirnov. MR0638360

[22] N. N. Ural′tseva, *Hölder continuity of gradients of solutions of parabolic equations with boundary conditions of Signorini type*, Dokl. Akad. Nauk SSSR **280** (1985), no. 3, 563–565. MR775926

[23] N. N. Ural′tseva, *On the regularity of solutions of variational inequalities*, Uspekhi Mat. Nauk **42** (1987), no. 6(258), 151–174, 248. MR933999

[24] N. N. Ural′tseva, *C^1 regularity of the boundary of a noncoincident set in a problem with an obstacle*, Algebra i Analiz **8** (1996), no. 2, 205–221. MR1392033

[25] N. N. Ural′tseva and A. B. Urdaletova, *Boundedness of gradients of generalized solutions of degenerate nonuniformly elliptic quasilinear equations* (Russian, with English summary), Vestnik Leningrad. Univ. Mat. Mekh. Astronom. **vyp. 4** (1983), 50–56. MR725829

Darya Apushkinskaya

Arshak Petrosyan

Henrik Shahgholian

Credits

Opening photo is courtesy of Norayr Matevosyan.
Figures 1 and 2 are from Nina Uraltseva's personal archive (Figure 2 photograph by Nina Alovert).
Figures 4–6 are courtesy of Arshak Petrosyan.
Figure 7 is courtesy of Sophia Nazarova.
Photo of Darya Apushkinskaya is courtesy of Darya Apushkinskaya.
Photo of Arshak Petrosyan is courtesy of Arshak Petrosyan.
Photo of Henrik Shahgholian is courtesy of Henrik Shahgholian.

The Early Career Section offers information and suggestions for graduate students, job seekers, early career academics of all types, and those who mentor them. Angela Gibney serves as the editor of this section assisted by the Early Career Intern Katie Storey. Next month's theme will be advice from the BIG Math Network. For all the Early Career articles that have appeared so far listed by topic, see `www.angelagibney.org/the-ec-by-topic`.

In Celebration of Women's History Month

Advice from our Advisor: Fan Chung

Sinan G. Aksoy

Fan and the World of Mathematics

As I reached for my wallet to offer my share for dinner, my advisor stopped me and said, "Listen, Sinan. We're going to have a lot of dinners together here. Let's just decide now that I'm going to pay for every single one of them. Your advisor pays for dinner. Once you have students of your own, then it will be your turn. OK?" Thanking her, I put my wallet away. I began to walk back toward my hotel and Fan said "Meet you here again on Wednesday?"

We had just landed in Taipei for a semester-long visit at National Taiwan University. In the following months, I would complete the first major result of my thesis. Fan and I would pour over dozens of drafts of our first joint paper. She would help break my habit of writing overly dense sentences (*"If you try to say too much all at once, you'll end up saying nothing at all!"*), impart the importance of a compelling and succinct introduction (*"You know, many people won't read your paper beyond the intro and theorem statement..."*), and push me to strengthen our main result, even when I was certain we were finally done (*"It's normal to have dozens of 'final drafts' before actually converging!"*). Through all this, I'd begin to feel a sense of security and optimism about my prospects as a career mathematician. But before all that, my advisor was establishing a different type of routine for this chapter of our collaboration: she was making sure that, in a place where I knew no one else, I'd have a friend with whom to regularly eat dinner.

These are the types of interactions that come to mind when I reflect on what my advisor taught me. In this article, I and Fan's other students recount lessons we learned from her, organizing them into three "axioms" for flourishing in the mathematical world. As will soon be made clear, Fan Chung showed us the role of a good advisor was not limited to the classroom. Sure, she would walk you through the spectral proof of Szemerédi's Regularity Lemma. But

Sinan G. Aksoy is a scientist at Pacific Northwest National Laboratory. His email address is `sinan.aksoy@pnnl.gov`.

DOI: https://dx.doi.org/10.1090/noti2441

she would also counsel you on how to handle aggressive questions during math talks, how to rebound when another researcher publishes the theorem you were working on before you, and how to navigate the social complexities that arise in collaboration. All the while, she was your friend. Perhaps underlying Fan's advising style is the recognition that a myriad of sometimes-fragile things must happen to grow and sustain a healthy career as a mathematician. Fan's best advice was given, often by example, on how to thrive within the *world of mathematics*. Our accounts of her "advising axioms" are by no means comprehensive, since, as put by Josh Cooper:

> Fan Chung has been a tremendous source of inspiration for generations of mathematicians working in so many areas it is hard to even categorize them. In addition, she is a font of profoundly good advice. Fellow graduate students and I never knew quite what to think about Fan's penchant for waxing philosophical in class, in discussions in her office, and in research talks. Much to my surprise, these moments of casual commentary and impressionistic meta-cognition about mathematics have stuck with me over the years and have become integral to my own research advisement! I hear myself repeat her words to students frequently, and I often find myself wishing that I followed her advice more closely myself.

Indeed, for many of us, both in our recollections and through our continued interactions with her, Fan's guidance is ongoing: "Even now, more than a decade after finishing my degree, Fan is still interested in hearing what I am doing and offers suggestions on different opportunities I should explore," says Steve Butler. "For Fan, being an advisor to her students is more than getting a dissertation completed, it involves setting students up for long-term success and is a lifetime commitment." As a first step towards this success, Fan acclimated her students to the sometimes-dizzying and failure-prone nature of mathematics research, via a problem selection approach we dub the "velcro ball method."

Axiom 1: Use the Velcro Ball Research Method

On Fan's bookshelf, a plastic desk sign tucked in the corner reads:

> *A creative mess is better than idle tidiness.*

Sometimes overwhelmed, I found myself staring at this sign—the words "creative mess" glaring—when listening to Fan launch into *yet another* open problem during our weekly meeting. As noted by several of Fan's collaborators and students, Fan advocated working on multiple problems simultaneously. As put by past student Franklin Kenter:

> Fan was a proponent of the "velcro ball method." Instead of dictating what problem to work on or what direction to take, Fan would expose her students to a multitude of different problems. During individual research meetings, the students take the reins and discuss their progress. If they "forget" to bring up a certain point or problem, then the "velcro ball" didn't stick. Time to throw another!

The velcro ball method was not without an initial adjustment period. Sam Spiro recounts:

> Almost every time I met with her, Fan would disregard whatever we were working on last week, and instead give me a completely new problem to work on. At first I didn't make progress on any of these problems, which made me feel like I was failing Fan. Eventually I realized that Fan was by no means expecting me to solve all of these problems: if I could solve one in ten of them, I would be making good progress. And indeed, before long, she threw out a problem that I actually made some progress on, and eventually I ended up solving it. After this happened a few more times, I really began to appreciate Fan's methodology.

Many of Fan's former students report using Fan's frenzied approach to problem selection. As a practical matter, sometimes retreating from a problem gives us much needed headspace: "That way, if you get stuck, there is something else to think about to allow your subconscious to chew on the sticking point," notes Josh Cooper. As a tool for teaching, Franklin Kenter found her approach elicits introspection within his students: "Currently, I use this approach with my undergraduate students; albeit, with more guidance. Ultimately, a student project is what they make of it, and it is best if the student has interest and takes ownership in the specific topic." In my own case, I've internalized Fan's velcro ball method as a norm for maintaining productivity in a multidisciplinary research environment. For national laboratory scientists like myself, pivoting between multiple projects on a daily basis is a necessity. In a setting where I may interact with chemists, biologists, and power grid engineers within a single day, Fan's training keeps me comfortable and grounded in what might otherwise feel like a chaotic environment.

Axiom 2: The Math Career Graph Is an Expander—Find Your Path!

Just as the velcro ball method prompted us to sift through options and reflect on the problems *we liked*, Fan encouraged us to reflect more broadly on rich options available to us as early-career mathematicians. With her own varied career spanning academia and Bell Labs serving as proof, Fan sought to empower us with frequent reminders that mathematicians are uniquely positioned to pursue a plethora of career paths: finance, government, industry, teaching, academia, and more. To be clear, Fan didn't feel her job was to steer you towards any one of these—or any subfield within mathematics for that matter. Ross Richardson recounts:

> Despite being the driver of a number of distinct research programs, in my tenure I never observed Fan to push any of her students into a particular line of research or force them to advance or advocate for her projects. She unflappably supported me in each direction I pursued. While she is available to students, she understands the value of allowing students to motivate and pace themselves. Her active role as advisor is one of connecting and empowering her students.

Instead of choosing your path for you, Fan instead asked you to start by committing (firmly, for now) to a path. Once you had, Fan assuaged graduation fears by acknowledging she had no intention of getting in your way. Franklin Kenter recalls:

> "Find a job, and the thesis will write itself" was something Fan would often repeat. The point here is that Fan was very mindful of her students' career goals and provided opportunities for her students to see those career goals. Not every student has the same desires; some want to teach, some want to work in tech, others want to work in national labs, and so on. Each of these directions requires a slightly different emphasis, but they all require research experience nonetheless. Typically, if one had enough research experience to secure *their* job, then they had enough for a thesis!

Franklin describes his appreciation for this attitude, having been both on the giving and receiving end:

> As an undergraduate project advisor, I take this philosophy to heart. For instance, a student once informed me they were going to focus on studying for their professional exams instead of working on our project for one whole month. I could have been annoyed or even threatened their grade, but in the end I should be as supportive of their goal as possible. They passed with flying colors, and it made the final months much more pleasant as an advisor—even resulting in a submission. Indeed, "they found their job, and the thesis wrote itself."

Lastly, Fan's career advice also acknowledged the broader social contexts in which we work. As one example of such, Olivia Simpson reflects on how Fan made a name for herself as a female mathematician while also working at the same institutions as another renowned mathematician, her husband Ron Graham:

> I think she also understood that the playing field was a little different for me as a woman. I'm not sure she ever explicitly expressed this to me, but I felt a nurturing from her that went beyond what I thought an advisor relationship would be. I remember once asking her what it was like starting out, working with Ron at Bell Labs, and she said that it was "quite hard to not be in his shadow." This really helped me to understand that she helped pave an important path for women in mathematics.

It is doubtless Fan's role as a trailblazing female mathematician—including being among an early group of women to get tenure from Ivy League universities—paved a path for many women in mathematics. Her success also underscores her simple advice to young women interested in math: *don't be intimidated!* Not only does a confident mindset overcome the all-too-common fear barrier in mathematics, but it also puts us at ease during an activity Fan cherished: collaborating with others.

Axiom 3: Collaboration Means Family

The first time I heard Fan say, "You know, collaboration is a much closer relationship than most ordinary friendships," I didn't really get it. Over the years, I've come to appreciate how far this viewpoint is from hyperbole. Much of Fan's advice stressed the importance of, as put by Ross Richardson, "how we communicate, promote, and socialize our work as part of a delicate series of social interactions." Continuing, Ross writes:

> If there is one point she makes explicitly more than any other, it is the importance of the social aspect of mathematics, and the intense and joyful friendships that come from collaboration. This emphasis is unnecessary, however—it is clear in how much energy she puts into her

Figure 1. Fan Chung, Ron Graham, and many of their past students at the Networked Life conference in 2016.

> collaborations that they are the highlight of her mathematical life.

To Fan, collaboration means wholly committing your time and energy to your collaborators. Fan was available at seemingly any hour of the day; "an emailed question on a Friday night could lead to an hour-long conversation on Saturday morning," recalls former student Josh Tobin. Above all, Fan advocated for garnering trust with collaborators by behaving *generously*. For example, Richardson recalls Fan's willingness to share credit:

> I remember her handing me a draft latex file containing an unfinished attack on the Erdős unit distance problem. It was still raw and contained new unpublished ideas, but she gave it to me freely based on my interests to see if I could push it forward. I was floored that she would hand it over, and it stayed with me as a mark of the respect and trust she had in her students and in what she expected to be part of any collaboration.

As put by Alex Tsiatas, this intellectual generosity is a reflection of the fact that "Fan practiced research without ego." Another facet of this generosity is acknowledging the contributions of others. Distilled to three words, Josh Cooper summarizes Fan's advice in this regard as: "*Don't burn bridges!* Math is about community, and so requires care in addressing colleagues. It costs nothing to include another coauthor, but not doing so can lead to all kinds of headaches. In fact, after a paper is written, it is wise to take the stance that each person involved probably did 75% of the work." As my own network of collaborators has grown, I increasingly take Fan's attitude towards collaboration as essential. The undertaking of any collaboration requires granting trust and showing intellectual vulnerability: trust that everyone will see through the (often years-long, arduous) process of paper drafting, submission, referee reports, and revisions, trust that everyone will maintain mutual respect should they make mistakes or contribute significantly more or less than their coauthors, and trust in our ability to effectively co-steward and gracefully share credit for ideas which form the basis of our careers. I view Fan's generosity practices as ways of honoring these types of trust.

Through these practices, Fan advocated cultivating a family of collaborators throughout one's career. For Fan, this sometimes manifested in the near-literal sense of the word "family." Steve Butler realized this early in graduate school:

> When I was in graduate school I moved a significant distance from San Diego because of my wife's work. So I started sleeping in my office several nights a week to limit my commuting. Eventually word of this got to my advisor (Fan) and instead of ignoring the situation or kicking me out of my office, Fan offered up a room in her basement that I could stay in when I needed. Over the course of the next several years I spent hundreds of nights there, and became like a part of the family.

In fact, a number of Fan's students feel familial bonds with her. "She considers me and my family as a part of her extended family," says Linyuan Lu, fondly recalling how "she gave gifts to my children and would take us to have Dim Sum with her at San Diego's Emerald Restaurant." Similarly, Mark Kempton was struck by how "Fan always took an interest in my family after I got married and had kids." I too feel the same bond with Fan.

Nevertheless, as family, collaborators must also be willing to show tough love when necessary. "Fan didn't hold back with her criticism," recalls Mark Kempton. Linyuan Lu similarly echoes, "When something goes wrong, she is not afraid of pointing out my errors!" In particular, Fan's students quickly become acutely aware of her uncompromising standards for the tone, flow, and presentation of math talks. Paul Horn recounts a common experience of tough love:

> I thought highly of my talks at the time and so it was surprising to me when, after the talk, everyone left the room except for Fan and Jeff Remmel and she absolutely obliterated my talk. She deconstructed it from beginning to end, pointing out myriad mistakes. I was stung—I thought I had done well. But, as I processed, I also realized she was absolutely right on every point. That day, I started to completely rethink how I plotted out my talks and slides and today, whatever flaws my talks may have, they are 1000 times better thanks to her.

Linyuan Lu similarly reports benefiting from Fan's criticism of his job talk, crediting his job offer at the University of South Carolina in 2004 to the "weeks of training and practice Fan and Ron's mom provided on pronunciation and presentation."

In conclusion, Fan Chung's advising axioms are aimed at protecting and nourishing the joy we share by engaging in mathematics together. From her problem solving approach, to her career guidance, to her collaboration ethics and beyond, her advising has profoundly impacted her students' careers, affecting our research, teaching, and management practices in diverse and substantive ways. Whether it be Paul Horn emulating the "pride she took in our accomplishments and opportunities she afforded her graduate students," Steve Butler passing on "Fan's compassion and caring," or Jake Hughes and Alex Tsiatas applying her collaboration philosophies to "foster an inclusive, diverse, high-performing team" in industry settings, we are in agreement with Josh Cooper when he says: "Fan's thoughtful influence on her students and colleagues will have as lasting an impact on mathematics as her theorems." I submit there is much to be gained by internalizing her axioms, for students, educators, and all those inhabiting the mathematical world alike.

Sinan G. Aksoy

Credits

Figure 1 is courtesy of Todd Kemp.
Photo of Sinan G. Aksoy is courtesy of Sinan G. Aksoy.

Celebrating Karen Parshall as an Advisor

Karen Parshall is the Commonwealth Professor of Mathematics and History at the University of Virginia. Her extensive research focuses on the history of nineteenth- and twentieth-century mathematics. She was named an inaugural Fellow of the American Mathematical Society in 2012 and a Fellow of the American Association for the Advancement of Science in the Section on Mathematics in 2020. In 2018, she received the Albert Leon Whiteman Memorial Prize of the American Mathematical Society "for her outstanding work in the history of mathematics, and in particular, for her work on the evolution of mathematics in the United States and on the history of algebra, as well as for her substantial contribution to the international life of her discipline through students, editorial work, and conferences."[1] Here, her graduate students Della Dumbaugh, Patti Hunter, Sloan Despeaux, Deborah Kent, and Laura Martini (organized in order of completion of their PhD) offer reflections on their experiences while working with Karen Parshall.

[1]*The citation for this prize offers an overview of Parshall's scholarly work along with biographical and autobiographical insights. See "2018 Albert Leon Whiteman Prize Announcement,"* Notices of the American Mathematical Society *65 (4), 2018, 472–474.* `https://www.ams.org/journals/notices/201804/rnoti-p472.pdf`.

DOI: https://dx.doi.org/10.1090/noti2445

Della Dumbaugh

When I arrived at the University of Virginia as a graduate student in the fall of 1988, I planned to study pure mathematics, with some combination of algebra and number theory. One rainy Friday evening that first semester I stumbled on a copy of Carl Boyer's *A History of Mathematics* at the local independent bookstore in Charlottesville. I couldn't put it down that weekend. I decided to take a class that spring in the history of mathematics. This meant listening to Karen Parshall talk about the history of calculus, learning about and reading primary sources, and writing papers on a broad range of topics, including the history of the solution to the cubic. I was hooked.

This class led to a joint project with Karen exploring the American mathematical community as it took shape in the late nineteenth- and early twentieth-centuries. This entailed me driving to Karen's lovely home on Coleman Drive, to the sunny solarium where we combed through early editions of the *Bulletin of the New York* and, later, *American Mathematical Society* and meticulously recorded information about mathematicians, talks, conferences, and institutions. These details ultimately combined to identify a growing, vibrant community of mathematicians. As we worked, we talked about our observations. Who was this Leonard Dickson who kept giving talks and writing papers? Why were there so few women recorded on these early pages of the *Bulletin*? What was going on at the University of Chicago? I listened to Karen talk more broadly about these observations and queries that arose along the way. Looking back, these long afternoons form some of my most treasured moments with Karen. I had the chance to hear her think out loud about the early American mathematical community as new ideas unfolded before us.

Sometime later, I made an appointment with Karen, drove out to Coleman Drive, handed her a single piece of paper with an outline of a dissertation on Leonard Dickson and his work in the theory of algebras, and asked her if she would take me on as a student. She said yes. That moment, I suppose, was the beginning of my work with her as a graduate advisor. Through Karen's expert guidance, that piece of paper eventually grew into a 237-page dissertation. How did that happen? That evolution hinged on what I consider the two sterling features of Karen Parshall as PhD advisor: her commitment to weekly meetings and her focus on writing. The former taught consistency and the latter attention to detail. To this day, I am never far from my current research project and I am unafraid to print out my written work and take a red pen to it. But these skills did not come easily. Meeting with Karen to discuss Bruno Latour or Thomas Kuhn could leave me frazzled and

Della Dumbaugh is a professor of mathematics at the University of Richmond and editor of the American Mathematical Monthly. *Her email address is* `ddumbaugh@richmond.edu`.

frustrated (but I always went back). It could be deflating to hand her twenty pages of work one week and have her hand it back to me the next week with so much red ink I could barely find the original text. Over time, however, I came to appreciate these practices. I gradually learned to ask more informed questions as I read demanding texts. (It also helped when Patti Hunter joined the research group so she and I could discuss the texts all week long in advance of the meeting with Karen.) I also grew more accustomed to the red ink and, gradually, I learned to reach for stronger verbs and write in the active voice. Lo and behold, the red ink subsided. I remember writing my parents during this time and searching for stronger verbs as I expressed my personal thoughts on lavender stationary. Karen had worked her magic. Years passed before I realized how much of herself she had given me in the process. When was the last time someone read ten or fifteen pages of your writing and offered you genuine, thoughtful feedback? That is work. It is a labor of love that advances you personally and the professional community more broadly.

Karen helped me learn to appreciate the writing process and, in particular, to value the opportunity to reconsider and revise my thoughts. She showed me how to keep both the details and the broader context in focus. She also taught me to give talks in a specified time frame ("if your talk is scheduled for 20 minutes, people stop listening at 21 minutes"). I remain grateful to Karen for believing in me before I believed in myself and for her consistent advising strategies that allowed me to grow into a scholar and pursue an immensely fulfilling life in mathematics.

Della Dumbaugh

Patti Hunter

As a Master's Degree student in Mathematics at the University of Virginia, I took a History of Mathematics course from Karen my second year, at the recommendation of a good friend, Della Dumbaugh. "Karen will challenge you to read carefully and write clearly, and her course will open your eyes to new vistas in mathematics," Della insisted. She was right and it was a delightful course—one of the last, I thought, I would take as a graduate student, since I intended to find a job teaching high school mathematics. About a year and a half later, while teaching at a small college as a sabbatical replacement, I registered for the Joint Mathematics Meetings and noticed a talk by Karen on the program. Sliding into a seat as the talk began, I caught Karen's eye, wondering whether she would remember me—her look of recognition and delight emboldened me to talk with her after the lecture, and she suggested that we have coffee. A few weeks later I submitted my application to pursue doctoral work with Karen, back at U.Va.

Over the next few years, Karen would always greet my weekly knock on her door for our meetings with that look of delight, no matter how busy she was with her teaching, scholarship, or leadership in the university's departments of history and mathematics where she holds appointments. Karen invested in us as her students as part of her calling to contribute to the community of historians of mathematics. She taught us—by example—that the people of the community are important. The subjects of our research are important, and we ourselves, as students, teachers, and scholars, are important. Her own research uncovered the crucial aspects of the formation and sustenance of scholarly communities, and I suspect her investment in her students emerged in part from what she learned in her research about the important role played by mentors in advancing those communities.

As a beneficiary of Karen's commitment to the advancement of knowledge, and to the nourishing of communities that produce and disseminate that knowledge, I am grateful for the investment she made in my own professional work.

Patti Hunter

Sloan Despeaux

With every passing year, I am more impressed with the work and commitment Karen put into being my advisor. I once heard that completing a PhD is the most self-absorbing thing a person can do, and it is true that during my path to a PhD in the history of mathematics under Karen, I only thought about my own efforts. What I did not consider is the time, energy, and direction she devoted to me every day.

Patti Hunter is a professor of mathematics and vice provost at Westmont College. Her email address is phunter@westmont.edu.

Sloan Despeaux is a professor of mathematics at Western Carolina University. Her email address is despeaux@email.wcu.edu.

I remember our first discussion of an article she had assigned me to read. I dutifully rattled off a summary of what the article said, but Karen quickly challenged me to analyze its argument. This challenge was the beginning of my historiographical training. Since then, Karen has taught me big things like asking intriguing questions and balancing mathematical detail with wider historical context, but she has also taught me little but important things like not jiggling change in my pocket while I give a talk. Every one of these lessons took time, which was in short supply for Karen as a professor in two different departments (history and mathematics) at the University of Virginia.

As the years go by, I also appreciate more and more how Karen prepared me to succeed in a mathematics department. While I groaned at the time about all of the graduate mathematics courses she required me to take (on top of French and British history and her seminar), I now realize how important it was for me to feel as much a mathematician as a historian. Through her foresight and planning, I sidestepped obstacles I did not even realize were there.

Sloan Despeaux

Deborah Kent

While there are only a few pithy words of advice I recall from Karen—about job negotiations: "Don't be milquetoast" and on juggling many tasks: "You can't expect the luxury of doing only one thing at a time"—her example embodied advice for academic endeavors. I have witnessed firsthand her process of writing a book from start to finish, including remarkable perseverance in the face of tedious tasks like making an index or proofreading a bibliography. Sometimes, what's left is simply to do the work. I also observed the value of working with colleagues one enjoys both personally and professionally. From crudité with the resident pet basset hound, through coconut cake dessert, her dinner parties were—like her research projects—precisely planned, meticulously organized, and successfully accomplished. The faculty book club she hosted and the dedicated weekly correspondence with her own PhD advisor likewise communicated the value of collegial community.

In Karen's publications, I read about the tripartite goal of a research university: teaching, research, and the training of future researchers. Under Karen's supervision, I lived this. She had envisioned and designed a path to graduate work in History of Mathematics well before I arrived at the mathematics department at the University of Virginia. This program involved all the standard requirements for mathematics PhD students—entrance requirements, coursework, two general exams, two language exams, a proposal, a thesis defense—in addition to teaching duties, archive work, and ongoing training in historiography. Although the process sometimes felt taxing, I have many times appreciated both the thoroughness and the practicality of this preparation for the rigors of academic life. Karen intentionally equipped her students with essential tools for being an historian of mathematics in a mathematics department.

Deborah Kent

Laura Martini

In 1998, after finishing my *laurea* degree at the University of Siena, I moved from Italy to Virginia on a traveling fellowship for advanced studies in the History of Mathematics to specifically study under Karen's supervision. I then became her first foreign PhD student in her History of Mathematics program at the University of Virginia.

While I was undertaking PhD-related tasks in a second language for the first time, Karen went out of her way to provide extra advice and guidance. During my course of studies Karen served as a model both as a researcher and a teacher: I have witnessed and learned from her writing process of articles and books, her preparation for talks and lectures, and her delivery of seminars and classes on the history of mathematics and history of science.

I had the privilege to be advised in an academic environment characterized by high standards and directed by Karen's methodology of meticulous organization, clarity, and rigorous historiography. From Karen I learned the value of academic community: her interactions with her own PhD adviser (historian Allen Debus—sadly, her other advisor, mathematician Yitz Herstein, had passed away before my

Deborah Kent is a reader in History of Mathematics at the University of St Andrews, Scotland. Her email address is dk89@st-andrews.ac.uk.

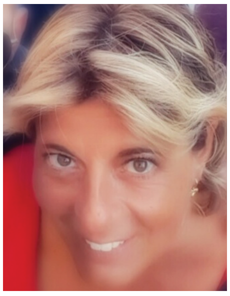
Laura Martini is currently working for a multinational company. Her email address is lauramartinisiena@gmail.com.

time with Karen) and her colleagues both professionally and personally provide an example of academic excellence.

I will never forget my first birthday away from home when Karen organized a surprise party at her own place as well as the warm and welcoming dinner for my father visiting from Italy.

Although the path of graduate work in the history of mathematics Karen had envisioned and designed for me sometimes felt burdensome, I always recognized her genuine goal of training her students to become knowledgeable and successful professionals.

I have often treasured the rigor and the dedication that Karen required from me as her student: these qualities have proved fundamental in the development of my professional career also outside of the academic world.



Laura Martini

Credits

Photo of Della Dumbaugh is courtesy of the University of Richmond.
Photo of Patti Hunter is courtesy of Patti Hunter.
Photo of Sloan Despeaux is courtesy of Sloan Despeaux.
Photo of Deborah Kent is courtesy of Radina Droumeva.
Photo of Laura Martini is courtesy of Laura Martini.

Interview with Jennifer Chayes[2]

Anthony Bonato

I first met Jennifer Chayes at the 2012 Workshop on Algorithms and Models for the Web Graph conference in Halifax. She gave a keynote talk, and I chatted with her as she set up her presentation. My first impression was of her cool confidence and the force of her intellect.

Jennifer is one of the leading researchers in network science, working at the interface of mathematics, physics, computational science, and biology. She is the Managing

Figure 2. Jennifer Chayes.

Director of Microsoft Research New England and New York City. She is highly awarded, being a Fellow of the American Mathematical Society, a Fellow of the Association for Computing Machinery, recipient of the Anita Borg Institute Women of Vision Leadership Award, and winner of the Society for Industrial and Applied Mathematics John von Neumann Lecture Prize. She was also recently made Microsoft Technical Fellow.

This interview was conducted in September 2017.

AB: *What was your first mathematical memory as a child?*

JC: I was four years old or so. I used to visit a neighbor—I started going there because a very nice woman gave cookies to my brothers and me. When we were choosing cookies, I heard her husband and daughter, who was a high-school math teacher, doing math problems. I thought it sounded cool, so I started asking them if they could give me puzzles. They probably thought I was weird, but they gave me puzzles, and I loved doing them.

There hadn't been any math in my household; my father was a pharmacist, and my mom couldn't add fractions (although she's very smart). But my neighbors sounded like they were having so much fun. They then started making up little word problems for me. I didn't know algebra or anything, although I did know how to count.

I liked it, and I found it very fun. They loved projective geometry, and they would give me things to work on at my level.

AB: *Was there a person or teacher who influenced your early scientific career before university?*

JC: In seventh grade, I took Euclidean geometry, and our teacher taught us how to prove things. It was an honors class for kids good in math. He taught us about logic: statements,

---

Anthony Bonato is a professor of mathematics at Ryerson University. His email address is abonato@ryerson.ca.

[2]*Chapter 3 of* Limitless Minds: Interviews with Mathematicians *by Anthony Bonato,* https://bookstore.ams.org/mbk-118.

DOI: https://dx.doi.org/10.1090/noti2442

converses, contrapositives, how to properly conclude things, and so on. I loved it, and for me it was magic.

This sometimes happens with great teachers. He obviously understood a lot—not all of my teachers were like that. In eighth grade, I asked my teacher why I couldn't put a square root in the denominator. My teacher said to me, "You can't put radicals in the denominator, just like you can't put bananas in the refrigerator!" While the year before I had someone teaching me formal logic. They were a huge variance in the quality, enthusiasm, and understanding that my math teachers had.

My seventh grade teacher was very passionate, and he would get excited when there were kids in the class like me who fell in love with the subject.

AB: *Your undergraduate was in biology and physics, and then you completed a doctorate in mathematical physics at Princeton. What led you to Princeton, and how did you end up working with your supervisor?*

JC: Theoretical physics was more stylish at the time than mathematical physics. People said mathematical physics wouldn't get me a job. But things have changed, with mathematicians now caring a lot about that topic, which is at the forefront of mathematics. I liked proving theorems, and I liked physics.

I thought I was going to be a particle physicist first, but it wasn't nearly as mathematical as it is now. Then I took a class from Elliott Lieb who was doing beautiful work on atomic physics. I liked the class, but I wasn't as wild about atomic physics. I asked him if he would do statistical physics with me as his student and he agreed.

I also worked with the supervisor of my ex-husband Lincoln Chayes, who was Michael Aizenman. I was co-advised by Elliott and Michael. The work with Elliott was more analytical, and the things with Michael were more probabilistic.

I knew Princeton had very good mathematical physics at the time. Even though I was not a mathematics undergraduate (I was one or two courses shy of such a degree), I loved mathematics. The summer before my senior year, I met Barry Simon at Princeton. Barry thought it was implausible that this couple coming from Wesleyan would go into mathematical physics at Princeton as they only accept one or two students in that area a year. He tried to have us apply to other graduate schools. I left feeling depressed, but I learned later that Barry was on the admissions committee and pushed to get us in.

Figure 3. Elliott Lieb.

AB: *Your work is in many areas, ranging from graph limits, to phase transitions, to modeling complex networks. How would you describe your research to a non-mathematician?*

JC: It might appear as if I am doing several different things, but I have a set of lenses through which I see the world. Many mathematicians have similar lenses, and, for many of us, the lenses are set early. In graduate school, I did work on random surfaces and percolation that helped set these lenses.

I try to see networks and geometric structures as representations of something going on in a certain system. Much of my early work was on percolation, which is like coffee percolating. A passageway is either open or closed to a liquid. Not everyone's grains of coffee are in the same configuration and yet there are some bulk properties similar to my espresso and yours. It's like taking draws from a distribution.

I tend to study systems with randomness, which is common in the real world. After I got to Microsoft about fifteen years ago, people were just starting to talk about the internet and the World Wide Web as structures that one could understand. I took insights from percolation and phase transitions, and I used that to model the internet and the web, and then later social networks. I also see networks in computational biology that I do. There are omic networks, where you have genomic or proteomic data, where many

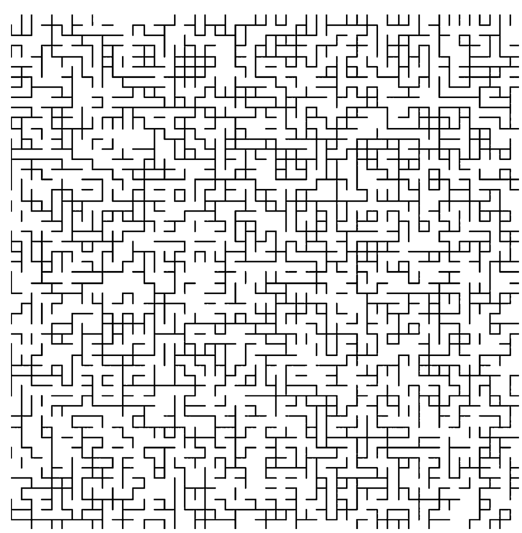


Figure 4. Percolation theory studies the connectivity of networks.

times you don't observe the whole network. For example, when you study an evolutionary tree, all you see are the leaves. In biological data, you don't see every gene that has been activated or every protein in a cell. You try to infer the missing parts of a network. These inference problems are often related to machine learning problems. There are nice implications there: if I recognize a protein as important that no one else has recognized, then that could be a drug target.

Phase transitions were things I also studied in graduate school, and, for me, these have become a metaphor for the world. We just underwent a phase transition in November 2016! In phase transitions, when you vary the system, you see both quantitative and qualitative changes. Examples are water boiling or freezing, or certain types of magnets, which are magnetized or not at certain temperatures. They also happen in graphical representations of other problems. This is how I went from a mathematics department to Microsoft. There were problems people studied in computer science that have graphical representations and that undergo phase transitions from tractable to intractable. There is a precise mathematical correspondence there, and I started to use equilibrium statistical physics to study things that were happening in theoretical computer science.

More recently, I've used non-equilibrium statistical physics methods, which happens in a system with a driving force of some sort. I am using this to provide insight into how deep neural nets work.

AB: *What research topics are you working on now? You can be more technical here if you like.*

JC: I am working on several different things. I am working on graph limits, which is something we invented twelve years ago with László Lovász, Christian Borgs, Vera Sós, and Katalin Vesztergombi. Christian and I have continued to work on the topic up to the present. We first did graph limits for dense graphs, where every node is connected to a positive fraction of all other nodes. Most networks in the real world are sparse, however. For example, Facebook

Figure 5. Jennifer Chayes with husband and collaborator Christian Borgs at Microsoft.

keeps growing, but I'm not friends with a positive fraction of other nodes. So that's a sparse graph.

In the last five years or so, we've developed two very different theories for graph limits of sparse graphs, and in particular sparse graphs with long tails like the Facebook graph or power-law graphs. We have one that is a static theory, a kind of L_p theory, where things are integrable, but they may not have a second moment. We also have a time-dependent theory that the statisticians like that models the progression of these networks.

Another thing I am working on is how to do A/B testing on networks. For example, we do A/B testing when the outcome of one group is getting a drug and another getting a placebo. Or the Microsoft homepage might roll out a different version to one percent of their traffic to see if they like that version more. But suppose I was studying people getting a flu shot with treated flu virus or a placebo flu shot. If your children got a real flu shot and you received a placebo shot, and none of you got the flu, it would not be sound to conclude the flu shot had no influence. There's interference in the network because members of your family are interacting with each other.

So how do you do an A/B test on a network? We have methods to draw correct inferences. We are excited about it since it has many practical applications for the experimental design of tests on networks of interacting entities.

Another big project we are starting is with Stand Up To Cancer. They are a wonderful foundation that has raised about six hundred million dollars. For our project, they raised about fourteen million dollars. They bring together groups of researchers to study certain classes of questions. Usually, they bring together biologists and oncologists. Recently, they've started what they call convergent projects, where they bring mathematicians, physicists, and computer scientists together with oncologists and biologists.

Our project is called Convergence 2.0, and we are studying cancer immunotherapy. We are applying machine learning and network analysis to try to understand why certain people respond favorably to cancer immunotherapy while others don't. These are very complex problems involving genomes and your T-cell profile. Everyone has a different T-cell profile; there is a neat combinatorial trick your body does to come up with a unique T-cell profile. We will work with people at about ten different institutions on problems around this. I also have a new physics-based theory of deep learning, which I mentioned earlier, although it's been non-rigorous up to this point. But in equilibrium statistical physics people are only now proving the results rigorously. There is little understanding of why these neural nets work, so I think it's worthwhile even to do non-rigorous work that gives us conjectures to attempt to prove.

AB: *Congratulations on becoming a Microsoft Technical Fellow.*

JC: In the industry, it's a big thing. It is the equivalent of a corporate vice president, but it is much more technical. Microsoft has over one hundred thousand employees, and under thirty Technical Fellows, so I am excited about it.

It wasn't just a promotion, but I have an additional responsibility. I have a lab in New York, one in Boston, and a small group in Israel, but I just got a new lab up in Montreal. It's a company that we acquired about eight months ago called Maluuba, which was roughly half research and half development, focusing on machine reading and comprehension, dialogue, reinforcement learning, and other topics in machine learning. I am also super excited about getting personally involved in the Montreal AI hub: Canada in general and Montreal, in particular, is at the forefront of AI, and I think this will continue. You have an amazing government, both federally and provincially in Quebec, which is supporting the Montreal region. We are going to be growing there, and I am thrilled to have a group there.

AB: *Maybe someday you will start something in Toronto?*

JC: Maybe. Like the investment in Montreal, the Canadian government is also making a big investment in Toronto with the Vector Institute. One of my postdocs is going there. There are three institutes in AI funded by the Canadian government. With these investments, instead of a brain drain, Canada is creating remarkable groups that can have an outsized influence on a dominant field.

AB: *What advice would you give to young people, especially young women, on pursuing a career in mathematics and STEM?*

JC: First, there is something I tell women even before they go to university: it's not sufficiently well publicized that STEM fields are creative and collaborative. We tend to see pictures of solitary guys sitting at computer terminals.

What I do is creative. I imagine worlds and prove theorems about them. Sometimes they have an impact on the real world; for example, they may help with cancer therapy. I have amazing collaborations, and I work in teams. And I have societal impact.

I think it is an easier life than being creative in other fields. I thought about becoming an artist when I was younger, but then I probably would have had to do a day job and come home exhausted trying to make art.

I would also say that for whatever reason, women tend to be less confident than men are. Part of that is not seeing as many role models. As a professor, I would interact with super talented women undergraduates, who didn't think they were good enough to go to graduate school. Women often don't realize that everyone is working hard; if you are doing well, it's also because of talent.

Women tend to take themselves out of the running before they should. Your self-assessment of your ability is not a reliable signal. If you like STEM, or if you had a teacher who tells you that you are good at it or you're passionate about it, then you should follow it. Reach out and network. STEM professions tend to lead to really satisfying careers.

Figure 6. "I imagine worlds and prove theorems about them." Jennifer Chayes.

AB: *You were a child of Iranian immigrants to the US. What effects do you think the travel ban is doing to mathematical and scientific research in the US right now?*

JC: I'm thrilled that Microsoft brings in Iranian interns, going through the necessary steps hiring them. We don't think about nationality when we are hiring someone. Many Iranians have been my interns and postdocs, so I was concerned by the travel ban. I have a young colleague at Yale whose family was stuck outside of the country owing to the ban. His wife and child couldn't get back in, and he couldn't leave because he had to teach.

I also have DACA [Deferred Action for Childhood Arrivals] students in New York City who are very scared now. I think we are a country of immigrants, and our greatest talent and vibrancy comes from that. Embracing immigrants is so fundamental to what it is to be America. I am concerned.

AB: *I'd like to close with looking forward. What would you say are some of the major directions for mathematics in the future (or in your own program)?*

JC: Whenever there are things that work well with little understanding of why, then I believe that there is some mathematics to be formulated and proved. In my position, I'm witnessing much of what is happening in deep learning—from image recognition to speech recognition to machine reading and comprehension. We see all these unexpected breakthroughs. These are high-dimensional random problems.

What is it about the structure of deep neural net algorithms that is finding useful information in these sparse high-dimensional structures? Answering that will involve many areas of mathematics, and a great deal of new mathematics will be developed.

I believe that the problems the world brings us guide us in the development of new mathematics.

Credits

Figure 2 is licensed under Creative Commons Attribution-ShareAlike 4.0 International (CC BYSA 4.0).
Figure 3 is courtesy of the Archives of the Mathematisches Forschungsinstitut Oberwolfach.
Figure 4 is licensed under Creative Commons Attribution-ShareAlike 3.0 Unported (CC BY-SA 3.0).
Figure 5 is licensed under Creative Commons Attribution 2.0 Generic (CC BY 2.0).
Figure 6 is licensed under Creative Commons Attribution-ShareAlike 3.0 International (CC BYSA 3.0).

Interview with Maria Chudnovsky[3]

Anthony Bonato

Maria Chudnovsky is a leading mathematician working in the field of graph theory. She was born in St. Petersburg, Russia, in 1977, and moved to Israel with her family at the age of thirteen. Maria studied mathematics at Technion in Haifa. She completed her PhD at Princeton University in 2003, supervised by Paul Seymour, where she is now a professor.

Maria has the most famous PhD dissertation in the recent history of graph theory. She proved, in joint work with Neil Robertson, Paul Seymour, and Robin Thomas, the "Strong Perfect Graph Theorem" (SPGT), which was first posed by Claude Berge back in the 1960s. Research on SPGT and perfect graphs resulted in hundreds of papers and partial solutions before its resolution. There are about 700 citations on MathSciNet® for the search "perfect graphs" up to 2006 when the 178-page proof was published in the *Annals of Mathematics*. I vividly remember the excitement surrounding the announcement of the proof of SPGT, and how it sent ripples throughout the discrete mathematics community and beyond.

Maria is a giant in her field. For her work on SPGT, she won the Fulkerson Prize in 2009. She holds a MacArthur Foundation Fellowship (or "Genius" grant), and her research is funded by the National Science Foundation. Maria is also unique among mathematicians I know of for appearing in not one, but two television commercials: one for TurboTax and one for Comfortpedic.

This interview was conducted in May 2016.

Figure 7. Maria Chudnovsky.

AB: *How did you first become interested in mathematics?*

MC: I don't remember a time when I wasn't. Math was always easy and fun, and everything else was hard. It was a very natural thing for me. It was always my favorite subject. I remember the pain of learning to read, and I don't remember the pain of learning to count.

AB: *Did anyone play a role in inspiring your interest in mathematics?*

MC: My dad loved math. He was an engineer, but as a kid he loved mathematics. Probably he said enough things to get me interested.

I was lucky as I had very good teachers. I went to a special mathematics school in St. Petersburg, Russia. I was born in Russia, and my family moved to Israel when I was thirteen. I went to this school from the ages seven to thirteen, where math was the most important thing in the world, and the best thing you could be was to be good at math. I also had many good teachers who made things beautiful and made things interesting. From everything I heard in school, I never doubted there was anything more interesting than math.

AB: *Can you tell us something about your experience at Technion? In particular, what was the environment like there, and how did it help lead you to Princeton?*

MC: I started at Technion in the eleventh grade, where I began going to a Math Circle that was led by a Masters student in applied mathematics. It was a fun experience. He would solve Math Olympiad problems with us. If he attended an advanced mathematics lecture, he would tell us

[3]*Chapter 4 of* Limitless Minds: Interviews with Mathematicians *by Anthony Bonato*, `https://bookstore.ams.org/mbk-118`.

DOI: https://dx.doi.org/10.1090/noti2443

about it, making it digestible for us high school kids. It was also a social thing; it was where I met most of my friends. That was huge in my life. It was when I realized you could be a mathematician by profession.

Unlike in the United States, in Israel, you have to declare your major right away. You don't apply to a university, but, instead, you apply to a department. If I hadn't gone to this Math Circle, then I wouldn't have applied to the math department.

Both my parents were engineers, and it was always clear that I should go into engineering or computer science, as I was good in mathematics. This is ironic, as the math I do is on the border of computer science; many of my colleagues work in computer science departments.

I have to say, a lot of it was social. We were a group of friends, all very interested in math. We talked to each other about the things we learned, how pretty or nice something was. That reinforced my conviction that this is what I want to do in my life. If I could keep going like this, it would be great.

I applied only to three places for my doctorate. One didn't accept me.

AB: *They are probably regretting that decision now!*

MC: It was clear that Princeton was the right choice for me.

AB: *Why did you choose to study graph theory?*

MC: I knew I was going to study discrete math because as an undergraduate I thought it was a pretty area and I had a good intuition for it. When you like something, it is never clear if you first like it and become good at it, or you are first good at it and then like it. I certainly had this connection with discrete math.

At Technion, I also did a Masters in discrete math. All the places I applied to had strong discrete math programs. At Princeton, my research would have been graph theory, and, in other places, it would be other topics. I ended up at Princeton, so I became a graph theorist.

AB: *How did you come to work on the Strong Perfect Graph Theorem (SPGT) in your doctoral work?*

MC: I showed up at Princeton, and I knew I wanted to work with Paul Seymour. That was the problem he was interested in at the time. I was lucky at the time that I didn't understand what was appropriate and what wasn't. I came up to him and said: "Can I work with you on this?" He said "Sure."

When he saw I was contributing, I became part of the group. I don't know if when I first approached him, he thought it was a little bit strange.

Figure 8. Paul Seymour.

AB: *Andrew Wiles famously spoke about a great eureka moment when he settled Fermat's Last Theorem. Did you have a similar "aha" moment when settling SPGT with your collaborators?*

MC: I was working with Paul Seymour at 5:45 p.m. (we usually work until 6:30 p.m.). We knew we were near the end of the proof, and there was one last step left. And then we saw it. We saw why A implies B.

We looked at each other and said "That's it. We are done; we can go home early."

AB: *When you settled something as important as SPGT, were you confident the proof was correct?*

MC: With all proofs, there are many levels of confidence. At first, you see a solution, and that is very good. But it is a huge proof so that you may have overlooked something. Then you sit down and write notes. After you have done that, you are more confident. And then you sit down and write a paper, and you feel even more confident. And then you start giving talks, and people think about what you said. And there is the refereeing process.

By now, we are pretty confident. With such a large proof, I don't know how to tell one hundred percent that it is true. Probably someone by now would have found a mistake, as it is something that people care about.

AB: *Structure plays a major role in the work you do, ranging from perfect graphs to claw-free graphs, or analysing other graph families. What do you think is the importance of structure in graph theory?*

MC: First, to me personally, structure is very satisfying. You are not just answering one question; you are seeing some huge, global phenomenon going on. This is how I like the world to be, where I can completely understand what is going on. If I have a question, then I can go and look at this systematic picture that I have in mind, and I can find out the answer to this question. That is one kind of answer to your question.

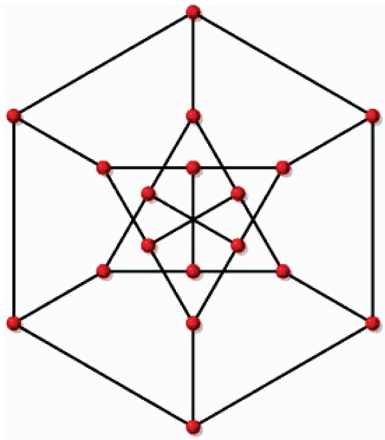


Figure 9. An example of a perfect graph. A graph is perfect if for every induced subgraph, the clique and chromatic numbers are equal.

Another answer is that it is surprising that some properties give you a kind of structure. There are many different properties of graphs you can think about. Some are little things: maybe they happen, or maybe they don't. Some have a huge influence on the graph.

In some sense, this is the strongest kind of theorem. You have some property, and it happens if and only if there is a certain structure. Somehow, that tells you a lot.

Not all properties allow you to prove a beautiful structure theorem. Some are just yes or no questions. When I give talks, I sometimes distinguish between properties and structure. It is remarkable and satisfying that with the property of a graph being perfect, you can understand the structure.

AB: *What inspires your mathematical ideas?*

MC: Things that appeal to me aesthetically. It could be a problem that seems beautiful or a concept that seems beautiful. Or it could be someone else's proof that seems beautiful, and I want to see what else I can do with it.

AB: *The notion of beauty comes up often in our discipline. On that note, mathematics is sometimes called a science or art. What is your view?*

MC: I think in the middle. Beauty is the guiding motivation. There is math that is motivated by physics, chemistry, or engineering. That is somehow separate. In much of math, you are just looking for the most beautiful thing you can think of. And only that determines if something is interesting or not. Beauty is also subjective. What I think of as beautiful someone else might think is ugly.

AB: *What advice would you give to young people, especially young women, who want to study mathematics?*

MC: I can give advice that is not exactly my own. When you start graduate school or anything big, you feel like that there is no way you are going to succeed. And there are setbacks: maybe you tried to solve a problem and you didn't, or you get a bad grade on an exam, or you attend a class you didn't understand.

You might then say to yourself that since I didn't understand this, I should be doing something else. That's not the right approach. What one shouldn't do is quit. It's not wrong to think about quitting, but I think one should take a very long time to consider the situation before quitting. Don't let your self-doubt scare you too much. Just accept that everyone has their moments when they feel like a complete misfit. Just keep pushing.

When you do something creative, ninety percent of the time you fail. If you are failing much of the time, you are not going to feel good about yourself much of the time. But then you succeed, and it more than makes up for it! You have to accept this as part of the creative lifestyle.

AB: *Besides mathematics, what are your interests or hobbies?*

MC: I like art. I don't produce it, but I like seeing art.

I have a two-and-a-half-year-old, and he is a full-time hobby. I am not one of these people who does math and then has another thing that is a close second. I would say my job is my hobby.

AB: *Graph theory is a robust discipline with so many directions, such as structural graph theory, probabilistic graph theory, topological graph theory, and applications through network science. What do you think are the major directions in the field?*

MC: That's a very good question that I wish I knew the answer to, as I would work in that direction. I think applications are becoming huge. I think applications are slightly different from the things I do since in applications the graph you are looking at is very large. The kind of things I do are deterministic things. What is needed in

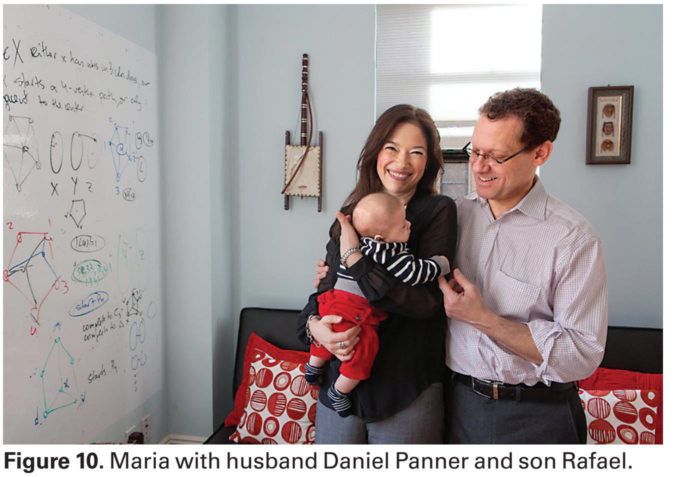


Figure 10. Maria with husband Daniel Panner and son Rafael.

applications are more like if you assemble ten percent of your information about the graph, then what can you say with high probability? I think there are many people doing beautiful theoretical research that's vaguely or not vaguely motivated by that approach.

I would like to take the classical questions I've worked on and translate them into this language. We used to prove that every vertex of this set is adjacent to *every* vertex of another set. Instead, we can think about if *many* vertices of one set are adjacent to many vertices of another. I would look for an analogue or translation like that.

Yesterday I was on the train, and I saw someone with a t-shirt with a graph on it. And I thought, how nice. It was a Princeton Computer Science t-shirt!

Credits

Figures 7 and 8 are courtesy of the Archives of the Mathematisches Forschungsinstitut Oberwolfach.

Figure 10 is courtesy of Emon Hassan/*The New York Times*/Redux.

Interview with Lisa Jeffrey[4]

Anthony Bonato

I met Lisa Jeffrey in Ottawa, where we worked together on the NSERC Evaluation Group for Mathematics & Statistics. Lisa comes off as modest and reticent, which reminds me of the quote by Stephen Hawking: "Quiet people have the loudest minds." We walked back to the hotel from dinner one evening and discussed her field of symplectic geometry, which was largely a mystery to me. She described the symplectic camel, and I knew then I had to learn more from her.

Lisa is a professor at the University of Toronto whose research focuses on symplectic geometry and mathematical physics. She is highly acclaimed, winning the Krieger-Nelson Prize and the Coxeter-James Prize from the Canadian Mathematical Society. She gave a prestigious Noether Lecture this year. Lisa is also a Fellow of the American Mathematical Society and a Fellow of the Royal Society of Canada.

This interview was conducted in December 2017.

AB: *Where were you born and what did your parents do?*

LJ: I was born in Fort Collins, Colorado. My (Scottish) father was doing his PhD in forest hydrology at Colorado State University. My (Canadian) mother also worked in forestry and forest pathology—she now is very much involved in the environmental movement.

Figure 11. Lisa Jeffrey.

My parents met in Calgary working for the Canadian government. She became research officer, which at that time wasn't something that very many women did. We moved to Canada six months after I was born.

AB: *What is your first mathematical memory? That is, a time in your youth where you had a vivid memory of something related to math.*

LJ: In grade 8 or 9, I did a project on the Goldbach conjecture (which states that every even integer greater than two can be expressed as the sum of two primes). I just stated that it was. I obviously didn't prove it!

Another mathematical memory came from the time we lived in northern Norway for a year and a half when I was nine and ten. My stepfather got a job there. I was in the Norwegian public school, and my mother arranged that I would take math classes at a grade level two years higher than my own. The teacher (Mr. Saeboe) was very helpful and encouraging. I was in grade 6 math, and there was an exam at the end of the year and, apparently, I got the highest score of anyone in the county. This is not a large-scale achievement (Norway is a small country—population four million—subdivided into twenty counties), but maybe this went to my head.

AB: *How did you decide to choose mathematics as a major in university?*

LJ: I was a physics major, and I switched to mathematics in graduate school. After having done a physics undergraduate degree, I was awarded a Marshall Scholarship for study in the UK. I did parts II and III of the Mathematical Tripos at

[4] *Chapter 8 of* Limitless Minds: Interviews with Mathematicians *by Anthony Bonato*, https://bookstore.ams.org/mbk-118.

DOI: https://dx.doi.org/10.1090/noti2444

Cambridge. That brought me up to the level where I could think about continuing with mathematics. If you are a physics major, then you take quite a lot of math anyway, but I wouldn't have been equipped to start a math PhD without my two years at Cambridge.

In my undergraduate years, I took real analysis, complex analysis, algebra, differential equations, and a course on Riemann surfaces; I took two courses on mathematical physics that were basically analysis (we followed a text by Ivar Stakgold which dealt with topics like the Fredholm alternative).

There were posters of historical mathematicians in the undergraduate physics and math library when I was an undergraduate. There were two female figures: Sofia Kovalevskaya and Emmy Noether. Both of them were role models for me.

Figure 12. Sofia Kovalevskaya (1850–1891) and Emmy Noether (1882–1935).

AB: *How did you come to be Michael Atiyah's doctoral student? What was his style of supervision?*

LJ: We met weekly, and he always had an hour allocated for each student. There were always many ideas from him, and he provided a lot of starting points. By the way, Ruth Lawrence was two years ahead of me (also working with Atiyah), although she was six years younger than I was.

In my first year, Ed Witten had just written his paper on quantum field theory and the Jones polynomial. There was a seminar in Oxford that fall about this material. Ruth Lawrence edited the notes, and quite a lot of material in Atiyah's book *The Geometry and Physics of Knots* was based on that seminar. That was the point of departure for my thesis: I was working on Chern-Simons gauge theory.

AB: *What is symplectic geometry/topology?*

LJ: One of the standard examples people use to describe the field is the symplectic camel. Anyone in the Christian tradition will have heard the quote from the Bible: "It is

Figure 13. Sir Michael Atiyah.

easier for a camel to go through the eye of a needle than for a rich man to get into heaven." Now imagine you have a ball on one side of a plane and there is a small, circular hole in the plane. How would you get the ball through the hole and onto the other side?

If it were a question of volume, then you would squeeze out the ball and make it long and narrow. Then you would thread it through the hole. But to preserve the symplectic structure, that's not good enough. The radius of the ball would have to be smaller than the hole's radius. This example is discussed in the book *Introduction to Symplectic Topology* by McDuff and Salamon. Symplectic structure is the natural mathematical home for classical mechanics. It is the natural home for Newton's laws of motion, which can be rephrased as Hamilton's equations.

Noether's theorem (for any symmetry there is a conserved quantity—for example, symmetry under rotation corresponds to conservation of angular momentum) goes back to Emmy Noether and is a fundamental principle in symplectic geometry and Hamiltonian mechanics.

AB: *What are you working on now?*

LJ: The fundamental group of a space is the set of loops in the space, where you can deform the loops but not cut them. For example, the plane has a trivial fundamental group as you can always shrink any loop to a point. If you puncture the plane, then the fundamental group is no longer trivial, as a loop around the puncture cannot shrink to

Figure 14. In symplectic geometry, you cannot squeeze a ball through the eye of a needle unless the radius of the ball is small enough.

a point. Basically, the fundamental group is classified by the winding number, which counts the number of times the loop goes around the hole. So, the fundamental group of the punctured plane is the set of integers, counting the number of times and which direction you wind around the hole.

The space I've worked on is representations of the fundamental group into some other group such as the circle group. In the case of the punctured plane, this would be one copy of the circle group as you just have to say where the generator of the group goes (it goes to some point on the unit circle). I've worked on other more complicated examples that come up often. There was a groundbreaking paper of Atiyah and Bott in 1982 where they studied the space of representations of the fundamental group of a 2-manifold. A 2-manifold would be a torus or a 2-dimensional sphere, or anything you get by gluing these structures together (classified by the number of the holes, which is called the genus).

AB: *Does your research interact much directly with physics?*

LJ: I published a paper about five years ago that was from my PhD thesis that did have to do with physics. As a result of that, I spoke at a physics conference, Theory Canada 9, at Wilfrid Laurier University in Waterloo. Many of the talks there were straight physics. In hindsight, it would have been better if I had rephrased my talk in physics-language.

AB: *I've been thinking a lot lately about diversity in mathematics, and why women constitute only about twenty percent of mathematics departments in Canada. What are your thoughts on this? How can we change this culture?*

LJ: In our department, fifteen percent of our faculty are women, and the same percentage holds among our graduate students. I wish the numbers were higher. I don't understand the graduate percentage, as often our admissions committees are composed of women. It's not a matter of any discrimination, but we just have fewer women applicants. I don't think our field has as much gender disparity as in engineering or perhaps physics. How to change the culture? When I was in high school, there was a lot of attention paid to the question of why girls were dropping out of math class and what to do about it. Now there is much discussion of how the school system is failing boys. There are books on the "war against boys." So, somehow, the focus has changed.

Many people think that women are overrepresented in universities, but this is not true in STEM fields. Medical and law schools will typically have more women than men. People need to be reminded that there are still issues getting women to study STEM. The problems in the education system encountered by women have not disappeared.

Figure 15. Matilde Marcolli.

By the way, the University of Toronto and the Perimeter Institute just hired Matilde Marcolli from Caltech. This is fabulous news. Matilde is going to be a role model, splitting her time between both Toronto and the Perimeter Institute.

AB: *What is your advice to young people (especially young women) who are considering studying mathematics at university/grad school?*

LJ: The important thing is to understand that math leads in many directions, not just academic ones. People with mathematical training will be highly employable. One of my best students finished his PhD, got a postdoc, and within eighteen months he had a programming (non-academic) job at a physics research institute. He had no trouble at all getting a good job.

AB: *I always close looking forward. What would you say are some of the major directions for mathematics in the future?*

LJ: There are many questions related to the work of Nigel Hitchin on Higgs bundles (the same Higgs associated with the Higgs boson). That work has major ramifications in many different directions, including to representations of the fundamental group that I discussed earlier. That work is an outgrowth of the 1982 paper of Atiyah and Bott, but it takes things in a different direction.

Credits

Figure 11 is courtesy of the University of Toronto, Scarborough.
Figure 12 is courtesy of Wikimedia Commons.
Figure 13 is licensed under Creative Commons Attribution-ShareAlike 2.0 Germany (CC BYSA 2.0 DE).
Figure 14 is courtesy of Shutterstock.
Figure 15 is courtesy of the Archives of the Mathematisches Forschungsinstitut Oberwolfach.

"Brilliance and Generosity of Heart": Elizabeth Meckes (1980–2020)

Della Dumbaugh

Introduction

The mathematical community suffered an incalculable loss at the end of 2020 when Elizabeth Elder Meckes passed away from colon cancer at the age of 40½. This article offers a brief introduction to her life, a personal reflection Elizabeth wrote when Karen Uhlenbeck was awarded the Abel Prize, a response to Elizabeth's comments by Uhlenbeck, a tribute from Rafe Mazzeo, professor of mathematics and department chair at Stanford University, and a collection of thoughts to consider as the mathematical community moves forward with insights gleaned from Meckes' life.

Brief Biography

Elizabeth Meckes grew up in Cincinnati, Ohio where her parents, Richard Elder and Katherine Tepperman Elder, served on the chemistry and biology faculty, respectively, at the University of Cincinnati. A talented flutist, Elizabeth played in many school and community ensembles, including the Cincinnati Symphony Youth Orchestra. She attended Case Western Reserve University (CWRU), completing her undergraduate degree in mathematics in 2001 and a master's degree the following year. She met Mark Meckes at a high school band camp and the two married when she graduated in 2001. She traveled to Palo Alto to pursue a PhD in mathematics at Stanford from 2002–2006, working under the direction of Persi Diaconis. In her thesis on an "Infinitesimal version of Stein's method" she developed a "powerful tool for solving problems" [1]. On her personal website, Meckes described Diaconis as "fun" and encouraged readers to "talk to him sometime." Diaconis described Meckes as one who "pursued whatever mathematical problem she was tackling with all of her being. She was at the top of her field, driven, giving, and full of life" [10]. When she graduated from Stanford, she received the American Institute of Mathematics Five-Year Fellowship which provided (as the name suggests) five years of research support for an outstanding new PhD in pure mathematics. Her research considered questions in probability and analysis. As she put it, she tended "to be most interested in situations in which probability arises naturally in other fields, e.g. differential geometry, convex geometry, and number theory" [3]. Her 27 papers and two books have left a "considerable mathematical legacy."[1]

She spent a year at Cornell and then she and her husband accepted assistant professor positions at Case Western Reserve University in Cleveland, Ohio in 2007. They published their first joint paper that year. They collaborated on several other papers and published what she referred to as "a textbook for a first rigorous course in linear algebra (imaginatively titled)" *Linear Algebra* with Cambridge University Press in 2018 [3, 4]. All the while, Meckes maintained a commitment to work on behalf of all vulnerable groups of faculty, particularly in

Della Dumbaugh is a professor of mathematics at the University of Richmond and editor of the American Mathematical Monthly. *Her email address is* ddumbaugh@richmond.edu.

Reflecting on Elizabeth Meckes for the Case Western Reserve University community, Dean Joy Ward noted that "I didn't have a great amount of time to get to know Elizabeth before she and her family left for Oxford, but even through just a few conversations I had the opportunity to witness the extent of her brilliance and the generosity of her heart*" [10, our emphasis]. The author would like to thank Mark Meckes for his thoughtful help with this article, particularly his willingness to check details and provide insights into Elizabeth's early life.*

For permission to reprint this article, please contact: reprint-permission@ams.org.

DOI: https://dx.doi.org/10.1090/noti2435

[1] *Durrett [1] offers a beautiful overview of Elizabeth Meckes' work along with this analysis of her contributions.*

her role as chair of the Executive Committee of the CWRU College of Arts and Sciences. The following year she published her own 224-page book, *The Random Matrix Theory of the Classical Compact Groups*, with Cambridge [8]. Supported by a Simons Foundation grant, she was spending the 2020–2021 academic year with Jon Keating's random matrix theory group at the Mathematical Institute at the University of Oxford.

In the preface of *The Random Matrix Theory of the Classical Compact Groups*, Elizabeth acknowledged the contributions of her family when she wrote "[i]f my writing helps to illuminate the ideas I have tried to describe, it is because I got to talk it out first at the breakfast table" [8]. In a beautiful tribute to Elizabeth in "Eigenvalue is not a dirty word: My mathematical collaborations with Elizabeth Meckes," Mark Meckes pointed out that they were always joined by their children Juliette and Peter at that breakfast table [9]. In particular, he noted that their children were very much a part of their mathematical lives, both acquiring passports as babies to travel to conferences in Banff and joining them at various conferences and seminars around the world throughout their young lives.

For Meckes, outreach and mathematical writing went hand in hand. She wrote a two-part article for girls in the *Girls' Angle Bulletin*, a bimonthly publication with the aim of connecting K–12 students with professional mathematicians. Meckes intended for the first installment to introduce the idea of axiomatic probability and the law of large numbers [5]. The second installment offered what Meckes described as an "exposition" of the central limit theorem [6].

Perhaps not surprisingly then, Elizabeth collected her thoughts on paper when Karen Uhlenbeck was awarded the Abel Prize in 2019 [7]. In particular, this prize prompted Meckes to consider the question "Does it matter [that women in mathematics have female role models]?" To explore this query, she reflected on her personal journey as a mathematician in a broader community with dedicated initiatives to advance women in mathematics. She outlined the evolution of her thinking, which began with her initial skepticism and, after several observations and experiences, led to a more expansive view. That progression alone tells us Elizabeth Meckes maintained a willingness to reflect on and (re)evaluate her thoughts over time. In this case, however, she shared her personal analysis with us. She ultimately answered her own query with three definitive words: "Yes, it matters."

"Some personal thoughts I wrote down when Karen Uhlenbeck was awarded the Abel Prize" by Elizabeth Meckes

There's this fantastic quote from the great science fiction author Octavia Butler, about the lack of black women science fiction authors who could have served as role models for her: "Frankly, it never occurred to me that I needed someone who looked like me to show me the way. I was ignorant and arrogant and persistent (*sic*) and the writing left me no choice at all." I might have written that. When I was a mathematics student, I was so firmly convinced that I had what it took that it never occurred to me to care whether there were female role models around. This week, it was announced that Karen Uhlenbeck, professor emerita of mathematics at UT Austin, was awarded this year's Abel prize. The Abel prize is the most prestigious award there is for a senior mathematician (much closer to a Nobel prize for mathematics than the usually mentioned Fields medal), and Professor Uhlenbeck is the first woman to receive it. Now, in that rarified stratosphere of Abel-prize recognized mathematical brilliance, there is a woman we can look up to. Does it matter?

I was a student at the turn of the last century, when the mathematics community, along with many other male-dominated professions, had noticed that the expected influx of women somehow never seemed to materialize, and was looking for more active ways to bend the curve. In my experience, this meant mostly the appearance of "women in mathematics" groups, whose very existence I found vaguely offensive: how dare anyone tell me I needed extra support? As a graduate student, I read a piece in a professional publication by a famous (white, male) mathematician, offering the advice to people in underrepresented groups that, when choosing a graduate program, they should find one that had faculty in their demographic, to serve as role models. And again, my reaction was "How dare he?" How dare he suggest that I should have turned down a place at Stanford because there were no women on the faculty? Like Ms. Butler, it never would have occurred to me to do such a thing, and for the record, I loved Stanford and will always be grateful for everything I got out of my time there. First and foremost, a wonderful advisor who led me to beautiful mathematics.

The thing is, I don't know what Ms. Butler said next after that fantastic quote. But my career didn't end with being a graduate student at a famous university; life went on. I got exactly the kind of job I wanted: a tenure-track assistant professorship at the same institution where I had been an undergraduate, together with a prestigious fellowship that allowed me to focus on research. At first, I was way too focused on proving the next theorem to notice or care about any forms of gender bias around me. But little by little, I began to notice things. The students who complained that they were too intimidated to ask me questions because it all seemed too easy for me (isn't the professor supposed to know the material well?). The fact that I developed the habit of never stopping to breathe while making a point in a meeting, lest someone start talking over me. The aggressive questions during and after some of my talks. And I started to see why having peers or mentors who understood these experiences had some value.

I met Karen Uhlenbeck once. It was at the program "Women and Mathematics" at the Institute for Advanced Study. Every year, they run a two-week workshop on a current research topic for female students and postdocs, with four senior women giving a week's worth of lectures each.

In 2014 I was invited to be one of the lecturers. I felt that I didn't need a support group because of my gender, but I did recognize that maybe it did matter to at least some young women, seeing people who looked like them at the front of the room, and maybe that could be me. So I went and it was wonderful. And as much as I'd seen it as a vaguely altruistic gesture on my part, having the opportunity to meet and talk to the senior women organizers affected me in ways I didn't expect. I remember in particular talking to Karen and to Dusa McDuff, another towering figure of current mathematics. I never would have believed it, and my younger self would have been incensed at the very idea, but interacting with them propped up a little part of my self-image that I hadn't realized was sagging.

In March 2017, the *Notices of the American Mathematical Society* featured on its cover a head shot of Andrew Wiles, who was awarded the Abel prize in 2016. Professor Wiles proved Fermat's last theorem, a conjecture that had stood for over 350 years; such an achievement certainly merits all the awards we can throw at it. But I bet I wasn't the only woman who noticed that that issue of the *Notices* had a large picture of a famous male mathematician, that the other two names on the front cover belonged to men, and that there was a tiny, colorful banner declaring that it was Women's History Month. Next time, it will be Karen Uhlenbeck on the cover and that little banner about Women's History Month won't feel like a sad little bone thrown to the people who don't really matter. It will feel like a celebration of something real. Yes, it matters.

Karen Uhlenbeck's Response to Elizabeth's Reflections: "Go for it, gal!"

I was saddened to hear of the untimely death of Elizabeth Meckes, whose contributions to a range of subjects in probability are substantial. I am reminded that, when asked what were the greatest difficulties I have had in my career, I answer "ill health." I am indeed sad for her, her family, her students, and the profession.

I met Elizabeth only the once in 2014, when I was still peripherally involved with the WAM program at IAS. When I was her age, I knew or knew quite a bit about all the women in mathematics, and I am heartened to say that there are enough that I do not know even a tenth! So I did not know Elizabeth well at all. I was asked to write a response to her comments on my receiving the Abel Prize in 2019.

My first response was "Go for it, gal!" No one does anything creative or important by following people like themselves. Her initial reaction, on being urged to find a department with role models, of wanting to do the mathematics first, was very healthy. Mathematics culture also has its positive aspects. Most mathematicians encourage good students, quickly become involved with students' mathematical interests, abilities, and individual talents, and are able to put aside issues of language, gender, race, culture, and social class. This does not mean these differences are easily overcome, but I believe many mathematicians now believe that they can be and try to act on this belief. I am not sure this was true 60 years ago when I was a student. Instead of being told to find a role model, it was off and on explained to me that women could or should not do mathematics. I paid no more attention to this than Elizabeth did to the advice she got.

Role models are important. I read all the nonfiction that Virginia Woolf wrote, and it is extensive. I learned about the difficulties of being a woman and of being ill but also of the possibilities and pleasures of ambition. I admired the famous male mathematicians of the past and present. I particularly recall reading Andre Weil's autobiography *The Apprenticeship of a Mathematician*. He tells the tale of getting a lot of mathematics done while incarcerated as a conscientious objector. I never thought the ideal life was necessary or even conducive to doing good mathematics, but I did always envy these particular figures for the intellectual life they were exposed to as children.

I also had the privilege of taking one course by a woman professor in my eight years in college and graduate school. Cathleen Morawetz taught second semester complex variables during my first year in graduate school at NYU. With the arrogance of youth, I was critical of her hair, her clothes, her teaching, and her mathematics (too applied). Later on when life got difficult, she became a beacon of success despite imperfection! I remember thinking: "If Cathleen can do it, so can I." I have always hoped to be such a figure for younger mathematicians.

I was asked to add an example of difficulties I faced and how I overcame them. The truth is, the mathematics was the easy part. As a woman, I was often isolated and asked to do things professionally that I had not seen a woman (and sometimes even a man) do. In my mid-career, this was very difficult. I could treat students the way I had been taught as a student, but how to function as a professor in an elite mathematics department, or how to chair a governing board? As I matured, I simply learned to always insist on talking it through and sharing a job with other mathematicians, students, and staff. In particular, Dan Freed and I undertook many joint projects from writing a book through forming a research group to helping start the Park City Mathematics Institute. When Orit Davidovitch came to me as a graduate student wanting a program of invited women speakers, I assigned her the job, providing suggestions and support. The Distinguished Women Lecture Series at the University of Texas is now a model for other programs. Later I came to rely heavily on a group of women friends. I started out a loner and turned to collaboration as a solution to managing new challenges.

The Women and Mathematics program at the Institute for Advanced Study, where I met Elizabeth, was an earlier project. When I was offered the chance to start this, the offer of staff support was the real enticement. I also immediately asked my collaborator Chuu-Lian Terng, who was at the time president of the Association for Women in Mathematics, to share the job. This program has been important to both of us, as it is for most of the participants. Research has shown that group behavior varies a lot when the participants are mostly male or mostly female. For those of us entranced by the beauty of mathematics, it is startling to experience this "difference based on gender of participants" in a mathematical setting. Meeting many women mathematicians, after decades of functioning in a primarily male environment, was, among other things, simply fun. In addition for me, the existence of younger women like Elizabeth, and the even larger number of women of the age to be her students, is a justification for my own less than straight path. And I am sorry to have missed the opportunity to know Elizabeth better.

Tribute from Rafe Mazzeo, Chair, Department of Mathematics, Stanford University

I knew Elizabeth during and after her time at Stanford, and though her subject is not close to mine, I attended a few research talks she gave. Given the difference in fields, I might not have learned much, but in fact I did learn quite a lot—she had a very insightful and deep understanding of her subject and a great ability to make it come alive for listeners, even for probabilistic amateurs such as myself. I am heartbroken for her husband Mark and her children, as well as her many close friends. It is extremely bitter that two great women mathematicians associated with Stanford, Elizabeth and Maryam Mirzakhani, both passed away so tragically and prematurely. Elizabeth and Maryam had so much more mathematics to give the world.

Concluding Thoughts: Value Added

In a 2018 interview with *The Daily* at Case Western Reserve University, Elizabeth acknowledged that "I think a lot about the fact that I'm a research mathematician—that's a really important part of my professional identity, but I'm also a professor and I teach students. I think a lot about what my value added is—what am I giving students that they couldn't get from just picking up a book?" [2]. Her value added came from living her life with "brilliance and generosity of heart." She advanced the discipline of mathematics *and* the people who pursued mathematics—students and colleagues.[2] Despite her initial reluctance about programs that focus on women in mathematics ("how dare you?"), Elizabeth came to recognize the positive impact of these experiences for women at all levels in the profession. Unfortunately, however, her life shows that even women who earn PhDs at Stanford and receive prestigious awards that allow them to pursue a research year at Oxford still feel the need to hold their breath when they speak for fear someone will interrupt them. So there is room for improvement. The recent issues of the *Notices* celebrating Women's History Month in 2019, 2020, 2021, and now 2022, have taken steps in the right direction. These issues have all offered more than a little banner in the corner of the cover of the publication. They celebrate rich contributions of women in mathematics, including the sterling life of Elizabeth Meckes. She matters. They matter.

[2] *As her colleague at Case Western Reserve observed, the university "is a better place because of Elizabeth's service and commitment to raising up her colleagues" [10].*

References

[1] Rick Durrett, *Obituary: Elizabeth Meckes 1980–2020*, Institute of Mathematical Sciences, February 23, 2021. `https://imstat.org/2021/02/23/obituary-elizabeth-meckes-1980-2020/`

[2] *5 questions with… research mathematician, associate professor Elizabeth Meckes*, The Daily, Case Western University, March 9, 2018.

[3] Elizabeth Meckes, Case Western Reserve University Website. `https://case.edu/artsci/math/esmeckes/`

[4] Elizabeth Meckes and Mark Meckes, *Linear Algebra*, Cambridge: Cambridge University Press, 2018.

[5] Elizabeth Meckes, *The Laws of Probability: What Makes a Coin Fair?*, Girls Angle Bulletin **9** (October/November 2015), no. 1, pp. 10–11. `http://www.girlsangle.org/page/bulletin-archive/GABv09n01E.pdf`

[6] Elizabeth Meckes, *The Laws of Probability: Zooming In*, Girls Angle Bulletin **9** (December/January 2016), no. 2, pp. 6–8. `http://www.girlsangle.org/page/bulletin-archive/GABv09n02E.pdf`

[7] Elizabeth Meckes, *Some personal thoughts I wrote down when Karen Uhlenbeck was awarded the Abel Prize.* `https://case.edu/artsci/math/esmeckes/Uhlenbeck_Abel_prize`

[8] Elizabeth Meckes, *The Random Matrix Theory of the Classical Compact Groups*, Cambridge: Cambridge University Press, 2019.

[9] Mark Meckes, *Eigenvalue is not a dirty word: My mathematical collaborations with Elizabeth Meckes*, Southeastern Probability Conference, May 2021. `https://www4.math.duke.edu/media/watch_video.php?v=34UGRH3WDANH`

[10] *Remembering Professor of Mathematics Elizabeth Meckes*, The Daily, Case Western University, December 20, 2020.

Della Dumbaugh

Credits

Photo of Della Dumbaugh is courtesy of the University of Richmond.

An Unlikely Cohort

Disrupting the Stereotypes about Mathematicians and their Careers

Taryn Butler Lewis, Tasha R. Inniss, Monica Jackson, and Calandra Tate Moore

> ***We must reject not only the stereotypes that others hold of us, but also the stereotypes that we hold of ourselves.***
> *—Shirley Chisholm*

Introduction

Euphemia Lofton Haynes (1943), Evelyn Boyd Granville (1949), Marjorie Lee Browne (1949), Sadie Gasaway (1961), Georgia Caldwell Smith (1961 posthumously), Gloria Conyers Hewitt (1962), Thyrsa Anne Frazier Svager (1965), Vivienne Malone-Mayes (1966), Shirley M. McBay (1966), Eleanor Dawley Jones (1966), Annie M. Watkins Garraway (1967), Geraldine Darden (1967), Mary Lovenia Deconge-Watson (1968), Etta Zuber Falconer (1969), Elayne Arrington (1974), Rada Higgins McCreadie (1974), Fern Y. Hunt (1978), Karolyn Ann Morgan (1978), Fannie Ruth Gee (1979), Emma R. Fenceroy (1979).[1]

What do all of these people have in common? They are all mathematicians and Black women. These are the first 20 Black women to earn doctoral degrees in mathematics in the United States.[2]

When you picture a mathematician, what image comes to mind? Chances are a vision of an African American woman is not your first image. We are all familiar with the stereotype of a mathematician.

Since 1943 when the first Black woman, Dr. Euphemia Lofton Haynes, earned her doctorate, Black women continued to excel in the mathematical sciences, even if they did not earn a Ph.D. Case in point, the famous book, *Hidden Figures*, highlighted Katherine Johnson who served as a research mathematician at NASA[3] [Shetterly, 2016]. Did you know that she is considered the first African American to pursue graduate studies in mathematics; integrated West Virginia University; and pursued a master's degree in mathematics there, but did not earn the degree?[4] Yet she is one of the most famous mathematicians of our time given her groundbreaking research in the space race of the 1950s. She is also "the first female mathematician to be awarded the highest civilian honor in the U.S., the Presidential Medal of Freedom", which she was awarded by President Barack Obama in 2015 [Walker, 2017]. In addition, she and the other "computers" from NASA received a Congressional Medal of Honor in 2019.[5]

Taryn Butler Lewis is the VP of Operations at Metron Aviation, Inc. Her email address is tarynblewis@gmail.com.

Tasha R. Inniss is the associate provost for research and tenured associate professor of mathematics at Spelman College. Her email address is Tinniss@spelman.edu.

Monica Jackson is the deputy provost and dean of faculty at American University. Her email address is monica@american.edu.

Calandra Tate Moore is the Video, Image, Speech, and Text Analytics Research team lead at the Department of Defense. Her email address is caltatemoore@gmail.com.

[1] http://www.math.buffalo.edu/mad/wohist.html; https://www.mathad.com

Communicated by Notices *Associate Editor Asamoah Nkwanta.*

For permission to reprint this article, please contact: reprint-permission@ams.org.

DOI: https://dx.doi.org/10.1090/noti2431

[2] *During this same timeframe, Black women were also "firsts" for earning doctoral degrees in mathematics education (e.g., Louise Nixon Sutton* https://en.wikipedia.org/wiki/Louise_Nixon_Sutton*).*

[3] https://www.nasa.gov/content/katherine-johnson-biography

[4] https://www.nytimes.com/2020/02/24/science/katherine-johnson-dead.html

[5] https://www.nasa.gov/feature/hidden-figures-honored-at-us-capitol-for-congressional-gold-medal

Though African American women have made an impact with their work in mathematics (in industry, government, and academia), the numbers of those who earn doctoral degrees in mathematics remain very low. According to the Mathematical and Statistical Sciences Annual Survey administered by the American Mathematical Society,[6] no more than 1% of all mathematics/statistics doctoral degrees in a given year were awarded to Black women (US Citizens or Permanent Residents) between 1990 and 2017. If we were to consider raw numbers during these years, the highest number ever awarded in an individual year was 14 (in 2008 and 2010). The average number is seven. Thus, the data reveals that Black women who earn doctoral degrees in the mathematical sciences are very rare.

Given these statistics, it is remarkable that a few universities have been effective at recruiting African American women in their graduate programs in the mathematical sciences, and providing an environment in which they are able to thrive as mathematicians. One such institution is the University of Maryland at College Park. During the 1990s, it is widely believed that the department had the largest number of Black students pursuing degrees in the mathematical sciences in the entire country.

Here is another list: Angela Grant, Asia Wyatt, Calandra Tate Moore, Danielle Middlebrooks, Hatshepsitu Tull, Jhacova Williams, Joycelyn Wilson, Karamatou Yacoubou-Djima, Kimberly Sellers, Kimberly Weems, Monica Jackson, Naiomi Cameron, Roselyn Marsa Abbiw-Jackson, Shelby Wilson, Sherry Scott, Stacey O. Nicholls, Tamara Singleton-Goyea, Taryn Butler Lewis, Tasha R. Inniss, and Toni Watson. These are the 20 Black women who earned either a Master's or Doctoral degree or both in mathematics from the University of Maryland, College Park (UMCP) from the mid-1990s to 2020. It is interesting to note that in the first year that Black women earned doctoral degrees from UMCP (2000), there were three that finished at the same time! [Kellogg, 2001]. This is extremely rare, especially given that the statistics reflect that there might not be three in the entire country who earn Ph.Ds. in the mathematical sciences in a given year. Just one year later, in 2001, Howard University conferred Ph.D.s to four African American women.[7]

We the authors are among those who earned advanced degrees from UMCP during the twenty-year time period between 2000 and 2020. To capture as many of the stories of the Black women as possible, we developed an informal questionnaire[8] and reached out to our colleagues to gather background information and their thoughts on pursuing an advanced degree in mathematics and a career in the mathematical sciences. We were interested in their personal histories, career trajectories, and reasons they decided to pursue mathematics and an advanced degree in the mathematical sciences at UMCP. Additionally, we sought to understand the factors that contributed to their success. We asked questions such as: "Who were your inspirations to pursue and attain an advanced degree in the mathematical sciences?"; "As a student at UMCP, who do you feel supported you in your pursuit of an advanced degree in the mathematical sciences?"; "While matriculating at UMCP, did you feel a sense of community?"; and "While at UMCP, what was your area of study?" From those who responded to the questionnaire, 80% (16/20) studied applied mathematics at UMCP. Later in this article, we will discuss the breadth of careers that resulted from studying different areas in the mathematical sciences.

From the questionnaire responses, it is interesting to note that **75%** of us are the first in our families to earn an advanced degree in mathematics and the majority (**90%**) of us earned our undergraduate degrees in mathematics at a Historically Black College or University (HBCU). Not only are we disrupting educational systems that use stereotypes to limit who can succeed in math, we are also disrupting the perception about the academic potential of students from HBCUs.

According to the National Science Foundation report, *Women, Minorities, and Persons with Disabilities in Science and Engineering*, in the top 15 baccalaureate institutions for Black or African American doctorate recipients in the sciences between 2015–2019, nine of them are HBCUs, with Spelman College and Howard University ranking first and second, respectively[9] [NSF NCSES, Table 7–8]. For some, this might be a startling statistic given the fact that HBCUs only make up a small percentage (3%) of all institutions of higher education (IHEs).[10] In their study and publication, "What Makes the Difference? Black Women's Undergraduate and Graduate Experiences in Mathematics," Borum and Walker [2012] stated "it is evident that for women who attended HBCUs, smaller class sizes and a more nurturing environment contributed to them proceeding and receiving their doctorate in mathematics." All four of us (authors) attended HBCUs as undergraduate students. Based on our experiences, we hypothesize that there are several factors that contributed to our success and that of our community of Black women mathematicians from UMCP. First, many of us attribute the supportive community for Black graduate students at UMCP to the former department chair, Dr. Raymond Johnson, who was the first African American to earn a Ph.D. in mathematics from Rice University.[11] It

[6] https://www.ams.org/profession/data/annual-survey/annual-survey

[7] www.jstor.org/stable/3134128

[8] *Surveys of the UMCP Math Sistahs*

[9] https://ncses.nsf.gov/pubs/nsf21321

[10] https://uncf.org/the-latest/the-numbers-dont-lie-hbcus-are-changing-the-college-landscape

[11] https://news.rice.edu/news/2020/task-force-hosts-conversation-rices-first-black-student-raymond-johnson

Figure 1. Black women who pursued advanced degrees in the mathematical sciences at UMCP.

should be noted that Dr. Johnson was also the only African American chair of a mathematics department at a Predominantly White Institution (PWI) in the country at the time. As chair of the Department of Mathematics at UMCP, he was committed to increasing the diversity of the graduate student body and was intentional about recruiting strong students, particularly from HBCUs. He understood the systemic or structural racism that is present in Institutions of Higher Education (IHEs), especially in Science, Technology, Engineering or Mathematics (STEM) departments. He endeavored to create a productive learning community for Black students, instill a "sense of belonging," [Herzig, 2006] and implement structures that facilitated equitable treatment of all students. It is clear that Dr. Johnson was effective at achieving these goals because **90%** of the respondents said that they felt a sense of community in the department at UMCP.

In addition to a sense of community with other Black women, the other factors we exuded that contributed to our success was our doggedness/perseverance, advocacy, and resilience. The rigorous academic training and mentoring that we received during our undergraduate education (primarily at HBCUs) coupled with high expectations of mentors motivated and inspired us to persist to the finish line of attaining advanced degrees in the mathematical sciences. Since most of us engaged in undergraduate research and participated in summer Research Experiences for Undergraduates (REUs), we had that foundation on which to build, to develop, and to complete theses and dissertations. At UMCP, we had the opportunity to be advised by professors who were also supportive and had confidence in our abilities to earn mathematics graduate degrees. We caution the reader against "blaming" anyone who did not earn an advanced degree in mathematics because most often the reason is not due to a lack of talent, but the structural racism they encountered at PWIs.

We accepted education as the means to rise above limitations that a prejudiced society endeavored to place upon us.

—Evelyn Boyd Granville, 2nd African American Woman to earn a Ph.D. in Mathematics[12]

Now let's consider the prevailing notion about what mathematicians do. Often when one shares that someone is pursuing a degree in mathematics, the first question posed is "What will you do...teach?" We feel teaching is a noble profession, and as a matter of fact, **60%** of the respondents spent their entire career or some of it in academia. The beautiful and remarkable thing about the mathematical sciences is that there is a breadth of career options in different sectors. Thirty-five percent (**35%**) of respondents have or have had careers in government and **30%** in industry.

The following three sections focus on the experiences and the personal histories of three UMCP mathematicians

[12] `https://www.azquotes.com/author/46214-Evelyn_Boyd_Granville`

because each one represents a different career sector: industry, government, and academia.

Mathematics Careers in Industry

Taryn Butler Lewis, M.A., PMP
VP of Operations, Metron Aviation, Inc.
MA '98, Applied Mathematics
(Operations Research), UMCP

I was raised in southern Prince George's County, Maryland, as the only child of a federal government employee and an elementary school teacher. They both stressed good grades and good behavior, and that they expected me to go to college; but they didn't push me in any particular career direction. I somewhat fell into STEM on my own. I remember enjoying science and math classes in school but never considered majoring in either. I loved sports and wanted to be a sports commentator, one of the few females in that role in the 1980s. STEM felt more like a hobby than anything else to me. I did not personally know any Black women in STEM at the time.

After high school, I attended Morgan State University (MSU) on an undergraduate honors scholarship. I planned to earn a degree in Communications (remember, I wanted to talk about sports). After learning that I had to take only one math class and spending my freshman year in liberal arts classes, I changed my major to mathematics because I could not fathom not taking more math classes. That was a decision that I am so happy I made. MSU's math department was filled with faculty and students who challenged me and supported me along the way.

While at MSU, I was encouraged by professors, classmates, and family to consider graduate school, and I decided during my senior year that I would pursue an advanced degree in math at UMCP. I received a fellowship/teaching assistantship to attend UMCP and chose pure math as my study area, with plans to pursue a career in cryptology. My second choice was actuarial science, but I was uninterested in the real analysis that I would need to be successful in that field.

I was very familiar with UMCP and the campus because my mother grew up within walking distance of the university—in a small, then all-Black neighborhood called Lakeland. After my mom graduated from high school, she chose to earn her undergraduate and graduate degrees in elementary education from Bowie State University because she didn't feel UMCP was an option for her and most college-bound Blacks in her circle chose to attend HBCUs. Note that the first Black UMCP undergraduate student, Hiram Whittle, was admitted in 1951,[13] but didn't graduate. The first Black UMCP undergraduate to obtain a four-year degree was a woman, Elaine Johnson, in 1959.[14] The first Black UMCP graduate student, the then-future U.S. Congressman from Maryland, Parren J. Mitchell,[15] graduated in 1952. One of my mom's younger siblings did eventually go on to earn an undergraduate degree from UMCP.

With my family's experience growing up in the shadows of UMCP, attending UMCP and receiving a fellowship gave me a great sense of satisfaction and a sense of pride. I joined UMCP's math department at a time when several other women and men of color were joining the department, thanks to the efforts of Dr. Raymond Johnson. This definitely made the introduction to graduate school easier. After one year, I found my pure math courses to be anything but enjoyable. While I appreciated this area, I did not relish living in an abstract world. I enjoyed learning how to apply math to real-world problems. I changed my study area to applied math and chose operations research (OR) as my area of concentration. I opted to pursue a master's degree and selected Professor Michael O. Ball as my thesis advisor. I found his research and work in air traffic management intriguing. The more I worked with him and met his other students and peers, the more I knew I had made the right choice.

I started my career working for a few years for defense contractors on asset scheduling projects. These projects provided great learning experiences and an introduction to what's possible with a degree in applied math/OR. Eventually, I accepted an opportunity at Metron Aviation, a small, boutique government contractor with whom my UMCP advisor collaborated on air traffic management problems. This became the place where I learned a lot about R&D, analysis, business, government contracting, and project management.

Working at Metron Aviation introduced me to mathematicians and OR analysts working in a variety of roles. The CEO had an advanced degree in OR, and the CEOs of the parent company had mathematics PhDs. My colleagues and I worked on interesting global air traffic management problems in roles such as prototype software development, requirements development, development of concepts of operations, analytics (before the term became popular), and data analysis. We developed innovative solutions that were deployed around the world. Some of us used this experience to segue into leadership positions. This was a defining moment in my professional career and the type of professional experience that taught me about the union of R&D and business. I moved on to work for a couple of other government contracting companies after this, further developing my business and leadership skills.

I also spent nearly three years working for the Institute for Operations Research and Management Sciences (INFORMS) leading the development and oversight of products, programs, and activities that promoted the field

[13] *Hiram Whittle, UMCP—University Libraries Digital Collections*

[14] *Elaine Johnson, UMCP—University Libraries Digital Collections*

[15] *Parren J. Mitchell UMCP College of Behavioral & Social Sciences*

of operations research, analytics and its related areas. My primary focus was finding ways to provide membership value for industry members through mentoring, professional development, and leadership opportunities. However, I was also responsible for supporting our academic audience, which included students and faculty, by providing opportunities to showcase their research and learn from others. INFORMS provided a platform for all of these individuals to demonstrate their thought leadership in operations research, analytics, and the like. Prior to joining the staff at INFORMS, I was a member. I joined while in graduate school, based on the recommendation of Professor Ball, and remained a member throughout my career. Membership in professional organizations can help those in industry stay current, and develop and grow their professional reputation.

As a Black woman in STEM, I was always the only Black woman in the room. There were occasionally Black men, but that was rare. I routinely dealt with issues such as microaggressions, being marginalized, not considered for growth opportunities, sexual harassment—essentially, all the things you hear about from women who work in majority-male environments. Getting a seat at the table is always challenging. Speaking up or demanding to be heard is frequently interpreted as being confrontational, angry, emotional, or some other negative adjective used to describe Black women in the workplace. If I was quiet, this would be misinterpreted as a lack of interest or understanding. It has been a stressful place to be, but I've persevered.

My experiences helped me to develop a passion for encouraging and supporting young Black women and men to pursue STEM degrees. I believe we deserve not just a seat at the table, but also an opportunity to participate in the discussions at the table. We all have something to offer. Studies have shown that increased diversity in STEM generates more innovation and improves the success of companies who are intentional about being inclusive.[16, 17] To this very day, I experience some of the same slights that I did over twenty years ago, but I've learned to challenge the status quo and stand up for myself and those that I lead.

Industry, usually thought of as for-profit, corporate-like organizations, is not for everyone; but it can be a rewarding career choice. The need to generate profits or satisfy stockholders and key stakeholders can sometimes obscure the good work of employees and lead to work that is unfulfilling or lacks deeper purpose. Not-for-profit, non-academic organizations are also an option. There are many organizations that benefit from the mathematician skill-set. Ultimately, it is extremely important to find organizations that mirror your values, provide continuous learning opportunities, and have a reputation for doing meaningful work while also maintaining financial health and operational excellence. You are not looking for the perfect company (it does not exist), but instead for a company for which you are proud to work.

[16] *4 Ways Diversity Is Directly Linked to Profitability,* Entrepreneur, *February 14, 2020*

[17] *Getting Serious About Diversity: Enough Already with the Business Case,* Harvard Business Review, *Nov–Dec 2020*

Many other Black UMCP women followed the industry career path as well, but for a wide variety of reasons. Jhacova Williams parlayed a UMCP Masters in mathematics into a Ph.D. in economics, following her desire to marry STEM and economic and community policies, which she is able to do as an Associate Economist at the RAND Corporation. Shelby Wilson is using her Ph.D. in a career as a Senior Data Scientist at Johns Hopkins Applied Physics Laboratory. Hatshepsitu Tull is using her degree as the Sr. Manager of Administration in the family business and as an adjunct math professor. Part-time teaching is very common among industry professionals with STEM degrees and is a good way to give back and also stay current in the chosen subject. Other career possibilities include starting a business that provides consulting services to other businesses or the government, and doing research and development in hopes of innovating the next big solution. Create the company for which you would want to work!

Corporate environments can be challenging, so you always need to remember who you are, maintain your moral and ethical standards, and lean not unto your own understanding (Proverbs 3:5-6). It helps to have sources of motivation that keep you in a positive space and keep you on your God-given path. In addition to prayer and surrounding myself with people who genuinely have my best interests at heart, I have a few inspirational quotes that I find uplifting:

- "I am more than enough"—unknown
- "When someone shows you who they are, believe them the first time."—Maya Angelou
- "Sweet are the uses of adversity."—William Shakespeare

One of the lessons that I grew up with was to always stay true to yourself and never let what somebody else says distract you from your goals.

—Mrs. Michelle Obama

Most of my career in industry has been spent supporting the government (federal and local). While there may be similarities in the type of work, government agencies have vastly different strategic goals and objectives, so being a government contractor is a different experience from being a government employee. As such, Calandra Tate Moore explains her experience in the government sector in the next section.

Mathematics as Federal Service

Calandra Tate Moore, Ph.D.
Video, Image, Speech, and Text Analytics Research Team Lead, Department of Defense (DoD)
MS '03, PhD '07, Applied Mathematics (Statistics and Natural Language Processing), UMCP

I was raised just outside the city limits of Zachary, Louisiana. I have always loved math and over the years, teachers would comment on my mathematical abilities; yet I became a math major haphazardly. This is partly attributed to the fact that I had such a limited view of what it meant to major in math or even further, what such a career path would entail. Additionally, growing up in a small town where being "smart" often equates to, "you should be a doctor" and having an older cousin whom I looked up to in medical school; seeds were planted to become a doctor. In fact, I wanted to be an anesthesiologist, and thus enrolled as a pre-medicine major at Xavier University of Louisiana, an HBCU known for sending the most African Americans to medical school [Hannah-Jones, 2015]. After being convinced by a fellow student and undergraduate faculty advisor to consider changing majors, a brief stint as math pre-med major followed, but eventually becoming a full-fledged mathematics major.

It was in college that I learned about the underrepresentation of minorities and women in STEM disciplines. This sparked my desire to pursue an advanced degree in mathematics with the intention of a career in the academy. I participated in multiple programs designed to bridge this gap such as the Summer Mathematics Program for Women held at Carleton College and the Mellon Minority Undergraduate Fellows Program at Emory University, which is sponsored by the Social Science Research Council via the Mellon Foundation. In my junior year, I received a graduate fellowship supported by the U.S. Army Research Laboratory, which initially unbeknownst to me was a recruiting initiative. This fellowship propelled an unintended more than 15-year career in federal service. Although my tenure in government began with a scholarship acceptance, I have had a rewarding career as a mathematician across various government agencies to include time as a visiting scientist in the Department of Mathematical Sciences at the U.S. Military Academy and currently as Research Team Lead for the video, image, speech and text analytics group in Computer and Analytic Sciences Research at the DoD.

In order for the U.S. to ensure production of a sufficient number of STEM experts, it has been stated that the government needs to take more actions to motivate US students [Hossain & Robinson, 2012]. One way in which the government does this outside of funding STEM- related educational and scholarship opportunities is through active recruitment and formal career programs for federal service implemented across many government agencies. According to Go Government,[18] there are nearly 17,000 federal employees in the mathematics field with the largest employers consisting of the Departments of the Army, Navy, and Air Force.[19]

Government mathematicians tackle some of the most challenging problems applicable to our nation's operations, defense, and security. And although the U.S. Department of Defense accounts for about 81% of mathematicians employed by the Federal Government,[20] they are prevalent across many other agencies. The National Institute of Standards and Technology (NIST), for example, has long employed mathematicians to develop standards and measurement techniques, including history-making mathematician Fern Hunt,[21] who was listed above as one of the first twenty Black women to receive a doctoral degree in mathematics. Katherine Johnson, as mentioned earlier, was one of the pioneering mathematicians at NASA where successful space shuttle operations and data processing fundamentally rely on mathematics. With the rise in analytic needs due to increasingly large amounts of available data, government labs, research offices, and agencies are hiring mathematicians, statisticians, and computational scientists at expanding rates.

A significant cohort of UMCP African American women mathematicians have begun careers in government through either fellowship or post-doctoral positions. Some have then transitioned into either industry or academic appointments, while others have made lifelong careers as public servants. Nonetta Pierre is a signals analyst who originally pursued federal service because it was in line with her husband's career path. That trail developed into research and leadership positions in cryptography, risk management, and systems engineering. Danielle Middlebrooks is currently a postdoc at a government lab working in theory, analysis, and modeling of networks. A great opportunity with the chance to collaborate with highly esteemed colleagues in her field led to her post. Government work offered a great intersection between industry and academia. Tamara Goyea realized her government laboratory of choice provided the best probability to utilize math skills for solving real-world problems while continuing research in mathematics. Her interest lies in the realm of data science, modeling and simulation, and data visualization. Lastly, Valerie Nelson became dissatisfied with the private sector and its underlying goal of making money. She also wanted to return to school while working to achieve life balance. As an applied mathematician, she's pursued assignments ranging from department head for math, cryptanalysis and

[18] https://gogovernment.org/career-guides/mathematics/

[19] https://www.federaljobs.net/Occupations/gs-1500_jobs.htm

[20] https://studentscholarships.org/salary/484/mathematicians.php?p=2

[21] https://www.thehistorymakers.org/biography/fern-hunt

data science to technical director for the cybersecurity office of cryptographic solutions.

"There's no greater challenge and there is no greater honor than to be in public service."

—Condoleezza Rice,
former United States Secretary of State

"When you have good ideas, you need to follow through, and if somebody tells you it's not your turn, but you're sure you're right – then you got to be unbought and unbossed."

—filmmaker Shola Lynch, producer of the documentary, "Chisholm '72: Unbought & Unbossed"

Similar to my initial plan to pursue a career in academia, many mathematicians who receive advanced degrees choose this career pathway. In the next section, Monica Jackson shares her personal history and about being a research mathematician and climbing the ranks in academia.

Careers as Academic Mathematicians

Monica Jackson, Ph.D.
Deputy Provost and Dean of Faculty, American University
PhD '03, Applied Mathematics (Statistics), UMCP

I was born and raised in Kansas City, Missouri. I come from a family of educators. Both of my parents taught high school; my mother taught english and my father taught history. My aunt and cousins were educators too. Therefore education was always important in my family. There was not much diversity in my hometown. In my neighborhood we were one of very few Black families. I attended a private high school for girls in Kansas City where there were only two other Black students in my class. Therefore it was important for me to attend an HBCU for college. I was anxious to just "blend in". My love for math started at an early age. I recall doing math races with my dad when I was young. I would use a calculator and he would use his brain. I was fascinated that he could beat a calculator! I was only somewhere around 8 years old. But I knew then that I would not do anything but math.

I attended Clark Atlanta University (CAU), an HBCU, for undergraduate school and received a B.S. in mathematics. My older brother was nearby studying engineering at Georgia Tech at the same time. So it seemed like the perfect place. CAU truly became my family. I stayed there for graduate school and obtained a Master's degree in Applied Mathematics. To this day, I am very connected to the faculty at CAU, who have remained my mentors. My decision to attend UMCP was based on the recommendation from the faculty at CAU. Until I attended CAU, I had little exposure to Black mathematicians. They became my role models and I valued their opinions. Two other students in the math department at CAU enrolled at UMCP the same year that I did. In fact, the department chair at CAU found us all housing together. And he spoke with Dr. Raymond Johnson, the chair of the department at UMCP, about us and informed him that we were coming. The baton was passed and Dr. Johnson became an instrumental part in our success at UMCP. Dr. Johnson was one of the first Black mathematicians to chair a department at a primarily white institution. His recruiting and mentoring efforts resulted in a large cohort of Black students attending UMCP during this period. At one point, there were about 30 Black students in the program. Dr. Johnson mentored us heavily and ensured that we knew one another and supported each other.

I graduated from UMCP with a Ph.D. in Applied Mathematics and Computational Science. After leaving Maryland, I decided to do a postdoc at Emory University in the department of Biostatistics, where I spent two years. I became interested in statistics at UMCP and was eager to do more applied work in public health. I worked under a spatial statistician. I studied disease surveillance with applications to developing, investigating methods for detecting cancer clusters, global clustering patterns, and developing simulation algorithms for spatially correlated data. My research still focuses in this area. I also now study health disparities. I knew I wanted a career as an academic mathematician. Therefore, I only applied for academic jobs. The autonomy of an academic career where I could choose the hours of my work, who I worked with, and on what topics appealed to me and allowed me to be creative. I have spent 17 years at American University in the Department of Mathematics and Statistics. I am a tenured professor and recently I was promoted to full professor. I have over 25 publications, 8 grants totaling over 800K. I have spent sabbaticals at UCLA, the National Institutes of Health, and Statistical and Applied Mathematical Sciences Institutes.

Three years ago, I decided to move into administration. It was not a career path I was seeking. Instead it found me. I was on sabbatical when the Dean of the College of Arts and Sciences at American University asked me to become the Associate Dean of Undergraduate Studies at the time. I have always loved being a faculty member and never even considered doing anything else. I truly believe it is the best job on the planet. But after many conversations, I decided to give administrative work a try. I surprised myself at how quickly I picked up this new role. My mathematics background equipped me with the critical thinking skills and the analytical mind that was crucial for this role. Recently, I was promoted to Deputy Provost and Dean of Faculty at American University, which is a critical position at my institution. This role requires that I oversee all faculty matters, including hiring and grievances, as well as consult with the Provost on various university concerns.

Despite my transition to academic leadership, I have maintained an active research agenda even through the last three years as I served as a full-time administrator. Since becoming Deputy Provost last July, I have received three federal grants, published three manuscripts (one focused on the early stages of the pandemic in NYC when it was the epicenter in the U.S.) and published the first edition of my statistics textbook (which has already been adopted at two universities). I also continue to engage with the broader research community. This past year I gave four research presentations and currently serve major roles in professional societies that include the American Statistical Association and the American Mathematics Society.

Of the experiences that I am most proud are my co-organization of the Conference for African American Researchers in the Mathematical Sciences (CAARMS); establishment of AU's first Research Experience for Undergraduates (REU), the Summer Program in Research and Learning (SPIRAL) in which students and faculty from across the country spend eight weeks conducting scientific research at AU- (I participated in a similar program as an undergraduate student at CAU which fueled my support for REUs); and my previous role as Associate Dean of Undergraduate Studies in the College of Arts and Sciences, which was instrumental in developing my passion for academic leadership positions.

I have succeeded in academia but like any career, it has had its challenges. I found that true success in academia requires mentoring at all levels and all stages of your career. It is important for an early career mathematician to find mentors that are supportive and can provide critical feedback to help a faculty member manage the many expectations and workload of an academic mathematician. While the autonomy of an academic career is appealing, it does require a person to be self-motivated to succeed. I am grateful to have had excellent mentors and a supportive family throughout my career, even while I was an undergraduate. This helped pave the path that I am on now.

Other Black women mathematicians from UMCP share similar stories and passion for life as an academic as I do. There is a theme around the reasons why we selected to go into academia. Karamatou A. Yacoubou Djima who is currently an Assistant Professor of Mathematics at Amherst College feels as if the mix of research and teaching in academia is ideal for her. Joycelyn Wilson always wanted to teach on the collegiate level and is doing precisely that in her role as Senior Mathematics Instructor at Spelman College. Also at Spelman College, Naiomi Cameron, who was recently promoted to Full Professor, really wanted research and teaching to be major portions of her job. Both Kimberly S. Weems, Associate Professor of Mathematics at North Carolina Central University, and Kimberly Sellers, Professor of Statistics at George Washington University, indicated that flexibility was one of the main reasons they selected a career in academia. They enjoy the flexibility of teaching, conducting research, and mentoring students. One of my co-authors, Tasha R. Inniss, who is currently serving as Associate Provost for Research at Spelman College is the only one of us who has spent time in all three sectors during her career. She initially chose academia because of her passion for teaching and desire to mentor other women of color in STEM. She parlayed her experiences with grant writing and serving on proposal review panels into a rotation at the National Science Foundation (NSF). Subsequent to that, she worked as an inaugural director at a non-profit organization. It is evident where her heart lies because she is now back in academia at the place where she earned tenure, making contributions in academic leadership.

Lessons Learned: Strategy of Supporting and Being Supported

Disrupting the stereotypes of mathematicians may not have come easy nor been the declared intention of the women featured in this article. Yet, with various starting points and trajectories, it has been precisely what they have done. So, for those considering pursuing a career in mathematics, this cohort of women brings multiple perspectives of valuable measures for overcoming hurdles, roadblocks, and challenges that may arise. It is possible to create safe spaces and support networks to thrive in math departments and mathematical careers.

When surveyed, almost all the women attributed network and relationship building with classmates as key to feeling connected. They noted that identifying with peers going through similar experiences helped them not to feel alone in their pursuits. Specifically, many mentioned the power of this very community and network of Black women mathematicians from UMCP. Regardless of the distance between cohorts, there always seemed to be a steady stream of support flowing between these women and connecting them for life.

This unlikely cohort credited family and friends as key to inspiring them in the pursuit and attainment of their advanced degrees. Also, finding appropriate mentors with whom they could meet regularly to discuss career goals and options was important. It should be noted for those following this path that allies may not necessarily be within your department, so don't hesitate to seek outside support. Never be afraid to be your own advocate and don't let others determine your path for you.

The Black women mathematicians from UMCP contributed many words of wisdom that we can all keep in mind. Be prepared to sacrifice and understand there will be hurdles, but don't let those steer you away from pursuing your goals and dreams. Learn from those experiences and let them motivate you. Practice being patient with yourself; it takes time to learn. Mathematics isn't easy. Take good care of your mental and physical health at all stages, they come first and impact your ability to do well professionally, so balance is vital. Have an outlet you enjoy such as cooking, exercising, juggling, anything you can do on a daily or consistent basis that allows you to decompress. Set boundaries for yourself and show others how to treat you. Tell people your value, show them your value, and then tell them your value again. If you're not invited to the decision-making table, invite yourself; you deserve to be there!

Lastly, early support has proven to be a strong indicator of survival in graduate work and beyond. The impact and influence of HBCUs in producing strong math majors who go on to graduate school cannot be overstated. Out of the number of Black women from UMCP who responded to the questionnaire, **90%** of us are graduates of a Historically Black College or University. These include Clark Atlanta University, Dillard University, Howard University, Morgan State University, Prairie View A&M University, Spelman College, and Xavier University of Louisiana. The AMS Task Force Report[22] stated as one of its findings "[HBCUs] have an outsized influence on the production and the support of Black mathematicians, and providing outstanding models of successful mentoring".

Conclusion

Systemic racism has resulted in many barriers and challenges that Black women have had to overcome. This brilliant and amazing cohort of women has exerted mounds of energy constantly trying to prove that they belong and are qualified. African American mathematician, Erica J. Graham, in "How (and How Not) To Be an Anti-Racist in Mathematics",[23] identifies many forms of racism in mathematics, including perfectionism; defensiveness of power structure; paternalism, and censorship of those who cause discomfort. Graham also highlights that we must resist being pushed from the spaces that afford even marginal privilege by a white supremacy culture. As underrepresented minorities, we endure regular traumas in this space, all for the love of mathematics. While our experiences are unique, this shared love of mathematics and STEM formed the perspectives of the Black women mathematicians from UMCP.

We, this unlikely cohort of Black women mathematicians, demonstrate that we are not the stereotypes placed upon us. We provide proof that HBCUs produce future educators, leaders, researchers, and scientists in academia, government, and industry. Whether we are the first to earn advanced degrees in our families or continuing educational traditions, we represent and carry forward the intelligence and abilities that are frequently denied to Black women. We acknowledge that we are special and unique, but that was not our goal. We just love mathematics and the many doors it has opened and continues to open for us.

[22] *Toward a Fully Inclusive Mathematics Profession:* https://www.ams.org/about-us/understanding-ams-history

[23] *How (and How Not) To Be an Anti-Racist in Mathematics, Erica J. Graham, MAA Focus, April/May 2021*

We are grateful to those who paved the way before us, who encouraged us, who gave us a shoulder to cry on, and who were patient with us when we just couldn't solve that mixed integer programming model or get our code to run. We hope that the perspectives of the Black women mathematicians from UMCP encourage young Black women (and men) to realize their mathematical potential and become the future of this field. In closing, we leave the reader with this poem by Dr. Dionne L. Price, an HBCU alumna who is the first African American to earn a Ph.D. in Biostatistics from Emory University, and currently serves as the Director of the Division of Biometrics IV in the Office of Biostatistics at the Food and Drug Administration. She is also the first African American to be elected President of the American Statistical Association (President-Elect 2022, President 2023).

The Journey [24]
I am that woman and that woman is me,
Acquiring knowledge, I've learned is the key,
I stand on the foundation planted long ago,
I'm rooted in numbers and love how they flow
Now is the present and data abounds,
Quantitative inclinations will need to astound,
And as we look to tomorrow and days to come,
May we solve global challenges, and still have some fun.
I am that woman and that woman is me.

References

[1] American Mathematical Society, Mathematical and Statistical Sciences Annual Survey, 2018, accessed May 19, 2021. https://www.ams.org/profession/data/annual-survey/annual-survey

[2] Viveka O. Borum and Erica N. Walker, *What Makes the Difference? Black Women's Undergraduate and Graduate Experiences in Mathematics*, Journal of Negro Education **81** (2012), no. 4, 366–378.

[3] Bettye Anne Case and Anne M. Leggett, eds., *Complexities: Women in Mathematics*, Princeton University Press, Princeton, NJ, 2005.

[4] G. Finley, *Early success, Encouragement Add Up to Math Ph.Ds*, Diverse Issues in Higher Education **19** (2002), no. 14, 8.

[5] Nikole Hannah-Jones, *A Prescription for More Black Doctors: How does tiny Xavier University in New Orleans manage to send more African-American students to medical school than any other college in the country?*, The New York Times (September 9, 2015). https://www.nytimes.com/2015/09/13/magazine/a-prescription-for-more-black-doctors.html

[6] Abbe H. Herzig, *Women Belonging in the Social Worlds of Graduate Mathematics*, unpublished manuscript, 2006.

[7] M. Hossain and M. G. Robinson, How to Motivate US Students to Pursue STEM (Science, Technology, Engineering and Mathematics) Careers (ED533548), ERIC, 2012. https://files.eric.ed.gov/fulltext/ED533548.pdf

[24] https://mathematicallygiftedandBlack.com/honorees/dionne-price/, *by permission of the author*

[8] Shelly M. Jones and Veronica Martins, *Women Who Count*, American Mathematical Society, 2019.

[9] Alex P. Kellogg, *A University Beats the Odds to Produce Black Ph.D.'s in Math*, Chronicle of Higher Education **47** (2001), no. 23, A14–A15.

[10] Report of the Task Force for Understanding and Documenting the Historical Role of the AMS in Racial Discrimination, *Toward a Fully Inclusive Mathematics Profession*, American Mathematical Society, 2021, accessed June 1, 2021. `https://www.ams.org/about-us/understanding-ams-history`

[11] Margot Lee Shetterly, *Hidden Figures: The American Dream and the Untold Story of the Black Women Mathematicians Who Helped Win the Space Race*, Harper Collins Publishers, New York, 2016.

[12] E. N. Walker, *Excellence and Devotion: Black Women in Mathematics in the United States*, in: J. Beery, S. Greenwald, J. Jensen-Vallin, M. Mast (eds.), Women in Mathematics, Association for Women in Mathematics Series, vol 10, Springer, 2017.

Taryn Butler Lewis

Tasha R. Inniss

Monica Jackson

Calandra Tate Moore

Credits

Figure 1 is courtesy of the authors.

Photo of Taryn Butler Lewis is courtesy of the author.

Photo of Tasha R. Inniss is courtesy of Furery Reid/Spelman College.

Photo of Monica Jackson is courtesy of American University/Jeff Watts.

Photo of Calandra Tate Moore is courtesy of the author.

Tenured Women at Berkeley Before 1980

Cathy Kessel

One way to measure the progress of women in mathematics is to look at their presence or absence at elite institutions. Table 1 lists mathematics departments that are often considered the "top ten," the first woman tenured in each department before 1980, and the year when the department granted its first PhD to a woman. Of the institutions listed in Table 1, UC Berkeley (UCB) is unique in having had more than one tenured woman in its mathematics department before 1980.[1]

In this article, I recount the history of those women and some notable contemporaries, illustrating how attitudes about women affected their participation in mathematics with regard to education, research, and especially employment. Attitudes about women were (and are) instantiated in a variety of ways, including customs, policies, and individual actions. In this article, examples of each appear in the national events and local actions associated with changes for women in academia, in mathematics, and at Berkeley.

institution	first tenured woman before 1980	first woman PhD
Chicago	1930 Mayme Logsdon	1908
UC Berkeley	1931 Pauline Sperry	1911
Stanford	1969 Mary Sunseri	1928
Caltech	1971 Olga Taussky Todd	1964
MIT	1978 Michèle Vergne	1930
Columbia	–	1886
Harvard	–	1917*
Michigan	–	1914
Princeton	–	1972
Yale	–	1895

*Radcliffe College

Table 1. Women in mathematics departments. Dates of first PhDs and MIT entry from [16]. Other sources posted at `https://works.bepress.com/cathy_kessel`.

Table 2 shows how the composition of the UCB tenure-track faculty changed over the decades.

academic year	total	women	percent women
1928–29	15	2	20
1938–39	19	2	11
1948–49	27	2	7
1958–59	38	1	3
1968–69	81	0	0
1972–73	68	0	0
1982–83	69	3	4
1991–92	62	2	3
Spring 2020	57.5	10.5*	18

*One appointment is half time in physics. One is a teaching professor.

Table 2. Tenure-track faculty in UCB mathematics department.

Cathy Kessel is a consultant who lives in Berkeley, CA. Her email address is `cbkessel@earthlink.net`.

A version of this article with more detailed references is posted on the author's website, `https://works.bepress.com/cathy_kessel`.

[1]*Here, I use "tenured" to refer to full and associate professors. At Berkeley, the common understanding was that full and associate professors were tenured. However, tenure was not included in Berkeley's regulations until 1958 [14]. I use "tenure-track" or "professorial rank" to refer to professorial positions that are tenured or tenure-eligible and distinguish them from positions such as research associate, lecturer, and instructor.*

Communicated by Notices *Associate Editor Laura Turner.*

For permission to reprint this article, please contact: `reprint-permission@ams.org`.

DOI: https://dx.doi.org/10.1090/noti2448

Overview and Major Historical Sources

In her three volumes on women scientists in America from early times to the present, Margaret Rossiter has created extraordinarily detailed and readable narratives that synthesize a vast amount of complex information, helping us to see the context in which the people represented in Tables 1 and 2 lived.

In the early 1900s, a large percentage of women mathematicians were employed at women's colleges but were expected to resign after marriage [17]. Those who were already married were not given equal opportunity for employment [17]. Similar phenomena were the "marriage bars" that arose in the late 1800s, expanded during the Depression, and lasted until the 1950s. These were policies of firing women who married and not hiring women who were already married. Primarily affected were white US-born women, who predominated in occupations such as teaching and clerical work, rather than Black and foreign-born women in occupations such as manufacturing, waitressing, and domestic service.

In the 1920s, universities began to establish anti-nepotism rules in order to avoid being a "dumping ground" for patronage appointees from state governments. These were applied primarily to restrict rather than prohibit the employment of married women when their husbands were on the faculty, thus they were, in a sense, less restrictive than marriage bars. Anti-nepotism rules generally allowed a married woman to hold a faculty position elsewhere (such a position might be found at women's colleges which had begun to show a "modest tolerance" toward retaining women who married) or hired as a research associate at the same university as her husband [17]. The practice of tenure began in the 1930s and was adopted by the American Association of University Professors in 1941, along with the recommendation that no full-time faculty member be kept untenured for more than seven years. At coeducational institutions, this combination of tenure practices and anti-nepotism rules tended to work to the advantage of men, helping to preserve them from female competition during and after the Depression [17]. In the case of married couples, implementation of anti-nepotism rules tended to result in tenure-track positions for husbands and non-tenure-track position for wives [18], even sometimes depriving wives of tenure.[2] Among the rare exceptions was the astronomer Cecilia Payne-Gaposchkin. In 1956, she became one of the first female full professors at Harvard. Her husband was a research astronomer at the Harvard College Observatory [18].

Of the 228 American women who earned mathematics doctorates before 1940, 40% were married to men with PhDs in mathematics and 21% to men with PhDs in other fields. We know this, and much more, from the work of Judy Green and Jeanne LaDuke [9, 10]. In the supplementary material for their remarkable book *Pioneering Women in American Mathematics: The Pre-1940 PhD's*, Green and LaDuke have collected and updated material on each of these women, collecting information on all doctorate-earners of that period who were born in the US or who earned their doctorates from a US institution.[3] PhDs from the next 20 years are the subject of Margaret Murray's *Women Becoming Mathematicians: Creating a Professional Identity in Post–World War II America*.

The rise, fall, and rise of the percentages in Table 2 is part of a larger pattern for women in mathematics (see Figure 1), which, as Murray points out, reflects social, political, and economic trends, first the Depression, then the onset of World War II. She notes that the decline in women's share of PhDs occurred during the period from 1945 to 1955, "just as mathematics entered a period of unforeseen power, prestige, and prosperity" in the United States [15]. Although the number of PhDs granted to women increased, they were part of a much larger total due to the post-war influx of graduate student men in the sciences. In Table 2, this period of mathematical prosperity is reflected in the post-1950s increases in its first column (the total number of tenure-track faculty in the UCB mathematics department).

Details about these increases are given in *Mathematics at Berkeley: A History* written by Calvin Moore, a long-time UCB faculty member and former dean and department chair. In particular, its increases in faculty members coincide with its increased focus on research and the appointment of Griffith Evans as chair in 1936. Under Evans, who served as chair until 1949, the department became one of

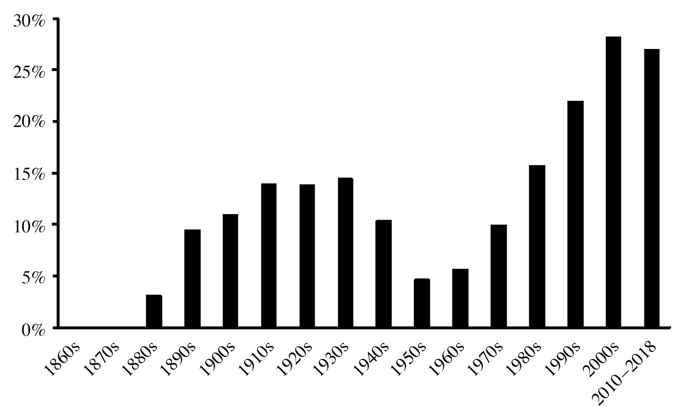


Figure 1. Percentage of US mathematics PhDs awarded to women by decade. Sources: [15] (1862–1994); Survey of Earned Doctorates (1995–2019).

[2] *An example from the 1950s: Josephine Mitchell and Lowell Schoenfeld were both professors in mathematics at the University of Illinois. She had tenure, he did not. After they married, she was informed that she would not be reappointed [18, see also 15].*

[3] *They include, for example, US-born Mary Winston Newson (PhD Göttingen 1897) and Chinese-born Shu Ting Hsia (PhD Michigan 1930).*

the top ten departments in the United States, and its ranking moved further upward thereafter [14].

Another view of the department comes from the work and biography of statistician Elizabeth Scott, who was a UCB faculty member from 1950 to 1988. She served on campus and national committees on women, and published research on a wide range of topics, including equity in academia.

Additional information comes from other Berkeley graduates and faculty members. Among other things, these accounts sometimes give the rationales and reasoning put in writing when hiring (or not hiring) female mathematicians, detailing local responses to policies such as anti-nepotism or affirmative action. In the 1970s, many of these accounts appeared in the *Newsletter of the Association for Women in Mathematics* [1].

1911: The First Female PhD at Berkeley

Before 1890, most PhD programs in the United States did not allow women to enroll, although they could sometimes participate by making special arrangements [17]. Even when women could enroll, finding a PhD advisor may have been an obstacle. Anecdotes[4] and records of advisees (see Appendix) suggest that faculty members were sometimes unwilling to advise students from particular demographic groups or unsuccessful in doing so. The first PhD in mathematics from a US university awarded to a woman went to Winifred Edgerton in 1886. The first African American man known to have been awarded a US mathematics doctorate was Elbert Frank Cox, who earned his in 1925. The first African American woman, Euphemia Lofton Haynes, earned her doctorate in 1943, almost two decades later. Of the 228 women in Green and LaDuke's study, only one (Shu Ting Hsia, PhD 1930) seems to be of Asian heritage and none are said to be Latina or Indigenous.

In 1911, Annie Dale Biddle became the first woman and third person to earn a PhD from the Berkeley mathematics department [10]. She was the first of seven women who earned PhDs at Berkeley before 1940.

After receiving her PhD, Biddle taught mathematics for a year as an instructor at the University of Washington in Seattle. She returned to Berkeley; married in 1912; and taught as a teaching fellow (1914–16), assistant in mathematics (1916–17), associate in mathematics (1920–23), and instructor (1924–33). In 1933, she was one of four instructors considered for non-reappointment because the department had decided to concentrate more on its graduate program [10]. Thus, the impetus that led to hiring Griffith Evans to build up the department also led to Biddle's termination. Her file noted that she would not be destitute because she had married a practicing attorney. In contrast, the three male instructors were retained, one because he had a wife and children to support [14].

Biddle was far from being the only woman with a PhD who held a variety of non-tenure-track positions. And Berkeley was far from being the only university that employed female doctorates in such positions [9, 17].

1923 and 1924: The First Two Female Professors

In 1923, Pauline Sperry became the first woman to attain a tenure-track position in mathematics at the University of California, six years after she had been hired as an instructor. She was one of the few women with a mathematics doctorate earned before 1940 whose primary employment was a professorial position at a PhD-granting institution [9, 10].

Sperry was joined by female colleagues, Sophia Levy in 1921, and Emma Lehmer in 1940.

Unlike Sperry, who earned a PhD in mathematics from the University of Chicago, Levy's doctorate was from the UCB astronomy department and her undergraduate degree was also from UCB. In 1921, she joined the mathematics department as an instructor, advanced to assistant professor in 1924,[5] and became a full professor in 1949. Along with Annie Biddle and the three other instructors, she and a male assistant professor were considered for termination in 1933. In the end, both assistant professors were retained in order to allow Evans to make the decision about their termination. At that time, her file noted that she was the sole support of her ailing mother [14]. After his retirement, she married John Hector McDonald, thus avoiding the effects of Berkeley's anti-nepotism rule.

The anti-nepotism rule shaped Emma Lehmer's life in a different way. After her father-in-law Derrick Norman Lehmer retired in 1937, Emma's husband Dick was able to join the UCB mathematics department as one of the tenure-track faculty. However, Emma Lehmer never had a tenure-track position, thus does not raise the count in Table 2. Here we see a different accommodation to anti-nepotism rules. Dick Lehmer held several research appointments, then a faculty position at Lehigh University until his father retired. Emma Lehmer, on the other hand, did not hold a professorial position. In his account of the UCB department, Calvin Moore explains that because Lehmer's husband held a faculty appointment, "the university's nepotism regulations did not permit her to hold a faculty position except for some short-term visiting positions to meet teaching needs."

Those "teaching needs" occurred during World War II (see the online exhibit "The Lehmers at Berkeley"). During this period, male faculty members often left for government

[4]*For example, in the 1960s Vivienne Malone Mayes was admitted to graduate school but could not enroll in one professor's course. "He didn't teach Blacks. And he believed the education of women was a waste of the taxpayer's money" [6].*

[5]*An* in memoriam *article gives different dates: 1923 and 1925.*

science projects or military service and universities sometimes rescinded their anti-nepotism rules, reinstating them after the war was over [18].

In contrast to Levy and Lehmer, Pauline Sperry did not marry. For many years, she shared a residence in Berkeley with her close friend Alice Tabor, who was a member of the German department and, like Sperry, had a PhD from the University of Chicago. Sperry and Tabor initiated Berkeley's Faculty Women's Club in 1919 [10].

Education and scholarly activities: A comparison. Sperry (1885–1967), Levy (1888–1963), and Lehmer (1906–2007) came to Berkeley at different times, with different credentials, and different past experiences.

Sperry came from the East Coast. She attended Olivet College and later Smith College, then earned a PhD from the University of Chicago in 1916 with a thesis in projective differential geometry. After a year as an assistant professor at Smith, she joined the UCB faculty as an instructor, eventually rising to the rank of associate professor. At Berkeley, she taught graduate courses in differential geometry and supervised five PhD students—more than any pre-1940 woman US PhD except Anna Pell Wheeler. During World War II, she taught a course on navigation. She published an article based on her dissertation, gave talks at AMS meetings, and wrote textbooks on spherical and plane geometry. She served as vice-chair, then chair of the Northern California section of the MAA [10].

Levy was born in Alameda, California—a few miles from the university. There, she majored in astronomy as an undergraduate and wrote her dissertation in astronomy. She contributed to a National Academy of Sciences publication and wrote numerous items in the *Lick Observatory Bulletin*. She was deeply engaged in secondary teacher preparation, serving on regional and state committees on mathematics education. During World War II, she taught courses and wrote a textbook on the mathematics of antiaircraft gunnery. She cofounded the Northern California section of the MAA and served as its secretary, vice-president, president, and sectional governor [14].

Lehmer was born Emma Trotskaya in Samara, Russia and came to Berkeley (and the US) for college in 1924. There she assisted Derrick Lehmer on a number theory project and met his math major son Dick. After Emma finished her bachelors degree in mathematics and Dick returned to Berkeley from a year of graduate school at the University of Chicago, they married. They went to Brown University where Emma enrolled in the masters program and Dick in the doctoral program.[6] After they completed their degrees in 1930, they traveled for Dick's employment to various locations (California Institute of Technology, Stanford University, Institute for Advanced Study, Cambridge University, Lehigh University), and returned to Berkeley in 1940 [14]. Emma was an active researcher, an author or co-author of 56 papers, and a Russian translator for the American Mathematical Society. In 1968, she and her husband founded the West Coast Number Theory meeting—"a comfortable, friendly, and informal environment where [young people] can find their way into the real world of mathematics" [5].

Each of these women led an active intellectual life, but Emma Lehmer's circumstances, which included a mathematician spouse who was in the same field and opportunities for travel and interaction with other researchers, seem to have helped her to become part of a mathematical research community in a way that Pauline Sperry and Sophia Levy did not. For example, she said of the year that she and her husband spent at the Institute for Advanced Study.

> I had a one-year-old girl. It was a rare occasion when I could get a babysitter and come to somebody's lecture. I did manage a few, but not very many that year. We had people come to the house, and we saw a lot of [number theorist H. S.] Vandiver. He did not have anybody else in number theory to talk to really. He was a constant visitor. I got quite a lot of inspiration and wrote a paper as a consequence of my talking to him.

The Lehmers' reputation for hospitality at Berkeley and their founding of the West Coast Number Theory meeting suggests that Emma Lehmer's interaction with Vandiver was far from being an isolated incident in her life.

The highest academic ranks awarded to Lehmer, Sperry, and Levy were, respectively, occasional lecturer, associate professor, and professor. This ordering corresponds to their degree of involvement in activities related to undergraduate education: textbooks for service courses, MAA governance, and, in Levy's case, teacher education.

This correspondence may not be a coincidence. Within US mathematics, research and teaching had begun to separate in the 1920s, and this rift intensified over the decades. It had a gendered aspect: research and the AMS were associated with men. Women were more welcome as officers and as contributors in the MAA, whose purview included teaching, history, and scholarship [15].

Attitudes about women's academic employment. During the period when Sperry, Levy, and Lehmer were educated, few women mathematicians and scientists were faculty members at major universities. Of those few, only a small percentage advanced beyond the rank of associate professor [9, 17]. Senior women scientists advised their juniors to get satisfaction from work, not advancement. Some claimed—"publicly at least," as Rossiter notes—to have

[6] *Some speculations: Financial considerations may have played a role in Emma's decision not to get a PhD. Gertrude Stith, the other woman enrolled in the graduate mathematics program at Brown at that time, could not complete her degree there for financial reasons and went to the University of Illinois where she had obtained an assistantship [10]. Lehmer may also have been aware that her employment prospects were limited not only by anti-nepotism rules but also by being Jewish.*

been quite happy with their treatment and unaware of differences in status.[7]

Sperry, Levy, and Lehmer seem to have behaved similarly. Lehmer's unpublished essay "On the Advantages of Not Having a PhD" says the first advantage is "lower expectations. If one happens to discover something new, one's peers are surprised and generous in their praise." A short biography of Lehmer based on a 1996 interview concludes, "Not being a particularly competitive person, Emma did not miss the prestige of holding a faculty position."

When interviewed in the 1980s, members of the Berkeley Women's Faculty Club did not describe Sperry, Levy, or other faculty women of their era as expressing frustration about academic rank. "I think they felt embattled. . . . but [I] did not ask if they felt hurt or demeaned" said an English professor colleague, although she did say that Sperry "would barge up and down the hall, saying 'Damn Professor So-and-So!'"

Rossiter remarks that for women professors at major universities, such as Sperry and Levy, promotions were gifts from colleagues, not necessary consequences of good work [17]. It was better not to think about discrimination and avoid being labeled as an ingrate or troublemaker. Promotion could draw resistance and criticism from colleagues and others. An extreme case from 1936 (described in more detail later in this article) was the German physicist Hertha Sponer's appointment as full professor at Duke University which elicited a letter from a physicist on the opposite coast.

Even if an entire mathematics department supported an appointment, it might fail. For example, at University of Michigan in the 1930s, the appointment of William Claytor, the third Black person to earn a PhD in mathematics, was fully supported by the department. However, as faculty member Raymond Wilder put it, "the administration was simply afraid," and the appointment was not made. In the 1940s, the appointment of Black statistician David Blackwell at Berkeley was stymied by the department chair's wife who said that she would not accept him at social functions.[8]

1950–1953: Then There Were Three

Due to her refusal to take the newly established "loyalty oath" requiring employees to swear that they were not members of the Communist Party, Pauline Sperry left the mathematics department in 1950 [7]. Despite her departure, in 1951 three of the remaining 26 tenure-track faculty members were female: Sophia Levy, and the two assistant professors Elizabeth Scott and Evelyn Fix [7].

Like Sophia Levy, Scott had been an undergraduate at UCB and her doctorate was from its astronomy department. Her official advisor was Robert Trumpler, an astronomer, although her biographer Golbeck says, "it was clear" that Jerzy Neyman, a statistician, acted as a co-advisor. Between 1939 (when she received her bachelor's degree) and 1949 (when she received her doctorate), she held a variety of appointments, including that of research assistant in Neyman's statistics lab.

Evelyn Fix earned her bachelor's and master's degrees in mathematics at the University of Minnesota, before working as a high school mathematics teacher, secretary, and school librarian in Seattle from 1934 to 1941. During that time, she attended UCB summer courses. She then moved to Berkeley to work in Neyman's statistics lab and received her doctorate (with Neyman as her advisor) in 1948.

Neither Scott nor Fix married, thus avoiding direct consequences of the anti-nepotism rule. Fix shared a house with F. N. David, who had been a student of Neyman's in London and was, among other accomplishments, the founding chair of the UC Riverside statistics department. All three women were well regarded as statisticians and became fellows of the Institute of Mathematical Statistics, David in 1946, and Fix and Scott in 1951 while they were still assistant professors.

1954–1959: And Then There Was One

In 1954, Sophia Levy retired, ending the mathematics department's involvement with K–12 education [14]. In the following year, Elizabeth Scott moved to the newly established statistics department, serving as its chair from 1968 to 1973 [7]. The entry for 1958 shown in Table 2 suggests that Evelyn Fix remained in the mathematics department until academic year 1958–59. Her obituary indicates that she became a professor of statistics in 1963.

Zero

After one hundred years of its existence, four women had been professors in the Berkeley mathematics department. Of these, only the first (Pauline Sperry) had earned her doctorate at an institution other than Berkeley. Levy's and Scott's PhDs were in astronomy and Fix's in statistics, although granted by the mathematics department.

As indicated in Table 3, Scott left the mathematics department in 1955 and became a member of the newly created statistics department. Table 3 would have been considerably shorter had the statistics department been established earlier, as Neyman had wished [14].

[7] *Similar themes occur in Cathleen Morawetz's account of a conversation with Olga Taussky Todd around 1968, "[I]t was an opportunity for her to put away her wonderful smile and air her complaints. Her greatest difficulties had come from being both Jewish and a woman. Her early year in Bryn Mawr had been difficult, and not having a regular position at Caltech rankled within her. But her beloved work in mathematics saved her" [6].*

[8] *In 1954, after he established an independent UCB statistics department, Jerzy Neyman hired Blackwell who became the first Black tenured UCB faculty member. In 1965, Blackwell became the first Black member of the National Academy of Sciences.*

year	names of female professors
1923	Sperry
1924	Sperry, Levy
1951	Levy, Scott, Fix
1954	Scott, Fix
1955	Fix
1963	Fix promoted to professor of statistics

Table 3. Female tenure-track faculty in UCB mathematics department.

The mathematics department felt Scott's influence in various ways. For example, the construction of Evans Hall, a building named in honor of Griffith Evans and designed to house the mathematics and statistics departments, began in 1968. Scott ensured that every floor had toilet facilities for women, not without struggle [7]. A less concrete influence was her work on equity.

In 1969, the faculty Committee on Policy observed that, among other things, "it is surprising that so few women—only 15 at the present time—achieve the rank of full professor at Berkeley." (Statistics for leading universities show this was not surprising at all [18, 19].)

At Berkeley, a Subcommittee on the Status of Academic Women (CSAW) was appointed, cochaired by Elizabeth Scott. The subcommittee's report, produced a year later, displayed the pre-1970 statistics shown in Table 2 and stated that,

> 45 women are appointed to ladder [tenure-track] positions which carry Senate membership and that the proportion of women in the Senate is less than it has been at any time since the 1920s. This fact alone warrants quick action to ensure that conditions leading to such a situation be rectified.

The report recommended rescinding the anti-nepotism rule, establishing paid maternity leave, and "an ultimate goal of having a representation of qualified women faculty at each rank at least in rough proportion to the number of women trained in that field." For mathematics, this "ultimate goal" would have been at least 5% (see Figure 1).

The Berkeley CSAW report was one tributary in the flood of reports on the status of academic women that appeared across the United States in the spring of 1970, "just in time to be reprinted in congressional hearings on discrimination on campuses" which became the basis of Title IX [18].

1970: Affirmative Action

Between the appointment of the Berkeley CSAW and the completion of its report, a landmark event occurred. Bernice Sandler, a psychologist who had raised two children while teaching part-time at the University of Maryland, had completed her doctorate in clinical psychology and applied for full-time positions there in 1969. She was rejected for these and other positions for reasons such as "coming on too strong for a woman" or being "not really a professional . . . just a housewife who went back to school." Her husband, who was a lawyer, identified this behavior as sex discrimination. Upon investigation, Sandler discovered that sex discrimination was illegal in some situations, but not at educational institutions in general. However, because the University of Maryland was a federal contractor, its sex discrimination could lead to the termination or nonrenewal of its federal grants. The pattern of discrimination against women in professorial positions was "industry-wide," so Sandler could and did file class-action complaints with the US Department of Labor against numerous universities, including the entire University of California system and the University of Wisconsin [18]. Sandler explained in 1997:

> Because these were administrative charges filed with a federal agency rather than a lawsuit filed in court, it was not necessary for me to be an attorney. There were no special forms to fill out. Individuals did not need to be named; the charges were filed on behalf of all women in higher education.

In May 1971, the first *AWM Newsletter* communicated information about Sandler's complaints to mathematicians.

> The basis of the complaints is not a law, but rather Executive Order 11246, amended by Executive Order 11375 (effective October 1968), forbidding discrimination by Federal contractors because of sex (as well as race, color, religion or national origin). There is no exclusion for educational institutions. Discrimination is not illegal—it can simply lead to cancellation of existing contracts or failure to make new grants. The contractors must not only not practice discrimination, but must have an affirmative action plan if necessary to remedy the effects of past discrimination. . . . HEW [the Department of Health, Education, and Welfare] has been designated as the compliance agency responsible for the enforcement of the executive order for all university contracts.

Among other things, HEW demanded that anti-nepotism rules be rescinded.

1971: Anti-nepotism Rules Crumble

In 1971, the American Association of University Professors (AAUP) revived its committee on women (Committee W) which had been discontinued in 1928. An outcome of the committee's early actions was the official policy statement, "Faculty Appointments and Family Relationship,"

> calling for the rescinding of laws and institutional regulations which subject faculty members to any automatic exclusion from academic employment solely on the grounds of being related to a member of the same family on the faculty of an institution.

This statement was endorsed by the AAUP in April 1971 and by the Association of American Colleges in June 1971.

Several major universities then quickly rectified the situations of wives who had held untenured positions for years [19]. For example, Mary Ellen Rudin and her husband, both mathematicians, although with different specialties, had come to the University of Wisconsin in 1959. This, her husband wrote,

> turned out to be exactly the right kind of place for us—the right kind of city and the right kind of Mathematics Department. There is no point in describing in detail what we did for the next 33 years. I taught my classes, had graduate students, worked with colleagues, wrote papers and books, exactly what a professor is supposed to do. Ellen did the same, first as a part-time temporary lecturer, until she was suddenly promoted to a full professorship . . . (the anti-nepotism rules, which were actually never a law, had fallen into disrepute).

Berkeley's anti-nepotism rule was rescinded in 1971, according to Margaret Rossiter, who cites evidence from Scott's files [19]. Susan Graham, a new hire in computer science, was about to marry Michael Harrison, who was already on the faculty. Her appointment was initially disapproved, but the decision was reversed a month later after the anti-nepotism rule was changed. However, this change had no effect on regularizing the situation of Emma Lehmer because, according to Calvin Moore, both Lehmers were "virtually at the age of mandatory retirement." But, this objection did not apply to Lehmer's younger colleague Julia Robinson.

As a graduate student, Julia Robinson worked with Elizabeth Scott in Neyman's statistics lab during World War II (an opportunity to gain research experience, although, unlike Lehmer, not in her chosen field).[9] Like Emma Lehmer, she had a husband in the UCB mathematics department who was in her field (mathematical logic). Again like Lehmer, her husband's opportunities for interaction with other researchers seem to have helped her to become part of a mathematical research community. For example, Robinson's husband and her advisor Alfred Tarski attended weekly "logic lunches" in the Faculty Club's main dining room. Because the main dining room was restricted to men, Robinson could not attend.[10] Instead, she learned from her husband about a question that became part of her thesis, and later about a conjecture that stimulated her approach to Hilbert's 10th problem, both posed by Tarski at lunch.

Robinson was younger than Lehmer—and had a PhD. Moreover, her husband had taken early retirement in 1971. Like Lehmer, she had worked as a temporary lecturer in mathematics, although between 1960 and 1975, not during World War II. How this was consistent with the UCB anti-nepotism rule is unclear. Perhaps the rule was waived in order to allow her to teach or run a seminar on the significant contributions that she made toward a solution of Hilbert's 10th problem during the 1960s. Or perhaps she was unpaid, as the "volunteer professor" Maria Goeppart Mayer was at the University of Chicago physics department during the 1950s [18].

If the Berkeley mathematics department immediately offered a full professorship to Julia Robinson in 1971 as Wisconsin did for Mary Ellen Rudin, this is a well-kept secret. Instead, something rather curious happened four years later.

Hiring Regulations Change

Before 1970, faculty hiring occurred via the "old-boy network." In 2016, Susan Ervin-Tripp, a psychologist hired by Berkeley in 1958 and member of Scott's CSAW, remarked in her oral history,

> It's hard to believe but they didn't advertise jobs. There was no public advertising of positions in the old days before 1970. It was considered inappropriate to apply for a job. I can't remember how I came as a visitor, whether somebody wrote them to recommend me or what. You weren't supposed to apply for a job. It was sort of like an arranged marriage. For instance, one of the reasons that I knew about this was that Dan Slobin told me how he had been hired [in 1964]. The [UCB] chair of psychology called, I guess it was probably Roger Brown in the Harvard social relations department and said, "Have you got any good men?" This sounds funny. Dan got hired without giving a job talk and before he'd even chosen a thesis topic. Isn't that amazing? [laughter] So he was promised this job. He did a fast thesis basically so he could come. [laughter]

[9] *A remembrance from Scott in the November 1985* Notices *describes the maneuver that allowed Robinson to be paid for her work in the statistics lab despite the anti-nepotism rule.*

[10] *In 1969, after an angry letter from Elizabeth Scott, faculty women were no longer excluded from the Faculty Club [7].*

In 1989, Saunders Mac Lane, a very prominent mathematician at the University of Chicago, described how new graduates were matched with jobs:

> all the active [research] mathematicians . . . had pretty shrewd ideas as to the level of mathematical activity at many schools, and they also had quite detailed (but perhaps mistaken) knowledge of the qualities of their own current products. So when they heard that Oberlin College, or the women's college of North Erehwon, or the University of W had a vacancy, they knew which of their graduates would be an appropriate candidate there, and they acted accordingly.

"Acting accordingly" sometimes involved calling the head of a department with a vacancy to recommend one's student. This could (and did) result in women not being recommended for jobs at top departments. (As Mac Lane put it, "Chicago did not normally send its women PhDs to universities anxious to acquire research hot-shots.")

Other types of employment constraints are illustrated by the experiences of Dorothy Bernstein, who graduated from Brown University in 1939 and became the first woman president of the MAA in 1979. When looking for her first job, she consulted a well-informed person at Brown:

> [H]e took out a map of the United States, covered the region west of the Mississippi and said, "You can't get a job there, because you are a woman." Then he covered the part south of the Ohio River and said, "And, you can't get a job there because you are Jewish." That left the Northeast quadrant. [6]

Expectations that women would not do research after their dissertations may have helped to reinforce the practice of not recommending them for positions at research-intensive departments—creating a vicious circle (see Table 1). According to Mac Lane:

> In this period [1931–1960], women were encouraged to study for the PhD degree at Chicago, and there was a role model on the staff to help and support them (Mayme I. Logsdon[11]). But these women students were not really expected to do any substantial research after graduation; the doctorate was it, and in many cases the thesis topic was chosen to suit. . . . I might add that for some of the men-students there was the same low level of research expectations—but not for all.

Like Logsdon, Mary Sunseri was a professor at a top department. She earned her masters degree from Stanford in mathematics in 1940; taught at San Jose State University for a year; and returned to Stanford as a faculty member, becoming an associate professor in 1969 and a professor in 1979. She taught only undergraduate calculus and mathematical analysis courses. She won awards for teaching and retired in 1986. Like Sperry and (especially) Levy, the duties of Logsdon and Sunseri reflect the association of women with education rather than research.

In the 1970s, affirmative action was intended to replace old-boy hiring practices with a system in which jobs were advertised and hiring was based on applications and interviews. Percentages of women hired or in the applicant pool were compared with percentages of qualified women (e.g., women with PhDs). In general, the transition to this new system was neither smooth nor immediate (see Rossiter's book on women scientists after 1972). Although its administration had an academic assistant for affirmative action and another for the status of women, Berkeley's transition was no exception. This is illustrated below for the case of the mathematics department.

Women Lecturers and Graduate Students at UCB: 1968–1980

Some well-known women who were lecturers in the UCB mathematics department between 1968 and 1980 were:

- Mary Gray (AWM president 1971–73, fellow of the AMS, fellow of the American Statistical Society)
- Lenore Blum (AWM president 1975–1978, fellow of the AMS)
- Jill Mesirov (AWM president 1989–1991, fellow of the AMS)
- Chuu-Lian Terng (AWM president 1995–1997, fellow of the AMS)
- Ruth Charney (AWM president 2013–2015, fellow of the AMS, AMS president 2021–2022)
- Karen Uhlenbeck (MacArthur fellow 1983, American Academy of Arts and Sciences fellow 1985, NAS 1986, fellow of the AMS, Abel Prize 2019)
- Michèle Vergne (American Academy of Arts and Sciences fellow 1998, fellow of the AMS).[12]

Although Berkeley's anti-nepotism rule was rescinded in 1971 and affirmative action was required because the CSAW report had documented patterns of discrimination in the mathematics department, none of these former lecturers ever became members of the UCB tenure-track faculty [3]. According to Calvin Moore, in 1971 an assistant professor position was offered to Uhlenbeck, who declined, and

[11] *After being widowed in 1910, Logsdon studied at the University of Chicago. She earned her PhD at the age of 40 and was a Chicago faculty member until 1946. She did not remarry. She taught a required undergraduate survey course required of all undergraduates, served as a dean from 1923 to 1927, and was head of a graduate dormitory [10, 13]. Until 1982, she was the only woman at Chicago to hold a rank above instructor [10].*

[12] *This list of honors is not exhaustive.*

to Michèle Vergne in 1972. Vergne accepted, but was not immediately able to take up duties, then resigned [14].

Some reasons for Berkeley's lack of success in hiring women are given in an April 1974 letter from faculty member Morris Hirsch to his colleagues:

> There are two different causes for this state of affairs. One is that too few of us want any Affirmative Action; many, in fact, consider it bad policy ("You mean we should hire inferior mathematicians?").
>
> A second cause is the Dean's insistence that we recruit only within narrowly specified fields. This virtually rules out the possibility of hiring women or minority mathematicians since there are relatively very few of them.

Another reason might have been the atmosphere for women, which was "incredibly horrible—for the women instructors as well as the students" [11]. Many students (both male and female) did not pass their qualifying exams or complete their degrees. Competition among graduate students may have been intensified by the unusually high ratio of graduate students to tenure-track faculty [14]. Among other things (such as sexist comments), one incident harked back to Annie Biddle's termination in 1933. A married female graduate student became pregnant and her teaching stipend was reduced because she didn't "need as much money" [11]. Recall that Biddle's file said she would not be destitute because she had married a practicing attorney.

In 2020, Chuu-Lian Terng commented, "During the time I was an instructor at UC Berkeley, the atmosphere for women was far from ideal." She added, "So it was very fortunate that I was part of the friendly and supportive differential geometry group led by S. S. Chern." This remark suggests the effect that individuals and subfields may have, resulting in quite different experiences within the same department or within mathematics.

Another effect on women's experiences may be their academic positions. As evidenced by the list above, the department had no problem hiring women as lecturers—offers were made and some women accepted. However, tenure-track positions (which were higher status and longer term) were another matter. In 1970, a memo from the UCB chancellor's office reminded deans and department chairs that sex discrimination in employment was illegal. In response, mathematics department chair John Addison asserted "we think we have gone out of our way to make sure women are not discriminated against," noting that three women (Lenore Blum, Julia Robinson, Karen Uhlenbeck) had been hired as lecturers and the full-time non-academic staff of 20 was entirely female.[13] In response to the vice chancellor's question, "Do women get appointed only as lecturers—not as regular ladder [tenure-track] members?," he said the department was "hopeful" that the women appointed as lecturers would be promoted to assistant professors [7]. They weren't.

1975: A New Hire at Berkeley

In 1975, Marina Ratner was hired as an assistant professor, one of the tenure-track faculty.[14] According to one account, the mathematics department established a "Committee W" which was charged with searching for and recommending women and minority candidates. Faculty member Rufus Bowen, who was aware of Ratner's work, brought it to the committee's attention. Her work, which was in ergodic theory, was viewed with favor, and her training in Moscow during the Cold War complemented rather than replicated the training of the existing UCB ergodic theorists [14]. At the time, this situation was described somewhat differently by faculty member Robion Kirby in a letter to the student newspaper *The Daily Californian*:

> Apparently she is the best woman candidate, for a special committee searched hard. In research she is well qualified, and a few years ago we would have been lucky to get her. Now competition is sharper. There is at least one man (I think several) whose research looks significantly better (many of those voting for [Ratner] agree with this).

In response, faculty member David Goldschmidt's letter to the *Daily Cal* noted that:

> the two individuals in question are in completely separate mathematical specialties and that there is no one in our department who is even able to read both sets of papers, much less to give a competent technical evaluation of the work. In fact, there may well be no such individual anywhere in the world. . . . among those of our colleagues who are competent to comment technically on the work of [Ratner], opinion was unanimous that there were no better qualified people available in her field. This evaluation was supported by outside letters.

Hiring Ratner involved an unusual extra step. Faculty member Stephen Smale described it in a letter written to

[13]*The first staff member, Sarah Hallum, had been hired in 1936 as a part-time secretary and stenographer while a graduate student in mathematics. She eventually became a full-time staff member, obtained a masters degree in mathematics, and retired in 1975 [14].*

[14]*It may be worth noting that Ratner was the first woman assistant professor at the UCB mathematics department who had a child.*

department chair Maxwell Rosenlicht and published in the *AWM Newsletter*:

> On Feb. 27, the department voted 26 to 7 to offer regular appointments to Drs. Marina Ratner and Robert [sic] Stanley. The normal procedure would be for you [the chair] to process these appointments. In fact, your letter I mentioned imposes a completely new obstacle to the appointment of Ratner without precedent in the department history. In the name of affirmative action procedures (what irony) you poll the department with the following question: "I believe that Ratner is superior to, or at least as well qualified as, the other leading candidates for the pure mathematics position." with boxes marked yes and no and space for reasons. The Stanley appointment is not mentioned in your letter.

The latter two letters suggest that for some faculty members the question was not whether Ratner was the best-qualified candidate available in her field but whether she was the best candidate available in some broader category. As another faculty member put it: "The problem is that, while there are many competent women mathematicians, there are very few outstanding ones and no 'super stars.'"

This sort of slippage was not unique. Two examples from earlier eras illustrate how criteria for hiring women could shift from "qualified for the position" to "best qualified in some broader arena." The first comes from 1936. Hertha Sponer (a German refugee then reputed to be the third best woman physicist in the world) had been hired as a full professor at Duke University. The president of Duke received a letter from a Caltech physicist who was concerned about Sponer and "the policy of bringing women into a university department of physics." His rationale: Finding a female physicist as accomplished as Lise Meitner or Marie Curie was unlikely; and young men were "drawn into the graduate department by the character of the *men* on its staff, rather than the character of its women" [17].

The second example begins in the 1950s. Although the mathematicians in Caltech's small department were anxious to hire Olga Taussky Todd, the Caltech trustees apparently required assurances that she was "considered the leading living woman mathematician in the world." This assurance was repeated in 1963 when Taussky Todd was granted tenure—not as a professor, but as a research associate.[15] In 1971, Taussky Todd's public display of acceptance stopped after she encountered press coverage about a young assistant professor of English who was the first woman on Caltech's faculty. "She went straight to the administration and had her rank changed to professor" [6].

1976: A Sometime Lecturer Becomes a Full Professor

In 1975, Saunders Mac Lane, a distinguished professor at the University of Chicago and past president of the American Mathematical Society, successfully nominated Julia Robinson for membership in the National Academy of Sciences [6]. Department chair John Kelley then "seize[d] this opportunity," as Calvin Moore puts it, to make her a professor.

But, since the advent of affirmative action, the department (and the University of California system) had been under pressure to hire more women for several years (see, e.g., [7]). Why wasn't Robinson already a professor? Responses varied. Years later, some members of the department described the chair's action as "seizing the opportunity to challenge the university's nepotism rule." However, this rule had been rescinded in 1971. Moreover, Robinson's husband had retired by 1973.[16] In 1968, responses to this question were "vague reasons alluding to [Robinson's] health, nepotism rules, and some linearly ordered list of logicians" [4]. In 1970, the department chair (a logician) said that were it not for the nepotism rules, the department "might well have appointed her to a regular faculty position long ago" (Addison as quoted in [7]).

The inconsistency in these responses suggests that some department members did not favor having Robinson join the tenure-track faculty—or perhaps couldn't imagine it. Depending on the questioner, it seemed that any answer would do as long as it justified not appointing her. Apparently, some faculty members did not see her as outstanding or a superstar. This suggests why the department chair might have decided that a useful precursor to a tenured appointment would be an NAS membership. And this was not just any NAS membership—Robinson was the first woman to be elected in mathematics.

In any case, appointing Robinson as a professor after her election seems to have avoided the awkwardness that occurred after Ratner's hire. Perhaps her NAS membership reassured UCB mathematicians outside Robinson's field, just as statements that Taussky Todd was the best living woman mathematician seem to have reassured the Caltech trustees.

Robinson was not the only woman to be offered a professorship after being elected to the NAS. In 1956, Maria Goeppart Mayer, "volunteer professor" at the University of Chicago, was elected to the NAS; three years later, she and her husband accepted professorships at UC San Diego in physics and chemistry, respectively. In 1963, she became the second woman to receive the Nobel prize in physics.

[15] *Although Caltech had "no fixed or stated policy" on nepotism, her husband had been hired as a tenured professor when she was hired as a research associate in 1957 [8].*

[16] *One source says he retired in 1971, another says 1973.*

Unfortunately, this variant of what the historian Margaret Rossiter calls the "Madame Curie strategy"—hiring women with exceptional qualifications[17] that are recognizable by outsiders and nonspecialists—is often impractical. It is also unfair, unless the same standard is applied to all.

Concluding Remarks

The small sample in this article illustrates a variety of changes in the lives of women in the mathematical sciences at Berkeley. As the decades passed and Berkeley's prestige grew, its emphasis on research increased and connection with K–12 education dwindled. Its statistics and logic programs took root; statisticians Elizabeth Scott and Evelyn Fix became tenured faculty members, first in the mathematics department, then in the statistics department.

As evidenced in this narrative, employment barriers were not uniform for all women in all mathematics departments in all capacities. Before 1971, Mayme Logsdon at the University of Chicago and Mary Sunseri at Stanford achieved professorial rank in roles that deemphasized research, as did Sophia Levy at Berkeley. Before and after 1971, many women were hired as lecturers, secretaries, and technical typists. In a few cases, barriers to advancement seem to have been procedural: researchers Mary Ellen Rudin and Olga Taussky Todd quickly became full professors in 1971.

Summarizing the employment situation for women in science in the 1920s and 1930s, Rossiter said:

> although most of the barriers to women's advancement that one finds documented are administrative or procedural, at root they were cognitive and perceptual. [18]

As evidenced by responses to affirmative action in the 1970s at Berkeley and elsewhere [19], federal and university regulations may not immediately change the number of women on the faculty, nor faculty hearts and minds. University anti-nepotism rules were an important factor in dampening women's participation. But, their removal did not erase perceptions built on decades of experience.

This narrative illustrates two perceptions of women in mathematics. The rare woman who was labeled as outstanding in some broad category, e.g., "best woman mathematician," might obtain a professorship at a top school such as Berkeley or Caltech. If not, she might be hired for teaching, either as a lecturer or in a tenure-track position with no research expectations. The continued paucity of women in elite departments (see tables in [2]) suggests that these limited perceptions of women in mathematics have been slow to change.

[17] *Recall that Curie is the only person to have received Nobel prizes in two scientific fields.*

	PhD Year	Name	Adviser
1	1911	Annie Dale Biddle Andrews (1885–1940)	D. N. Lehmer
2	1918	Mary Helen Sznyter Sagal (1893–1975)	J. H. McDonald
3	1920	Elsie Mcfarland Buck (1897–1984)	J. H. McDonald
4	1921	Nina M. Alderton Moore (1890–1973)	D. N. Lehmer
5	1932	Emma Whiton McDonald (1886–1948)	D. N. Lehmer
6	1933	Dorothy Brady (1903–1977)	J. H. McDonald
7	1935	Andrewa Noble (1908–1993)	D. N. Lehmer
8	1940	Virginia Wood Wakerling (1915–1997)	J. H. McDonald
9	1941	Elizabeth Sherman Arnold (1915–1992)	J. H. McDonald
10	1948	Evelyn Agnes Fix (1904–1965)	Jerzy Neyman*
11	1948	Louise Hoy Chin Lim (1922–1985)	Alfred Tarski
12	1948	Julia Bowman Robinson (1919–1985)	Alfred Tarski
13	1949	Esther Seiden (1908–2014)	Jerzy Neyman
14	1950	Wanda Montak Szmielew (1918–1976)	Alfred Tarski
15	1953	Anne Davis Morel (1920–1984)	Alfred Tarski
16	1957	Mary I. Hanania Regier (1926–2020)	Elizabeth Scott*
17	1959	Kathleen Baxter O'Keefe (1923–2012)	Abraham Seidenberg

*In 1955, Neyman and Scott became members of the newly formed statistics department.
Sources: Pre-1940: [10]; 1940s and 1950s: `https://womenbecomingmathematicians.net/db/`

Appendix. Women granted PhDs by the Berkeley mathematics department before 1960.

ACKNOWLEDGMENTS. Thanks to Charity Hirsch, Morris Hirsch, Margaret Murray, Anne MacLachlan, Laura Turner, and two anonymous reviewers for comments on earlier versions of this article.

The author was a graduate student at Berkeley from 1975 to 1978.

References

[1] *AWM Newsletter* Archive. https://www.drivehq.com/folder/p8755087.aspx

[2] G. Benkart, K. Lauter, and S. Wiegand, *AWM at 50 and beyond*, Notices Amer. Math. Soc. **68** (2021), no. 3, 387–397. https://www.ams.org/journals/notices/202103/rnoti-p387.pdf

[3] L. Blum, *A brief history of the Association for Women in Mathematics: The presidents' perspectives*, Notices Amer. Math. Soc. **38** (1991), no. 7, 738–754. https://www.ams.org/journals/notices/199109/199109FullIssue.pdf

[4] L. Blum, *Review of Julia: A life in mathematics*, American Mathematical Monthly **105** (1998), no. 10, 964–972.

[5] J. Brillhart, *Emma Lehmer 1906–2007*, Notices Amer. Math. Soc. **54** (2007), no. 11, 1500–1501. https://www.ams.org/notices/200711/tx071101500p.pdf

[6] B. Case and A. Leggett (eds.), *Complexities: Women in Mathematics*, Princeton University Press, Princeton, NJ, 2005. MR2118372

[7] A. Golbeck, *Equivalence: Elizabeth L. Scott at Berkeley*, CRC Press, Taylor & Francis Group, Boca Raton, FL, 2017.

[8] J. Goodstein, *Olga Taussky Todd*, Notices Amer. Math. Soc. **67** (2020), no. 3, 345–353. https://www.ams.org/journals/notices/202003/rnoti-p345.pdf, https://dx.doi.org/10.1090/noti2038

[9] J. Green and J. LaDuke, *Pioneering Women in American Mathematics: The Pre-1940 PhD's*, American Mathematical Society and London Mathematical Society, Providence and London, 2009.

[10] J. Green and J. LaDuke, *Supplementary material for Pioneering Women in American Mathematics: The Pre-1940 PhD's*, 2016. https://www.ams.org/publications/authors/books/postpub/hmath-34-PioneeringWomen.pdf

[11] C. Henrion, *Women in mathematics: The addition of difference*, Indiana University Press, Bloomington and Indianapolis, 1997.

[12] Emma Lehmer, On the advantages of not having a PhD, in: *The Lehmers at Berkeley*, online exhibit, n. d. https://bancroft.berkeley.edu/Exhibits/Math/dhla.html

[13] S. Mac Lane, *Mathematics at the University of Chicago: A brief history*, in: A century of mathematics in America, part II, P. Duren, R. A. Askey, and U. C. Merzbach (eds.), History of Mathematics, vol. 2, American Mathematical Society, Providence, RI, 1989. Republished in 2012. https://celebratio.org/MacLane_S/article/459/

[14] C. Moore, *Mathematics at Berkeley: A history*, A K Peters, Wellesley, MA, 2007.

[15] M. Murray, *Women becoming mathematicians: Creating a professional identity in post-World War II America*, MIT Press, Cambridge, MA, 2000.

[16] M. Murray, *A celebration of women in mathematics at MIT*, Notices Amer. Math. Soc. **56** (2009), no. 1, 42–47. https://www.ams.org/notices/200901/tx090100042p.pdf

[17] M. Rossiter, *Women scientists in America: Struggles and strategies to 1940*, Johns Hopkins University Press, Baltimore, 1989. Original work published 1982.

[18] M. Rossiter, *Women scientists in America: Before affirmative action*, Johns Hopkins University Press, Baltimore, 1995.

[19] M. Rossiter, *Women scientists in America: Forging a new world since 1972*, Johns Hopkins University Press, Baltimore, 2012.

[20] *Report of the Subcommittee on the Status of Academic Women on the Berkeley Campus*, ERIC ID: ED042413, 1970. https://eric.ed.gov/?id=ED042413

Cathy Kessel

Credits

Author photo is courtesy of Anne MacLachlan.

Grace Alele-Williams

Nigerian Mathematician of Many Firsts—Breaking Down Barriers and Opening Paths

Karin-Therese Howell and Nancy Ann Neudauer

Grace Awani Alele-Williams is a woman of many firsts. She is the first woman in Nigeria to receive a doctorate in any field and the first woman appointed to be the Vice-Chancellor of an African university. She is a champion of numerous women's causes, paving the way to make the road easier for women who came after her. She believes that in being the first, it was essential to demonstrate that women could succeed in these roles. So she did not cower when faced with opposition, but rather was a force to be reckoned with, exposing and overturning corruption and cultism, developing robust programs for both in-service teachers and university students, changing how school mathematics was taught in Nigeria, building up a new university both in terms of facilities and programs, and confronting restrictions on women in the workplace arising from societal norms and employer policies. She is among the very few mathematicians who have made significant contributions to mathematics education at all levels, from elementary to university. More remarkably, some of the textbooks and ideas about teacher training that Alele-Williams developed are still in use in Nigeria today. But while her impact and contributions might be well known in mathematics education circles in Nigeria, they are not as well known in other parts of Africa or the world. This article outlines the life of Grace Alele-Williams and offers insight into her ground-breaking work in education and work practices for women in Nigeria in the twentieth century.

Karin-Therese Howell is an associate professor of mathematics at Stellenbosch University. Her email address is kthowell@sun.ac.za.

Nancy Ann Neudauer is the Thomas and Joyce Howell Professor of Science and a professor of mathematics at Pacific University, and is the Associate Secretary of the MAA. Her email address is nancy@pacificu.edu.

Communicated by Della Dumbaugh.

For permission to reprint this article, please contact: reprint-permission@ams.org.

DOI: https://dx.doi.org/10.1090/noti2432

Figure 1. Grace Awani Alele-Williams.

Background

Born Grace Awani Alele on December 16, 1932, in Warri (now Delta State), an oil hub and former capital of the province in Nigeria, she was the last of five siblings. Her family valued education and her mother devoted much of her time to teaching all five children herself. During her primary school education, Grace's mother moved her to three different schools by the age of 10, trying to ensure that she received the best education possible. Despite suffering through many illnesses as a child, at the age of about 12, Grace gained entry into the prestigious Queens College, Lagos (then the capital of Nigeria) and lived as a boarder for secondary school, allowed to visit her family only once a year. At 18, she began undergraduate studies as one of 10 women amongst 400 students at the University College of Ibadan (now the University of Ibadan). She studied with many students who would later work their way up to having careers as policy makers and in senior positions in education. For example Bola Ige, Chinua Achebe and Akin Mabogunje were her contemporaries.[1] She chose to study mathematics owing to her interest in the subject and the joy she experienced solving problems and working through proofs. She often discussed mathematics with her cousin, who also maintained a keen interest in the discipline and told Grace from a young age that when writing proofs, every statement had to be supported with a reason. Her cousin also had a pet monkey, which was an added attraction of these visits. At this time, University College Ibadan did not award its own degrees.[2] Consequently, Alele-Williams received her Honours Degree in Mathematics in 1954, as an external degree from the University of London [10].

Perhaps inspired by her mother's dedication to education, Alele-Williams began her career as a teacher at Queen's School, Ede, Osun State, not far from Ibadan. During this time, she made a long-lasting impact on her students. Working with other teachers from Britain (who taught Arts and Sciences) and Nigeria (who taught Home Economics, Religion, and Physical Education), they produced a "large cadre of girls who subsequently became leading professionals in various sectors of Nigerian society [10]." Thus, right from the very start, Alele-Williams contributed to educating young women.

Figure 2. Regions of Nigeria, with locations from Alele-Williams' life and career highlighted.

[1]*Bola Ige was a lawyer and prominent Nigerian politician. Chinua Achebe was a novelist and poet who is regarded as the most dominant figure in modern African literature. Akin Mabogunje was a geographer and the first African to be elected as a Foreign Associate of the United States National Academy of Sciences.*

[2]*Alele-Williams was in the second set of students of this first university in Nigeria, along with Chinua Achebe and several others who rose to prominence.*

Out of Africa

Alele-Williams secured a Nigerian government grant to study at the University of Vermont, where she also worked as a graduate assistant, with the goal of becoming a secondary school teacher. The cold weather in Vermont combined with the stifling experience of segregation in the rural setting [2] prompted Alele-Williams to leave Vermont after finishing her Master's Degree in Education in 1959. Her exposure to mathematics and science education in the United States, however, inspired her to question the education system in Nigeria, and she decided to continue beyond her Master's degree. In the United States, she had witnessed active learning alongside formal lecture and a focus on understanding instead of just memorization. Universal education was also different from the system in Nigeria, where access varied widely from region to region, and had sometimes been reserved for the sons of chiefs. Most of her previous experience in Nigeria was in a system dominated by British influence. She writes about this in her PhD thesis and highlights the fact that contributions of British-trained Nigerians dominated the education system in Nigeria at the time. She believed that armed with a PhD, she would be better equipped to introduce changes. She recognized that this credential was necessary for her to lead such a charge [10]. At this point, the vast post-Sputnik support for mathematics provided her with the opportunity to attend the University of Chicago, Columbia, or Harvard, funded by a graduate fellowship. Thus as a young woman of 25, she had the educational choice of a lifetime for any student from Africa or otherwise. She chose Chicago.

The University of Chicago gave Alele-Williams education in and access to experts in comparative education, especially through its Center for Comparative Education,[3] and a distance and lens through which she could view and study the educational system in Nigeria. In her dissertation, she observed that the newly independent African states "envision education as a means of fostering economic growth, expanding social amenities and inculcating in the masses the ideals of democratic nationhood [1]." She pointed out that education had become an important instrument of social change, in contrast with the colonial period, during which educational activities were externally motivated and focused on the training of the present generation of African political leadership. Alele-Williams looked at the history

[3]*The Center for Comparative Education (CCE) is an interdisciplinary research center within the Department of Education of the University of Chicago that was launched with the purpose of bringing social science faculty outside of the field of education and a cross-cultural flavor into the Department of Education. This was at a time when comparative education was gaining stature as an educational specialty. Its founding director, Arnold C. Anderson, chairman of Alele-William's advisory committee, wrote in 1966, "We at Chicago do believe that there are certain essentials of a sound program, and we give high priority to features that others would regard as idiosyncratic" [15].*

and geography of education in Nigeria, introduced by British missionaries to spread the Christian gospel in the East and West, but kept out of the North because the prevailing Islamic administration banned proselytizing and establishing schools. Educational policy promoted the education of the chiefs' sons and members of the ruling class, resulting in an educational system throughout Nigeria that was not uniform. As she described it, the schools and universities had "created cleavage between the elite and the masses [1]."

Alele-Williams argued in her dissertation that the indigenous nationalist movement to create an independent Nigerian state (encouraged by foreign-educated Nigerians who resented aims for Nigerian development in British terms) sought to replace native authorities with local governments, establish a new social order, and expand educational and health facilities. American-trained Nigerians, she noted, helped the nationalist movement whereas the British-trained Nigerians were less attuned to the masses and had a stake in the status quo. Alele-Williams wanted to be a part of the transition to a new universal educational system.

The First Firsts

Alele-Williams completed her PhD in 1963—the first Nigerian woman awarded a doctorate—with a thesis entitled, *Dynamics of Education in the Birth of a New Nation: Case Study of Nigeria*. And Nigeria was a new nation at this time, gaining independence in late 1960, but not, as it turned out, a stable government until many decades later. Independence for Nigeria was finally achieved, and Alele-Williams began to forge the path ahead for women—for herself and for many, many women to follow. She returned to Nigeria in December of 1963 to take a postdoctoral position at the University of Ibadan and to marry Babatunde Abraham Williams. Williams had completed his Master's Degree in 1954 at the University of Illinois and in 1963 he was a Senior Lecturer in political science at the University of Ibadan, where Adele-Williams had earned her Honors Degree.

In 1965, after two years of postdoctoral work in Ibadan, Alele-Williams was appointed as a Lecturer in the Faculty of Education at the University of Lagos where her husband also secured a position. She was promoted to Senior Lecturer in 1968 and to Associate Professor of Mathematics Education in 1974, becoming the first female in Nigeria to hold this position. Even though she had been in the university just as long as her husband, and she was an Associate Professor there, she did not have the same rights as he and other men did. When her husband was laid off in 1975, they were told to vacate the campus apartment they occupied with their five children. Her petition to retain their apartment based on her position as an Associate Professor was originally denied, but then granted on appeal, leading to a new "points system" for all employees that allowed women to be treated more equally, and also made it possible for a woman to retain accommodation.

Alele-Williams was building her own identity and independence at the same time as Nigeria was moving towards its independence. As a newly independent nation, Nigeria was establishing new systems and structures, including a new educational system, which opened the door for Alele-Williams to work toward universal education. Could it be that leaving the colonial system, and such strict adherence to the British mores, also opened the door for women, including Alele-Williams, to push traditional boundaries and limitations for women?

Contributions to the Educational System and Teacher Training

Alele-Williams believed that students should take an active role in learning mathematics and discovering concepts—ideas embraced today in the form of inquiry-based learning. These ideas are still sparse in African schools and universities across the continent, where more formal lecture is the norm. Her work emphasised the importance of student understanding, as opposed to just memorizing mathematical methods. These ideas were revolutionary in the 1960s and 70s (see [5]). In her thesis she also expressed the view that Nigeria would need scientists to drive economic activity to ensure graduates have employment opportunities.

When Alele-Williams finished her doctorate and returned to Nigeria, she was able to become a participant in a new series of mathematics workshops, held in Entebbe and Mombasa. These workshops were part of the African Mathematics Programme (AMP) under the leadership of MIT professor Ted Martins, who made several visits to Africa during this time. The AMP has its roots in the SMSG (School Mathematics Study Group), an American initiative focused on the reform of mathematics education [5]. The aim of the Programme was to consider changes in education in Africa with the view that a more lasting type of aid to Africa might take the form of assistance to educational institutions and programs [4]. This philosophy of aid to Africa persists today, with the African Institute of Mathematical Sciences (AIMS) providing graduate degrees in six countries to pan-African students from over 30 countries, grounded in the belief that a robust background in mathematics can prepare Africans to solve their own challenges and problems.

From 1963 to 1975, the AMP organized annual eight-week writing workshops in Africa that produced the *Entebbe Modern Mathematics series*. These workshops included participants from many African countries, including Ethiopia, Ghana, Kenya, Liberia, Malawi, Nigeria, Sierra Leone, Uganda, Tanzania, and Zambia. Alele-Williams captured the contributions when she wrote,

> The Entebbe Mathematics Series have sometimes been dubbed American but this is to ignore the valuable contribution of the African participants, who feel keenly the African origin of the series. Moreover the whole exercise has provided an international forum for teaching and learning, unprecedented in the annals of education. Africans, working with Europeans and Americans, have produced mathematics texts good enough for use anywhere in the world. Mutual benefits have been derived by all concerned and the project has clearly contributed to international understanding [6].

With these words, Alele-Williams staked a claim for the African contribution to this series. As is often the case, the Americans and Europeans were credited with saving the Africans. The reality, however, was that the Africans were full participants, bringing their knowledge and experience to the workshops, developing and shaping the *Entebbe Modern Mathematics Series* (see [7]).

The AMP workshops produced at least 67 volumes of materials covering mathematics education, including primary school, teacher training, secondary, and sixth form levels (the secondary mathematics of the final two years, preparing university-bound students for their A-level exams). The aim was to provide support for teachers in both the methodology of teaching mathematics and in the content itself. Later, videos were made as additional resources for teachers [11].

Initially, a limited number of schools adopted these materials in order to test the educational development of the students against those using the standard curricular materials of that time. The standard curriculum mostly focused on arithmetic, while the revised modern mathematics included new topics like set theory, geometry, probability and complex numbers. As Alele-Williams noted, these latter subjects were already included in European and American instruction. In what was referred to as the *Lagos Experiment*, schools would offer one experimental class with the others taught as traditional classes. In one Lagos school, this meant 15 traditional grade one classes ran alongside one experimental class. Parents demanded their children have access to the experimental class to learn the modern mathematics with the hope that it would improve their future options. As a result, the materials were soon widely adopted.

Although the AMP had redesigned the curriculum with care and thought, some serious obstacles arose. In particular, some teachers were not adequately qualified to teach the new material, particularly in certain regions of the country. To address this issue, Alele-Williams published the *Modern Mathematics Handbook for Teachers* in 1974 to help both new and in-service teachers learn the methods and the topics from the *Entebbe Modern Mathematics series*. Her awareness of the challenges facing the educational system in Nigeria were clear in a report she wrote in 1976 ([4]): "Teaching the teachers mathematics is a relatively simple task but changing their attitude and practice is harder. Several years of hard work are still necessary before we can truly claim that modern mathematics has come to stay." Alele-Williams understood that it was not only about producing new training materials but also about equipping teachers with content knowledge and confidence to teach the content. In fact, Alele-Williams is still fighting for better training for teachers, and in 2017 she sued the government for better funding to produce quality teachers so that secondary students could compete globally [16].

Despite the extensive work that went into creating the new curriculum, the teaching of this "modern mathematics" in the schools was short lived. In a 2004 interview, Alele-Williams commented on her role in the project: "I tried to review the teaching of mathematics in schools, to make sure that the teachers understood the new concept which was already in use in Europe and America. I think we made an appreciable progress. But one of the saddest days of my life was the day the federal commissioner announced in 1978 that modern mathematics was abolished in schools." [9] The reform was also criticised as unsuitable for the populace. This was likely because some teachers were not adequately trained and parents did not understand the modern mathematics and were not equipped to help their children, especially once it made its way out of the initial *Lagos Experiment* schools where it was tested and into the village schools. The introduction of modern mathematics throughout Nigeria may have followed a similar path as the introduction of "new math" in the United States, where teachers, who were not prepared to teach it, found it challenging, and parents were baffled. The quote of Alele-Williams from the previous paragraph perhaps extends beyond Nigeria, "Teaching the teachers mathematics is a relatively simple task but changing their attitude and practice is harder." Consequently, modern mathematics was abolished in Nigeria. A task force was established to investigate how to redesign the curriculum. The changes were implemented beginning in 1981.

In 1974, Alele-Williams was appointed Director of the Institute of Education of the University of Lagos, where she served until 1985. In this role, she introduced many non-degree courses and certificate programs. In particular, these programs helped older women working as elementary school teachers to improve their training, opportunities that were not available to them earlier (see [12], for example). Alele-Williams's educational ideas were not limited to K-12 students or to university students who were pre-service teachers. She aimed to improve the lives of in-service women teachers too. From 1979 to 1985, she also served as the chair of the Lagos State Curriculum Review Committee and Lagos State Examinations Board.

Late Career and Widespread Recognition

As Professor of Mathematics Education at the University of Lagos, where she remained until 1985, Alele-Williams received many honors and awards, including becoming a Fellow of the Mathematical Association of Nigeria and a Fellow of the Nigerian Academy of Education, and receiving the Merit Award for Bendel State. These awards recognized her contributions to the education system in Nigeria.

The year 1985 brought another first to Alele-Williams. In this year, she was appointed as the Vice-Chancellor of the University of Benin in Benin, Nigeria, and she became the first woman to hold this position at an African University. At the time, Nigeria was still an extremely patriarchal society, with few accustomed to a woman serving in a leadership role. She was not deterred, however, revealing financial irregularities and calling attention to neglected student facilities functioning without water and to unfinished campus buildings. She resolved these problems, setting the university back on track. It was not an easy time to occupy this office—not only did some colleagues try to undermine her, but Nigeria was also at the height of militaristic rule and the tertiary education system was struggling and fraught with secret cults[4] associated with inciting violence and creating havoc on campuses, including trying to suppress student protest movements demanding democracy. Alele-Williams was a skilled administrator and her courage and ingenuity are credited with limiting the cultism in her university which sent "ripples of change across institutions of higher learning all over the country [6]." These ripples of change were in the form of quelling, at least for a time, cultism at universities across the country. She was also demonstrating that a woman could be an effective (and tough) high-level university administrator.

One of Alele-William's former students, speaking of her time as Vice-Chancellor on the occasion of her 80$^{\text{th}}$ birthday, said,

> Professor Alele-Williams did it with grace, guts and grit. As the first woman to be appointed the Vice-Chancellor of a Nigerian university, the cynicism before her takeover in Benin was ear-splitting! From calmly and firmly defusing sponsored "alutas," rumour-mongering, scary shadow-boxing, sabotage and all, her time at the University of Benin from 1985 to 1991 qualifies as a Golden Age. Mama Grace Alele-Williams opened her doors to everyone, treated students with respect, listened to what they had to say, encouraged academic freedom, victimized no one on account of holding contrary views and made the University of Benin a true place of learning. And she did all this at the height of military rule [14].

During her time as Vice-Chancellor, she aimed to advance other women and used the criticism of colleagues (and the publicity it brought) to further her initiatives. For example, she introduced modern computer facilities, degree courses in computer science, and diplomas in the Faculties of Science and Medicine. Her contributions echo her early life, building her own identity at a time when Nigeria was finding its way to independence and sharing this process with others. Alele-Williams was once again building something—this time an institution—against the backdrop of an unsettled situation.

> I saw it as an opportunity to show that women too could rise up to the occasion. Also, I knew what the weight of the expectations of the women was. They were eager to see how things would go and I was not going to let them down [6].

Alele-Williams wanted to use her positions to give women confidence to pursue their interests. Many years later she admitted that her excitement about serving as Vice-Chancellor had more to do with opening up the field for women than anything else. This highlights one aspect of her legacy, the imprint she made on individual lives. She also had a more collective impact. She served as a member of the African Union Commission on Women in Mathematics in Africa and as the Vice-President of the Third World Organisation for Women in Science. She was the recipient of the very prestigious Officer of the Order of the Niger (OON) in 1987 that honors Nigerians who have rendered service to the benefit of the nation.

Her focus on developing and improving education in Africa reached beyond the University of Lagos and the University of Benin, and even beyond Nigeria. She served on a global level in many capacities during this time, including as a member of the governing council at the United Nations Educational, Scientific, and Cultural Organization, and as a consultant to UNESCO and the Institute of Educational Planning.[5] She was also Vice-President of the World Organization for Early Childhood Education and later became President of the Nigerian branch. In these positions she advocated for the alleviation of poverty in communities, the education of girls, and gender equality.

[4] *These cults were secret confraternities within higher education established in 1952 by idealistic students—including Nobel laureate Wole Soyinka—to rebel against middle-class elitism. In the 1960s and 1970s several breakaway groups formed rivalries. They are now banned in Nigeria.*

[5] *This is an arm of UNESCO that aims to support educational policy, planning and management through various programs.*

Following her time as Vice-Chancellor, Alele-Williams served on the board of directors of Chevron-Texaco, Nigeria and HIP, an Asset Management Company in Lagos. In November 1994, she was invited to give the Distinguished Annual Lecture at the National Institute for Policy and Strategic studies in Kuru, a conglomerate of small semi-developed villages, hamlets, and households on the Jos Plateau in north-central Nigeria. During this address, she spoke passionately about the role of tertiary education in bringing about cohesion and development in Nigeria [13]. On that occasion, she explained her approach to mathematics that focused on problem solving. Her visionary ideas at the time remain remarkably relevant today. There is no denying that graduates with a strong background in mathematics, skilled in problem-solving and creative thinking, are needed for Nigeria's economic development.

On 28 February 2014, Alele-Williams walked with the aid of crutches to receive the Centenary Award in Nigeria, awarded by the President, Goodluck Jonathan. The Centenary awards honored a hundred Nigerians on the occasion of 100 years as an amalgamated nation. It was widely reported in the media that she received deafening applause as she made her way to the podium, reflecting how admired she was by the broader community.

In addition to the Centenary Award, Alele-Williams received two honorary degrees in recognition of her contributions, one by the University of Benin in 2017 and the other by the University of Ibadan in 2018. These awards testify to the national recognition of the importance of her contribution to the education system in Nigeria.

In Conclusion

Grace Awani Alele-Williams achieved many of her "firsts" for Nigerian women, and sometimes for African women, as Nigeria was becoming a new nation. In this broader time of transition for the country as a whole, she contributed to the transformation of education at all levels, to the building of a university, and to the breaking down of boundaries for women.

Alele-Williams's contributions to the primary and secondary mathematics education system in Nigeria include working toward a system of universal education, bringing what was referred to as "modern mathematics" to Nigeria, and writing a handbook for mathematics teachers (see, for example, [2,3]). Many of the educational programs she introduced are still in use today in several African countries.

Alele-Williams not only had an impressive career and broke down boundaries for women in Nigeria, but she also did this while raising five children with Babatunde, and later devoting time to her nine grandchildren. She is known for mentoring women and giving advice on navigating careers while balancing family responsibilities.

Although Alele-Williams holds the beacon of many firsts for women in Nigeria, including being the first woman awarded a doctorate in any field, the first to hold the position of Associate Professor, and then the first to be Vice Chancellor, being the first was not her goal. Rather, her "firsts" formed part of a much larger and broader aim she had for Nigeria, where more women would occupy senior positions. This is revealed in the following excerpt from a 2004 talk she gave on Gender Dignity at Lagos State University.

> As long as we are celebrating a woman vice chancellor because she is the first or a woman chief judge because she is the first, then we have not arrived. We look forward to the time when we will have many women in such positions and we will be celebrating so many of them.

Grace Alele-Williams is regarded by many as the mother of Nigerian academia. Her contributions extend across decades of mathematics education. She was not only skilled in mathematics education, but also in administration and in deftly tackling corruption. Throughout her career, she faced many challenges, but always maintained a balance of kindness, availability, and fierce courage. Grace seems like an apt name.

References

[1] Grace Awani-Alele, *Dynamics of Education in the Birth of a New Nation: Case Study of Nigeria*, PhD Thesis, University of Chicago, 1963.

[2] Grace Alele Williams, *Nigerian Pilot* (2 August 2018).

[3] G. Alele Williams, *The Entebbe Mathematics Project*, International Review of Education / Internationale Zeitschrift für Erziehungswissenschaft / Revue Internationale de l'Education **17** (1971), no. 2, Educational Trends in Some Developing Countries / Tendenzen des Bildungswesens in Einigen Entwicklungsländern / Tendances Pedagogiques Dans Quelques Pays en Voie de Development, 210–214.

[4] G. Alele-Williams, *The development of a modern mathematics curriculum in Africa*, The Arithmetic Teacher **23** (1976), no. 4, 254–261.

[5] M. J. Atkinson, *Facilitating Constructive Change in Secondary Mathematics Classrooms in Zimbabwe*, MSc (Education) Dissertation, Simon Fraser University, July 1977.

[6] M. Taire, *G. Alele-Williams: Mathematician who dealt with cultism at UNIBEN*, Amin News, Special Report (14 April 2018).

[7] B. J. Wilson, *Mathematics education in Africa*, in: R. Morris and M. S. Aurora (eds.), Studies in Mathematics Education: Moving into the twenty-first century, vol. 8, UNESCO, Paris, 1992.

[8] *The Iconic Academic Achievements of Grace Alele-Williams*, Amazons Watch (8 August 2016). `https://www.amazonswatchmagazine.com/arts-academia/the-iconic-academic-achievements-of-grace-alele-williams/`

[9] *Power of Grace*, interview in THISDAY Online, November 16, 2004.

[10] *Grace Awani Alele-Williams*, Legacy Way, 14 February 2019.

[11] G. Alele-Williams, *Dynamics of Curriculum Change in Mathematics—Lagos State Modern Mathematics Project*, West African Journal of Education **XVII** (1974), no. 2.

[12] G. Alele-Williams, *Education and Status of Nigerian Women*, Nigerian women and development, 1988.

[13] G. Alele-Williams, *Education and Government in Northern Nigeria*, Présence Africaine, Pre-Colloquium on "Black Civilization and Education" (3e TRIMESTRE 1973), pp. 156–177.

[14] *Professor Grace Alele-Williams at 80: A Salute to Beauty, Excellence, Substance and Toughness!*, Facebook (18 December 2012). `https://www.facebook.com/notes/kennedy-emetulu/professor-grace-alele-williams-at-80-a-salute-to-beauty-excellence-substance-and/10151563034389027/`

[15] A. C. Anderson, *The University of Chicago program in comparative education*, Int. Rev. Educ. **12** (1966), 80–91. `https://doi.org/10.1007/BF01417067`

[16] D. Adesulu, *Shettima, Dickson, Tinubu, Alele-Williams sue for quality education*, Vanguard (27 October 2017). `https://www.vanguardngr.com/2017/10/shettima-dickson-tinubu-alele-williams-sue-quality-education/`

Karin-Therese Howell

Nancy Ann Neudauer

Credits

Figure 1 is courtesy of The Journal in Nigeria.

Figure 2 is courtesy of Burmesedays. Licensed under CC-BY-3.0.

Photo of Karin-Therese Howell is courtesy of the author.

Photo of Nancy Ann Neudauer is courtesy of Pacific University.

OPINION

Increasing the Number of Women Faculty at R1 Schools: Research-based Strategies that Work

Jenna P. Carpenter

Note: The opinions expressed here are not necessarily those of Notices.

The Issue

Despite decades of effort, we still don't attract and retain all of the female faculty talent that exists in fields like mathematics, engineering, and science. But there are things your department can do to change this. There is a large body of research and more than two decades of work by the NSF ADVANCE Program which has shed light on issues related to faculty diversity, along with strategies for addressing them. Campbell University, where I am the founding Dean of the School of Engineering and a past NSF ADVANCE grant principal investigator, is a good example. Today our engineering faculty is 62% women. Eighty percent of the mechanical engineering faculty are women. One hundred percent of our administrators are women. Compare these percentages to the data for women faculty in mathematics, below, and note that diversity in engineering lags behind that of mathematics. Moreover, our ten faculty in engineering currently are on six National Science Foundation grants totaling $7.6 million, and have obtained four additional grants from foundations. Our faculty have won awards for best papers, have obtained competitive fellowships, and have received national recognition for their research.

So what does the research say about how we can increase the number of women faculty in mathematics? One persistent underlying issue is unconscious, implicit, or unintentional bias [9, 11]. Unconscious bias is just that. It is bias that we all have but of which we are unaware. Research suggests that it comes from stereotypes in our culture (not our identity). That means men and women have similar unconscious biases about women in mathematics.

How exactly does unconscious bias fuel systemic inequities for women faculty in mathematics? One of the reasons that implicit bias is damaging is the fact that it is not intentional. Most people's conscious values support women in mathematics, so people assume that they are treating women fairly. Yet research studies confirm that we engage in many behaviors that treat women unfairly, discourage women, and drive them away from STEM fields like mathematics [11]. We simply are not aware of the collective, real impact of our unconscious biases on our everyday actions, words, attitudes, and decisions. This impact is often described in terms of micro-insults, micro-inequities, and micro-aggressions. Plainly put, these are "small" things that we say and do which suggest that women are not good at math, not interested in math, that it is not appropriate for women to pursue math, and that women don't belong in

Jenna P. Carpenter is dean and professor in the School of Engineering at Campbell University. Her email address is carpenter@campbell.edu.

DOI: https://dx.doi.org/10.1090/noti2434

math. Because these things are small when viewed individually, people are tempted to dismiss them as "harmless" or "unimportant." But the collective impact of these daily dings and disses over multiple years is significant. A continual undercurrent of negative behavior creates a chilly and unwelcoming climate for women in mathematics. When it comes to faculty, the collective effect erodes women's authority, power, confidence, and effectiveness in the mathematics workplace. It also erodes our respect for women and view of women as experts in mathematics. So, it shouldn't be a surprise that research shows women in fields like mathematics have to repeatedly prove themselves at work and that they aren't viewed as competent or liked [11]. Yet research also shows that both of these attributes – being viewed as competent and being liked – are important to success in the workplace, influencing favorable evaluations, recommendations for special opportunities, and raises [9]. As a result, qualified women get passed over for scholarships, fellowships, jobs, grants, promotions, administrative and leadership roles, and other opportunities [20]. Virginia Valian, Distinguished Professor at Hunter College and the CUNY Graduate Center, and author of the seminal book *Why So Slow: The Advancement of Women* [6] refers to this collective impact as the accumulation of disadvantage. This is one of the main reasons that we struggle to attract and retain women faculty in mathematics departments. The farther up the academic ladder we go, the fewer women there are.

Let's Start with the Data

The Mathematical and Statistical Sciences Annual Survey collects data on the number of male and female faculty, department heads, and assistant/associate/full professors. The most recent data set as of this writing (from 2017-2018) is shown below [1,2]:

	% Men Hired	% Women Hired
Doctoral Math Institution	72%	28%
Masters & Bachelors Institution	65%	35%
Statistics & Biostatistics	67%	33%

	% Department Chair - Men	% Department Chair - Women
Math Public	83%	17%
Math Private	75%	25%
Applied Math	68%	32%
Statistics	80%	20%
Biostatistics	79%	21%
Masters Institution	74%	26%

	% Tenured (Men/ Women)	% Full Professor (Men/ Women)	% Tenure-Track (Men/ Women)	% Non-Tenure Track (Men/ Women)
Doctoral Math Institution	79%/ 21%	88%/ 12%	73%/ 27%	70%/ 30%
Masters Institution	70%/ 30%	76%/ 24%	66%/ 34%	62%/ 38%
Bachelors Institution	67%/ 32%	73%/ 27%	61%/ 39%	65%/ 35%
Statistics & Biostatistics	66%/ 34%	78%/ 22%	66%/ 34%	53%/ 47%

This data shows that women occupy more non-tenure track positions [1,2] more lower-level tenure-track positions [2], and fewer leadership positions [2]. The more prestigious the institution (doctoral versus bachelors, say) the worse the gaps in the data between men and women are. The less applied and interdisciplinary the program (pure mathematics versus biostatistics, say), the worse the gaps in the data are.

Research-based Strategies and Best Practices

So, what can we do? One area that is ripe for improvement is faculty hiring. Research shows that unconscious bias permeates how we advertise positions, how we form search committees, as well as how we recruit candidates, filter applicant pools, evaluate candidates, and conduct interviews [3]. Fortunately, there is a large volume of research that highlights how unconscious bias negatively impacts recruitment and retention of women faculty, as well as research-based strategies for addressing these problems [11, 19]. There is also a solid body of best practices from other institutions and NSF ADVANCE programs [3, 4, 5] that can be adapted anywhere. It is also important to focus on the climate and culture in the department since research shows that unconscious bias continues to impact women after they are hired [16, 17, 18]. Let's start at the beginning, with recruiting new faculty.

Recruiting

The hiring process starts well before you talk with any candidates. Departments may assume that including the required EEOC statement in their advertisement will be sufficient to attract a more diverse pool of applicants. However, there are many other things you can do to attract a more diverse cohort of applicants. First, make sure that your ad clearly states your commitment to diversity, beyond the required EEOC language. A well-thought-out statement that genuinely reflects your department's and institution's commitment to diversity can help you attract a more diverse applicant pool [19]. Next, faculty position advertisements should avoid a long wish list of desired experience and skills that you really don't expect any single candidate to meet. Why? Research suggests that men tend to overestimate their skills and achievements, so they will apply for a position if they meet most (but not all) of the requirements. They tend to assume that they know enough to get by and can figure out whatever they don't know once they get on the job. Women, on the other hand, tend to undervalue their skills and achievements. Women may underestimate their abilities, predict they won't do well, and don't consider themselves ready for promotions and opportunities [11, 17] even when they are well-qualified. Consequently, women may not apply for a job unless they clearly meet all of the desired skills and experience listed in the ad. Women may assume that these qualifications are just that, minimum requirements for the job. So, instead of that long wish list, state in your advertisement that you are looking for candidates who are interested in an accurate list of required experience and skills. For example, instead of saying that a candidate must have taught course X, say that you are looking for candidates interested in teaching course X. Instead of saying that candidates must work in research area Y, say that you are looking for candidates interested in working in research areas related to Y. These may seem like small changes, but they matter. It means that women candidates are more likely to see themselves as qualified and apply for your position, versus taking themselves out of the pool before they ever apply.

Next, think carefully about the breadth of requirements, experience, and background outlined in your advertisement. Women (due to years of accumulation of disadvantage) often have more varied and less traditional career paths [19]. If your search committee assumes that these women aren't qualified, they will be overlooking some fantastic candidates. Why? Research shows that faculty are poor at predicting who has potential and who does not [7], even though they believe themselves to be skilled at such. Research also suggests that faculty may be biased in favor of candidates from doctoral institutions that are similar to their own, using the prestige of the candidate's doctoral institution as a proxy for the quality of the candidate [19]. Instead of narrow categories of experience and background, focus on broader ranges of experience and expertise that truly are required for the position.

Once you have your written your job ad, do more than just post it in the usual places. If you look where you've always looked, you are likely to find the same type of candidates you've always found. Do targeted recruiting. Reach out to national organizations, like the Association for Women in Mathematics (AMW), and other groups focused on supporting traditionally underrepresented groups in mathematics. Reach out to institutions with LSAMP Programs, GEM consortium institutions, Historically Black Colleges and Universities, Minority-Serving Institutions, and Hispanic-Serving Institutions. Email graduate advisors in PhD programs and emphasize your interest in attracting diverse candidates. Send your advertisement to women mathematicians at other institutions and ask them to distribute them to their networks. Use a combination of personal emails, advertisements, national organization newsletters, and listserv posts to cast as broad a net as possible [4].

Search Committees

The next area where unconscious bias plays a large role is with the search committee. The first overarching principle with all of the recommendations is to make sure your search committee and department understand that the goal is to hire the most qualified candidate, not to hire, say, a woman. As we noted above, research shows that because of unconscious bias, faculty are not able to ascertain the most qualified candidate, even though we think that we can [7, 11, 20]. Unless we make a conscious and intentional effort to do otherwise, we tend to hire people who align with our stereotypes. The second overarching principle is to help the search committee and department understand that diversity increases innovation, creativity, productivity, and critical analysis. These attributes position your department to be more successful in teaching, research, and securing grant funding, as well as attracting and mentoring quality graduate students.

Where do we start? First, make sure your search committee is itself diverse. The search committee is a primary source of unfounded bias in searches. A diverse search committee results in more diverse hires [15, 19]. Second, your search committee should complete unconscious bias training before they begin their work, write the advertisement, or look at any applications. Hiring an external expert to do several training sessions with your search committee is a good strategy because training that is poorly done can do more harm than good [19]. Reminding the search committee of potential biases at the evaluation stage can also reduce the impact of bias [19]. Third, appoint a trained diversity advocate as a full-fledged member of your search committee [4]. This person should not be a woman faculty member, underrepresented minority faculty member, or staff member. Preferably, the advocate should be someone the department faculty respect and view as having influence. They can help the committee watch out for bias in your discussions, deliberations, language, and decisions. It is a given that your search committee will engage in unconscious bias during the search. A trained diversity advocate can help you catch yourselves and redirect your thoughts and actions to avoid biased decisions and actions.

Next, make sure that your search committee decides on evaluation criteria for the candidates before looking at any applications [19]. These criteria should be the agreed-upon items on which all candidates will be evaluated. Ideally, these criteria should align verbatim with your advertisement (hidden criteria are a major source of implicit bias). Once you have settled on the criteria, create a rubric to be used by every search committee member in evaluating every candidate. Insist that evaluations be backed up with evidence from the application materials (and make sure your rubric includes this evidence) [19]. Search committee members should not use unstated criteria or rationales for evaluating candidates. Why? We filter the applications through our biased lenses, so if we aren't forced to focus on the criteria we agreed upon and the facts from the application materials, bias will sneak in and skew our evaluations [11].

Sound like a lot of work? Yes, it is. But this approach helps your search committee base their evaluations on objective information, versus (unconsciously biased) opinions. It will also help them pick the higher-performing candidate [11], again, because it helps them navigate around the unconscious biases that taint their evaluations. While faculty may insist that they are fair and objective, as we noted above, research studies show that we are not [14]. For example, there are studies where two identical CVs are submitted, the only difference being the candidate's names, and yet the male-named candidate is consistently rated as more qualified than the female-named candidate [18]. Similar studies show that candidates with white-sounding names are rated higher than candidates with ethnic-sounding names [15]. Other research studies have looked at identical application materials for a graduate program. Both men and women faculty at research-intensive institutions rate applicants as significantly more qualified, suggest a higher stipend, and offer more career mentoring to the applicants with male-sounding names versus female-sounding names, even though the application materials were identical aside from the name [11, 18]. Your search committee can avoid these evaluation pitfalls by using stated and agreed upon criteria backed up by observable and documented facts from application materials, all documented in your candidate evaluation rubric. This approach keeps your search committee from (unconsciously) shifting the criteria, and the weight they give to those criteria, to justify their (unconsciously) biased evaluations.

What else can your search committee do to avoid bias? Your committee should be made aware of the fact that letters of recommendation written for men are longer, contain more references to their CV, discuss their publications, and present the candidates as colleagues and researchers. On the other hand, letters of recommendation written for women are shorter, contain more (irrelevant) references to their personal lives (like "she has two kids"), describe women as teachers and students, use fewer standout adjectives, and contain more doubt raisers – hedges, faint praise, and irrelevancies (like "she is friends with my wife") [19]. The search committee also needs to understand that the research on teaching evaluations has found that evaluations are consistently biased in favor of men and against women [12]. In one such study, a male faculty member, Dr. Martin, and a female faculty member, Dr. Mitchell, each taught sections of the same online course. The courses were identical except for the instructor. The male faculty member received higher evaluations in every category, including non-instructor specific categories like course materials and technology that were unrelated to the instructor's demeanor, ability, or attitude [12].

Another key to avoiding bias is to make sure your search committee takes their time when reviewing applications. Research shows that when faculty hurry through reviewing CVs, applications, or grant proposals, they have to rely on mental shortcuts (in order to go faster) and these mental shortcuts rely on unconscious biases and their associated stereotypes [10]. When faculty take their time, they are less likely to be biased. Lastly, when it comes to creating short lists of candidates, start with an empty short list. Then have the search committee go through the list of candidates and look for reasons to put candidates onto the short list, instead of looking for reasons to toss their names out of the pool. Again, this is a small change in thinking but one that makes a difference. It helps us retain candidates who look interesting versus throwing them out because we haven't rated them as perfect. Because our unconscious biases lead us to devalue women's contributions, intelligence, potential, and accomplishments [8], this approach helps short-circuit some of our biased assessments. While it may seem unsurprising, research suggests that when we have a more diverse short list, we are more likely to hire diverse candidates [19].

Interview Process

Let's assume that you have successfully advertised and selected a more diverse candidate pool using the research-based best practices above. What about the interview process? Be sure to use clear, proactive communication throughout the entire interview (and hiring) process. Use the same (think template) communication with every candidate (to avoid unintentionally biased responses).

Structured elements help you avoid bias. These include informing all candidates about institutional policies, like spousal hiring, modified duty policies, and family leave policies, as well as information on local schools, as part of the interview. Don't make candidates ask questions about these items. Don't make assumptions about what candidates may or may not be interested in. And when you bring candidates to campus, make sure they have an opportunity to get to know the campus community and interact with people outside your department who are not on your search committee. It gives candidates a chance to ask questions about the department, the institution, and the community that might impact their decision to accept a position but that they might not feel comfortable asking you. What is the reputation of the department elsewhere on campus? How robust is the institution's budget? How good are the local schools?

Another way to use a structured approach is to create a standard, comprehensive offer package before you decide on your final candidates. Do not offer candidates the bare minimum and expect candidates to negotiate to get more. Why? When you hire a candidate, you have your best opportunity to secure for them a competitive salary and the resources they need to be successful. It is unwise to try to hire a candidate as cheaply as possible and think that a windfall will arrive in your department budget down the road. What kind of message are you sending to your prime candidate when you make a poor offer? Also don't insist that your candidate negotiate with you to get better terms. As a society, we tend to regard negotiation as an appropriate activity for men. The result is that male candidates tend to be more experienced negotiators, they are more likely to have mentors who can advise them on how and what to negotiate for, and we tend to respond more positively to men's efforts to negotiate. On the other hand, in our culture it is viewed as socially inappropriate for women to negotiate. Therefore, women tend to reach adulthood with less experience in negotiation and are hesitant to do so. Also, because there are fewer senior women in mathematics, women candidates are less likely to have a woman mentor to advise them on how to negotiate, and they are less likely to know what they can ask for, and how and when to ask. And even if women do try to negotiate, we are not likely to respond to them favorably because they are acting against our stereotype of appropriate female behavior [13, 16]. Research suggests that this difference in negotiation is one of the reasons that women faculty make less money than comparable men faculty and that female hires are less successful [16]. If we don't give tenure-track faculty the resources and support they need to meet the goals and expectations we have for them, then we shouldn't be surprised if they don't succeed.

After the Hire

It isn't enough to focus solely on hiring. Unconscious bias continues to impact women after they are hired [11]. Unchecked bias can create a general culture and a climate that ranges from unwelcoming and unsupportive to outright hostile. Therefore, it is crucial to make sure that you follow up all of your hard work during the search and hiring process with strategies to reduce the impact of unconscious bias after the hire. For all of the reasons noted above, unconscious bias influences how we make teaching assignments and utilize teaching evaluations, distribute resources, provide support, and offer opportunities, as well as to our tendency to overburden women with low value service activities. Research shows that academia is based on a masculinized model of success that tends to be competitive and hierarchal, versus collaborative and egalitarian [13]. This lends itself to an unsupportive climate for many faculty. To address this means you will have to meaningfully engage your department in finding better ways to support and mentor all of your faculty. A large number of research-based strategies and practical advice for improving your workplace climate, such as addressing unconscious bias, offering bias training, creating a sense of belonging, using fair and consistent management practices, and promoting diversity can be found in [11].

Conclusion

We can make real progress in diversifying our faculty. There is no shortage of research-based best practices and resources to help us navigate around our biases and do a more successful job of recruiting and evaluating candidates and mentoring faculty. But if we want to hire and retain a more diverse and qualified faculty, we have to be intentional and proactive. We have to be willing to devote the time, effort, and resources necessary to address the issues. Once we do, we are likely to find that our department is not only more diverse, but also a more supportive, innovative, and pleasant place for everyone to work. This, in turn, will help us attract and retain even more diverse and outstanding faculty in the future.

References

[1] A. L. Golbeck, T. H, Barr, and C. A. Rose, *Report on 2017-2018 Academic Recruitment, Hiring and Attrition*, Notices of the American Mathematical Society **67** (2020), no. 2, 235–239.

[2] A. L. Golbeck, T. H. Barr, and C. A. Rose, *Fall 2018 Departmental Profile Report*, Notices of the American Mathematical Society **67** (2020), no. 10, 1615–1621.

[3] K. O'Meara and D. Culpepper, *How Bias Emerges in Academic Hiring: A Research Brief for Faculty Search Committees*, Inclusive Hiring Pilot Materials, ADVANCE Program, University of Maryland College Park. `https://www.advance.umd.edu/sites/advance.umd.edu/files/5.%20Bias%20in%20Hiring%20Handout.pdf`

[4] Guide to Best Practices in Faculty Search and Hiring – Columbia University. https://provost.columbia.edu/sites/default/files/content/BestPracticesFacultySearchHiring.pdf

[5] Handbook (and toolkit) of Best Practices for Faculty Searches – University of Washington. https://www.washington.edu/diversity/faculty-advancement/handbook/

[6] V. Valian, *Why So Slow? The Advancement of Women*, MIT Press, Cambridge, Massachusetts, 1999.

[7] C. Wenners and A. Wold, *Nepotism and sexism in peer-review*, Nature **387** (1997), 341–343.

[8] M. E. Heilman, A. S. Wallen, D. Fuchs, and M. M. Tamkins, *Penalties for success: Reaction to women who succeed in male-gender-typed tasks*, Journal of Applied Psychology **89** (2004), no. 3, 416–427, DOI 10.1037/0021-9010.89.3.416.

[9] J. C. Williams, *The 5 Biases Pushing Women Out of STEM*, Harvard Business Review, March 24, 2015.

[10] M. Bendick, Jr., and A. P. Nunes, *Developing the research basis for controlling bias in hiring*, Journal of Social Issues **68** (2012), no. 2, 238–262, DOI 10.1111/j.1540-4560.2012.01747.x.

[11] C. Corbett and C. Hill, *Solving the Equation*, AAUW, Washington, DC, 2015.

[12] K. M. W. Mitchell and J. Martin, *Gender Bias in Student Evaluations*, Political Science and Politics **51** (2018), no. 3, 648–652, DOI 10.1017/S104909651800001X.

[13] C. Pritlove, C. Juando-Prats, K. Ala-leppilampi, and J. A. Parsons, *The good, the bad, and the ugly of implicit bias*, The Lancet **399** (2019), no. 10171, 502–504, DOI 10.1016/S0140-6736(18)32267-0.

[14] K. S. Lyness and M. S. Heilman, *When fit is fundamental: Performance evaluations and promotions of upper-level female and male managers*, Journal of Applied Psychology **91** (2006), no. 4, 777–785, DOI 10.1037/0021-9010.91.4.777.

[15] M. Asplund and C. G. Welle, *Advancing Science: How Bias Holds Us Back*, Neuron **99** (2018), no. 4, 635–639, DOI 10.1016/j.neuron.2018.07.045.

[16] L. Babcock and S. Lacschever, *Women don't ask: Negotiation and the gender divide*, Princeton University Press, Princeton, NJ, 2003.

[17] A. Bandura, *Self-efficacy mechanism in human agency*, American Psychologist **37** (1982), no. 2, 122–147, DOI 10.1037/0003-066x.37.2.122.

[18] R. E. Steinpreis, K. A. Anders, and D. Ritzke, *The impact of Gender on the Review of the Curricula Vitae of Job Applicants and Tenure Candidates: A National Empirical Study*, Sex Roles **41** (1999), no. 7-8, 509–528, DOI 10.1023/A:1018839203698.

[19] K. O'Meara, D. Culpepper, and L. Templeton, *Nudging Toward Diversity: Applying Behavioral Design to Faculty Hiring*, Review of Educational Research **90** (2020), no. 3, 311–348, DOI 10.3102/0034654320914742.

[20] E. Reuben, P. Sapienza, and L. Zingales, *How Stereotypes Impair Women's Careers in Science*, Proceedings of the National Academy of Sciences **111** (2014), no. 12, 4403–4408, DOI 10.1073/pnas.1314788111.

Jenna P. Carpenter

Credits

Author photo is courtesy of Campbell University.

COMMUNICATION

Working at Their Full Potential: The Impact of the AMS Birman Fellowship

Scott Hershberger

One hundred fifty-nine men. One woman. Such was the gender breakdown of the faculty of Stevens Institute of Technology when Joan Birman became an assistant professor of mathematics there in 1968. Three years later, she arrived at Princeton University as a visiting assistant professor mere days before the first woman to earn a PhD in mathematics at Princeton departed for a job.

In the decades since, the representation of women at all levels of mathematics education and research has improved. But gender gaps persist: At US four-year colleges and universities, women make up 32% of all full-time mathematics and statistics faculty and hold 23% of tenured positions; in mathematics departments at PhD-granting institutions, those figures fall to 24% and 15%, respectively [1]. (Across all science, engineering, and health disciplines, by comparison, women occupy 38% of faculty positions and 31% of tenured positions at four-year colleges and PhD-granting institutions [2].) And even when women do attain faculty positions, they often face challenges that their male colleagues do not.

Blatant discrimination against women mathematicians, while on the decline, still hinders the progress of some in the field. Others experience indirect slights that accumulate over time, as their contributions to research, teaching, and community-building tend to be undervalued or overlooked. And outside of work, women are more likely than men to spend substantial time caregiving for relatives [3]. All these factors can diminish the amount of time and energy that women are able to dedicate to research.

Scott Hershberger is the communications and outreach content specialist at the AMS. His email address is slh@ams.org.

DOI: https://dx.doi.org/10.1090/noti2439

According to Birman, for those who plan to have children, the most prominent obstacle is that "women's biological clock is very different from that of men." The birth of a child often coincides with a critical professional juncture. In 1970, the average age of first-time mothers in the US was 21. By 2016, it had risen to 26. Among college-educated women it had reached 30, an age at which mathematicians following a typical path are working in a post-doctoral or early tenure-track position [4], [5]. "A woman mathematician's instincts may tell her that attention not given to her children when they are infants may be irreplaceable, whereas time lost to mathematics can be made up later," Birman says. "Unfortunately, the mathematical community has tended to judge such decisions harshly, concluding that the woman slowed down because she was lacking in drive and in scholarship."

Joseph and Joan Birman in 2008.

Birman herself followed a nontraditional path in the field (which she does not recommend). After receiving her bachelor's degree, she worked in industry, married, and had three children before returning to graduate school in mathematics. With her husband Joseph on the faculty at New York University, she received free tuition, and the couple could afford to hire babysitters so that Joan could take classes part-time. After passing her qualifying exams, she received a fellowship that enabled her to focus full-time on her dissertation.

Today Birman is well known for her pioneering work on braids and knot theory. Among her numerous recognitions, she was elected last year to the National Academy of Sciences. Her intellect, passion, and perseverance were key to her success. But she remains humble, acknowledging that in her early- and mid-career years, she received a level of support from her husband and male mathematicians such as Wilhelm Magnus, Ralph Fox, and Lipman Bers that was uncommon for women in mathematics in earlier generations.

After retiring, Birman sought to give back and lift up a new generation of women mathematicians. In 2017, she and her husband established the AMS Joan and Joseph Birman Fellowship for Women Scholars. The annual $50,000 fellowship gives mid-career mathematicians extra research support. Crucially, the application process takes personal circumstances into consideration to ensure that the fellowship will make a real difference in recipients' trajectories.

So far, four mathematicians have been awarded Birman Fellowships. True to Joan Birman's goal, all have had exceptional research records but faced unique obstacles. "The first four awardees were just terrific. They were exactly the kind of candidates that Joe and I had in mind," she says. With the fellowship, they could handle life's hurdles, both expected and unanticipated, while carrying out ambitious research plans.

Lillian Pierce, the 2019–2020 Birman Fellow, spent some eight years either pregnant or nursing during the first decade of her career. She likens the long-term effect on her mathematics research of being a gestational parent to running a marathon while carrying an enormous pack. "The marathon is hard enough. A fellowship like the Birman can make a huge difference," she says. "Using its flexible resources allows women to creatively outsource the weight of that enormous pack and get back to running the marathon in a way that shows their full strength."

A Change of Plans

When applications opened for the inaugural Birman Fellowship, Margaret Beck was nearing a sabbatical semester. She wanted to extend the sabbatical to last a full year, but to do so she needed funding for the second semester. So she applied, not expecting anything to come of it.

Margaret Beck, the 2018–2019 Birman Fellow.

At the time, Beck was an associate professor at Boston University. Studying dissipative partial differential equations, she was interested in exploring how a model's topological features can affect the stability of its solutions. When she found out that she had won the fellowship for the 2018–2019 academic year, "I was happily shocked and really excited."

Beck spent the fall 2018 semester as a research professor at the Mathematical Sciences Research Institute. "It was great. I got a lot done that semester," she says. "It was really nice to be away in a new environment where I wasn't as distracted by either everyday things in my home department or everyday things at my house." Meanwhile she was looking forward to the spring semester, which she planned to spend in Australia collaborating with Robert Marangell at the University of Sydney.

But while still in Berkeley, Beck had to take her one-year-old daughter Esme to the emergency room. Esme was diagnosed with a rare condition whose treatment required medicine that was new, expensive, and difficult to obtain—ruling out travel to Australia. Instead, Beck returned with her daughter to Boston, still planning to collaborate with Marangell as best she could remotely.

Unfortunately, life interfered once again when her mother fell deeply ill. Beck spent the majority of the semester with her and remembers being "terrified" that she would need to return the fellowship money. But supporting women through career disruptions is the point of the fellowship. Built into it is the flexibility to adapt to changing circumstances. Joan Birman reiterates that the selection committee "got everything right."

"I honestly don't know what I would have done if I hadn't been on the fellowship," Beck says. "If I'd been teaching two classes and trying to be with my mother, I really have no idea how that would have worked." Although her year did not go according to plan, the fellowship "enabled me to survive during that time, and it enabled my research to survive—even though it was on pause."

The following summer, Beck attended mathematics conferences alone for the first time since the birth of her daughter. And during the pandemic, Esme's father was a stay-at-home dad, giving Beck time to dive back into research. Among other projects, she resumed her remote collaboration with Marangell. The two are studying the Maslov index, a topological invariant connected with stability problems in partial differential equations. Last year, Beck was promoted to a full professor in recognition of her continuing excellence in research.

"The Mountain We Needed to Climb"

Many mathematicians attribute their trajectory in the profession to the influence of a particularly inspiring mentor. For Lillian Pierce, that mentor was Elias Stein, a harmonic analyst at Princeton who was her first college mathematics professor and her PhD advisor. Stein taught and wrote about mathematics with patience, high standards, and clarity, Pierce says, and without him she would not be a mathematician. "It would be hard to overstate the impact that Eli has had on my time in mathematics."

As Pierce began her career, analysis and number theory often seemed to be treated as separate disciplines. But Pierce saw more connecting the two than isolating them, and she spent her post-doctoral years exploring those links.

In 2018, she became the Nicholas J. and Theresa M. Leonardy Associate Professor of Mathematics at Duke University. Still actively collaborating with Stein, Pierce embarked with him on a project to write the first book on discrete operators in harmonic analysis. When she applied to the Birman Fellowship in late 2018, she had three children, including a nursing infant. She proposed to use the funds to fly frequently for short visits with Stein to accommodate his age and her children. But then, in December of that year, Stein passed away from lymphoma.

As Pierce mourned her friend, mentor, and collaborator, the fellowship committee wrote to her, asking if she wanted to reframe her application. "I had an instantaneous feeling that my application had crumbled, and I accepted that," she says. But further reflection helped her see how the fellowship could still benefit her long-term plans. She proposed using the funds to buy out teaching so she could focus on the book, and to fly collaborators to visit her so that she would not have to leave her children as often.

With no teaching, the first semester of the fellowship was "rejuvenating," Pierce says. "I tackled a series of chapters in the middle of the book that Eli had referred to as the mountain we needed to climb. I was back to full speed on the project."

Lillian Pierce, the 2019–2020 Birman Fellow.

One concept that Pierce explored during that time was "superorthogonality," which generalizes orthogonality from pairs of functions to any even number of functions. Building on her collaboration with Stein, she studied five types of superorthogonality, finding unexpected connections between current problems in analysis, number theory, and even algebraic geometry.

When the pandemic hit, Pierce had not yet taken the research trips that she had planned. The flexibility of the Birman Fellowship allowed her to postpone using the remaining funds. And in a year when optimism was hard to come by, the fellowship continued to help her. "I'm remembering the lesson that when external circumstances improve, I will again be able to work at my full potential," she says. "I feel extremely lucky to have had this generous, flexible fellowship buoying me during this time period."

Now a full professor, Pierce looks forward to finishing the book. And as travel becomes possible again, she will use the remainder of the fellowship funds for visits with collaborators.

A Long-sought Proof

While the pandemic impacted the tail end of Pierce's fellowship, it left a mark on Karin Melnick's entire fellowship year. Melnick, an associate professor at the University of Maryland, College Park, works on differential geometric aspects of rigidity and has long been interested in the Lorentzian Lichnerowicz conjecture, a statement about conformal transformations of compact Lorentzian manifolds.

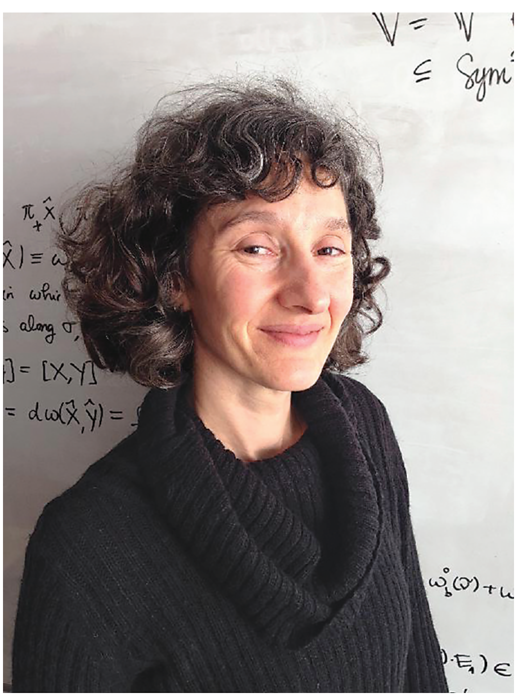
Melnick won the AMS Centennial Fellowship for the 2012–2013 academic year and received tenure in 2014.

Three years later she had a baby. Caring for her child "took away from the time that I had for work, though certainly my colleagues were supportive," she says. "I just didn't have as much time to do research."

As her baby became a toddler, Melnick began ramping up her research again and finished a major project. She applied for the Birman Fellowship with the goal of continuing to build that momentum through research travel and course buyouts. But COVID-19 forced her to scrap plans to visit collaborators. Instead, she bought out an additional course, eliminating her teaching load entirely for the fall 2020 semester. That opportunity proved even more valuable than in a typical semester, since "I was spared some of the chaos" of online teaching, she says. While working virtually with collaborators, she reached a milestone in her work. As her fellowship year drew to a close, she neared completion of a proof of the Lichnerowicz conjecture for 3-dimensional manifolds, valid under certain minor assumptions.

"This would be the first time it's been proved in a given dimension," she says. "It feels very good. It's exciting because it gives some encouragement that the general conjecture is really true, and it could be that we're closer to having the tools to prove it."

An unexpected benefit of the fellowship was a new friend. When Melnick's award was announced, Pierce reached out to her to offer congratulations. The number theorist and the geometer found that they had other things in common, sparking a longer email conversation. "That meant a lot to me," Melnick says.



Karin Melnick, the 2020–2021 Birman Fellow.

In the future, she hopes to keep working on the Lichnerowicz conjecture while expanding into new areas. One of her new projects stems from the Zimmer program, which seeks to understand the actions of large semisimple Lie groups and their discrete subgroups on smooth manifolds. Melnick is classifying certain actions in this context and also wants to understand when these actions preserve differential geometric structures.

Melnick says that the boost provided by the fellowship will benefit her research career for years to come. "It's not just another star on the CV, but also the confidence that comes from being recognized and the extra research time that comes from having money to buy out teaching."

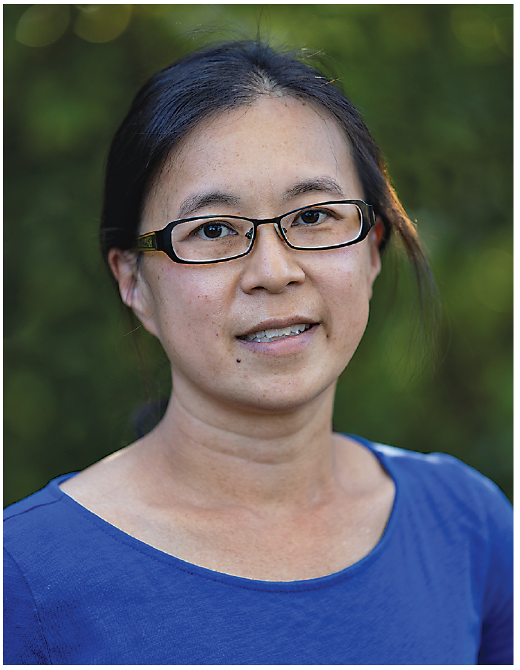
Unrecognized Work and Undervalued Contributions

The first four recipients of the Birman Fellowship have experienced firsthand the challenges associated with being pregnant and raising children while pursuing mathematics research. They also point out that many other roadblocks still exist for women in the field.

Despite the advances of recent decades, some women still experience direct discrimination from other mathematicians. Beck occasionally finds herself providing support to graduate students whose male peers are treating them poorly. Carrying that weight can consume a lot of emotional energy, she says. But open discrimination is just one aspect of a subtler imbalance: "My feeling is that women are much more likely to take on the unrecognized work that has to be done in any department just to make things function well." Various forms of community-building work take away from the time Beck and others have to conduct their research. "I have many male colleagues who are wonderful, great people, but somehow, on average, I don't think the genders tend to do equal amounts of work in that regard," she says.

When it comes to research, too, women's talent and contributions are sometimes undervalued, Melnick says. "I think [this occurs] maybe because of a shortage of role models or because of the dynamics of how mathematicians interact with each other and assess each other," she says. "It's unfortunately possible for especially younger female mathematicians to undervalue their own talent." A similar bias extends to students' perceptions of their professors: In course evaluations on RateMyProfessors, female mathematics professors are praised as "brilliant" or "genius" about 40% as often as their male colleagues [6].

The factors pushing women away from careers in mathematics begin early in life. According to Helen Wong, the current Birman Fellow, the typical experience of a student—attending lectures, completing homework, and taking exams—emphasizes individual work, often resulting in a misunderstanding about the nature of mathematics research. "People don't realize that collaboration is so



Helen Wong, the 2021–2022 Birman Fellow.

prevalent and so important in mathematics, and women from the get-go are turned off by the idea of working alone," Wong says.

On top of it all, studies show that the negative impact of the COVID-19 pandemic on researchers has been disproportionately felt by women, especially mothers. By the summer of 2020, single mothers in academia lost an average of 90 minutes of research time per day, compared to an average loss of 60 minutes per day for single fathers [7]. The effects of this lost time will compound in the coming years, underscoring the importance of awards like the Birman Fellowship for mathematicians whose personal circumstances hinder their research aspirations.

Moving Forward After a Pandemic Year

In high school, Helen Wong didn't know that math beyond calculus existed. As an undergraduate, she considered majoring in chemistry, economics, and even music. But a summer mathematics program at the University of California, Berkeley revealed the joys of a career in mathematics and introduced her to topology, the subject she fell in love with. Wong earned her PhD in mathematics from Yale University in 2007 and is now an associate professor at Claremont McKenna College.

For Wong, receiving a fellowship named for Joan Birman is meaningful on multiple levels. In addition to being inspired by Birman's life story, Wong refers to Birman's results on braid groups often in her own research. Wong studies quantum topology, examining invariants such as those related to the Jones polynomial or Witten-Reshetikhin-Turaev quantum field theory. Her work has potential applications to quantum computation, where the motion of anyon qubits can be viewed through the lens of braid theory. In another project, Wong is working with biophysicists to analyze how proteins and DNA become knotted.

In March 2020, Wong suddenly found herself supervising Zoom school for her two daughters, then in preschool and 1st grade. She and her husband, a biostatistician, both had full-time jobs, and "there was no way that we could both work 9 to 5 and make sure the kids were okay." The parents did their best to share household duties, but both saw their research suffer. And on top of childcare, the shift to online instruction forced Wong to spend extra time and energy overhauling the classes she was teaching.

"Luckily, I had research collaborators that were basically in the same boat. They all had little kids at home," Wong says. Occasional meetings served to keep the thread of research alive, even if no one had time to make progress between meetings.

Like Beck, Wong did not think she had much of a chance to receive the Birman Fellowship—she only applied because her college encouraged her to do so. She had similar doubts about the Simons Fellowship, yet she received both awards for the 2021–2022 academic year. "I want to encourage women to apply for things even though they feel like they're not qualified," Wong says. During her fellowship year, she aims to resume her research with undergraduates as well as fly to visit collaborators on other projects. Although she loves teaching, she is using the fellowship to buy out her spring courses in order to create a year focused on research.

For Beck, Pierce, Melnick, and Wong, the Birman Fellowship is just the latest manifestation of support—whether institutional, financial, familial, or from colleagues—that has eased the burden of their proverbial packs and buoyed their careers in mathematics. While money alone cannot eliminate all the obstacles faced by women mathematicians, the early success of the fellowship makes Birman optimistic about its long-term impact: "I believe that, after enough time, and especially if it attracts additional support, the Birman Fellowship for Women Scholars stands a chance to make a real contribution to the matter of the paucity of women at the top levels of research in mathematics."

References

[1] A. Golbeck, T. Barr, and C. Rose, *Fall 2018 Departmental Profile Report*, Notices Amer. Math. Soc. **67** (2020), 1615–1621.

[2] National Center for Science and Engineering Statistics, *Women, Minorities, and Persons with Disabilities in Science and Engineering: 2021*, Special Report NSF 21–321, Alexandria, VA: National Science Foundation, last visited July 23, 2021. `https://ncses.nsf.gov/wmpd`

[3] AARP and National Alliance for Caregiving, *Caregiving in the United States 2020*, AARP, May 2020, last visited August 13, 2021. `https://doi.org/10.26419/ppi.00103.001`

[4] Q. Bui and C. C. Miller, *The Age That Women Have Babies: How a Gap Divides America*, The New York Times, Aug. 4, 2018, last visited June 23, 2021. `https://www.nytimes.com/interactive/2018/08/04/upshot/up-birth-age-gap.html`

[5] A. Golbeck, T. Barr, and C. Rose, *Report on the 2017–2018 Employment Experiences of the New Doctoral Recipients*, Notices Amer. Math. Soc. **67** (2020), 1207–1213.

[6] D. Storage, Z. Horne, A. Cimpian, and S. J. Leslie, *The Frequency of "Brilliant" and "Genius" in Teaching Evaluations Predicts the Representation of Women and African Americans across Fields*, PLOS ONE **11** (2016), no. 3, last visited June 23, 2021. `https://doi.org/10.1371/journal.pone.0150194`

[7] T. Deryugina, O. Shurchkov, and J. E. Stearns, *COVID-19 Disruptions Disproportionately Affect Female Academics*, National Bureau of Economic Research, Working Paper 28360 (2021), last visited June 23, 2021. `https://www.nber.org/papers/w28360`

Scott Hershberger

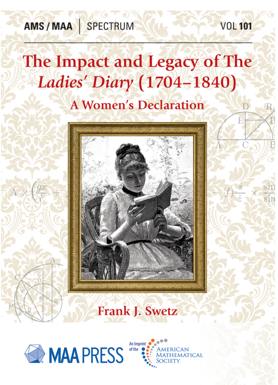
Credits

Photo of Joseph and Joan Birman is courtesy of Beth Birman.
Photo of Margaret Beck is courtesy of Greg Blotzer.
Photo of Lillian Pierce is courtesy of Lillian Pierce.
Photo of Karin Melnick is courtesy of A. Chatzistamatiou.
Photo of Helen Wong is courtesy of Claremont McKenna College.
Author photo is courtesy of Jiyoon Kang.

BOOK REVIEW

The Impact and Legacy of *The Ladies' Diary* (1704–1840): A Women's Declaration

Reviewed by Laura E. Turner



AMS/MAA Press, 2021, 169 pp.

The Impact and Legacy of* The Ladies' Diary *(1704–1840): A Women's Declaration
Frank J. Swetz

Recent years have seen a number of new scholarly and popular works highlighting the contributions, often unlikely and under-appreciated, of women to mathematics and describing the social and cultural conditions that helped and, more commonly, hindered them. Swetz's *The Impact and Legacy of* The Ladies' Diary *(1704–1840): A Women's Declaration*, which treats the activities of both professionals and amateurs involved in the consumption and creation of mathematics, forms part of this body of literature.

For those unacquainted with the periodical, *The Ladies' Diary: or, the Woman's Almanack* was published annually in London and appeared in print between 1704 and 1841. Almanacs sold well in England during this period. For people of limited means who could not afford books, they provided reading material and recreation. They also contained information about weather forecasts, tidal flow, and astrological predictions. In some senses this particular almanac was quite ordinary early in its history, though by design it lacked astrological content. That said, the *Diary* was only the third periodical dedicated to women, and the first to survive more than one year [Cos02, p. 50]. It was originally intended to provide such "genteel" subject matter as recipes, poems, household tips, health advice, and romantic stories, and to "entertain and provide diversion" through enigmas (riddles, often written in iambic pentameter). By 1708, however, it also included arithmetical problems, often stated in verse, which, according to the directions of its founder and first editor, should be "pleasant" but "not too hard" [Swe21, p. 21].

The *Diary* was the first widely-read popular periodical to contain this form of content, and in this sense, it was groundbreaking. The problems in it were immediately and enduringly popular with its readers (indeed, close to forty other almanacs and periodicals would contain mathematical question and answer sections in the eighteenth and nineteenth centuries [Des14, pp. 55–56]), who were invited to submit solutions and new problems for publication. Over time, the *Diary* became increasingly devoted to this topic. Because the mathematical questions were selected according to editors' interests, however, and because several later editors were faculty members at military academies, the problems became increasingly applied and technical (and difficult). As this occurred, women evidently submitted fewer mathematical solutions; the scope and level of the material had shifted beyond the limited education and training most received.[1] Ultimately, the *Diary*

Laura E. Turner is an assistant professor of mathematics at Monmouth University. Her email address is `lturner@monmouth.edu`.

Communicated by Notices *Book Review Editor Katelynn Kochalski.*

For permission to reprint this article, please contact: `reprint-permission@ams.org`.

DOI: https://doi.org/10.1090/noti2430

[1]*One notable exception is Mary Somerville, who distinguished herself as a prizewinner in an offspring of the* Diary [Swe21, p. 120]; *see also* [Ste20].

became a periodical for ladies in name alone and in 1841 it was merged with *The Gentleman's Diary, or The Mathematical Repository* (founded in 1741) to become *The Lady's and Gentleman's Diary*, which focused on the amusement and instruction of mathematics students.

By Swetz's own testament, the roots of his work on *The Ladies' Diary* began developing in the 1970s when he became acquainted with the periodical through the work of Teri Perl [Per77, Per79]. Against the backdrop of educational research on gender and mathematics during this period—which saw the posing of questions about why fewer (cisgender) women were drawn to study mathematics than (cisgender) men, how prevalent sex-linked differences in mathematical achievement actually were, and whether they were cognitive or attitudinal—*The Ladies' Diary* emerged as an apparent contradiction not only to the physiological hypotheses proposed by some researchers but also to the socialization of the proper English lady of the Georgian era.

All of this made the *Diary* something of a fascination to Swetz, and subsequent work on it by other scholars during the aughts further piqued his interest. He began featuring problems from the periodical within *Convergence*, the online Mathematical Association of America journal devoted to the history of mathematics and its use in teaching, and included them in a book of historical word problems. More recently, he wrote a feature article in *Convergence* [Swe18] describing the historical uniqueness and mathematical significance of the periodical for women and as a problem-solving resource, and placing it within its social, cultural, and mathematical contexts. His latest book is a continuation and expansion of this effort.

In this work, Swetz addresses a number of different themes surrounding the origin and evolution of *The Ladies' Diary*. His opening chapter sets the scene by depicting the gathering of three fictitious male university students, one of whom is a *Diary* reader, in a London coffeehouse in 1754. There, his friends poke fun at his interest in a periodical for ladies, but he encourages them to attempt some of the mathematical problems it contained and shows them an exercise proposed by a "Miss Maria A-t-s-n" (Mary Atkinson) for which he planned to submit a solution.[2] Through this exchange, which has one friend reject mathematics as "the stuff of tradesmen" and which depicts Atkinson as a suspected "bluestocking" and perhaps also "a prune, a dried up old spinster" [Swe21, p. 3] in the eyes of the gentlemen, Swetz introduces contextual factors that he explores in subsequent chapters, as well as the arguments at the core of his work: that women actively engaged in mathematics in spite of the social and cultural forces working against them, and that the *Diary* was an important context in which this engagement occurred.

In the second chapter, Swetz briefly provides background concerning English literacy rates and popular publications from the mid-seventeenth century on, as well as a cursory introduction to the *Diary* and its apparent value to its readers and subscribers. Notably, this value is assessed only in connection with the mathematical content of the periodical, a matter to which we will return below. At the close of the chapter he identifies three particular stimuli as having influenced the direction of *The Ladies' Diary* across its lifetime. These are the individual editors, the ladies for whom the journal was ostensibly intended, and British mathematical reforms and movements. The remainder of the book is devoted to exploring these factors.

The third chapter is where the body of Swetz's book begins. In it, one learns about *Diary* founder John Tipper's sincere desire to create a periodical that would be "useful and appealing to women" [Swe21, p. 18] by including the broad swath of content described above, and about the actions of the next editor, Henry Beighton, who took the journal in a more serious mathematical direction and broadened the audience to both sexes by openly soliciting male readers. Subsequent editors[3] also left their marks on the *Diary*. Swetz outlines their tenures, and in doing so he describes how the almanac was converted from its original design to one focused on problem solving.

In the fourth chapter, which discusses the attraction of enigmas and mathematical problems to *Diary* readers, some context concerning the history of the enigma and its popularity is provided, and modern readers are guided through the solution of an 1835 example.[4] Swetz also demonstrates the sorts and complexity of mathematical problems under different editors and then returns to the perspective of *Diary* readers, describing the informal problem-solving gatherings in which some participated; the use of pseudonyms as social protection and as a demonstration of wit; and an example of a seemingly

[2] *This is a real problem from the almanac, one to which Swetz later returns.*

[3] *These were Beighton's wife, Elizabeth, followed by Robert Heath, Thomas Simpson, Edward Rollinson, Charles Hutton, and Olinthus Gregory.*

[4] *It reads: "The regal crown shines bright with many a gem,*
But I oft bear a brighter diadem;
I own indeed no purple robes I wear,
But yet the lily's self is scarce more fair.
A friend, like great Medea, in dark times,
Am I to learning patron of all climes.
E'en Jove, when he was ruler of the sky,
Never received more sacrifice than I;
The lamb, which on the mead is sporting free,
Must soon be a burnt offering to me.
The mighty whale, the sov'reign of the main,
For me is captured, and for me is slain.
The bee may build his cells with instinct fine,
Those waxen homes must melt before my shrine.
And when the sun has fallen from the skies,
Then, clad in all my glory, I arise;
I scorn the day, but glory in the night,
And only shine when I alone am bright."

flirtatious exchange within its pages during a period of confining social limitations.

Having demonstrated the appeal of mathematical offerings to women *Diary* readers, in the fifth chapter Swetz turns to the opportunities for women to study mathematics before and during the lifetime of the periodical. Here, a background of mathematics in England during the sixteenth and seventeenth centuries is discussed, and the significance of English-language works written or translated during this period which advocated the use of mathematics by common people is stressed.[5] Although such works spurred the creation of public lectures on mathematics, local mathematical and scientific societies, and demand for formal instruction, women were generally excluded from these circles. The remainder of the chapter describes avenues through which some English ladies[6] were able to obtain (usually limited) training in arithmetic, and the ways in which those who did were viewed by society.

The focus of the sixth chapter is an assessment of the intellectual value possessed by the *Diary*. Here, emphasis is placed on the mathematical needs it fulfilled in providing guidance and feedback (via published solutions) vis-à-vis mathematical problem solving, and the scientific facts and explanations it gave readers through expository pieces, instructional essays and dialogues, and eventually a "question and answer" column.

In the seventh chapter, Swetz explores links between the *Diary* and broader mathematical concerns such as shoring up the foundations of Newton's theory of fluxions, finding a method to more precisely determine longitude at sea, and connecting more deeply with mathematical and scientific developments from mainland Europe. Some connections are more clearly documented than others; examples of *Diary* solutions utilizing Newton's fluxions are provided and the reader is told that after 1835 such methods were abandoned in favour of Leibniz's approach to differentiation and integration in a "shift to a more rigorous calculus [which] paralled that generally taking place in Britain at this time," but no explanation is given about how or if it related to the efforts to strengthen Newton's calculus cited at the beginning of the chapter. Additionally, a greater reliance on secondary literature could have provided a more nuanced picture of the connections between the *Diary* and concomitant mathematical developments. Albree and Brown [AB09, p. 32] assert, for example, that "[a]s challenging as many of [the mathematical questions in the *Diary*] were and as ingenious as some of their answers were, by their intent, they played no part in any research program or extended theory."

Chapter eight questions whether the *Diary* truly served the needs of women throughout its history. After returning to the actions of its editors, Swetz considers the women contributors themselves, who ranged in age and marital status. Some, such as teenage sisters Anna and Mary Wright of Cheshire and a Mrs. Mary Nelson, demonstrated considerable mathematical competence and poetic wit in the early years of the *Diary*.[7] Swetz uses such examples, along with statistical data about respondents to mathematical problems, to emphasize that problem solving in mathematics was important to women, who enjoyed and excelled at it in spite of the obstacles they faced.[8]

In the ninth and final chapter, which summarizes the effects and societal impact of the *Diary*, Swetz's focus is the published mathematical descendants it inspired in Britain and America. This brief chapter provides concluding remarks and is followed by an epilogue distinguishing Swetz's work on *The Ladies' Diary* from other studies of the periodical and highlighting questions for future consideration.

Several features of the book warrant particular attention. An examination of its fairly extensive bibliography reveals that not all works are cited, and at several points in the text, secondary sources are omitted from discussions in spite of considerable overlap in content. This makes it difficult for the reader to determine what is new in the present text and has the unfortunate effect of making the book appear somewhat disconnected from the relevant secondary literature, a heavier reliance on which could have provided additional important insights.

In exploring the "real worth" of *The Ladies' Diary* [Swe21, p. 10], for example, Swetz cites enthusiastic testimonies from readers concerning its mathematical content. Kathryn James, however, has pointed out that not all readers valued this material equally. Rather, many used it "primarily for notes on accounts, for paper for contracts, as a notebook, as a diary, just like the other gazettes and calendars and almanacs which filled the popular market" [Jam11, p. 15].[9] What is more, the significance of its literary content and its union of mathematics and poetry have been stressed by others and seem to deserve greater

[5] *Consider the cautionary view that, as phrased by Robert Recorde, "yf nombre be lackynge, it maketh men dumme, so that to most questions, they must answer mum"* [Swe21, p. 59].

[6] *A lady was a woman born to a respectable family. In the period treated here, her father might have been a clergyman, academic, or lawyer, for instance, or possibly a country squire with sufficient land or a merchant with sufficient wealth* [Swe21, p. 4].

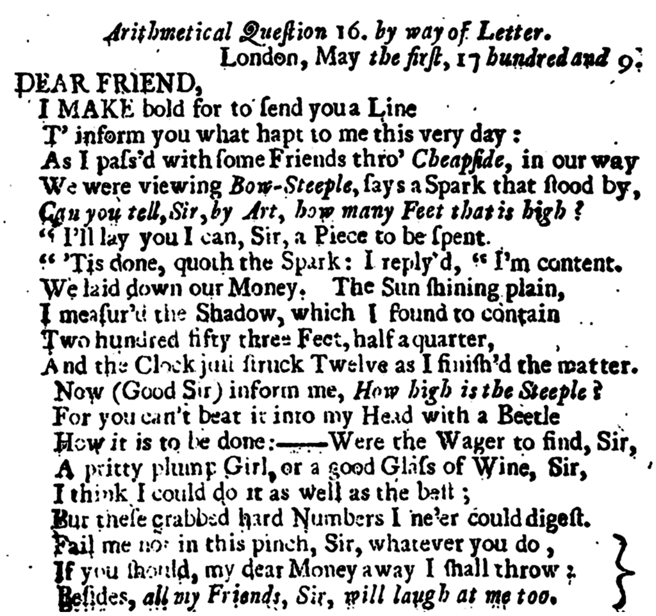
[7] [Cos00, p. 202] *hypothesizes that Mary Wright and Mary Nelson were the same person, though Swetz does not address this possibility.*

[8] *As Swetz notes, such data have been used by others but must be taken with a grain of salt: the assumption that all who submitted work under a woman's name were, indeed, women, is disputable* [Swe21, p. 118], [Mie08, p. 190], [AB09, p. 17].

[9] *Anna Miegón points to evidence implying that the* Diary *was used as a record book by some women. In* [Mie08, p. 116], *she describes the annotations of* Diary *numbers by Alice Le Neve and her mother between 1724 and 1774, which included records of boarding and other expenses (such as costs of mending shoes and purchasing new stockings), as well as rent paid by tenants and crops sown on family land.*

attention in summarizing the intellectual value of the *Diary*, particularly early in its history. Tipper, for instance, cited three problems from 1710 (including the "Bow-Steeple" problem of Figure 1) as

> good patterns how an arithmetical question should be composed: namely, to cloathe it with such delightful circumstances, as should egg us on to solving the most useful part. To heighten delight, whet the imagination, and sharpen invention all at once, to enlarge the capacity of the mind, and raise our pleasure to the highest pitch it is capable of. [Jam11, p. 15]



Arithmetical Queſtion 16. by way of Letter.
London, May *the firſt*, 17 *hundred and* 9.
DEAR FRIEND,
I MAKE bold for to ſend you a Line
T' inform you what hapt to me this very day:
As I paſs'd with ſome Friends thro' *Cheapſide*, in our way
We were viewing *Bow-Steeple*, ſays a Spark that ſtood by,
Can you tell, Sir, by Art, how many Feet that is high?
"I'll lay you I can, Sir, a Piece to be ſpent.
"'Tis done, quoth the Spark: I reply'd, "I'm content.
We laid down our Money. The Sun ſhining plain,
I meaſur'd the Shadow, which I found to contain
Two hundred fifty three Feet, half a quarter,
And the Clock juſt ſtruck Twelve as I finiſh'd the matter.
Now (Good Sir) inform me, *How high is the Steeple?*
For you can't beat it into my Head with a Beetle
How it is to be done:——Were the Wager to find, Sir,
A pritty plump Girl, or a good Glaſs of Wine, Sir,
I think I could do it as well as the beſt;
But theſe crabbed hard Numbers I ne'er could digeſt.
Fail me not in this pinch, Sir, whatever you do,
If you ſhould, my dear Money away I ſhall throw;
Beſides, *all my Friends, Sir, will laugh at me too.*

Figure 1. The "Bow-Steeple problem" of 1710. Note that it is directed at a gentleman, not a lady.

This coupling of mathematics and poetry is perhaps what led Swetz to remark that the periodical "projected an aesthetic rationalism" [Swe21, p. 114], but he neither defines this term nor provides explanations of this epistemological position or its cultural context.[10] The *Diary*, however, is emerging as significant in these very connections, particularly insofar as the roles of women are concerned. Jacqueline D. Wernimont has argued that part of the importance of the periodical to early modern and eighteenth-century studies stems from the fact that it serves as "a textual record of the *centrality of women to the development of a national and individual aesthetic rationalism that found pleasure and meaning in exploring math and poetry together* [emphasis added]" [Wer17, p. 338].

The spirit in which the book is written deserves comment. Swetz approaches *The Ladies' Diary* with the same fascination evident in his earlier works, and his enthusiasm for the subject is apparent throughout. Problems and their solutions are often presented as images from digitized copies of the *Diary*, demonstrating the original typography. His desire to introduce the reader to such facets of the *Diary* as "the wit, the verbal ostentation and posturing, the format, punctuation and emphasis of written statements, and the intellectual motivation" [Swe21, p. xv] further emphasizes that he still views this text as "a true mathematical treasure" [Swe18] for the reader to explore, appreciate, and enjoy. This spirit is also evidenced by his treatment of the enigmas and mathematical problems, which emerge in part as challenges for the reader to attempt. The book contains many examples thereof, often answered in the text, and adventurous readers will enjoy attempting the word puzzles and mathematical exercises contained in two appendices. There is also a third appendix containing three mathematical problems with worked solutions.

It is noteworthy, too, that this book is one of few histories treating the *Diary* throughout its entire lifetime, and in writing it Swetz has made this fascinating periodical and its history more accessible to a general audience. In fact, while the publisher describes the readership of the book as graduate students and researchers interested in the history of mathematics, with the possible exception of certain mathematical discussions (which are generally self-contained), this can likely be broadened to include undergraduates and others with a reasonably strong mathematical background. Many will enjoy reading this book, and those with an interest in the histories of women in mathematics and of mathematical periodicals are especially encouraged to pick it up.

[10] *Timothy J. Reiss has described aesthetic rationalism within the context of the "mathematization" of knowledge in early modern Europe through the replacement of the trivium with the quadrivium, and in particular of language with mathematics and a new rational method as a means of discovery. Through this shift, he argues, the "fictive imagination" (which produced such arts as poetry and literature) and mathematical practices were "wholly dependent on one another"* [Rei97, p. 16]. *He has characterized aesthetic rationalism within this context as a general effort to attain "depth with clarity, variety without confusion, and interest with pleasure"* [Rei97, p. 194], [Wer17, p. 338].

References

[AB09] Joe Albree and Scott H. Brown, *"A valuable monument of mathematical genius": The Ladies' Diary (1704–1840)* (English, with English and French summaries), Historia Math. **36** (2009), no. 1, 10–47, DOI 10.1016/j.hm.2008.09.005. MR2272883

[Cos00] S. A. Costa, *The "Ladies' Diary": Society, gender and mathematics in England, 1704-1754. Ph.D. Dissertation. Cornell University, Ithaca, NY*, PhD Thesis, 2000.

[Cos02] S. Costa, *The "Ladies' Diary": Gender, mathematics, and civil society in early-eighteenth-century England*, Osiris **17** (2002), no. 1, 49–73.

[Des14] Sloan Evans Despeaux, *Mathematical questions: a convergence of mathematical practices in British journals of the eighteenth and nineteenth centuries* (English, with English and French summaries), Rev. Histoire Math. **20** (2014), no. 1, 5–74. MR3245149

[Jam11] K. James, *Reading numbers in early modern England*, BSHM Bulletin: Journal of the British Society for the History of Mathematics **26** (2011), no. 1, 1–16.

[Mie08] A. Miegoń, *The Ladies' Diary and the emergence of the almanac for women, 1704-1753. Ph.D. Dissertation. Simon Fraser University, Burnaby, BC*, PhD Thesis, 2008.

[Per77] T. Perl, *The ladies' diary circa 1700*, The Mathematics Teacher **70** (1977), no. 4, 354–358.

[Per79] Teri Perl, *The Ladies' Diary or Woman's Almanack, 1704–1841* (English, with French and German summaries), Historia Math. **6** (1979), no. 1, 36–53, DOI 10.1016/0315-0860(79)90103-4. MR518839

[Rei97] T. J. Reiss, *Knowledge, discovery, and imagination in early modern Europe: The rise of aesthetic rationalism*, Vol. 15, Cambridge University Press, New York, 1997.

[Ste20] Brigitte Stenhouse, *Mary Somerville's early contributions to the circulation of differential calculus* (English, with English and French summaries), Historia Math. **51** (2020), 1–25, DOI 10.1016/j.hm.2019.12.001. MR4107782

[Swe18] F. J. Swetz, *The "Ladies' Diary": A True Mathematical Treasure*, Convergence (2018August).

[Swe21] F. J. Swetz, *The Impact and Legacy of The Ladies' Diary (1704–1840): A Women's Declaration*, Spectrum, MAA Press, Providence, 2021.

[Wer17] J. D. Wernimont, *Poetico-Mathematical Women and The Ladies' Diary*, The Palgrave Handbook of Early Modern Literature and Science, 2017, pp. 337–350.

Laura E. Turner

Credits

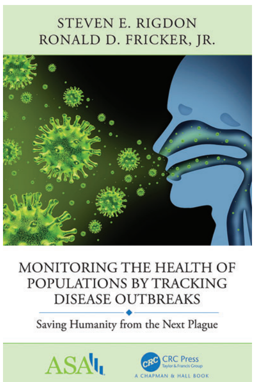
Figure 1 page scan is courtesy of Google Books.

Photo of Laura E. Turner is courtesy of Monmouth University.

BOOKSHELF

New and Noteworthy Titles on our Bookshelf

March 2022



Cover is courtesy of Taylor and Francis.

Monitoring the Health of Populations by Tracking Disease Outbreaks

Saving Humanity from the Next Plague

By Steven E. Rigdon and Ronald D. Fricker, Jr.

Given the health concerns surrounding COVID-19 and how our lives were impacted over the last two years (and for many of us, longer), the title of this book alone will likely draw many of us in immediately. Published shortly before COVID-19 was declared a pandemic, this book does not use the coronavirus as a motivating case, but it is easy to see how the models used to track other epidemics and pandemics, such as the Spanish flu, cholera, and bird flu, have been instrumental in battling COVID. This book focuses on how statistics is used to identify contagious diseases and can be applied to determine key factors that influence the development of a disease and can reduce its spread. It has two main parts: The first focuses on methods of monitoring diseases to be able to take action before a disease becomes widespread. The second is about how an epidemiologist works to understand the cause of a disease, presenting seven case studies to illustrate the issues.

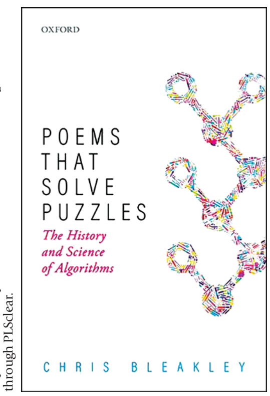
When a new statistical concept is introduced, such as the relationship between correlation and causation, the authors include a section that defines the concept and how it is generally used before discussing its particular application to the disease currently being presented. The text also uses many ideas from probability and statistics such as conditional probability, the chi-squared hypothesis test, and experimental design techniques.

This book is an ideal read for anyone with an interest in biostatistics or mathematical models of disease. It is packed with interesting graphs and figures, including a graph demonstrating the SIR (Susceptible-Infected-Recovered) model for the flu and how vaccinations and education campaigns can impact the number of infected individuals. It is a highly engaging and informative read. In the Preface there is a promise of a second edition which will include an analysis of COVID-19. I look forward to reading the second edition!

The Bookshelf is prepared monthly by Notices *Associate Editor Katelynn Kochalski.*

Appearance of a book in the Notices *Bookshelf does not represent an endorsement by the* Notices *or by the AMS.*

Suggestions for the Bookshelf can be sent to `notices-booklist@ams.org`.



Reproduced with permission of the Oxford Publishing Limited through PLSclear.

Poems that Solve Puzzles

The History and Science of Algorithms

By Chris Bleakley

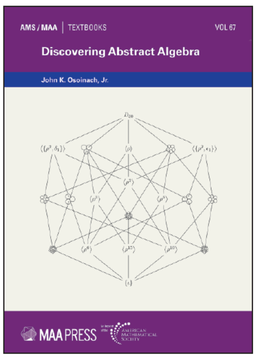
Poems that Solve Puzzles is a thorough investigation into the history of algorithms. Examples of algorithms from as far back as Mesopotamia are given, as well as some algorithms we all use for things such as sorting or deciding how to run our errands most efficiently. Bleakley gives instances, such as weather forecasting, where the algorithmic idea was in place long before the computer technology required to implement it efficiently existed.

Throughout the book, we see how mathematical concepts like simulation, modeling, and networking work together with algorithms. While a good deal of time is spent discussing Turing and his groundbreaking contributions, this book gives ample pages to more modern advances in computer science. Topics such as the birth of the internet, Amazon's personalized recommendations (that are often eerily accurate!), and Google's Page Rank algorithm are discussed. There is even a chapter on IBM's *Jeopardy* playing robot, which in the wake of the passing of Alex Trebek, will surely bring back some fond memories for many readers.

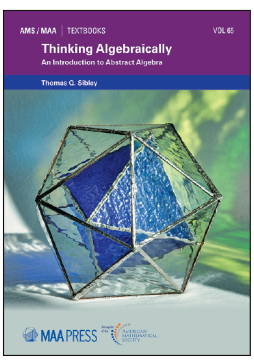
This is an interesting read written for a general audience. Bleakley does not assume any math or computer science background, clearly defining technical terms when they are first used. While you do not have to be a mathematician to enjoy this book, any mathematician will recognize many names mentioned, such as Archimedes, Ada Lovelace, and John Von Neumann, as having made significant contributions to the development of the implementation of algorithms. It is an enjoyable read for anyone curious about how algorithms developed and were implemented throughout history.

AMS BOOKSHELF

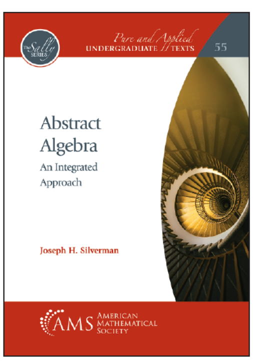
The AMS Book Program serves the mathematical community by publishing books that further mathematical research, awareness, education, and the profession while generating resources that support other Society programs and activities. As a professional society of mathematicians and one of the world's leading publishers of mathematical literature, we publish books that meet the highest standards for their content and production. Visit **bookstore.ams.org** to explore the entire collection of AMS titles.



AMS/MAA Textbooks, Volume 67, 2021, 199 pp. (TEXT/67).



AMS/MAA Textbooks, Volume 65, 2021, 478 pp. (TEXT/65).



Pure and Applied Undergraduate Texts, Volume 55, 2021, 556 pp. (AMSTEXT/55).

A Friendly Introduction to Abstract Algebra
by Ryota Matsuura

Discovering Abstract Algebra
by John K. Osoinach, Jr.

Thinking Algebraically
An Introduction to Abstract Algebra
by Thomas Q. Sibley

Abstract Algebra
An Integrated Approach
by Joseph H. Silverman

How do you teach undergraduate abstract algebra? Your answer surely depends on to whom you are teaching it and your educational goals. Tom Sibley, in his excellent MAA text, explains that he wants to teach his students to "think algebraically." Joe Silverman, in his beautiful new AMS textbook, wants his students to understand the skeletal axiomatic structures (and maps preserving those structures) underlying the mathematical objects with which they are familiar when specific details are stripped away and similarities are highlighted. Which is precisely what Sibley means by his title. Both authors transcend the groups-first versus rings-first debate and move to a presentation we might call "structure-first." Technically both introduce groups first, but both also introduce rings and fields early on and go back and forth emphasizing common structural features.

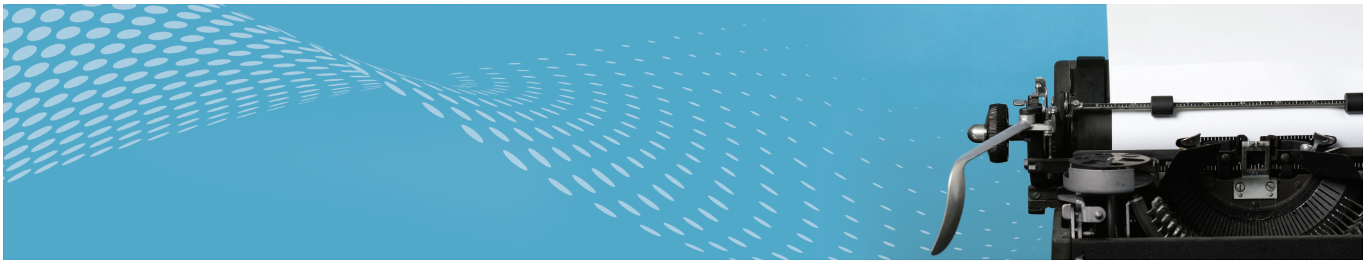
Despite the harmony of pedagogical approaches, Silverman and Sibley have meaningful differences of emphasis. Silverman is primarily interested in exploring the properties of algebraic structures. His content coverage is deeper and goes further than Sibley's. And he recognizes that a good course is built on an underlying narrative that knits everything together, propels development, and makes the course a coherent story. He knows that a good story contains episodes of surprise and wonder. Silverman calls these "punchlines," by which he means results whose depth and beauty reward the explorer for the, sometimes sterile-seeming, work of abstraction and generalization. His chapters beautifully build towards denouements. Sibley also builds a compelling narrative but he is more invested in connecting his story to ideas his students already know. He expends considerable energy connecting abstract algebra to the algebra his students learned in high school. Sibley doesn't get as far as Silverman, but he trades that coverage for giving his reader a clear idea of how and why we got to where we are.

I think of the Silverman and Sibley texts as centering the *content* of algebra in contrast to the books by John Osoinach and Ryota Matsuura, which emphasize the *process* of learning algebra. The books by Osoinach and Matsuura are informed by current pedagogical theory. Both use a form of inquiry-based pedagogy that is rooted in extensive experience and familiarity with examples before abstraction occurs. Osoinach is a committed IBLer, his goal is that students construct all the proofs themselves but he is extraordinarily adept at making it all feel natural and organic. For example, before presenting the definition of a group he carefully notes exactly which algebraic properties one needs to solve $ax=b$ in a variety of contexts. The definition of a group falls right out of the analysis. Matsuura teaches a course that does not assume that students have already been introduced to proof-writing. Matsuura's mantra, formed from his decades of studying teaching and learning, is "experience before formality." He is very skilled at giving students structured space to play with examples, to formulate concepts and conjectures, and to uncover the insights and connections in examples that lead them forward while simultaneously introducing proof-writing. (His book will be released in summer 2022.)

All of these authors want their students to learn to think algebraically. All are extraordinary pedagogues. All have profound ideas about what it means to understand algebra and how their particular students, and yours, might get to that understanding.

The AMS Bookshelf is prepared bimonthly by AMS Acquisitions Specialist for MAA Press titles Stephen Kennedy. His email address is `skennedy@amsbooks.org`.

SHORT STORIES



Circles Great and Small

Danny Calegari

The Olympic logo consists of five interlocking rings, representing the five continents of the world (the "continents" here are Europe, Africa, Asia, the Americas, and Oceania; apparently Antarctica does not qualify, possibly because the International Olympic Committee is prejudiced against penguins). But in the Middle Ages the Europeans only knew of three continents, and the Olympic logo looked as shown in Figure 1.

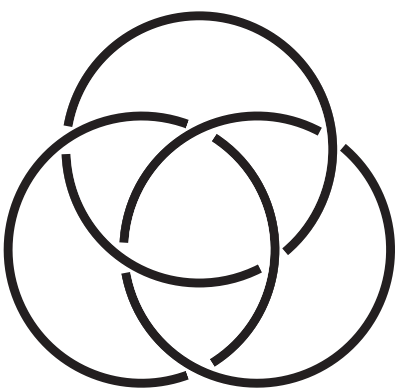


Figure 1. The Olympic logo, circa 1370.

Actually, Figure 1 is the *Borromean rings*, so named for the Borromeo family of medieval Milanese bankers on whose coat of arms they appeared. From a topological point of view, these rings form a 3-component link L with a very interesting property: although each pair of rings by themselves form a 2-component unlink, the three of them together are nontrivially linked. One way to see this is to pick (any) two of the components which we can call L_1 and L_2 and observe that the fundamental group of $S^3 - L_1 \cup L_2$ is free on two generators, say x_1 and x_2, represented by little loops that go once around each component. Then the third component represents the conjugacy class of the commutator $x_1 x_2 x_1^{-1} x_2^{-1}$ in $\pi_1(S^3 - L_1 \cup L_2)$ rather than the trivial element, as it would if L were the 3-component unlink.

One of the attractive things about the Borromean rings—or, to speak more precisely, about this specific image (technically: this *projection*)—is the psychological contradiction between the (apparent) roundness (and hence, one feels, simplicity) of the rings, and the over–under complexity of their interaction. The rings *look* perfectly round in the figure, but one feels this *must* be some sort of optical illusion, like Escher's staircase, or Penrose's tribar. *Real* perfectly round rings could not link in 3-dimensional space in that precise way.

Or could they? Can one find a configuration of three round disjoint circles in three-dimensional space which is isotopic to the Borromean rings?

As far as I know this question was first asked, and answered, by Mike Freedman and Richard Skora. Their argument is rather lovely, and we shall give it shortly. The argument proves a rather general fact about round links, as follows. Let L be *any* n-component link in S^3 made from round circles; we call L a *round link*. Note that any two round circles in S^3 are isotopic; thus the components, individually, are unknots. Furthermore, there are only two

Danny Calegari is a professor of mathematics at the University of Chicago. His email address is dannyc@math.uchicago.edu.

For permission to reprint this article, please contact: reprint-permission@ams.org.

DOI: https://doi.org/10.1090/noti2433

possibilities for the 2-component sublinks: two disjoint round circles in S^3 either form an unlink, or they are isotopic to the *Hopf link* (see Figure 2).

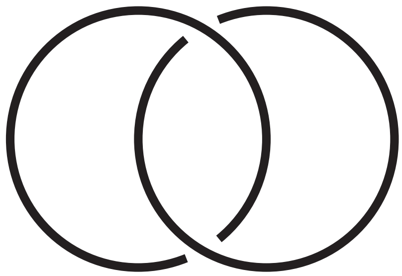


Figure 2. The Hopf link is the only nontrivial link of two round circles.

The Hopf link is distinguished from the unlink by the fact that the components have a nontrivial (algebraic) linking number. One way to define this linking number is to think of the 3-sphere S^3 as the boundary of the unit ball B^4 in 4-dimensional Euclidean space. An oriented knot in S^3 bounds an embedded oriented surface in B^4 (actually, it already bounds such a surface in S^3—a so-called *Seifert surface*). If two knots K, K' bound embedded surfaces R, R' in B^4 we can perturb these surfaces slightly so that they intersect in general position, and then the *linking number* of K and K' is the (algebraic) intersection number of R and R'. This intersection number can be defined at the level of (relative) homology classes, and therefore does not depend on the choice of surfaces R and R'.

Thus, the Hopf link is nontrivial, and so is any round link that has a pair of algebraically linked components. However, Freedman and Skora show that *every round link with pairwise unlinked components is trivial!* Here is the reason. A *round* circle L_i in S^3 bounds a rather obvious embedded surface in B^4, namely if we think of L_i as the intersection of S^3 with a flat plane Π_i in $\mathbb{R}^4$, then Π_i intersects B^4 in a smooth disk D_i bounding L_i. Since components L_i, L_j are pairwise disjoint, the disks D_i, D_j intersect transversely if at all (since otherwise the intersection is contained in a positive-dimensional affine subspace that must intersect S^3 somewhere). If they were to intersect, then their algebraic intersection number would be ± 1 and therefore the linking numbers of L_i and L_j would be ± 1, contrary to hypothesis. So a round link L as above bounds a family of disjoint *totally geodesic* disks D_i in the 4-ball.

Now consider the intersection of these D_i with the ball tB^4 obtained from B^4 by scaling it by some $t \in (0, 1)$. Each D_i intersects the boundary tS^3 in a collection of round circles, or (for finitely many discrete values of t) a single point. Thus this family of intersections gives an *isotopy* of the link L which shrinks the circles one by one (keeping them disjoint) until they shrink down to a point and disappear. This isotopy shows that each component can be unentangled from all the others, and that L is the unlink (in particular, the Borromean rings—whose components are pairwise unlinked—is not a round link).

This example may make one think that round links are uninteresting, but this is not at all true. A rather striking and beautiful theorem of Genevieve Walsh (that we shall discuss shortly) concerns links whose components are not only round circles but are *geodesics* in the round metric on S^3—i.e., they are *great circles* in S^3. Call such links *great circle links*. Note that every pair of great circles in S^3 forms a Hopf link. Conversely, *every* round link whose components are pairwise linked is actually isotopic to a great circle link! To see this, recall that for two round circles L_i, L_j in S^3 to be linked is equivalent to the flat planes Π_i, Π_j in $\mathbb{R}^2$ they lie in to intersect transversely at some point in the interior of B^4. Now consider the intersections of the Π_i with the balls tB^4 obtained from B^4 by scaling it by some $t \in (1, \infty)$. This gives a family of round links that deform by an isotopy, and in the limit as $t \to \infty$ the components all become great circles (this argument is due to Bill Thurston).

By associating to each great circle L_i in S^3 the projectivization of the plane Π_i it spans in $\mathbb{R}^4$ one obtains an equivalence between great circle links and configurations of *skew lines* (i.e., arrangements of disjoint straight lines in $\mathbb{RP}^3$). The number of arrangements of n skew lines for small n was determined by Julia Drobotukhina-Viro and Oleg Viro and is equal to $1, 1, 2, 3, 7, 19, 74$ for $n = 1, \dots, 7$. The case $n = 1$ corresponds to the unknot, and $n = 2$ the Hopf link. The two links for $n = 3$ are illustrated in Figure 3.

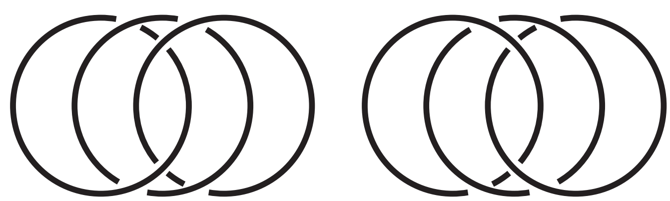


Figure 3. There are two great circle links with three components up to isotopy; they are mirror images of each other.

Walsh shows that every great circle link is *fibered*; i.e., $S^3 - L$ is a fiber bundle over S^1. Here is the proof. Let's think of S^3 (conformally) as $\mathbb{R}^3$ union infinity. After a conformal transformation we can make L_1 into the z-axis in $\mathbb{R}^3$. Put cylindrical coordinates z, r, θ on $\mathbb{R}^3$, so that (z, r, θ) corresponds to the point $(r\cos(\theta), r\sin(\theta), z)$ in (x, y, z) coordinates. Then the map $(z, r, \theta) \to \theta$ is a fibration of $\mathbb{R}^3 - L_1$ to S^1, whose fibers P_θ are the radial half-planes with constant θ-coordinate. The projection of these half-planes to the xy plane is the set of radial lines emanating from the origin.

The projection of every other component L_j to the xy plane is either an ellipse or a degenerate segment (if the circle is contained in a vertical plane); since great circles pairwise link, L_j links L_1 once so the projection of each L_j must be a nontrivial ellipse winding around the origin. Thus every component but L_1 is transverse to the foliation by P_θ, and the projection $S^3 - L \to S^1$ to the θ coordinate is a fibration, qed.

One beautiful application due to Walsh is to give an infinite family of examples of 3-manifolds that do not fiber over the circle, but are *virtually fibered* (i.e., they admit a finite sheeted cover which fibers). A *2-bridge link* is a knot or link that can be arranged in S^3 in such a way that projection to the x-axis (say) has exactly two maxima and two minima. These are equivalent to the so-called *rational links* (named by Conway) and are classified by a rational number p/q. See Figure 4 for an example.

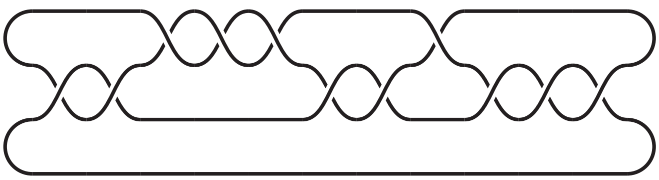


Figure 4. The 2-bridge knot $K_{p/q}$ associated to $p/q = 37/85$ with continued fraction expansion $[2, 3, 2, 1, 3]$.

It is unusual for a 2-bridge link complement to be fibered. In fact, a theorem of David Gabai says $S^3 - K_{p/q}$ is fibered if and only if p/q has a continued fraction expansion of the form $[\pm 2, \pm 2, \dots, \pm 2]$.

Walsh shows that the link complement $S^3 - K_{p/q}$ is finitely covered by a great circle link complement. For simplicity we explain the case q odd. Think of S^3 as the unit sphere in $\mathbb{C}^2$ with coordinates z and w. Let γ be the great circle obtained by intersecting S^3 with $\mathbb{R}^2 \subset \mathbb{C}^2$. If we pick coprime integers p and q (q odd), the map $\phi_{p/q} : (z, w) \to (e^{2\pi i/q} z, e^{2\pi i p/q} w)$ is an isometry of S^3 of order q, and the orbit of γ under the cyclic group C generated by $\phi_{p/q}$ is a great circle link L with q components. The quotient S^3/C is a Lens space M, and the great circle link L projects to a knot $K' \subset M$.

Complex conjugation $(z, w) \to (\bar{z}, \bar{w})$ acts as an involution on S^3 with fixed point set γ. It normalizes C, and conjugates $\phi_{p/q}$ to $\phi^{-1}_{p/q}$, and therefore descends to an involution on M with fixed point set K'. The quotient of M by complex conjugation is S^3 again, and so the map $M \to S^3$ is a branched cover in which K' projects to the branch locus $K \subset S^3$, the 2-bridge knot $K_{p/q}$.

In his celebrated 1982 paper in the *Bulletin of the AMS*, Thurston posed the question of whether every finite-volume hyperbolic 3-manifold has a finite cover that fibers over the circle. This question became known as the *virtual fibration conjecture*, despite the fact that it was posed as a question and not a conjecture. Thurston wrote, "(t)his dubious-sounding question seems to have a definite chance for a positive answer" but it is fair to say that many 3-manifold topologists were for a long time far more skeptical than Thurston. Personally, I found Walsh's proof of virtual fibration for rational link complements to be a watershed moment in my thinking on this conjecture, and in fact between 2009 and 2012 a series of papers by Daniel Wise and Ian Agol together proved the conjecture in full generality. A truly Olympian achievement!

AUTHOR'S NOTE. Walsh's paper is "Great circle links and virtually fibered knots" and appeared in Topology **44** (2005), no. 5, 947–958. Freedman–Skora's paper is "Strange actions of groups on spheres," J. Diff. Geom. **25** (1987), no. 1, 75–98. I'm grateful to Genevieve Walsh for feedback on an early version of this note.

Danny Calegari

Credits

Figures 1–4 and photo of Danny Calegari are courtesy of Danny Calegari.

Calls for Nominations & Applications

Leroy P. Steele Prize for Lifetime Achievement

About this Prize

The *Steele Prize for Lifetime Achievement* is awarded for the cumulative influence of the total mathematical work of the recipient, high level of research over a period of time, particular influence on the development of a field, and influence on mathematics through PhD students. The amount of this prize is US$10,000.

Next Prize: January 2023

Nomination Deadline: March 31, 2022

Nomination Procedure: `www.ams.org/steele-prize`

Nominations can be submitted between February 1 and March 31. Nominations for the Steele Prize for Lifetime Achievement should include a letter of nomination, the nominee's CV, and a short citation to be used in the event that the nomination is successful. Nominations will remain active and receive consideration for three consecutive years.

Leroy P. Steele Prize for Mathematical Exposition

About this Prize

The *Steele Prize for Mathematical Exposition* is awarded for a book or substantial survey or expository research paper. The amount of this prize is US$5,000.

Next Prize: January 2023

Nomination Deadline: March 31, 2022

Nomination Procedure: `www.ams.org/steele-prize`

Nominations can be submitted between February 1 and March 31. Nominations for the Steele Prize for Mathematical Exposition should include a letter of nomination, a complete bibliographic citation for the work being nominated, and a brief citation to be used in the event that the nomination is successful. Nominations will remain active and receive consideration for three consecutive years.

Leroy P. Steele Prize for Seminal Contribution to Research

About this Prize

The *Steele Prize for Seminal Contribution to Research* is awarded for a paper, whether recent or not, that has proved to be of fundamental or lasting importance in its field, or a model of important research.

Special note: The Steele Prize for Seminal Contribution to Research is awarded according to the following six-year rotation of subject areas:

1. Open (2025)
2. Analysis/Probability (2020)
3. Algebra/Number Theory (2021)
4. Applied Mathematics (2022)
5. **Geometry/Topology (2023)**
6. Discrete Mathematics/Logic (2024)

Next Prize: January 2023

Nomination Deadline: March 31, 2022

Nomination Procedure: `www.ams.org/steele-prize`

Nominations can be submitted between February 1 and March 31. Nominations for the Steele Prize for Seminal Contribution to Research should include a letter of nomination, a complete bibliographic citation for the work being nominated, and a brief citation to be used in the event that the nomination is successful.

Fellows of the American Mathematical Society

The Fellows of the American Mathematical Society program recognizes members who have made outstanding contributions to the creation, exposition, advancement, communication, and utilization of mathematics.

AMS members may be nominated for this honor during the nomination period which occurs in February and March each year. Selection of new Fellows (from among those nominated) is managed by the AMS Fellows Selection Committee, comprised of 12 members of the AMS who are also Fellows. Those selected are subsequently invited to become Fellows and the new class of Fellows is publicly announced each year on November 1.

Learn more about the qualifications and process for nomination at `www.ams.org/profession/ams-fellows`.

AMS COMMUNICATION

One Teen's Journey from Local Math Club to Professional Publication

Scott Hershberger

On a long flight from Boston to Atlanta, eighth grader Milena Harned was pondering a geometry problem involving perimeter bisectors and angle bisectors of quadrilaterals. As she looked at some examples, she noticed an intriguing pattern. By the time the plane landed, Harned had made a breakthrough that would lead not to an "A" on a homework assignment, but to her first publication in a professional, peer-reviewed mathematics journal.

"Perimeter bisectors, cusps, and kites"[1] appeared in the October 2021 issue of the *International Journal of Geometry*, when Harned was just 16. Publishing a new result at such a young age sets Harned apart from other teens with a knack for mathematics. Yet her achievement is a natural next step in her education—she has long explored open-ended mathematics problems outside of school.

In fifth grade, Harned joined Girls' Angle,[2] a Cambridge, Massachusetts-based club where girls learn math through investigation and discovery. The students follow their curiosity and creativity, guided by mentors who have proven original mathematical theorems.

"It was one of the most invaluable experiences I've had," Harned says. "It definitely got me to pursue some [less common] topics in math, especially at a younger age than I would have ever considered to do them."

Girls' Angle gave Harned the opportunity not just to hone her problem-solving skills, but also to practice explaining her work through writing. In the span of two and a half years, she authored or co-authored 10 articles in the *Girls' Angle Bulletin*, the organization's bimonthly magazine. It bridges high school and professional mathematics, providing a venue for students who go above and beyond to share their work. For a decade, the AMS has published the magazine at a subsidized cost through its print shop in Pawtucket, Rhode Island, as a service to the community.

As her mathematical skills grew, Harned also received private lessons from Girls' Angle founder and president Ken Fan, who describes her as a "fountainhead" of new ideas.

Figure 1. Milena Harned published her first professional, peer-reviewed mathematics paper at the age of 16.

Scott Hershberger is the communications and outreach content specialist at the AMS. His email address is `slh@ams.org`.

[1] `https://ijgeometry.com/product/milena-harned-perimeter-bisectors-cusps-and-kites/`

[2] `http://www.girlsangle.org/`

DOI: https://dx.doi.org/10.1090/noti2446

"She's actually the only student I've ever had who I did not have to set an agenda for at all," he says. "She's really quite phenomenal."

Proving a New Theorem

Harned's first article in the *Bulletin*, which she wrote in sixth grade, discussed the volume of a regular n-simplex (the generalization of a tetrahedron). Her subsequent articles about a Fibonacci-like sequence and counting all possible ways to play a game of Nim garnered her two entries[3] in The On-Line Encyclopedia of Integer Sequences.

Fan saw a clear progression in the sophistication of Harned's ideas and the clarity of her exposition as the middle schooler continued writing up her work. Harned agrees that publishing in the *Bulletin* played a key role in helping her reach the professional level.

"It helped me realize that [...] in order to write something up, I need to know everything about it. [...] That was a really difficult process at first," she says. "Learning how to convey my thoughts really helped me get a more well-rounded understanding of math."

In seventh grade, Harned began the work that culminated in her first professional publication. Out of curiosity, she looked at lines that divide the perimeter of a polygon in half. The set of these perimeter bisectors defines a curve called an envelope—but depending on the polygon, the envelope might or might not be continuous. After analyzing some examples, Harned was able to prove that the only triangles with continuous envelopes are equilateral ones. So she moved on to convex quadrilaterals.

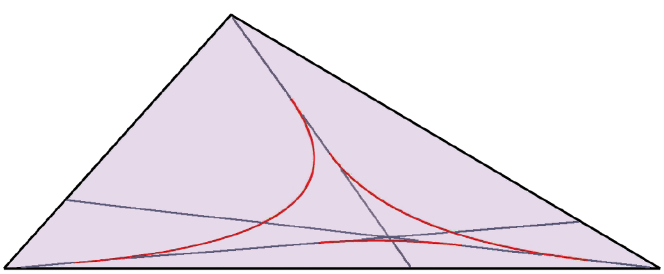


Figure 2. The envelope (in red) of a scalene triangle is not continuous.

The four-sided case turned out to be more complicated. Harned knew that the envelope would be continuous if all the angle bisectors were also perimeter bisectors. But classifying all possible quadrilaterals with this property eluded her until that flight to Georgia, nearly a year after she began working on the problem.

Her key realization was that any continuous envelope must have an odd number of sharp cusps, which corresponded to the perimeter bisectors that pass through vertices. In the case of quadrilaterals, that meant that one perimeter bisector must pass through two vertices. A few technical details remained before the proof would be complete, but "in the moment it just felt correct," she says.

Later that summer, she filled in the final missing piece, arriving at her new theorem: In a given convex quadrilateral, every angle bisector also bisects the perimeter if and only if the quadrilateral is either a kite with three congruent acute angles or a rhombus.

An International Journey

Before formally writing up her results, Harned had to shift her attention to a more pressing challenge: starting her first year of high school. Not until the COVID-19 lockdown did she have the chance to return to her research.

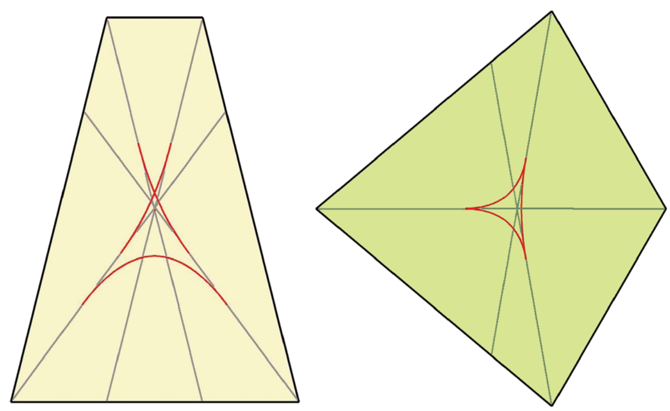


Figure 3. A trapezoid (left) and a kite (right) with their envelopes (red) and perimeter bisectors (grey). The kite has a continuous envelope with three cusps, and one perimeter bisector passes through a pair of vertices.

Although writing articles in the *Girls' Angle Bulletin* had prepared Harned for this larger project, it was easy to lose sight of her passion for mathematics in the midst of the day-to-day work.

"There was a period of time when writing it up that I was like, 'Okay, I just want this done. I want to get this over with,'" she says. "But it was really important for me to ground myself and think, 'You know, I really love this stuff. I need to remember that.'"

By July 2020, her paper was complete, so she sent it to the *International Journal of Geometry* for review. Throughout the writing process, she had benefited from feedback from Fan. Now, mathematicians she had never met would judge whether her work was worthy of publication.

"For the first month or so I would check my email almost every day, thinking, 'Is [the decision] here yet?'" Harned says. "Then I sort of just put it out of my mind, let myself not worry about it."

Soon afterward, she moved to Switzerland to enroll in an International Baccalaureate program, skipping 10th grade entirely. In February of 2021, she learned that her paper had been accepted pending minor revisions.

[3] *A286983 and A289329*

Harned was overjoyed, as was her mentor. "For me, it's thrilling—it feels better than the feeling I have when I publish one of my own papers," Fan says. "I'm excited for her that she's starting to get professional recognition."

A Bright Future

During the pandemic, Girls' Angle meetings have shifted to an online format, but the time difference between Massachusetts and Switzerland has prevented Harned from participating for now. She hopes to serve as a club mentor once she returns to the US for college. She plans to major in mathematics and then pursue a PhD in the field.

"If there's some sort of lab where I can apply the math I know to help some people create something new [...] I think that would be something I'd look to do," she says. "Math is my passion, but I also want to use it in a way that will help the world."

Harned's paper is the first peer-reviewed publication by a Girls' Angle member, but Fan is confident it will not be the last. Harned is now collaborating with another member on a paper about intersections of lines in the plane. Fan attributes these results, as well as novel findings published by Harned and others in the *Girls' Angle Bulletin*, in large part to the club's emphasis on open-ended exploration rather than math competitions.

According to Fan, the traditional metrics of math competitions, which reward fast, accurate thinking, would likely not recognize Harned's talent. "She actually thinks so fast that she often stumbles on herself," says Fan. "Her mode of operation is to plow ahead and just worry about [errors] as they come. [...] She doesn't seem to mind seeing a lot of wrong things." Yet this approach is perfectly suited to the creative problem-solving needed to prove new theorems, he says.

Harned's trajectory is a testament to the importance of mentorship in cultivating the next generation of mathematicians.

"I don't think I would have ever considered math as something more than a school subject that I got good grades in had I not gone [to Girls' Angle]," Harned says. "I got to better understand how I think, and that's really helped me in my schoolwork and in the math I do as I've gotten older—just to understand my own mind."

Scott Hershberger

Credits

Figure 1 is courtesy of Tony Capozzi.
Figures 2 and 3 are courtesy of Milena Harned.
Author photo is courtesy of Jiyoon Kang.

Mathematics People

Gupta Awarded Ramanujan Prize

Neena Gupta

Neena Gupta of the Indian Statistical Institute was awarded the 2021 DST–ICTP–IMU Ramanujan Prize for young mathematicians in developing countries. She was honored for her "outstanding work in affine algebraic geometry and commutative algebra, in particular for her solution of the Zariski cancellation problem for affine spaces." According to the prize committee, her work "shows impressive algebraic skill and inventiveness."

Gupta received her PhD from the Indian Statistical Institute (ISI) in 2011. She was a visiting scientist at ISI Kolkata (2012–2014) and a visiting fellow at the Tata Institute of Fundamental Research (TIFR) Mumbai in 2012 before joining the faculty at ISI. She was honored with the 2014 Young Scientists Award of the Indian National Science Academy for her solution of the Zariski cancellation problem. Her honors also include the 2019 Shanti Swarup Bhatnagar Prize, the 2014 Ramanujan Prize of the University of Madras, the inaugural Professor A. K. Agarwal Award of the Indian Mathematical Society, and the B. M. Birla Science Prize in Mathematics (2017). Gupta tells the *Notices*: "I have been always supported by my family and teachers in my life. I owe a lot to them, especially my father and my PhD supervisor Professor Amartya K. Dutta. I am also grateful to my husband and my parents-in-law for supporting my dreams."

The DST–ICTP–IMU Ramanujan Prize is awarded to researchers from developing countries who are under the age of forty-five for outstanding research in a developing country in any branch of the mathematical sciences. The prize is administered by the Abdus Salam International Centre for Theoretical Physics (ICTP), the Department of Science and Technology (DST) of the Government of India, and the International Mathematical Union (IMU). It carries a cash award of US$15,000.

—From a DST–ICTP–IMU announcement

Bhatt Receives Clay Research Award

Bhargav Bhatt

Bhargav Bhatt of the University of Michigan has been named the recipient of the 2021 Clay Research Award of the Clay Mathematics Institute (CMI). He was recognized for "his groundbreaking achievements in commutative algebra, arithmetic algebraic geometry, and topology in the p-adic setting."

According to the prize citation, "his profound contributions include the development, in joint work with M. Morrow and P. Scholze, of a unified p-adic cohomology theory (prismatic cohomology) and, in joint work with J. Lurie, a p-adic Riemann–Hilbert functor. Striking applications of this work include Bhatt's resolution of long-standing problems in commutative algebra, in particular concerning the Cohen–Macaulay property and Kodaira vanishing up to finite covers. These results have in turn fueled startling progress on the minimal model program in mixed characteristic."

Bhatt received his PhD from Princeton University in 2010. He became a postdoctoral assistant professor at the University of Michigan in 2010 and was a member of the Institute for Advanced Study from 2012 to 2014. He is currently Frederick W. and Lois B. Gehring Professor at Michigan. His honors include a Packard Fellowship (2015–2021), the Compositio Prize (2016), a Simons Investigator Award (2019–2024), and a New Horizons Prize in Mathematics (2021). He will be a plenary speaker at the International Congress of Mathematicians in St. Petersburg in 2022. He is a Fellow of the AMS.

—From a CMI announcement

Kayal Awarded Infosys Prize

Neeraj Kayal of Microsoft Research Lab in Bangalore, India, has been awarded the 2021 Infosys Prize in Mathematical Sciences for his outstanding contributions to computational complexity. According to the prize announcement, his "extensive, innovative work on algebraic computation includes the development of deep lower bound techniques proving limitations of this natural model, as well as designing efficient algorithms for reconstruction and equivalence of such algebraic circuits."

Kayal received his PhD in theoretical computer science from the Indian Institute of Technology in 2007. He did postdoctoral research at the Institute for Advanced Study in Princeton and at Rutgers University. Since 2008, he has been working with the Microsoft Research Lab India as a researcher. With M. Agrawal and N. Saxena, he was awarded the Gödel Prize and the AMS Delbert Ray Fulkerson Prize, both in 2006, for their discovery of the AKS primality test. He received the Young Scientist Award of the Indian National Science Academy in 2012.

—From an Infosys announcement

Hans Schneider Prize Awarded

Pauline van den Driessche

Nicholas J. Higham

Pauline van den Driessche of the University of Victoria and **Nicholas J. Higham** of the University of Manchester have been awarded the 2022 Hans Schneider Prize of the International Linear Algebra Society. Van den Driessche received her PhD from the University College of Wales in 1964 and joined the University of Victoria in 1965, where she is now professor emerita. Her research involves aspects of stability in biomathematical models and matrix analysis; mathematical biology, especially models in epidemiology and ecology; and matrix analysis, especially stability and combinatorial matrix analysis. Her honors include the 2007 Krieger–Nelson Prize of the Canadian Mathematical Society, the inaugural Olga Taussky-Todd Lectureship at the 2007 International Congress on Industrial and Applied Mathematics, the 2013 David H. Turpin Gold Medal of the University of Victoria, and the CAIMS Research Prize of the Canadian Applied and Industrial Mathematics Society. She is a Fellow of the Society for Industrial and Applied Mathematics (SIAM). She is a member of the scientific management committee at the Centre for Disease Modeling at York University and a member of the scientific research board of the American Institute of Mathematics (AIM).

Higham received his PhD in 1985 from the University of Manchester under the supervision of George Hall. He became an appointed lecturer at Manchester in 1985 and has been Richardson Professor of Applied Mathematics since 1998. He has held visiting positions at Cornell University and the Institute for Mathematics and its Applications, University of Minnesota. He received a Royal Society–Wolfson Research Merit Award in 2003. His honors include the 1999 Junior Whitehead Prize, the 2008 Fröhlich Prize, and the 2019 Naylor Prize and Lectureship, all from the London Mathematical Society; the 2020 IMA Gold Medal of the Institute of Mathematics and its Applications; and the 2021 George Pólya Prize for Mathematical Exposition from SIAM. He was elected a Fellow of the Royal Society in 2007 and was awarded a Royal Society Research Professorship in 2018. He is a Fellow of the Society for Industrial and Applied Mathematics (SIAM), a Fellow of the Association for Computing Machinery (ACM), and a Member of Academia Europaea.

The Hans Schneider Prize in Linear Algebra is awarded by the International Linear Algebra Society for research, contributions, and achievements at the highest level of linear algebra and is awarded for an outstanding scientific achievement or for lifetime contributions.

—From Schneider Prize announcements

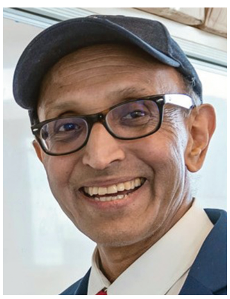
Prizes of Australian Mathematical Society

The Australian Mathematical Society has awarded several prizes for 2021.

Serena Dipierro

Serena Dipierro of the University of Western Australia (UWA) was awarded the Australian Mathematical Society Medal for her "outstanding contributions to the area of analysis and PDEs, with a special focus on the theory of nonlocal operators and free boundary problems." Her work "aims at establishing regularity properties and geometric features of the interfaces occurring in phase transitions. In addition to their mathematical interest, such questions arise naturally in applications to physics, engineering, mathematical finance, and population dynamics."

Dipierro received her PhD in 2012 from Scuola Internazionale Superiore di Studi Avanzati (SISSA) in Trieste. She held postdoctoral positions at the University of Chile and the University of Edinburgh and faculty positions at University of Melbourne and Università di Milano. She has been the recipient of a Humboldt Fellowship. She has served as the head of the Department of Mathematics and Statistics at UWA, as a council member of the Australian Mathematical Society, and as secretary of the Women in Mathematics Special Interest Group. The medal is awarded to a member of the Society under the age of forty for distinguished research in the mathematical sciences. She tells the *Notices*: "I am a very curious person in general and I love reading books, traveling, and visiting new places. Besides doing math, I also enjoy outdoor activities, and in particular kayaking and hiking."


Mathai Varghese

Mathai Varghese of the University of Adelaide was awarded the George Szekeres Medal for his "significant contributions to geometric analysis and to mathematical physics. Among the highlights are his co-invention of projective and fractional index theory, which has recently been generalized to certain infinite dimensional manifolds and for the Mathai–Quillen formalism in index theory and topological field theories. He is also renowned for his research in string theory, T-duality in a background flux with a change of topology and novel applications to condensed matter physics." He received his PhD from the Massachusetts Institute of Technology (MIT) under the supervision of Daniel G. Quillen. He is director of the Institute for Geometry and its Applications and Elder Professor of Mathematics at Adelaide, as well as adjunct professor in the Mathematical Sciences Institute at Australian National University. He has been a research fellow of the Clay Mathematics Institute and a visiting scientist at MIT (2000–2001), an ARC Senior Research Fellow at the University of Adelaide (2001–2005), and a senior research fellow at the Erwin Schrödinger Institute (2006). He was awarded the Australian Mathematical Society Medal in 2000, the ARC Discovery Outstanding Researcher Award in 2013, and an Australian Laureate Fellowship in 2017. He served as editor of the *Proceedings of the AMS* from 2008 to 2016. He is a fellow of the Australian Academy of Science, of the Australian Mathematical Society, and of the Royal Society of South Australia. Varghese tells the *Notices*: "I like to read popular science books, especially written by famous scientists. I also like to travel and taste cuisine all over the world (when this was possible!)."

Sarah Dart of the Queensland University of Technology received the 2021 Award for Teaching Excellence. She was honored for her use of technology "to support learning of mathematics for large and diverse student cohorts, including development of worked example videos to improve problem-solving skills, and implementation of personalized emails to foster an effective learning environment when transitioning to university." Dart received her PhD in 2018 from the Queensland University of Technology by investigating red blood cell shape and deformability from a numerical modeling perspective. She has been recognized by the Australasian Association for Engineering Education (2019) and Australian Awards for University Teaching (2020) and with a Senior Fellowship of the Higher Education Academy and a Vice Chancellor's Award for Excellence (2017). Her research interests are in engineering and mathematics education, educational technology, and academic development. She enjoys playing netball and going running with friends on weekends.


Sarah Dart

The Gavin Brown Prize is given for outstanding and innovative research published by members of the Society. The awardees for 2021 are the following:

Mike Meylan (University of Newcastle), **Luke Bennetts** (University of Adelaide), **Johannes Mosig** (Rasa Technologies, Berlin), **W. Erick Rogers** (Naval Research Laboratory, Stennis Space Center, Mississippi), **Martin Doble** (Polar Scientific, United Kingdom), and **Malte Peter** (University of Augsberg) for their paper "Dispersion relations, power laws, and energy loss for waves in the marginal ice zone," *Journal of Geophysical Research: Oceans* **123** (2018).

Brett Parker of the Australian National University for his paper "Holomorphic curves in exploded manifolds: Virtual fundamental class," *Geometry and Topology* **23** (2019).

—From Australian Mathematical Society announcements

Wallenberg Academy Fellows Announced

The Wallenberg Academy Fellowship Program has announced its new Fellows for 2021. The following individuals whose work involves the mathematical sciences were selected.


Hannes Thiel

Hannes Thiel of Kiel University, Germany, is a scholar in mathematics whose work will contribute to identifying mathematical objects. He received his PhD in mathematics in 2012 from the University of Copenhagen and has held positions at the University of Münster, the Fields

Institute in Toronto, and Dresden University of Technology. His current work deals with the classification and structure of operator algebras and, more specifically, C^*-algebras. He tells the *Notices* that he grew up in Potsdam, Germany, and remembers the fall of the Berlin Wall when he was six years old: "The atmosphere was exhilarating; it was surreal." He likes to read (particularly science fiction) and watch movies and television series. He is also involved in projects to raise awareness of difficulties faced by people with disabilities, especially visual disabilities.


Laura Donnay

Laura Donnay of the Vienna University of Technology works in mathematical physics and will use the fellowship to further develop mathematics for describing black holes. She is currently investigating in particular a newly discovered and intriguing infinite set of symmetries that appear close to black hole event horizons. Donnay received her PhD in 2016 from the University of Brussels. She held a postdoctoral research position at Harvard University from 2016–2017 and was a Black Hole Initiative Fellow and Postdoctoral Researcher at Harvard from 2017 to 2019. She has been awarded the Start-Preis of the Austrian Ministry for Science and the Marie Sklodowska-Curie Fellowship from the European Commission.

Silvia De Toffoli of Princeton University will explore mathematics' fallibility and human weaknesses. She holds a PhD in mathematics from the Technical University of Berlin as well as a PhD in philosophy from Stanford University. Her work focuses on philosophy of mathematics and epistomology.

—From Wallenberg Academy announcements

Neiger and Pernet Receive Best Paper Award

Vincent Neiger of the University of Limoges and **Clément Pernet** of Université Grenoble Alpes have been chosen to receive the 2021 Best Paper Award of the *Journal of Complexity* for their joint paper, "Deterministic computation of the characteristic polynomial in the time of matrix multiplication," *Journal of Complexity* **67** (2021). The prize of US$4,000 will be divided between the awardees.

—Erich Novak
Editor in Chief, Journal of Complexity

2020 Rosenthal Prize Awarded

Doug O'Roark, executive director of Math Circles of Chicago, was awarded the 2020 Rosenthal Prize for Innovation and Inspiration in Math Teaching for his lesson "Towers and Dragons," in which "students discover a stunning connection between paper folding and a classic disc-moving puzzle." He received a cash prize of US$25,000. **Lauren Siegel,** director of the MathHappens Foundation in Austin, Texas, was named runner-up for her lesson, in which "students learn to appreciate ratios by making their own calipers and applying them to objects, photos, and geometric figures." She received a cash prize of US$5,000. The prizes are awarded by the National Museum of Mathematics (MoMath) and are designed to recognize and promote hands-on math teaching in upper elementary and middle school classrooms.

—MoMath announcement

Rhodes Scholars 2022

The Rhodes Trust has announced the names of the American scholars chosen to receive the 2022 Rhodes Scholarships. The scholars will spend two to three years studying at the University of Oxford. The value of the scholarships averages approximately US$75,000 per year. The names and brief biographies of the scholars whose work involves the mathematical sciences follow.

Nicholas Hayes of Long Valley, New Jersey, is a senior at the University of Alabama, where he majors in applied mathematics and German. He also did an intensive course in Swahili language and culture in Tanzania as a Boren Scholar. An ultramarathoner, he was named the outstanding junior at the University of Alabama on the basis of scholarship, leadership, and service. Hayes edited an undergraduate science journal, interned at the National Oceanic and Atmospheric Administration in fisheries science, and has published in academic journals in politics and biology. He has also published poetry and translates between English and Swahili. Hayes will do the MSc in mathematical sciences and the MSt in linguistics, philology, and phonetics at Oxford.

Michael Y. Cheng of Wynnewood, Pennsylvania, is a Harvard College senior concentrating in history and mathematics concurrently with a master's degree in computer science. From an immigrant household, he struggled with English and received special language training until he was sixteen, yet he began his studies at Drexel University while still in high school. Cheng's career interests are in energy technology and policy. He has researched perovskite solar panels in Taiwan, urban development policy in Argentina, and the history of energy transitions worldwide. He taught himself to swim by watching YouTube videos before walking onto the Harvard varsity men's lightweight crew. He was elected as a junior to Phi Beta Kappa. Cheng plans to do the MSc in energy systems and the MSc in political theory research at Oxford.

Elizabeth Guo of Plano, Texas, is a senior at Harvard College, where she majors in physics. Elizabeth's undergraduate research explores the intersection of science and the law. As an intern at the US Department of Commerce, Elizabeth's work helped inform the incoming president's strategic plan. She currently serves as a news executive of the *Harvard Crimson* and is a member of the Harvard College Honor Council. She was elected to Phi Beta Kappa as a junior. While at Oxford, she plans to pursue an MSc in mathematical and theoretical physics and an MSc in social science of the Internet.

—From a Rhodes Trust announcement

Credits

Photo of Neena Gupta is courtesy of the Indian Statistical Institute.
Photo of Bhargav Bhatt is courtesy of Katie Grillaert.
Photo of Nicholas J. Higham is courtesy of Thomas Higham.
Photo of Mathai Varghese is courtesy of John Montesi.
Photo of Hannes Thiel is courtesy of Eusebio Gardella.

Mathematics Opportunities

Listings for upcoming mathematics opportunities to appear in Notices *may be submitted to* `notices@ams.org`.

Early-Career Opportunity

AMS-Simons Travel Grants

The AMS-Simons Travel Grants are administered by the AMS with support from the Simons Foundation. Each grant provides an early-career mathematician with $2,500 funding for two years to be used for research-related travel expenses. Applicants must be located in the United States (or be US citizens employed outside the United States) and must have completed the PhD within the last four years. The department of the awardee will also receive a small amount of funding to help enhance its research environment.

The application period for 2022 opened February 1, 2022, and runs through **11:59 pm Eastern time on March 31, 2022**. Up to seventy awardees will be chosen from the set of applicants, and the earliest date for supported travel would be July 1, 2022. To learn more about eligibility and how to apply, visit `https://www.ams.org/AMS-SimonsTG`.

—AMS Programs Department

Early-Career Opportunity

MAA Project NExT

MAA Project NExT (New Experiences in Teaching) is a year-long professional development program for new(ish) or recent PhDs in the mathematical sciences. The program is designed to connect new faculty with expert teachers and leaders in the mathematics community and address the three main aspects of an academic career: teaching, research, and service.

MAA Project NExT Fellows join an active community of faculty who have become award-winning teachers, innovators on their campuses, active members of the MAA, and leaders in the profession.

MAA Project NExT welcomes applications from new(ish) and recent PhDs in postdoctoral, tenure-track, and visiting positions. We particularly encourage applicants from underrepresented groups, including women and minorities. Applications for the 2022 cohort of MAA Project NExT Fellows are due on **April 15, 2022**, and can be found at `projectnext.maa.org`.

The AMS is one of the sponsors of MAA Project NExT.

—Project NExT announcement

Early-Career Opportunity

NSF CAREER Awards

The National Science Foundation (NSF) Faculty Early Career Development (CAREER) Program supports early-career faculty members who have the potential to serve as academic role models in research and education and to lead advances in the missions of their departments or organizations. Activities pursued by early-career faculty members should build a firm foundation for a lifetime of leadership in integrating education and research. The deadline for proposals is **July 25, 2022**. See the website `https://www.nsf.gov/funding/pgm_summ.jsp?pims_id=503214`.

—NSF announcement

Early-Career Opportunity

CAS–TWAS Scholarship

The Chinese Academy of Sciences (CAS) and the World Academy of Sciences (TWAS) offer the CAS–TWAS Chinese Scholarship to scholars to pursue a doctoral degree in China. The 2022–2023 program is now accepting applications from international scholars for the 2022 academic session. The deadline date for applications is **March 31, 2022**. See the website `https://ascholarship.com/cas-twas-scholarship-cas-twas-presidents-fellowship`.

—From a CAS–TWAS announcement

The most up-to-date listing of NSF funding opportunities from the Division of Mathematical Sciences can be found online at `www.nsf.gov/dms` *and for the Directorate of Education and Human Resources at* `www.nsf.gov/dir/index.jsp?org=ehr`. *To receive periodic updates, subscribe to the DMSNEWS listserv by following the directions at* `www.nsf.gov/mps/dms/about.jsp`.

Classified Advertising

Employment Opportunities

CHINA

Tianjin University, China
Tenured/Tenure-Track/Postdoctoral Positions at the Center for Applied Mathematics

Dozens of positions at all levels are available at the recently founded Center for Applied Mathematics, Tianjin University, China. We welcome applicants with backgrounds in pure mathematics, applied mathematics, statistics, computer science, bioinformatics, and other related fields. We also welcome applicants who are interested in practical projects with industries. Despite its name attached with an accent of applied mathematics, we also aim to create a strong presence of pure mathematics.

Light or no teaching load, adequate facilities, spacious office environment and strong research support. We are prepared to make quick and competitive offers to self-motivated hard workers, and to potential stars, rising stars, as well as shining stars.

The Center for Applied Mathematics, also known as the Tianjin Center for Applied Mathematics (TCAM), located by a lake in the central campus in a building protected as historical architecture, is jointly sponsored by the Tianjin municipal government and the university. The initiative to establish this center was taken by Professor S. S. Chern. Professor Molin Ge is the Honorary Director, Professor Zhiming Ma is the Director of the Advisory Board. Professor William Y. C. Chen serves as the Director.

TCAM plans to fill in fifty or more permanent faculty positions in the next few years. In addition, there are a number of temporary and visiting positions. We look forward to receiving your application or inquiry at any time. There are no deadlines.

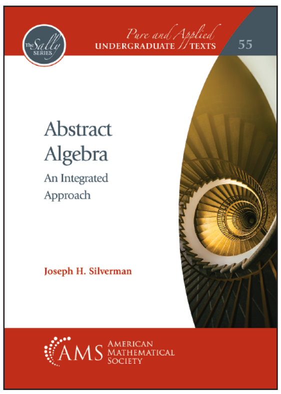
Please send your resume to `mathjobs@tju.edu.cn`.

For more information, please visit `cam.tju.edu.cn` or contact Mr. Albert Liu at `mathjobs@tju.edu.cn`, telephone: 86-22-2740-6039.

07

New Books Offered by the AMS

Algebra and Algebraic Geometry



Abstract Algebra
An Integrated Approach

Joseph H. Silverman, *Brown University, Providence, RI*

This abstract algebra textbook takes an integrated approach that highlights the similarities of fundamental algebraic structures among a number of topics. The book begins by introducing groups, rings, vector spaces, and fields, emphasizing examples, definitions, homomorphisms, and proofs. The goal is to explain how all of the constructions fit into an axiomatic framework and to emphasize the importance of studying those maps that preserve the underlying algebraic structure. This fast-paced introduction is followed by chapters in which each of the four main topics is revisited and deeper results are proven.

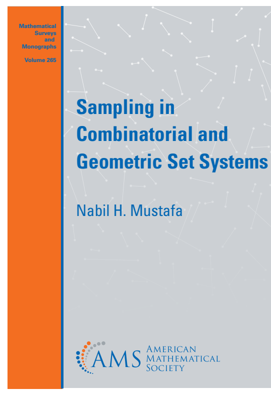
The second half of the book contains material of a more advanced nature. It includes a thorough development of Galois theory, a chapter on modules, and short surveys of additional algebraic topics designed to whet the reader's appetite for further study.

This book is intended for a first introduction to abstract algebra and requires only a course in linear algebra as a prerequisite. The more advanced material could be used in an introductory graduate-level course.

Pure and Applied Undergraduate Texts, Volume 55
May 2022, 567 pages, Softcover, ISBN: 978-1-4704-6860-6, LC 2021045594, 2010 *Mathematics Subject Classification*: 12–XX, 13–XX, 16–XX, 20–XX, List US$49, **AMS members** US$39.20, MAA members US$44.10, Order code AMSTEXT/55

bookstore.ams.org/amstext-55

Discrete Mathematics and Combinatorics



Sampling in Combinatorial and Geometric Set Systems

Nabil H. Mustafa, *ESIEE Paris, Marne-la-Vallée, France*

Understanding the behavior of basic sampling techniques and intrinsic geometric attributes of data is an invaluable skill that is in high demand for both graduate students and researchers in mathematics, machine learning, and theoretical computer science. The last ten years have seen significant progress in this area, with many open problems having been resolved during this time. These include optimal lower bounds for epsilon-nets for many geometric set systems, the use of shallow-cell complexity to unify proofs, simpler and more efficient algorithms, and the use of epsilon-approximations for construction of coresets, to name a few.

This book presents a thorough treatment of these probabilistic, combinatorial, and geometric methods, as well as their combinatorial and algorithmic applications. It also revisits classical results, but with new and more elegant proofs.

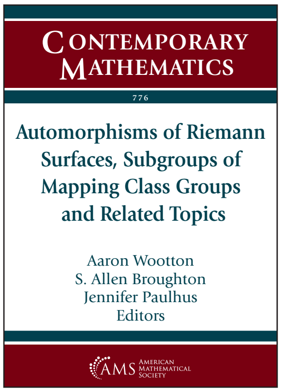
This item will also be of interest to those working in geometry and topology.

Mathematical Surveys and Monographs, Volume 265
April 2022, 251 pages, Softcover, ISBN: 978-1-4704-6156-0, LC 2021040893, 2010 *Mathematics Subject Classification*: 68Q87, 52C45, 05D40, 03D32, 11K38, List US$125, **AMS members** US$100, MAA members US$112.50, Order code SURV/265

bookstore.ams.org/surv-265

New in Contemporary Mathematics

Analysis



Automorphisms of Riemann Surfaces, Subgroups of Mapping Class Groups and Related Topics

Aaron Wootton, *University of Portland, OR*, **S. Allen Broughton**, *Rose-Hulman Institute of Technology, Terre Haute, IN*, and **Jennifer Paulhus**, *Grinnell College, IA*, Editors

Automorphism groups of Riemann surfaces have been widely studied for almost 150 years. This area has persisted in part because it has close ties to many other topics of interest such as number theory, graph theory, mapping class groups, and geometric and computational group theory. In recent years there has been a major revival in this area due in part to great advances in computer algebra systems and progress in finite group theory.

This volume provides a concise but thorough introduction for newcomers to the area while at the same time highlighting new developments for established researchers. The volume starts with two expository articles. The first of these articles gives a historical perspective of the field with an emphasis on highly symmetric surfaces, such as Hurwitz surfaces. The second expository article focuses on the future of the field, outlining some of the more popular topics in recent years and providing 78 open research problems across all topics. The remaining articles showcase new developments in the area and have specifically been chosen to cover a variety of topics to illustrate the range of diversity within the field.

This item will also be of interest to those working in geometry and topology.

Contemporary Mathematics, Volume 776
April 2022, approximately 351 pages, Softcover, ISBN: 978-1-4704-6025-9, LC 2021029308, 2010 *Mathematics Subject Classification*: 30Fxx, 14Hxx, 20H10, 20B25, 11G32, 57K20, List US$125, AMS members US$100, MAA members US$112.50, Order code CONM/776

`bookstore.ams.org/conm-776`

New in Memoirs of the AMS

Analysis

The Brunn-Minkowski Inequality and a Minkowski Problem for Nonlinear Capacity

Murat Akman, *University of Connecticut, Storrs, CT*, **Jasun Gong**, *Fordham University, Bronx, NY*, **Jay Hineman**, *Data Analytics, Durham, NC*, **John Lewis**, *University of Kentucky, Lexington, KY*, and **Andrew Vogel**, *Syracuse University, NY*

Memoirs of the American Mathematical Society, Volume 275, Number 1348
February 2022, 115 pages, Softcover, ISBN: 978-1-4704-5052-6, 2010 *Mathematics Subject Classification*: 35J60, 31B15, 39B62, 52A40, 35J20, 52A20, 35J92, List US$85, AMS members US$68, MAA members US$76.50, Order code MEMO/275/1348

`bookstore.ams.org/memo-275-1348`

Differential Equations

The Yang-Mills Heat Equation with Finite Action in Three Dimensions

Leonard Gross, *Cornell University, Ithaca, NY*

This item will also be of interest to those working in mathematical physics.

Memoirs of the American Mathematical Society, Volume 275, Number 1349
February 2022, 111 pages, Softcover, ISBN: 978-1-4704-5053-3, 2010 *Mathematics Subject Classification*: 35K58, 35K65; 70S15, 35K51, 58J35, List US$85, AMS members US$68, MAA members US$76.50, Order code MEMO/275/1349

`bookstore.ams.org/memo-275-1349`

Instability, Index Theorem, and Exponential Trichotomy for Linear Hamiltonian PDEs

Zhiwu Lin, *Georgia Institute of Technology, Atlanta, GA* and **Chongchun Zeng**, *Georgia Institute of Technology, Atlanta, GA*

Memoirs of the American Mathematical Society, Volume 275, Number 1347
February 2022, 136 pages, Softcover, ISBN: 978-1-4704-5044-1, 2010 *Mathematics Subject Classification*: 35B35, 37K45; 35P05, 47A10, List US$85, AMS members US$68, MAA members US$76.50, Order code MEMO/275/1347

bookstore.ams.org/memo-275-1347

Geometry and Topology

Tits Polygons

Bernhard Mühlherr, *Universität Giessen, Germany* and **Richard M. Weiss**, *Tufts University, Medford, MA*
with an Appendix by Holger P. Petersson

Memoirs of the American Mathematical Society, Volume 275, Number 1352
February 2022, 114 pages, Softcover, ISBN: 978-1-4704-5101-1, 2010 *Mathematics Subject Classification*: 20E42, 51E12, 51E24, List US$85, AMS members US$68, MAA members US$76.50, Order code MEMO/275/1352

bookstore.ams.org/memo-275-1352

Sutured ECH is a Natural Invariant

Çağatay Kutluhan, *University at Buffalo, NY*, **Steven Sivek**, *Imperial College, London, United Kingdom*, and **C. H. Taubes**, *Harvard University, Cambridge, MA*

Memoirs of the American Mathematical Society, Volume 275, Number 1350
February 2022, 125 pages, Softcover, ISBN: 978-1-4704-5054-0, 2010 *Mathematics Subject Classification*: 53D40, 57M27, List US$85, AMS members US$68, MAA members US$76.50, Order code MEMO/275/1350

bookstore.ams.org/memo-275-1350

Number Theory

On the Asymptotics to all Orders of the Riemann Zeta Function and of a Two-Parameter Generalization of the Riemann Zeta Function

Athanassios S. Fokas, *University of Cambridge, United Kingdom* and **Jonatan Lenells**, *Royal Institute of Technology, Stockholm, Sweden*

Memoirs of the American Mathematical Society, Volume 275, Number 1351
February 2022, 114 pages, Softcover, ISBN: 978-1-4704-5098-4, 2010 *Mathematics Subject Classification*: 11M06, 30E15; 33E20, List US$85, AMS members US$68, MAA members US$76.50, Order code MEMO/275/1351

bookstore.ams.org/memo-275-1351

New AMS-Distributed Publications

Algebra and Algebraic Geometry

Espaces de Configuration Généralisés - Espaces Topologiques *i*-acycliques - Suites Spectrales Basiques

Alberto Arabia, *Université Paris Diderot-Paris 7, France*

A note to readers: This book is in French.

This memoir presents a new approach to generalized configuration spaces of a locally compact space M. The approach is two-fold. The first part applies only to i-acyclic spaces, which class contains noncompact contractible spaces, and, if X is i-acyclic, contains also the open subspaces of X and the products $X \times M$ by any space M. The second part describes a procedure which extrapolates cohomological properties of configuration spaces of i-acyclic spaces X to general topological spaces M.

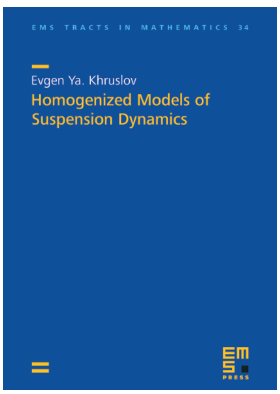
This item will also be of interest to those working in geometry and topology and number theory.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Mémoires de la Société Mathématique de France, Number 170
November 2021, 248 pages, Softcover, ISBN: 978-2-85629-934-0, 2010 *Mathematics Subject Classification*: 55R80, 20–XX, 20C30, 18G40, 55–XX, 55R20, 11B73, List US$75, AMS members US$60, Order code SMFMEM/170

`bookstore.ams.org/smfmem-170`

Differential Equations



Homogenized Models of Suspension Dynamics

Evgen Ya. Khruslov, *B. Verkin Institute for Low Temperature Physics and Engineering, National Academy of Sciences, Ukraine*

This book studies the motion of suspensions, that is, of mixtures of a viscous incompressible fluid with small solid particles that can interact with each other through forces of non-hydrodynamic origin. In view of the complexity of the original (microscopic) system of equations that describe such phenomena, which appear both in nature and in engineering processes, the problem is reduced to a macroscopic description of the motion of mixtures as an effective continuous medium.

The focus is on developing mathematical methods for constructing such homogenized models for the motion of suspensions with an arbitrary distribution of solid particles in a fluid. In particular, the results presented establish that depending on the concentration of the solid phase of the mixture, the motion of suspensions can occur in two qualitatively different modes: that of frozen or of filtering particles.

This book, one of the first mathematically rigorous treatises on suspensions from the viewpoint of homogenization theory, will be useful to graduate students and researchers in applied analysis and partial differential equations as well as physicists and engineers interested in the theory of complex fluids with microstructure.

A publication of the European Mathematical Society (EMS). Distributed within the Americas by the American Mathematical Society.

EMS Tracts in Mathematics, Volume 34
November 2021, 288 pages, Hardcover, ISBN: 978-3-98547-009-9, 2010 *Mathematics Subject Classification*: 35Q35, 76M50, 35B27, 76T20, List US$69, AMS members US$55.20, Order code EMSTM/34

`bookstore.ams.org/emstm-34`

Meetings & Conferences of the AMS

March Table of Contents

The Meetings and Conferences section of the *Notices* gives information on all AMS meetings and conferences approved by press time for this issue. Please refer to the page numbers cited on this page for more detailed information on each event. *Paid meeting registration is required to submit an abstract to a sectional meeting.*

Invited Speakers and Special Sessions are listed as soon as they are approved by the cognizant program committee; the codes listed are needed for electronic abstract submission. For some meetings the list may be incomplete. Information in this issue may be dated.

The most up-to-date meeting and conference information can be found online at `www.ams.org/meetings`.

Important Information About AMS Meetings: Potential organizers, speakers, and hosts should refer to `https://www.ams.org/meetings/meetings-general` for general information regarding participation in AMS meetings and conferences.

Abstracts: Speakers should submit abstracts on the easy-to-use interactive Web form. No knowledge of LaTeX is necessary to submit an electronic form, although those who use LaTeX may submit abstracts with such coding, and all math displays and similarly coded material (such as accent marks in text) must be typeset in LaTeX. Visit `www.ams.org/cgi-bin/abstracts/abstract.pl`. Questions about abstracts may be sent to `abs-info@ams.org`. Close attention should be paid to specified deadlines in this issue. Unfortunately, late abstracts cannot be accommodated.

Associate Secretaries of the AMS

Central Section: Georgia Benkart, University of Wisconsin–Madison, Department of Mathematics, 480 Lincoln Drive, Madison, WI 53706-1388; email: `benkart@math.wisc.edu`; telephone: 608-263-4283.

Eastern Section: Steven H. Weintraub, Department of Mathematics, Lehigh University, Bethlehem, PA 18015-3174; email: `steve.weintraub@lehigh.edu`; telephone: 610-758-3717.

Southeastern Section: Brian D. Boe, Department of Mathematics, University of Georgia, 220 D W Brooks Drive, Athens, GA 30602-7403; email: `brian@math.uga.edu`; telephone: 706-542-2547.

Western Section: Michel L. Lapidus, Department of Mathematics, University of California, Surge Bldg., Riverside, CA 92521-0135; email: `lapidus@math.ucr.edu`; telephone: 951-827-5910.

Meetings in this Issue

Meetings & Conferences of the AMS

IMPORTANT information regarding meetings programs: AMS Sectional Meeting programs do not appear in the print version of the *Notices*. However, comprehensive and continually updated meeting and program information with links to the abstract for each talk can be found on the AMS website. See `https://www.ams.org/meetings`.

Final programs for Sectional Meetings will be archived on the AMS website accessible from the stated URL.

New: Sectional Meetings Require Registration to Submit Abstracts. In an effort to spread the cost of the sectional meetings more equitably among all who attend and hence help keep registration fees low, starting with the 2020 fall sectional meetings, you must be registered for a sectional meeting in order to submit an abstract for that meeting. You will be prompted to register on the Abstracts Submission Page. In the event that your abstract is not accepted or you have to cancel your participation in the program due to unforeseen circumstances, your registration fee will be reimbursed.

Charlottesville, Virginia

University of Virginia

March 11–13, 2022

Friday – Sunday

Meeting #1175

Southeastern Section
Associate Secretary for the AMS: Brian D. Boe

Program first available on AMS website: January 27, 2022
Issue of *Abstracts*: Volume 43, Issue 2

Deadlines

For organizers: Expired
For abstracts: Expired

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Moon Duchin, Tufts University, *Title to be announced* (Einstein Public Lecture in Mathematics).
Laura A Miller, University of North Carolina at Chapel Hill, *Title to be announced*.
Betsy Stovall, University of Wisconsin-Madison, *Title to be announced*.
Yusu Wang, University of California, San Diego, *Topological and geometric analysis of graphs*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Advances in Difference, Differential, Fractional Differential and Dynamic Equations with Applications, **Muhammad Islam** and **Youssef Raffoul**, University of Dayton.

Advances in Infectious Disease Modeling: From Cells to Populations, **Lauren Childs**, **Stanca Ciupe**, and **Omar Saucedo**, Virginia Tech.

Advances in Operator Algebras, **Ben Hayes** and **David Sherman**, University of Virginia.

Algebraic Combinatorics and Category Theory in Topological Data Analysis, **Woojin Kim**, Duke University, **Alex McCleary**, Ohio State University, **Amit Patel**, Colorado State University, and **Facundo Mémoli**, Ohio State University.

Categorical Structures in Hopf Algebras and Representation Theory, **Agustina Czenky**, University of Oregon, **Julia Plavnik**, Indiana University, and **Guillermo Sanmarco**, Universidad Nacional de Córdoba / Iowa State University.

Celebrating Diversity in Mathematics, **Lauren Childs**, Virginia Tech, **Sara Maloni**, University of Virginia, and **Rebecca R.G.**, George Mason University.

Combinatorial Methods in Geometric Group Theory, **Tarik Aougab**, Haverford College, **Marrissa Loving**, Georgia Institute of Technology, and **Priyam Patel**, University of Utah.

Commutative Algebra, **Eloísa Grifo**, University of Nebraska - Lincoln, and **Keri Sather-Wagstaff**, Clemson University.

Curves, Jacobians, and Abelian Varieties, **Andrew Obus**, Baruch College (CUNY), **Tony Shaska**, Oakland University, and **Padmavathi Srinivasan**, University of Georgia.

Homotopy Theory, **Julie Bergner** and **Nick Kuhn**, University of Virginia.

Integrable Probability, **Leonid Petrov**, University of Virginia, and **Axel Saenz**, University of Warwick.

Interactions Between Noncommutative Ring Theory and Algebraic Geometry, **Jason Gaddis**, Miami University (Ohio), and **Robert Won**, George Wasington University.

Knots and Links in Low-Dimensional Topology, **Thomas Mark**, University of Virginia, and **Allison Moore**, University of California Davis.

Knots, Skein Modules and Categorification, **Rhea Palak Bakshi** and **Józef H Przytycki**, George Washington University, **Radmila Sazdanovic**, North Carolina State University, and **Marithania Silvero**, Universidad de Sevilla.

Knot Theory and its Applications, **Hugh Howards** and **Jason Parsley**, Wake Forest University, and **Eric Rawdon**, St Thomas University.

Large Cardinals and Forcing Axioms, **Brent Cody**, Virginia Commonwealth University, and **Victoria Gitman**, City University of New York.

Mathematical Modeling of Problems in Biological Fluid Dynamics, **Laura Miller**, University of North Carolina at Chapel Hill, and **Nick Battista**, The College of New Jersey.

Mathematical String Theory, **Ilarion Melnikov**, James Madison University, **Eric Sharpe**, Virginia Tech, and **Diana Vaman**, University of Virginia.

Multiparameter Persistence in Theory and Practice, **Håvard Bjerkevik**, TU Graz, and **Ezra Miller** and **Margaret Regan**, Duke University.

Probabilistic Methods in Geometry and Analysis, **Fabrice Baudoin** and **Li Chen**, University of Connecticut.

Recent Advances in Graph Theory and Combinatorics, **Neal Bushaw**, Virginia Commonwealth University, and **Martin Rolek** and **Gexin Yu**, College of William and Mary.

Recent Advances in Harmonic Analysis, **Amalia Culiuc**, Amherst College, **Yen Do**, University of Virginia, and **Eyvindur Ari Palsson**, Virginia Tech.

Recent Advances in Mathematical Biology, **Junping Shi**, College of William & Mary, **Zhisheng Shuai**, University of Central Florida, and **Yixiang Wu**, Middle Tennessee State University.

Recent Advances in PDEs and Applications, **Khai Nguyen**, North Carolina State University, and **Loc Nguyen**, University of North Carolina at Charlotte.

Recent Advances on Wave-based Imaging and Inverse Problems, **Yiran Wang**, Emory University, and **Yang Yang**, Michigan State University.

Recent Progress on Singular and Oscillatory Integrals, **Betsy Stovall**, University of Wisconsin-Madison, and **Joris Roos**, University of Massachusetts Lowell.

Representation Theory of Algebraic Groups and Quantum Groups: A Tribute to the Work of Cline, Parshall and Scott (CPS), **Chun-Ju Lai** and **Daniel K. Nakano**, University of Georgia, and **Weiqiang Wang**, University of Virginia.

Representation Theory of Algebras and Related Combinatorics, **Markus Schmidmeier**, Florida Atlantic University, and **Khrystyna Serhiyenko**, University of Kentucky.

Special Sets of Integers in Modern Number Theory, **Cristina Ballantine**, College of the Holy Cross, and **Hester Graves**, Center for the Computing Sciences.

Spectral Theory of Ergodic Quantum Systems, **Rui Han**, Louisiana State University, and **Ilya Kachkovskiy**, Michigan State University.

Structural and Extremal Graph Theory, **Guangming Jing**, Augusta University, **Zhiyu Wang**, Georgia Institute of Technology, and **Xingxing Yu**, Georgia Insitute of Technology.

Tensors and Complexity, **Visu Makam**, Institute for Advanced Study, and **Rafael Oliveira**, University of Waterloo.

The Role of Mathematics in Computer Vision, **Thomas Y. Chen**, Academy for Mathematics, Science, and Engineering.

Topics in Convexity and Probability, **Steven Hoehner**, Longwood University, and **Mark Meckes** and **Elisabeth Werner**, Case Western Reserve University.

Trends in Teichmüller Theory, **Thomas Koberda** and **Sara Maloni**, University of Virginia, and **Giuseppe Martone**, University of Michigan.

Vertex Algebras and Geometry, **Marco Aldi**, Virginia Commonwealth University, **Michael Penn**, Randolph College, and **Nicola Tarasca** and **Juan Villarreal**, Virginia Commonwealth University.

Youth and Enthusiasm in Arithmetic Geometry and Number Theory, **Evangelia Gazaki** and **Ken Ono**, University of Virginia.

Medford, Massachusetts

Tufts University

March 19–20, 2022

Saturday – Sunday

Meeting #1176

Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub

Program first available on AMS website: January 27, 2022
Issue of *Abstracts*: Volume 43, Issue 2

Deadlines

For organizers: Expired
For abstracts: Expired

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Daniela De Silva, Barnard College, Columbia University, *Title to be announced*.

Enrique R. Pujals, Graduate Center, CUNY, *Title to be announced*.

Christopher T Woodward, Rutgers University, New Brunswick, *Title to be announced*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Analysis on Homogeneous Spaces, **Jens Christensen**, Colgate University, **Matthew Dawson**, CIMAT, Mérida, México, and **Fulton Gonzalez**, Tufts University.

Analytic Methods in Arithmetic Statistics, **Robert Hough**, State University of New York at Stony Brook, and **Robert J. Lemke Oliver**, Tufts University.

Automorphisms of Riemann Surfaces, Subgroups of Mapping Class Groups and Related Topics, **S. Allen Broughton**, Rose-Hulman Institute of Technology, **Jen Paulhus**, Grinnell College, and **Aaron Wootton**, University of Portland.

Combinatorial Methods in Commutative Algebra, **Alessandra Costantini**, Oklahoma State University, and **Gabriel Sosa Castillo**, Colgate University.

Crossroads: Ergodic Theory, Harmonic Analysis, and Combinatorics, **Daniel Glasscock**, University of Massachusetts Lowell, **Andreas Koutsogiannis**, Aristotle University of Thessaloniki, Greece, and **Joris Roos**, University of Massachusetts Lowell.

Discrete and Convex Geometry, **Undine Leopold** and **Egon Schulte**, Northeastern University, and **Pablo Soberón**, Baruch College, City University of New York.

Equivariant Cohomology, **Jeffrey D. Carlson**, Imperial College London, and **Loring Tu**, Tufts University.

Gauge Theory, Geometric Analysis, and Low-Dimensional Topology, **Paul Feehan**, Rutgers University, and **Thomas G. Leness**, Florida International University.

Geometric Dynamics and Billiards, **Boris Hasselblatt** and **Zbigniew Nitecki**, Tufts University, and **Kathryn Lindsey**, Boston College.

Homological Methods in Commutative Algebra, **Janet Striuli**, Fairfield University and National Science Foundation, and **Oana Veliche**, Northeastern University.

Inverse Problems and Their Applications, **Youssef Qranfal**, Wentworth Institute of Technology.

Linear Algebraic Groups: their Structure, Representations, and Geometry., **George McNinch**, Tufts University, and **Eric Sommers**, University of Massachusetts.

Macdonald Theory and Beyond: Combinatorics, Geometry, and Integrable Systems, **Daniel Orr**, Virginia Tech, and **Joshua Jeishing Wen**, Northeastern University.

Mathematical Methods for Ecology and Evolution in Structured Populations, **Olivia Chu**, Princeton University, **Daniel Cooney**, University of Pennsylvania, and **Chadi Saad-Roy**, Princeton University.

Mathematical Modeling in Biology and Medicine, **Arkadz Kirshtein**, Tufts University, and **Navid Mohammad Mirzaei**, University of Massachusetts.

Mathematics in Security and Defense, **Lubjana Beshaj** and **Paul Goethals**, United States Military Academy.

Mathematics of Data Science, **Vasileios Maroulas**, University of Tennessee Knoxville, and **James M. Murphy** and **Abiy Tasissa**, Tufts University.

Moduli Spaces in Algebraic and Tropical Geometry, **Ignacio Barros Reyes**, Université Paris-Saclay, France, **Noah Giansiracusa**, Bentley University, and **Montserrat Teixidor i Bigas**, Tufts University.

Quantum Probability, Orthogonal Polynomials, and Special Functions, **Maxim Derevyagin** and **Ambar Sengupta**, University of Connecticut.

Subgroups in Nonpositive Curvature, **Carolyn Abbott**, Brandeis University, and **Ivan Levcovitz**, **Kim Ruane**, **Lorenzo Ruffoni**, and **Genevieve Walsh**, Tufts University.

Symmetries of Polytopes, Maps, and Graphs, **Gabe Cunningham**, University of Massachusetts Boston, and **Mark Mixer**, Wentworth Institute of Technology.

West Lafayette, Indiana

Purdue University

March 26–27, 2022

Saturday – Sunday

Meeting #1177

Central Section
Associate Secretary for the AMS: Georgia Benkart

Program first available on AMS website: February 3, 2022
Issue of *Abstracts*: Volume 43, Issue 2

Deadlines

For organizers: Expired
For abstracts: Expired

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Christine Berkesch, University of Minnesota, *Title to be announced*.

Matthew Edward Hedden, Michigan State University, *Knot theory and complex curves*.

Brian Street, University of Wisconsin-Madison, *Maximal Subellipticity*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.

Algebraic Geometry, **Donu Arapura**, **Deepam Patel**, and **K.V. Shuddhodan**, Purdue University.

Analysis and Probability in Sub-Riemannian Geometry, **Jeremy Tyson**, University of Illinois at Urbana-Champaign, and **Jing Wang**, Purdue University.

Analysis of Nonlinear Evolution Equations, **John Holmes**, Wake Forest University, **Ryan Thompson**, The University of North Georgia, and **Feride Tiğlay**, The Ohio State University.

Analytical, Computational, and Data-Driven Approaches in Fluid Dynamics, **Aseel Farhat**, Florida State University, **Vincent Martinez**, CUNY Hunter College, and **Ali Pakzad**, Indiana University Bloomington.

A Women in Analysis Research Network Event, **Donatella Danielli**, Arizona State University, and **Irina Mitrea**, Temple University.

Combinatorial Algebra and Geometry, **Christine Berkesch**, University of Minnesota, and **Laura Matusevich** and **Aleksandra Sobieska**, Texas A&M University.

Combinatorial Techniques in Commutative Algebra, **Giulio Caviglia**, Purdue University, and **Jay Schweig**, Oklahoma State University.

Combinatorics and Representations of Noncommutative Algebras, **Jason Gaddis**, Miami University, and **Daniele Rosso**, Indiana University Northwest.

Commutative Algebra and Connections with Algebraic Geometry, **Claudia Polini**, University of Notre Dame, and **Bernd Ulrich**, Purdue University.

Complex Geometry, **Laszlo Lempert**, Purdue University, **Chi Li**, Rutgers University, **Sai-Kee Yeung**, Purdue University, and **Yuan Yuan**, Syracuse University.

Computational and Applied Algebraic Geometry, **Taylor Brysiewicz** and **Parker Edwards**, University of Notre Dame.

Fully Nonlinear Partial Differential Equations, **Farhan Abedin**, University of Utah, and **Fernando Charro**, Wayne State University.

Gaussian and non-Gaussian Stochastic Analysis, **Cheng Ouyang**, University of Illinois at Chicago, **Takashi Owada**, Purdue Univeristy, and **Samy Tindel**, Purdue University.

Geometric Topology in the Middle Dimensions, **James F. Davis**, Indiana University, and **Mark Powell**, Durham University.

Geometry of Measures and Metric Spaces, **Matthew Badger**, University of Connecticut, **Guy C. David**, Ball State University, and **Lisa Naples**, Macalester College.

Group Theory and Logic, **Meng-Che "Turbo" Ho**, California State University, Northridge, **Julia F. Knight**, University of Notre Dame, and **D.B. McReynolds** and **Thomas Sinclair**, Purdue University.

Harmonic Analysis, **Shaoming Guo** and **Brian Street**, University of Wisconsin-Madison.

Higher Structures in Topology, Geometry and Physics, **Ralph Kaufmann**, Purdue University, **Martin Markl**, Czech Academy of Sciences, and **Alexander Voronov**, University of Minnesota.

Integrability, Symmetry and Physics, **E. Birgit Kaufmann** and **Oleksandr Tsymbaliuk**, Purdue University.

Low-dimensional Topology, **Matthew Hedden**, Michigan State University, **Juanita Pinzón-Caicedo**, University of Notre Dame, and **Lev Tovstopyat-Nelip**, Michigan State University.

Mathematical Foundation of Data Science in Scientific Computing, **Senwei Liang**, Purdue University, **Lizuo Liu**, Southern Methodist University, and **Haizhao Yang**, Purdue University.

Mathematical Methods for Inverse Problems, **Isaac Harris** and **Peijun Li**, Purdue University.

Mathematics of Complex Systems in Biology, **Alexandria Volkening** and **Ning Wei**, Purdue University.

Modeling and Forecasting Complex Turbulent Systems, **Nan Chen**, University of Wisconsin-Madison, and **Di Qi**, Purdue University.

Model Theory and its Applications, **Saugata Basu**, Purdue University, **Philipp Hieronymi**, University of Bonn, and **Margaret E. M. Thomas**, Purdue University.

Multiplicative Ideal Theory in Honor of the Career of William Heinzer, **Evan Houston**, University of North Carolina - Charlotte, and **Alan Loper**, Ohio State University.

Nonlinear Algebra with Applications to Statistics, **Aida Maraj**, University of Michigan, and **Sonja Petrović**, Illinois Institute of Technology.

Nonlinear Partial Differential Equations From Variational Problems and Complex Fluids, **Tao Huang**, Wayne State University, and **Changyou Wang**, Purdue University.

Numerical Linear Algebra, **Lothar Reichel**, Kent State University, **Jianlin Xia**, Purdue University, and **Qiang Ye**, University of Kentucky.

Optimization, Complexity, and Real Algebraic Geometry, **Saugata Basu** and **Ali Mohammad Nezhad**, Purdue University.

Quantum Algebra and Quantum Topology, **Shawn Cui**, Purdue University, **Julia Plavnik**, Indiana University, and **Tian Yang**, Texas A&M University.

Random Growth Models, **Christopher Janjigian**, Purdue University, **Firas Rassoul-Agha**, University of Utah, and **Timo Seppalainen**, University of Wisconsin - Madison.

Recent Developments in Automorphic Forms and Representations of p-adic Groups, **David Goldberg**, **Baiying Liu**, and **Freydoon Shahidi**, Purdue University.

Recent Developments in Commutative Algebra, **Jennifer Kenkel**, University of Michigan, and **Linquan Ma** and **Uli Walther**, Purdue University.

Recent Developments in High Order Numerical Methods for Partial Differential Equations, **Zheng Sun**, The University of Alabama.

Recent Developments in Operator Algebras, **Roy Araiza**, University of Illinois, and **Rolando de Santiago**, **Thomas Sinclair**, and **Andrew Toms**, Purdue University.

Recent Developments of Variational Methods in Deterministic and Stochastic Systems, **Yuan Gao**, Purdue University, **Tao Luo**, Shanghai Jiao Tong University, and **Nung Kwan Yip**, Purdue University.

Recent Progress of Efficient and Robust Schemes for Compressible Navier-Stokes Equations, **Chen Liu** and **Xiangxiong Zhang**, Purdue University.

Recent Trends in Graph Theory, **Adam Blumenthal**, Westminster College, and **Katherine Perry**, Soka University of America.

Spectral Estimation and Optimization, **Mark Ashbaugh**, University of Missouri, and **Richard Laugesen**, University of Illinois.

Stability in Topology, Arithmetic, and Representation Theory, **Jeremy Miller**, Purdue University, **Peter Patzt**, University of Oklahoma, and **Andrew Putman**, University of Notre Dame.

The Interface Between Nonlinear PDEs, Harmonic Analysis, and Quantitative Geometric and Functional Inequalities, **Emanuel Indrei** and **Victor Lie**, Purdue University.

The Interface of Harmonic Analysis and Analytic Number Theory, **Theresa Anderson**, Purdue University, **Robert Lemke Oliver**, Tufts University, and **Eyvindur Palsson**, Virginia Tech University.

Topics in Algebraic and Geometric Topology, **David Ben McReynolds** and **Sam Nariman**, Purdue University.

Virtual JMM 2022

Now meeting virtually, PDT (hosted by the American Mathematical Society)

April 6–9, 2022

Wednesday – Saturday

Meeting #1174

This meeting includes the annual meetings of the AMS, Association for Women in Mathematics (AWM), and National Association of Mathematicians (NAM), winter meeting of Association for Symbolic Logic (ASL), and sessions/events by them and Society for Industrial and Applied Mathematics (SIAM), American Statistical Association (ASA), Consortium for Mathematics and its Applications (COMAP), International Linear Algebra Society (ILAS), Julia Robinson Mathematics Festival (JRMF), Mathematical Sciences Research Institute (MSRI), Spectra, and Transforming Post-Secondary Education in Mathematics (TPSE).

Associate Secretary for the AMS: Georgia Benkart, University of Wisconsin-Madison

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: Expired

For abstracts: Expired

The 2022 JMM will take place virtually April 6–9. Program subject to changes due to the change to a virtual format. Watch `https://www.jointmathematicsmeetings.org/jmm` for further details.

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/national.html`.

Joint Invited Addresses

Marianna Csőrnyei, University of Chicago, *The Kakeya Needle Problem for Rectifiable Sets* (AWM-AMS Noether Lecture).

Dave Kung, Charles A. Dana Center, The University of Texas at Austin, *Why the Math Community Struggles with Equity & Diversity - and Why There's Reason for Hope* (MAA-SIAM-AMS Hrabowski-Gates-Tapia-McBay Lecture).

Kavita Ramanan, Brown University, *Interacting Stochastic Processes on Random Graphs* (AAAS-AMS Invited Address).

Lauren K Williams, Harvard University, *The positive Grassmannian and amplituhedron* (MAA-AMS-SIAM Gerald and Judith Porter Public Lecture).

Talithia Ann Williams, Harvey Mudd College, *The Power of Talk: Engaging the Public in Mathematics* (JPBM Communications Award Lecture).

AMS Invited Addresses

Anna Gilbert, Yale University, *Metric representations: Algorithms and Geometry* (von Neumann Lecture).

Tyler J. Jarvis, Brigham Young University, *Restoring confidence in the value of mathematics* (AMS Lecture on Education).

Daniel Reuben Krashen, University of Pennsylvania, *Field patching and algebraic structures*.

Dan Margalit, Georgia Institute of Technology, *Mixing surfaces, algebra, and geometry* (AMS Maryam Mirzakhani Lecture).

Gaston Mandata N'Guerekata, Morgan State University, *An invitation to periodicity*.

Hee Oh, Yale University, *Euclidean lines on hyperbolic manifolds* (AMS Erdős Memorial Lecture).

Jill Pipher, Brown University, *Regularity of solutions to elliptic operators and elliptic systems* (AMS Retiring Presidential Address).

Karen Smith, University of Michigan, *Resolutions of Singularities and Rational Singularities* (AMS Colloquium Lectures: Lecture I).

Karen Smith, University of Michigan, *Measuring Singularities* (AMS Colloquium Lectures: Lecture II).

Karen Smith, University of Michigan, *Extremal Singularities* (AMS Colloquium Lectures: Lecture III).

Eitan Tadmor, University of Maryland, *Emergent behavior in collective dynamics* (AMS Josiah Willard Gibbs Lecture).

Invited Addresses of Other JMM Partners

Jeremy Avigad, Carnegie Mellon University, *The promise of formal mathematics* (ASL Invited Address).

Robert Q. Berry, III, University of Virginia, *Interest Convergence: An analytical viewpoint for examining how power dictates policies and reforms in mathematics* (NAM Cox-Talbot Address).

Peter Cholak, University of Notre Dame, *Ramsey like theorems on the rationals* (ASL Invited Address).

Pauline van den Driessche, University of Victoria, B.C., Canada, *Sign Patterns Meet Dynamical Systems* (ILAS Invited Address).

Qiang Du, Columbia University, *Analysis and Applications of Nonlocal Models* (SIAM Invited Address).

Monica Jackson, American University, *Spatial data analysis for public health data* (NAM Claytor-Woodard Lecture).

Franziska Jahnke, University of Münster, *Decidability and definability in unramified henselian valued fields* (ASL Invited Address).

Autumn Kent, University of Wisconsin - Madison, *Families* (Spectra Lavender Lecture).

Xihong Lin, Harvard University, Broad Institute of MIT and Harvard, *Learning from COVID-19 Data on Transmission, Health Outcomes, Interventions and Vaccination* (ASA Committee of Presidents of Statistical Societies Lecture).

Sandra Müller, Technical University of Vienna, *Lower Bounds in Set Theory* (ASL Invited Address).

Lynn Scow, California State San Bernardino, *Semi-retractions and the Ramsey Property* (ASL Invited Address).

Erik Walsberg, University of California Irvine, *Model theory of large fields* (ASL Invited Address).

Invited Addresses of Other Organizations

Karl-Dieter Crisman, Gordon College, *Mersenne Matters: Mathematics, Music, Monotheism, and More* (ACMS Guest Speaker).

Nicolas Fillion, Simon Fraser University, *Trust but Verify: What Can We Know About the Reliability of a Computer-Generated Result?* (SIGMAA on the Philosophy of Mathematics (POM SIGMAA) Guest Lecture).

Edray Herber Goins, Pomona College, *Addressing Anti-Black Racism in Our Departments* (Project NExT Lecture on Teaching and Learning).

Heather Price, North Seattle College, *Climate Justice Integrated Learning in STEM* (SIGMAA Environmental Mathematics Guest Speaker).

Adrian Rice, Randolph-Macon College, *Beyond the Strength of a Woman's Physical Power: Mathematics, Machines, and the Mind of Ada Lovelace* (SIGMAA on the History of Mathematics (HOM SIGMAA) Guest Speaker).

AMS Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://jointmathematicsmeetings.org/meetings/abstracts/abstract.pl?type=jmm`.

Some sessions are cosponsored with other organizations. These are noted within the parenthesis at the end of each listing, where applicable.

Abraham Robinson's Nonstandard Methods in Mathematics and Its Applications, **Matt Insall**, Missouri University of Science and Technology, **Peter Loeb**, University of Illinois at Urbana-Champaign, and **Malgorzata Marciniak**, City University of New York.

Advances in Coding Theory, **Katie Haymaker**, Villanova University, **Hiram Lopez**, Cleveland State University, and **Beth Malmskog**, Colorado College.

Advances in Operator Algebras, **Rolando de Santiago**, Purdue University, **Adam Fuller**, Ohio University, **Lara Ismert**, Embry-Riddle Aeronautical University, and **Pieter Spaas**, University of California, Los Angeles.

Advancing Data Privacy-Preserving Methodologies, **Claire Bowen**, Urban Institute.

Algebraic and Bijective Methods in Permutation Enumeration, **Sergi Elizalde**, Dartmouth College, **Bridget Tenner**, DePaul University, and **Justin Troyka** and **Yan Zhuang**, Davidson College.

A Match Made in the Stacks: Mathematician and Librarian Collaborations, **Anya Bartelmann**, Princeton University, and **Samuel Hansen**, University of Michigan.

Analysis and Differential Equations at Undergraduate Institutions, **John Ross**, Southwestern University, **Mihai Stoiciu**, Williams College, and **Scott Zimmerman**, The Ohio State University at Marion.

Analysis in Metric Spaces (a Mathematics Research Communities Session), **Chris Gartland**, Texas A & M University, **Silvia Ghinassi**, University of Washington, **Ilmari Kangasniemi**, Syracuse University, and **Ryan Alvarado**, Amherst College.

Analysis of and Recent Advances in Difference, Differential and Dynamic Equations with Applications, **Raegan Higgins** and **Ozkan Ozturk**, Texas Tech University.

Applications of Mathematical Models and Dynamical Systems in Biology, **Yang Li**, University of Cincinnati, **Hongying Shu**, Shaanxi Normal University, and **Xiang-Sheng Wang**, University of Louisiana at Lafayette.

Applied Combinatorial Methods, **Sinan Aksoy**, Pacific Northwest National Laboratory, **Bill Kay**, Oak Ridge National Laboratory, and **Stephen Young**, Pacific Northwest National Laboratory.

A Showcase of Number Theory at Undergraduate Institutions, **Ricardo Conceicao**, Gettysburg College, **Lindsay Dever**, Bryn Mawr College, and **Eva Goedhart**, Williams College.

Asymptotic Behavior of Evolution Equations, **Jin Liang**, Shanghai Jiao Tong University, **Nguyen Minh**, University of Arkansas Little Rock, **Gaston N'Guerekata**, Morgan State University, and **Ti-Jun Xiao**, Fudan University.

Bifurcations of Difference Equations and Discrete-time Competitive and Cooperative Population Models, **Arzu Bilgin**, Recep Tayyip Erdogan University, and **Toufik Khyat**, Texas Tech University.

Collaborative Undergraduate Research: Experiences with CURM, **Kathryn Leonard**, Occidental College.

Combinatorial Applications of Computational Geometry and Algebraic Topology (a Mathematics Research Communities Session), **Stephen Gillen**, University of Pennsylvania, and **Sam Simon**, Simon Fraser University.

Combinatorial Approaches to Topological Structures and Applications, **Emilie Purvine** and **Cliff Joslyn**, Pacific Northwest National Laboratory.

Commutative Algebra, **Eloisa Grifo**, University of Nebraska-Lincoln, **Keri Sather-Wagstaff**, Clemson University, and **Janet Vassilev**, University of New Mexico.

Competing Foundations for Mathematics: How Do We Choose? (Sponsored by POMSIGMAA), **Jeff Buechner**, Rutgers University, **Bonnie Gold**, Monmouth University, and **Kevin Iga**, Pepperdine University.

Complex Adaptive Systems and Evolutionary Models in Biology and Psychology, **Jun Chen**, **Yun Kang**, **M. Gabriela Navas-Zuloaga**, and **Lucero Rodriguez**, Arizona State University.

Current Advances in Computational Biomedicine, **Heiko Enderling**, H. Lee Moffitt Cancer Center & Research Institute, **Niels Halama**, German Cancer Research Center, **Viviana Risca**, Rockefeller University, and **Nek Valous**, National Center for Tumor Diseases.

Distance Problems in Continuous Discrete and Finite Field Settings, **Abdul Basit**, Iowa State University, **Steven Miller**, Williams College, **Eyvindur Palsson** and **Sean Sovine**, Virginia Tech, and **Charles Wolf**, University of Rochester.

Dynamics of Infectious Diseases: Ecological Models Across Multiple Scales (a Mathematics Research Communities Session), **George Lytle**, University of Montevallo, and **Zhuolin Qu**, University of Texas, San Antonio.

Early Career Number Theory Research with Combinatorics, Modular Forms, and Basic Hypergeometric Series, **Christopher Jennings-Shaffer**, University of Denver, and **Ali Uncu**, University of Bath.

Engaging Students Through Modeling Hands-on Projects and Innovative Exploratory Approaches, **Rachel Grotheer**, Wofford College, **Joel Kitty**, Centre College, **Alison Marr**, Southwestern University, **Alex McAllister**, Centre College, and **Stephen Walk**, St. Cloud State University.

Evolution Equations and Their Asymptotic Behavior, **Gisele Mophou**, Universite des Antilles en Guadeloupe, **Gaston N'Guerekata**, Morgan State University, and **Mahamadi Warma**, George Mason University.

Explicit Methods for Modularity, I (Sponsored by Simons Collaboration on Arithmetic Geometry Number Theory and Computation), **Eran Assaf**, Dartmouth, **Edgar Costa**, Massachusetts Institute of Technology, **Brendan Hassett**, Brown University, and **David Roe**, Massachusetts Institute of Technology.

Finding Needles in Haystacks: Approaches to Inverse Problems Using Combinatorics and Linear Algebra (a Mathematics Research Communities Session), **Shahla Nasserasr**, Rochester Institute of Technology, **Emily Olson**, Millikin University, and **Sam Spiro**, University of California San Diego.

Fusion Categories and Their Applications in Physics, **Colleen Delaney**, Indiana University, and **Corey Jones**, North Carolina State University.

Geometric and Topological Combinatorics, **Anton Dochtermann**, Texas State University, **Bennet Goeckner** and **Gaku Liu**, University of Washington, and **Steven Klee**, Seattle University.

Geometric Group Theory, I (Associated with AMS Maryam Mirzakhani Invited Address), **Carolyn Abbott**, Brandeis University, **Mladen Bestvina**, University of Utah, and **Dan Margalit**, Georgia Tech University.

Geometric Measure Theory, **Theodora Bourni** and **Vyron Vellis**, University of Tennessee, Knoxville.

Geometry in the Mathematics of Data Science, **Tim Doster**, **Tegan Emerson**, and **Henry Kvinge**, Pacific Northwest National Laboratory.

Heat Content Exit Time and Geometric Analysis, **Patrick McDonald**, New College of Florida, and **Jeffrey Langford**, Bucknell University.

History of Mathematics, **Sloan Despeaux**, Western Carolina University, **Deborah Kent**, University of St. Andrews, **Jemma Lorenat**, Pitzer College, and **Daniel Otero**, Xavier University.

Hopf Algebras and Tensor Categories, **Siu-Hung Ng**, Louisiana State University, **Julia Plavnik**, Indiana University, and **Henry Tucker**, University of California, Riverside.

If You Build It They Will Come: Presentations by Scholars in the National Alliance for Doctoral Studies in the Mathematical Sciences, **David Goldberg**, Purdue University, and **Phil Kutzko**, University of Iowa.

Innovative and Effective Ways to Teach Linear Algebra, **Sepideh Stewart**, University of Oklahoma, **Gil Strang**, Massachusetts Institute of Technology, **David Strong**, Pepperdine University, and **Megan Wawro**, Virginia Tech.

Inquiry-based Teaching and Learning, **Volker Ecke**, Westfield State University, **Parker Glynn-Adey**, University of Toronto at Scarborough, **Mel Henriksen**, Wentworth Institute of Technology, **Nathaniel Miller**, University of Northern Colorado, **Lee Roberson**, University of Colorado-Boulder, **Christine von Renesse**, Westfield State University, **Mami Wentworth**, Wentworth Institute of Technology, and **Nina White**, University of Michigan.

Intersections of Geometric Analysis and Mathematical Physics, **Xianzhe Dai** and **A'kos Nagy**, University of California, Santa Barbara.

Knots, Links, 3-manifolds,... and 4-manifolds, **Christopher Davis**, University of Wisconsin, **Shelly Harvey**, Rice University, and **Carolyn Otto**, University of Wisconsin Eau Claire.

Knot Theory in Dimension Four, **Jeffrey Meier**, Western Washington University, **Maggie Miller**, Stanford University, and **Patrick Naylor**, Princeton University.

Latinxs in Combinatorics, **Laura Escobar**, Washington University in St. Louis, **Pamela E. Harris**, Williams College, and **Andres R. Vindas Melendez**, MSRI & UC Berkeley.

Little School Dynamics: Cool Research at Primarily Undergraduate Institutions, **Kimberly Ayers**, Carroll College, **Han Li**, Wesleyan University, **David McClendon**, Ferris State University, **Andy Parrish**, Eastern Illinois University, and **Ami Radunskaya**, Pomona College.

Low-dimensional Manifolds, **Catherine Pfaff**, Queen's University, **Rachel Roberts**, Washington University in St Louis, and **Jennifer Schultens**, University of California, Davis.

Mathematical and Conceptual Foundations of Physics, **David Weisbart**, University of California Riverside, and **Adam Yassine**, Bowdoin College.

Mathematical Modeling of Biological Processes, **Dawit Denu**, Georgia Southern University, **Sedar Ngoma**, SUNY Geneseo, and **Rachidi Salako**, The Ohio State University.

Mathematical Modeling of Population Dynamics Across Scales: From Immuno-epidemiology to Multilevel Selection, **Daniel Cooney**, University of Pennsylvania, and **Chadi Saad-Roy**, Princeton University.

Mathematical Models for Biomolecular and Cellular Interactions, **Daniel Cruz**, Georgia Institute of Technology, and **Margherita Ferrari**, University of South Florida.

Mathematical Models of Diseases: Analysis and Computation, **Xuming Xie** and **Najat Ziyadi**, Morgan State University.

Mathematical Tools for Computer Vision Problems, **Anna Grim**, Brown University, **Patricia Medina**, Yeshiva College, and **Marilyn Vazquez**, Ohio State University.

Mathematics and New Media, **Mohamed Omar**, Harvey Mudd College, and **Michael Penn**, Randolph College.

Mathematics and Sports, **Russell Goodman**, Central College, and **Hope McIlwain**, Mercer University.

Mathematics and the Arts, **Karl Kattchee**, University of Wisconsin-La Crosse, **Doug Norton**, Villanova University, and **Anil Venkatesh**, Adelphi University.

Mathematics Through the Informational Lens, **Chid Apte**, **Rachel Bellamy**, **Charles Bennett**, **Kenneth Clarkson**, **John Cohn**, **Payel Das**, **Lior Horesh**, **Jon Lenchner**, **JR Rao**, **John Smolin**, **Mark Squillante**, **Yuhai Tu**, and **Chai Wah Wu**, IBM Research.

Modular Forms and Combinatorics, **Madeline Dawsey**, University of Texas at Tyler, **Larry Rolen**, Vanderbilt University, **Robert Schneider**, University of Georgia, and **Ian Wagner**, Vanderbilt University.

New Problems in Several Complex Variables (a Mathematics Research Communities Session), **Sean Curry**, Oklahoma State University, **Zhenghui Huo**, Duke Kunshan University, **Valentin Kunz**, University of Manchester, and **Kevin Palencia Infante**, Northern Illinois University.

Noncommutative Algebra and Noncommutative Invariant Theory, **Ellen Kirkman**, Wake Forest University, and **Robert Won** and **James Zhang**, University of Washington.

Nonlinear Evolution Equations Stability and Long Time Behavior of Solutions, **Ezzinbi Khalil**, and **Gaston N'Guerekata**, Morgan State University.

Number Theory at Non-PhD Granting Institutions, **Harris Daniels**, Amherst College, **Alia Hamieh**, University of Northern British Columbia, **Steven Miller**, Williams College, **Naomi Tanabe**, Bowdoin College, and **Enrique Trevino**, Lake Forest College.

Numerical Methods and Deep Learning for PDEs, **Wei Guo**, Texas Tech University, and **Chunmei Wang**, University of Florida.

Partial Differential Equations and Complex Variables, **Hyunkyoung Kwon**, University at Albany, and **Bingyuan Liu**, The University of Texas Rio Grande Valley.

Partition Theory and Related Topics, **Dennis Eichhorn**, University of California, Irvine, **William Keith**, Michigan Technological University, and **Brandt Kronholm**, University of Texas, Rio Grande Valley.

Perfectoid Spaces, **Shanna Dobson**, California State University, Los Angeles.

Polymath Jr: Mentoring and Learning, **Kira Adaricheva**, Hofstra University, **Zhanar Berikkyzy**, Fairfeld University, **Johanna Franklin**, Hofstra University, **Seoyoung Kim**, Queens University, **Steven Miller**, Williams College, **Adam Sheffer**, Baruch College, and **Yunus Zeytuncu**, University of Michigan-Dearborn.

Presenting Research Mathematics Through Visual Storytelling: Slides Without Words and Equations, **Henry Adams**, **Justin O'Connor**, **Kyle Salois**, **Brittany Story**, and **Ciera Street**, Colorado State University.

Quadratic Forms, Theta Functions and Modularity, **Allison Arnold-Roksandich**, United States Department of Defense, **Gene Kopp**, Purdue University, and **Kate Thompson**, United States Naval Academy.

Quantitative Literacy and Society, **Mark Branson**, Stevenson University, **Catherine Crockett**, Point Loma Nazarene University, **Gizem Karaali**, Pomona College, **Kathryn Knowles**, Texas A&M-San Antonio, and **Samuel Tunstall**, Trinity University, San Antonio TX.

Quantum Categorical Structures in Mirror Symmetry, **Nathaniel Bottman**, Max Planck Institute for Mathematics, **Sheel Ganatra**, University of Southern California, **Alexei Oblomkov**, University of Massachusetts, Amherst, and **Abigail Ward**, Massachusetts Institute of Technology.

Quaternions, **Terrence Blackman**, Medgar Evers College - City University of New York, and **Johannes Familton** and **Chris McCarthy**, Borough of Manhattan Community College - City University of New York.

Random Matrix Theory and its Applications, **Kyle Luh** and **Sean O'Rourke**, University of Colorado Boulder, and **Tom Trogdon**, University of Washington.

Random Polynomials and Related Models, **Sean O'Rourke**, University of Colorado Boulder, and **Noah Williams**, Appalachian State University.

Reaction Diffusion Models with Applications in Spatial Ecology, **Jerome Goddard II**, Auburn University Montgomery, and **Ratnasingham Shivaji**, University of North Carolina Greensboro.

Real World Applications of Mathematics, **Vinodh Chellamuthu**, Dixie State University, and **Darren Narayan**, Rochester Institute of Technology.

Recent Advances in Fluids and Related Models, **Theodore Drivas**, Stony Brook, and **Hussain Ibdah** and **Huy Nguyen**, University of Maryland.

Recent Advances in Mathematical Biology Ecology and Epidemiology, **Lale Asik**, University of the Incarnate Word, and **Ummugul Bulut**, Texas A&M University San Antonio.

Recent Advances in Packing, **Joseph Iverson**, Iowa State University, **John Jasper**, South Dakota State University, and **Dustin Mixon**, The Ohio State University.

Recent Developments in Nonlocal Modeling and Analysis, **James Scott**, University of Pittsburgh, **Tadele Mengesha**, University of Tennessee, and **Xiaochuan Tian**, University of California, San Diego.

Recent Progress in Function Theory and Operator Theory, **Alberto Condori**, Florida Gulf Coast University, **Elodie Pozzi**, St Louis University, **William Ross**, University of Richmond, and **Alan Sola**, Stockholm University.

Research in Mathematics by Undergraduates and Students in Post-baccalaureate Programs, **Darren Narayan**, Rochester Institute of Technology, **Christopher O'Neill**, San Diego State University, **Khang Tran**, California State University, Fresno, **Mark Daniel Ward**, Purdue University, and **John Wierman**, Johns Hopkins University (AMS-SIAM).

Rethinking Number Theory, **Heidi Goodson**, Brooklyn College City University of New York, **Allechar Serrano Lopez**, Harvard University, **Christelle Vincent**, University of Vermont, and **McKenzie West**, University of Wisconsin-Eau Claire.

Scalar Curvature and Convergence, **Brian Allen**, University of Hartford, **Lan-Hsuan Huang**, University of Connecticut, and **Raquel Perales**, Universidad Nacional Autonoma de Mexico.

Several Complex Variables Geometric PDE and CR Geometry, **Anne-Katrin Gallagher**, Gallagher Tool & Instrument, Redmond, WA, and **Bernhard Lamel** and **Nordine Mir**, Texas A&M University at Qatar.

Skein Theory and Quantum Algebra, **Rhea Bakshi**, The George Washington University, **Wade Bloomquist**, Georgia Institute of Technology, and **Vijay Higgins**, University of California Santa Barbara.

Statistics and Machine Learning Using Topology and Geometry, **Austin Lawson** and **Vasileios Maroulas**, University of Tennessee Knoxville, **Farzana Nasrin**, University of Hawaii at Manoa, and **Christopher Oballe**, University of Notre Dame.

Stochastic Models in Studying Biological Systems, **Shusen Pu**, Vanderbilt University, and **Alexander Strang**, University of Chicago.

Structured Polynomial Systems In Mathematics and Its Applications, **Taylor Brysiewicz**, Max Planck Institute for Mathematics in the Sciences, and **Frank Sottile**, Texas A&M University.

The Mathematics of Decisions, Elections and Games, **Michael Jones**, American Mathematical Society - Mathematical Reviews, **David McCune**, William Jewell College, and **Jennifer Wilson**, Eugene Lang College The New School.

The Mathematics of RNA and DNA, **Johannes Familton** and **Chris McCarthy**, Borough of Manhattan Community College City University of New York.

The Teaching and Learning of Undergraduate Ordinary Differential Equations, **Chris Goodrich**, The University of New South Wales, **Viktoria Savatorova**, Central Connecticut State University, **Itai Seggev**, Wolfram Research, and **Beverly West**, Cornell University.

Topics and Generalizations in Geometric Group Theory, **John Bergschneider**, **Bikash Das**, and **Opal Graham**, University of North Georgia.

Topics in Extremal Combinatorics, **Cory Palmer**, University of Montana, and **Amites Sarkar**, Western Washington University.

Transient Probabilities of Random Processes, Duality Theory and Gambler's Ruin Probabilities, **Alan Krinik** and **Randall Swift**, Cal Poly Pomona.

Undergraduate Research Activities in Mathematical and Computational Biology, **Timothy Comar**, Benedictine University, and **Hannah Highlander**, University of Portland.

Weave Reality into Your Differential Equations Course with Modeling, **Vinodh Chellamuthu**, Dixie State University, **Rikki Wagstrom**, Metropolitan State University, **Tracy Weyand**, Rose-Hulman Institute of Technology, and **Brian Winkel**, SIMIODE.

AAAS Special Sessions

Stochastic Processes on Networks, **Oanh Nguyen**, University of Illinois at Urbana-Champaign, and **Kavita Ramanan**, Brown University.

ASA Special Sessions

Statistical issues of COVID-19 Data, **Xihong Lin**, Harvard University and Broad Institute of MIT.

ASL Special Sessions

Model-theoretic Classification Program, **Artem Chernikov** and **Nicholas Ramsey**, University of California, Los Angeles.

AWM Special Sessions

Celebrating the Mathematical Contributions of the AWM, **Donatella Danielli**, Arizona State University, **Kathryn Leonard**, Occidental College, **Michelle Manes**, University of Hawaii at Manoa, and **Ami Radunskaya**, Pomona College.

Mathematics in the Literary Arts and Pedagogy in Creative Settings, **Shanna Dobson**, California State University, Los Angeles, and **Elizabeth Donovan**, Murray State University.

Women and Gender Minorities in Symplectic and Contact Geometry and Topology, **Orsola Capovilla-Searle**, Duke University, **Dahye Cho**, Stony Brook University, and **Angela Wu**, University of College, London.

Women in Computational Topology, **Brittany Fasy**, Montana State University, and **Lori Ziegelmeier**, Macalester College.

Women in Geometry, **Catherine Searle**, Wichita State University, **Elizabeth Stanhope**, Lewis and Clark University, and **Guofang Wei**, University of California, Santa Barbara.

Women in Mathematical Biology, **Christina Edholm**, Scripps College, **Maryann Hohn**, Pomona College, **Amanda Laubmeier**, Texas Tech University, **Carrie Manore**, Los Alamos National Laboratory, and **Heather Zinn-Brooks**, Harvey Mudd College.

Women in Topology, **Kristine Bauer**, University of Calgary, **Anna Marie Bohmann**, Vanderbilt University, **Angelica Osorno**, Reed College, **Carmen Rovi**, MPIM and University of Heidelberg, and **Sarah Yeakel**, University of California, Riverside.

Women of Color in Combinatorics, **Zhanar Berikkyzy**, Fairfield University, and **Shanise Walker**, University of Wisconsin Eau Claire.

COMAP Special Sessions

COMAP's Mathematical Modeling Contests: Sharing Experiences and Benefits, **Amanda Beecher**, Ramapo College of New Jersey, **Steve Horton**, US Military Academy (Emeritus), and **Kathleen Snook**, COMAP.

ILAS Special Sessions

Matrix Analysis and Applications I, **Mohsen Aliabadi**, Iowa State University, and **Luyining Gan** and **Tin-Yau Tam**, University of Nevada, Reno.

The Interplay of Matrix Analysis and Operator Theory, **Kelly Bickel**, Bucknell University, **Meredith Sargent**, University of Arkansas, **Ryan Tully-Doyle**, California Polytechnic, San Luis Obispo, and **Hugo Woerdeman**, Drexel University.

The Inverse Eigenvalue Problem for a Graph, Zero Forcing, Throttling and Related Topics, **Mary Flagg**, University of St. Thomas, and **Hein Van der Holst**, Georgia State University.

MSRI Special Sessions

Combinatorial and Homological Methods in Commutative Algebra, **Jennifer Biermann**, Hobart and William Smith Colleges, and **Selvi Kara**, University of Utah.

Frame Theory and Applications, **Roza Aceska**, Ball State University, and **Yeon Kim**, Central Michigan University.

Lie Group Actions in Differential Geometry, **Carolyn Gordon**, Dartmouth College, **Meera Mainkar**, Central Michigan University, **Tracy Payne**, Idaho State University, and **Cynthia Will**, University of Cordoba (Argentina).

Metric Geometry and Topology, **Christine Escher**, Oregon State University, and **Catherine Searle**, Wichita State University.

Resistance Distance and Other Metrics on Graphs and Networks, **Emily Evans**, Brigham Young University, and **Amanda Francis**, Mathematical Reviews, American Mathematical Society.

Tensor Modeling and Optimization, **Anna Ma**, University of California, Irvine, **Deanna Needell**, University of California, Los Angeles, and **Jing Qin**, University of Kentucky.

The MSRI African Diaspora Joint Mathematics Workshop (ADJOINT), **Caleb Ashley**, Boston College, and **Edray Goins**, Pomona College.

The MSRI Undergraduate Program, **Rebecca Garcia**, Sam Houston State University, and **Pamela E. Harris**, Williams College.

NSF Special Sessions

Outcomes and Innovations from NSF Undergraduate Education Programs in the Mathematical Sciences, Part 1, **Michael Ferrara**, **Sandra Richardson**, **John Haddock**, **Lee Zia**, **Mindy Capaldi**, and **Elise Lockwood**, Division of Undergraduate Education, National Science Foundation.

SIAM Minisymposium

Advances in Mathematical Biology, **Shilpa Khatri**, **Roummel Marcia**, and **Erica Rutter**, University of California Merced.

Advancing Racial Equity in Applied Mathematics, **Ron Buckmire**, Occidental College, **P. Seshaiyer**, George Mason University, and **Suzanne Sindi**, University of California Merced.

Graduate Research in Industry and in National Laboratory Internships, **Nicole Buczkowski** and **Hayley Olson**, University of Nebraska-Lincoln.

Lessons Learned: The Future of Online and Hybrid Modalities in Education and the Workplace (A SIAM ED session), **Manuchehr A. Aminian**, Cal Poly Pomona, and **Alvaro Ortiz**, Georgia Gwinnett College.

Mathematics of Complex Systems, **Heather Zinn Brooks**, Harvey Mudd College, **Alexander P. Hoover**, University of Akron, **Mason A. Porter**, University of California Los Angeles, **Alice Schwarze**, University of Washington, and **Alexandria Volkening**, Purdue University.

Nonlocal and Fractional Problems in Analysis and PDEs, **Marta Lewicka**, University of Pittsburgh, and **Petronela Radu**, University of Nebraska-Lincoln.

Quantum Algorithms, **Lin Lin**, University of California, Berkeley, and **Nathan Wiebe**, University of Toronto.

Sensitivity Analysis and Uncertainty Quantification for Scientific and Biological Models, **Ralph Smith**, North Carolina State University.

SIGMAA Special Sessions

Lightning Talks in Environmental Mathematics, **Russ deForest**, Pennsylvania State University, **Gordon Bower**, Excelsior Statistics, **Amanda Beecher**, Ramapo College of New Jersey, **Jacci White**, Saint Leo University, and **Eric Marland**, Appalachian State University.

Math Circle Outreach Activities that Engage Diverse Audiences, **Lauren Rose**, Bard College, and **James Taylor**, Math Circles Collaborative of New Mexico.

Mathematical Knowledge for Teaching High School and College Calculus Courses, I (Sponsored by SIGMAA on Mathematical Knowledge for Teaching), **James Madden**, Louisiana State University, **Carl Olimb**, Augustana University, and **Jennifer Whitfield**, Texas A&M University.

Programs that Support Student Research - SIGMAA on Undergraduate Research, **Allison Henrich**, Seattle University, **Kate Kearney**, Gonzaga University, and **Nicolas Scoville**, Ursinus College.

Spring Western Virtual Sectional Meeting

Now meeting virtually, PDT (hosted by the American Mathematical Society)

May 14–15, 2022

Saturday – Sunday

Meeting #1178

Western Section
Associate Secretary for the AMS: Michel L. Lapidus

Program first available on AMS website: March 24, 2022
Issue of *Abstracts*: Volume 43, Issue 3

Deadlines

For organizers: Expired
For abstracts: March 15, 2022

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Advances in Functional Analysis and Operator Theory (Code: SS 13A), **Michel L. Lapidus**, University of California, Riverside, **Marat V. Markin**, California State University, Fresno, and **Igor Nikolaev**, St. John's University.

Algebraic, Combinatorial, and Optimization Methods for Kuramoto and Power-flow Equations (Code: SS 14A), **Rob Davis**, Colgate University, **Julia Lindberg**, University of Wisconsin-Madison, and **Tianran Chen**, Auburn University at Montgomery.

Algebraic Logic (Code: SS 17A), **Nick Galatos**, University of Denver, and **José Gil Férez**, Chapman University.

Commutative Algebra (Code: SS 6A), **Tái Huy Há**, Tulane University, and **Selvi Kara**, University of Utah.

Computational Topology and Applications (Code: SS 7A), **Hitesh Gakhar** and **Miroslav Kramar**, University of Oklahoma.

Enumerative and Extremal Problems in Chromatic Graph Theory (Code: SS 18A), **Stephen Hartke**, University of Colorado Denver, and **Hemanshu Kaul**, Illinois Institute of Technology.

Factorization and Arithmetical Properties of Commutative Rings and Monoids (Code: SS 3A), **Scott Chapman**, Sam Houston State University, and **Jim Coykendall**, Clemson University.

Finite groups, their representations, and related structures (Code: SS 4A), **Robert Boltje**, University of California Santa Cruz, and **Alexander Hulpke**, Colorado State University.

Fractal Geometry and Dynamical Systems (Code: SS 10A), **Sangita Jha**, National Institute of Technology Rourkela, India, **Mrinal Kanti Roychowdhury**, University of Texas Rio Grande Valley, and **Saurabh Verma**, Indian Institute of Information Technology Allahabad.

Geometric and Functional Inequalities and Applications to PDEs (Code: SS 15A), **Joshua Flynn**, **Guozhen Lu**, and **Jianxiong Wang**, University of Connecticut.

Interactions Between Probability and Statistics (Code: SS 16A), **Kayvan Sadeghi** and **Terry Soo**, University College London.

Mathematical Advances in Bayesian Statistical Inversion and Markov Chain Monte Carlo Sampling Algorithms (Code: SS 9A), **Nathan Glatt-Holtz**, Tulane University, **Justin Krometis**, Virginia Tech, and **Cecilia Mondaini**, Drexel University.

Q-series, Number Theory and Quantum Topology (Code: SS 11A), **Chris Jennings-Shaffer** and **Shashank Kanade**, University of Denver, and **Robert Osburn**, University College Dublin.

Ramsey Theory of Infinite Structures (Code: SS 22A), **Dana Bartosova**, University of Florida, and **Natasha Dobrinen**, University of Denver.

Recent Advances on the Langlands Program (Code: SS 2A), **Kwangho Choiy**, Southern Illinois University, **Melissa Emory**, University of Toronto, and **Ralf Schmidt**, University of North Texas.

Recent progress in numerical methods for PDEs (Code: SS 5A), **Muhammad Mohebujjaman**, Texas A&M International University, and **Leo Rebholz**, Clemson University.

Recent Trends in Semigroup Theory (Code: SS 21A), **Michael Kinyon**, University of Denver, and **Ben Steinberg**, City College of New York.

Research in Mathematics by Graduate Students (Code: SS 12A), **Marat V. Markin** and **Khang Tran**, California State University, Fresno.

Rethinking the Preparation of Mathematics GTAs for Future Faculty Positions (Code: SS 8A), **Michael Jacobson**, University of Colorado, Denver.

Some Modern Developments in the Theory of Vertex Algebras (Code: SS 19A), **Florencia Orosz Hunziker**, **Shashank Kanade**, and **Andrew Linshaw**, University of Denver.

Zero-dimensional Dynamics: Algebraic and Topological Aspects (Code: SS 20A), **Ronnie Pavlov** and **Scott Schmieding**, University of Denver.

Grenoble, France

AMS-SMF-EMS Joint International Meeting

Université de Grenoble-Alpes

July 18–22, 2022

Monday – Friday

Meeting #1168

Associate Secretary for the AMS: Brian D. Boe
Program first available on AMS website: Not applicable
Issue of *Abstracts*: Not applicable

Deadlines

For organizers: Expired

For abstracts: To be announced

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/internmtgs.html`.

Invited Addresses

Andrea Bertozzi, University of California, Los Angeles, USA, *Title to be announced*.
Peter Bühlmann, ETH Zürich, Switzerland, *Title to be announced*.
Maria Chudnovsky, Princeton University, USA, *Title to be announced*.
Alessio Figalli, ETH Zürich, Switzerland, *Title to be announced*.
Vincent Lafforgue, Université de Grenoble Alpes & CNRS, France, *Title to be announced*.
Peter Sarnak, Institute for Advanced Study (IAS), Princeton, USA, *Title to be announced*.
Claire Voisin, Collège de France, Paris, France, *Title to be announced*.
Simone Warzel, Technische Universität München (TUM), Munich, Germany, *Title to be announced*.

Special Sessions

Advances in Functional Analysis and Operator Theory, **Marat V. Markin**, California State University, Fresno, USA, **Igor Nikolaev**, St. John's University, USA, **Jean Renault**, Universite d'Orleans, France, and **Carsten Trunk**, Technische Universitat Ilmenau, Germany.

Algebraic Geometry (Associated with Plenary Speaker Claire Voisin), **Radu Laza**, Stony Brook University, USA, **Catriona Maclean**, Grenoble, France, and **Claire Voisin**, Paris, France.

Automorphic Forms, Moduli Spaces, and Representation Theory (Associated with Plenary Speaker Vincent Lafforgue), **Jean-François Dat**, Sorbonne Université, France, and **Bao-Chau Ngo**, University of Chicago, USA.

Classical and Quantum Fields on Lorentzian Manifolds, **Dietrich Häfner**, Université Grenoble Alpes, France, and **Andras Vasy**, Stanford University, USA.

Combinatorial and Computational Aspects in Topology, **Eric Samperton**, University of Illinois, USA, **Saul Schleimer**, University of Warwick, United Kingdom, and **Greg McShane**, Université Grenoble-Alpes, France.

Contact Geometry, **David E. Blair**, Michigan State University, USA, **Gianluca Bande**, Universita degli Studi di Cagliari, Italy, and **Eric Loubeau**, Université de Bretagne Occidentale, France.

Deformation of Artinian algebras and Jordan type, **Anthony Iarrobino**, Northeastern University, USA, **Pedro Macias Marques**, Universidade de Evora, Portugal, **Maria Evelina Rossi**, Universita degli Studi di Genova, Italy, and **Jean Valles**, Universite de Pau et des Pays de l'Adour, France.

Deformation Spaces of Geometric Structures, **Sara Maloni**, University of Virginia, USA, **Andrea Seppi**, Université Grenoble Alpes, France, and **Nicolas Tholozan**, Ecole Normale Superieure de Paris, France.

Derived Categories and Rationality, **Matthew Ballard**, University of South Carolina, USA, **Emanuele Macrì**, Université Paris-Saclay, France, and **Patrick McFaddin**, Fordham University, USA.

Differential Geometry in the Tradition of Élie Cartan (1869 - 1959), **Vincent Borelli**, Université Claude Bernard, **Bogdan Suceavă**, California State University, Fullerton, USA, **Mihaela B. Vajiac**, Chapman University, USA, **Joeri Van der Veken**, KU Leuven, Belgium, **Marina Ville**, Université de Tours, France, and **Luc Vrancken**, Université Polytechnique Hauts-de-France, Valenciennes, France.

Drinfeld Modules, Modular Varieties and Arithmetic Applications, **Tuan Ngo Dac**, CNRS Université Claude Bernard Lyon 1, France, **Matthew Papanikolas**, Texas A&M University, USA, **Mihran Papikian**, Pennsylvania State University, USA, and **Federico Pellarin**, Université Jean Monnet, France.

Fractal Geometry in Pure and Applied Mathematics, **Hafedh Herichi**, Santa Monica College, USA, **Maria Rosaria Lancia**, Sapienza Universita di Roma, Italy, **Therese-Marie Landry**, University of California, Riverside, USA, **Anna Rozanova-Pierrat**, CentralSuplec, Universite Paris- Saclay, France, and **Steffen Winter**, Karlsruhe Institute of Technology, Germany.

Functional Equations and Their Interactions, **Guy Casale**, IRMAR, Université de Rennes 1, France, **Thomas Dreyfus**, IRMA, Université de Strasbourg, France, **Charlotte Hardouin**, IRMAR, Université de Toulouse 3, France, **Joel Nagloo**, CUNY, New York, USA, **Julien Roques**, Institut Camille Jordan, Université de Lyon 1, France, and **Michael Singer**, North Carolina State University, Raleigh, USA.

Graph and Matroid Polynomials: Towards a Comparative Theory, **Emeric Gioan**, LIRMM, France, **Johann A. Makowsky**, Israel Institute of Technology- IIT, Israel, and **James Oxley**, Louisiana State University, USA.

Groups and Topological Dynamics, **Nicolas Matte Bon**, University of Lyon, France, **Constantine Medynets**, United States Naval Academy, USA, **Volodymyr Nekrashevych**, Texas A&M University, USA, and **Dmytro Savchuk**, University of South Florida, USA.

Group Theory, Algorithms and Applications, **Indira Chatterji**, Université de Nice, France, **Francois Dahmani** and **Martin Deraux**, Institut Fourier, Université Grenoble, Alpes, France, and **Delaram Kahrobaei**, CUNY and NYU, USA.

History of Mathematics Beyond Case-Studies, **Catherine Goldstein**, CNRS, IMJ-PRG, France, and **Jemma Lorenat**, Pitzer College, USA.

Integrability, Geometry, and Mathematical Physics, **Luen-Chau Li**, Pennsylvania State University, USA, and **Serge Parmentier**, Universite Claude Bernard Lyon 1, France.

Inverse Problems, **Hanna Makaruk**, Los Alamos National Laboratory (LANL), USA, **Robert Owczarek**, University of New Mexico, Albuquerque and Los Alamos, USA, **Tomasz Lipniacki**, Polish Academy of Sciences, Poland, and **Piotr Stachura**, Warsaw University of Life Sciences-SGGW, Poland.

Low-Dimensional Topology, **Paul Kirk**, University Bloomington, USA, **Christine Lescop**, CNRS, Institut Fourier, Université Grenoble Alpes, France, and **Jean-Baptiste Meilhan**, Institut Fourier, Université Grenoble, Alpes, France.

Mathematical Challenges in Complex Quantum Systems (Associated with Plenary Speaker Simone Warzel), **Alain Joye**, Institut Fourier, Université Grenoble Alpes, France, **Jeffrey Schenker**, Michigan State University, USA, **Nicolas Rougerie**, Université Grenoble-Alpes and CNRS, France, and **Simone Warzel**, Zentrum Mathematik, TU München, Germany.

Mathematical Knowledge Management in the Digital Age of Science, **Patrick Ion**, University of Michigan, Ann Arbor, USA, **Thierry Bouche**, Université Grenoble-Alpes, France, and **Stephen Watt**, University of Waterloo, Canada.

Mathematical Physics of Gravity, Geometry, QFTs, Feynman and Stochastic Integrals, Quantum/Classical Number Theory, Algebra, and Topology, **Michael Maroun**, AMS-MRC Boston, USA, and **Pierre Vanhove**, EMS/SMF CEA Paris Saclay, France.

Modular Representation Theory, **Pramod N. Achar**, Louisiana State University, USA, **Simon Riche**, Universite Clermont Auvergne, France, and **Britta Spath**, Bergische Universitat Wuppertal, Germany.

Percolation and Loop Models (Associated with Plenary Speaker Hugo Duminil-Copin), **Ioan Manolescu**, University of Fribourg, Switzerland.

Quantitative Geometry of Transportation Metrics, **Florent Baudier**, Texas A&M University, USA, **Dario Cordero-Erausquin**, Sorbonne Universite, France, **Alexandros Eskenazis**, University of Cambridge, United Kingdom, and **Eva Pernecka**, Czech Technical University in Prague, Czech Republic.

Recent Advances in Diffeology and their Applications, **Jean-Pierre Magnot**, Université d'Angers, France, and **Jordan Watts**, Central Michigan University, USA.

Rough Path and Malliavin Calculus, **Fabrice Baudoin**, University of Connecticut, USA, **Antoine Lejay**, University of Lorraine, France, and **Cheng Ouyang**, University of Illinois at Chicago, USA.

Spectral Optimization, **Richard S. Laugesen**, University of Illinois at Urbana Champaign, USA, **Enea Parini**, Aix Marseille University, France, and **Emmanuel Russ**, Grenoble Alpes University, France.

Statistical Learning (Associated with Plenary speaker Peter Bühlmann), **Christophe Giraud**, Paris Saclay University, France, **Cun-Hui Zhang**, Rutgers University, USA, and **Peter Bühlmann**, ETH Zürich, Switzerland.

Sub-Riemannian Geometry and Interactions, **Luca Rizzi**, CNRS, Institut Fourier, Grenoble, France, and **Fabrice Baudoin**, University of Connecticut, USA.

El Paso, Texas

University of Texas at El Paso

September 17–18, 2022

Saturday – Sunday

Meeting #1179

Central Section
Associate Secretary for the AMS: Georgia Benkart

Program first available on AMS website: August 5, 2022
Issue of *Abstracts*: Volume 43, Issue 3

Deadlines

For organizers: February 22, 2022
For abstracts: July 26, 2022

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Caroline Klivans, Brown University, *Title to be announced*.
Brisa Sanchez, Drexel University, *Title to be announced*.
Alejandra Sorto, Texas State University, *Title to be announced*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Algebraic Structures in Topology, Logic, and Arithmetic (Code: SS 4A), **John Harding**, New Mexico State University, and **Emil D. Schwab**, The University of Texas at El Paso.

Banach Fixed Point Theorem: 100thyear Celebration (Code: SS 5A), **Parin Chaipunya**, King Mongkut University of Technology, Thailand, and **Mohamed A. Khamsi**, **Osvaldo Mendez**, and **Julio C. Urenda**, The University of Texas at El Paso.

Eliiptic and Parabolic PDEs in Complex Fluid and Free Boundary Problems (Code: SS 6A), **Alaa Haj Ali**, Arizona State University, and **Hengrong Du**, Vanderbilt University.

High-Frequency Data Analysis, Complex Datasets, and Applications (Code: SS 3A), **Maria Christina Mariani** and **Michael Pokojovy**, The University of Texas at El Paso, **Ambar Sengupta**, University of Connecticut, **Osei K. Tweneboah**, Ramapo College of New Jersey, and **Maria Pia Beccar Varela**, The University of Texas at El Paso.

Ordered Structures (Code: SS 2A), **Piotr Wojciechowski**, University of Texas at El Paso.

Topics in Applied Analysis (Code: SS 1A), **Behzad Djafari-Rouhini**, University of Texas at El Paso, and **Gisele Goldstein** and **Jerome Goldstein**, University of Memphis.

Amherst, Massachusetts

University of Massachusetts-Amherst

October 1–2, 2022

Saturday – Sunday

Meeting #1180

Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub

Program first available on AMS website: August 18, 2022
Issue of *Abstracts*: Volume 43, Issue 4

Deadlines

For organizers: March 1, 2022
For abstracts: August 16, 2022

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Melody Chan, Brown University, *Title to be announced*.
Steven J. Miller, Williams College, *Title to be announced*.
Tadashi Tokieda, Stanford University, *Title to be announced*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.

Algebraic and Analytic theory of Elliptic Curves (Code: SS 1A), **Alina Cojocaru**, University of Illinois, Chicago, **Seoyung Kim**, Grand Valley State University, **Steven J. Miller**, Williams College, and **Jesse A Thorner**, University of Florida.

Game-Theoretic and Agent-Based Approaches to Modeling Biological and Social Systems (Code: SS 7A), **Olivia Chu**, Dartmouth College, and **Daniel Cooney**, University of Pennsylvania.

Iwasawa Theory (Code: SS 6A), **Robert Pollack**, Boston University, **Anwesh Ray**, University of British Columbia, and **Tom Weston**, University of Massachusetts.

Lagrangian and Legendrian Submanifolds (Code: SS 2A), **Dani Alvarez-Gavela**, Massachusetts Institute of Technology, and **Mike Sullivan**, University of Massachusetts.

Non-Abelian Hodge Theory and Minimal Surfaces (Code: SS 4A), **Robert Kusner**, **Charles Ouyang**, and **Franz Pedit**, University of Massachusetts.

Nonlinear waves and Applications: a Celebration of Dimitri Frantzeskakis 60th Birthday (Code: SS 5A), **Ricardo Carretero**, San Diego State University, and **Panos Kevrekidis**, University of Massachusetts.

Ramsey Theory (Code: SS 3A), **Louis DeBiasio**, Miami University, and **Gábor Sárközy**, Worcester Polytechnic Institute and Alfréd Rényi Institute of Mathematics.

Chattanooga, Tennessee

University of Tennessee at Chattanooga

October 15–16, 2022

Saturday – Sunday

Meeting #1181

Southeastern Section
Associate Secretary for the AMS: Brian D. Boe

Program first available on AMS website: September 1, 2022
Issue of *Abstracts*: Volume 43, Issue 4

Deadlines

For organizers: March 15, 2022
For abstracts: August 23, 2022

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs/sectional.html.

Invited Addresses

Giulia Saccã, Columbia University, *Title to be announced*.
Chad Topaz, Williams College, *Title to be announced*.
Xingxing Yu, Georgia Institute of Technology, *Title to be announced*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Active Learning Methods and Pedagogical Approaches in Teaching College Level Mathematics (Code: SS 6A), **Hashim Saber**, University of north Georgia.

Applied Knot Theory (Code: SS 1A), **Jason Cantarella**, University of Georgia, **Eleni Panagiotou**, University of Tennessee at Chattanooga, and **Eric Rawdon**, University of St Thomas.

Boundary Value Problems for Differential, Difference, and Fractional Equations (Code: SS 9A), **John R Graef** and **Lingju Kong**, University of Tennessee at Chattanooga, and **Min Wang**, Kennesaw State University.

Enumerative Combinatorics (Code: SS 10A), **Miklós Bóna** and **Vince Vatter**, University of Florida.

Geometric and Topological Generalization of Groups (Code: SS 4A), **Bikash C Das**, University of North Georgia.

Nonstandard Elliptic and Parabolic Regularity Theory with Applications (Code: SS 2A), **Hongjie Dong**, Brown University, and **Tuoc Phan**, University of Tennessee, Knoxville.

Probability and Statistical Models with Applications (Code: SS 5A), **Sher Chhetri**, University of South Carolina, Sumter, and **Cory Ball**, Florida Atlantic University.

Quantitative Approaches to Social Justice (Code: SS 7A), **Chad Topaz**, Williams College.

Special Session on Combinatorial Commutative Algebra (Code: SS 8A), **Michael Cowen**, **Hugh Geller**, **Todd Morra**, and **Sean Sather-Wagstaff**, Clemson University.

Structural and Extremal Graph Theory (Code: SS 3A), **Hao Huang**, Emory University, and **Xingxing Yu**, Georgia Institute of Technology.

Salt Lake City, Utah

University of Utah

October 22–23, 2022

Saturday – Sunday

Meeting #1182

Western Section
Associate Secretary for the AMS: Michel L. Lapidus

Program first available on AMS website: September 8, 2022
Issue of *Abstracts*: Volume 43, Issue 4

Deadlines

For organizers: March 22, 2022
For abstracts: August 30, 2022

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Algebraic Combinatorics and Applications in Harmonic Analysis (Code: SS 3A), **Joseph Iverson** and **Sung Y. Song**, Iowa State University, and **Bangteng Xu**, Eastern Kentucky University.

Approximation Theory and Numerical Analysis (Code: SS 2A), **Vira Babenko**, Drake University, and **Akil Narayan**, University of Utah.

Building Bridges Between Commutative Algebra and Nearby Areas (Code: SS 5A), **Benjamin Briggs** and **Josh Pollitz**, University of Utah.

Commutative Algebra (Code: SS 4A), **Adam Boocher**, University of San Diego, **Eloísa Grifo**, University of California, Riverside, and **Jennifer Kenkel**, University of Michigan.

Extremal Graph Theory (Code: SS 1A), **József Balogh**, University of Illinois, and **Bernard Lidický**, Iowa State University.

Fractal Geometry, Dimension Theory, and Recent Advances in Diophantine Approximation (Code: SS 9A), **Alexander M. Henderson**, University of California, **Machiel van Frankenhuijsen**, Utah Valley University, and **Edward K. Voskanian**, The College of New Jersey.

Free Boundary Problems Arising in Applications (Code: SS 14A), **Mark Allen**, Brigham Young University, **Mariana Smit Vega Garcia**, Western Washington University, and **Braxton Osting**, University of Utah.

Geometry and Representation Theory of Quantum Algebras and Re- lated Topics (Code: SS 6A), **Mee Seong Im**, United States Military Academy, West Point, **Bach Nguyen**, Xavier University of Louisiana, and **Arik Wilbert**, University of Georgia.

Graphs and Matrices (Code: SS 11A), **Emily Evans**, **Mark Kempton**, and **Ben Webb**, Brigham Young University.

Higher Topological and Algebraic K-Theories (Code: SS 18A), **Agnès Beaudry**, (University of Colorado Boulder, **Jonathan Campbell**, Duke University, and **John Lind**, California State University, Chico.

Inverse Problems (Code: SS 12A), **Hanna Makaruk**, Los Alamos National Laboratory, and **Robert Owczarek**, University of New Mexico.

Knotted Surfaces and Concordances (Code: SS 15A), **Mark Hughes**, Brigham Young University, **Jeffrey Meier**, Western Washington University, and **Maggie Miller**, Princeton University.

Mathematics of Collective Behavior (Code: SS 10A), **Daniel Lear** and **Roman Shvydkoy**, University of Illinois at Chicago.

PDEs, Data, and Inverse Problems (Code: SS 7A), **Jared Whitehead**, Brigham Young University.

Recent Advances in Algebraic Geometry and Commutative Algebra in or Near Characteristic p (Code: SS 8A), **Bhargav Bhatt**, University of Michigan, and **Karl Schwede**, University of Utah.

Recent Advances in the Theory of Fluid Dynamics (Code: SS 17A), **Elaine Cozzi**, Oregon State University, and **Magdalena Czubak**, University of Colorado Boulder.

Recent Advances of Numerical Methods for Partial Differential Equations with Applications (Code: SS 16A), **Joe Koebbe** and **Jia Zhao**, Utah State University, and **Yunrong Zhu**, Idaho State University.

Recent Developments in Inverse Problems for PDEs and Applications (Code: SS 20A), **Loc Nguyen**, University of North Carolina at Charlotte, **Dinh-Liem Nguyen**, Kansas State University, and **Fernando Guevara Vasquez**, University of Utah.

Several Complex Variables: Emerging Applications, Connections, And Synergies (Code: SS 13A), **Jennifer Brooks**, Brigham Young University, and **Dusty Grundmeier**, Harvard University.

Topics in Graphs, Hypergraphs and Set Systems (Code: SS 19A), **John Engbers**, Marquette University, **David Galvin**, University of Notre Dame, and **Cliff Smyth**, The University of North Carolina at Greensboro.

Boston, Massachusetts (JMM 2023)

John B. Hynes Veterans Memorial Convention Center, Boston Marriott Hotel, and Boston Sheraton Hotel

January 4–7, 2023

Wednesday – Saturday

Meeting #1183

Associate Secretary for the AMS: Steven H. Weintraub
Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 44, Issue 1

Deadlines

For organizers: To be announced

For abstracts: To be announced

Submit Your Proposals for AMS Special Sessions at the 2023 Joint Mathematics Meetings

All members of the mathematics community are invited to submit proposals for American Mathematical Society (AMS) Special Sessions at the 2023 Joint Mathematics Meetings (JMM). If you have a topic that you would like to explore in a special session, now is the time to put your great idea into motion.

The 2023 JMM will be held January 4–7, 2023 in Boston, MA. On behalf of the American Mathematical Society, Prof. Steven H. Weintraub (`shw2@lehigh.edu`), the AMS Associate Secretary responsible for the AMS program at this meeting, solicits proposals for AMS Special Sessions for this meeting. Proposals that reflect the full spectrum of interests of the mathematical community are welcome.

A special session is a collection of talks devoted to a single area of mathematics or a single topic. Special sessions can be proposed by teams of organizers.

Please go to https://meetings.ams.org/math/jmm2023/cfs.cgi and provide the following information:

1. the title of the session
2. the name, affiliation, and email address of each organizer, with one organizer designated as the contact person for all communication about the session
3. a brief description of the topic of the proposed special session
4. a *sample* list of speakers whom the organizers plan to invite (It is not necessary to have received confirmed commitments from these potential speakers.)
5. either the primary two-digit MSC (Mathematics Subject Classification) number that most closely matches the topic — see http://www.ams.org/mathscinet/msc/msc2020.html
   or one of the following new code numbers adopted for topics:
   - 101: Teaching and learning
   - 102: Recreational mathematics
   - 103: Professional development and professional concerns
   - 104: Wider issues

   — see http://www.ams.org/journals/notices/202010/rnoti-p1602.pdf

The deadline for submission of proposals is **March 31, 2022**. Late proposals will not be considered. No decisions will be made on proposals until after the submission deadline has passed. For questions about using the submission form, contact meet@ams.org.

Organizers are encouraged to read the *AMS Manual for Special Session Organizers* at www.ams.org/meetings/specialsessionmanual.html in its entirety.

Some key information:

Special sessions will in general be allotted between 5 and 10 hours in which to schedule speakers. To enable maximum movement of participants between sessions, organizers must schedule each session speaker for either (a) a 20-minute talk with 5-minute discussion and 5-minute break or (b) a 45-minute talk with 5-minute discussion and 10-minute break. A special session may include any combination of 20-minute and 45-minute talks that fits within the time allotted to the session, but all talks must begin and end at the scheduled time.

The number of special sessions in the AMS program at the JMM is limited, and because of the large number of high-quality proposals, not all can be accepted. Please be sure to submit as detailed a proposal as possible for review by the Committee on Special Sessions and Contributed Paper Sessions. Decisions will be made on acceptance and scheduling of sessions by early June 2022. At that time, contact organizers will be notified whether their proposal has been accepted. If so, they will be informed of their session's schedule and will be sent additional information about organizational details.

We look forward to reviewing your proposals.

Atlanta, Georgia

Georgia Institute of Technology

March 18–19, 2023

Saturday – Sunday

Meeting #1184

Southeastern Section
Associate Secretary for the AMS: Brian D. Boe

Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

The scientific information listed below may be dated. For the latest information, see https://www.ams.org/amsmtgs sectional.html.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at https://www.ams.org/cgi-bin/abstracts/abstract.pl.

Advanced Topics in Graph Theory and Combinatorics (Code: SS 1A), **Songling Shan**, Illinois State University, and **Guangming Jing**, Augusta University.

Groups, Geometry, and Topology (Code: SS 4A), **Dan Margalit** and **Yvon Verberne**, Georgia Institute of Technology.

Recent Developments in Commutative Algebra (Code: SS 5A), **Florian Enescu**, Georgia State University, and **Thomas Polstra**, MSRI and University of Virginia.

Recent Developments on Analysis and Computation for Inverse Problems for PDEs (Code: SS 2A), **Dinh-Liem Nguyen**, Kansas State University, and **Loc Nguyen** and **Khoa Vo**, University of North Carolina at Charlotte.

Topology and Geometry of 3- and 4-Manifolds (Code: SS 3A), **Siddhi Krishna**, Georgia Institute of Technology and Columbia University, **Miriam Kuzbary**, Georgia Institute of Technology, and **Beibei Liu**, Max Planck Institute for Mathematics and Georgia Institute of Technology.

Spring Eastern Virtual Sectional Meeting

Hosted by the American Mathematical Society

April 1–2, 2023

Saturday – Sunday

Meeting #1185

Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub

Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

Cincinnati, Ohio

University of Cincinnati

April 15–16, 2023

Saturday – Sunday

Meeting #1186

Central Section
Associate Secretary for the AMS: Georgia Benkart, University of Wisconsin-Madison

Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

Fresno, California

California State University, Fresno

May 6–7, 2023

Saturday – Sunday

Meeting #1187

Western Section
Associate Secretary for the AMS: Michel L. Lapidus

Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: October 4, 2022
For abstracts: March 7, 2023

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Advances by Scholars in the Pacific Math Alliance (Code: SS 22A), **Andrea Arauza Rivera**, California State University, East Bay, **Mario Banuelos**, California State University, Fresno, and **Jessica De Silva**, California State University, Stanislaus.

Advances in Functional Analysis and Operator Theory (Code: SS 6A), **Michel L. Lapidus**, University of California, Riverside, **Marat V. Markin**, California State University, Fresno, and **Igor Nikolaev**, St. John's University.

Algebraic Structures in Knot Theory (Code: SS 4A), **Carmen Caprau**, California State University, Fresno, and **Sam Nelson**, Claremont McKenna College.

Algorithms in the Study of Hyperbolic 3-manifolds (Code: SS 26A), **Robert Haraway, III** and **Maria Trnkova**, University of California, Davis.

Analysis of Fractional Differential and Difference Equations with its Application (Code: SS 20A), **Bhuvaneswari Sambandham**, Dixie State University, and **Aghalaya S. Vatsala**, University of Louisiana at Lafayette.

Artin-Schelter Regular Algebras and Related Topics (Code: SS 27A), **Ellen Kirkman**, Wake Forest University, and **James Zhang**, University of Washington.

Combinatorics Arising from Representations (associated with the Invited Address by Sami Assaf) (Code: SS 16A), **Sami Assaf**, University of Southern California, **Nicolle Gonzalez**, University of California, Los Angeles, and **Brendan Pawloski**, University of Southern California.

Complexity in Low-Dimensional Topology (Code: SS 14A), **Jennifer Schultens**, University of California, Davis, and **Eric Sedgwick**, DePaul University.

Data Analysis and Predictive Modeling (Code: SS 8A), **Earvin Balderama**, California State University, Fresno, and **Adriano Zambom**, California State University, Northridge.

Inverse Problems (Code: SS 5A), **Hanna Makaruk**, Los Alamos National Laboratory, and **Robert Owczarek**, University of New Mexico, Albuquerque and University of New Mexico, Los Alamos.

Math Circle Games and Puzzles that Teach Deep Mathematics (Code: SS 13A), **Maria Nogin** and **Agnes Tuska**, California State University, Fresno.

Mathematical Biology: Confronting Models with Data (Code: SS 21A), **Erica Rutter**, University of California, Merced.

Mathematical Methods in Evolution and Medicine (associated with the Invited Address by Natalia Komarova) (Code: SS 1A), **Natalia Komarova** and **Jesse Kreger**, University of California, Irvine.

Methods in Non-Semisimple Representation Categories (Code: SS 11A), **Eric Friedlander**, University of Southern California, Los Angeles, **Julia Pevtsova**, University of Washington, Seattle, and **Paul Sobaje**, Georgia Southern University, Statesboro.

Recent Advances in Mathematical Biology, Ecology, Epidemiology, and Evolution (Code: SS 10A), **Lale Asik**, Texas Tech University, **Khanh Phuong Nguyen**, University of Houston, and **Angela Peace**, Texas Tech University.

Research in Mathematics by Early Career Graduate Students (Code: SS 7A), **Doreen De Leon**, **Marat Markin**, and **Khang Tran**, California State University, Fresno.

Scientific Computing (Code: SS 19A), **Changho Kim**, University of California, Merced, and **Roummel Marcia**.

The Use of Computational Tools and New Augmented Methods in Networked Collective Problem Solving (Code: SS 18A), **Mario Banuelos**, California State University, Fresno, **Andrew G. Benedek**, Research Centre for the Humanities, Hungary, and **Agnes Tuska**, California State University, Fresno.

Women in Mathematics (Code: SS 12A), **Doreen De Leon**, **Katherine Kelm**, and **Oscar Vega**, California State University, Fresno.

Zero Distribution of Entire Functions (Code: SS 9A), **Tamás Forgács** and **Khang Tran**, California State University, Fresno.

Buffalo, New York

University at Buffalo (SUNY)

September 9–10, 2023

Saturday – Sunday
Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

Omaha, Nebraska

Creighton University

October 7–8, 2023

Saturday – Sunday
Central Section
Associate Secretary for the AMS: Georgia Benkart, University of Wisconsin-Madison
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

Albuquerque, New Mexico

University of New Mexico

October 21–22, 2023

Saturday – Sunday
Western Section
Associate Secretary for the AMS: Michel L. Lapidus
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

Auckland, New Zealand

December 4–8, 2023

Monday – Friday
Associate Secretary for the AMS: Steven H. Weintraub
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

San Francisco, California (JMM 2024)

Moscone West Convention Center

January 3–6, 2024

Wednesday – Saturday
Associate Secretary for the AMS: Michel L. Lapidus
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

San Francisco, California

San Francisco State University

May 4–5, 2024

Saturday – Sunday
Western Section
Associate Secretary for the AMS: Michel L. Lapidus
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

Palermo, Italy

July 23–26, 2024

Tuesday – Friday

Associate Secretary for the AMS: Brian D. Boe

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced

For abstracts: To be announced

Riverside, California

University of California, Riverside

October 26–27, 2024

Saturday – Sunday

Western Section

Associate Secretary for the AMS: Michel L. Lapidus

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced

For abstracts: To be announced

Washington, District of Columbia (JMM 2026)

Walter E. Washington Convention Center and Marriott Marquis Washington DC

January 4–7, 2026

Sunday – Wednesday

Associate Secretary for the AMS: Georgia Benkart

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced

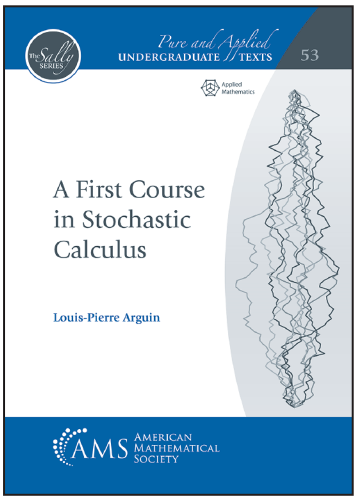
For abstracts: To be announced

American Mathematical Society
Distribution Center

35 Monticello Place,
Pawtucket, RI 02861 USA

NEW RELEASES from the AMS

= Textbook
= Applied Mathematics

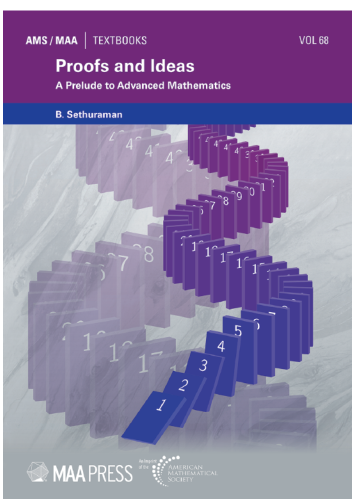


A First Course in Stochastic Calculus

Louis-Pierre Arguin, *Baruch College, City University of New York, NY, and Graduate Center, City University of New York, NY*

Louis-Pierre Arguin offers an exceptionally clear introduction to Brownian motion and to random processes governed by the principles of stochastic calculus. This is achieved by emphasizing numerical experiments using elementary Python coding to build intuition and adhering to a rigorous geometric point of view on the space of random variables.

Pure and Applied Undergraduate Texts, Volume 53; 2022; approximately 277 pages; Softcover; ISBN: 978-1-4704-6488-2; List US$85; AMS members US$68; MAA members US$76.50; Order code AMSTEXT/53



Proofs and Ideas

A Prelude to Advanced Mathematics

B. Sethuraman, *California State University, Northridge, CA, and Krea University, Sri City, India*

Proofs and Ideas serves as a gentle introduction to advanced mathematics for students who previously have not had extensive exposure to proofs. It is intended to ease the student's transition from algorithmic mathematics to the world of mathematics that is built around proofs and concepts.

AMS/MAA Textbooks, Volume 68; 2021; approximately 339 pages; Softcover; ISBN: 978-1-4704-6514-8; List US$85; AMS members US$63.75; MAA members US$63.75; Order code TEXT/68

facebook.com/amermathsoc
@amermathsoc

Photos by Kate Awtrey, Atlanta Convention Photography

Free shipping for members in the **USA** (including Puerto Rico) and **Canada**.

AMS AMERICAN MATHEMATICAL SOCIETY
Advancing research. Creating connections.

Discover more titles at **bookstore.ams.org**