The Olympic logo consists of five interlocking rings, representing the five continents of the world (the “continents” here are Europe, Africa, Asia, the Americas, and Oceania; apparently Antarctica does not qualify, possibly because the International Olympic Committee is prejudiced against penguins). But in the Middle Ages the Europeans only knew of three continents, and the Olympic logo looked as shown in Figure 1.

Actually, Figure 1 is the Borromean rings, so named for the Borromeo family of medieval Milanese bankers on whose coat of arms they appeared. From a topological point of view, these rings form a 3-component link $L$ with a very interesting property: although each pair of rings by themselves form a 2-component unlink, the three of them together are nontrivially linked. One way to see this is to pick (any) two of the components which we can call $L_1$ and $L_2$ and observe that the fundamental group of $S^3 - L_1 \cup L_2$ is free on two generators, say $x_1$ and $x_2$, represented by little loops that go once around each component. Then the third component represents the conjugacy class of the commutator $x_1 x_2 x_1^{-1} x_2^{-1}$ in $\pi_1(S^3 - L_1 \cup L_2)$ rather than the trivial element, as it would if $L$ were the 3-component unlink.

One of the attractive things about the Borromean rings—or, to speak more precisely, about this specific image (technically: this projection)—is the psychological contradiction between the (apparent) roundness (and hence, one feels, simplicity) of the rings, and the over-under complexity of their interaction. The rings look perfectly round in the figure, but one feels this must be some sort of optical illusion, like Escher’s staircase, or Penrose’s tribar. Real perfectly round rings could not link in 3-dimensional space in that precise way.

Or could they? Can one find a configuration of three round disjoint circles in three-dimensional space which is isotopic to the Borromean rings?

As far as I know this question was first asked, and answered, by Mike Freedman and Richard Skora. Their argument is rather lovely, and we shall give it shortly. The argument proves a rather general fact about round links, as follows. Let $L$ be any $n$-component link in $S^3$ made from round circles; we call $L$ a round link. Note that any two round circles in $S^3$ are isotopic; thus the components, individually, are unknots. Furthermore, there are only two
possibilities for the 2-component sublinks: two disjoint round circles in $S^3$ either form an unlink, or they are isotopic to the Hopf link (see Figure 2).

Figure 2. The Hopf link is the only nontrivial link of two round circles.

The Hopf link is distinguished from the unlink by the fact that the components have a nontrivial (algebraic) linking number. One way to define this linking number is to think of the 3-sphere $S^3$ as the boundary of the unit ball $B^4$ in 4-dimensional Euclidean space. An oriented knot in $S^3$ bounds an embedded oriented surface in $B^4$ (actually, it already bounds such a surface in $S^3$—a so-called Seifert surface). If two knots $K$, $K'$ bound embedded surfaces $R, R'$ in $B^4$ we can perturb these surfaces slightly so that they intersect in general position, and then the linking number of $K$ and $K'$ is the (algebraic) intersection number of $R$ and $R'$. This intersection number can be defined at the level of (relative) homology classes, and therefore does not depend on the choice of surfaces $R$ and $R'$.

Thus, the Hopf link is nontrivial, and so is any round link that has a pair of algebraically linked components. However, Freedman and Skora show that every round link with pairwise unlinked components is trivial! Here is the reason. A round circle $L_i$ in $S^3$ bounds a rather obvious embedded surface in $B^4$, namely if we think of $L_i$ as the intersection of $S^3$ with a flat plane $\Pi_i$ in $\mathbb{R}^4$, then $\Pi_i$ intersects $B^4$ in a smooth disk $D_i$ bounding $L_i$. Since components $L_i, L_j$ are pairwise disjoint, the disks $D_i, D_j$ intersect transversely if at all (since otherwise the intersection is contained in a positive-dimensional affine subspace that must intersect $S^3$ somewhere). If they were to intersect, then their algebraic intersection number would be $\pm 1$ and therefore the linking numbers of $L_i$ and $L_j$ would be $\pm 1$, contrary to hypothesis. So a round link $L$ as above bounds a family of disjoint totally geodesic disks $D_i$ in the 4-ball.

Now consider the intersection of these $D_i$ with the ball $tB^4$ obtained from $B^4$ by scaling it by some $t \in (0, 1)$. Each $D_i$ intersects the boundary $tS^3$ in a collection of round circles, or (for finitely many discrete values of $t$) a single point. Thus this family of intersections gives an isotopy of the link $L$ which shrinks the circles one by one (keeping them disjoint) until they shrink down to a point and disappear. This isotopy shows that each component can be unentangled from all the others, and that $L$ is the unlink (in particular, the Borromean rings—whose components are pairwise unlinked—is not a round link).

This example may make one think that round links are uninteresting, but this is not at all true. A rather striking and beautiful theorem of Genevieve Walsh (that we shall discuss shortly) concerns links whose components are not only round circles but are geodesics in the round metric on $S^3$—i.e., they are great circles in $S^3$. Call such links great circle links. Note that every pair of great circles in $S^3$ forms a Hopf link. Conversely, every round link whose components are pairwise linked is actually isotopic to a great circle link! To see this, recall that for two round circles $L_i, L_j$ in $S^3$ to be linked is equivalent to the flat planes $\Pi_i, \Pi_j$ in $\mathbb{R}^2$ they lie in intersect transversely at some point in the interior of $B^4$. Now consider the intersections of the $\Pi_i$ with the balls $tB^4$ obtained from $B^4$ by scaling it by some $t \in (1, \infty)$. This gives a family of round links that deform by an isotopy, and in the limit as $t \to \infty$ the components all become great circles (this argument is due to Bill Thurston).

By associating to each great circle $L_i$ in $S^3$ the projectivization of the plane $\Pi_i$ it spans in $\mathbb{R}^4$ one obtains an equivalence between great circle links and configurations of skew lines (i.e., arrangements of disjoint straight lines in $\mathbb{R}\mathbb{P}^3$). The number of arrangements of $n$ skew lines for small $n$ was determined by Julia Drobotukhina-Viro and Oleg Viro and is equal to $1, 1, 2, 3, 7, 19, 74$ for $n = 1, \ldots, 7$. The case $n = 1$ corresponds to the unknot, and $n = 2$ the Hopf link. The two links for $n = 3$ are illustrated in Figure 3.

Figure 3. There are two great circle links with three components up to isotopy; they are mirror images of each other.

Walsh shows that every great circle link is fibered; i.e., $S^3 - L$ is a fiber bundle over $S^1$. Here is the proof. Let’s think of $S^3$ (conformally) as $\mathbb{R}^3$ union infinity. After a conformal transformation we can make $L_1$ into the $z$-axis in $\mathbb{R}^3$. Put cylindrical coordinates $z, r, \theta$ on $\mathbb{R}^3$, so that $(z, r, \theta)$ corresponds to the point $(r \cos(\theta), r \sin(\theta), z)$ in $(x, y, z)$ coordinates. Then the map $(z, r, \theta) \to \theta$ is a fibration of $\mathbb{R}^3 - L_1$ to $S^1$, whose fibers $R_\theta$ are the radial half-planes with constant $\theta$-coordinate. The projection of these half-planes to the $xy$ plane is the set of radial lines emanating from the origin.
The projection of every other component $L_j$ to the $xy$ plane is either an ellipse or a degenerate segment (if the circle is contained in a vertical plane); since great circles pairwise link, $L_j$ links $L_1$ once so the projection of each $L_j$ must be a nontrivial ellipse winding around the origin. Thus every component but $L_1$ is transverse to the foliation by $F_2$, and the projection $S^3 - L \to S^1$ to the $\vartheta$ coordinate is a fibration, qed.

One beautiful application due to Walsh is to give an infinite family of examples of 3-manifolds that do not fiber over the circle, but are virtually fibered (i.e., they admit a finite sheeted cover which fibers). A 2-bridge link is a knot or link that can be arranged in $S^3$ in such a way that projection to the $x$-axis (say) has exactly two maxima and two minima. These are equivalent to the so-called rational links (named by Conway) and are classified by a rational number $p/q$. See Figure 4 for an example.

Figure 4. The 2-bridge knot $K_{p/q}$ associated to $p/q = 37/85$ with continued fraction expansion $[2, 3, 2, 1, 3]$.

It is unusual for a 2-bridge link complement to be fibered. In fact, a theorem of David Gabai says $S^3 - K_{p/q}$ is fibered if and only if $p/q$ has a continued fraction expansion of the form $[\pm 2, \pm 2, \ldots, \pm 2]$.

Walsh shows that the link complement $S^3 - K_{p/q}$ is finitely covered by a great circle link complement. For simplicity we explain the case $q$ odd. Think of $S^3$ as the unit sphere in $\mathbb{C}^2$ with coordinates $z$ and $w$. Let $\gamma$ be the great circle obtained by intersecting $S^3$ with $\mathbb{R}^2 \subset \mathbb{C}^2$. If we pick coprime integers $p$ and $q$ ($q$ odd), the map $\phi_{p/q} : (z, w) \to (e^{2\pi i/q}z, e^{2\pi i/p}w)$ is an isometry of $S^3$ of order $q$, and the orbit of $\gamma$ under the cyclic group $C$ generated by $\phi_{p/q}$ is a great circle link $L$ with $q$ components. The quotient $S^3/C$ is a Lens space $M$, and the great circle link $L$ projects to a knot $K' \subset M$.

Complex conjugation $(z, w) \to (\bar{z}, \bar{w})$ acts as an involution on $S^3$ with fixed point set $\gamma$. It normalizes $C$, and conjugates $\phi_{p/q}$ to $\phi_{p/q}^{-1}$ and therefore descends to an involution on $M$ with fixed point set $K'$. The quotient of $M$ by complex conjugation is $S^3$ again, and so the map $M \to S^3$ is a branched cover in which $K'$ projects to the branch locus $K \subset S^3$, the 2-bridge knot $K_{p/q}$.

In his celebrated 1982 paper in the *Bulletin of the AMS*, Thurston posed the question of whether every finite-volume hyperbolic 3-manifold has a finite cover that fibers over the circle. This question became known as the virtual fibration conjecture, despite the fact that it was posed as a question and not a conjecture. Thurston wrote, “(t)his dubious-sounding question seems to have a definite chance for a positive answer” but it is fair to say that many 3-manifold topologists were for a long time far more skeptical than Thurston. Personally, I found Walsh’s proof of virtual fibration for rational link complements to be a watershed moment in my thinking on this conjecture, and in fact between 2009 and 2012 a series of papers by Daniel Wise and Ian Agol together proved the conjecture in full generality. A truly Olympian achievement!