From Barth Sextics to Minimal Surfaces: Printing Mathematics in 3 Dimensions

Scott Hershberger

Mathematics abounds with mind-bending shapes. Take the Barth Sextic—it’s a sextic surface in complex 3-dimensional projective space with 65 singular points. To visualize a corresponding surface in real 3-dimensional space, you would need a computer program to generate images. But what if you could actually create the physical object?

Silviana Amethyst did just that, turning equations and pixels into plastic through 3D printing. The result was “this really cool star shape that jumped out and grabbed me by my eyeballs,” said Amethyst, an assistant professor of mathematics at the University of Wisconsin–Eau Claire. (Disclosure: Amethyst is the author’s cousin.) “It was really satisfying to put my fingers through it and play with it and look at the axes of rotation that it exhibited.” Manipulating the object in real life also revealed features hidden on the computer screen, such as a cube lurking within the overall icosahedral symmetry.

And Amethyst didn’t stop there. Through a years-long iterative process, she designed a watermelon-sized version of the Barth Sextic with 20 snap-together components, all containing LED lights. Each 12-centimeter “cone” took six hours to print. Then came a flurry of painstaking soldering and wiring to assemble the algebraic light fixture. Finally, two of her undergraduate students (and later Amethyst herself) programmed the lights to bring to life the twofold, threefold, and fivefold rotational symmetries of the Sextic. The result was a mesmerizing confluence of mathematics and art, making algebraic geometry and group theory tangible in a way both mathematicians and nonmathematicians could appreciate.

Amethyst is part of a growing community of mathematicians using 3D printing in their work. (You might recall that a 3D-printed dynamical system graced the cover of the Notices in December 2020 [1].) With this rapidly advancing technology, researchers can actually hold in their hands the...
“Mathematics is so directly related to 3D printers, so I think that it’s mathematicians who have to be pioneers in introducing it to our students and bringing it to the next generation,” said Maria Trnkova, a lecturer at the University of California, Davis, and an organizer of this year’s Short Course. “There are so many opportunities, but on the other side, there are real mathematical problems that can be solved in order to advance [3D printing].”

Layer Upon Layer of Mathematics

Most traditional manufacturing techniques are subtractive—hence the tongue-in-cheek maxim that to make a sculpture like Michelangelo’s David, you just have to remove all the material that doesn’t look like David. 3D printing, by contrast, encompasses a group of additive manufacturing techniques in which objects are built up from scratch, layer by layer. A major advantage of additive manufacturing is its versatility for producing complex designs. “Because you are building up an object from nothing, you can have much more complex geometry, such as internal features or voids, that traditional manufacturing techniques cannot create,” explained Bethany Weeks during the Short Course. Weeks is an engineer at Boeing.

3D printing has current and future applications in fields as disparate as aerospace engineering and orthodontics, she said. Regardless of the particular use, the connections to mathematics begin at a fundamental level: Creating a computer model of an object to fabricate requires a quantitative understanding of the relevant geometry, whether defined parametrically, implicitly, or iteratively. So it’s no surprise that since the technology’s early days, some innovators have seen the potential uses for 3D printing in mathematics.

The first commercial 3D printer launched in the late 1980s. According to Henry Segerman, the first 3D print of
a mathematical object was likely a trefoil knot created by Stewart Dickson in 1989. Segerman, an associate professor of mathematics at Oklahoma State University, is a leader in the visualization of mathematics through 3D printing. In the late 1990s, a trickle of other people pursued similar projects, he said. Then, soon after desktop 3D printers hit the market in the mid-2000s, the field took off.

Much of Segerman’s work explores mechanisms that move in unexpected ways, like non-planar arrangements of three linked gears. He began the Short Course with a series of other examples from across mathematics: a Klein bottle opener (Bathsheba Grossman), a spiky Perko knot (Laura Taalman), a projection of a hypercube (Grossman), a Hilbert cube (Carlo Séquin), and more.

So how do you go about making a 3D-printable model of a mathematical object? An immediate challenge, even for a shape as simple as the skeleton of a cube, is that “you cannot 3D print a one-dimensional line—you have to have some thickness to it,” Segerman said. Choosing how to add thickness is far from the only decision in the process.

Mathematicians must also select from a gamut of software tools to make their models. Parametrization-driven options include Mathematica and Sage, while OpenSCAD is script-based. More sophisticated software such as Rhinoceros and Blender allow for a mixture of scripting and mouse manipulation. Each one has strengths and weaknesses, so a single project might move through a sequence of tools. Before hitting the printer, a model must be “sliced” into instructions for layers, infill, and temporary support structures in an application such as Cura.

Then there’s the question of the type of printer and print material. For at-home machines, polylactic acid (PLA) is a popular low-cost material, though it can be brittle and sensitive to heat. Thermoplastic polyurethane (TPU) is soft and elastic but more difficult to print. Amethyst used PLA for all the Barth Sextics pictured in this article, turning to TPU for a smaller electronic version. More expensive appliances can handle photopolymers, metal, and even ceramics.

"There’s such a wide variety of things to print and ways to go about achieving the model," Amethyst said. "There’s no one right tool; there’s no one right way to 3D print."

### 3D Printing for Research and the Classroom

Between the lectures and workshops, this year’s Short Course buzzed with the exchange of creative ideas. Many of the 30 participants were eager to discuss their successes in incorporating 3D printing into research and teaching.

Jane McDougall, an associate professor at Colorado College, turned to the technology for the first time recently to bring her research into the physical world. A complex analyst, McDougall was searching for new minimal surfaces with her student Sohair Abdullah (now a graduate student in physics at the University of Wisconsin–Madison). In 2017, the pair succeeded in creating a new minimal surface that was triply periodic—periodic along three axes in space.

McDougall later enlisted the help of physics major Nate Hohner to bring the equations out of the screen and into her hands. McDougall wrote a Mathematica program to create a model of the Rosette minimal surface with a thickness and a mesh. Hohner then refined the code for printing. Upon first holding the surface in summer 2021, "I was overjoyed—you look at it with incredulity, like ‘This now exists in physical form,’” McDougall said. "It does feel more real, in a way,” than its computer counterpart.

This print and prints of other surfaces, beyond merely representing completed research, informed the direction of McDougall’s ongoing investigations. "When you’re holding

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1The surface is described on [https://janemcdougall.org/](https://janemcdougall.org/)
To better understand the problem, math major Ian Miller (now a graduate student at the University of Colorado, Boulder) printed a bunch of tetrahedra plus a sphere missing a cap. He proceeded to cram as many tetrahedra in the sphere as he could. When arranged in the form of an icosahedron, 20 tetrahedra can fit inside, a fact that Miller easily confirmed. Miller could almost get a 21st tetrahedron to fit—but no matter how he jiggled the shapes, the extra one still jutted out slightly. The evidence was suggestive that 20 is the maximum, but not conclusive. It remains unknown whether some precise arrangement allows 21 or 22 tetrahedra to fit. (23 cannot fit because their volume exceeds the volume of the sphere.)

Despite the copious prep time needed, teaching such a hands-on course is worth it to Sterling. “It’s very rewarding—the students are more enthusiastic than [in] any [other] class I’ve taught.”

Like Sterling, Short Course organizer Andrew Yarmola can testify to the impact of 3D printing in the mathematics classroom. When students came to his office hours last semester struggling to visualize a hyperbolic paraboloid, Yarmola, who is an instructor at Princeton University, handed them the actual surface. “They came off saying they had a much better understanding of what it looks like. The task of sketching it […] became much easier” for them, he said.
Failures and Possibilities

Ask any mathematician about their experiences with 3D printing, whether they’ve produced a handful of prints or hundreds, and one word will inevitably emerge: failure.

The first time McDougall tried to fabricate the Rosette minimal surface, the filament produced "a combination of spaghetti and printout." Amethyst's initial attempt at a Barth Sextic broke right away because the connections at the singular points were too small. And “I don’t think I had any student,” Sterling said, “[who] didn’t fail in one way or another.”

Often, ambitious mathematical designs push the technology to its limits. Segerman said that current printers lack the precision to fully realize some of his ideas. Conversely, people who wish to create large objects are hindered by issues of scale, including the size of the machines themselves.

Still, researchers are rapidly expanding the capabilities of 3D printing. Elisabetta Matsumoto, an assistant professor of physics at Georgia Tech, spoke at the Short Course about an innovation made possible by differential geometry. Using a composite material that swells anisotropically when submerged in water, Matsumoto and her team printed surfaces that were initially flat, but curved into the shape of a catenoid, a helicoid, or even a calla lily within minutes [2]. Doing so required modeling the mean curvature and Gaussian curvature of the desired surface, she explained.

With the nearly endless possibilities to interweave mathematics and 3D printing, and the falling costs for universities and individuals to own and operate printers, more mathematicians should enter the field, Trnkova said. Yarmola emphasized that getting started is not difficult, with or without coding. In addition to ample online resources, he pointed to the annual Bridges conference2 for mathematical art and to local makerspaces that thrive in many cities.

Amethyst encouraged those interested to simply dive in. “The first step is the hardest,” she said. "Just print your first object, and it'll start to flow from there. […] Make mistakes, make garbage, and eventually you’ll make beautiful artwork.”

References


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