the effects of long standing inequities. Acknowledging and responding to the forces that have blocked these goals is our responsibility. Our efforts must be broad and intentional in all our activities. Experience shows that without continued vigilance and effort, the system reverts to old patterns. Pursuing, refining, and improving our activities, looking for new ways to move forward, sharing our experiences, and supporting each other leaves us hopeful that MSRI will help the broader mathematical community meet the challenge of creating and sustaining inclusive communities.

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Credits

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The Community of AIM SQuaREs

Estelle Basor

1. Overview

The American Institute of Mathematics (AIM) was founded to advance mathematical knowledge by fostering collaborations among diverse groups of researchers. AIM's primary scientific programs are week-long focused workshops and small research groups called Structured Quartet Research Ensembles (SQuaREs). Approximately 750 mathematicians per year visit AIM for either a workshop or a SQuaRE.

The SQuaREs program supports collaborations of four to six researchers to work on hard and interesting mathematics. This program is unique among all the major mathematical institutes in that it allows not only for small groups to work together, but also for repeated visits and intense collaboration. Most often the groups meet three times for a week over a three-year period. While these groups are each individually very small communities, together they make up a broad and vibrant community of AIM SQuaRE participants.

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Often we host four or five SQuaRE groups at once. On Monday morning we ask participants to introduce themselves and tell us something about the mathematics of their SQuaRE. Participants from other SQuaREs often ask questions about the mathematics, and one SQuaRE group may discover that a participant from another group may be helpful to them during the week. For AIM staff, this is one of the most interesting and enjoyable parts of the week, listening to the wide variety of mathematical topics and sensing the enthusiasm in the participants.

After the morning introductions, each group is assigned a seminar room to use as a base while at AIM. But all participants share breakfast, eat lunch together, and enjoy happy hour together each evening, and on Tuesday evening, there is a joint reception. Sometimes schedules are such that concurrent SQuaREs one year return at the same time the following year. Other times the set of SQuaREs are entirely different groups.

2. A Brief History

In keeping with our goal of increasing collaboration in and advancing mathematics, AIM initiated the SQuaRE program in the fall of 2007. The original motivation for the SQuaRE program was to provide the means for researchers to continue, in a more organized fashion, collaborations that began at previous AIM workshops. Now, however, many of the SQuaRE groups are new collaborative efforts that have arisen independently of the AIM workshop program.

We have hosted over 475 SQuaRE weeks since our program began. In the first few years of the program we accepted only six to ten new SQuaRE projects each year, thinking that our steady state would be about 30 groups per year. But as the program gained in popularity, we increased the number so that now we are closer to hosting 50 to 60 SQuaRE groups each year. These groups are funded by our institute grant from NSF and also with supplemental funding from the Simons Foundation. The program is highly competitive with an acceptance rate of about 25 percent.

The AIM Scientific Board selects both workshops and SQuaREs. The proposal for a SQuaRE includes all the participants and usually a description of what each might contribute to the project. Often we find that a couple of the participants have worked together before, or that the participants know of each other's work, but have never met in person. After someone has the idea that it would be beneficial to have these particular people work together, the meetings of the SQuaRE help them build a collaborative community. The AIM staff suggests to a new group that they use some time for planning and organizing. We remind them that they are not meeting only once, but multiple times. We know that often research collaboration begins with good intentions, but then things get in the way. The

Figure 1. SQuaRE participants at AIM.

feature of returning motivates the community to evolve and grow.

3. Some Highlights

The rich and varied topics of SQuaREs range over many areas of pure and applied mathematics. While significant progress on research problems is the most common outcome of a SQuaRE, there are other important consequences of a SQuaRE collaboration.

Over time we have noticed that the SQuaRE format provides a particularly good way for junior and senior faculty to interact. On average, SQuaRE groups produce about one paper per year. For junior faculty, this can help lay the foundation for a good solid research program. When choosing SQuaRE projects, we pay particular attention to diversity. Many of our SQuaRE groups consist entirely of women and have their beginnings in other communities that support diversity, such as Women in Numbers. We encourage faculty from non-research schools to apply for and participate in SQuaREs. We have also noticed unusual blends of expertise at SQuaREs. The group may, for example, include mathematicians and medical researchers. The format allows for these individuals to interact in real time, face-to-face, and to make real progress.

To illustrate some of the topics and outcomes, here are some highlights of past SQuaREs.

- The SQuaRE **Understanding the mathematical underpinning of medical imaging** was an example of how a unique set of participants can come together. This group included experts in convolution gridding, sampling, coding, numerical analysis, and neuroscience. They considered how data inherent to the scanning process, such as resonant frequencies and signal decay rates (which are currently used only to provide contrast), can be useful in diagnosing a condition or measuring a response to treatment. Much more information can be extrapolated from the same scan, such as temperature, blood flow, diffusion, structure, and physiology. Their goal was to develop nonconventional image reconstruction techniques to get this information faster and clearer than ever before possible.
- The SQuaRE **New connections between link homologies and physics** consisted of mathematical physicists, experts in topological recursion, and a number theorist. The main purpose of this SQuaRE was to find new homological theories for knots and links and analyze their properties through new perspectives. These perspectives include understanding superpolynomials (knot and link invariants) and developing computational tools to derive explicit forms for them. Various algebraic curves associated to knots arise as classical limits of recursion relations for knot polynomials and in some cases, recursion relations for knots encode information about invariants. This SQuaRE was led by a senior mathematician and had several young, junior members.
- Multiplying matrices is among the most frequently performed algorithmic tasks in the world: it is the workhorse of scientific computation. Understanding the complexity of matrix multiplication is a central question in algebraic complexity theory that has led to new insights and conjectures in the representation theory of finite groups and the algebraic geometry of tensors. In the SQuaRE **Fast matrix multiplication via representation theory of finite groups,** the goal was to find new ideas for matrix multiplication algorithms using ideas from representation theory. The naïve method of multiplying two n by n matrices takes $O(n^3)$ steps. The best known lower bound is $O(n^2)$ since any algorithm must at least read all $2n^2$ entries of the input matrices. Currently the best known algorithm takes $O(n^{2.373\ldots})$ steps. Closing this gap is a major open problem in algebraic complexity theory and was the ultimate goal of this SQuaRE.
- The SQuaRE **Formation of shocks** spawned many new ideas for understanding singularity formation in solutions to wave equations and related equations. The SQuaRE participants discovered a new regime of initial conditions for which they were able to prove stable shock formation for a large class of wave equations. The initial data are allowed to be small or large and must be approximately (but not exactly) symmetric. The results imply, as special cases, stable shock formation in nearly plane symmetric solutions to the compressible Euler equations (both relativistic and non-relativistic) under the assumption of vanishing vorticity. The SQuaRE members produced two monographs, new summer school courses at MIT and Cambridge for graduate students, and one of the participants was awarded an NSF Career award.
- An example of a very applied and practical SQuaRE was **Mathematical modeling, simulation, and optimal design for agricultural water management** that brought mathematicians, engineers, and a numerical analyst together to try to help solve some of the water issues facing California growers. The SQuaRE had its beginning in a "Sustainability Problems" workshop that had representatives from Driscoll's Berries. The goal was to devise planting strategies to reduce water use and maintain profit. The SQuaRE group has incorporated recharge basin models into their program and has a very robust

program that includes almost all important variables for the decision making process, i.e., How much irrigation is optimal? What crops should be planted and how should they be rotated? etc. The work of this SQuaRE and a previous workshop became the subject for an NSF "Science Nation" video, which was featured on the NPR Science News Hour website.

- The SQuaRE **Geometry of physics of the new topological recursion** was focused on computing topological invariants for certain manifolds using ideas from mathematical physics. This SQuaRE brought together a unique set of experts in various areas and blended together topics from both pure and applied mathematics. This SQuaRE grew out of an AIM workshop, but because the SQuaRE generated so many new topics, questions, and ideas, the SQuaRE in turn led to another workshop. At the workshop **Spectral data for Higgs bundles**, a group of participants sketched a paper with new ideas, which later positively settled a conjecture of Gaiotto that had been outstanding for years. Thus while the workshops were originally thought to be a starting place for many of the SQuaRE groups, sometimes the opposite is also true. A SQuaRE may generate a robust set of ideas and these then become the focus of an AIM workshop.
- The SQuaRE **Computations with explicit reduction theories** focused on computing the cohomology of the general linear group of rings of integers of quadratic fields, with the hope of computing higher K-groups. The group had remarkable success. For many specific cases, they computed the ranks of the cohomology, made progress in computing the K-groups, and as a by-product of their first paper, developed a new conjecture relating the Euler characteristic of the associated locally symmetric spaces to class numbers. They wrote three significant papers and commented that "Those of us on the computational geometry side learned about automorphic forms, L-functions, and their connections with the cohomology of arithmetic groups. Those of us coming from the automorphic forms side learned about state-of-the-art techniques in the study of perfect forms and computational geometry. We established new research connections that allowed distinct research groups to form a greater whole."
- Wolbachia is a reproductive parasite that infects arthropod species, including mosquitoes, all over the world. The parasite interferes with the reproductive mechanisms of its host, by, for example, inducing cytoplasmic incompatibility and male killing. What if this common infection could be used as a biological control tool to fight mosquito-borne diseases, such as West Nile virus (WNv)? This idea was the focus of the SQuaRE **Wolbachia infections in mosquitos carrying West Nile virus**, during which the group introduced and analyzed a 12-dimensional dynamical system. The rigorous analysis of the Wolbachia model revealed, among other things, that under certain reasonable assumptions, the model has multiple steady states in which Wolbachia-infected mosquitoes could coexist with small numbers of uninfected mosquitoes. This enabled the group to assess the potential of Wolbachia infection to eradicate WNv via its effect on WNv replication in Wolbachia-infected mosquitoes. Notably, the expression obtained for the basic reproduction number suggests that the Wolbachia infection substantially reduces WNv replication in mosquitoes, and that WNv would be eradicated if, at the steady state, the overwhelming majority of mosquitoes are infected with Wolbachia.

4. Participant Reactions

The participant reaction to SQuaREs has been overwhelmingly positive. The most frequent comment is that the SQuaRE program allows for participants to work together in person for a week at a time and that there is absolutely no substitute for the person-to-person interaction despite the myriad of electronic means of communication available for long-distance collaboration. Another frequent observation is that, since the SQuaRE program allows for participants to return, the need to plan for a second or third year of a SQuaRE is a great incentive to keep a research program alive. The repetition of the SQuaRE week really binds the participants as a small community. Here are some representative reactions from participants.

> *For me, the SQuaRE program has led to some of the most productive weeks in my career by far. The opportunity to work in a focused environment, without distractions and especially to collaborate with more senior colleagues in such an environment where they are not distracted by other responsibilities or projects is truly invaluable. Taking part in this SQuaRE has had enormous value to me both in the results it produced and in the collaborative relationships it fostered.*

> *I find the SQuaRE program perfect for the intensive collaboration. It provides an optimal setting for distraction-free focus on the ongoing research. In addition, members of our SQuaRE group came from different places in the world, therefore making it hard to effectively collaborate. Meeting together regularly here in San Jose solved that problem. Since this is our last meeting, it was really interesting to see how progress was made through the years, and new offsprings of the original plans came out of this. Therefore, I strongly support having multiple yearly SQuaRE meetings.*

This has been phenomenal. This has greatly influenced my directions and I love working with my SQuaRE mates and am constantly learning from them.

The SQuaRE really functioned well and served an unforeseen purpose: to make researchers of totally different backgrounds to be able to scientifically communicate and to formulate possible theorems. Without the AIM meeting the language barrier would have prevented the two sides (physics and mathematics) to communicate.

This SQuaRE program is likely to be a turning point in my mathematical career. Not only did it open a new set of research projects but it also allowed me to strengthen the relation with my collaborators. I think this collaboration will be for life.

You have clearly mastered the elements of creating an environment conducive to intense periods of productive collaboration. Our group has made massive progress with our last two AIM visits.

The opportunity to focus intensively on a single problem for several days and to have the insights of all participants focused together made for a very productive and enjoyable week. For me this SQuaRE has been much more beneficial and productive than other conferences and workshops that I have participated in.

SQuaREs are a remarkable opportunity for us to meet as a group and bring together our individual talents. I assume others have found this to be an "Almost Unique" (if this is an allowable mathematical concept) opportunity within U.S. NSF Institutions.

My experience at AIM was very fruitful and productive. The interaction with the homotopy theorists have helped me greatly in sharpening my ideas on relation of generalized cohomology theories to physics. That also helped me make my proposals much more tangible. I would say that the program at AIM was a turning point in my research.

5. Other SQuaRE Program Features

SQuaREs complement existing research opportunities

It is clear that the SQuaRE program provides a valuable resource for members of the mathematical community to collaborate in a focused, organized, and companionable way.

Figure 2. SQuaRE seminar room discussion.

This is especially valuable in present times, when research funds are scarce. SQuaRE groups work on hard, important, ambitious problems and because the group returns, there is an impetus to keep working throughout the year (even when not at AIM). It is hard for established researchers, even those who have their own individual grants, to bring a group of colleagues together multiple times to work on a project for an intense, uninterrupted period.

In some ways, being a part of a SQuaRE project mimics having a mini-three-year individual investigator grant. This opportunity is reflected in the number of excellent proposals to the program, and the success is reflected in the number of results and papers produced by the groups.

SQuaREs provide an accessible intense research experience

The SQuaRE program provides a practical way for these small groups to collaborate. One of our recent young SQuaRE participants remarked that while it was difficult for him to spend months away from his home institution at a semester-long program and away from family obligations, it was not difficult to come to AIM for a week each year. There is no administrative work required of any of the participants. The AIM staff take care of invitations, reservations, all logistics, and provide a comfortable research environment for the small groups.

For more information about the program and for a complete list of our past groups and papers, please see `https://aimath.org/pastsquares/`.

We hope in the future that our SQuaRE program will continue to flourish and the creation of these tiny communities, which are all part of the larger SQuaRE community, will play an important role in creating mathematical leaders, making the mathematical community more diverse, and advancing mathematical knowledge.

As a final note, this article was written before the pandemic. For nearly two years, all AIM programs were virtual. Some of our SQuaRE groups decided to postpone until they could come back in person and others met virtually. In addition to the virtual SQuaRE program, we hosted virtual workshops and to help provide additional opportunities for collaboration, AIM began a new program that allows larger groups of at least 40 people with a common research interest to work together virtually. These are called AIM Research Communities (ARC), and it is expected that they will have a long lifetime and grow over time.

For more information about the ARCs, please see `https://aimath.org/programs/research communities/`.

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Credits

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