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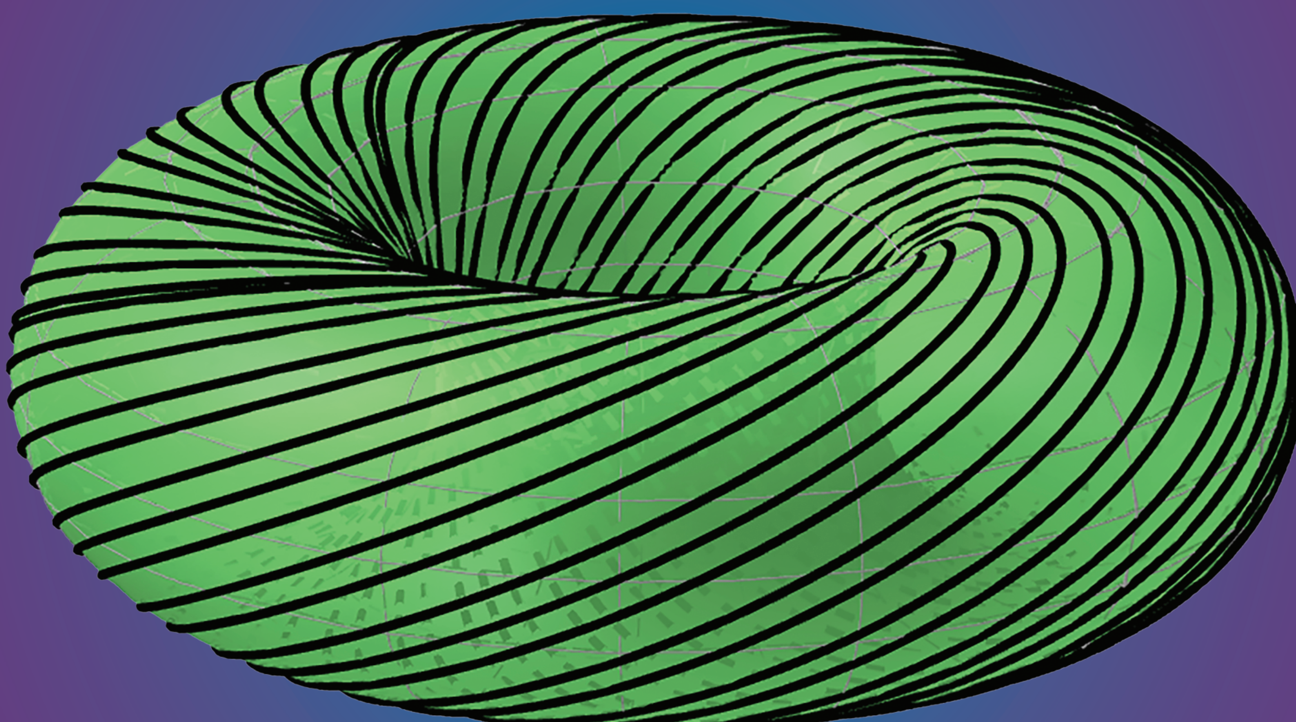

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Notices of the American Mathematical Society

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Notices
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LETTERS TO THE EDITOR

Establishment of the Arab Physical Society

As a regular reader of *AMS Notices* for decades, I would like to share the news of the establishment of the Arab Physical Society (ArPS, `https://www.arabphysicalsociety.org/`). On Thursday, April 7th, 2022, the Arab Physical Society was formally launched in a day-long online event. The event was marked by presentations by a Fields Medallist (Edward Witten, 1990), Nobel Laureates (Gerard 't Hooft, 1999, Takaaki Kajita, 2015, and Roger Penrose, 2020), Wolf Prize Winners (Pablo Jarillo-Herrero, 2020), along with Dirac Medallists (Edward Witten, 1985, John Ellis, 2005, and Charles Kane, 2012). The eminent speakers covered topics of contemporary interest from physics, mathematical physics, and applied mathematics. Jordan's Prince Hassan Ben Talal, a long-time supporter of science, delivered the opening address. Over 1,700 people attended the event.

On June 7th, 2021, the Arab Physical Society (ArPS) was registered as a non-governmental and non-profit organization. The newly created organization already has four hundred members. The mission of the ArPS is to "Promote excellence and creativity in the field of physics for the benefit of the Arab region and humanity, it encourages scientific and research collaboration among researchers and students in the Arab region." Along with physics, related areas including mathematical physics, applied mathematics, and some areas of engineering are within the domain of ArPS. Like its counterparts, ArPS has clear-cut plans to organize scientific and educational activities; establish peer-reviewed journals; and importantly promote interaction among individuals, institutions, and organizations. In order to achieve these goals, ArPS has established a strong organizational structure through its Governing Council, Advisory Committee, and Focal Point. From the onset, ArPS is paying attention to diversity and gender. This is evident by the fact that each of the aforementioned governing bodies have seven members of which three are women. The Advisory Committee comprises of members from institutions in Egypt, Jordan, Libya, Morocco, Palestine, Qatar, and Syria. The founding president of ArPS is Shaaban Khalil, a high-energy physicist. He is the director of the Fundamental Physics Center at the Zewail City of Science and Technology in Egypt. The ArPS is, according to its website, also keen to foster diversity "to ensure that everyone has the same opportunities, regardless of gender, ethnicity, religion and culture." Any person, regardless of nationality or domicile, can join the ArPS. A nominal fee of fifty euros is charged for annual membership. Founding of the Arab Physical Society is of immense interest to the scientific community worldwide, particularly to the diaspora from numerous countries residing in the Middle East. The number of such residents is over thirty-five million. The science news from the region of the Middle East has been encouraging. As an example, we note that the 2022 King Faisal International Prize in the field of mathematics was awarded to Martin Hairer and Nader Masmoudi of Tunisia. Masmoudi also holds the post of a distinguished professor of mathematics at the Center for Stability, Instability, and Turbulence in the New York University's Abu Dhabi campus in UAE. Arab diaspora (numbering eight million outside of the Middle East) can look forward to contributing to the Middle Eastern region through the ArPS.

Further Reading

[1] S. A. Khan, *Ahmed Hassan Zewail (1946–2016)*, Current Science **111** (2016), no. 5, 936–937.

[2] S. A. Khan, *2022 King Faisal International Prize for Science and Medicine*, Current Science **122** (2022), no. 5, 508.

[3] H. Barcelo and S. Kennedy, *Maryam Mirzakhani: 1977–2017*, Notices Amer. Math. Soc. **65** (2018), no. 10, 1221–1247.

[4] S. A. Khan, *Maryam Mirzakhani (1977–2017)*, Current Science **113** (2017), no. 5, 982–983.

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**We invite readers to submit letters to the editor at* `notices-letters@ams.org`.

A WORD FROM...

Jennifer J. Quinn, President of the MAA

The opinions expressed here are not necessarily those of the Notices *or the AMS.*

Courtesy of Mark O. Martin.

I believe myself to be an optimist. Yet, I cannot overlook issues in the mathematics community and the academy that need improvement. No one person has a solution—especially not me. A first step is awareness and a willingness to engage.

When you scan the mathematics environment, what do you see? Admittedly, impressions are biased by personal experiences. I am a full professor at a primarily undergraduate, urban-serving, public institution and president of a sister mathematics association. I have observed:

- **Declining membership in professional organizations.** Membership concerns have been a frequent topic of discussion by the societies involved in the Conference Board of the Mathematical Sciences.[1] In her 2018 State of the AMS report, Catherine Roberts wrote "we saw an increase in the number of regular members for the first time in at least eight years. Although our total membership numbers continued to decline (as is being experienced across all professional societies, so this is not unique to the AMS), numbers decreased at a slower rate than in previous years."[2]
- **A greater reliance on non-tenure track faculty to support the educational duties at institutions of higher learning.** Using the most recent data available (2016), AAUP found that 73% of instructional positions at all US institutions of higher education were off the tenure track.[3] This erosion in the tenure system weakens protections for academic freedom. Typically, non-tenure track faculty cost less, teach more, and have fewer protections in their positions.
- **Lack of recognition of contributions outside of traditional scholarship.** Peer-reviewed publications remain the gold standard for achievement—even at some institutions that provide minimal support for scholarship. Successfully fulfilling academic responsibilities, typically to include scholarship, teaching, and service, should lead to promotion. Academic responsibilities temporarily shifted during the once-in-a-century disruption to education due to COVID-19. The Herculean efforts in instruction that allowed universities to continue offering courses uninterrupted were rewarded with a pat on the back, a recommendation of more self-care, and for many faculty, a pause on their tenure clock. The loss of research progress during lockdown disproportionately affected women and faculty of color.[4] Institutional understanding and short-term flexibility are a start but continued emphasis on achievement as measured through traditional scholarship fails to recognize the emergency shift of academic responsibilities undertaken at the behest of these institutions during a global pandemic.

Jennifer J. Quinn is the president of the MAA and a professor at the University of Washington Tacoma. Her email address is `jjquinn@uw.edu`.

[1]*See for example the May 2019 or December 2020 CBMS agendas available at* `https://www.cbmsweb.org/council-meeting-materials/`.

[2]*C. Roberts,* Report of the Executive Director: State of the AMS, 2018, *Notices Amer. Math. Soc.* ***66*** *(2019), no. 7, 1115.*

[3]Data Snapshot: Contingent Faculty in US Higher Ed, *published October 11, 2018,* `https://www.aaup.org/news/data-snapshot-contingent-faculty-us-higher-ed#.YjjES-rMJD9`.

[4]*A. Tugend,* On the Verge of Burnout Covid19's impact on faculty well-being and career plans, *Chronicle of Higher Education, 2020.*

- **An exponential growth in publications.** The number of journal articles indexed annually by Mathematical Reviews has doubled over the last 19 years with a growth rate of 3.6%.[5] Changes in the professoriate have not kept pace. According the 2015 CBMS Survey of Undergraduate programs, the number of full-time faculty in mathematics and statistics departments grew from 28,500 to 33,500 faculty over the same approximate period, between 2000 and 2015.[6]
- **An environment that is not welcoming to everyone.** Climate surveys confirm that harassment, discrimination, and bias persist in academia.[7] Despite an increased emphasis on inclusion, humanity, anti-racism statements, codes of conduct, and welcoming environment policies, the underlying systems are flawed and need change.

I see these five issues as the result of people, either individually or institutionally, giving too much or too little value to the subject under consideration. When a mathematician chooses not to join a professional organization, they are making a determination of worth. It could be because of a generational shift (the iTunes mentality of buying a single song rather than the whole album), a technological shift (increased access to information on the internet previously only available through membership), or unfamiliarity with the myriad ways associations support mathematicians and advance the understanding of mathematics and its impact on the world. When institutional leaders lean heavily on non-tenure track faculty for financial flexibility, they are valuing the bottom line over long term commitments to and stability of their faculty. When academic culture supports a publish-or-perish paradigm, it overvalues peer-reviewed scholarship, sometimes to the exclusion of achievements in public scholarship, teaching innovation, mentorship, institution building, or service to various communities. It pushes aspiring professionals to seek minimal-publishable-units, perhaps fueling the proliferation of publications. Finally, when an institutional culture intentionally or unintentionally perpetuates an unwelcoming environment, it undervalues its members and denies them equitable opportunities to learn, use, and contribute to the mathematical sciences.

Academia is a strange and hierarchical entity—rooted in privilege and intentional exclusion. Prior to 1773, graduates of Harvard were arranged "according to the dignity of birth, or to the rank of [their] parents."[8] To this day, everything and everyone is ranked according to some metric of merit and those not at the apex can be made to feel less than or unwelcome. Now we rank students by test scores or GPA, faculty by publications or citation indices, and institutions by Carnegie Classification or status on the annual lists of "Best Universities and Colleges."

Initially, I thought the problem lay in the hierarchy itself,[9] but on reflection that was naive. Proposing an alternate structure was unrealistic—especially when experts in organizational behavior say hierarchies are inevitable in complex societies[10] and philosophers have yet to identify an unambiguous example of a self-organizing (non-hierarchical) society.[11] Organizational psychologist Harold Leavitt said, "We cling to hierarchies because our place in a hierarchy is, rightly or wrongly, a major indicator of our social worth."[12] So eliminating the hierarchy is out.

The Carnegie Foundation for the Advancement of Teaching has done something interesting; it is changing the hierarchy. Its classification system was never intended as an indicator of institutional prestige, only a tool for educational-research comparison. Recently in partnership with the American Council on Education, they rethought the classifications. Beginning is 2023, the Social and Economic Mobility Classification will be launched to recognize and reward institutions committed to and succeeding in creating better futures for diverse, inclusive student populations. US secretary of education Miguel

[5] *E. Dunne,* Looking at the Mathematics Literature, *Notices Amer. Math. Soc.* ***66*** *(2019), no. 2, 227–230,* `https://www.ams.org/journals/notices/201902/rnoti-p227.pdf`.

[6] *R. Blair, E. K. Kirkman, and J. W. Maxwell,* Statistical Abstract of Undergraduate Programs in the Mathematical Sciences in the United States Fall 2015 CBMS Survey, *AMS, 2018, p. 32,* `https://www.ams.org/profession/data/cbms-survey/cbms2015-Report.pdf`.

[7] *E. B. Stolzenberg, M. K. Eagan, H. B. Zimmerman, J. Berdan Lozano, N. M. Cesar-Davis, M. C. Aragon, and C. Rios-Aguilar,* Undergraduate teaching faculty: The HERI Faculty Survey 2016–2017, *Los Angeles: Higher Education Research Institute, UCLA, 2019.*

[8] `https://www.harvard.edu/about-harvard/harvard-history/#1600s`

[9] *I said as much in the December 2021–January 2022* MAA Focus *President's Message: "Don't Let Us Get Sick."*

[10] *B. Sutton,* Hierarchy is Good. Hierarchy is Essential. And Less Isn't Always Better, *eCorner, Stanford University, April 7, 2016,* `https://ecorner.stanford.edu/articles/hierarchy-is-good-hierarchy-is-essential-and-less-isnt-always-better/`.

[11] *K. A. Appiah,* Digging for Utopia, *New York Review of Books, December 16, 2021,* `https://www.nybooks.com/articles/2021/12/16/david-graeber-digging-for-utopia/`.

[12] *H. J. Leavitt,* Top Down: Why Hierarchies are Here to Stay and how to Manage Them More Effectively, *Harvard Business Press, 2005, p. 35.*

A. Cardona said of this change, "Colleges and universities need to reimagine themselves around inclusivity and student success, not selectivity and reputation."[13]

Similarly, I propose rethinking the meaning of 'merit' in the academy so we can impart greater worth to everyone within the hierarchy. In practice, determination of merit becomes a self reinforcing cycle—what you reward, you get more of. If respect in the hierarchy is only earned through publication, then we get more publication accompanied by a proliferation of predatory publication practices.[14] If contributions essential to your institution's mission and core values are ignored or devalued, who will engage in the necessary work?

Amartya Sen argues that measuring merit in fixed, absolute terms reflects values and priorities of the past and often comes in conflict with contemporary objectives and views of a good society.[15] To promote mathematics and welcome more practitioners, I ask that we work together to modernize our disciplinary metrics and match merit to mission.

To create improved systems, reimagined around inclusivity and not selectivity, will require hard work. Modernizing disciplinary metrics can impel progress towards diversity, equity, and inclusion; safeguard academic freedom; and support career promotion pathways for non-tenure track faculty. More people might take up the work, if they knew that their contributions would be valued. Change can occur locally at the individual and institutional level or nationally through professional organizations, as evidenced by the partnership between the Carnegie Foundation and the American Council on Education. When you are ready to be part of enacting change, you can find support and resources through associations like AMS or MAA.[16] If you find these resources valuable and you don't already belong, please consider joining.

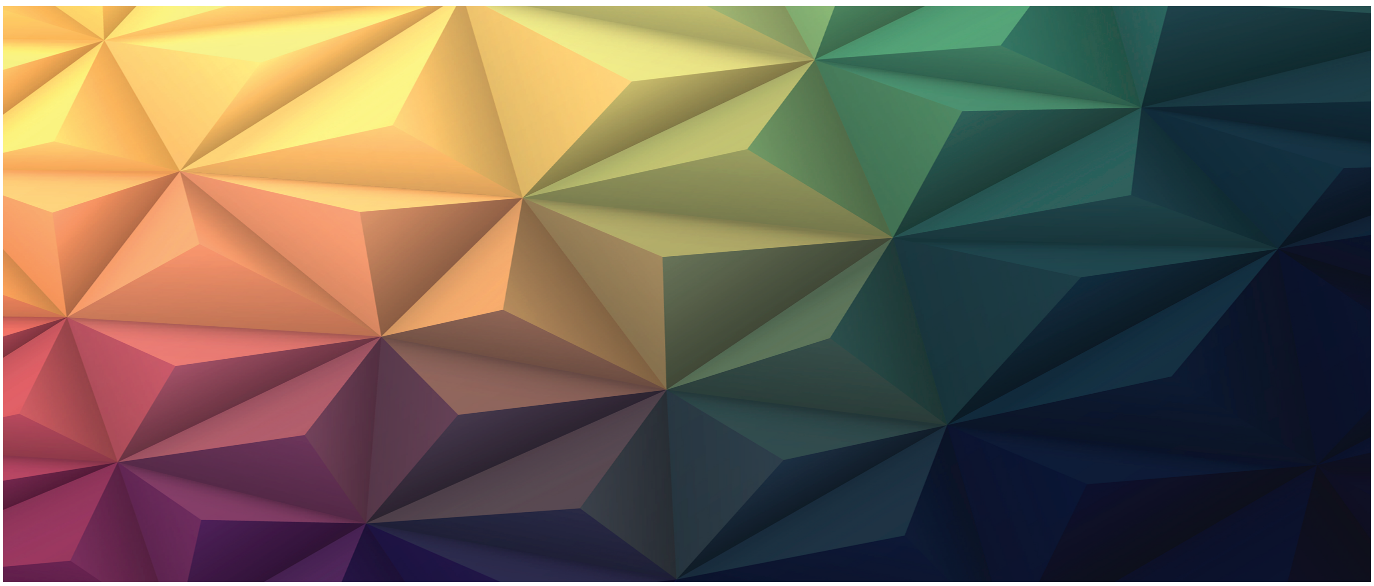
[13] *K. Mangan,* New Carnegie Classification Will Reflect Social and Economic Mobility, *Chronicle of Higher Education, February 9, 2022.*

[14] *D. S. Chawla,* Hundreds of 'predatory' journals indexed on leading scholarly database, *Nature, February 8, 2021,* https://www.nature.com/articles/d41586-021-00239-0.

[15] *A. Sen,* Merit and Justice, *In: K. J. Arrow, et al., Meritocracy and Economic Inequality, Princeton: Princeton University Press, 2000.*

[16] *See for instance, the statements and guidelines from the MAA at* https://www.maa.org/programs-and-communities/professional-development/committee-on-faculty-and-departments *or the AMS at* https://www.ams.org/about-us/governance/policy-statements/sec-ams-policystatements.

A.T.V.'s for (Geometric) Off-roading: A Gentle Introduction to Arithmetic Toric Varieties



Patrick K. McFaddin

1. Introduction

A fundamental problem in mathematics is how to determine whether a given finite collection of polynomials has a common solution, and how to quantify and qualify the shape of this solution set in a meaningful way. Indeed, linear algebra is the study of polynomials of degree one, and the theory of quadratic forms investigates polynomials of degree two. In general, this is the staring point of *algebraic geometry* and the theory of *algebraic varieties*. Over the past few centuries, algebraic geometers have developed a wide array of tools called *invariants* for studying such geometric objects. Invariants are pieces of data (numbers, sets, groups, vector spaces, categories) associated to mathematical objects which produce the same output on objects which are effectively identical, i.e., those which are *isomorphic* in the appropriate sense. In other words, invariants are those associations which are invariant under isomorphism.

Invariants help us distinguish algebraic varieties which have distinct geometric properties, greatly improving our ability to classify them. To study varieties, it is necessary to study their invariants, how to effectively compute them, and what information they reflect about the algebra, geometry, and arithmetic of a given variety. Often one gains a great deal of insight by computing invariants on large classes of particularly nice varieties. Toric varieties defined over the complex numbers have proved to be an extremely useful class of geometric objects to test the flexibility and robustness of a slew of algebro-geometric invariants. Our understanding of divisor class groups, Picard groups, algebraic K-theory, derived categories, moduli problems,

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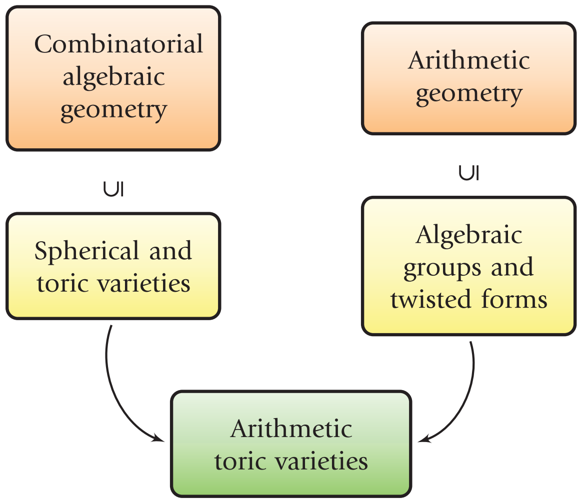
geometric invariant theory, and the minimal model program has been greatly improved by investigating these invariants for toric varieties [CLS11].



To investigate the utility and limitations of these invariants in the arithmetic case, i.e., for varieties over non-algebraically closed fields, one aims to find a similarly useful class of varieties. Simply taking the naive, analogous definition of toric varieties over arbitrary fields yields a theory which is too similar to the complex case and fails to faithfully reflect arithmetic. Instead, one looks to *twisted forms* of toric varieties, which we call *arithmetic toric varieties*. For such varieties, the combinatorial nature of complex toric varieties persists, but additional Galois-theoretic analysis plays a crucial role. The study of arithmetic toric varieties is then reduced to understanding combinatorial objects (fans, cones, etc.) and how Galois groups associated to certain field extensions act on these objects. In this way, arithmetic toric varieties serve as a convenient bridge between arithmetic and combinatorial algebraic geometry, which we hope to exploit to study derived invariants. By veering off of the geometrically beaten path with our A.T.V.'s, we hope to gain deeper insight into various invariants and their limitations in reflecting arithmetic data. Here we highlight two properties of interest in the arithmetic setting: the existence of *rational points* and *rationality*. The first is the problem of determining whether a collection of polynomials admits a common solution. The second focuses on how to parametrize the solution set of a collection of polynomials using only rational functions.

The purpose of this manuscript is to give a small sampling of the theory of arithmetic toric varieties, focusing on simple examples in low dimension using explicit polynomial equations. Even with this restricted "cheat-sheet" type viewpoint, there is hope that the reader may glean a general sense of the analysis of such objects. While some topics are only glossed over, and some terminology is defined in a colloquial manner, we hope that this manuscript serves as an invitation to the reader to look more deeply into these topics and ideas.

A second goal is to highlight the class of arithmetic toric varieties as the most natural testing ground to study the limitations of algebro-geometric invariants in the arithmetic setting. There is already a rich history of such work, including the analysis of del Pezzo surfaces and Severi-Brauer varieties. Using more recent work which systematically investigates arithmetic toric varieties using Galois (and non-ablelian) cohomology [Dun16, ELFST14, MP97], we hope to gain new insights into this class of objects. We point out one particular success story in this line of investigation concerning the invariant given by the *coherent derived category*. It has been found that unless the derived category has a fine decomposition into very simple pieces, it does not generally reflect rationality properties well. This yields a broader understanding of the derived category as an arithmetic invariant. On the other hand, one might share the author's disappointment that the derived category is not better suited to this task. C'est la vie! Our hope is that the class of arithmetic toric varieties will be used to probe other invariants in an analogous fashion.

Organization. We begin by recalling some basic terminology from the theory of algebraic varieties and their arithmetic, including a discussion of algebraic groups and twisted forms. In Section 3, we discuss toric varieties defined over the complex numbers, introducing the combinatorial description of these geometric objects in low dimension. In Section 4, arithmetic toric varieties are introduced and examples are provided. In Section 5, we report on recent results which provide insight into the use of arithmetic toric varieties in studying the limitations of the derived category as an arithmetic invariant of smooth projective varieties.

Notation and conventions. All rings are associative with identity and all vector spaces are finite-dimensional. A *monoid* is a set together with an associative binary operation which admits an identity element. Given a field k, a k-*algebra* is a ring which also admits the structure of a k-vector space, and these operations are compatible. A *division ring* (resp., *division k-algebra*) is a ring (resp., k-algebra) in which every non-zero element has a multiplicative inverse. Given two objects A and B in the same category, we use $\operatorname{Hom}(A, B)$ to denote the collection of all morphisms (i.e., structure preserving functions) from A to B. Throughout, we let $\mathbb{C}$ denote the field of complex numbers, and $k \subseteq \mathbb{C}$ a Galois extension of fields.

2. Varieties, Groups, and Twisted Forms

Let us begin by giving an overview of varieties defined over the field of complex numbers. An *affine* $\mathbb{C}$-*variety* is a subset of $\mathbb{C}^n$ of the form

$$\mathbf{V}(I) = \{\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{C}^n \mid f(\mathbf{a}) = 0 \text{ for all } f \in I\}$$

for some ideal $I \trianglelefteq \mathbb{C}[x_1, \dots, x_n]$. Given a finite set of polynomials $f_1, .., f_m \in \mathbb{C}[x_1, \dots, x_n]$, we similarly define the variety given by the set of simultaneous zeros of all the f_i, i.e., $\mathbf{V}(f_1, \dots, f_n) = \{\mathbf{a} \in \mathbb{C}^n \mid f_i(\mathbf{a}) = 0 \text{ for all } f_i\}$. This coincides with the variety associated to the ideal generated by the f_i, and any variety is given by the vanishing of finitely many polynomials, a consequence of the Hilbert Basis Theorem [Eis95, Thm. 1.2].

Conversely, given any affine $\mathbb{C}$-variety $V \subseteq \mathbb{C}^n$, we obtain *the defining ideal* of V

$$\mathbf{I}(V) = \{f \in \mathbb{C}[x_1, \dots, x_n] \mid f(\mathbf{a}) = 0 \text{ for all } \mathbf{a} \in V\}.$$

The quotient ring $\mathbb{C}[V] \coloneqq \mathbb{C}[x_1, \dots, x_n]/\mathbf{I}(V)$ is the *coordinate ring* of V. Notice that if $W \subseteq V$ is an inclusion of affine $\mathbb{C}$-varieties, then $\mathbf{I}(V) \subseteq \mathbf{I}(W)$. The constructions $\mathbf{I}$ and $\mathbf{V}$ are are inverses of one another if we restrict to a nice class of ideals. Indeed, for any ideal $I \trianglelefteq \mathbb{C}[x_1, \dots, x_n]$, we have

$$\mathbf{I}(\mathbf{V}(I)) = \sqrt{I} = \{f \in \mathbb{C}[x_1, \dots, x_n] \mid f^m \in I \text{ for some } m \in \mathbb{Z}\}$$

by Hilbert's Nullstellenstatz [Eis95, Thm. 1.6]. This result, which crucially relies on the fact that $\mathbb{C}$ is algebraically closed, yields a bijective correspondence between affine varieties and radical ideals (i.e., ideals satsifying $I = \sqrt{I}$). In other words, the geometric object, given by solution sets of polynomials, completely determines the underlying algebraic data, and vice versa.

The natural maps between affine $\mathbb{C}$-varieties are given by polynomial functions. If $V \subseteq \mathbb{C}^n$ and $W \subseteq \mathbb{C}^m$ are affine $\mathbb{C}$-varieties, a *morphism* or *regular function* $\varphi : V \to W$ has both geometric and algebraic descriptions:

1. Geometric: A function $\varphi : V \to \mathbb{C}$ is *regular* if there is a polynomial $F \in \mathbb{C}[x_1, .., x_n]$ such that $\varphi(x) = F(x)$ for all $x \in V$. A function $\varphi : V \to W$ is *regular* if there exist regular functions $\varphi_1, .., \varphi_m : V \to \mathbb{C}$ such that $\varphi(x) = (\varphi_1(x), \dots, \varphi_m(x))$ for all $x \in V$.
2. Algebraic: A regular function $V \to W$ is equivalent to the data of a $\mathbb{C}$-algebra homomorphism $\mathbb{C}[W] \to \mathbb{C}[V]$. If $\varphi : V \to W$ is a regular function, the corresponding $\mathbb{C}$-algebra homomorphism is defined as $\varphi^*(f) = f \circ \varphi$, for $f \in \mathbb{C}[W]$.

An *isomorphism* of affine $\mathbb{C}$-varieties is a regular map which has a regular inverse. Equivalently, an isomorphism of affine $\mathbb{C}$-varieties V and W is given by an isomorphism of $\mathbb{C}$-algebras $\mathbb{C}[W] \xrightarrow{\sim} \mathbb{C}[V]$. A general $\mathbb{C}$-variety is a space which locally looks like an affine $\mathbb{C}$-variety, i.e., each point has a neighborhood which is isomorphic to an affine $\mathbb{C}$-variety. If V and W are $\mathbb{C}$-varieties, a function $\varphi : V \to W$ is *regular* if for each $x \in V$ and any affine open subset $V \subseteq \mathbb{C}^m$ containing $\varphi(x)$, there is a neighborhood $x \in U \subseteq V$ such that $\varphi(U) \subseteq V$ and $\varphi : U \to V$ is regular. This is analogous to the definition of a manifold, which is locally Euclidean space.

In the case where the base field is not algebraically closed, the Nullstellensatz no longer holds, and the geometric data of the variety is not enough to fully determine its underlying algebraic structure. For this reason, our definition of k-variety centers on algebraic data. For our purposes, we adopt (a simplified version of) the approach given in [Bor91, Spr98] using *k-structures* on varieties.

A *k-structure* on a $\mathbb{C}$-algebra A is a k-subalgebra $A_k \subseteq A$ such that the homomorphism $\mathbb{C} \otimes_k A_k \to A$ is an isomorphism. If A is a $\mathbb{C}$-algebra with k-structure A_k and $J \trianglelefteq A$, then J is *defined over k* if $J \cap A_k$ generates J as an ideal. If $\varphi : A \to B$ is a homomorphism of $\mathbb{C}$-algebras with k-structures A_k and B_k, then f is *defined over k* if $\varphi(A_k) \subseteq B_k$. An affine $\mathbb{C}$-variety $V \subseteq \mathbb{C}^n$ is *defined over k* if there exist $f_1, \dots, f_m \in k[x_1, \dots, x_n]$ such that $V = \mathbf{V}(f_1, \dots, f_m)$. From this perspective, k-varieties are given by $\mathbb{C}$-varieties whose defining polynomials are elements of $k[x_1, \dots, x_n]$, and we use this perspective onward.

Definition 2.1 (Affine k-variety). An *affine k-variety* is a finite collection of polynomials

$$X = \{f_i \in k[x_1, \dots, x_n]\}.$$

Such a collection gives rise to the following algebraic and geometric data:

1. Algebraic: The defining ideal $\mathbf{I}(X) = (f_1, \dots, f_m)$ and coordinate ring $\mathbb{C}[X]$, which admits a k-structure given by $k[X] := k[x_1, \dots, x_n]/\mathbf{I}(X) \subseteq \mathbb{C}[X]$. Relative to this k-structure, the ideal $\mathbf{I}(X)$ is defined over k.
2. Geometric: The set of *k-rational points* (or simply *k-points*) is given by
$$X(k) := \{\mathbf{a} \in k^n \mid f_i(\mathbf{a}) = 0 \text{ for all } f_i\} \subseteq k^n.$$
More generally, for any field extension $k \subseteq F$, the set of *F-points* is given by
$$X(F) := \{\mathbf{a} \in F^n \mid f_i(\mathbf{a}) = 0 \text{ for all } f_i\}.$$

Notice that the solution set of the defining equations of a variety is just one piece of information that we can extract. The ability to consider the set of solutions over extensions of k allow us to probe deeper questions about the arithmetic of varieties.

We may refer to a variety by describing its set of points, but will often remind the reader of its full variety structure when necessary, i.e., we write $X = \{f_i = 0\}$ to denote a variety X defined by $\{f_i\}$. In certain cases, e.g., the pointless conic of Example 2.4, varieties may have no points at all, yet still provide interesting examples of "geometric objects," via their algebraic and arithmetic data. In these cases (especially in Section 4) it is necessary to include the full variety structure.

The sets given by the collection of common zeros of finitely many polynomials are the closed sets in a topology on k^n called the *Zariski topology*. If we view a polynomial $f \in k[x_1, .., x_n]$ as a function $k^n \to k$, then the Zariski topology is the coarsest topology in which all polynomials are continuous [Kem11, Rmk 3.3(g)]. Algebraic varieties are often described in terms of sets together with this topology. For instance, the set k^n together with its Zariski topology is the variety $\mathbb{A}_k^n$ called *affine n-space*. We can also describe it using a (quite trivial!) polynomial equation.

Example 2.2 (Affine space). Consider the k-variety $\mathbb{A}_k^n$ whose defining polynomial is the 0 polynomial in $k[x_1, \dots, x_n]$. Since every element of k^n vacuously satisfies this equation $0 = 0$, the k-points of $\mathbb{A}_k^n$ are given by $\mathbb{A}_k^n(k) = k^n$. We will often denote this variety simply by k^n, e.g., $\mathbb{C}^n$ denotes the variety $\mathbb{A}_{\mathbb{C}}^n$ and $\mathbb{R}^n$ denotes $\mathbb{A}_{\mathbb{R}}^n$.

Note that any affine variety X can be viewed as a *subvariety* of affine space. This is easiest to see on points. Indeed, a point of X is a simultaneous solution of its defining equations. But such solutions are elements of $k^n = \mathbb{A}_k^n$ by definition.

Example 2.3 (Circle group). Consider the $\mathbb{R}$-variety $X = \{x^2 + y^2 - 1 = 0\}$. Then $X(\mathbb{R})$ is given by the unit circle in $\mathbb{R}^2$. For that reason, we denote this variety by S^1. It also has the structure of a group, which comes from the action of rotation matrices on the coordinates x, y of the ambient space $\mathbb{R}^2 = \mathbb{A}_{\mathbb{R}}^2$, and we call this variety the *circle group*. We will encounter a few other *algebraic groups* below (Example 2.5 and Subsection 2.1).

Example 2.4 (Pointless real conic). Consider the $\mathbb{R}$-variety $X = \{x^2 + y^2 + 1 = 0\}$. Then $X(\mathbb{R}) = \emptyset$, so there exist $\mathbb{R}$-varieties which have no $\mathbb{R}$-points, a fact we all remember from solving equations in grade school! Even though this variety has no points, it is still useful to view it as a one-dimensional geometric object, i.e., a curve. This is the benefit of remembering its defining equation (i.e., its full structure as a variety). This reflects the arithmetic complexity of $\mathbb{R}$ versus that of $\mathbb{C}$, particularly the fact that $\mathbb{C}$ is algebraically closed while $\mathbb{R}$ is not. We expect arithmetically complicated fields to admit many examples of such varieties.

Example 2.5 (Multiplicative group). Consider the $\mathbb{R}$-variety $X = \{xy - 1 = 0\}$. The set of $\mathbb{R}$-points of X is given by a hyperbola in $\mathbb{R}^2$. Projecting this hyperbola onto the x-axis gives us $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. This variety is called the *multiplicative group* and is an algebraic group, i.e., it simultaneously has the structure of an affine variety and a group. When we wish to remember its structure as a variety (i.e., its defining equation), we will denote it by $\mathbb{G}_m$ or $\mathbb{G}_{m,\mathbb{R}}$. We will also denote this variety by $\mathbb{R}^*$, since this set defines its $\mathbb{R}$-points.

The defining equation of $\mathbb{G}_{m,\mathbb{R}}$ is equivalent to $y = x^{-1}$, and so this variety is often described by the polynomial ring $\mathbb{R}[x, x^{-1}]$. This gives an intuitive understanding of how polynomial functions define our varieties. When we restrict our attention to non-zero elements of $\mathbb{R} = \mathbb{A}_{\mathbb{R}}^1$, both x and $\frac{1}{x}$ are well-defined functions which satisfy the relation $x \cdot \frac{1}{x} = 1$. Conversely, the one-dimensional geometric space on which both x and $\frac{1}{x}$ are well-defined is given by the non-zero elements of $\mathbb{R}$. We also realize $\mathbb{G}_{m,\mathbb{R}}$ as an open subset of the affine line. In the Zariski topology, the polynomial function $f(x) = x$ has zero set $\{0\}$, which is a closed set. Its complement is then the open set $\mathbb{R}^*$, a *quasi-affine variety*.

We may equally well view X as a variety over $\mathbb{Q}$ or $\mathbb{C}$, or any field k; they all contain 0 and 1, and this is all that is needed for its defining equation. In these cases, we write $\mathbb{G}_{m,\mathbb{Q}}$, $\mathbb{G}_{m,\mathbb{C}}$, or $\mathbb{G}_{m,k}$ but will also use $\mathbb{Q}^*$, $\mathbb{C}^*$, and k^*, respectively. We also refer to the multiplicative group as the *one-dimensional torus*. This moniker is discussed in Section 3.

Definition 2.6. Let X and Y be affine k-varieties. A *morphism* $X \to Y$ is given by the data of a $\mathbb{C}$-algebra homomorphism $\varphi : \mathbb{C}[Y] \to \mathbb{C}[X]$ which is defined over k, i.e., φ satisfies $\varphi(k[Y]) \subseteq k[X]$. A morphism $X \to Y$ is an isomorphism if the corresponding map of $\mathbb{C}$-algebras is an isomorphism defined over k.

A morphism $X \to Y$ of k-varieties defined over k induces a function $X(F) \to Y(F)$ on F-points for any $k \subseteq F$, which makes senses even if the domain has no points (by including the empty set as a subset). A nice class of examples is provided by *twisted-linear subvarieties* of Severi-Brauer varieties [GS06, §5.1]. If the domain has a point, the image of this point is then a point in the codomain. This is a useful fact that can be used to argue the nonexistence of morphisms between k-varieties, e.g., any $\mathbb{R}$-variety which has $\mathbb{R}$-points cannot admit a morphism to the pointless $\mathbb{R}$-conic in Example 2.4.

Examples 2.3 and 2.4 above show us the difficulty in determining when a k-variety admits rational points. Indeed, the circle group and pointless conic have nearly identical defining equations, but starkly contrasting algebro-geometric properties; one has lots of $\mathbb{R}$-points and a group structure while the other has neither.

A related problem is to determine when a given variety contains a dense open subset which is isomorphic to an open subset of affine space $\mathbb{A}_k^n$. Such varieties are called *rational*, and this property is equivalent to having a parametrization by rational functions. Such varieties have lots of rational points, corresponding to all the points of $\mathbb{A}_k^n$. Again, S^1 and the pointless conic are quite distinct in this regard. The variety S^1 is rational (stereographic projection onto a line is one way to see this), while the pointless conic is not rational since it does not have any rational points.

Determining whether a given variety is rational (or unirational, stably rational, retract rational) is a long-standing problem in arithmetic and algebraic geometry. Recent work has invoked the use of sophisticated (co)homological techniques, including Chow groups, unramified cohomology, and derived categories (see [AB17, Pir18] for surveys on these topics). In Section 5 we provide a discussion on a particular success story in the theory of derived categories.

2.1. Algebraic groups.

Since our main objects of interest are certain varieties which admit a group structure (and partial compactifications thereof), we give a formal definition of algebraic groups, their morphisms, and additional examples.

Definition 2.7. An *algebraic group* or *group variety* over k is a k-variety G which also admits the structure of a group. In other words, an algebraic group is a variety together with the following regular maps (morphisms in the category of varieties)

1. group multiplication $m : G \times G \to G$
2. inclusion of the identity element $e : * \to G$
3. inversion $\mathrm{inv} : G \to G$.

Here, $*$ denotes the variety which consists of a single point, e.g., $\{x = 0\} \subseteq \mathbb{A}^1_k$. These maps are then required to satisfy additional compatibility properties, mimicking the usual group axioms and which we leave to the reader. We note that if G is an algebraic group, its set of points $G(k)$ has a group structure (in the usual sense of abstract groups), but this does not fully define the algebraic group structure on G. Given algebraic groups G and H, an *algebraic group morphism* $\varphi : G \to H$ is a regular map which preserves the algebraic group structure, i.e., a homomorphism of groups.

Example 2.8 (General linear group). Consider the polynomial ring $R = k[x_{11}, x_{12}, \dots, x_{nn}, y]$ in $n^2 + 1$ variables, and let $X = (x_{ij})$ be the matrix of indeterminates. The determinant of X is a polynomial, so that $\det(X)y - 1 \in R$. Let $\mathrm{GL}_{n,k}$ denote the affine k-variety defined by this polynomial. Notice that its k-points $\mathrm{GL}_n(k)$ are given by pairs (M, d) where M is an invertible $n \times n$ matrix with entries in k and $d = \det(M)^{-1}$. Since invertibility of a matrix is equivalent to its determinant being non-zero, the defining equation recovers the usual definition encountered in group theory. Moreover, this description realizes $\mathrm{GL}_{n,k}$ as a subvariety of $\mathbb{A}^{n^2+1}_k$.

Example 2.9 (Algebraic tori). It was mentioned above that $\mathbb{G}_{m,k} = k^*$ is also an algebraic group. In fact, we can realize it as a subvariety of $\mathrm{GL}_{n,k}$. Indeed, if we take our matrix of variables X, to be the diagonal matrix $\mathrm{diag}(x_{11}, 1, \dots, 1)$, then the equation $\det(X)y = 1$ becomes $x_{11}y = 1$. But this is exactly the defining equation of $\mathbb{G}_{m,k} = k^*$ (apart from the names of our indeterminants). More generally, $\mathbb{G}^n_{m,k} = \mathbb{G}_{m,k} \times \cdots \times \mathbb{G}_{m,k}$ is the subvariety of $\mathrm{GL}_{n,k}$ where we take our matrix of variables to be $X = \mathrm{diag}(x_{11}, x_{22}, \dots, x_{nn})$.

Example 2.10 (Determinant, special linear group). The determinant map $\det : \mathrm{GL}_{n,k} \to \mathbb{G}_{m,k}$ defines a morphism of algebraic groups and has kernel $\mathrm{SL}_{n,k}$, the affine k-variety given by $\det(X) = 1$. If we consider the k-points of each of these algebraic groups, we recover our usual determinant map $\det : \mathrm{GL}_n(k) \to k^*$, which has kernel $\mathrm{SL}_n(k)$.

2.2. Twisted forms.

For $k \subseteq \mathbb{C}$, any k-variety may be considered as a $\mathbb{C}$-variety simply by viewing the defining equations as having coefficients in $\mathbb{C}$. We denote this $\mathbb{C}$-variety by $X_{\mathbb{C}}$. The process of enlarging the field of coefficients of a given variety is called *base extension* or *extension of scalars*.

Example 2.11. Let $\mathbb{R}^* = \{uv - 1 = 0\}$ be the real multiplicative group and let $S^1 = \{x^2 + y^2 - 1 = 0\}$ be the circle group. As $\mathbb{R}$-varieties, these are non-isomorphic. Indeed, one may consider the invariant given by the number of connected components of $\mathbb{R}$-points. On the other hand, we note that the defining equations yield isomorphic varieties when we view them as polynomial equations over $\mathbb{C}$, i.e., after *extending scalars*. Indeed, the equation $uv - 1 = 0$ over $\mathbb{C}$ defines the variety $\mathbb{G}_{m,\mathbb{C}} = \mathbb{C}^*$, and we have an isomorphism $\mathbb{C}^* \cong S^1_{\mathbb{C}}$ given by $u \mapsto (x + iy)$ and $v \mapsto (x - iy)$.

Since varieties defined over $\mathbb{C}$ are completely determined by their geometric structure, we view $X_{\mathbb{C}}$ as a geometric "shadow" of X, and use its nicer geometric properties to extract information about X. For this reason, properties of k-varieties and morphisms which hold over $\mathbb{C}$ are often labelled as *geometric*. In the above example, $\mathbb{R}^*$ and S^1 are *geometrically isomorphic* since they are isomorphic over $\mathbb{C}$.

How can we get a handle on systematically studying examples of geometrically isomorphic varieties? One solution is to parametrize the collection of all such varieties in a way that reduces the study of k-varieties X to understanding their geometric avatar $X_{\mathbb{C}}$, and the arithmetic relationship between k and $\mathbb{C}$. This can be achieved using the theory of *Galois cohomology*. Given a Galois field extension $k \subseteq \mathbb{C}$, the *Galois group* $\mathrm{Gal}(\mathbb{C}/k) = \{\sigma \in \mathrm{Aut}(\mathbb{C}) \mid \sigma|_k = \mathrm{id}_k\}$ is the set of those field automorphisms of $\mathbb{C}$ which fix k pointwise. If X is a k-variety, the action of the Galois group on $\mathbb{C}$ induces an action on $X_{\mathbb{C}}$, where an automorphism $\sigma \in \mathrm{Gal}(\mathbb{C}/k)$ is applied to the coefficients of the defining polynomial equations.

Example 2.12. Consider the $\mathbb{R}$-variety $X = \{x^2 + y^2 = 0\}$. The $\mathbb{R}$-points of X are given by $X(\mathbb{R}) = \{(0, 0)\}$. If we consider X as a $\mathbb{C}$-variety, then we get a factorization of its defining equation. That is, $X_{\mathbb{C}} = \{(x + iy)(x - iy) = 0\}$. This variety consists of two pieces, given by $\{x = iy\}$ and $\{x = -iy\}$. The Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}/2$ acts on $X_{\mathbb{C}}$ via conjugation, and this action swaps its two components. It also acts on the $\mathbb{C}$-points, e.g., $(i, -1) \mapsto (-i, -1)$.

Definition 2.13 (Twisted form). Let X be a k-variety (resp., k-algebra). A *twisted form* of X is a k-variety (resp. k-algebra) Y such that $X_{\mathbb{C}} \cong Y_{\mathbb{C}}$. We let $\mathrm{Tw}(X)$ denote the set of isomorphism classes of twisted forms of X. This set is *pointed*, i.e., it comes with a canonical distinguished element given by X itself.

Example 2.14. As we saw in Example 2.11, the variety S^1 is a twisted form of $\mathbb{R}^*$. Indeed, we have $S^1_{\mathbb{C}} \cong \mathbb{C}^* \cong (\mathbb{R}^*)_{\mathbb{C}}$. This is a complete list, so that $\mathrm{Tw}(\mathbb{R}^*) = \{\mathbb{R}^*, S^1\}$, and the distinguished element is $\mathbb{R}^*$.

There is a natural (in the technical sense of the word) way to parametrize the collection of twisted forms of a given variety. This comes from the theory of profinite group cohomology and its application in the case of Galois groups associated to field extensions provides an invaluable tool in the study of twisted forms.

Theorem 2.15 ([Ser79, Ser02]). *There is a bijection of pointed sets* $\mathrm{Tw}(X) \cong H^1(k, \mathrm{Aut}(X))$.

Let us roughly define $H^1(k, \mathrm{Aut}(X))$, following [Jah00]. Given a finite group G, a *G-group M* is a group with an action of G such that $g \cdot (ab) = (g \cdot a)(g \cdot b)$ for all $a, b \in M$ and $g \in G$, so that the group structure of M is compatible with the group action of G. A *cocycle* from G to M is a function $\varphi : G \to M$ such that $\varphi(gh) = \varphi(g)g \cdot \varphi(h)$. This may be viewed as a twisted version of a group homomorphism, where the twisting occurs when moving an application of φ past g. Two cocycles φ and ψ are *cohomologous* if there is an element $b \in M$ so that $\psi(g) = b^{-1}\varphi(g)g \cdot b$ for all $g \in G$ (a "twisted conjugate"). This defines an equivalence relation on cocycles, and the set of equivalence classes is denoted $H^1(G, M)$. This set is pointed by the *trivial cocycle* given by $\varphi : G \to M$ via $\varphi(g) = e$ for all $g \in G$. We extend this definition to *profinite* groups, i.e., (projective) limits of finite groups, by taking the colimit of inflation maps over all open normal subgroups of G. Given a Galois extension $k \subseteq \mathbb{C}$, the group $\mathrm{Gal}(\mathbb{C}/k)$ is profinite. If X is a k-variety, the automorphism group $\mathrm{Aut}(X_{\mathbb{C}})$ of $X_{\mathbb{C}}$ is a $\mathrm{Gal}(\mathbb{C}/k)$-group, so we can consider its first group cohomology $H^1(\mathrm{Gal}(\mathbb{C}/k), \mathrm{Aut}(X_{\mathbb{C}}))$, which we denote by $H^1(k, \mathrm{Aut}(X))$ as above.

The advantage of the above result is that we have a purely algebraic and functorial method to parameterize the collection of all twisted forms of a given variety (as long as it is nice enough; Theorem 2.15 holds for quasi-projective varieties where $\mathrm{Aut}(X)$ is an algebraic group). This makes analysis of such twisted forms much more systematic.

Example 2.16. A classical example of the above framework is the bijective correspondence between Severi-Brauer varieties and central simple algebras. A *Severi-Brauer variety X* is a twisted form of projective space, i.e., $X_{\mathbb{C}} \cong \mathbb{P}^{n-1}_{\mathbb{C}}$. A *central simple algebra A* is a twisted form of a matrix algebra, i.e., $A \otimes_k \mathbb{C} \cong M_n(\mathbb{C})$. The automorphism group of both the matrix algebra $M_n(\mathbb{C})$ and projective space $\mathbb{P}^{n-1}_{\mathbb{C}}$ is the projective linear group $\mathrm{PGL}_n(\mathbb{C})$. Thus, we get bijections

$$\mathrm{Tw}(M_n(\mathbb{C})) \cong H^1(k, \mathrm{PGL}_n) \cong \mathrm{Tw}(\mathbb{P}^{n-1}_{\mathbb{C}}),$$

and therefore a bijection between Severi-Brauer varieties and central simple algebras. This assoication has an explicit description in low dimension (see Example 4.6).

3. Toric Varieties

We discuss the beautiful class of toric varieties via low-dimensional examples. Our goal is to put forth the systematic combinatorial description of these varieties using cones and fans. The standard references for this material are [Ful93, CLS11]. Throughout this section we only work over the field $\mathbb{C}$, and so will almost solely work with rational points. As such, we let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ denote the multiplicative group $\mathbb{G}_{m,\mathbb{C}}$, which we also call the *one-dimensional algebraic torus*. More generally, we let $(\mathbb{C}^*)^n = \mathbb{G}^n_{m,\mathbb{C}}$ denote the n-dimensional algebraic torus. The reason we use the term "torus" in this context (at the risk of ignoring the true etymology) is that the topological circle is a deformation retract of $\mathbb{C}^*$. Since products of the topological circle are called tori, it is natural to view products of $\mathbb{C}^*$ as algebraic tori.

Definition 3.1 (Toric variety). A *torus* is any algebraic group T isomorphic to $\mathbb{G}^n_{m,\mathbb{C}} = (\mathbb{C}^*)^n$. A *toric variety* is a variety X that contains a torus T as a dense open subset so that the natural group multiplication $T \times T \to T$ extends to an action $T \times X \to X$ of T on X.

The fact that X contains a torus as a dense open set allows us to view toric varieties as (partial) *compactifications* of tori, i.e., varieties which look nearly identical to $(\mathbb{C}^*)^n$, but which have some or all of the holes filled in. It turns out that our most basic examples of algebraic varieties are toric.

Example 3.2 (Affine space). Affine n-space has the structure of a toric variety. We have a dense open inclusion $(\mathbb{C}^*)^n \subseteq \mathbb{C}^n = \mathbb{A}^n_{\mathbb{C}}$. The action of $(\mathbb{C}^*)^n$ is via scaling, i.e., for $(\lambda_1, \dots, \lambda_n) \in (\mathbb{C}^*)^n$ and $(a_1, \dots, a_n) \in \mathbb{A}^n_k$, we have $(\lambda_1, \dots, \lambda_n) \cdot (a_1, \dots, a_n) = (\lambda_1 a_1, \dots, \lambda_n a_n)$. This clearly extends the group multiplication of $(\mathbb{C}^*)^n$.

Example 3.3 (Projective space). Projective n-space is defined via an equivalence relation on punctured affine space $\mathbb{A}^{n+1}_{\mathbb{C}} \setminus \{\mathbf{0}\}$, identifying any two points which are scalar multiples of one another. Points of $\mathbb{P}^n_{\mathbb{C}}$ are denoted $[x_0 : x_1 : x_2 : \cdots : x_n]$, so that $[x_0 : \cdots : x_n] = [\lambda x_0 : \cdots : \lambda x_n]$ for any $\lambda \in \mathbb{C}^*$. Two points of $\mathbb{A}^{n+1}_{\mathbb{C}}$ which lie on the same line yield the same point of $\mathbb{P}^n_{\mathbb{C}}$, so we view projective space as the collection of lines in $\mathbb{A}^{n+1}_{\mathbb{C}}$ passing through the origin.

Projective space is a k-variety since it is locally an affine variety. Indeed, any point $[x_0 : \cdots : x_n]$ with $x_0 \neq 0$ is equal to $[1 : X_1 : X_2 : \cdots : X_n]$, where $X_i = \frac{x_i}{x_0}$. The set of points of this form defines a copy of $\mathbb{A}^n_{\mathbb{C}}$ with coordinates given by $X_1, \ldots, X_n$. Proceeding with $x_1 \neq 0$, $x_2 \neq 0$, etc., we arrive at a union $\mathbb{P}^n_{\mathbb{C}} = \mathbb{A}^n_{\mathbb{C}} \cup \cdots \cup \mathbb{A}^n_{\mathbb{C}}$ of $n+1$ copies of $\mathbb{A}^n_{\mathbb{C}}$ which cover $\mathbb{P}^n_{\mathbb{C}}$. This description holds for any base field k, and this defines the (points of the) variety $\mathbb{P}^n_k$.

Projective space compactifies affine space. Indeed, in the case of the projective line we can write $\mathbb{P}^1_{\mathbb{C}} = \{[a : 1] \mid a \in \mathbb{C}\} \cup \{[1 : b] \mid b \in \mathbb{C}\} = \mathbb{A}^1_{\mathbb{C}} \cup \mathbb{A}^1_{\mathbb{C}}$. The first set in this union is only missing the point $[1 : 0] \in \mathbb{P}^1_k$ since any point of the form $[1 : b]$ with $b \neq 0$ can already be written as $[a : 1]$, via $[1 : b] = [\frac{1}{b}1 : \frac{1}{b}b] = [\frac{1}{b} : 1]$. The projective line is thus a union $\mathbb{P}^1_{\mathbb{C}} = \{[a : 1] \mid a \in \mathbb{C}\} \cup \{[1 : 0]\}$. The first subset is $\mathbb{A}^1_{\mathbb{C}}$ and the second is the "point at ∞." The real projective line $\mathbb{P}^1_{\mathbb{R}}$ has points which are given by the circle. The complex projective line is given by the Riemann sphere (the one-point compactification of $\mathbb{C}$). The action of the torus $\mathbb{C}^*$ on $\mathbb{P}^1_{\mathbb{C}}$ is given by $\lambda \cdot [x : y] = [\lambda x : y]$, extending the action of $\mathbb{C}^*$ on itself.

Example 3.4 (Products)**.** Products of toric varieties are toric varieties, stemming from the fact that products of tori are tori. If X and Y are toric varieties, then we have dense inclusions $(\mathbb{C}^*)^n \subseteq X$ and $(\mathbb{C}^*)^m \subseteq Y$. Thus, we also have a dense inclusion $(\mathbb{C}^*)^{n+m} \cong (\mathbb{C}^*)^n \times (\mathbb{C}^*)^m \subseteq X \times Y$. One checks that the induced action extends the group multiplication of $(\mathbb{C}^*)^{n+m}$. In particular, the variety $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ is a toric variety. See also Figure 2, as well as Examples 4.7 and 4.9.

3.1. Extracting combinatorial data.

The amazing fact about toric varieties is that their geometry can be encoded combinatorially via *fans*, i.e., collections of cones in affine/Euclidean space. This comes from considering their *characters* and *cocharacters*, which are simply algebraic group morphisms (see Definition 2.7) to and from the one-dimensional torus $\mathbb{G}_{m,\mathbb{C}} = \mathbb{C}^*$.

Definition 3.5. Let G be an algebraic group. A *character* of G is an algebraic group morphism $\chi : G \to \mathbb{C}^*$. We let $\widehat{G} = \mathrm{Hom}(G, \mathbb{C}^*)$ denote the set of all characters of G. A *cocharacter* (sometimes called a *one-parameter subgroup*) of G is an algebraic group morphism $\lambda : \mathbb{C}^* \to G$. We let $\Lambda(G) = \mathrm{Hom}(\mathbb{C}^*, G)$ denote the collection of all cocharacters of G.

Proposition 3.6. *Let $T = (\mathbb{C}^*)^n$ be a torus*

1. *For each $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$, we have a character $\chi^m : (\mathbb{C}^*)^n \to \mathbb{C}^*$ defined by $\chi^m(t_1, \ldots, t_n) = t_1^{m_1} \cdot t_2^{m_2} \cdots t_n^{m_n}$. All characters of T are of the form χ^m for some $m \in \mathbb{Z}^n$, yielding a group isomorphism $\widehat{T} \cong \mathbb{Z}^n$.*
2. *For each $u = (b_1, \ldots, b_n) \in \mathbb{Z}^n$, we have a cocharacter $\lambda^u : \mathbb{C}^* \to (\mathbb{C}^*)^n$ defined by $\lambda^u(t) = (t^{b_1}, \ldots, t^{b_n})$. All cocharacters of T are of the form λ^u for some $u \in \mathbb{Z}^n$, yielding a group isomorphism $\Lambda(T) \cong \mathbb{Z}^n$.*

Remark 3.7. Given a torus $T = (\mathbb{C}^*)^n$, it is customary to use M and N to denote $\widehat{T}$ and $\Lambda(T)$, respectively. By the very definition of these *lattices*, duality yields a pairing $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$. If compatible bases are chosen for both M and N, this is just given by the dot product.

Remark 3.8. Characters $\chi : T \to \mathbb{C}^*$ define regular functions on T, and *rational functions* (regular functions defined on open dense subsets) on any toric variety $X \supseteq T$. Similarly, cocharacters $\lambda : \mathbb{C}^* \to T$ define curves in T and any toric T-variety X. The geometry of X is thus easier to extract from subsets of $N = \Lambda(T)$, while the algebraic properties of X are more easily extracted from subsets of $M = \widehat{T}$.

3.2. One-dimensional toric varieties (toric curves).

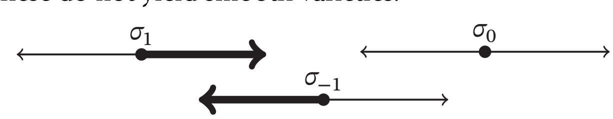
Let us exhibit the process for producing toric varieties from subsets of $N = \Lambda(T)$, beginning with the case of curves. Our focus will be on *smooth* or *nonsingular* varieties. If $T = \mathbb{C}^*$ is the one-dimensional torus, then $M = \widehat{T} \cong \mathbb{Z}$ and $N = \Lambda(T) \cong \mathbb{Z}$.

Step 1. Choose a *cone* in $N_{\mathbb{R}} \coloneqq N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}$.

Definition 3.9. Given an $\mathbb{R}$-vector space V, a *cone* in V is is a subset of the form

$$\sigma = \{\alpha_1 v_1 + \cdots + \alpha_s v_s \mid \alpha_i \in \mathbb{R}^{\geq 0}\}.$$

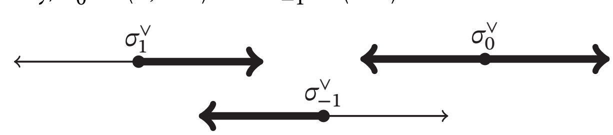
We can view cones as "vector half-spaces." For certain technical reasons, we only consider those cones which are *strongly convex*, *rational*, and *polyhedral* [Ful93, §1]. In dimension one, there are only three such cones σ_1, σ_0, and σ_{-1} in $N_{\mathbb{R}} = \mathbb{R}$, given by the non-negative real numbers, $\{0\}$, and the non-positive real numbers, respectively. We view these as those cones generated by 1, 0, and -1, respectively. While other cones do exist (e.g., the cone generated by 2), these do not yield smooth varieties.



Step 2. Compute the *dual cones* in $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}$.

Definition 3.10. Given a cone $\sigma \subseteq N_{\mathbb{R}}$, its *dual cone* is $\sigma^\vee = \{v \in M_{\mathbb{R}} \mid \langle v, u \rangle \geq 0 \text{ for all } u \in \sigma\}$.

The dual cones of σ_1, σ_0 and σ_{-1} are given below. We view these cones as being generated by powers of x, obtained by exponentiating the generators of σ_i. That is, $\sigma_1^\vee$ is generated by x^1 since $1 \in \mathbb{R} = M \otimes \mathbb{R}$ generates σ_1. Similarly, $\sigma_0^\vee = \langle x, x^{-1} \rangle$ and $\sigma_{-1}^\vee = \langle x^{-1} \rangle$.



Step 3. Having utilized the vector spaces $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$ for cone geometry, we now produce functions on our resulting toric variety, i.e., elements of M. For each cone $\sigma \subseteq N$, we can form the *monoid* $S_\sigma = \sigma^\vee \cap M \subseteq M$. In our one-dimensional case, these are given by

$$S_{\sigma_1} := \{1, x, x^2, \dots\} \cong \mathbb{N},$$

$$S_{\sigma_0} = \{\dots, x^{-2}, x^{-1}, 1, x, x^2, \dots\} \cong \mathbb{Z},$$

$$S_{\sigma_{-1}} = \{1, x^{-1}, x^{-2}, \dots\} \cong \mathbb{N}.$$

Step 4. The *monoid algebra* $\mathbb{C}[S_{\sigma_i}]$ associated to S_{σ_i} generalizes the common notion of the group algebra or group ring. These algebras have basis given by the elements of S_{σ_i}, so that a general element of $\mathbb{C}[S_{\sigma_i}]$ is given by a finite sum $\sum a_g g$ with $g \in S_{\sigma_i}$ and $a_g \in \mathbb{C}$. Addition is given component-wise and multiplication is the usual convolution product. In our case

$$\mathbb{C}[S_{\sigma_1}] = \mathbb{C}[\mathbb{N}] = \mathbb{C}[x],$$

$$\mathbb{C}[S_{\sigma_0}] = \mathbb{C}[\mathbb{Z}] = \mathbb{C}[x, x^{-1}],$$

$$\mathbb{C}[S_{\sigma_{-1}}] = \mathbb{C}[\mathbb{N}] = \mathbb{C}[x^{-1}].$$

Step 5. We determine generators and relations for our $\mathbb{C}$-algebras which yield polynomial equations, and in turn define affine varieties. We have

$$\begin{aligned} V_{\sigma_1} &:= \operatorname{Spec} \mathbb{C}[S_{\sigma_1}] = \operatorname{Spec} \mathbb{C}[x] = \mathbb{A}^1_{\mathbb{C}}, \\ V_{\sigma_0} &:= \operatorname{Spec} \mathbb{C}[S_{\sigma_0}] = \operatorname{Spec} \mathbb{C}[x, x^{-1}] \\ &= \operatorname{Spec} \mathbb{C}[x, y]/(xy-1) = \mathbb{C}^*, \\ V_{\sigma_{-1}} &:= \operatorname{Spec} \mathbb{C}[S_{\sigma_{-1}}] = \operatorname{Spec} \mathbb{C}[x^{-1}] = \mathbb{A}^1_{\mathbb{C}}. \end{aligned}$$

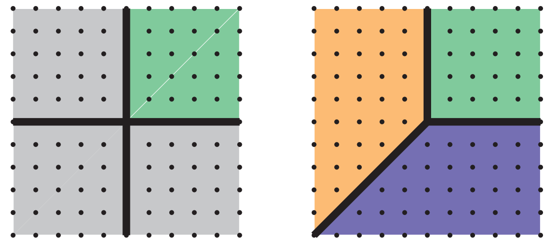
Step 6. We build new toric varieties from these affine ones by gluing. This is prescribed by the data of a collection of cones and the way in which they fit together.

Definition 3.11. A *fan* Σ in $N_{\mathbb{R}}$ is a collection of cones such that

1. each face of a cone in Σ is also a cone in Σ
2. the intersection of two cones in Σ is a face of each.

In our case, we take $\Sigma = \{\sigma_{-1}, \sigma_0, \sigma_1\}$, and we realize σ_0 as the intersection $\sigma_1 \cap \sigma_{-1}$. The fan Σ defines a toric variety $X(\Sigma)$ which is built from the affine pieces V_{σ_i} and the gluing data is encoded in the combinatorics of the fan structure. This defines the variety $\mathbb{P}^1_{\mathbb{C}}$ as the union of affine spaces from Example 3.3:

$$\begin{array}{ccccc} V_{\sigma_{-1}} & \supseteq & V_{\sigma_0} & \subseteq & V_{\sigma_1} \\ \| & & \| & & \| \\ \mathbb{A}^1_{\mathbb{C}} & \supseteq & \mathbb{C}^* & \subseteq & \mathbb{A}^1_{\mathbb{C}} \end{array}$$



(A) The fan for $\mathbb{A}^2_{\mathbb{C}}$ is given by a single cone, generated by $(1,0)$ and $(0,1)$. These correspond to the variables x and y in the lattice M.

(B) The fan for $\mathbb{P}^2_{\mathbb{C}}$. Note that it contains the cone for $\mathbb{A}^2_{\mathbb{C}}$. In fact, you can see the three copies of $\mathbb{A}^2_{\mathbb{C}}$ which give an affine cover described in Example 3.3.

Figure 1.

3.3. Two-dimensional toric varieties (toric surfaces).

In dimension 2, our torus is $T = (\mathbb{C}^*)^2$, so that $M \cong N \cong \mathbb{Z}^2$.

Example 3.12 (Affine plane, Figure 1A). Consider the cone σ in $N_{\mathbb{R}} \cong \mathbb{R}^2$ given by the first quadrant, i.e., generated by $(1,0)$ and $(0,1)$. The dual cone $\sigma^\vee$ is then given by the first quadrant in $M_{\mathbb{R}} \cong \mathbb{R}^2$ since this is the collection of vectors which have non-negative dot product with elements of σ. Intersecting this cone with M gives the generators x, y, so that the associated affine toric variety is given by $V_\sigma = \operatorname{Spec} \mathbb{C}[x, y] = \mathbb{A}^2_{\mathbb{C}}$.

Example 3.13 (Projective plane, Figure 1B). The fan given by three cones $\sigma_1 = \langle(1,0),(0,1)\rangle$, $\sigma_2 = \langle(0,1),(-1,-1)\rangle$, and $\sigma_3 = \langle(-1,-1),(1,0)\rangle$, as shown in Figure 1B, defines the variety $\mathbb{P}^2_{\mathbb{C}}$. Indeed, each cone gives a copy of $\mathbb{A}^2_{\mathbb{C}}$, and these are precisely the three affine patches described in Example 3.3. These are glued to one another according to the combinatorics of the cone intersections.

Example 3.14. Consider the fan generated by the four cones depicted in Figure 2. This yields the variety $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, which comes from the fact that products of fans correspond to products of the associated toric varieties. That is, given fans $\Sigma, \Delta \subseteq N_{\mathbb{R}}$, the fan $\Sigma \times \Delta := \{\sigma \times \tau \mid \sigma \in \Sigma, \tau \in \Delta\} \subseteq N \times N$ defines the toric variety $X(\Sigma \times \Delta) = X(\Sigma) \times X(\Delta)$. Since $\mathbb{P}^1_{\mathbb{C}}$ has an associated fan consisting of two cones (the positive and negative real lines) given in Step 6 above, the description of the fan of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ follows.

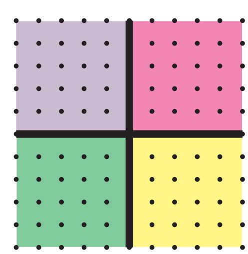
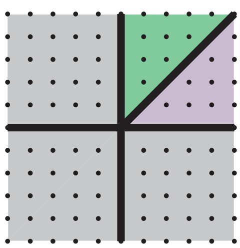


Figure 2. The fan for $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ consists of 4 cones, corresponding to the product of the fans which define $\mathbb{P}^1_{\mathbb{C}}$ as described in Example 3.14.

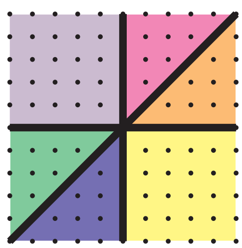
Example 3.15 (Blowups, Figures 3A and 3B). Recall that the *blowup* $\mathrm{Bl}_{\mathbf{0}}(\mathbb{A}^2_{\mathbb{C}})$ of $\mathbb{A}^2_{\mathbb{C}}$ at the origin is a subvariety of $\mathbb{A}^2_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ given by the equation $xw_1 = yw_0$, where (x, y) are coordinates on $\mathbb{A}^2_{\mathbb{C}}$ and $[w_0 : w_1]$ are coordinates on $\mathbb{P}^1_{\mathbb{C}}$. Covering this variety by its affine pieces $\{w_0 \neq 0\} = \mathbb{A}^2_{\mathbb{C}} \times \mathbb{A}^1_{\mathbb{C}} \cong \mathbb{A}^3_{\mathbb{C}}$ and $\{w_1 \neq 0\} \cong \mathbb{A}^3_{\mathbb{C}}$, our defining equation becomes $y = x\frac{w_1}{w_0}$ and $x = y\frac{w_0}{w_1}$ on these respective sets. We can thus generate our corresponding $\mathbb{C}$-algebras by x and $\frac{w_1}{w_0} = x^{-1}y$ on the first affine subset (since their product determines y) and by y and $\frac{w_0}{w_1} = xy^{-1}$ on the second (since their product determines x).

Consider the fan given in Figure 3A. A simple system of equations gives a description of the corresponding dual cones as $\langle x, x^{-1}y \rangle$ and $\langle y, xy^{-1} \rangle$, which are precisely the same generators that arise in the blowup construction! We see from this example that blowups can be described by subdivision of a cone.

The *del Pezzo surface of degree 6* over $\mathbb{C}$ is the blowup of $\mathbb{P}^2_{\mathbb{C}}$ at three non-colinear points. As such, we can recover its fan by staring with the fan of $\mathbb{P}^2_{\mathbb{C}}$ (see Figure 1B), and dividing each of its three cones. This results in the fan given in Figure 3B.



(A) The fan for the blowup of $\mathbb{A}^2_{\mathbb{C}}$ at the origin. The blowup construction has the effect of subdividing cones.



(B) The fan for the del Pezzo surface of degree 6, which is the blowup of $\mathbb{P}^2_{\mathbb{C}}$ at 3 non-colinear points. This fan is obtained from the fan of $\mathbb{P}^2_{\mathbb{C}}$ by subdividing each of its three cones.

Figure 3.

4. Arithmetic Toric Varieties

The advantage of working with toric varieties over $\mathbb{C}$ is the combinatorially encoded geometry, which allows one to effectively compute various algebro-geometric invariants using the data of fans. On the other hand, these varieties are geometrically and arithmetically simple since they are rational varieties and $\mathbb{C}$ is algebraically closed. We treat the appropriately analogous class of varieties over $k \subseteq \mathbb{C}$. The computationally effective understanding of the geometry of complex toric varieties allows one to corner arithmetic aspects of this new class of varieties defined over arbitrary fields. We revisit Example 2.11 to motivate a more general notion of "torus."

Example 4.1 (Non-isomorphic tori of equal dimension). The real one-dimensional torus $\mathbb{R}^* = \{uv = 1\}$ and the circle group $S^1 = \{x^2 + y^2 - 1 = 0\}$ are non-isomorphic $\mathbb{R}$-varieties but become isomorphic after extending scalars to $\mathbb{C}$. The isomorphism $\mathbb{C}^* \xrightarrow{\sim} S^1_{\mathbb{C}}$ is given by $u \mapsto x + iy$ and $v \mapsto x - iy$.

This example encourages us to enlarge our definition of "algebraic tori" to include those varieties which become isomorphic to $(\mathbb{C}^*)^n$ after base extension, i.e., *twisted forms* of the algebraic tori $(\mathbb{C}^*)^n = \mathbb{G}^n_{m,\mathbb{C}}$.

Definition 4.2 (Arithmetic toric variety). A *k-torus* (also called a $\mathbb{G}^n_{m,k}$*-torsor*) is a group variety T such that $T_{\mathbb{C}} \cong (\mathbb{C}^*)^n = \mathbb{G}^n_{m,\mathbb{C}}$. If T is a k-torus such that $T \cong (k^*)^n$, then we call T a *split torus*. An *arithmetic toric variety* is a variety with a faithful action of a torus T with dense open orbit. Given an arithmetic toric variety X with torus T, we say X is *split* if T is a split torus.

Remark 4.3. Example 4.1 shows that both $\mathbb{R}^*$ and S^1 are $\mathbb{R}$-tori, where the former is a split torus.

As in the case of toric varieties over $\mathbb{C}$, arithmetic toric varieties should be viewed as (partial) compactifications of algebraic tori (in this more general sense). The study of arithmetic toric varieties is relatively new, having been taken up over the past few decades. They were first used as tools to study general tori via their compactifications [Kun82, Kun87, Vos98]. A systematic classification and analysis via Galois cohomology was only treated within the last decade or two [Dun16, ELFST14]. The K-theory of tori and arithmetic toric varieties was also studied in [MP97], and the analysis therein was thematically aligned with the theory of non-commutative motives.

Example 4.4 (Split toric varieties). Over any field k, the split tori $\mathbb{G}^n_{m,k} = (k^*)^n$ provide a geometrically rich class of tori. The analysis of $(k^*)^n$-toric varieties via the associated combinatorial data of fans and cones is nearly identical to the complex case. One example of this phenomenon arises via the analysis of the divisor class group and Picard group. If X is a split toric variety with torus $T = \mathbb{G}^n_{m,k}$, there are canonical isomorphisms $\mathrm{Cl}(X) \cong \mathrm{Cl}(X_{\mathbb{C}})$ and $\mathrm{Pic}(X) \cong \mathrm{Pic}(X_{\mathbb{C}})$ [Dun16, §4].

Example 4.5 (Real projective space). The variety $\mathbb{P}^1_{\mathbb{R}}$ admits the structure of an arithmetic toric variety for two non-isomorphic tori, $\mathbb{R}^*$ and S^1. Viewed as an $\mathbb{R}^*$-toric variety, $\mathbb{P}^1_{\mathbb{R}}$ is a split arithmetic toric variety. When we extend scalars to $\mathbb{C}$, both tori become isomorphic to $\mathbb{C}^*$, and $(\mathbb{P}^1_{\mathbb{R}})_{\mathbb{C}} \cong \mathbb{P}^1_{\mathbb{C}}$.

Example 4.6 (Pointless real conic). The variety $X = \{x^2 + y^2 + z^2 = 0\}$ is an arithmetic toric variety with torus $S^1 = \{x^2 + y^2 = 1\}$, where the S^1-action is given by rotation matrices. Note that the pointless conic of Example 2.4 is an affine open subset in X. The variety X is not a toric variety for the split torus $\mathbb{R}^*$. Again, the torus S^1 is a twisted form of $\mathbb{C}^*$, and $X_{\mathbb{C}} \cong \mathbb{P}^1_{\mathbb{C}}$. This gives our first example of a (truly) non-split arithmetic toric variety. That is, no choice of torus gives X the structure of a split toric variety.

The above example arises naturally in the study of *central simple algebras*. Indeed, the variety X given above is the *Severi-Brauer variety* $\mathrm{SB}(\mathbb{H}) = \{x^2 + y^2 + z^2 = 0\}$ associated to Hamilton's quaternion $\mathbb{R}$-algebra $\mathbb{H}$. This is the four-dimensional $\mathbb{R}$-algebra given by

$$\mathbb{H} = \langle 1, i, j, k \mid i^2 = j^2 = -1, ij = -ji = k \rangle.$$

In fact, to any *quaternion* k-algebra $(a, b)_k := \langle 1, i, j, k \mid i^2 = a, j^2 = b, ij = -ji = k \rangle$ we have the *associated conic* $C(a, b) = \{ax^2 + by^2 = z^2\}$, which is a one-dimensional k-subvariety of $\mathbb{P}^2_k$ and a twisted form of $\mathbb{P}^1_{\mathbb{R}}$ [GS06]. Note that $\mathbb{H} = (-1, -1)_{\mathbb{R}}$ and $M_2(\mathbb{R}) = (1, 1)_{\mathbb{R}}$.

The associated conic is a special case of the general construction of the Severi-Brauer variety associated to a central simple algebra A, encountered in Example 2.16 via Galois cohomology. Such varieties are all examples of arithmetic toric varieties, although one needs to additionally specify a torus to completely determine their toric structure (just as in the case of $\mathbb{P}^1_{\mathbb{R}}$).

Example 4.7 (Weil restriction). Given a $\mathbb{C}$-variety X, we may view it as an $\mathbb{R}$-variety of twice the dimension by rewriting the defining equations of X using an $\mathbb{R}$-basis of $\mathbb{C}$. This process restricts our field of scalars to a subfield, and is called the *Weil restriction of scalars* [Vos98, §3.12]. For instance, taking $\{1, i\}$ as an $\mathbb{R}$-basis of $\mathbb{C}$, where $i = \sqrt{-1}$, any function $f \in \mathbb{C}[z]$ can be written in terms of the variables x_1, x_2 given by $z = x_1 + x_2 i$, and thus may be viewed as an element of $\mathbb{R}[x_1, x_2]$. We use the notation $\mathsf{R}_{\mathbb{C}/\mathbb{R}}(X)$ to denote the Weil restriction. Since the 0-polynomial in $\mathbb{C}[z]$ (the defining equation of $\mathbb{A}^1_{\mathbb{C}}$) corresponds to the 0-polynomial in $\mathbb{R}[x_1, x_2]$ in this basis, we have $\mathsf{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{A}^1_{\mathbb{C}}) = \mathbb{A}^2_{\mathbb{R}}$.

The Weil restriction is best described using points. If X is a $\mathbb{C}$-variety and $k \subseteq \mathbb{C}$, then the k-variety $\mathsf{R}_{\mathbb{C}/k}(X)$ has k-points given by

$$\mathsf{R}_{\mathbb{C}/k}(X)(k) = X(k \otimes_k \mathbb{C}) = X(\mathbb{C}),$$

i.e., the $\mathbb{C}$-points of X. Each $\mathbb{C}$-point must be specified using a k-basis of $\mathbb{C}$, so that $X(\mathbb{C})$ has dimension $[\mathbb{C} : k] \cdot \dim_{\mathbb{C}}(X(\mathbb{C}))$ as a k-variety. This is consistent with our example of the Weil restriction of the affine line above. More generally, we have $\mathsf{R}_{\mathbb{C}/k}(\mathbb{A}^n_{\mathbb{C}}) = \mathbb{A}^{n[\mathbb{C}:k]}_{\mathbb{R}}$.

Weil restrictions of tori are tori. For instance, $\mathsf{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$ has $\mathbb{R}$-points given by $\mathsf{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})(\mathbb{R}) = \mathbb{G}_{m,\mathbb{C}}(\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}) = \mathbb{C}^*$. Since $[\mathbb{C} : \mathbb{R}] = 2$, we have $\mathbb{C} = \mathbb{R}^2$ as sets, and taking units gives $\mathbb{C}^* = (\mathbb{R}^*)^2$. Extending scalars to $\mathbb{C}$, we obtain $(\mathbb{C}^*)^2$, and thus $\mathsf{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$ is a twisted form of $\mathbb{G}^2_{m,\mathbb{C}} = (\mathbb{C}^*)^2$. Similarly, one can show that $\mathsf{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{P}^1_{\mathbb{C}})$ is a twisted form of $\mathbb{P}^1 \times \mathbb{P}^1$.

This gives us many new examples of arithmetic toric varieties. The varieties $\mathsf{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{A}^n_{\mathbb{C}})$ and $\mathsf{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{P}^n_{\mathbb{C}})$ are arithmetic toric varieties with torus $T = \mathsf{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}^n_{m,\mathbb{C}})$. More generally, if X is a complex toric variety with torus T, then $\mathsf{R}_{\mathbb{C}/k}(X)$ is an arithmetic toric variety with torus $\mathsf{R}_{\mathbb{C}/k}(T)$.

Example 4.8 (Norm tori, [Vos98, p. 53]). A related construction is given by the *norm-one torus*. For $k \subseteq \mathbb{C}$ a Galois extension, the Weil restriction $\mathsf{R}_{\mathbb{C}/k}(\mathbb{G}_{m,\mathbb{C}})$ has k-points given by $\mathbb{G}_{m,\mathbb{C}}(k \otimes \mathbb{C}) = \mathbb{G}_{m,\mathbb{C}}(\mathbb{C}) = \mathbb{C}^*$. There is a natural map $N : \mathbb{C}^* \to k^*$ given by the field norm

$$z \mapsto \prod_{\sigma \in \mathrm{Gal}(\mathbb{C}/k)} \sigma(z).$$

This induces an algebraic group morphism $\mathsf{R}_{\mathbb{C}/k}(\mathbb{G}_{m,\mathbb{C}}) \to \mathbb{G}_{m,k}$. Its kernel is denoted $\mathsf{R}^{(1)}_{\mathbb{C}/k}(\mathbb{G}_{m,\mathbb{C}})$, called the norm-one torus.

The circle group S^1 arises as such a torus. If we apply the above construction in the case $k = \mathbb{R}$, real points of $\mathsf{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^*)$ are $\mathbb{C}^*$, and the field norm $\mathbb{C}^* \to \mathbb{R}^*$ is given by the product of a non-zero complex number and its conjugate $z \mapsto z\bar{z} = |z|^2$. This induces a group variety morphism $\mathsf{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}}) \to \mathbb{G}_{m,\mathbb{R}}$. Thus $\mathsf{R}^{(1)}_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^*)$ consists of all those complex numbers of modulus 1. This yields the unit circle S^1.

Example 4.9 (Products [Dun16, Ex. 5.8 and 6.9]). Just as in the split case (Example 3.4), products of arithmetic toric varieties are arithmetic toric varieties. This stems from the fact that products of twisted forms of tori are twisted forms of tori. We have already encountered one example of a twisted form of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ with its standard torus $(\mathbb{C}^*)^2$ in Example 4.7, given by $\mathsf{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{P}^1_{\mathbb{C}})$ with torus $\mathsf{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$. Let us dive further into this analysis for examples of products.

Of course, we have the split example $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$ with torus $(\mathbb{R}^*)^2$. We can use this same variety together with the non-split tori $S^1 \times \mathbb{R}^*$ and $S^1 \times S^1$. Let $X = \{x^2 + y^2 + z^2 = 0\}$ be the (projectivized) pointless conic encountered in Example 4.6. Then we have the arithmetic toric variety $X \times \mathbb{P}^1_{\mathbb{R}}$ with torus $S^1 \times \mathbb{R}^*$ or $S^1 \times S^1$. We can also replace both factors of $\mathbb{P}^1_{\mathbb{R}}$, to obtain $X \times X$ with torus $S^1 \times S^1$. This gives all possible twisted forms of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$.

Example 4.10 (See [Dun16, ELFST14]). Arithmetic toric varieties are twisted forms of split toric varieties, so they may be described via Galois cohomology, as in Theorem 2.15 and Example 2.16. If X is an arithmetic toric variety over k and T is the split torus of the split toric variety $X_{\mathbb{C}}$, the Galois group $\mathrm{Gal}(\mathbb{C}/k)$ acts on $X_{\mathbb{C}}$, so its elements can be viewed as automorphisms of $X_{\mathbb{C}}$. Since a toric variety is completely determined by its associated fan $\Sigma \subseteq N_{\mathbb{R}} = \Lambda(T)_{\mathbb{R}}$, these automorphisms of $X_{\mathbb{C}}$ determine automorphisms of Σ. This gives $\mathrm{Aut}(\Sigma)$ the structure of a $\mathrm{Gal}(\mathbb{C}/k)$-group, and the particular form X can be associated to a cocycle $\beta \in H^1(k, \mathrm{Aut}(\Sigma))$. This still leaves the choice of torus for the arithmetic toric variety structure on X, which is determined by a cocycle $\alpha \in H^1(k, {}^{\beta}T)$, where ${}^{\beta}T$ denotes the twisted form of T corresponding to the cocycle β. In general, any twisted form of X corresponds to a cocycle $\gamma \in H^1(k, T \rtimes \mathrm{Aut}(\Sigma))$.

The above cohomological description shows that the study of arithmetic toric k-varieties may be reduced to understanding split toric varieties $X_{\mathbb{C}}$ together with the action of the Galois group $\mathrm{Gal}(\mathbb{C}/k)$ on the fan associated to $X_{\mathbb{C}}$. In many cases, the automorphism group of the fan is easy to identify. For instance, the automorphism group of the fan of $\mathbb{P}^2_{\mathbb{C}}$ (Figure 1B) is S_3 given by permuting each of the three (maximal) cones. The automorphism group of the fan associated to $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ (Figure 2) is D_4, the dihedral group of order 8. And the automorphism group of the fan associated to dP6 (Figure 3B) is D_6, the dihedral group of order 12.

5. A.T.V.'s Are What Derive Us

Arithmetic toric varieties provide a wonderful class of objects on which to test the capabilities of certain invariants to reflect arithmetic and geometric data. One interesting invariant is given by the *coherent derived category* $\mathsf{D}^{\mathrm{b}}(X) = \mathsf{D}^{\mathrm{b}}(\mathsf{coh}(X))$, a triangulated category which probes the geometry of a variety through sheaves of modules. This should be viewed as a globalization of the study of rings via their modules. An affine variety X may be recovered from its defining ideal $\mathbf{I}(X)$ or associated coordinate ring $R = k[x_1, \dots, x_n]/\mathbf{I}(X)$ (Defn. 2.1). In this case, $\mathsf{D}^{\mathrm{b}}(X)$ has objects given by chain complexes of finitely-generated R-modules, where two complexes are identified if they are quasi-isomorphic, i.e., if their associated homology groups are isomorphic in all degrees. For a general variety, one considers the category of coherent $\mathcal{O}_X$-modules.

As an invariant, the derived category sits between algebra and topology. Viewing the derived category as an algebraic invariant, our interest lies in its decompositions into indecomposable pieces in the vein of vector spaces, modules, and representations. On the other hand, this category is non-abelian, and its triangulated structure allows one to use more topological approaches in the study of these decompositions. One such decomposition is given by an *exceptional collection*.

Definition 5.1. Let X be a k-variety and D a division k-algebra. An object $E \in \mathsf{D}^{\mathrm{b}}(X)$ is *D-exceptional* if (1) $\mathrm{End}(E) = D$, concentrated in degree 0, and (2) $\mathrm{Hom}(E, E[n]) = 0$ if $n \neq 0$ (here $E[n]$ denotes the shift of the complex E by n). An exceptional object is *étale* if D is a field extension of k. A totally ordered set $\{E_1, \dots, E_n\}$ of exceptional objects in $\mathsf{D}^{\mathrm{b}}(X)$ is an *exceptional collection* if $\mathrm{Hom}(E_j, E_i[n]) = 0$ for all n whenever $j > i$. It is *full* if the only triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(X)$ which contains every E_i is all of $\mathsf{D}^{\mathrm{b}}(X)$.

Remark 5.2. Throughout the literature, the term "exceptional" often means "k-exceptional," as we have defined above. This more restrictive definition is appropriate in the case where k is algebraically closed. The definition given here is a strict generalization. If k is algebraically closed, these definitions coincide since there are no non-trivial division algebras over such fields.

From the algebraic perspective, such a collection is analogous to decomposing a vector space via an (semi-)orthonormal basis. Indeed, the objects E_i should be viewed as basis elements of $\mathsf{D}^{\mathrm{b}}(X)$, which are semi-orthonormal relative to the (categorified) bilinear form $\mathrm{Hom}(-,-)$. From the topological viewpoint, such a collection provides a decomposition of the derived category into a collection of subcategories with maps in only one direction. One may view such maps as the instructions for gluing these subcategories together, analogous to a simplicial/cell complex.

The use of derived categories to study questions of rationality in the arithmetic setting was motivated by the success in the geometric case for rationality of three- and four-dimensional varieties. The use of $\mathsf{D}^{\mathrm{b}}(X)$ in determining whether a variety admits rational points was motivated by a question of H. Esnault. This was answered in the negative in [AAFH19]. Related work was carried out in [AKW17], showing that twisted forms cannot always be disinguished by the derived category, even in nice situations in low dimension. Examples in both cases were non-toric. Thus, it turns out the derived category is generally not an appropriate invariant to determine if a collection of polynomials has a common solution, although there is evidence that higher-categorical invariants might do the trick.

To better understand the derived category of an arithmetic toric variety, we would like to leverage the combinatorial tools available over $\mathbb{C}$. The following theorem shows that the Galois-theoretic philosophy described above holds for the derived category. Indeed, the existence of an exceptional collection on a k-variety X is guaranteed by the existence of an exceptional collection on $X_{\mathbb{C}}$ where the Galois group action permutes the elements of the exceptional collection (*stable*).

Theorem 5.3 (Descent for exceptional collections, [BDM19, Thm. 1.3])**.** *Let $k \subseteq \mathbb{C}$ be a Galois extension and let X be a smooth projective k-variety. Then X admits a (full, strong) exceptional collection if and only if $X_{\mathbb{C}}$ admits a* $\mathrm{Gal}(\mathbb{C}/k)$*-stable (full, strong) exceptional collection.*

Recently, arithmetic toric varieties have been successfully used to determine the limitations of the derived category in the arithmetic setting in regards to questions of rationality. In particular, while the existence of a k-exceptional collection is sufficient to determine that an arithmetic toric variety is rational, the existence of a general exceptional collection is not sufficient to determine whether a variety is even *retract rational*, a much weaker property.

Theorem 5.4 ([BDLM, Thm. 1 and 2]). *(1) Let X be a smooth projective arithmetic toric variety over a field k with $X(k) \neq 0$. If $\mathsf{D}^b(X)$ admits a full k-exceptional collection, then X is k-rational. (2) There exists a smooth threefold X defined over $\mathbb{Q}$ which admits a full étale exceptional collection but is not k-rational.*

With this success story in hand, it is our hope that other invariants may be investigated using the class of arithmetic toric varieties. Indeed, such varieties are so well-suited to act as a proving ground for analyzing geometric invariants and their ability to faithfully reflect arithmetic information.

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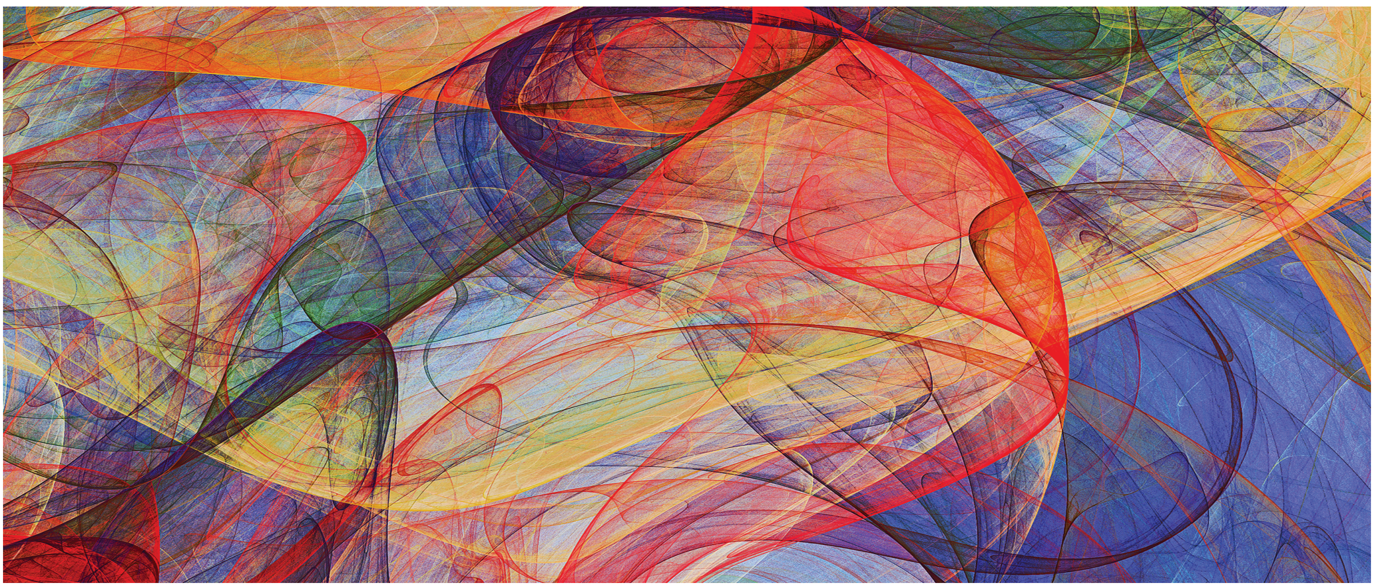
K-stability: The Recent Interaction Between Algebraic Geometry and Complex Geometry



Chenyang Xu

Solution spaces of polynomial equations in $\mathbb{C}^n$ or its natural compactification $\mathbb{CP}^n$ form the basic objects to study in algebraic geometry. Because such a space, called *an algebraic variety*, has an underlying topology inherited from the Euclidean topology of the complex spaces, it can also be studied through complex differential geometry. The interplay between these two kinds of geometries has been investigated by some of the greatest minds in mathematical history, started from Abel, Jacobi, Riemann, Weierstrass, Enriques, Lefschetz, Hodge, Weil, Chern, Kodaira, to more recently Serre, Mumford, Griffiths, Siu, Deligne, Yau, Mori, Kollár, Donaldson, Demailly, Tian, Voisin etc, and it leads to monumental progress in mathematics.

In this article, we want to convince the reader as with previous landmark theorems, in the last decade, there is another major step forward along this direction. The research originates from the searching of Einstein metrics on a manifold, with a root from physics. This topic has prevailed in geometry for more than a century, and on a complex variety, one should consider Kähler metrics. The most challenging case is when the variety has a positive Chern class, called *a Fano variety*. The research results in purely algebro-geometric theorems such as the construction of moduli spaces parametrizing Fano varieties, which is far beyond the original scope of the field. In fact, an astonishingly huge amount of mathematical topics, including geometric analysis, metric geometry, pluripotential theory, geometric invariant theory, non-archimedean geometry, higher dimensional geometry etc., are linked together by one theme.

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The central notion is called *K-stability*. The term was coined by Tian, following Mabuchi's definition of K-energy, where "K" stands for the first letter of the German word "Kanonisch." It has been known for decades that the solution of the Kähler-Einstein Problem has striking consequences in algebraic geometry, as exhibited famously in Yau's work. The relation between the existence of canonical metrics on algebraic objects and its stability from the Geometric Invariant Theory has also been intensively investigated. In the more recent episode, to understand the Kähler-Einstein Problem of Fano varieties, one crucial shift in viewpoint is the realization that one has to go beyond the Geometric Invariant Theory and invoke the tools provided by the Minimal Model Program. This bridges two previously well-studied areas. Discovering it not only sheds new light on longstanding central questions and eventually leads to their complete solutions, but also points toward unseen research directions.

In this article, we will discuss some of the main ideas involved in K-stability theory stemmed from different parts of mathematics, with a focus on the recently founded algebraic part of the story.

Canonical metrics and stability. A smooth complex variety X admits a class of metrics which are called *Kähler*. More precisely, for any Riemannian metric g, there is an associated 2-form $\omega(X,Y) = g(JX,Y)$ where J is the complex structure on the tangent bundle TX. The Kähler condition is equivalent to saying that ω is a closed 2-form. While a given Kähler metric ω is analytic, its class $[\omega] \in H^2(X,\mathbb{R})$ is a topological invariant. For a Kähler form ω, one can attach the Ricci form $\mathrm{Ric}(\omega)$. It is also a closed two form and a remarkable fact is that its class $[\mathrm{Ric}(\omega)]$ is the first Chern class $c_1(X)$.

Around the 50s, two fundamental questions became central in complex geometry. The first one is called the Calabi Conjecture, which claims

Given a compact Kähler manifold (X,ω) together with a 2-form R representing $c_1(X)$, one can always find a Kähler form $\tilde{\omega}$ such that $[\omega]=[\tilde{\omega}]$ and $\mathrm{Ric}(\tilde{\omega}) = R$.

This conjecture was proved in Yau's famous work in the late 70s. The second question, called the *Kähler-Einstein Problem*, first asked by Kähler in the 30s and later advertised by Calabi, considers whether there exists a Kähler metric satisfying the Einstein equation. This requires the existence of a Kähler form whose class is proportional to $c_1(X)$, and that the Kähler-Einstein Problem is equivalent, to ask whether the equation $c_1(X) = \lambda\cdot[\omega]$ ($\lambda = -1, 0$ or $+1$) can be lifted to the level of form, i.e.,

Does there always exist a Kähler form ω_{KE} on X, such that $\mathrm{Ric}(\omega_{\mathrm{KE}}) = \lambda\cdot\omega_{\mathrm{KE}}$?

A solution ω_{KE} yields a *Kähler-Einstein metric*.

When $\dim_{\mathbb{C}} X = 1$, the Kähler-Einsten metric is given by the Poincaré Uniformization Theorem.

One immediately sees that when $\lambda = 0$, X always has a Kähler-Einstein metric by the solution of the Calabi Conjecture, since one can choose the form representing $c_1(X)$ to be 0. It was also proved, by Aubin and Yau independently, when $\lambda = -1$, X admits a Kähler-Einstein metric. These results have tremendous influence on people's understanding of algebraic varieties with $c_1(X)$ being zero or negative.

When $\lambda = 1$, X is called *Fano* (named after Italian mathematician Fano). In this case, the Kähler-Einstein Problem becomes more subtle as there is no definitive answer. The first obstruction was found in the late 50s, when Matsushima showed that a Fano manifold X has a Kähler-Einstein metric would require $\mathrm{Aut}(X)$ to be reductive. This can be applied to, e.g., the blow up X of a point on $\mathbb{CP}^2$ to conclude it does not admit a Kähler-Einstein metric. In the early 80s, Futaki found a deeper obstruction which is the vanishing of $\mathrm{Fut}(v)$ for any vector field v on X. Here $\mathrm{Fut}(v)$ is a numerical invariant that we will discuss more later.

In the late 80s, Tian proved that for a compact complex Fano surface X, $\mathrm{Aut}(X)$ being reductive is the necessary and sufficient condition for admitting a Kähler-Einstein metric. However, the proof was a case-by-case study and there was no reason to believe this would be the general situation.

In fact, Yau had a profound speculation that for a Fano manifold X, having a Kähler-Einstein metric should be equivalent to deeper algebraic properties of X, namely certain kind of stability. There are two reasons to make the speculation. Firstly, based on the solution of the Calabi Conjecture, it was believed that a variant of C^0-estimate would be the key analytic input, but such an estimate should be governed by algebraic data. Secondly, prior to the Kähler-Einstein Problem, this philosophy of canonical metric/stability correspondence had already appeared in a different setting: by the work of Donaldson and Uhlenbeck-Yau, a vector bundle on a compact Kähler manifold admits a Hermitian-Einstein metric if and only if it is slope polystable, which is now called *the Hitchin-Kobayashi Correspondence*. Here the Hermitian-Einstein on a vector bundle has a similar nature to the Kähler-Einstein metric. However, the non-linear feature of varieties makes the Kähler-Einstein Problem manifestly harder, and to deal with the difficulty, there is a major shift of viewpoint in regard to how to understand the stability condition in algebraic geometry.

In the late 90s, Tian [Tia97] proposed the precise statement and defined the notion of *K-stability*: For a given Fano manifold X, Tian considers $\mathbb{C}^*$-equivariant families

$\mathcal{X} \to \mathbb{C}$, such that the fiber $\mathcal{X}_t \cong X$ for any $t \neq 0$, and the special fiber X_0 is *mildly singular* (we will discuss this technical assumption later), so that one can define *the (generalized) Futaki invariant* $\mathrm{Fut}(\mathcal{X}) := \mathrm{Fut}(X_0, v)$ where v is the vector field on X_0 induced by the $\mathbb{C}^*$-action. Then Tian showed that X admits a Kähler-Einstein metric implies $\mathrm{Fut}(\mathcal{X}) \geq 0$ for all $\mathcal{X}$ and the equality holds only when $\mathcal{X} \cong X \times \mathbb{C}$. This latter condition posted on X, which is of a global nature, is called *K-polystablity*.

Later Donaldson [Don02] considered a similar setting of $\mathbb{C}^*$-equivariant degenerations but for all polarized projective varieties (X, L) with no assumptions on the degeneration, and defined the Futaki invariant *algebraically*. Under this setting, one can extend the study to the existence of a constant scalar curvature metric, but we will only focus on Fano varieties in this article.

One major step forward in the field is that the converse direction of Tian's theorem is also true.

Theorem (Yau-Tian-Donaldson Conjecture). *Let X be a Fano variety, then X admits a Kähler-Einstein metric if and only if X is K-polystable.*

The Yau-Tian-Donaldson Conjecture was first proved for smooth Fano manifolds by Chen-Donaldson-Sun [CDS15] and Tian [Tia15], following the strategy called the Cheeger-Colding-Tian theory in Riemannian geometry.

Later based on many earlier works of understanding the geometry of the space of Kähler metrics, notably by Mabuchi, Tian, Donaldson, Chen, Eyssidieux, Guedj, Zeriahi, Berndtsson, Darvas, and many others, Berman-Boucksom-Jonsson [BBJ21] proceeded along a completely different route, namely the variational approach, which focuses on (the completion of) the space of Kähler metrics. The argument was extended by Li-Tian-Wang and later to all singular cases by Li [Li22]. It relies on far less differential geometry input than the Cheeger-Colding-Tian theory. However, on the algebraic side it has to assume a stronger condition called *uniform K-stability*.

Therefore, it remains to verify the subtle equivalence between uniform K-stability and K-stability, which is an algebraic question. Built on the powerful machinery developed in the Minimal Model Program, the equivalence is eventually established by Liu-Xu-Zhuang [LXZ21], as part of the project to construct the moduli space of Fano varieties. (The easier direction that the existence of Kähler-Einstein metric implies K-polystability was achieved earlier in [Ber16].) In the rest of this article, we will survey these ideas, with a focus on the algebraic theory.

For now let us explain why this kind of condition is called *stability*. The term "stability" used in algebraic geometry is often related to the construction of moduli spaces. While the connection between K-stability and moduli space is a profound topic that we will extensively discuss later, we first compare it with a more classical notion in the moduli theory, which is the *geometric invariant theory (GIT) stability*.

As far as we know, the notion of stability in algebraic geometry first implicitly appeared in Hilbert's work on invariant theory, and later was explicitly coined by Mumford in the geometric invariant theory, which considers the setting of a reductive group G acting on a projective variety M. To form the quotient space, which is the 'moduli space' parametrizing orbits, one has to throw away some *unstable* points on M. To make the setting pragmatical, one often also considers a very ample line bundle L over M such that the action on G can be lifted to L as linear maps $L_x \to L_{g(x)}$ for any x. Then the Hilbert-Mumford criterion says that to check whether $x \in M$ is stable, one only needs to consider for any one parameter subgroup $\mathbb{C}^* \subset G$, the sign of the weight of the $\mathbb{C}^*$-action on $L|_{x_0} \cong \mathbb{C}$ where $x_0 := \lim_{t\to 0} t \cdot x$.

Now let X be a Fano variety, and $[X]$ the corresponding point in the Hilbert scheme M, under an embedding $X \to \mathbb{P}^N$ induced by $|-rK_X|$ for some r. Let $\mathbb{C}^*$ be any subgroup of $\mathrm{PGL}_{\mathbb{C}}(N+1)$. The closure $\overline{\mathbb{C}^* \cdot [X]}$ yields a morphism $\mathbb{C} \to M$, and the pull-back of the universal family induces a $\mathbb{C}^*$-equivariant family $(\mathcal{X}, \mathcal{L})$ over $\mathbb{C}$. This is precisely the notion of a *test configuration* (with index r). Then the Futaki invariant for a test configuration $\mathcal{X}$ can be considered as the weight of a $\mathbb{C}^*$-action on the restriction of a line bundle, namely the *CM line bundle*, over the point $[X_0]$ where $[X_0] := \lim_{t\to 0} t \cdot [X]$. There is a deeper reason to adopt the GIT stability into the consideration, proposed by Donaldson, following the famous work of Atiyah-Bott. We first recall Kempf-Ness's interpretation of stability: let G be the complexification of a compact group K, and assume a projective manifold (M, L) admits a G-action with a K-invariant norm $\|\cdot\|$ on L. Define the function

$$f : G/K \to \mathbb{R}, \quad x \to \log \|g \cdot \hat{x}\|,$$

where $\hat{x}$ is a non-zero lift of x. This function is convex along any geodesic $\mathbb{C}^* \to G/K$, therefore f has a minimum on G/K if and only if the limit $\lim_{t\to\infty} f'(e^{it\xi} \cdot x)$ at infinity is positive for any $\xi \in \mathrm{Lie}(K)$. The latter is precisely the weight of the $\mathbb{C}^*$-action on $L|_{x_0}$, so combining with the Hilbert-Mumford criterion, we know

$$x \text{ is stable if and only if } f \text{ has a minimum.}$$

In the Kähler-Einstein Problem, one can view the space $\mathcal{H}$ of Kähler metrics with the same class as an infinite dimensional analogue of symmetric spaces, and the existence of the Kähler-Einstein metric is equivalent to the existence of a minimum for certain geometric functional, e.g., the Mabuchi functional, on $\mathcal{H}$. For a test configuration $(\mathcal{X}, \mathcal{L})$, the pullbacks ω_t of the (normalized) Fubini-Study metric along $X_t \subseteq \mathbb{P}^N$ gives a ray in $\mathcal{H}$. The derivative of the Mabuchi functional on ω_t at infinity is the Futaki invariant. This clear analogy makes the comparison of Kähler-Einstein Problem with the finite dimensional GIT fruitful.

However, there are two fundamental differences between the K-stability and the standard GIT: since we consider all $\mathcal{X}$, a priori we need to take into account all sufficiently large r; moreover, the CM line bundle is not ample. Therefore, the usual geometric invariant theory fails to apply here. With the benefit of hindsight, it is the more intrinsic Minimal Model Program theory which remedies the study of K-stability as an algebro-geometric subject and enables people to go beyond the GIT.

Fano varieties. As one of the three building blocks of an arbitrary variety, the class of Fano varieties has been the central subject in higher dimensional geometry for more than a century. The only Fano manifold in dimension one is $\mathbb{CP}^1$. In dimension two, it is a classical result that there are 10 families of them, namely blowing up m general points on $\mathbb{CP}^2$ for $0 \le m \le 8$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$. For dimension three, in the 80s Iskovskikh and Mori-Mukai classified that there are 105 smooth families. Fano manifolds often have rich geometry, as they could have many interesting birational models. However, this also makes the study of them intricate. Moreover, from the view of Minimal Model Program, one needs to also consider Fano varieties with mild singularities. In particular, these singular Fano varieties naturally appear as degenerations of Fano manifolds.

Example. To see a basic example, we look at the family given by the equation

$$(tx_0^d + \sum_{i=1}^{n} x_i^d = 0) \subset \mathbb{CP}^n \times \mathbb{C} \ \ (2 \le d < n). \qquad (1)$$

If $t \neq 0$, we get a Fano manifold X_t isomorphic to the Fano Fermat hypersurface $\sum_{i=0}^n x_i^d = 0$; and if $t = 0$, the Fano variety X_0 is a cone over a Fano hypersurface of one dimensional lower, with a singularity at $(1, 0, \dots, 0)$. Therefore, this family of Fano varieties is trivial over $\mathbb{C}^*$, but the algebraic structure changes over $t = 0$. We note that as a comparison, this phenomenon can not occur when $c_1(X) \le 0$. Moreover, let $f: Y \to X_0$ be the blow up at $(1, \dots, 0)$. The exceptional divisor is E, and a simple calculation yields

$$f^*(\omega_{X_0})((n-d-1)E) = \omega_Y.$$

Definition. Let E be a divisor over X, i.e., E is a divisor on a normal birational model $\mu: Y \to X$. We define *the log discrepancy along* E to be

$$A_X(E) = (\text{the coefficient of } K_{Y/X} \text{ along } E) + 1.$$

We say X has *Kawamata log terminal (klt)* singularities if $A_X(E) > 0$ for any divisor E over X. It is known it suffices to check $A_X(E) > 0$ for divisors E appearing on a fixed log resolution.

In particular, X_0 in (1) is a klt Fano variety.

Admittedly, it is not easy to see at the first glance why klt singularities are important. Nevertheless, built on four decades' experience accumulated by people working in the Minimal Model Program theory, it becomes clear that, from various viewpoints in higher dimensional geometry, klt varieties form the right class of singular varieties to be studied.

For instance, one non-trivial property, established in [LX14], is that klt Fano varieties are closed under degeneration: Let $X^\circ \to \Delta^\circ$ be a family of klt Fano varieties over a punctured disc $\Delta^\circ = \Delta \setminus \{0\}$, then after a possible base change of Δ°, we can fill in the limit with a klt Fano variety.

New ideas from algebraic geometry. The exploration of the connection between K-stability and the MMP was initiated in the work in Odaka [Oda13] and Li-Xu [LX14] around the early 2010s. After a small gap of a few years, it started to blossom around 2015. The meeting of these two subjects brings up many new insights to both sides. One main advantage is that this algebraic method provides a stronger tool to treat Fano varieties with singularities, which are currently beyond the reach of the original metric geometry approach.

There are *five* fundamental new ideas coming out to advance the study of K-stability via higher dimensional geometry. These new ideas are related to each other. Nevertheless, they are somewhat different. In my later discussions, I will elaborate three of them:

1. Various equivalent characterizations of K-stability.
2. The uniqueness of K-polystable degeneration.
3. The higher rank finite generation.

I will only briefly mention the last two:

4. The connection of the K-stability of fibers and the positivity of the CM line bundle on the base, via Harder-Narashimhan filtration.
5. The local K-stability theory via the normalized volume function.

Valuative criterion of K-stability. One fundamental development of the algebraic theory of K-stability, underlying the further progress in various directions, is the equivalent description of the notions of K-stability, using valuations over the function field $K(X)$.

Let X be a Fano variety with klt singularities, a prime divisor E yields a valuation ord_E given by the vanishing multiplicity along E for any meromorphic function, i.e., for any meromorphic function f,

$$\mathrm{ord}_E(f) \coloneqq \mathrm{mult}_E(\mathrm{div}(f)).$$

Definition. Let E be a prime divisor over X, i.e., E is a prime divisor on a normal birational model $\mu: Y \to X$. Let $\dim H^0(-mK_X) = N_m$, we can choose a basis

$\{s_1, \dots, s_{N_m}\}$ compatible with the filtration on $H^0(-mK_X)$ induced by ord_E, i.e., for any λ, the subspace

$$\begin{aligned} & \mathcal{F}_E^{\lambda} H^0(-mK_X) \\ = \quad & \{s \in H^0(-mK_X) \mid \mathrm{ord}_E(s) \geq \lambda\} \end{aligned} \tag{2}$$

is generated by s_i of $\{s_1, \dots, s_{N_m}\}$ with $s_i \in \mathcal{F}_E^{\lambda} H^0(-mK_X)$. We define

$$S_m(E) := \lim_{m\to\infty} \frac{1}{mN_m} \sum_{i=1}^{N_m} \mathrm{ord}_E(s_i),$$

and the *expected vanishing order*

$$S_X(E) = \frac{1}{(-K_X)^n} \int_0^{\infty} \mathrm{vol}(\mu^*(-K_X) - tE) dt.$$

Then $S_X(E) = \lim_{m\to\infty} S_m(E)$. We also denote *the stability threshold* by

$$\delta(X) = \inf_E \frac{A_X(E)}{S_X(E)},$$

where E runs through over all prime divisors over X.

Theorem (Valuative criterion). *We have the following equivalent characterization of notions in K-stability.*

1. *X is K-semistable if and only if $\delta(X) \geq 1$;*
2. *X is uniformly K-stable if and only if $\delta(X) > 1$; and*
3. *X is K-stable if and only if $A_X(E) > S_X(E)$ for any E.*

The Statement 1 is proved by Fujita [Fuj19] and Li [Li17], which we will explain below. A similar argument is used to prove Statement 2 in [Fuj19], as well as Statement 3 in [Fuj19, Li17] with a technical assumption, which was later removed by Blum-Xu [BX19].

Before explaining the ideas of proving the valuative criterion, to exemplify its strength, we first give an application to the Bishop-Gromov type Comparison Theorem for Kähler manifolds.

Example (Fujita). Let X be an n-dimensional K-semistable Fano variety, e.g., X admits a Kähler-Einstein metric. Then $(-K_X)^n \leq (n+1)^n$.

This can be seen in the following way. Let $\mu: Y \to X$ be the blow up of a smooth point $x \in X$ with the exceptional divisor E. Then

$$H^0(\mu^*(-mK_X) - tmE) = H^0(\mathcal{O}_X \otimes \mathfrak{m}_x^{[tm]}),$$

which implies that

$$\mathrm{vol}(-\mu^* K_X - tE) \geq (-K_X)^n - t^n.$$

The valuative criterion implies that

$$n(-K_X)^n \geq \int_0^{((-K_X)^n)^{\frac{1}{n}}} \left((-K_X)^n - t^n\right) \mathrm{d}t.$$

Therefore, $(-K_X)^n \leq (n+1)^n$.

What is striking about the valuative criterion is that K-stability notions are checked by looking at valuations which are apparently different from test configurations as in the original definition. A connection is observed by Boucksom-Hisamoto-Jonsson: for any irreducible component F of the special fiber X_0 of a test configuration $\mathcal{X}$, since $\mathcal{X}$ is birational to $X \times \mathbb{A}^1_t$, $K(\mathcal{X}) \cong K(X)(t)$, the restriction of ord_F on the subfield $K(X) \subseteq K(\mathcal{X})$ is of the form $a \cdot \mathrm{ord}_E$ for some divisor E over X, as long as F is not from $X \times \{0\}$.

A deeper result, proved in [LX14], says that started with any test configuration $(\mathcal{X}, \mathcal{L})$ we can use the Minimal Model Program techniques to run a process to obtain a new normal test configuration $\mathcal{Y}$ such that $-K_{\mathcal{Y}}$ is relatively ample and the central fiber X_0 is also a klt Fano variety. These test configurations are called *special*. Moreover, up to a base change $\mathbb{C} \xrightarrow{z^d} \mathbb{C}$,

$$\mathrm{Fut}(\mathcal{X}, \mathcal{L}) \geq \mathrm{Fut}(\mathcal{Y}, -K_{\mathcal{Y}}) := \mathrm{Fut}(\mathcal{Y}).$$

As a consequence, verifying K-stability on all test configurations as in Donaldson's definition, is equivalent to restricting over special test configurations. This subclass is smaller than the one allowed in Tian's definition, therefore, for Fano varieties, Donaldson's definition is indeed equivalent to Tian's.

Given a non-trivial special test configuration $\mathcal{X}$, let $a \cdot \mathrm{ord}_E$ be the valuation on $K(X)$ induced by the unique component X_0 as above. One can calculate

$$\mathrm{Fut}(\mathcal{X}) = a(A_X(E) - S_X(E)),$$

and we immediately conclude that if $\delta(X) \geq 1$, then X is K-semistable.

There are two different approaches to prove the converse direction. The first one was given in [Fuj19, Li17]. It needs the notion of Ding stability, which is algebraically defined by Berman [Ber16], by looking at the sign of the *Ding invariant* $\mathrm{Ding}(\mathcal{X}, \mathcal{L})$ for all test configurations $(\mathcal{X}, \mathcal{L})$. Berman-Boucksom-Jonsson [BBJ21] and Fujita [Fuj19] observed that the Minimal Model Program process in [LX14] also can be used to show testing Ding stability notions on general test configurations and special test configurations are equivalent. As a consequence, one immediately sees that K-stability notions are equivalent to the corresponding Ding-stability ones, as Futaki invariants are identical to Ding invariants on special test configurations.

Then Fujita makes the key observation that the definition of Ding invariant can be extended from test configurations to any linearly bounded (decreasing) multiplicative filtration $\mathcal{F}^{\bullet} := (\mathcal{F}^t R)_{t\in\mathbb{R}}$ on the anti-canonical ring $R = \bigoplus_m H^0(-mK_X)$, and Ding semistability implies $\mathrm{Ding}(\mathcal{F}^{\bullet}) \geq 0$ for any such filtration $\mathcal{F}^{\bullet}$. For a divisor E over X, one can define the filtration $\mathcal{F}_E^{\bullet}$ by (2). Moreover, we have

$$A_X(E) - S_X(E) \geq \mathrm{Ding}(\mathcal{F}_E^\bullet).$$

Therefore, K-semistability, which is equivalent to Ding semistability, implies that $\delta(X) \geq 1$.

Complements and log canonical places. The second approach reinterprets special test configurations using valuations with a concrete geometric description, following [BLX19]. For this we need to first introduce the concept of *complements* defined by Shokurov.

Definition. For a Fano variety X, a $\mathbb{Q}$-divisor D is an N-complement, if $D = \frac{1}{N}\Gamma$ for some divisor $\Gamma \in |-NK_X|$ and (X, D) is log canonical.

We say D is a $\mathbb{Q}$-complement, if it is an N-complement for some N.

At first sight it is merely a technical concept. However, the following deep result, conjectured by Shokurov and proved by Birkar, sheds light on Fano varieties: there exists a uniform N which only depends on $\dim(X)$, such that X always has an N-complement. This is surprising since n-dimensional Fano varieties are unbounded.

Below, we will explain how complements play an important role in the study of K-stability, through the construction of the *basis type divisor*. For a sufficiently divisible m, we say D is a m-basis type divisor if

$$D = \frac{1}{m \cdot N_m}\big(\mathrm{div}(s_1) + \cdots + \mathrm{div}(s_{N_m})\big),$$

where $N_m = \dim H^0(X, -mK_X)$ and $\{s_1, \ldots, s_{N_m}\}$ forms a basis of $H^0(X, -mK_X)$. This concept was introduced by Fujita-Odaka, and they considered

$$\delta_m(X) = \inf_D \mathrm{lct}(X, D) = \inf_D \inf_E \frac{A_X(E)}{\mathrm{ord}_E(D)}$$

where D runs through over all m-basis type divisors, and E runs through over all prime divisors over X. Together with Jonsson-Blum's work, they show

$$\lim_{m\to\infty} \delta_m(X) = \delta(X). \tag{3}$$

We note that this gives us a way to verify K-stability for explicit Fano varieties, by estimating $\delta_m(X)$.

For a Fano variety X and a $\mathbb{Q}$-complement D, we say a prime divisor E over X is *a log canonical place* of D if $A_X(D) = \mathrm{ord}_E(D)$. Then using (3), one can show that

$$\text{if } \delta(X) \leq 1, \delta(X) = \inf_E \frac{A_X(E)}{S_X(E)},$$

where E runs through over all log canonical places of a $\mathbb{Q}$-complement.

An observation made in [BLX19] is that any divisor E arises from a special test configuration only if it is a log canonical place of a $\mathbb{Q}$-complement. In particular, if $\delta(X) < 1$, then there exists a special test configuration $\mathcal{X}$ with

$$\mathrm{Fut}(\mathcal{X}) = A_X(E) - S_X(E) < 0,$$

i.e., X is K-unstable. Additionally, by using the minimum norm function $\|\mathcal{X}\|_{\mathrm{m}}$ introduced by Dervan, a more precise relation can be given when $\delta(X) \leq 1$:

$$\delta(X) \coloneqq \inf_E \frac{A_X(E)}{S_X(E)} = 1 + \inf_{\mathcal{X}} \frac{\mathrm{Fut}(\mathcal{X})}{\|\mathcal{X}\|_{\mathrm{m}}}. \tag{4}$$

Then [BLX19] applied Birkar's theorem on the existence of bounded complements to investigate log canonical places of $\mathbb{Q}$-complements, and proved that there exists a *uniform N*, which only depends on $\dim(X)$, such that if E is a log canonical place of $\mathbb{Q}$-complement, it is indeed a log canonical place of an N-complement D. As all N-complements form a bounded family, their log canonical places are toroidal divisors over a bounded family of log resolutions. Both $A_X(E)$ and $S_X(E)$ are deformation invariant, so we can assume $\delta(X) = \lim_{E_i} \frac{A_X(E_i)}{S_X(E_i)}$ for a sequence of lc places E_i of N-complements, which is toroidal over *the same* log resolution. Thus after a rescaling of ord_{E_i} and passing to a subsequence, there exists a limiting quasi-monomial valuation v and it satisfies that $\delta(X) = \frac{A_X(v)}{S_X(v)}$. (There are straightforward generalization of various definitions and invariants from divisors to quasi-monomial valuations v over X.)

Higher rank finite generation. In the above discussion, the valuation v has its toroidal coordinates the limit of the ones of a rescaling of ord_{E_i}. In particular, it could be $\mathbb{Q}$-linearly independent. In other words, v could be a quasi-monomial valuation of higher rational rank. For any valuation v, its values on $R = \bigoplus_m H^0(-mK_X)$ form a monoid Φ contained in $\mathbb{R}_{\geq 0}$. For any $\lambda \in \Phi$, we define $I_\lambda \subseteq R$ to be the ideal consisting of sections $s \in R$ with $v(s) \geq \lambda$; and $I_{>\lambda} \subseteq R$ to be the ideal of sections s with $v(s) > \lambda$. Then the *associated graded ring* of R with respect to v is

$$\mathrm{gr}_v(R) \coloneqq \bigoplus_{\lambda\in\Phi} I_{\geq\lambda}/I_{>\lambda}.$$

To further advance the theory, we need another major recipe which is the following finite generation theorem, proved by Liu-Xu-Zhuang [LXZ21].

Theorem (Higher rank finite generation). *Let X be a Fano variety with $\delta(X) \leq 1$. Let v be the quasi-monomial valuation constructed above satisfying that $\delta(X) = \frac{A_X(v)}{S_X(v)}$. Then the associated graded ring $\mathrm{gr}_v(R)$ is finitely generated.*

In the above theorem, if v arises from a divisor, the finite generation easily follows from the work of Birkar-Cascini-Hacon-McKernan [BCHM10], as v is a log canonical place of a $\mathbb{Q}$-complement. However, it does not always hold that

$\mathrm{gr}_v(R)$ is finitely generated if a higher rational rank valuation v is merely assumed to be a log canonical place of a $\mathbb{Q}$-complement. Therefore, finer properties have to been extracted from the quasi-monomial valuation v computing $\delta(X)$, and the proof needs various new recipes, especially deep results on the boundedness of Fano varieties.

Revisit stability. As we already emphasized, roughly speaking, *stability* can be thought of as a condition to pick out the *optimal* degeneration for a family. As seen in Kempf-Ness's classical theorem, the theory often can be extended to associating every unstable object an "optimal destabilization." An example is to degenerate an unstable sheaf to the direct summand of the graded sheaves of its Harder-Narasimhan filtration. The notion is coined into the term Θ-*stratification* by Halpern-Leistner.

Our finite generation theorem yields such a theory for K-stability. In fact, one can show from it that there exists a divisorial valuation $a \cdot \mathrm{ord}_E$ sufficiently close to v, with $\delta(X) = \frac{A_X(E)}{S_X(E)}$ and E induces a special test configuration. As a consequence, we have

Theorem (Optimal Destabilization Theorem). *Let X be a Fano variety with $\delta(X) \leq 1$. Then there exists a divisor E which induces a special test configuration and satisfies $\delta(X) = \frac{A_X(E)}{S_X(E)}$. In particular, if X is K-stable then it is uniformly K-stable.*

In particular, X is K-stable if and only if $\delta(X) > 1$.

The "optimal" in our theorem is with respect to the minimum norm as (4) shows. Székelyhidi explored the optimal degeneration with respect to L^2-norm for toric varieties. Nevertheless, for a general Fano variety, the minimum norm fits better into our machinery. To help the reader to understand the features of different approaches, we would like to give a panoramic comparison in each approach of how the stability is applied to the degeneration process.

> Philosophically speaking, a key point to studying K-stability is understanding how to get the *limit* of a sequence of Fano varieties X_t with extra conditions, or similarly a sequence of test configurations of a fixed Fano variety, which approximates the infimum of the normalized Futaki invariants.

In the Cheeger-Colding-Tian theory, one starts with a Fano manifold X without a Kähler-Einstein metric, and it suffices to construct a test configuration destabilizing X. Donaldson introduced the idea of using an auxiliary boundary divisor and conic metrics along it. More precisely, fix any smooth divisor $\Gamma \in |-mK_X|$, one can find a sequence $\beta_i \in (0, \frac{1}{m})$ with $\beta = \lim_i \beta_i$ such that $(X, \beta_i \cdot \Gamma)$ admits a conical Kähler-Einstein metric ω_i, but $(X, \beta \cdot \Gamma)$ does not. After passing through a subsequence, the *Gromov-Hausdorff limit* X_∞ of the conical Kähler-Einstein Fano manifolds (X, ω_i) exists as a metric space. Then it took a large endeavor to show that X_∞ is indeed the underlying space of the Chow limit when one uses an orthonormal basis (with respect to ω_i) of $|-NK_X|$ (for some $N \gg 0$) to embed X into a projective space. Moreover if we take Γ_∞ to be the Chow limit of Γ, then $(X_\infty, \beta \cdot \Gamma_\infty)$ has a conical Kähler-Einstein metric. Then we obtain a test configuration degenerating X to X_∞ with a negative Futaki invariant, which implies X is K-unstable. However, for now this kind of analytic argument is restricted to the case when X is smooth.

The variational approach focuses on the space of smooth Kähler metrics and its various completions. The existence of a Kähler-Einstein metric is equivalent to the Mabuchi (or Ding) functional having a critical point. Following the infinite dimensional GIT picture, one should concentrate on the functional along *geodesic rays* by the analogue of the Hilbert-Mumford criterion in this setting. For simplicity, here we restrict ourselves to the case that the automorphism group is finite. It is shown that the existence of a critical point of the Ding functional follows from the *coercivity*, which means for any geodesic $U : \mathbb{R} \to \mathcal{E}^1$ in the space of Kähler metrics $\mathcal{E}^1$ with finite energy plurisubharmonic (psh) potentials, the Ding functional $\mathbf{D}$ grows at least linearly. After establishing the convexity of $\mathbf{D}$ along the geodesic ray, it is sufficient to understand the growth $\frac{1}{t}\mathbf{D}(U_t)$ when $t \to \infty$, where a translation dictionary between the archimedean and non-archimedean setting is established. We note that the space of non-archimedean metrics contains all test configurations, so we have a chance to connect the algebraic condition of K-stability and the analytic condition of the existence of Kähler-Einstein metrics.

The positivity of Ding invariants on non-archimedean metrics (with finite energy) implies coercivity, as when $t \to \infty$,

$$\lim_{t\to\infty} \frac{\mathbf{D}(U_t)}{t} \geq \mathrm{Ding}(\phi) > 0$$

for a non-archimedean metric ϕ attached to the geodesic ray U. Moreover, while ϕ might not come from a test configuration, it can be approximated by a sequence of non-archimedean metrics ϕ_i coming from test configurations (constructed by Demailly's regularization) such that the Ding invariants satisfy $\lim_{i\to\infty} \mathrm{Ding}(\phi_i) \leq \mathrm{Ding}(\phi)$. However, to have positivity of the limiting Ding invariant $\mathrm{Ding}(\phi)$, positivity of $\mathrm{Ding}(\phi_i)$ is not enough, but we have to assume uniform Ding stability.

As we mentioned before, uniform K-stability is known to be the same as uniform Ding stability. It remains to show the equivalence between K-stability and uniform K-stability, which is addressed by the Optimal Destabilization Theorem, obtained with deep results from the

Minimal Model Program. In other words, the algebro-geometric approach provides the necessary compactness result.

In the work of stability, this kind of study on the limit of a family of Fano varieties is clearly related to moduli spaces. In fact, the progress of our understanding on the fundamental concepts of K-stability intertwines with the construction of moduli spaces of K-polystable Fano varieties. We will continue to discuss the latter topic.

Moduli of Fano varieties. It is one of the most fundamental questions in algebraic geometry to construct moduli spaces of varieties. For instance, for $g \geq 2$ the moduli space $\mathcal{M}_g$ of curves and its compactification $\overline{\mathcal{M}}_g$ are among the most studied objects in algebraic geometry. Built on the machinery of the Minimal Model Program and progress on understanding the moduli problem of higher dimensional varieties, KSB (Kollár-Shephard-Barron) stability gives a satisfying theory to generalize the construction of $\overline{\mathcal{M}}_g$ to higher dimensions. In particular, an appropriate definition of families of arbitrary dimensional varieties (or even log pairs) was given by Kollár. As the output, we get a projective scheme which parametrizes KSB stable varieties X, whose canonical class K_X is ample.

The opposite case when $-K_X$ is ample had been mysterious for a long time. It is trivial when $\dim(X) = 1$ as $X = \mathbb{CP}^1$. However, in higher dimensions, it is much trickier. As (1) shows, an isotrivial family X_t could jump to X_0 with a different complex structure, which does not occur for a KSB stable family. So it has been well understood for a long time that the Minimal Model Program itself is unlikely to be sufficient to provide a satisfactory moduli theory for Fano varieties. However, in the example (1), the central fiber is K-unstable, so there is a chance that if we restrict ourselves to Fano varieties with K-stability assumptions, this pathology does not occur.

Various results from the analytic side, especially "the partial C^0-estimate" in the works of Tian and Donaldson-Sun, suggest that the condition on the existence of Kähler-Einstein metrics could sort out a class of Fano varieties to be parametrized by well-behaved moduli spaces. As K-stability is the corresponding algebraic condition, it is natural to hope that it yields a robust moduli theory. Speculation is also inspired by the successful role that the Hitchin-Kobayashi correspondence plays in the study of compact moduli spaces parametrizing them. However, for families of K-stable Fano varieties it was far from clear how to use the definition of K-stability itself to actually establish the list of properties needed to construct the moduli space. This could be the reason that such a topic did not attract people's attention in the decade right after the inventions of the notion. Li-Xu [LX14] studied families of Fano varieties from the perspective of K-stability and provided a hint that K-stability may lead to an algebraic moduli theory. Nevertheless, only until people have obtained enough fundamental knowledge on K-stability, e.g., the valuative criterion, we could start to get definitive results.

The main theorem is the following.

Theorem (K-moduli)**.** *Fix a positive integer n, and a rational number V. Then*

1. *There exists an Artin stack $\mathfrak{X}^{\mathrm{kss}}_{n,V}$ of finite type, which parametrizes families of n-dimensional K-semistable Fano varieties $X \to T$, with $(-K_{X_t})^n = V$ for any $t \in T$.*
2. *The stack $\mathfrak{X}^{\mathrm{kss}}_{n,V}$ admits a proper good moduli space $\mathfrak{X}^{\mathrm{kss}}_{n,V} \to X^{\mathrm{kps}}_{n,V}$, whose points correspond to K-polystable Fano varieties. Moreover, the CM line bundle Λ_{CM} is ample on $X^{\mathrm{kps}}_{n,V}$.*

The proof of the above theorem is a combination of many people's contributions. We will explain below the content of the K-moduli Theorem, and sketch the circle of ideas involved in the proof.

Example. Consider cubic hypersurfaces

$$X \subset \mathbb{P}^{n+1} \quad (n \geq 2). \tag{5}$$

It is not hard to verify K-stability for some special cubic hypersurface, e.g., the Fermat $X = (x_0^3 + \cdots + x_{n+1}^3) = 0$. As we will see this implies that there is a Zariski open locus of the parametrizing space, such that any corresponding cubic hypersurface is K-semistable.

It is shown by Odaka-Spotti-Sun for $n = 2$, Liu-Xu for $n = 3$, and Liu for $n = 4$, that in these cases, the GIT stability of cubic hypersurfaces is exactly the same as K-stability. In other words, the K-moduli is given by the GIT moduli space.

***K*-moduli stack.** By definition, the *K-moduli stack* $\mathfrak{X}^{\mathrm{kss}}_{n,V}$ admits a universal family $\mathfrak{U}^{\mathrm{kss}}_{n,V} \to \mathfrak{X}^{\mathrm{kss}}_{n,V}$, such that for any scheme T, all pull-back families under morphisms in $\mathrm{Hom}(T, \mathfrak{X}^{\mathrm{kss}}_{n,V})$ precisely correspond to all isomorphic classes of families of n-dimensional K-semistable Fano varieties with volume V over T.

To prove the first part of the K-moduli theorem, we first need to show the boundedness which says that there exists a uniform $M = M(n, V)$ such that for any n-dimensional K-semistable Fano variety, $-MK_X$ is very ample, i.e., the linear system $|-MK_X|$ induces an embedding of X into a same projective space $\mathbb{CP}^N$. This was first proved by Jiang, who reduced the question to Birkar's proof of the Borisov-Alexeev-Borisov Conjecture.

We consider (X, f) for X with an embedding $f : X \to \mathbb{P}^N$ as above. These pairs are parametrized by a locus Z of the Hilbert scheme $\mathrm{Hilb}(\mathbb{P}^N)$. It suffices to prove Z is locally closed in $\mathrm{Hilb}(\mathbb{P}^N)$, since

$$\mathfrak{X}^{\mathrm{kss}}_{n,V} = [Z/\mathrm{PGL}_{\mathbb{C}}(N+1)], \tag{6}$$

as two embeddings of X differ by an element in $\mathrm{PGL}_{\mathbb{C}}(N+1)$.

The locus in $\mathrm{Hilb}(\mathbb{P}^N)$ parametrizing Fano subvarieties of $\mathbb{P}^N$ is locally closed, so it remains to prove for a family of Fano varieties $X \to T$, the locus $U \subseteq T$ parametrizing K-semistable fibers is open. One way to get this is using the fact that the function $t \to \min\{\delta(X_t), 1\}$ is constructible and lower semi-continuous. This was first proved in [BLX19] for $\delta(X_{\bar{t}})$ instead of $\delta(X_t)$, where $\bar{t}$ is the geometric point given by t. Then Zhuang verified $\delta(X_{\bar{t}}) = \delta(X_t)$.

K-moduli space. From the above discussion, we see that for any fixed positive number $\delta_0 \in (0,1]$, we still get an Artin stack of finite type if we replace the K-semistability condition for $\mathfrak{X}^{\mathrm{kss}}_{n,V}$ by $\delta(X_t) \geq \delta_0$. However, a much more delicate property which distinguishes $\mathfrak{X}^{\mathrm{kss}}_{n,V}$ is that it admits a space which parameterizes points up to orbit closure equivalence, i.e., the S-equivalence after Seshadri, called *the good moduli space*.

One difficulty in constructing moduli space of Fano varieties is the fact that a Fano variety X may have a positive dimensional automorphism group $\mathrm{Aut}(X)$. This contrasts to the case of the KSB stability, where the moduli is a separated Deligne-Mumford stack for which one can apply Keel-Mori's theorem to conclude the existence of a coarse moduli space. When a stack has points with positive dimensional stabilizer groups, it usually does not admit any coarse moduli space. Nevertheless, a replacement construction initiated by Alper, called the *good moduli space*, provides a good framework to treat this more complicated case, at least in characteristic 0.

Definition. Let $\mathfrak{X}$ be a stack. We say X is a good moduli space of $\mathfrak{X}$, if the quasi-compact morphism $\phi\colon \mathfrak{X} \to X$ satisfies

1. The push-forward functor ϕ_* is exact on quasi-coherent sheaves;
2. There is an isomorphism $\mathcal{O}_X \to \phi_*(\mathcal{O}_{\mathfrak{X}})$.

Example. Let $\mathbb{C}^*$ act on $\mathbb{C}$ by $t \cdot x \to tx$, then the stack $[\mathbb{C}/\mathbb{C}^*]$ admits a good moduli space, which is a point.

However, if we replace $\mathbb{C}$ by $\mathbb{CP}^1$, with the action $t \cdot [x_0, x_1] \to [x_0, tx_1]$, then the stack $[\mathbb{CP}^1/\mathbb{C}^*]$ has no good moduli space.

The definition of good moduli space is simple, however it has strong implications. For a quotient stack $[Z/G]$ if it admits a good moduli space, it implies that for any $z \in Z$, the closure $\overline{G \cdot z} \subset Z$ has a unique minimal orbit $G \cdot z^*$. Moreover, the stabilizer of z^* is reductive. For $\mathfrak{X}^{\mathrm{kss}}_{n,V}$ (see (6)), this means that for any K-semistable Fano variety X, there is a unique K-polystable Fano variety Y which is S-equivalent to X, and $\mathrm{Aut}(Y)$ is reductive. The latter was Matsushima's theorem when Y is a smooth Kähler-Einstein Fano manifold.

A prototype example of good moduli space is given by $\mathfrak{X} = [\mathrm{Spec}(A)/G]$ where G is a reductive group acting on an affine variety $\mathrm{Spec}(A)$ (over characteristic 0), where the good moduli space is $X = \mathrm{Spec}(A^G)$. More generally, for a polarized projective variety (Y, L) with a reductive group action, then one can take $\mathfrak{X} = [Y^{\mathrm{ss}}/G]$ where Y^{ss} is the GIT semistable locus and the good moduli space of $\mathfrak{X}$ is the GIT quotient $X = Y/sslashG$. Nevertheless, there is no known way to interpret the K-moduli problem as a GIT problem, so we have to rely on a more general abstract method. Fortunately, a valuative criterion of the existence of good moduli space, which is a version of Keel-Mori's theorem for Artin stacks, is established in [AHLH18]. Applying their work, built on earlier analysis in [LWX21] and [BX19], the existence of the good moduli space and its separatedness are proved in [ABHLX20].

While the existence of the good moduli space $X^{\mathrm{kps}}_{n,V}$ should justify that K-stability gives the right notion of moduli theory of Fano varieties, arguably the most remarkable property of $X^{\mathrm{kps}}_{n,V}$ is its properness.

> Let $X^\circ \to \Delta^\circ$ be a family of K-semistable Fano varieties over a punctured curve, then after a base change of Δ°, there exists a unique K-polystable Fano filling.

For a family whose general fibers are smooth with Kähler-Einstein metrics, the existence of the limit is given as the deepest consequence from the Cheeger-Colding-Tian theory. For the general case, the compactness result essentially relies on the Optimal Destabilization Theorem.

Another fundamental property of $X^{\mathrm{kps}}_{n,V}$ is its projectivity, i.e., $X^{\mathrm{kps}}_{n,V}$ is a subscheme of $\mathbb{P}^N$ for some N. The CM line bundle Λ_{CM}, defined in various works with growing generality and eventually by Paul-Tian in its current form, gives a candidate of an ample bundle, because over the locus parametrizing smooth Kähler-Einstein Fano manifolds the positivity of Λ_{CM} can be seen analytically via a metric on Λ_{CM}, whose curvature form is the Weil-Petersson metric. However, as usual there is essential difficulty to extend the analytic argument to the locus parametrizing singular Fano varieties. Later [CP21] initiated an algebraic way to attack the positive problem, and their method was completed by [XZ20].

Explicit K-stable Fano varieties. With the vast progress on K-stability theory, it is natural to study explicit examples.

The valuative criterion provides a way to verify K-stability of a Fano variety X by estimating $\delta(X)$. Using $\delta(X) = \lim_m \delta_m(X)$, Abban-Zhuang invent an approach of applying inversion of adjunction to reduce the estimate of the log canonical thresholds $\delta_m(X)$ to low dimensional varieties. As a result, they prove any degree d smooth hypersurface X in $\mathbb{P}^{n+1}$ is K-stable if $n+2-n^{1/3} \leq d \leq n+1$.

Following Tian's work in dimension 2 which completes the Kähler-Einstein Problem for smooth Fano surfaces, in a recent joint work of Araujo-Castravet-Cheltsov-Fujita-Kaloghiros-Martinez-Garcia-Shramov-Suss-Viswanathan, they go through the list of 105 families in Iskovskikh-Mori-Mukai classification of smooth Fano threefolds, and determine in each family whether a general member is K-(semi,poly)stable. The arguments involve many different techniques.

K-moduli spaces provides many explicit examples of moduli spaces. Mabuchi-Mukai and Spotti-Odaka-Sun gave explicit descriptions of the K-moduli spaces which compactify the moduli spaces parametrizing smooth K-stable Fano surfaces. In higher dimensions, there is only a small number of cases for which we have a complete explicit description of the K-moduli space. We have seen in (5) for cubic hypersurfaces of dimension at most 4, the K-moduli spaces are identical to the GIT moduli spaces. The same statement is expected to hold for cubic hypersurfaces in any dimension. However, for hypersurfaces of a degree larger than 3, the K-moduli usually is not the same as the GIT moduli. While we expect all smooth Fano hypersurfaces are K-(poly)stable, there is not even a conjectural picture on what should appear as their limits.

It is also interesting to consider the case of log pairs, for which all the previous main theorems still hold. A research project, led by Ascher-DeVleming-Liu, studies moduli spaces whose general members parametrize K-semistable log Fano pairs (X, tD) for a Fano variety X, $D \in |-rK_X|$ and a constant $0 \leq t < \frac{1}{r}$. When one varies t, the K-moduli spaces are connected by wall-crossings. This framework is applied to $(X, D) = (\mathbb{P}^3, \text{quartic K3})$. The K-stability moduli theory provides interpolating moduli spaces between the Satake compactification of the moduli of quartic K3 surfaces which corresponds to $t \to 1$ and the GIT moduli space which corresponds to $t \to 0$. As a consequence, they confirm a conjecture of Laza-O'Grady, who study these moduli spaces from a different perspective.

Analogous problems. Given the fundamental importance of Kähler-Einstein metrics, it is probably not surprising that people try to develop similar studies in other geometric settings. Many parts of our discussion above can be extended accordingly.

For any Fano variety X, we know it does not necessarily have a Kähler-Einstein metric. As a replacement, people study whether it admits a more general kind of canonical metrics, namely *Kähler-Ricci soliton*. Combining the works of Han-Li and Blum-Liu-Xu-Zhuang, *any* klt Fano variety X is known to admit a unique two-step degeneration process to a Kähler-Ricci soliton. More precisely, X first degenerates to (X_0, ξ_0) where ξ_0 is a vector field on X_0, such that (X_0, ξ_0) is K-semistable (in the sense of a Fano variety with a vector field). Moreover, (X_0, ξ_0) has a unique K-polystable degeneration (Y, ξ_Y), which admits a Kähler-Ricci soliton metric. When X is smooth, this degeneration of X to Y is also given by the Gromov-Hausdoff limit of the Kähler-Ricci flow, upon the Hamilton-Tian Conjecture solved by Tian-Zhang in dimension three, Chen-Wang and Bamler in a general dimension.

One can also consider the local setting. There has been a well-studied local analogue of Kähler geometry, namely the *Sasaki geometry*, which corresponds to the link of a cone singularity. The corresponding Yau-Tian-Donaldson Conjecture, which claims the existence of a *Sasaki-Einstein metric* on a Fano cone singularity $(x \in X, \xi)$ is equivalent to the K-polystability of $(x \in X, \xi)$, was proved by Collins-Székelyhidi when the singularity is isolated, and by Li and Huang in the general case.

A more challenging local question is to consider an *arbitrary* klt singularity $x \in X$. As the combination of a number of conjectural statements, made by Li and Li-Xu, the *Stable Degeneration Conjecture* predicts that any klt singularity $x \in X$ has a degeneration to a K-semistable Fano cone singularity $(x_0 \in X_0, \xi)$ and such a degeneration comes from the unique (up to rescaling) minimizer of the normalized volume function. We recall that the normalized function $\widehat{\mathrm{vol}}$, invented by Li, is defined on the space of valuations $\mathrm{Val}_{X,x}$ which consists of all valuations on $K(X)$ centered on x. There are many works, by Blum, Li, Liu, Xu, and Zhuang, which solved various parts of the Stable Degeneration Conjecture. As of the writing of this article, there is only one part remaining open. That is the claim that any minimizing valuation v of $\widehat{\mathrm{vol}}$, which is shown by Xu to be quasi-monomial, has a finitely generated associated graded ring. In other words, as the writing of this article, the local higher rank finite generation is still unknown!

What's next? I wish by now, with the complete solution of the Yau-Tian-Donaldson Conjecture for all Fano varieties, as well as the construction of the K-moduli spaces and many other results, it becomes clear that the interaction between the Kähler-Einstein Problem and the Minimal Model Program provides an exceptionally beautiful example to mathematician's dream of viewing mathematics as one unified discipline.

As an important chapter is about to be closed, there are good reasons to be optimistic about the future of the field. Many exciting directions remain unexplored. For instance, we lack a good understanding of the algebraic properties of polarized manifolds with a constant scalar curvature metric, whereas in differential geometry many deep results have been obtained for it as a natural extension of the Kähler-Einstein Problem. It is also tempting to find out whether the new K-stability perspective of higher dimensional geometry can shed light on the longstanding questions in the Minimal Model Program, e.g., termination of flips and the Abundance Conjecture. Nevertheless, as always, only time can tell how far the interaction will take us.

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Chenyang Xu

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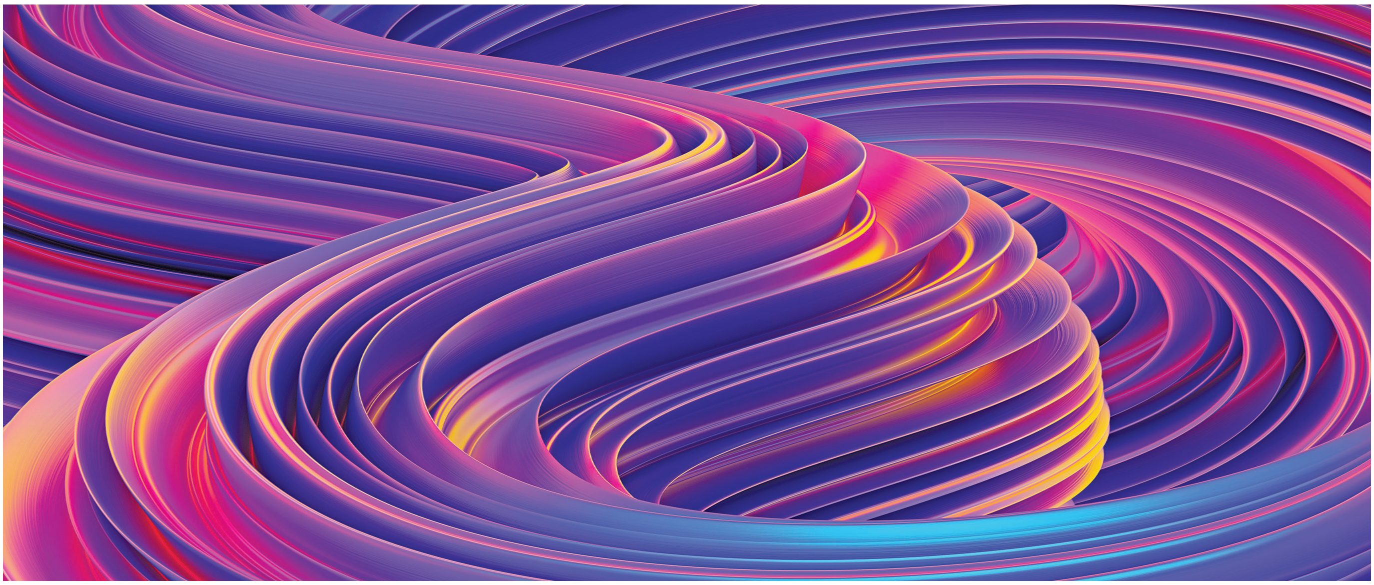
Foliations on Complex Manifolds



Carolina Araujo and João Paulo Figueredo

In these notes we survey some aspects of the theory of holomorphic foliations on complex manifolds. The origins of the theory go back to works of Darboux, Poincaré and Painlevé, where it was developed to study solutions of ordinary differential equations on $\mathbb{C}^2$. We briefly discuss some of the early works on this theory, mostly concerned with the local behavior of the leaves near the singularities. We then move the focus from local to global properties. Birational geometry has had a great influence on the development of a global theory of holomorphic foliations. After reviewing the Enriques-Kodaira classification of projective surfaces and explaining the general philosophy of the Mininal Model Program, we explore some of their recent counterparts for foliations.

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Foliations

A foliation $\mathcal{F}$ of dimension r on a differentiable manifold M^n is a decomposition of M into a disjoint union of immersed submanifolds of dimension r, called *leaves*, which pile up locally like fibers of a submersion. Formally, $\mathcal{F}$ is defined by an atlas $\{\varphi_i : U_i \to M\}$, with $U_i \subset \mathbb{R}^r \times \mathbb{R}^{n-r}$, and differentiable transition functions of the form

$$\varphi_{ij}(x,y) = \big(f(x,y), g(y)\big), \quad (x,y) \in \mathbb{R}^r \times \mathbb{R}^{n-r}.$$

The leaves of $\mathcal{F}$ are locally given by the fibers of the projection $U_i \to \mathbb{R}^{n-r}$. If the transition functions are C^∞, we say that $\mathcal{F}$ is a C^∞ foliation. The integer $n-r$ is called the codimension of the foliation $\mathcal{F}$. In the literature, the integer r is also referred to as the *rank* of the foliation.

Despite their simple local description, the existence of a foliation on a compact manifold M is subject to global topological constraints, and carries relevant information about the geometry of M. For instance, consider the 2-dimensional sphere S^2. If there were a foliation $\mathcal{F}$ of dimension one on S^2, then one could cook up a nonvanishing smooth vector field everywhere tangent to $\mathcal{F}$, contradicting the Poincaré-Hopf theorem. On the other hand, when $M = \mathbb{R}^2/\mathbb{Z}^2$ is a 2-dimensional torus, one can construct plenty of foliations of dimension one on M. For any choice of angle θ, the foliation by lines on $\mathbb{R}^2$ having slope equal to θ induces a foliation $\mathcal{F}_\theta$ of dimension one on the quotient $\mathbb{R}^2/\mathbb{Z}^2$. If θ is rational, then the leaves of $\mathcal{F}_\theta$ are homeomorphic to S^1. If θ is irrational, then the leaves are homeomorphic to $\mathbb{R}$, and dense in M by a theorem of Kronecker.

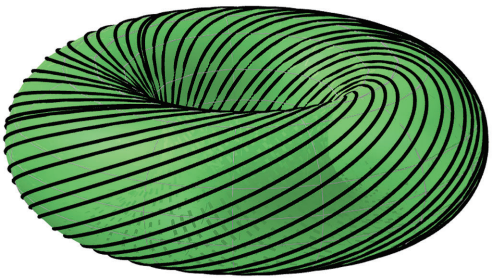

Figure 1. Foliation on the torus.

More generally, we have the following result by Thurston characterizing which closed manifolds admit a C^∞ foliation of codimension one in terms of the topological Euler characteristic.

Theorem 1 ([Thu76]). *Let M be a closed connected smooth manifold. Then M admits a C^∞ foliation of codimension one if and only if $\chi(M) = 0$.*

When X is a *complex manifold*, it is natural to consider *holomorphic foliations* on X. These are defined by atlases $\{\varphi_i : U_i \to X\}$, with $U_i \subset \mathbb{C}^r \times \mathbb{C}^{n-r}$, and biholomorphic transition functions of the form

$$\varphi_{ij}(x,y) = \big(f(x,y), g(y)\big), \ \ (x,y) \in \mathbb{C}^r \times \mathbb{C}^{n-r}.$$

The integer n is the complex dimension of X, r is the dimension of the holomorphic foliation, and $n-r$ its codimension. The geometric theory of holomorphic foliations was first introduced to better understand solutions of complex ordinary differential equations on the plane $\mathbb{C}^2$. In this context, it is natural to allow for singularities. Given a meromorphic function F on $\mathbb{C}^2$, consider the ODE

$$\frac{dy}{dx} = F(x,y). \tag{1}$$

By the theorem of existence and uniqueness of solutions of ordinary differential equations, the solutions of (1) induce a holomorphic foliation of dimension 1 on the complement of a closed subset of $\mathbb{C}^2$. Indeed, if $F(x,y)$ is holomorphic at a point $(x_0, y_0) \in \mathbb{C}^2$, then the ODE (1) admits a unique holomorphic solution $y = y(x)$ satisfying $y(x_0) = y_0$.

More generally, a singular holomorphic foliation $\mathcal{F}$ of dimension r on a complex manifold X is an equivalence class of holomorphic foliations of dimension r defined on the complement of a proper closed subset of codimension at least 2 in X. There is a minimal closed subset $\mathrm{Sing}(\mathcal{F})$ such that $\mathcal{F}$ can be extended to a holomorphic foliation of dimension r on $X \setminus \mathrm{Sing}(\mathcal{F})$. This closed set is called the *singular locus* of $\mathcal{F}$. Outside of $\mathrm{Sing}(\mathcal{F})$, the vectors that are tangent to the leaves of $\mathcal{F}$ form a sub-vector bundle of rank r of the tangent bundle T_X, which can be extended on the whole X to a coherent subsheaf $T_\mathcal{F}$ of T_X, called the *tangent sheaf* of $\mathcal{F}$. The subsheaf $T_\mathcal{F}$ is closed under the Lie bracket: if v and w are two local vector fields on $X \setminus \mathrm{Sing}(\mathcal{F})$ everywhere tangent to the leaves of $\mathcal{F}$, then their Lie bracket $[v,w]$ is also everywhere tangent to the leaves of $\mathcal{F}$. Conversely, we have the following classical theorem:

Theorem 2 (Frobenius). *There is a one to one correspondence between singular holomorphic foliation on X and saturated coherent subsheaves of the tangent bundle T_X that are closed under the Lie bracket.*

Much of the early work on singular holomorphic foliations focused on the behavior of the leaves near the singular locus. Let $\mathcal{F}$ be a singular holomorphic foliation on $\mathbb{C}^2$ with an isolated singularity at the origin, defined by a holomorphic vector field

$$v = p(x,y)\frac{\partial}{\partial x} + q(x,y)\frac{\partial}{\partial y},$$

where p and q are holomorphic functions. Denote by $p_1(x,y) = ax+by$ and $q_1(x,y) = cx+dy$ the linear parts of p and q, respectively, and suppose that they are not both identically zero. Under some genericity assumptions, the local behavior of the leaves near the origin is controlled by the eigenvalues of the nonzero matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Denote by λ_1 and λ_2 the two eigenvalues of A, with $\lambda_2 \neq 0$, and suppose that λ_1/λ_2 is not real, or it is a positive real number that is not an integer nor the inverse of an integer. A fundamental theorem of Poincaré states that v is linearizable. This means that locally around the singularity, after a suitable analytic change of coordinates, the vector field v can be written in the form

$$v = \lambda_1 z\frac{\partial}{\partial z} + \lambda_2 w\frac{\partial}{\partial w}, \tag{2}$$

where z and w are the new coordinates. The vector field (2) can be easily integrated. On a punctured neighborhood of the origin, the leaves of the foliation are parametrised by

$$\gamma(t) = (c_1e^{\lambda_1 t}, c_2e^{\lambda_2 t}), \ t \in \mathbb{C}.$$

The ones given by the equations $z = 0$ and $w = 0$ accumulate in the origin.

If λ_1/λ_2 is a positive integer n or the inverse of a positive integer $1/n$, then Dulac proved that, in suitable analytic coordinates z and w, the vector field v can be written in the form

$$v = z\frac{\partial}{\partial z} + (nw + \mu z^n)\frac{\partial}{\partial w},$$

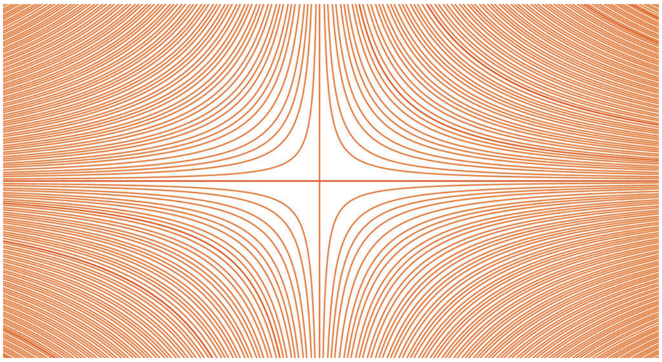

Figure 2. Singularity of a foliation on the plane.

where $\mu \in \mathbb{C}$. Again we see that there are leaves of the foliation on a punctured neighborhood of the origin that accumulate in $(0,0)$. More generally, let $p \in S$ be an isolated singularity for a foliation $\mathcal{F}$ of dimension 1 on a nonsingular complex surface S. A *separatrix* is a complex curve $C \subset S$ through p such that $C \setminus \{p\}$ is a leaf of $\mathcal{F}$ on a punctured neighborhood of p in S. By a theorem of Camacho and Sad [CS82], a separatrix always exists for foliations on surfaces.

In general, we say that a vector field v is in normal form if $v = v_s + v_n$, where $v_s = \lambda_1 z\frac{\partial}{\partial z} + \lambda_2 w\frac{\partial}{\partial w}$ is a semisimple vector field, v_n is a nilpotent vector field (i.e., the matrix associated to the linear part of v_n is nilpotent), and $[v_s, v_n] = 0$. We say that v is *analytically normalizable* if there is an analytic change of coordinates putting it in normal form. So we have seen that if $\lambda_1/\lambda_2 \notin \mathbb{R}_-$, then v is analytically normalizable. There exist examples of vector fields which are not analytically normalizable, such as $v = (z+w)\frac{\partial}{\partial z} + w^2\frac{\partial}{\partial w}$. In this case, after the formal change of coordinates

$$(Z, w) = \Big(z + \sum_{n\geq 1}(n-1)!\, w^n, w\Big),$$

v becomes $Z\frac{\partial}{\partial Z} + w^2\frac{\partial}{\partial w}$, which is in normal form. However, this formal change of coordinates is not holomorphic, since $\sum_{n\geq 1}(n-1)!\, w^n$ diverges. Questions about analytic linearization and normalization of local vector fields on the plane were very much studied in the twentieth century (see for instance [Mar81]).

Global Aspects

Now we move the focus from local to global properties of foliations. One example of a global invariant of a codimension one foliation $\mathcal{F}$ on $\mathbb{CP}^n$ is its *degree*. It is defined as the number of tangencies of a generic line in $\mathbb{CP}^n$ with $\mathcal{F}$. Codimension one foliations on $\mathbb{CP}^n$ of small degree were classified by Jouanolou in [Jou79]. In order to explain this classification, it is convenient to describe foliations using differential forms. The tangent sheaf of a codimension one (singular) foliation $\mathcal{F}$ on $\mathbb{CP}^n$ can be given by a polynomial one-form

$$\Omega = \sum_{i=0}^{n} P_i(x_0, \dots, x_n)dx_i, \tag{3}$$

where the P_i's are homogeneous polynomials of the same degree without common factors. This polynomial form is uniquely determined up to scalar, and the P_i's satisfy $\sum_{i=0}^{n} x_i P_i(x_0, \dots, x_n) = 0$. In the language of differential forms, the Frobenius' integrability condition translates into the condition

$$\Omega \wedge d\Omega = 0. \tag{4}$$

If $\mathcal{F}$ has degree d, then the homogeneous polynomials P_i in (3) have degree $d+1$. Using this description, it is not difficult to deduce that any codimension one foliation $\mathcal{F}$ on $\mathbb{CP}^n$ of degree $d = 0$ is induced by a linear projection $\mathbb{CP}^n \dashrightarrow \mathbb{CP}^1$. We say that $\mathcal{F}$ is induced by a pencil of hyperplanes containing a fixed linear subspace $\mathbb{CP}^{n-2}$, which is the singular locus of $\mathcal{F}$. In particular, the leaves of $\mathcal{F}$ are algebraic submanifolds of $\mathbb{CP}^n$.

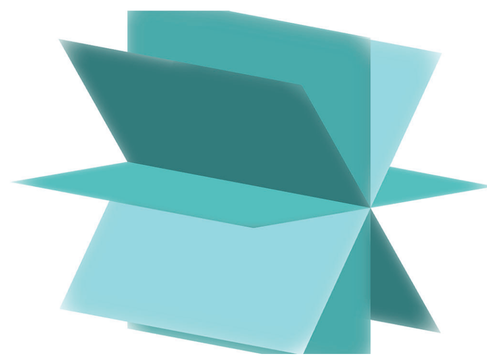

Figure 3. Foliation of degree 0 on $\mathbb{CP}^3$.

In general, the following result characterizes foliations with infinitely many algebraic leaves. It is often referred to as the Darboux-Jouanolou integrability theorem.

Theorem 3 ([Jou79, Théorème 3.3]). *Let $\mathcal{F}$ be a codimension one foliation on $\mathbb{CP}^n$ with infinitely many algebraic leaves. Then $\mathcal{F}$ has a first integral, i.e., it is induced by a rational map $f : \mathbb{CP}^n \dashrightarrow \mathbb{C}$. In particular, all the leaves of $\mathcal{F}$ are algebraic.*

The set $\mathrm{Fol}(d, n)$ of codimension one foliations of degree d on $\mathbb{CP}^n$ has a natural structure of a quasi-projective variety. When $n = 2$, the Frobenius' integrability condition (4) is automatic, and thus $\mathrm{Fol}(d, 2)$ is an open subset of the projective space of polynomial one-forms as in (3), where the P_i's are homogeneous polynomials of degree $d+1$ without common factors. When $d = 0$, we have seen above that $\mathrm{Fol}(0, n)$ can be identified with the Grassmannian parametrizing codimension 2 linear subspaces of $\mathbb{CP}^n$. Jouanolou also classified the next case, $d = 1$. When $n \geq 3$, he showed that $\mathrm{Fol}(1, n)$ has two irreducible components. One of these irreducible components corresponds to foliations given by homogeneous one-forms that depend only on three variables, after suitable projective change of coordinates. Geometrically, they are pullbacks under linear projections of foliations on $\mathbb{CP}^2$ induced by global vector fields. The other irreducible component of $\mathrm{Fol}(1, n)$ corresponds to foliations having a first integral of the form

$$f = \frac{\ell^2}{q} : \mathbb{CP}^n \dashrightarrow \mathbb{C},$$

where ℓ is a linear form and q is a quadratic form. Geometrically, they are induced by pencils of quadric hypersurfaces containing a double hyperplane. Later in [CN96], Cerveau and Lins Neto showed that $\mathrm{Fol}(2, n)$ has six irreducible components when $n \geq 3$, and described each of them. More recently, Da Costa, Lizarbe, and Pereira studied the case $d = 3$, $n \geq 3$ in [dCLP21]. They classify the 18 irreducible components of $\mathrm{Fol}(3, n)$ whose general member does not have a first integral. They also show that there are at least 6 components whose general member has a first integral.

Another global invariant of a foliation is the *algebraic dimension*, or *algebraic rank*. The *algebraic dimension* $r_a(\mathcal{F})$ of a foliation $\mathcal{F}$ on a complex projective manifold X is the maximum dimension of an algebraic subvariety Z through a general point of X that is tangent to $\mathcal{F}$. By this we mean that for every point $z \in Z \setminus \mathrm{Sing}(\mathcal{F})$, we have $T_z Z \subset T_z \mathcal{F}$. A foliation is said to be *purely transcendental* if its algebraic dimension is 0. It is said to be *algebraically integrable* if its algebraic dimension equals the dimension.

From the classification of codimension one foliations on $\mathbb{CP}^n$ of small degree, we observe a lower bound for the algebraic dimension in terms of the degree. Indeed, let $\mathcal{F}$ be a codimension one foliation on $\mathbb{CP}^n$ of degree d. If $d = 0$, then $\mathcal{F}$ is algebraically integrable. If $d = 1$, then $r_a(\mathcal{F}) \geq n - 2$, and this bound is attained when $\mathcal{F}$ is the pullback under a linear projection of a purely transcendental foliation on $\mathbb{CP}^2$ induced by a global vector field. In general, we have that

$$r_a(\mathcal{F}) \geq n - 1 - d.$$

This is a special case of Theorem 4, which gives a lower bound for the algebraic dimension of foliations in a more general context. Notice that the algebraic dimension makes sense for foliations on any complex manifold, while the notion of degree is particular to $\mathbb{CP}^n$ or other varieties covered by lines. As we shall see, the degree can be read off from a more general object that can be attached to any foliation, its *canonical class*. This notion has its origin in connection with birational geometry.

Birational Geometry

A central theme in algebraic geometry is the classification of complex projective varieties up to birational equivalence. Two projective varieties are said to be birationally equivalent if they have isomorphic dense open subsets. Examples of birational invariants of projective manifolds are the *genus* and *irregularity*. Given a complex projective manifold X, we consider the tangent bundle T_X of X, and its dual vector bundle $\Omega^1_X = T_X^\vee$. Let $\omega_X = \det(\Omega^1_X)$ denote the *canonical bundle* of X. It is the line bundle on X whose sections are the top holomorphic differential forms on X. The *genus* $p_g(X)$ of X is the dimension of the space of holomorphic global sections of the canonical bundle ω_X:

$$p_g(X) \coloneqq \dim \Gamma(X, \omega_X).$$

The *irregularity* $q(X)$ of X is the dimension of the space of holomorphic global sections of the cotangent bundle:

$$q(X) \coloneqq \dim \Gamma(X, \Omega^1_X).$$

Moreover, some special arithmetical properties of algebraic varieties turn out to be invariant under birational equivalence, making this notion fundamental also in connection with number theory and arithmetic geometry.

The *Minimal Model Program* (MMP for short) is an algorithmic surgery process designed to transform a given projective variety X into a simplest representative X' in its birational equivalence class. In this way, if one is interested in understanding a birational property of X, one can investigate it in the simpler model X'.

We start by reviewing the classical MMP for surfaces. It was established by the Italian school of algebraic geometry by the beginning of the 20th century, and reviewed in modern language in the 1960's, most notably by Kodaira, Zariski and Shafarevich. Given a smooth projective surface S, the blowup of a point $P \in S$ is a morphism $\pi : \tilde{S} \to S$ from a smooth projective surface $\tilde{S}$ that replaces the point $P \in S$ with the *exceptional curve* $C = \pi^{-1}(P) \cong \mathbb{CP}^1$, and restricts to an isomorphism between $\tilde{S} \setminus C$ and $S \setminus \{P\}$. Viewed as an element of $\mathrm{H}^2(\tilde{S}, \mathbb{Z})$, the exceptional curve C has self-intersection $C^2 = -1$. Conversely, Castelnuovo's contractibility theorem asserts that any curve C on a smooth surface S such that $C \cong \mathbb{CP}^1$ and $C^2 = -1$ is the exceptional curve of a blowup. Such a curve is called a (-1)-*curve*. It turns out that any smooth projective surface can be obtained from a distinguished representative of its birational equivalence class by a sequence of blowups. Such distinguished representatives are characterized by the property that they do not contain any (-1)-curve, and are classically called *minimal surfaces*. Figure 4 summarizes the classical MMP for surfaces. The MMP terminates after a finite number of steps because the second Betti number $b_2(S)$ drops by one every time we blow down a (-1)-curve.

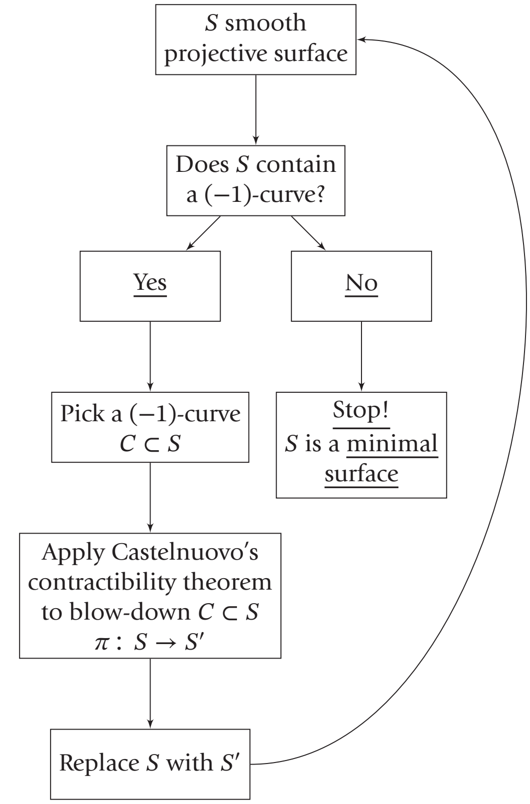

Figure 4. The classical MMP for surfaces.

In order to generalize this to higher dimensions, the question

$$\text{“Does } S \text{ contain a } (-1)\text{-curve?”} \tag{5}$$

must be rephrased in a way that it makes sense in any dimension. This is done with the aid of the *canonical class*.

Definition 1. Let X be a complex projective manifold X. The *canonical class* K_X of X is the first Chern class of the canonical line bundle of X:

$$K_X \coloneqq c_1(\omega_X) \in \mathrm{H}^2(X, \mathbb{Z}).$$

For any compact complex curve $C \subset X$, the intersection number $-K_X \cdot C$ measures the Ricci curvature of X along C. Usually, the sign of $-K_X \cdot C$ varies with the curve $C \subset X$. Varieties whose canonical class has a definite sign are very special, and play a distinguished role in algebraic geometry. Particularly important are *Fano varieties*. A *Fano variety* is a projective variety X with $-K_X > 0$ (i.e., $-K_X$ is ample).

Let X be a complex projective manifold. In order to generalize to higher dimensions the classical MMP for surfaces, question (5) is replaced with

$$\text{“Is the canonical class } K_X \text{ nef?”}$$

We say that K_X is *nef* if $K_X \cdot C \geq 0$ for every curve $C \subset X$. The goal of the MMP is to produce a finite sequence of elementary birational maps

$$X = X_0 \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow \cdots \dashrightarrow X_n = X',$$

ending with a variety X' in the same birational equivalence class of X, and satisfying exactly one of the following two conditions.

1. $K_{X'}$ is *nef*. Such a variety X' is called a *minimal model*.
2. There is a morphism $f : X' \to Y$ onto a lower dimensional variety whose fibers are Fano varieties. Such a morphism $f : X' \to Y$ is called a *Mori fiber space*.

The MMP involves making some choices, and the outcome is not unique in general. However, whether the MMP for X ends with a minimal model or a Mori fiber space depends only on the birational equivalence class of X. Next we introduce a fundamental birational invariant for projective manifolds, the *Kodaira dimension*, defined in terms of the rate of growth of holomorphic sections of the line bundle $\omega_X^{\otimes m}$ as m increases.

Definition 2. Let $\mathcal{L}$ be a line bundle on a complex projective manifold X. The *Iitaka dimension* $\kappa(\mathcal{L})$ of $\mathcal{L}$ is defined as follows. Consider the semigroup $\mathbb{N}(\mathcal{L})$ of nonnegative integers m for which $\mathcal{L}^{\otimes m}$ admits nonzero holomorphic sections. If $\mathbb{N}(\mathcal{L}) = \{0\}$, then we set $\kappa(\mathcal{L}) = -\infty$. If $\mathbb{N}(\mathcal{L}) \neq \{0\}$, then there is an integer κ such that, for $m \in \mathbb{N}(\mathcal{L})$ sufficiently large,

$$c_1 \cdot m^\kappa \ \leq \dim \Gamma(X, \mathcal{L}^{\otimes m}) \ \leq c_2 \cdot m^\kappa,$$

for suitable positive constants c_1 and c_2. We set $\kappa(\mathcal{L}) = \kappa$.

The *Kodaira dimension* of X is

$$\kappa(X) \coloneqq \kappa(\omega_X) \in \{-\infty, 0, \dots, \dim(X)\}.$$

It is a birational invariant for complex projective manifolds.

For surfaces, the elementary birational maps in the MMP are blowups of points. Surfaces that are birational to products $\mathbb{CP}^1 \times C$ are called *ruled surfaces*, and are characterized by the condition that $\kappa(S) = -\infty$. Surfaces with non-negative Kodaira dimension have a unique minimal model in their birational class. They can be divided into the following classes. This is known as the Enriques-Kodaira classification.

- $\kappa(S) = 0$. There are 4 classes.
  - *Enriques' surfaces*: $p_g(S) = q(S) = 0$.
  - *Bielliptic surfaces*: $p_g(S) = 0$, $q(S) = 1$.
  - *K3 surfaces*: $p_g(S) = 1$, $q(S) = 0$.
  - *Abelian surfaces*: $p_g(S) = 1$, $q(S) = 2$.
- $\kappa(S) = 1$. These surfaces are not ruled, and admit a fibration $f : S \to B$ onto a smooth curve whose generic fiber is an elliptic curve.
- $\kappa(S) = 2$. Most surfaces lie in this class. These are called *surfaces of general type*.

In higher dimensions, there are two types of elementary birational maps in the MMP, called *divisorial contractions* and *flips*. Divisorial contractions can be viewed as generalizations of the blowup of a point on a surface. Flips are birational maps $X \dashrightarrow X'$ that restrict to isomorphisms between the complements of *small* subsets of X and X', i.e., subsets of codimension ≥ 2. They have no parallel in surface theory. Another source of complication in higher dimensions is that the birational models X_i's may not be smooth, although they only have very mild singularities, called *terminal singularities*. In dimension three, the MMP was completed in [Mor88]. In higher dimensions a major breakthrough was achieved in [BCHM10], where it was proved that the flip surgery can always be performed. The major open problem in general is to show that there is no infinite sequence of flips. However, [BCHM10] establishes a weaker version of termination of flips, which allows them to prove the MMP in the following cases:

1. $\kappa(X) = \dim(X)$, in which case the MMP ends with a minimal model; and
2. X is *uniruled* (i.e., X is covered by rational curves), in which case the MMP ends with a Mori fiber space.

The *abundance conjecture* predicts that X is uniruled if and only if $\kappa(X) = -\infty$. What is currently known is that the condition that X is uniruled is equivalent to the condition that K_X is not *pseudo-effective*, i.e., K_X is not a limit of classes of effective divisors (with rational coefficients). We refer to [KM98] for an introduction to the MMP.

Canonical Class of Foliations

Similarly, there are relevant properties of holomorphic foliations $\mathcal{F}$ that depend only on the birational equivalence class of $\mathcal{F}$. The algebraic rank is an example of birational invariant for foliations. In recent years, techniques from birational geometry and the MMP have been successfully applied to the study of global properties of holomorphic foliations. This led, for instance, to a birational classification of codimension one foliations on surfaces similar to the Enriques-Kodaira classification, which we review below.

Starting with the tangent sheaf $T_{\mathcal{F}}$ of a foliation $\mathcal{F}$ on a complex projective manifold X, we can define its canonical class $K_{\mathcal{F}} \in \mathrm{H}^2(X, \mathbb{Z})$ and Kodaira dimension $\kappa(\mathcal{F}) \in \{-\infty, 0, \dots, \dim(X)\}$. This is analogous to the definition of the canonical class and Kodaira dimension of X starting with T_X. The Kodaira dimension of codimension one foliations on projective surfaces was first considered in [Men00]. Under restrictions on the singularities of $\mathcal{F}$, the Kodaira dimension $\kappa(\mathcal{F})$ is a birational invariant for foliations.

Motivated by the special role of Fano varieties in birational geometry, we introduce the following class of foliations.

Definition 3. A foliation $\mathcal{F}$ on a complex projective manifold is called a *Fano foliation* if $-K_{\mathcal{F}} > 0$ (i.e., $-K_{\mathcal{F}}$ is ample).

Example 1. Recall that $\mathrm{H}^2(\mathbb{CP}^n, \mathbb{Z}) \cong \mathbb{Z}$, where we choose the positive generator to be the cohomology class of a hyperplane section. Under this identification, $K_{\mathbb{CP}^n} = -(n+1) \in \mathbb{Z}$. Let $\mathcal{F}$ be a codimension one foliation of degree d on $\mathbb{CP}^n$. An easy computation shows that $K_{\mathcal{F}} = d - n + 1 \in \mathbb{Z}$. In particular, Fano foliations on $\mathbb{CP}^n$ are those with small degree, $d < n - 1$.

In a series of papers started in [AD13], the first named author and S. Druel have developed the theory of Fano foliations. They showed that the positivity of $-K_{\mathcal{F}}$ has an effect on the algebraicity of the leaves of $\mathcal{F}$. In order to measure the positivity of $-K_{\mathcal{F}}$, define the index $\iota(\mathcal{F})$ of a Fano foliation $\mathcal{F}$ on a complex projective manifold X to be the largest integer dividing $-K_{\mathcal{F}}$ in $H^2(X, \mathbb{Z})$. This is analogous to the index $\iota(X)$ of a Fano manifold X. The following result is a lower bound for the algebraic dimension $r_a(\mathcal{F})$ of Fano foliations in terms of the index, and a classification of the cases when this bound is attained.

Theorem 4 ([AD19, Corollary 1.6.]). *Let $\mathcal{F}$ be a Fano foliation of index $\iota(\mathcal{F})$ on a complex projective manifold X. Then $r_a(\mathcal{F}) \geq \iota(\mathcal{F})$, and equality holds if and only if $X \cong \mathbb{CP}^n$ and $\mathcal{F}$ is the pullback under a linear projection of a purely transcendental foliation on $\mathbb{CP}^{n-r_a(\mathcal{F})}$ with zero canonical class.*

Fano manifolds with large index have been classified. By a theorem of Kobayachi and Ochiai, the index $\iota(X)$ of a Fano manifold X satisfies $\iota(X) \leq \dim(X)+1$, equality holds if and only if X is a projective space, and $\iota(X) = \dim(X)$ if and only if X is a quadric hypersurface. Fano manifolds with $\iota(X) = \dim(X) - 1$ are called *del Pezzo manifolds*, and were classified by Fujita. Those with $\iota(X) = \dim(X) - 2$ were later classified by Mukai. We refer to [AC13] for a survey on the classification of Fano manifolds with large index, with many references. As a corollary of Theorem 4 above, we have a version of the Kobayachi-Ochiai's theorem for foliations: the index $\iota(\mathcal{F})$ of a Fano foliation $\mathcal{F}$ is bounded by the dimension, $\iota(\mathcal{F}) \leq r$. Moreover, equality holds if and only if $X \cong \mathbb{CP}^n$ and $\mathcal{F}$ is induced by a linear projection $\varphi : \mathbb{CP}^n \dashrightarrow \mathbb{CP}^{n-r}$. This means that, away from the center of the projection, where φ is not defined, the leaves of $\mathcal{F}$ are the fibers of φ. These foliations are precisely the ones of degree 0. The next cases have also been investigated. In analogy with the theory of Fano manifolds, Fano foliations $\mathcal{F}$ with index $\iota(\mathcal{F}) = r - 1$ are called *del Pezzo foliations*. Under restrictions on the singularities, there is a classification of del Pezzo foliations of rank $r \geq 3$, parallel to Fujita's classification of del Pezzo manifolds. We refer to [Fig22] for an account on the classification of del Pezzo foliations.

The birational classification of codimension one foliations on surfaces can be summarized as follows (see [Bru15] and [McQ08]). First of all, by a theorem of Seidenberg [Sei68], after a sequence of blowups, it is always possible to reduce the singularities of a foliation $\mathcal{F}$ on a surface. A point $p \in \mathrm{Sing}(\mathcal{F})$ is a *reduced singularity* if, locally around p, $\mathcal{F}$ is given by a vector field whose linear part has eigenvalues λ_1 and λ_2, with $\lambda_2 \neq 0$, and $\lambda_1/\lambda_2 \notin \mathbb{Q}_+$. This condition implies that there are only finitely many separatices through p. For foliations with reduced singularities, the Kodaira dimension $\kappa(\mathcal{F})$ is a birational invariant. We have the following birational classification of foliations with reduced singularities and Kodaira dimension $\kappa(\mathcal{F}) < 2$.

- $\kappa(\mathcal{F}) = 1$. Then $\mathcal{F}$ is birational to one of the following.
  - *Riccati foliation*: there is a rational fibration $S \to C$ with general fiber everywhere transverse to $\mathcal{F}$.
  - *Turbulent foliation*: there is an elliptic fibration $S \to C$ with general fiber everywhere transverse to $\mathcal{F}$.
  - Foliations induced by non-isotrivial elliptic fibrations.
  - Foliations induced by isotrivial fibrations of genus ≥ 2.
- $\kappa(\mathcal{F}) = 0$. After a ramified covering, $\mathcal{F}$ is birational to a foliation induced by a global holomorphic vector field with isolated zeroes.
- $\kappa(\mathcal{F}) = -\infty$. Either $\mathcal{F}$ is birational to a foliation on a ruled surface induced by a fibration by rational curves, or to a *Hilbert modular foliation*.

A Hilbert modular surface S is the minimal desingularization of the Baily-Borel compactification of the quotient of $\mathbb{H}^2 = \mathbb{H} \times \mathbb{H}$ by an irreducible lattice in $\mathrm{PSL}(2, \mathbb{R})^2$. The two projections $\mathbb{H}^2 \to \mathbb{H}$ induce two foliations on S with dense leaves, called *Hilbert modular foliations*. Hilbert modular foliations provide examples of a completely new behavior which is not seen in classical geometry. Namely, its canonical class $K_{\mathcal{F}}$ is pseudo-effective, while $\kappa(\mathcal{F}) = -\infty$. On the other hand, it is a remarkable theorem due to Miyaoka that foliations with non-pseudo-effective canonical class are uniruled (i.e., their leaves are covered by rational curves).

Much progress has been made toward the birational classification of foliations on higher dimensional varieties. Reduction theorems for foliations on threefolds have been established by Cano in codimension one, and by McQuillan and Panazzolo in dimension one. Building on works of McQuillan, versions of the MMP for foliations on threefolds were recently established by Cascini and Spicer. We refer to [CS21] for details and references about the MMP for codimension one foliations, and [CS20] for foliations of dimension one on threefolds.

Regular Foliations

We end these notes by discussing the classification problem for *regular foliations* on complex projective manifolds. These are holomorphic foliations $\mathcal{F}$ with $\mathrm{Sing}(\mathcal{F}) = \emptyset$. In view of Theorem 1, it is natural to ask for a characterization of projective manifolds that admit regular foliations. For surfaces, the first result in this direction is the following classification of regular foliations on minimal ruled surfaces by Gomez-Mont.

Theorem 5 ([GM89]). *Let $\pi : S \to C$ be a $\mathbb{CP}^1$-fibration, with $g(C) \neq 1$, and let $\mathcal{F}$ be a regular foliation of rank 1 on S. Then*

- *either $\mathcal{F}$ is induced by the fibration $\pi : S \to C$; or*
- *$\mathcal{F}$ is transverse to the fibration $\pi : S \to C$, and constructed by suspension of a representation $\rho : \pi_1(C) \to \mathrm{PSL}(2, \mathbb{C})$.*

Brunella extended this result, and completely classified regular foliations on complex projective surfaces S with Kodaira dimension $\kappa(S) < 2$. It is still not known whether there exist purely transcendental regular foliations on surfaces of general type. The following is a corollary of this classification.

Corollary 1. *Let $\mathcal{F}$ be a regular foliation of rank 1 on a smooth projective rational surface S. Then $\mathcal{F}$ is induced by a $\mathbb{CP}^1$-fibration $S \to \mathbb{CP}^1$.*

In higher dimensions, there is no classification of regular foliations. However, Touzet has conjectured the following generalization of Corollary 1 (see [Dru17, Conjecture 1.2]). From the point of view of birational geometry, a natural higher dimensional analog of the class of rational surfaces is that of *rationally connected varieties*. A complex projective manifold X is *rationally connected* if any two points of X can be connected by a rational curve. We refer to [Kol01] for an introductory discussion on this topic.

Conjecture 1 (Touzet). *Let $\mathcal{F}$ be a regular foliation on a rationally connected projective manifold X. Then $\mathcal{F}$ is induced by a fibration $X \to Y$.*

The class of rationally connected varieties includes that of Fano manifolds, for which the conjecture was verified in [Dru17]. While Touzet's conjecture is open already in dimension 3, for a regular codimension one foliation $\mathcal{F}$ on a projective threefold X, the MMP can be used to greatly reduce the problem. After a sequence of smooth blowdowns centered at smooth curves,

$$X \to X_1 \to X_2 \to \cdots \to X_n,$$

one reaches a smooth threefold X_n, together with a regular codimension one foliation $\mathcal{F}_n$ on X_n, such that either $K_{\mathcal{F}_n}$ is nef, or $X_n \to Y_n$ is a Mori fiber space with fibers tangent to $\mathcal{F}_n$. In the latter case, the second named author showed in [Fig21] that $\mathcal{F}_n$ is induced by a fibration by rational surfaces, and hence so is $\mathcal{F}$. This proves Touzet's conjecture for codimension one foliations on threefolds in some special cases, and reduces the problem to understanding regular foliations with nef canonical class.

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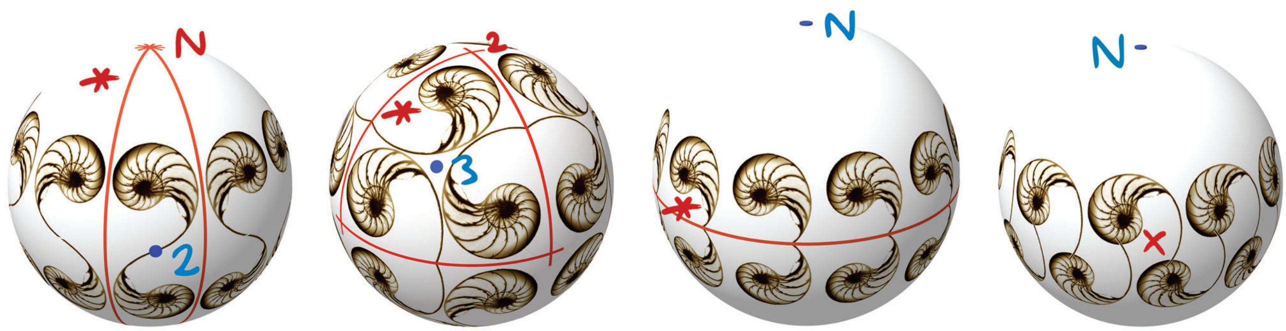
Conway's Mathematics After Conway



Alex Ryba with papers contributed by Philip Ehrlich, Richard Kenyon, Jeffrey Lagarias, James Propp, and Louis H. Kauffman

Mathematicians always queued to hear John Conway speak and delved into his writing. They expected to be entertained by beautiful mathematics and believed that they would emerge with valuable enlightenment. John welcomed this attention and considered it his duty to make his mathematics elegant. What really made his mathematics valuable was his wealth of insight. Wherever he worked, he opened up avenues for us to follow.

John challenged and inspired us in many different ways. His Game of Life has been investigated by thousands, while his *river* method provides a novel approach to the classical subject of quadratic forms. His *orbifold notation* was devised in collaboration with Bill Thurston. It gives a new way to describe symmetry patterns like those in the opening images above. The images are taken from the beautiful book, *The Symmetries of Things*, coauthored by John, Heidi Burgiel, and Chaim Goodman-Strauss. Group theorists, like me, are drawn to his paper "Monstrous Moonshine," written with his friend Simon Norton. John and Simon filled the paper with what were outlandish examples and provocative conjectures about the Monster group. Mathematicians immediately realized that the group was much more than a sporadic curiosity, as yet unconstructed. Richard Borcherds created his new theory of Vertex Algebras to prove the conjectures in John's paper. McKay and Sebbar aptly call Moonshine "21st century mathematics in the 20th century" [15]. John's influence can be seen across so many areas of mathematics that a complete account would fill many volumes, which would continue to expand forever. The three papers that follow present very different parts of mathematics where modern approaches are built on foundations laid by John.

The theory of surreal numbers was invented by John in the early 1970s. He expected this to become his most influential creation and was convinced that the theory would continue to evolve long into the future. Philip Ehrlich explains how this theory has progressed in the half century since its introduction.

John's algebraic approach to certain tiling problems is explained by Richard Kenyon, Jeffrey Lagarias, and James Propp. These tiling problems are mathematical questions. However, they used to be viewed as recreational because each required its own special trick. John's novel and very general approach showed that the problems belong to the field of Combinatorial Group Theory. John's vision across the whole breadth of mathematics allowed him to make many similar unexpected connections between fields.

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Knot theory was John's first research interest in mathematics. He was still in high school when he started to develop his Skein Theory. Louis Kauffman explains that the monumental 1983 paper of Vaughan Jones showed that the Jones polynomial satisfies a Conway type skein relation. A number of mathematicians were inspired to extend the Conway skein ideas to produce the Homflypt and Kauffman polynomials. This led directly to explosive progress in modern Knot Theory.

John saw himself as a link in a chain that stretched back through the mathematicians who created the fields where he worked and played. He particularly admired Georg Cantor, whose ordinals were uppermost in his mind when he worked on the surreal numbers. John, in turn, laid out new areas for others to cultivate after him. His legacy will be owned and continued by the mathematicians for whom he sowed so many seeds.

The Surreal Numbers and Their Aftermath

Philip Ehrlich

In 1970, as an outgrowth of his work on combinatorial games, J. H. Conway introduced a real-closed field, dubbed **No**, containing the reals and the ordinals, the arithmetic of the latter being the natural sums and products due to Hessenberg and Hausdorff rather than the usual non-commutative, non-associative sums and products of Cantor. Being a real-closed field containing the reals and the ordinals, **No** also contains a great many less familiar numbers including $-\omega$, $\omega/2$, $1/\omega$, $\sqrt{\omega}$ and $\omega - \pi$ to name only a few, where ω is the least infinite ordinal. Indeed, as Conway aptly quips, this particular real-closed field is so remarkably inclusive that it may be said to contain "All Numbers Great and Small" ([5, p. 3]). In this regard, **No** bears much the same relation to ordered fields that the ordered field $\mathbb{R}$ of real numbers bears to Archimedean ordered fields. Following D. E. Knuth, the members of **No** have come to be called *surreal numbers*.

While each surreal is an ordinary set, **No** is not. To address this, Conway formalizes his theory in NBG (*Von Neumann-Bernays-Gödel set theory*). Unlike standard *Zermelo-Fraenkel set theory* (ZF), whose sole entities are sets, NBG contains classes that are sets, as well as classes such as **No** and the class **On** of ordinals that are larger than any set, called *proper classes*.

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Against this set-theoretic backdrop, the relation between the inclusiveness of $\mathbb{R}$ and **No** may be made precise by the following result collectively due to Conway [5, pp. 42-43] and Ehrlich [7]: *whereas $\mathbb{R}$ is (up to isomorphism) the unique universally embedding Archimedean ordered field,* **No** *is (up to isomorphism) the unique universally embedding ordered field*, where an ordered field (Archimedean ordered field) A is said to be *universally embedding* if for each ordered subfield (Archimedean ordered subfield) B of A, whose universe is a set, and every ordered field (Archimedean ordered field) B' extending B, there is an embedding $f : B' \to A$ that is the identity on B. Thus, starting with a categorical characterization of the inclusiveness of $\mathbb{R}$ that makes use of the Archimedean axiom, one obtains a categorical characterization of the inclusiveness of **No** by simply deleting the Archimedean condition. On the basis of this observation, that Conway initially thought "too good to be true," Ehrlich (e.g., [7, 9]) has suggested that whereas $\mathbb{R}$ should merely be regarded as the *arithmetic continuum* (modulo the Archimedean axiom), **No** may be regarded as the *absolute arithmetic continuum* (modulo NBG).

Like the ordered set of reals, the ordered class of surreals may be constructed in a variety of ways (e.g., [2, 5, 9, 11]). In Conway's construction, which generalizes aspects of Dedekind's cut construction of $\mathbb{R}$ and von Neumann's construction of **On**, the members of **No** (or, more property speaking, *their vast array of equivalent representations*) are extracted from an antecedently, inductively defined partially ordered class of *games* vis-à-vis the following inductive definition.

Construction of Surreal Numbers

If L and R are two sets of surreal numbers such that no member of L is greater than or equal to any member of R, then there is a surreal number $\{L|R\}$. All surreal numbers are constructed in this way.

In accordance with his convention for games, which likewise are equivalence classes of representatives of the form $\{L|R\}$, Conway denotes each surreal $x = \{L|R\}$ by '$x = \{x^L|x^R\}$' where the x^L's and x^R's—the *left* and *right options* of x—are understood to range over the members of L and the members of R, respectively. Using this convention, Conway's construction of the surreal numbers, which is carried out in stages (called "days") indexed over the ordinals, can be informally described as follows.

On the 0th day, beginning with the empty set $\emptyset$ (of surreal numbers), Conway constructs the surreal number

$$0 = \{|\};$$

and on the 1st day, the surreal numbers

$$-1 = \{|0\} \qquad 1 = \{0|\};$$

and then on the 2nd day, the surreal numbers

$$-2 = \{|0,1\} \;\; -1/2 = \{-1|0\} \;\; 1/2 = \{0|1\} \;\; 2 = \{0,1|\},$$

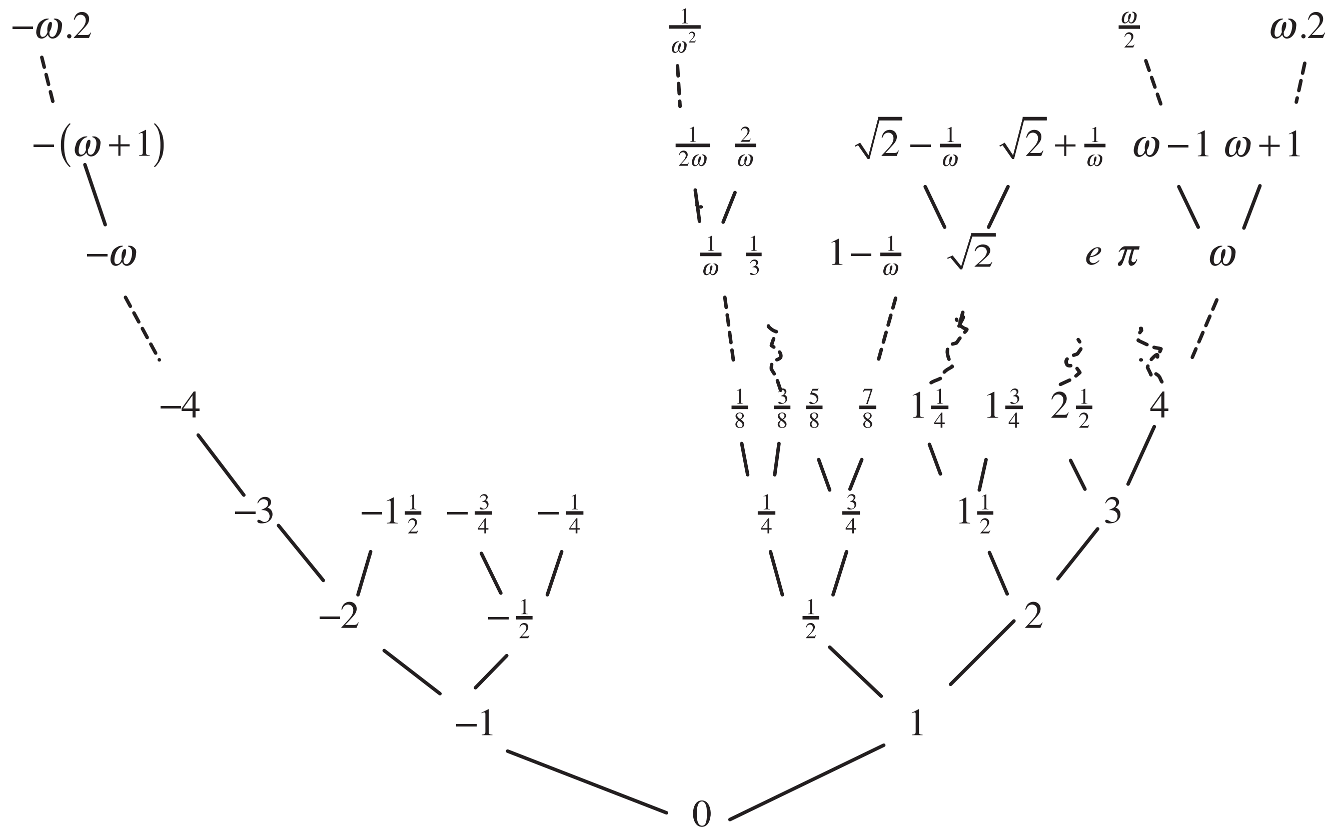

Figure 1. Some of the earliest created surreals.

and so on, each newly created surreal filling a cut in the ordered set of previously constructed surreal numbers. As is evident from the above, unlike Dedekind's cuts, the left and right sides of Conway's cuts may be empty. In fact, those surreal numbers having no right options, beginning with 0, 1, and 2, turn out to be **No**'s ordinals and those having no left options, beginning with $0, -1$ and -2, are the additive inverses of the ordinals, a surreal number and its additive inverse always being created on the same day. The surreal numbers other than the integers that emerge on finite days are the remainder of **No**'s dyadic rationals and those emerging on the ωth day are the non-dyadic real numbers as well as a host of other numbers including $-\omega = \{|\ldots, -n, \ldots, -1, 0\}$, $\omega = \{0, 1, \ldots, n, \ldots|\}$, $-1/\omega = \{-1, -1/2, \ldots - 1/2^n, \ldots|0\}$ and $1/\omega = \{0|\ldots, 1/2^n, \ldots, 1/2, 1\}$, to name a few. For a slightly broader glimpse of some of the earliest created surreal numbers, see Figure 1, which is essentially taken from [5].

As the above description of Conway's construction suggests, the system of surreal numbers has a rich algebraico-tree-theoretic structure that emerges from combining Conway's field operations with **No**'s structure as a tree. It is the marriage of these two components, which we now consider in turn, from which the appellations for surreal numbers as well as a host of other distinctive features of **No** accrue.

Using the familiar definitions of $=$ and $<$ in terms of the partial ordering $\geq$ that is anteriorly defined on games, Conway shows that **No** is an ordered field when $+, -$, and $\cdot$ are defined by the following inductive stipulations, where x^L, y^L, x^R, and y^R are understood to range over the left and right options of x and y.

Definition of $x + y$.

$$x + y = \left\{x^L + y, x + y^L | x^R + y, x + y^R\right\}.$$

Definition of $-x$.

$$-x = \left\{-x^R | -x^L\right\}.$$

Definition of xy.

$$\begin{aligned} xy \quad = \quad & \{x^L y + x y^L - x^L y^L, x^R y + x y^R - x^R y^R | \\ & x^L y + x y^R - x^L y^R, x^R y + x y^L - x^R y^L\}. \end{aligned}$$

Conway shows that for each surreal $x = \{L|R\}$, x is the *earliest created* surreal number lying between the members of L and the members of R, i.e., x is the earliest constructed $z \in$ **No** such that $L < \{z\} < R$. Appealing to this result a significant portion of the theoretical underpinnings of the above definitions of $+$ and $\cdot$ may be brought to the fore by the following variations on observations due to Conway.

Since $x = \{x^L | x^R\}$ and $y = \{y^L | y^R\}$, it follows that for all x^L, x^R, y^L and y^R:

$$(*) \quad x^L < x < x^R \text{ and } y^L < y < y^R.$$

Accordingly, for **No** to be an ordered additive group, it must be the case that

$$(**) \quad x^L + y, x + y^L < x + y < x^R + y, x + y^R,$$

for all x^L, x^R, y^L, and y^R. Therefore, since $x+y$ must lie between the two sets of inductively defined members of **No** specified in $(**)$ if **No** is to be an ordered group, the above definition of $x+y$ deems $x+y$ to be the earliest created surreal number consistent with that intended outcome. Similarly, for **No** to be an ordered field, it follows from $(*)$ that each of the differences $x - x^L, x^R - x, y - y^L, y^R - y$ must be positive and, hence, likewise each of the products $(x - x^L)(y - y^L)$, $(x^R - x)(y^R - y)$, $(x - x^L)(y^R - y)$ and $(x^R - x)(y - y^L)$. And so by applying the routine algebra of ordered fields to each of these products one obtains for all x^L, x^R, y^L and y^R:

$$\begin{aligned}(***) \quad & x^L y + x y^L - x^L y^L, x^R y + x y^R - x^R y^R < \\ & xy < x^L y + x y^R - x^L y^R, x^R y + x y^L - x^R y^L.\end{aligned}$$

Consequently, since xy must lie between the sets of inductively defined members of **No** specified in $(***)$ if **No** is to be an ordered field, the above definition of xy requires xy to be the earliest constructed surreal compatible with that desired end.

While the above observations reveal the incisive nature of Conway's cryptic-seeming inductive definitions, there is no a priori reason to believe that these definitions, however cleverly motivated, would lead to the existence of an ordered field, let alone one having the rich structure of **No**. That they do is one of Conway's most remarkable surreal discoveries.

As we mentioned above, in addition to its inclusive structure as an ordered field, **No** has a rich tree-theoretic structure, a *tree* being a partially ordered class $(A, <_A)$ such that for each $x \in A$ the class $\{y \in A : y <_A x\}$ of *predecessors* of x is a well-ordered set. This *simplicity hierarchical* (or *s-hierarchical*) *structure*, as it is sometimes called, is introduced by Conway by associating each surreal with a unique sequence of $+$'s and $-$'s indexed over an ordinal, called it's *sign-expansion*, but can be introduced more concisely as follows using the canonical representation of surreal numbers employed in Conway's treatment.

Each surreal x has a unique representation $\{L_x | R_x\}$, where (L_x, R_x) is a pair of (possibly empty) collectively exhaustive subsets of the set of all surreal numbers constructed at earlier stages of the construction. Using this representation, the *surreal number tree* $(\mathbf{No}, <_s)$ is obtained by stipulating that for all $x, y \in \mathbf{No}$, $x <_s y$ (read "x is *simpler than* y) if and only if $L_x < \{y\} < R_x$ and $x \neq y$.[1] $(\mathbf{No}, <_s)$ is in fact a *full binary tree*, that is, every member of **No** has two immediate successors and every chain in $(\mathbf{No}, <_s)$ of limit length (including the empty chain) has a unique immediate successor. In particular, for each surreal number x, $\{L_x | \{x\} \cup R_x\}$ and $\{L_x \cup \{x\} | R_x\}$ are the immediate successors of x, and if $(x_\alpha)_{\alpha<\beta}$ is a chain in $(\mathbf{No}, <_s)$ indexed over a limit ordinal β, then $\{\bigcup_{\alpha<\beta} L_{x_\alpha} | \bigcup_{\alpha<\beta} R_{x_\alpha}\}$ is the immediate successor of the chain, $\{\emptyset|\emptyset\}$ being the immediate successor of the empty chain.

As we alluded to above, among the striking s-hierarchical features of **No** is that every surreal number can be assigned a canonical "proper name" that is a reflection of its characteristic s-hierarchical properties. These *Conway names*, or *normal forms* as Conway calls them, are expressed as formal sums of the form

$$\sum_{\alpha<\beta} r_\alpha \omega^{y_\alpha},$$

where β is an ordinal, $(y_\alpha)_{\alpha<\beta}$ is a strictly decreasing sequence of surreals, and $(r_\alpha)_{\alpha<\beta}$ is a sequence of nonzero real numbers. Every such formal sum is in fact the Conway name of a surreal number, the Conway name of an ordinal being just its *Cantor normal form*.

In light of the above, we see that Figure 1 in fact offers a glimpse of the some of the earliest created members of $(\mathbf{No}, <_s)$ expressed in terms of their Conway names, where, for example, ω is the least infinite ordinal as well as *the simplest positive infinite number*, $-\omega$ is the additive inverse of ω as well as *the simplest negative infinite number* and $1/\omega$ is the multiplicative inverse of ω as well as *the simplest positive infinitesimal number*. It is worth noting, that being a real-closed field, the Conway names for **No**'s real algebraic numbers are determined solely by **No**'s algebraic structure, whereas the Conway names for the remaining surreals are fixed by algebraico-tree-theoretic considerations (e.g., [8, pp. 1244–1248]).

Conway observed that when the surreals are expressed in terms of their normal forms, **No** assumes the structure

[1] *As we mentioned above, Conway shows that each surreal number $\{L|R\}$ is the earliest created surreal number lying between its left and right options, the latter being a special case of his result for games, where, however, one cannot in general say "a game lies between its left and right options." In* [5, p. 23], *Conway dubs the earliest created such game (surreal number), the* simplest *such game (surreal number), naturally suggesting that for surreal numbers, as for games, "x is simpler than y" should be interpreted as x is constructed prior to y. In 1985 Conway agreed with the author that the simpler than relation for the surreals should be defined in terms of the predecessor relation in the tree rather than in terms of the created earlier than relation. This understanding was brought to the fore in* [8] *and has since emerged as the dominant interpretation of the simpler than relation in research on surreal numbers. Whereas a surreal number $\{L|R\}$ continues to be both the simplest and the earliest created surreal number lying between its left and right options, the meaning of "simpler than" has changed. For example, whereas $1/2$ is created earlier than ω, $1/2$ is not simpler than ω in the tree-theoretic sense (see Figure 1).*

of an ordered field of generalized formal power series with sums and products defined like polynomials and order defined lexicographically. In addition to making the surreals more tractable from an algebraic point of view, this permitted Conway to apply to **No** insights about such structures that accrue from the classical works of Hans Hahn (1907) and B. H. Neumann (1949) on the number systems and generalizations thereof that emerged from non-Archimedean geometry, and thereby relate the surreals to one of the roots of non-Archimedean mathematics.

Making use of Conway names, Conway also characterized the notion of *integer* appropriate to **No**. The discrete ring **Oz** of *omnific integers*, which extends **On**, consists of the surreal numbers whose Conway names have nonnegative exponents and integer coefficients when the exponent is 0. Every surreal number is distant at most 1 from some omnific integer, and **No** is **Oz**'s field of fractions.

Another striking s-hierarchical feature of **No** is that, much as the surreal numbers emerge from the empty set of surreal numbers by means of a transfinite induction that generates the entire spectrum of "numbers great and small," the inductive process of defining **No**'s arithmetic in turn generates the entire spectrum of ordered fields (ordered abelian groups) in such a way that an isomorphic copy of every such system either emerges as an *initial substructure* of **No** — a substructure A in which the tree-theoretic predecessors in A of each of its elements coincide with its predecessors in **No** — or is contained in a theoretically distinguished instance of such a system that does. In particular, as Ehrlich [8] showed, *every real-closed ordered field (divisible ordered abelian group) is isomorphic to an initial subfield (subgroup) of* **No**.

Since every real-closed field is isomorphic to an initial subfield of **No**, the underlying ordered field of every *hyperreal number system* — the nonstandand models of analysis employed in *Robinsonian* or *nonstandard analysis* — is isomorphic to an initial subfield of **No**. In fact, as Ehrlich [9] observed, **No** is isomorphic to the underlying ordered field of the richest hyperreal number system in NBG. On the other hand,"**No** is really irrelevant to nonstandard analysis," as Conway [5, p. 44] noted; and, vice versa. After all, whereas the transfer property of hyperreal number systems, a property not possessed by **No**, is central to the development of nonstandard analysis, the s-hierarchical structure of **No**, which is absent from hyperreal number systems, is central to the theory of surreal numbers. Of course, this does not preclude that down the line there might be cross-fertilization between the two theories.

However, while surrealists have thus far shown little interest in applying surreal numbers to nonstandard analysis or in providing an infinitesimalist approach to classical analysis based on surreal numbers more generally, a number of surrealists beginning with Norton, Kruskal, and Conway have fostered the idea of extending analysis to the entire surreal domain.

Building on work of B. H. Neumann, Conway observed that there is a notion of convergence in **No** for power series of infinitesimal surreals that can be expressed using Conway names, and that the various analytic functions could be defined on bounded portions of **No** using such power series whenever they converge in the appropriate sense. On the other hand, Conway originally expressed doubt that "reasonable" global definitions of exponentiation, logarithm, sine, and cosine could be defined on **No** [5, First Edition, p. 43]. Through the collective efforts of Kruskal, Norton, Gonshor, van den Dries, Ehrlich, and Kaplan, however, this doubt has been put to rest. Van den Dries and Ehrlich (2001) showed that **No** together with the Kruskal-Gonshor exponential function exp defined thereon [11] has the same elementary properties of the ordered field of real numbers with real exponentiation, and Ehrlich and Kaplan [10] have further shown that **No** has canonical sine and cosine functions which in turn lead to a canonical exponential function on **No**'s *surcomplex* counterpart **No**$[i]$ that extends exp.

Additional rudiments of analysis on the surreals have also been developed by Alling, Fornasiero, Rubinstein-Salzedo and Swaminathan, and Costin, Ehrlich and Friedman. Costin and Ehrlich, in particular, have developed a theory of integration (and differentiation) that extends the range of analysis from the reals to the surreals for a large subclass of *resurgent functions* that arise in applied analysis. The resurgent functions, which generalize the analytic functions, were introduced by Écalle in the early 1980s in connection with work related to Hilbert's 16th problem. Unlike nonstandard analysis, which provides an infinitesimalist approach to integration on the extended reals ($\mathbb{R} \cup \{\pm\infty\}$), surreal integration deals with integrals whose bounds and values need not be extended reals at all. For example, in the surreal theory (setting $e^x = \exp x$) we have

$$\int_0^\omega e^x dx = e^\omega - 1 = \omega^\omega - 1.$$

This work makes contributions towards realizing some of the analytic goals expressed by Kruskal and Norton in their unsuccessful early attempts to establish a theory of surreal integration as described by Conway in the Epilogue of [5, Second Edition].

Elements of asymptotic differential algebra — the subject that aims at understanding the asymptotics of solutions to differential equations from an algebraic point of view — have also been developed for the surreals. An *ordered differential field* is an ordered field K together with a *derivation* on K, i.e., a map $\partial : K \to K$ such that $\partial(a+b) = \partial(a) + \partial(b)$ and $\partial(ab) = \partial(a)b + a\partial(b)$ for all $a, b \in K$. Berarducci and Mantova (2018) constructed a derivation ∂_{BM} on **No** which has proven to have a number of desirable features, including $(\mathbf{No}, \partial_{BM})$ being universal with respect to a broad class of distinguished ordered differential fields. In their ICM talk [3], Aschenbrenner, van den Dries, and van der Hoeven outline the program they (along with Berarducci, Mantova, Bagayoko, and Kaplan) are engaged in for developing an ambitious theory of asymptotic differential algebra for all of **No**, though one that would require a derivation on **No** having compositional properties not enjoyed by ∂_{BM}. Such a program, if successful, would provide the most dramatic advance towards interpreting growth rates as numbers since the pioneering work of Paul du Bois-Reymond, G. H. Hardy, and Felix Hausdorff on "orders of infinity" in the decades bracketing the turn of the 20th century.

Work on the rates of growth of real functions, non-Archimedean geometry and Cantor's theory of the infinite are the primary sources of late nineteenth- and early twentieth-century non-Archimedean number systems. As the above remarks suggest, with his creation of the surreal numbers, Conway constructed a remarkable and profoundly original canonical framework for unifying not only these number systems but the reals and the underlying ordered fields of the hyperreal number systems to boot. With this, Conway joined the likes of Cantor, Dedekind, Hahn, and Robinson as one the foremost creators of systems of numbers great and small the world has ever known.

Conway's Tiling Groups

Richard Kenyon, Jeffrey Lagarias, and James Propp

John Conway was fascinated by tilings of the plane, from periodic tilings (the Conway criterion, orbifold notation for wallpaper groups) to aperiodic tilings (Penrose tilings, pinwheel tilings). Some of this work was described by Doris Schattschneider in her article "John Conway, Tilings, and Me" in the Summer 2021 special issue of the Intelligencer devoted to Conway's legacy. Less well-known is John's work on applying combinatorial group theory, a favorite tool of his, to the study of tiling problems in finite subregions of the plane. Conway and Lagarias' 1990 article "Tiling with Polyominoes and Combinatorial Group Theory" [6] opened a new door to the study of such problems. Thurston's 1990 article "Conway's tiling groups" added a geometric viewpoint on such invariants, introducing height functions, which we define below.

A *polyomino* is a connected union of a finite set of squares in an infinite square grid. We say that a collection of simply-connected polyominoes $T_1, \ldots, T_r$ (called *prototiles*) tiles a region R if we can write R as a union of polyominoes with disjoint interiors, each of which is a translate of one of the T_i. Recreational mathematics abounds in problems of the form "Do *these* prototiles tile *this* region?"; the classic problem of the genre is the Mutilated Checkerboard Problem, in which the prototiles are a 1-by-2 and a 2-by-1 rectangle (called dominos) and the region to be tiled is an 8-by-8 square from which two opposite 1-by-1 corner squares have been removed. The classic solution comes from a *coloring argument*, exploiting an alternating black-white coloring of the board: each domino covers one white square and one black square, but the mutilated checkerboard has unequal numbers of black and white squares, so no tiling exists. This argument was presented in Solomon Golomb's 1954 paper "Checker boards and polyominoes" which introduced the term "polyomino."

A more complicated problem of this kind, due to de Bruijn, calls for tiling a 6-by-6 square (or more generally a $(4m+2)$-by-$(4n+2)$ rectangle) with 1-by-4 and 4-by-1 prototiles; as in the case of the mutilated checkerboard, a naive area argument fails to solve the problem but a more sophisticated coloring argument (which the reader is invited to find) shows that no tiling exists.

Prior to 1990, the main ways to prove that a given tiling problem was unsolvable were to give a coloring argument, to conduct brute force examination of all possibilities, or to employ ad hoc methods that varied from problem to problem. The hope that a general method for solving such problems could be found was dashed by the work of Robert Berger, who in his 1966 article "The undecidability of the domino problem" showed that infinite tileability problems could be undecidable. For finite tilings, Leonid Levin showed in the 1973 paper "Universal search problems" that the class of finite tileability problems is NP-complete. Conway and Lagarias presented a new framework for tackling tiling problems that, while necessarily subject to the limitations imposed by these hardness results, went beyond what coloring arguments could do.

The basic idea, due to Conway, is to interpret paths in the square grid starting from the origin as elements of the free group F_2 on two generators A, U, where A ("across") represents a step to the right, A^{-1} a step left, U ("up") a step up, and U^{-1} a step down (backtracks naturally cancel under this correspondence). The free group product corresponds to concatenation of paths, translating one path

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to the end of the other. For simply-connected regions R, the counterclockwise boundary ∂R of R is described by a word w in F_2, unambiguous up to a choice of base point for the loop (or, equivalently, unique up to conjugation). Likewise, the boundary of each prototile T_i corresponds to a word u_i. The main insight is that if a simply-connected R can be tiled by simply-connected prototiles T_i, then the word w can be expressed as a product of conjugates of the words u_i; in other words, if R can be tiled, then w is trivial in the quotient group $G = F_2/N$, where N is the normal subgroup of F_2 generated by conjugates of the tiles. Following Thurston we call G the *Conway tiling group* associated to the given prototile set, and the associated necessary condition ("boundary criterion") for R to have a tiling is that the boundary word w for ∂R belong to N.

A simple example (see Figure 2) illustrates the boundary criterion.

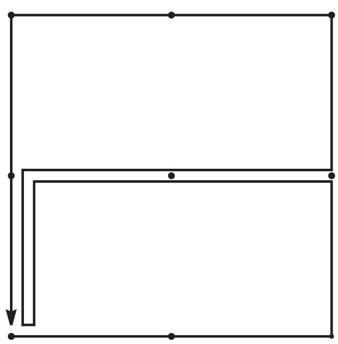

Figure 2. Composing boundary words, with conjugation.

Take R to be the 2-by-2 square $[0,2]\times[0,2]$, tiled by two 2-by-1 rectangles. We find that

$$w = A^2U^2A^{-2}U^{-2} = (A^2UA^{-2}U^{-1})\,U(A^2UA^{-2}U^{-1})U^{-1}$$

where the right side is the product of the boundary-words of the two constituent 2-by-1 rectangles, in which the second is conjugated by U.

It is not immediately clear that this algebraic criterion on tilings will be useful, since the word problem for groups is undecidable in general; however, there are many groups of geometric origin for which there are fast algorithms for decidability. The paper of Conway and Lagarias applied the group-theoretic condition to a tiling problem in the hexagonal lattice. Define a T_n-triangle as a polygon in a honeycomb grid composed of n rows of hexagons in which the ith row contains i hexagons ($1 \leq i \leq n$); for instance, Figure 3 shows T_n for the case $n = 9$.

The problem is to determine for which values of n the T_n-triangle can be tiled by copies of the T_2-triangle and the inverted T_2-triangle (as illustrated in Figure 3). The paper recast this as a problem in the square grid and then, using various algebraic and geometric arguments (including an invocation of the concept of winding number) showed that the values of n for which a tiling exists are the positive integers congruent to 0, 2, 9, or 11 (mod 12).

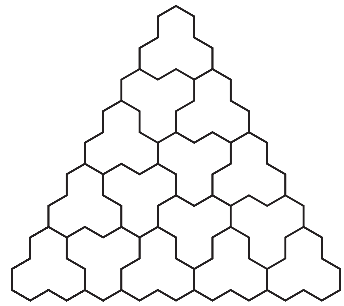

Figure 3. The T_9-triangle tiled by copies of the T_2-triangle and the inverted T_2-triangle.

The paper also showed that *no possible coloring argument can prove this congruence criterion*. Coloring arguments that prove the nonexistence of a solution to a tiling problem will also prove nonexistence of a solution to the associated "signed tiling" problem. But the signed version of the triangle tiling problem has a different answer: it can be solved whenever n is congruent to 0 or 2 (mod 3).

The paper more generally described a connection between the coloring approach to tiling problems and the boundary invariants approach, observing that coloring arguments are covertly group-theoretic. They can be phrased in terms of quotient groups of the commutator subgroup $C = [F_2, F_2]$. The group C contains N as a normal subgroup, and C/N is analogous to a homotopy group. The abelianization of C/N is then analogous to a homology group, and coloring invariants are homology invariants.

Thurston's article "Conway's tiling groups" [17] takes a more geometric viewpoint, introducing the idea of associating to each tiling of R a function from the set of lattice-points inside R that lie on the boundaries of tiles to the Conway tiling group. In the case where the Conway tiling group is an extension of $\mathbb{Z}^2$ by $\mathbb{Z}$ (as holds for domino tilings) these functions are called *height functions*. Thurston uses height functions to obtain a necessary and sufficient condition for a (simply-connected) region R in the square lattice to have a domino tiling. The notion of height functions played a crucial role in the study of random tilings that exploded in the 1990s. The third author reported on these developments in the Conway memorial issue of the Mathematical Intelligencer (Summer 2021).

Skein Theory and More

Louis H. Kauffman

Here we discuss two key contributions of John Conway to knot theory: rational tangles for knots and links, and skein theory for knots and links. Conway's Tangle Theorem associates a tangle to each rational number; two of these will have the same topological type if and only if the rational numbers are equal. The knots we obtain by closing these tangles have the same topological type if the continued fraction expansions of the two rational numbers differ by a reversal of order.

In Figure 4, we illustrate some of the features of the theory of tangles. You will see tangles T and S, one labeled with the continued fraction $[2,3,4] = 2 + 1/(3 + 1/4) = 2 + 4/13 = 30/13$ and the other labeled with the continued fraction $[4,3,2] = 4 + 1/(3 + 1/2) = 4 + 2/7 = 30/7$. It turns out by Conway's Tangle Theorem [4] that these fractions classify the topological type of the tangles. For tangle-type we keep the ends fixed and let the tangles move about. Thus these two tangles are not topologically equivalent. But, as the figure shows, they are related. They both close to the same rational knot, labeled $N(T) = N(S)$ in the figure. Now notice that both of these fractions have the same numerator (30) and as for the denominators, we have that $7 \times 13 = 91$, a number that leaves a remainder of 1 on division by 30. These are not accidents. If $[a_1, a_2, \cdots, a_n]$ is the continued fraction for the tangle T and $[a_n, a_{n-1}, \cdots, a_1]$ (obtained by reversing the order of the terms) is the continued fraction for the tangle S, then both T and S close to form the same rational knot or link.

There is a beautiful way to classify rational knots (closures of rational tangles) from their continued fractions. We can take the symbol $(4,3,2)$, up to such reversal, as an indicator of the rational knot in the figure. (There is a futher technicality to the classification. See [14].) In this way Conway developed a simple notation to indicate rational knots and then used it along with insertion in certain graphs to make a very efficient notation for knots that lets us indicate thousands of knots in the knot tables with great elegance.

Here is a quick introduction to the skein theory of John Conway [4]. In Figure 5, I indicate one knot or link diagram K_+ and another diagram K_-, where it is understood that these two diagrams differ only by the switch that is illustrated for a single crossing. The same figure indicates another diagram K_0 where the crossing has been replaced by two parallel arcs. This is called a *smoothing* of

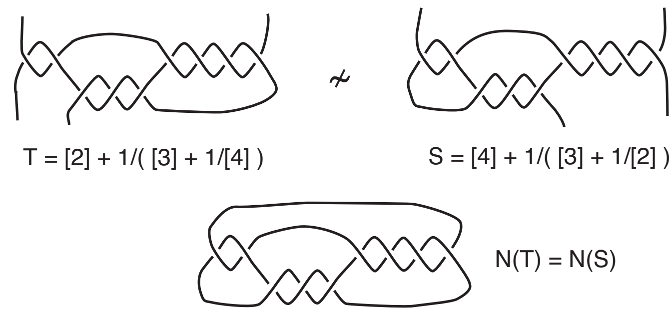

Figure 4. Continued Fractions and Rational Knots.

the crossing. The three diagrams K_+, K_-, K_0 taken together are called a *skein triple* and here is the key relationship for the *Alexander-Conway Polynomial* $\nabla_K(z)$ that is assigned to any oriented link diagram:

$$\nabla_{K_+} - \nabla_{K_-} = z\nabla_{K_0}.$$

Along with this *skein relation*, one has that

$$\nabla_O = 1$$

where O denotes an unknotted circle. If K and K' are topologically equivalent knots or links, then

$$\nabla_K = \nabla_{K'}.$$

This is the complete set of rules for finding the invariant $\nabla_K(z)$ for any oriented link K.

In Figure 6, we illustrate the simplest consequence of these axioms. The basic skein triple consists of U, U', V, where U and U' are unknots and V is a pair of unlinked circles. Each of U and U' evaluates to 1 and so their difference is 0. Thus we conclude that $z\nabla_V = 0$ and so $\nabla_V = 0$. In this way, an unlink V (of any number of components greater than 1) can be seen to receive the value 0 for its Alexander-Conway polynomial $\nabla_V(z)$.

In Figure 7, we indicate this situation. There T is a trefoil knot diagram and U is the result of switching one crossing in the diagram T. We can let $K_+ = T$, $K_- = U$ and $K_0 = L$, the link of two components illustrated in the top line of the figure. Thus we have

$$\nabla_T - \nabla_U = z\nabla_L$$

and

$$\nabla_L - \nabla_V = z\nabla_W.$$

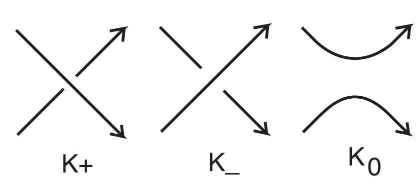

Figure 5. A SkeinTriple - three diagrams differing at one local site.

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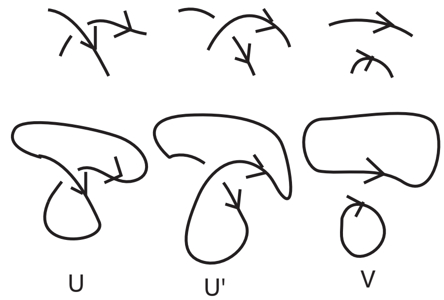

Figure 6. SkeinTriple for the UnLink.

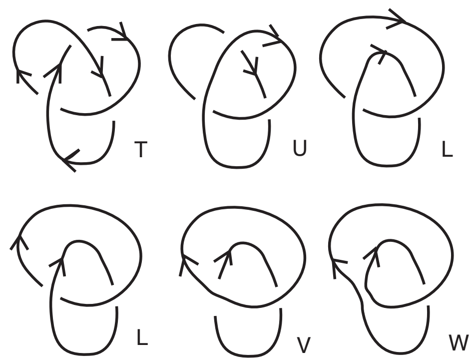

Figure 7. Trefoil Skein.

But we have that U is unknotted and V is unlinked, and W is unknotted. Furthermore we have checked that an unlink receives a zero polynomial. Thus we calculate that

$$\nabla_T - 1 = z\nabla_L$$

and

$$\nabla_L = z,$$

and so

$$\nabla_T = 1 + z^2.$$

Any knot or link diagram can be unknotted and unlinked by switching some of its crossings, just as we have done for the trefoil knot T and the Hopf link L. As a result, the Conway skein relation can be used to calculate the Alexander-Conway polynomial $\nabla_K(z)$ for any knot or link K. The remarkable fact is that, while there can be many different intermediate choices in this calculation, the answer is always unique and is a topological invariant of the link K.

One of the consequences of tangle theory and skein theory is that Conway was able to calculate invariants of knots and links very quickly. In Figure 8 we illustrate the well-known "Conway Knot" CK. This is an eleven crossing diagram of a non-trivial knot that has Alexander-Conway polynomial equal to 1. It is a smallest knot with this property. Is this knot the boundary of a disk embedded in four dimensional space? One says that a knot with this four dimensional property is *slice*. It was an open problem since

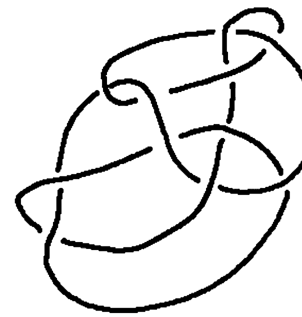

Figure 8. The Conway Knot of Alexander-Conway polynomial one.

Conway's work circa 1970 to determine whether the Conway knot is slice. This problem was recently resolved by Lisa Picarillo [16] in a stunning application of new invariants whose origins can be traced to Conway's original work. The knot is not slice.

Conway had a much more general notion of skein theory than the consequence of the one basic skein relation that we have quoted above. In this generalization, a knot or link K is placed in a "skein room" $\{K\}$ that represents its embedding in three dimensional space. Non-associative operations $\oplus, \ominus$ between skein rooms are defined so that

$$\{K_+\} = \{K_-\} \oplus \{K_0\}$$

and

$$\{K_-\} = \{K_+\} \ominus \{K_0\}.$$

Within a given room $\{K\}$, the knot or link K diagram can be deformed by ambient isotopy. When we examine a skein triple we take three representatives K_+, K_-, K_0 one from each of the rooms so that the representatives are exactly the same except in the places where they are switched or smoothed. We say that the $\{K_+\}$ room is *skein equivalent* to the concatenation $\{K_-\}\oplus\{K_0\}$. In this way, one can produce a *skein decompositon* of a knot or link. Refer to Figure 7 to see that we have

$$\{T\} = \{U\} \oplus \{L\},$$

$$\{L\} = \{V\} \oplus \{W\},$$

so that

$$\{T\} = \{U\} \oplus (\{V\} \oplus \{W\}).$$

This final skein decomposition of T expresses the trefoil knot in the skein as a composition of two unknots and an unlink. Every knot has such skein decomposition into unknots and unlinks. The non-associativity of the skein operations is crucial. Two knots or links are said to be *skein equivalent* if they have identical decompositions into unknots and unlinks. It is an open problem to this day to understand fully the skein equivalence classes of knots and links. In defining the skein, Conway opened a new area of topology.

In the context of the skein theory, the Alexander-Conway polynomial becomes a particular way to write down invariants of the skein so that

$$\nabla(A \oplus B) = \nabla(A) + z\nabla(B)$$

and

$$\nabla(A \ominus B) = \nabla(A) - z\nabla(B).$$

What was not obvious in the 1970s was the fact that there were other linear skein invariants than the Alexander-Conway polynomial and its multi-variable relatives. The most striking such invariant came in the wake of the Jones polynomial and is often called the Homflypt polynomial after its authors (in independent groups) Hoste, Ocneanu, Millett, Freyd, Lickorish, Yetter, Przytycki and Trawczk. The linear relation for Homflypt is

$$aP_{K_+} - a^{-1}P_{K_-} = zP_{K_0},$$

associating a Laurent polynomial $P_K(a,z)$ to an oriented knot or link K so that $P_K(a,z)$ is an invariant of the topological type of the knot. The Jones polynomial is a special case of the Homflypt polynomial. A key property of the Homflypt polynomial and the Jones polynomial is their ability to distinguish many knots from their mirror images. The background mathematical contexts that support these new skein polynomials involve many aspects of mathematical physics, Lie algebras and Hopf algebras. They are the background to more recent developments in Vassiliev invariants and link homology.

Certain aspects of skein theory came to light in relation to my own work. One was a model for the Alexander-Conway polynomial that used the work of Seifert from the 1930s [13]. Another is a state summation model for the Alexander-Conway Polynomial that is related to the original paper of J. W. Alexander [1, 12]. The state summation model is related to a state summation model (the Kauffman bracket state sum [13]) that I later discovered for the Jones polynomial. This state summation has a particularly simple form and a related unoriented skein expansion in the pattern shown below. The bracket can be seen as a special case of the so-called Kauffman two-variable polynomial denoted $L_K(a,z)$ with skein relation

$$L + L = z(L + L)$$

and

$$L = aL,$$

$$L = a^{-1}L.$$

The bracket polynomial [13] model for the Jones polynomial can be described by an unoriented skein expansion of crossings into *A-smoothings* and *B-smoothings* on a link diagram D via:

$$\langle \rangle = A\langle \rangle + A^{-1}\langle \rangle \quad (1)$$

with

$$\langle D \bigcirc \rangle = (-A^2 - A^{-2})\langle D \rangle \quad (2)$$

$$\langle \rangle = (-A^3)\langle \rangle \quad (3)$$

$$\langle \rangle = (-A^{-3})\langle \rangle. \quad (4)$$

In the sense of the Conway Skein Theory we have an unoriented skein with basic equation

$$\{ \} = \{ \} \oplus \{ \}.$$

By working with the equation at each crossing of a diagram for the knot, we obtain an unoriented skein decomposition where it is now understood that the operation $\oplus$ is neither commutative nor associative. Then the bracket polynomial becomes an evaluation on this unoriented skein satisfying the equation

$$\langle \{X\} \oplus \{Y\} \rangle = A\langle \{X\} \rangle + A^{-1}\langle \{Y\} \rangle.$$

Just as in the case of the oriented skein, this unoriented skein holds untapped mysteries that are slowly being revealed. It is a conjecture that the bracket polynomial detects the unknot, but it has been proved that the generalization of the bracket to a homology theory by Mikhail Khovanov does detect the unknot by work of Kronheimer and Mrowka. What else lies hidden in the oriented and the unoriented skeins for knots and links?

In Figure 9, we give a hint about the Khovanov homology. In that figure we illustrate all the states for the bracket state summation. This can also be construed as the full skein decomposition for the trefoil knot. Each diagram contributes a term to the bracket polynomial and the sum of these terms is the bracket polynomial. Khovanov examines this diagram of states and sees that it is a category. The objects of the category are the states themselves. The generating morphisms of the category are the arrows in the figure. Each arrow connects two states that differ by a smoothing at exactly one site, with the arrow going from the state with fewer B-smoothings to the state with one more B-smoothing. Khovanov defines his homology theory for knots by taking an appropriate homology theory for this category. Here we contact the roots of algebraic topology where the nerve of a category yields a simplicial structure and an appropriate functor from the category to a category of modules will send up rich possibilities of homological algebra. None of this would have come to pass if Conway had not found the skein.

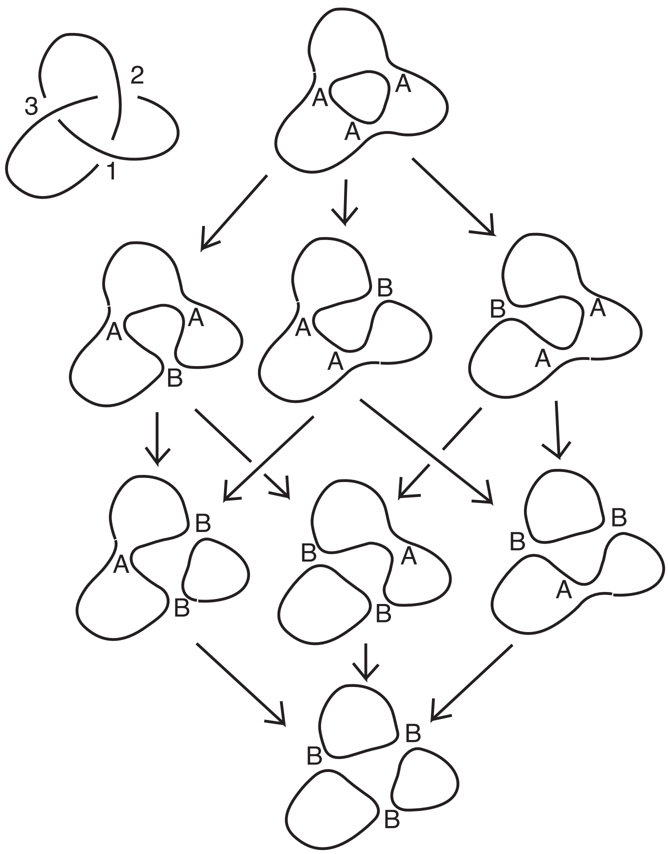

Figure 9. Bracket states and Khovanov Category - A category made from the states.

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Credits

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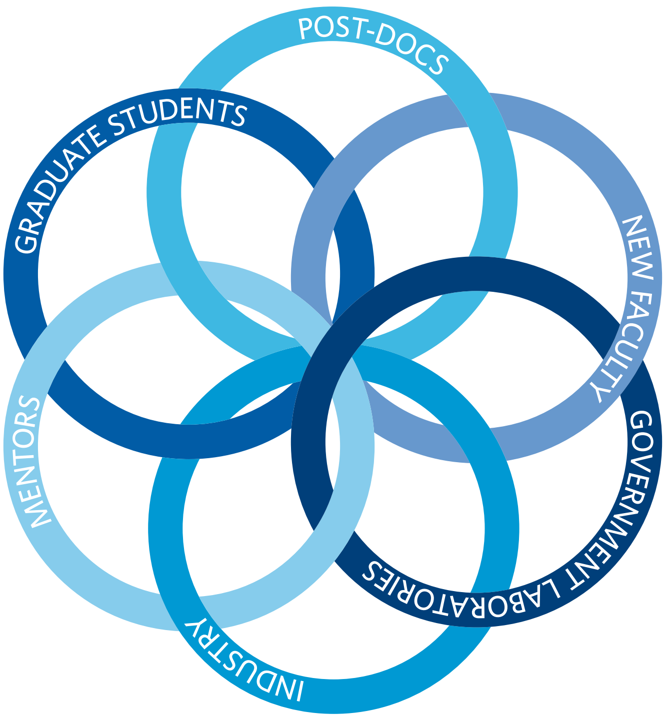
Figure 1 is courtesy of Philip Ehrlich.

Figures 2 and 3 are courtesy of Richard Kenyon, Jeffrey Lagarias, and James Propp.

Figures 4–9 are courtesy of Louis H. Kauffman.

The articles this month were curated by Early Career Intern Katie Storey with assistance from Angela Gibney, the editor of the section. Next month's theme will be Latinx History Month. To read all of the articles that have appeared, visit the AMS *Notices* Early Career Collection at `https://bit.ly/3aiZYBd`. Articles organized by topic are available at `https://www.angelagibney.org/the-ec-by-topic`.



More on Applying for Positions in Academia

Online and In-Person Interviewing for Tenure-Track Positions

Maria-Veronica Ciocanel and John T. Nardini

Showcasing the best version of yourself can be challenging during either in-person or virtual interviews! After a long process of writing job materials and sending them out to institutions, you may find yourself being invited for a screening or final-round interview for your dream position. First of all, congratulations! Your excitement may be short-lived, however, as you grow anxious about how to put your best foot forward and show the search committee and/or department that you are their ideal candidate. Interviews have traditionally been held in person, but virtual interviews have increased in popularity over the past few years (and are likely to continue being used in the future). In the following, we offer insights from our experiences with both types of interviews in recent years.

What In-Person Interviews and Virtual Interviews Have in Common

Try to get a schedule in advance

Try to get a schedule in advance It may seem obvious that having the interview schedule ahead of your visit would be helpful in preparing for meetings with individual faculty and campus leaders. However, your host and the search committee often have to juggle many time constraints and have to send reminders to encourage faculty to sign up for meeting with you. To avoid receiving your schedule the day you board your flight or when your first virtual meeting is about to start (this does happen!), we recommend asking for drafts of your schedule document every few days before your interview. You may even ask your host if they could share a Google document of the visit schedule, with updates in real time.

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A master list of questions and talking points

In preparing for the meetings you will have during your interview, we found it very useful to create a master list of questions that we might be asked, talking points to emphasize about ourselves, as well as questions that we ourselves would like to ask. This allowed us to customize our preparation for the program we were visiting by simply updating this document for other potential interviews. One approach is to organize your questions and talking points into sections: those about research or teaching, questions for faculty, for the chair, for the dean, or for students. Another approach is to organize your preparation by meeting if you know your schedule in advance: you can list out the relevant questions and talking points for each of your meetings. This was often a lifesaver when we needed a quick reminder of what it was important for us to learn while meeting with various people during the visit.

Ask for meetings with students and external faculty

Throughout your visit, you will likely meet with faculty in research areas related to yours, but also with junior and senior faculty in other fields. Take advantage of these meetings to both let faculty know what sets your scholarship and interests apart, but also to understand the ongoing research and teaching activities in the department, and to learn more about the geographical area you may be moving to. Before coming to campus, you can also ask to meet with specific faculty members who may otherwise not be on your schedule. For instance, you can ask to meet with faculty who started an impactful outreach program or DEI initiative in the department or in the community, or to meet with faculty in other departments, who could be potential collaborators. Your search committee will try to add these additional meetings to your schedule, if time permits.

If meetings with students are not already included in the schedule, we highly recommend requesting such a meeting, with either undergraduate or graduate students, or even with student members of a professional organization chapter that you would want to be involved with (such as AWM). Chatting with students can give you a useful perspective on their experience in the department. Don't expect discussions with students to be easy! We have found it inspiring to have conversations with students who wanted to see how aspiring new faculty intend to make meaningful contributions in their department.

Your scholarly talk

When receiving an invitation for an interview, it is important to pay careful attention to and to clarify what is expected of you. If giving a colloquium-style research talk, make sure you know what the expected time length is, and how much time to reserve for questions. Depending on the department and institution you are interviewing at, it may also be helpful to understand the background of the audience that is likely to attend your talk. In some cases, you may be asked to give an undergraduate-accessible research talk, or to give a teaching demonstration during a class in the department you are visiting. For some interdisciplinary programs or positions (such as mathematical or quantitative biology), you may be asked to give a chalk talk on your future research ideas in front of the search committee. Clarifying the details of these talks early on will help you best prepare for them.

One approach that we have found especially helpful is holding mock interview talks ahead of the actual interview. You could hold these with fellow students, postdocs, or faculty mentors in your current department. This will give you a chance to practice in a supportive environment, with a community that will give you valuable feedback or ask you questions that will mimic those that you will be asked in the actual interview. You may also find it advantageous to invite faculty that you know well from institutions that are similar to where you are interviewing, as they can give insight into how to align your material with what the institution values most.

Online Interviews

Virtual interviews (held over Zoom, Microsoft Teams, Webex, among others) are now often used for interviews for Postdoc and Visiting Assistant Professor positions. For tenure-track positions, search committees typically use virtual interviews to narrow their applicant pool down from 20–40 selected applications to their final 3–4 final candidates that will be invited for the final round interviews. In response to the Covid-19 pandemic, many final round interviews have been conducted virtually, including John's! Virtual Interviews present many challenges: technical difficulties are likely, your interviewers' background settings may be distracting, and conversations are hard to initiate. Fortunately, there are skills and habits you can start developing now to set yourself up for success on the virtual job market.

Practice speaking virtually

The best preparation for virtual interviews is to practice professional speaking in an online manner as much as possible. This may include giving talks at virtual conferences or joining an online group.

To get some practice, I (John) joined a local Toastmasters group nine months before interview season. Toastmasters is an international nonprofit educational organization that empowers its members by helping them develop public speaking and leadership skills. My Toastmasters group met weekly over Zoom; each week, its members can practice speaking by giving a prepared speech, hosting the meeting, responding to interview-type questions, or evaluating another speaker's presentation. I initially struggled performing each role online, but I gradually became more comfortable with each additional experience. By the time I began interviews for tenure-track positions, I was confident speaking online: in the face of technical difficulties, I asked my interviewers to repeat themselves; I laughed with interviewers when their pets jumped on screen; and I learned about my interviewers between meetings by initiating small talk about their hobbies and non-math interests.

Making a good virtual impression

It is important to be mindful about your appearance and sound during virtual interviews because your interviewers will form their first impression of you during your first two minutes on screen. Before an interview, you may want to ensure your room lighting and camera angle show a clear picture of yourself. Some strategies to consider include:

- Light should ideally evenly hit your face from a lamp, window, or lighting kit located behind your camera. Your face may be hard to see if your light source is coming from behind or above you.
- Eyeglasses can reflect your computer screen and block your eyes, which prevents your interviewers from making virtual eye contact with you. If you need glasses to see, you may want to angle your camera before the interview to avoid this.
- Look at your camera (and NOT at the screen) to maintain virtual eye contact. Place your camera level with or slightly above your eyes to provide the best angle of your face.
- A clean and warm background can help your personality shine through. We find that including things that matter to you, such as pictures of family or a favorite cartoon, can help your personality shine through.

Have a virtual conversation

Virtual interviews can feel one-sided and distant, but it is important to remember that they are actually a conversation! Some small changes on your end can turn a tough interview into a pleasant conversation.

- When speaking to a screen full of faces, it may be tempting to make statements like "I really admire how Dr. X has led program Y in the department, and I hope I can contribute," even if Dr. X is on the call. You might worry that addressing Dr. X directly will come off as too aggressive or intrusive; in reality, this statement distances you from the hiring committee. In a more successful interview, you can instead say "Dr. X, I really admire how you have led program Y in the department, and I hope I can contribute." As opposed to the first statement, this second statement turns the interview into a conversation.
- Virtual conversations often lose steam when one or two speakers dominate the airwaves; a good conversationalist will keep everyone involved. If you find an interviewer is not engaged during an online interview, we find that you can pull them back in with simple statements of the form "Dr. Z, I'm really curious to hear your thoughts on this topic." This should be done in moderation, however, in case Dr. Z is multi-tasking or preoccupied.

Incorporating these small changes into your virtual interviewing repertoire will allow you to comfortably converse with potential colleagues and can transform a day of hectic interviewing into a fun day of chatting about math, teaching, research, and scholarship. We recommend trying some of these changes during your next Zoom meeting!

Interviewing in Person

In-person interviews for academic jobs can feel extremely overwhelming. Most of them involve visiting the campus and department for 1–2 days, which includes giving one (or more) talks, meeting with many faculty members, the department chair, and (usually) a dean. Devoting ample time for preparation before you start your travels, such as creating a master list of questions or scheduling a mock in-person interview with your community (as mentioned above), can significantly reduce the stress associated with the interview. Certainly, the advantage is that you will get to potentially tour the campus, visit the department and see potential offices, as well as have meals with faculty and students. All of these experiences can paint a good picture of what your life in the department and at that institution may look like.

Being comfortable is key

The visit itself can feel like a marathon, both on an intellectual as well as a physical level. You will be moving a lot and having productive conversations with various people throughout most of your 1–2 interview days. It is therefore important to ask for short breaks between meetings as well as for some time before your talk(s) to collect your thoughts.

Since you are visiting in person, you will likely spend more time thinking about what you will wear in this professional interview setting. Regardless of the exact choice, we found that it was important to wear comfortable clothes and shoes, since the interview days involve plenty of walking. There are often no breaks between campus and dinner meetings, so be prepared to wear your outfit for the whole day. One of us can also attest that keeping some bandaids handy saved her from an uncomfortable situation during her interview!

Express an interest in the location

Some institutions may include a tour of neighborhoods or even a meeting with a realtor, all of which can give you an idea about places to live in the area. Even if your schedule does not include it, you could ask to see particular neighborhoods or visit a famous site at that location. This is not only a means for you to explore your potential new home, but also a way to show your particular interest in a position. While interviewing, it is challenging to keep in mind that programs and institutions worry that their short-list candidates may not actually come even if offered the position. Showing a genuine interest in exploring the location of an institution gives you one way to fight this worry.

The in-person scholarly talk

The in-person research or teaching talk may feel like the most stressful aspect of the academic job interview. Perhaps we can suggest a different and useful perspective on this, that one of us learned from a mentor. The hour during which you are giving your talk is actually the time when you are most in control. During that time, you are guiding others along your mathematical journey and through research results and scholarship that you are an expert in. Therefore, it is a time to enjoy and to show your enthusiasm for your work.

Finally, we must acknowledge that everything we suggest above takes a long time to prepare for. In our experience, it is very reasonable to expect that your research productivity will stagnate, as you make time to practice and research your potential future department. It is very useful to frame this work as making an investment in your professional development and network, which will pay off in the long run. In addition, we have found that having to present our research to mathematicians and students outside of our research areas during interviews and job talks really forced us to think about how to broadly present our work. Being ready to present your research in an accessible way and networking with colleagues at different institutions are skills that you will benefit from throughout the rest of your career.

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John T. Nardini

Credits

Photo of Maria-Veronica Ciocanel is by Laurie DeWitt, Pure Light Images.

Photo of John T. Nardini is courtesy of John T. Nardini.

Preparing for a Tenure-Track Job at an Urban Public College

Heidi Goodson and Diana Hubbard

There are many types of institutions that you may be considering when preparing for the academic job market. In this article, we aim to provide some insight into applying for a tenure-track position at an urban public college. Additionally, we hope to highlight some aspects of working at this type of college that may be different from other institutions in order to help you decide if this is the right type of academic environment for you.

We are both assistant professors at Brooklyn College, one of the four-year colleges in the City University of New York (CUNY) system. Brooklyn College has approximately 16,000 undergraduates and 2,500 graduate students, but despite this, in the mathematics department our classes are all relatively small. Like many other urban public colleges, our students primarily live off-campus. We mostly teach undergraduates, but we also support MA programs in education. Additionally, we are affiliated with the PhD-granting Graduate Center at CUNY, and we have the opportunity to work with graduate students. For us, Brooklyn College is the best of two academic worlds: we get both a liberal arts college experience on our campus and an R1 research community through our affiliation with the Graduate Center.

In many ways Brooklyn College is unique, but we will feature some aspects of working here that are generalizable to other urban public colleges. While we discuss some other ways that you can make your application stand out at a school like ours, your primary focus during graduate school and your postdoc should be on your research program and your teaching and these will be the most important things to highlight in your application materials.

Supporting the Needs of the Student Body

When applying to a job at any school, you will want to show that you have thought about how to teach, mentor, and support students. But different student bodies have different needs, and your application should reflect this. At Brooklyn College, for example, many of our students come from backgrounds that are underrepresented in higher education: many students are not native English speakers, come from low-income households, are first-generation college students, or are adult learners. Many commute to

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campus, and juggle additional responsibilities like full- or part-time jobs, or family and caregiving responsibilities. Our students are wonderful to work with; in fact, they are one of the best parts of our jobs. The diversity of their life experiences enrich the classroom, and they are bright, ambitious, and hard-working people. Many of our students are brilliant and would fit right in at any elite, expensive private institution. But often, they need instructors who understand and appreciate their life circumstances in order to provide appropriate support and mentoring. Demonstrating that you understand this can go a long way in showing that you're a good fit as an instructor at a public university.

Supporting these students can include using pedagogical techniques that foster a sense of community in the classroom, developing courses that teach marketable skills such as coding alongside mathematics, choosing low-cost course materials, and searching out and directing students to research or other opportunities that will suit their needs. It is important to use teaching and mentoring strategies that engage students who have a wide variation in their mathematical preparation. You will want to demonstrate that you have thought about your future students' strengths and needs in your teaching statement, diversity statement, and cover letter. But what to write? If such an opportunity arises, having meaningful experience teaching or mentoring students from communities that have been historically underrepresented and marginalized in higher education will help your application and will give you concrete, substantive ideas of what to say. However, if these sorts of experiences are not available to you, you can still engage thoughtfully with issues relating to diversity and equity in your mathematical life and in your application. There are a lot of great resources out there that can help you put in the work to learn and reflect on ways to support a diverse student body (see, for example, [GH3]).

Gaining some experience outside of academia is also a valuable way to support your future students and will help your application stand out. At Brooklyn College (and at most colleges), most students do not plan to pursue graduate work in mathematics. Experience outside of the standard academic track can be a real asset to your future department since you'll be better able to talk to students about jobs in the "real world." Departments are looking to hire faculty with broad experiences who can help mentor students or create connections between mathematics and other programs. These experiences include working at a non-academic job or doing an internship, completing actuarial exams, or participating in coding bootcamps and competitions. We recommend highlighting these experiences in your application materials.

Describing and Maintaining an Independent Research Program

At any institution, the search committee will include faculty members who are not in your area of research. At a school like ours it may be the case that no one in the department is an expert in your field, and you'll want to approach your research statement and job talk with this in mind. In your research statement, make it clear how your research fits into the overarching goals of your field, what your contributions are, and that you have ideas for what to work on next. It will be good to have a mixture of specific projects with clear outcomes and more general work that pushes the direction of the field. Additionally, your job talk is a chance to demonstrate that you can enthusiastically explain something difficult to a broad audience, which will give the committee a glimpse of what you will be like as an instructor.

You'll want to be able to demonstrate that you are a part of a larger research community so that the search committee will feel confident that you will be able to maintain your research program while on the tenure track. Between the two of us, before arriving at Brooklyn College, we had published research articles with collaborators and presented our work in a variety of venues. We co-organized seminars and special sessions at conferences, and mentored undergraduates on research projects. How can you gain these experiences as a graduate student or postdoc? The most important thing to do is to talk to people at and outside of your institution. Ask people in your field what opportunities are coming up that you should be aware of, and also check the websites of math institutes such as ICERM, MSRI, and PCMI regularly. Go to seminars, lunches, conferences, workshops, and summer schools, and make an effort to meet new people while you are there. Some conferences and workshops have opportunities for junior mathematicians to apply to speak or present a poster and many have funding that you can apply for.

An important part of an independent research program is applying for and being awarded grants. Grants support your research by funding your travel to conferences or to work with collaborators; helping you buy a computer, tablet, or other equipment; and allowing you to invite collaborators and seminar speakers to your campus. Some grants can be used to fund a course release, which would allow you to spend more focused time on your research. Large grants, like those awarded by the NSF, are great, but smaller grants also provide evidence that you are pursuing and successfully obtaining funding for your research. Small grants may be funded by external agencies and organizations, such as the AMS, the Simons Foundation, the AWM, and the AAUW. There may also be internal grants available at your current institution.

Writing a grant proposal can be challenging! Recent issues of the *Notices* have included articles with great advice on how to approach this type of writing (see, for example, [GH2, GH4]). We also recommend seeking advice from mentors and peers on this process.

Being Flexible and Wearing Many Hats

In our department there are a large number of part-time adjunct lecturers supporting our ability to offer the courses that our students need. Because of this, our full-time faculty members take on many additional roles to keep things running smoothly and to make improvements to better serve our students and community. Some of this work looks like what you may traditionally think of as "service," but not all. For example, a wide variety of courses have to be taught (within and outside of faculty members' areas of expertise), new courses must be developed to meet student demand, decisions need to be made about the curriculum, students must be selected for awards, departmental events need to be organized, the website must be kept up to date, job openings must be filled, and so on and so forth. Outside the department other work needs to be done as well, for instance in college-wide committees or in the faculty union if the institution has one.

It is important to protect research time, but it is also important to be willing to be flexible, wear several different hats, and contribute to the common effort of the department. We have found that much of this work is enjoyable and interesting: it can be a way to pursue intellectual interests outside of your research, collaborate with colleagues on a shared project, and contribute positively to the departmental atmosphere.

It will benefit your application if you can demonstrate that you would be willing and able to contribute to the department and the college in these various ways. Of course, as a grad student or postdoc, your primary focus should be on developing your research and teaching. However, you may want to take on some small, manageable responsibilities that show your ability to contribute to the community when the opportunity arises. For example, you could co-organize departmental activities such as colloquia and graduate student seminars, coordinate TAs and common final exam grading, or advise undergraduate students in a mathematical modeling competition. You can also seek opportunities outside of your department by, for example, serving as a judge at undergraduate research poster sessions or on panels for prospective graduate students.

Some Final Thoughts

We were both on the job market before the COVID-19 pandemic, and our experiences involved a lot of in-person opportunities and networking. We understand that the pandemic has made seeking out these sorts of opportunities much more challenging. We recommend reading other articles in addition to ours to get tips on how to navigate the job market in a more virtual world (see, for example, [GH1]).

Good luck in your job search!

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Heidi Goodson

Diana Hubbard

Credits

Photo of Heidi Goodson is courtesy of Heidi Goodson.
Photo of Diana Hubbard is courtesy of Nicholas Vlamis.

This Is What Success Feels Like: What I Learned from Applying for the NSF Postdoc Twice

Kim Klinger-Logan

Early one morning in January 2020 I was lying in bed trying to summon the will to exit my warm sheets and go downstairs. I usually check my email and the news to muster the energy to handle the chaos that results from juggling two dogs, a 9 month old, and oatmeal. There it was. *The* email. *The* NSF. But I had to be misreading something. "Congratulations" … since when do they congratulate you for applying … ? in January … ? I needed a rational, fully conscious and caffeinated person to read these words. I leapt from bed, calling my partner's name, and as I was rushing down to the stairs, my foot slipped out from under me and I slid down the entire flight. My partner was sure that I was holding the baby and it took a good 10 minutes before he could comprehend that nothing was wrong and

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we were just in some alternate universe where a new mom can also be an NSF Postdoctoral Fellow.

One year before, in January 2019, I was welcoming undergraduates to the beginning of a workshop that I had helped found and organize aimed at recruiting women and minority math majors. I was 6 months pregnant and there were no stairs to slide down in excitement. Instead, I was crying on the floor of the women's restroom of the math department after finding out that I did not get the NSF Mathematical Sciences Postdoctoral Research Fellowship (MSPRF) and, in fact, would probably not get any job as my husband was about to accept a job at a remote school in Kansas and I had no other offers. That day I wondered how I was supposed to show these young people that they could become mathematicians when I could not even tell myself that I was a successful mathematician.

When I think about the contrast between these two days, I am reminded of what of my advisor Paul Garrett used to tell me, "This is what success feels like." He never said this in regards to the "Congratulations" email you get while lying in bed. He was always talking about the other email. Success is trying and not quite succeeding but maybe learning something in between. So here is what I learned in 2019, and what I did differently a year later.

What is the NSF MSPRF and Why Should You Apply?

The NSF MSPRF is a postdoctoral fellowship taken at your research sponsor's institution for 2 years with no teaching or for 3 years, teaching one course per semester for the last two years (with part of your salary paid by the sponsoring department). Compared to many US postdocs the salary is a bit higher and the award has significantly more travel funding and less teaching.

There are many reasons why you should apply to the NSF MSPRF. First, there are 30–33 fellowships given each year and so it is seen as fairly prestigious. While you may personally not care about prestige, it will likely have a positive impact on your career.

Given how competitive it is you may wonder if applying is just not worth the effort. However, the effort is in fact what makes it worth it. Drafting the 5-page Project Description will force you to begin considering the direction of your research post-PhD. This document is less about what you have done and more about what you will do next. Writing the proposal will help you plan where you want your research to go and how you want to define it. It will likely form the basis of the research statement you write as a component of your job applications.

Finally, it is likely you will need to apply for grants as part of your job later on. Many NSF grant applications have similar evaluative criteria (which I will mention in more detail later on). While the requirements for other applications may not be exactly the same, framing a good proposal is a learned skill. Even if the projects you propose do not get funded and never come to fruition, you will get valuable practice at grant writing in a relatively low-risk yet high-reward setting.

The Application

If your first application is submitted while you are a graduate student you can apply at least one more time. It is also not rare to be awarded the fellowship on your second try.

Before I dive in, I should outline the main components to this NSF application (Note that these may change so please check the requirements each year.):

(a) Project Description—This is the meat of your application. It is to be a maximum of 5 pages (not including the references). The key emphasis should be on the "Intellectual Merit" and the "Broader Impacts" of your proposal which I will describe below.

(b) Project Summary—This is a one page summary of what is in (a). It has three main sections: "Proposal Summary," "Intellectual Merit," and "Broader Impacts."

(c) Biographical Sketch—This is essentially a CV in a very specific format.

(d) Letter of Sponsor Support—Unlike many other NSF grants, you will choose a "sponsoring scientist" to serve as your mentor who is required to write a letter that you submit with your application. This is not a letter of recommendation but should address how the sponsor plans to support you and your research during your fellowship.

(e) Letters of Recommendation—It is always good if you can have at least one of these coming from outside your home institution (though this is certainly not necessary).

(f) There is also a separate list of references and data management plan—these are not time consuming to complete but it is good to be aware of.

Your proposal needs to address two main criteria. These are the only dimensions on which it will be judged. The NSF describes these two components as follows and states that there are no weights assigned to the review criteria [K4].

Intellectual Merit: The Intellectual Merit criterion "encompasses the potential of the project to advance knowledge" [K3].

Broader Impacts: The Broader Impacts criterion encompasses the potential to "benefit society and contribute to the achievement of specific, decided societal outcomes" [K3].

Many people have written about what qualifies as Broader Impacts. Peter March's letter on Broader Impacts Review Criterion highlights some Broader Impacts often seen in successful proposals [K3]. There can be a fine line between a Broader Impact and "doing one's job" and I encourage you to see Max Lieblich's *Notices* article "What is Broader Impact?" [K2]. While this is aimed at faculty, he draws some helpful distinctions about what are and are not Broader Impacts.

When reading your materials, the reviewers are asked to evaluate them with respect to the 5 Merit Review Criteria (which can be found in full detail in [K1]). They address the potential of your proposal to advance knowledge and benefit society, the originality of your ideas, and the likelihood of you successfully completing what you propose.

What I Did the First Time Around

In this section I discuss the things I was glad I did the first time that I applied for the NSF MSPRF.

1. I started early.

The application is due the third Wednesday in October and I began writing my Project Description in early June. This is not strictly necessary; however, drafting and editing a good statement takes much longer than you likely think it is going to.

I recommend getting as many people to read your materials as possible. If you start this process early enough, you will have time to implement the suggestions of one person before sending it on to the next. The first people I had look at my application were other graduate students in my cohort. We made a schedule and rotated reviewing each other's proposals each week of the summer and most of the fall semester. It went something like "Week 1: draft NSF Project Description, Week 2: edit NSF Project Description, Week 3: draft NSF Project Summary, … " Having each document planned out with ambitious deadlines that someone else was holding me accountable for was immensely helpful.

If you are the only one in your cohort who is planning to apply to the NSF Postdoc, you can still do a version of this with friends at other institutions. Alternatively, you might set parallel schedules with friends who have other goals, e.g., who are applying for other jobs or other fellowships. No matter how disciplined you are, having an accountability buddy and external support is priceless. However, this tip is just meant to be helpful, and it is certainly not necessary to have peers to review your application or work on theirs with you. You should apply regardless of what your cohort is doing.

Once you have a draft of your materials, send them to your advisor and/or project sponsor. Ask around and find out if you know anyone who has been on an NSF review panel. I did not do this in my first round but I did on the second time around and I believe it made the difference between me not getting the award and me getting the award. This person may have a good eye for details that you should tweak in order to appeal to a panel.

2. I chose a sponsor whose work relates closely to the project I was proposing.

I was fortunate enough to have a clear idea of who I wanted my sponsor to be early on and asked him to be my sponsor during the summer before I first applied. My dissertation was a direct response to an area that he helped develop so it was natural that I would work with him. My sponsor was also an excellent and well-regarded mathematician with a variety of other research that I found interesting. While very busy, he was a person I found that I could work with and would have time and energy to devote to writing a good letter of support.

You want to choose a sponsor who you can talk to and will have time to work with you, but it is also helpful to have someone who is whose work is known in your field. It should be clear that your sponsor is capable of at least discussing your project with you. I was told early on that it is good to have your sponsor's work align well with yours and most of the projects I have seen funded support this theory. You may want your sponsor to be someone you or your advisor has published with, or perhaps your project cites your potential sponsor's work. When you write up your proposal, call attention to the relationship between your sponsor's work and the work you are proposing, and make the case that your sponsor will be equipped to support you.

When I asked my sponsor if he would serve in this role, I had never met him before and had no reason to believe he knew who I was. I did not have my full proposal written but I did give him a rough idea of what I was planning on working on and sent him my CV and other relevant information. I sent him a draft of my proposal not long after he agreed to be my sponsor and he did provide me with some helpful thoughts on the proposal but the work I proposed was my own.

3. I read other people's applications.

If you do not personally know anyone who has received this award, talk to your advisor and/or Director of Graduate Studies to see if any recent alumni have received it. You also likely have a faculty member or postdoc in your department who at least applied for this fellowship. (In fact, I wish I would have read more "unsuccessful" applications as well.) You can even view all of the previously funded projects and their PI's on the NSF's website; however, it is best if you have a personal connection with someone before asking to view their materials. Remember, these are someone's research ideas so sharing materials requires an element of trust. That said, many people are happy to send portions of their own materials if you ask nicely. For instance, a person that I had only spoken to once or twice at conferences generously shared her materials with me. However, do not share any of these materials with anyone else unless you were given explicit permission to do so.

While I did read a few accepted applications on my first round, I wish I had examined the proposals more critically and at varying points of my writing process. In particular, I should have revisited the applications I had access to after I thought my draft was "done." It is good to take a break from writing for a few weeks (if possible) to think about how your application will be perceived in a pile with the other applications you have—after all, that's essentially where it will be when it's reviewed. It is more likely you will be able to do this if you start early.

It's likely that (provided you have any proposals to read) the ones you have will be outside your area. In retrospect, I wish I had asked myself questions like: How did they format their document? How much time is spent on background material? How many projects and conjectures are proposed? How much of an outline are they providing for how they plan to attack each project? How does the proposal denote proposed versus completed work? What is the intellectual merit? How is project sponsor discussed?

Pay special attention to the verbiage used in successful applications and think about how your proposal will compare to the others if they were in the same group. For instance, characterizing your future work as "extending" or "generalizing" other work (whether yours or someone else's) without support or explicit details can make it seem like you really don't have anything novel to contribute.

4. I engaged in Broader Impacts.

Intuitively one ought to propose things you *will do* while you are on the fellowship that will have Broader Impacts. However, since you will likely be pursuing your fellowship at an institution you've never been to before, it is tough to say anything meaningful about either (a) what's possible in this new environment or (b) what impact such things might have.

However, like most things, the best indication of future success is past success. In graduate school, I was fortunate to be given support from my advisor and graduate program to develop programming to create community among graduate and undergraduate students with the goal of broadening participation for women and minorities. Getting involved in these projects early helped give me a clear perspective on what I enjoyed and what I saw my contributions to be. As a graduate student there are often lots of ways for you to get involved in whatever you are passionate about—AMS, AWM, and SIAM Student Chapters; REUs; Directed Reading Programs; Math Circles, Camps, and Festivals; adjunct teaching; etc.

This work should not distract you from your research, but it is important to be honest with yourself on both sides. Ask yourself: Are you doing so much that you do not have time or energy for research? If so, find a way to prioritize your research—that is the thing that will allow you to stick around to continue to make Broader Impacts. However, the tougher question may be: Are you really working on research and being productive with all of the time you do have? Contributing to your community will be part of your future job if you choose to stay in academia. First, service is a required and important component of many tenure application materials. Second, Broader Impacts are an important component of all NSF grant applications, and in many departments, you will be expected to apply for these grants regularly whether you want to or not. You might as well find out what you enjoy now while the stakes are relatively low. There is a "sweet spot" I try to achieve. If I do not have enough to do I get a bit directionless. On the other hand, if I say "yes" to every act of service asked of me I can run myself ragged and have no energy for research.

My Broader Impacts did not change significantly between my first application and my second. However, I did pare them down. In concrete terms, the Broader Impacts section in my first application occupied about half a page; and, in the second, it occupied a small paragraph that was closer to a quarter of a page. I included fewer items which were in my more distant past. This was tough, but to be honest, I did not engage in these activities for the NSF.

Women and minorities in general are asked to do far more service than others and they are often penalized for working on these projects rather than on research. We all have 5 pages for our Project Description and the more you say about your Broader Impacts the less you are saying about the Intellectual Merit. While program guidelines state that neither of these aspects is of greater weight, they are, in practice, not evaluated by similar standards of rigor. As I later became aware, while it is important that you have Broader Impacts, there is less of a hierarchical ranking about which are better or how many you need or what amount of time they should occupy. I am not providing a normative statement here but only a descriptive one.

What I Learned the Second Time Around

My sponsoring scientist and institution did not change between the first and second round and my relationship with my potential sponsor did not change much in the year between my first and second application.

In the year between my applications I had time to think about what I had proposed from a more detached perspective. As I mentioned previously, being able to come back to your application after taking time off is truly beneficial. What you saw a year ago as "as good as you could possibly do" is now kind of crappy. This is a measure of growth. The year between my applications was a year spent developing my ideas more deeply and adding new ideas to my application.

The major difference between my first and second application was the number of projects that I proposed. In the first round, I really only proposed one project. It had many steps and potential publications along the way but most of them were not very explicit. In my second round, I basically added a whole other aspect of my proposal that likely could have been its own application. Instead of having only one explicit conjecture, this proposal had five. In the second proposal the sectioning made very clear that there could be seven different publications from the proposal. While it is good to have some unifying theme or picture, the entire proposal does not have to work toward answering *one* question.

Between my two applications, I also sat down with someone who had served on NSF panels before. This person gave me details about the review process which helped me fine tune aspects of my application.

While having time for my ideas to develop was invaluable, it would have been worthless without also having my "failed" application. I had already done all the leg work that causes so much unacknowledged anxiety: contacting my sponsor and letter writers, formatting my biographical sketch, and, especially, writing a first draft.

Receiving a rejection when you put so much time and effort into something is difficult. However, the effort is exactly what makes the process worth it. I did not see it at the time, but all the work I put into the proposal was me being a successful mathematician. Putting all of my best ideas out there, pushing myself to learn more, trying to find answers to the questions that drive my research, and communicating that vision to my peers—that is what success feels like.

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Kim Klinger-Logan

Credits

Photo of Kim Klinger-Logan is courtesy of Shay Logan.

Pathways to a Career Outside of the Academic Silo

Elizabeth Munch

Whether we want to believe it or not, the world is changing. In particular with the rise of data science and high performance computing, there is no longer[1] a single career path to be followed by a person with a Mathematics PhD.

I earned my PhD from a traditional math department in a traditional university. I held a tenure-track position in a traditional math department, and still have a 30% appointment in one. I was expected to do only traditional math research. It was quickly made clear to me in my first years of a tenure-track job that I would need to conform to the template of what those institutions thought of as a "real mathematician," or I would fail. For me, satisfaction in my career came after my move to a new, interdisciplinary department, with all its freedoms and challenges. For some of you, the tenure-track position in a math department is your personal definition of success. I truly believe there is much good work to be done from that vantage point, and I wish you the best of luck. However, this advice is not for you. I'm writing to offer advice to those looking to make their mark outside of any single academic silo by looking for a career path in an interdisciplinary department, program, or institute.

No advice is unbiased. My advice comes, admittedly, with a heavy dose of survivor bias. Additionally, my axes of privilege, including being white, cis-gender, heterosexual, and having a big-name university attached to my PhD, have yielded a great deal of something-that-looks-like-luck over the course of my career. Finally, I only have experience in the American academic system; what I say may be wildly incorrect outside of the United States. Given these limitations, I invite you to sit with my advice, and if it doesn't fit for you, I encourage you to simply dismiss it.

Perhaps you are nearing the end of your PhD program in a math department and are working to envision where you can see yourself and your career in the next five years. Perhaps you are excited about the possibility of research in an interdisciplinary setting, but are trying to figure out how to get there or what that would look like. This advice is for you.

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[1] *Nor was there ever.*

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What Options are Available?

An interest in something-other-than-tenure-track-math does not necessitate leaving academia. There has been a steady rise in interdisciplinary departments and institutes inside of academic settings which are not "math departments" or even "applied math departments" per se, but instead are created to foster interdisciplinary collaborations. These fall roughly into two categories. The first, like the Department of Computational Mathematics, Science, and Engineering (CMSE) where I am at Michigan State University, is just that: a department. We have our own faculty, our own undergraduate and graduate students, our own tenure committee, our own self-determination.

Some of our faculty have PhDs in Mathematics, but others have PhDs in Physics, Biology, Statistics, and many flavors of Engineering. For me, a primary appeal of being our own department is that tenure expectations are tied to measures not usually celebrated in your run-of-the-mill math department. The expectations in my interdisciplinary department naturally align with the goals I have for my own work. I am expected to form interdisciplinary collaborations and publish across boundaries. I am expected to create and share open source code that could make the work we do available to others. I am expected to teach less, but to also bring in more grant funding to support graduate students. I am expected to find ways to communicate mathematics to colleagues without a graduate-level mathematics background. In short, the tenure expectations reflect what I already wanted to do. We are by no means the only such department: other examples include the Halıcıoğlu Data Science Institute at UC San Diego and the Department of Scientific Computing at Florida State University.

A second category of collaborative, interdisciplinary academic settings is that of non-departmental level institutes. Some examples of this include MIDAS at University of Michigan, and the Oden Institute at the University of Texas. However, rather than having tenure-granting status, these institutes bring together interdisciplinary-minded faculty, who are still granted tenure within traditional departments where their primary appointments are. One advantage of this arrangement is that a faculty member can retain a more traditional job title while still accessing a setting that actively fosters interdisciplinary collaboration. One challenge of this arrangement derives from its spread across campus, both in terms of physical office space and in terms of expectations. Then again, now that we live in a Zoom-infested world, the geographic challenges are nowhere near as insurmountable as they may have been even a few years ago. However, issues that should be thought about in advance include regular offering of courses; which department these are associated with; and how students who are spread across many degree programs don't get to interact with each other as much.

Use any and all resources to see what options are available to someone with a math PhD. Make use of your university's career office before you need it. They can help you build resumes and career documents long before you are setting them up in panic mode. The earlier in your PhD that you can get a sense of career options, the more time you have to seek out preparation and opportunities that will set you up for those positions. The first options that may come to mind for those trying to find alternatives to the usual career path are things like research positions in industry. Another exciting set of options are government research labs (e.g., Pacific Northwest or Sandia National Laboratories), which offer some potential for a middle ground between an industry and academic setting at the expense of limitations in terms of visa status.[2] Having no experience in those settings myself, I can do little more than encourage you to reach out to people in jobs that look like what you want at those places to understand more of what those career options look like.

Preparation Before You Go on the Market

Be active about obtaining experience and skills that will enable you to compete for the kind of job you want. For jobs in interdisciplinary departments or institutes, no amount of coding experience is too much. I won't wade into the which-language-is-better debate[3] because once you learn one computer language and start wrapping your brain around basic algorithmic and data structure tools, transferring your skills to whatever language your future job demands will be fairly straightforward. Even for the die-hard pure mathematicians out there, the ability to code can open so many research doors, particularly for automated testing and development of conjectures. Basic programming should be a required skill for every mathematician, but I digress. If possible, look into documentation that you have acquired these skills, such as online course certificates, that make a bigger impact on your CV than just saying you can, e.g., code.

The next thing to do is get out of the math building. Do you have a particular application area you are interested in? Start attending talks in that department. Maybe even try to attend an interdisciplinary conference. No, you likely won't understand everything in the talks, but you can start to get a sense of what people in the field are interested in; the problems they focus on; the vocabulary they use to convey their meaning. I have found that interdisciplinary research is often a matter of learning something close to a new language in order to translate across fields. We mathematicians are trained to have a precision in our language based on definitions we all have agreed upon. But, this concept is just as true in other fields. My favorite issue on this front has been trying to discuss "homology" with biologists. Sure, you're reading the *Notices* so if you know what

[2] *For US labs, and depending on the lab and project, there might be restrictions requiring US citizenship or permanent resident status.*

[3] *The answer is Python, don't @ me.*

"homology" means, you have a very specific definition in your head. But if I stand up and use the word in front of a bio conference, they think that I'm discussing the idea that bat wings and human arms are inherited from a common ancestor. To this end also, if you can find a grad student colleague in your applied field of interest, take them out for coffee. See if you can explain your research in a way that they can understand. See if they can explain theirs to you while you try to understand the big picture. In the future, developing collaborations might be tied to being able to have discussions like this with people who don't all start with the same vocabulary list and/or definition, so practice! If you can manage it, having a project (even a small one) across disciplinary boundaries can go a long way towards showing your potential for working with people in these sorts of fields and will definitely strengthen your CV. But, even if these discussions lead you to a place where you can give a sense to someone outside of the math department what it is you study and how it might be utilized, it will be time well spent.

Preparing Documents and Interviews

Again, due to my experience within the US academic system, I will focus on advice for the documents and interview experience for a non-traditional academic job. The documents required for many of these jobs will be the same as for a standard math department job: research, teaching, and DEI statements. The biggest unwritten difference comes in the research statement, where you need to ensure that you are writing for the correct audience. For instance, in my department, the faculty have PhDs in many different fields, so explaining the mathematical advancements I have made, and especially placing them in a context that they can understand, often requires a change in vocabulary and a view towards big picture explanations. I want to emphasize that this is possible without a loss of rigor, and practicing this skill has made my research stronger by giving me a broader perspective of where I want to go. As a bonus, interdisciplinary or not, you will surely use these skills on any grant application you submit. For the teaching statement, you should be sure to do some homework to get a sense for what classes are taught in the department to which you are applying, especially for these new programs that are coming into their adolescence with the development of new interdisciplinary courses and majors.

My advice when it comes to the interview is similarly related to communication. You will be talking to people with very different backgrounds. In advance of your interview, make sure you have prepared a few versions of your research "elevator pitch": a version for experts, a version for non-expert mathematicians, and a version for non-mathematicians. Having a solid place to start in these discussions can lead to very productive conversations. My advice for your job talk is similar, where I recommend giving your talk in a way that all but (at most) the middle third of the talk is accessible to non-experts. That way, you have people engaged at the beginning so that they can see the big picture, as well as the long game in your conclusions.

Questions You Should Be Asking

So you've gotten the interview! Congratulations! The biggest thing to do now is make sure you ask A LOT of questions about expectations for the position. Once you leave the standard departmental structure, there are many potential surprises as the program might not be run the same way your PhD-granting department was run. I'll speak to a few of the learning curves I experienced when transitioning from having a tenure-track job in a math department to my tenure-track job at CMSE.

The first major difference was graduate student training and recruitment. In mathematics, we generally accept a student to the department without an official advisor. We train them for two years or so by giving them a heavy course load to gain a broad mathematical foundation. Sometime in the second year, the student starts doing some reading with a potential advisor, and then assuming all parties agree, the official advisor relationship starts after that. In my department, however, students matriculate directly into an advisor's group. The upside of this arrangement is that my graduate students can start with focused research and training from the beginning. The downside is that if a student comes into graduate school the way I did (that is, with a very different view of the kind of research I wanted to do[4] than what I actually ended up doing), there is a bit more friction in terms of changing their direction. This is mitigated by having a collegial department where students switching among groups is viewed as a natural part of the graduate program rather than a failure on the part of the student or advisor.

To this end, the other major difference I found when moving to CMSE was a difference in how funding worked. Because mathematics departments often get a great deal of funding through service courses such as Calculus, student funding largely comes in the form of TAships. Because the funding is not tied directly to a funded grant, a student shows up in your office, and you start reading some things together that seem interesting and develop a research project from that. On the other hand, at CMSE we have a more limited allotment of TAships for students (although this is changing with the rise of the very popular data science courses), so I am expected to bring in more grant money in order to ensure funding for my group. This also means that student projects are more tightly aligned with the proposals that have been funded. I by no means view this as a bad thing, since it has enabled me to develop a clearer view of the big picture. Because I have the grant as a template, I can hand a new student a decently detailed research

[4] *I didn't even know that my research field existed until several years into grad school.*

project outline with key references and goals already in place, allowing them to get off the ground much faster.

Finally, a critical issue to explore in an interview is tenure expectations. In my experience, when working in an interdisciplinary space, unwritten rules and implicit assumptions about what it means to be a "successful" faculty member vary wildly across disciplines. To any extent possible, your goal should be to have a written understanding of expectations, particularly with respect to joint appointments, course load, funding, publication rate, and graduate students. This can be incredibly helpful towards guiding your focus over the tenure-track period, and ensure no one ends up surprised or angry at the end.

A Conclusion and a Call for Change

Moving to a non-traditional position was absolutely the best decision for my career. These sorts of positions are not for everyone, but building something new while pushing out the boundaries of what academia can be has been an incredibly gratifying experience. We need more mathematicians working across disciplines to bring the joy of mathematics and the training in logical thinking outside of the math department to ensure better numerical literacy, to create more informed citizens, and catalyze new knowledge in science. I know standing at the precipice of your PhD is a scary place to be when it's not clear where this path will take you next. Whether you end up inside or outside of academia (with a particular emphasis on the fact that taking jobs outside of academia is no less of a success than taking an academic job), know that your training gives you the potential to do great things! I wish you the best of luck on your exciting journey!

Elizabeth Munch

Credits

Photo of Elizabeth Munch is courtesy of Harley J. Seeley/MSU College of Natural Science.

John Horton Conway (1937–2020)

Alex Ryba with contributions by R. T. Curtis, Richard Borcherds, Manjul Bhargava, Dierk Schleicher, Jane Gilman, Aaron Siegel, and Louis H. Kauffman

Figure 1. John Horton Conway in August 2012 lecturing on FRACTRAN at Jacobs University Bremen.

John Horton Conway (1937–2020) was one of the great mathematicians of the past half century. John was a trailblazer with a sixth sense for the rich lodes where new mathematics hid, and when he intuited a prospect he worked very hard to reach the truth. The paths that he alone discerned and opened are now followed as the natural and simple approaches to wide expanses of mathematics. John was a daring mathematician, who tackled problems that others avoided. He was a generous mathematician, who was not satisfied until others saw what he saw in mathematics.

John was a trailblazer among mathematicians. John's mathematical legacy is so extensive that it is difficult for any individual to explain it fully. It is as if each of us has at some time stood with John on the peak of some high mountain, yet all the other heights which John scaled remain blurred by distance. Wikipedia states correctly that John was "active in" the theory of finite groups, knot theory, number theory, combinatorial game theory and coding theory, that he "made contributions" to recreational mathematics, and that his "major areas of research" also included geometry, geometric topology, algebra, analysis, algorithmics, and theoretical physics. (The Wikipedia author should have included logic, but however far the list extended, something would undoubtedly have been missed.)

John always reached the heights and changed the landscape, however broad the extent of his work in mathematics. His influence has been momentous and lasting. Aviezri Fraenkel concluded his review of *Winning Ways* with a prediction that has certainly proved true over more than thirty five years since it was written. He wrote "It is likely to remain an eminent leader in this field for many years to come." Don Knuth commented "Although John was a pure mathematician he covered so many bases that I've cited him more than 25 times for different

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contributions to *The Art of Computer Programming*. I expect the citations to continue long after his death, as happened to Elvis." John opened new paths where many of us have followed and worked. His vision was unexpected and unusual, and mathematicians responded to it with a resonance of attention, feeling, and imagination of their own. After his proof of Morley's theorem prompted a spate of competing proofs, John explained to me that he "liked to jump into an area and create enough of a splash to attract other mathematicians" and he "would then leave the water."

My own history with John Conway began in Cambridge in the early 1980s. At that time John had been in Cambridge for about a quarter of a century. He had come there at the age of eighteen, after being born and raised in Liverpool, having excelled in all subjects at the Holt High School, and having already started to investigate mathematical questions that went far beyond normal schoolwork. In Cambridge, he became a graduate student of Harold Davenport, and then a faculty member. He left Cambridge in 1986, accepting the John von Neumann Chair of Mathematics at Princeton University. John retired from his position at Princeton in 2013, but he never retired as a mathematician.

I was drawn towards John Conway and his ATLAS project, which was coming to its conclusion just as I was studying group theory as a graduate student of John Thompson. The ATLAS is a tabulation of facts about the finite simple groups. John explained that each sporadic simple group could only exist because of myriad unlikely number theoretic, combinatorial, and geometric coincidences. Any question of existence of a sporadic could be all too easily killed by the tiniest of oversights in the classification, which had been finished just recently. John held out hope that such oversights had been committed and that additional wonderful sporadics might have been missed. I recall sitting with John one night and running computer experiments to investigate a new group suggested by Richard Parker. John always liked to use calculation to hone his insight into problems. That night, he sat behind me watching the machine's answers. As dawn broke, we realized that the group was a large orthogonal group rather than a new sporadic—the hunt was in vain.

John stalked the Monster group, but always from a distance. As he explained his latest ideas about hyperbolic reflections or Lorentzian lattices, distant glimpses of the Monster would appear. In the summer of 1983, John seized on a loop double cover of the Golay code, built by Richard Parker. John saw that the loop offered a way around the sign problems that dogged Bob Griess's construction of the Monster. Within two weeks, John had a much simplified construction, albeit one that followed close to Bob's original path.

John liked to say that he had written eleven books, one of which had been translated into eleven languages. Three of his most important books were finished in the short period between 1982 and 1988. The very different subjects of these books demonstrate John's range as a mathematician and also show off different aspects of his personality. *Winning Ways*, published with Elwyn Berlekamp and Richard Guy in 1982, was a lighthearted humorous treatise on mathematical games and combinatorial game theory. Three years later, *the ATLAS of Finite Groups* appeared (coauthored by Rob Curtis, Simon Norton, Richard Parker, and Rob Wilson). Every mathematician whose work touches finite simple groups keeps a copy of John's ATLAS on their desk. After another three years, in 1988, John and Neil Sloane published *Sphere Packings, Lattices and Groups*. A review by Gian-Carlo Rota stated: "This is the best survey of the best work in the best fields of combinatorics written by the best people. It will make the best reading by the best students interested in the best mathematics that is now going on." John liked to quote this as the "best" review. However, even the *best review* understates the book's impact: it was not just a survey but a book filled with important and very clever original discoveries.

The Conway sporadic groups were received as a great result in 1968. John told me that after he found his groups, he discussed them at length with John Thompson. As Thompson applied a fairly straightforward representation theoretic argument, he realized that John Conway was not following. "You don't know any of this," said Thompson, apparently shocked. Thompson, who was the world's foremost group theorist, felt that it was worth giving John Conway a crash course in representation theory. John Conway and John Thompson had very different group theoretic interests, but each had tremendous respect for the other. Yet, John Conway had been very much a courageous outsider to group theory when he constructed his famous groups.

John was a daring mathematician. John was brave in that he tackled questions from which other mathematicians would shy away. He constructed his sporadic groups by finding the automorphism group of the Leech lattice, a problem that others avoided because there were no standard tools to apply. John also took on other types of orphan problems, including problems that held the threat of undecidability, problems where a solution seemed unlikely, and problems that had previously been solved. None of this mattered to John—if the problem attracted him he attacked it. He turned undecidable problems into new models of universal computation, created new techniques where methods had not existed, and obtained new insights into how some of the subtle theorems of mathematics work.

John liked packing and tiling questions, which are imposing because they are so far out of the scope of standard machinery. For instance, John and Sal Torquato discovered the best known packing of regular tetrahedra—the least spherical and trickiest to pack of the platonic solids. Before John's work, the impression was that anyone looking at these questions would be on their own and would have to live off their wits, competing on equal terms with Gauss and Euler and other great mathematicians of history. Particular instances of problems that are mathematical but outside the scope of mathematical techniques often become recreational puzzles. John relished this sort of challenge and approached these problems as a mathematician, always alert to general ideas and techniques.

The divide between recreational and research mathematics came up in connection with John more often than with most mathematicians. The difference between the two is something a mathematician knows when they see it, and many mathematicians are attracted to both. Yet, most mathematicians cannot bring the two together. John himself was an impartial judge of the many sides of his work. In his heart there was no difference between recreational and serious, professional mathematics—all that mattered to him was an interesting question to be explored. However, he was well aware that to much of the world there was a difference. He often said how grateful he was to be paid for what he would have otherwise done for free (mathematics). He wondered whether he looked like someone who was earning his pay since he was so obviously enjoying himself (rather than "working.") This apprehension ended with the fame from the "great" discovery of the Conway groups in 1968. Still, later, John was relieved to hear Frank Adams, upon whom John looked as an intimidatingly perfect model of a serious research mathematician, praise him as a mathematician with "fire in the belly." When John moved to Princeton, he was put at ease by the friendship of Bill Thurston. Their casual discussions quickly turned into orbifold notation, a piece of beautiful and very serious professional mathematics. Joe Kohn, a former department chairman at Princeton, remarked that John's playful approach revived a long-lost atmosphere in their common room.

Having attained "security" with the discovery of the Conway groups, John went back to studying partisan games and structures that could exhibit universal computation. He rapidly produced the Game of Life, his theory of combinatorial games, and within it the Surreal Numbers. John's play often led to discoveries that became important as new research areas, and John never stopped playing. Even at the time of ATLAS writing, he temporarily dropped all interest in the sporadic groups when the Look and Say sequence reached him. While it was only a "guess the next term" puzzle for other mathematicians in Cambridge and around the world, John noticed unexpected structure in the terms of the sequence. This appears in John's paper "Audioactive Decay," a paper that is considered recreational but it is full of imaginative mathematics.

John was always attracted by patterns and structures and would investigate these whatever their importance or origin. He could uncover mathematical structure even where such structure had no right to exist, as in his simple Doomsday Algorithm to determine the day of the week for any date in history. A much more important example, Monstrous Moonshine, was discovered with his ATLAS co-author Simon Norton. Their Moonshine conjectures gave structure to surprising numerological connections between the character theory of the Monster group and the theory of modular functions. Moonshine has heavily influenced research in both fields (including the Fields Medal-winning work of Richard Borcherds). Today, these conjectures play an important role in physics, inspiring a continuing wave of discoveries of truly surprising connections between group theory and analysis.

The Game of Life was John's most famous recreational discovery. Now it is fifty years old and continues to fascinate mathematicians and non-mathematicians, while beautiful patterns that come out of the game have moved it to an art form enjoyed by thousands. Life added an important impetus to the theory of cellular automata—it met John's goal of making a simple cellular automaton that could perform universal computation.

The Collatz conjecture (the $3N+1$ problem) must have been known to John when he was young. He would have wondered about a proof of the conjecture, but he also weighed up the idea that there is no proof. He considered more general problems of the same nature, and created his Fractran game, which he showed does encode universal computation.

The sequence of Subprime Fibs was one of John's more recent inventions—the name dates it to the 2008 financial crisis. He started with a pair of terms, traditionally 1 and 1, and constructed further terms by adding their two predecessors and dividing by the smallest prime factor of the sum, whenever it is not a prime itself. Whereas the normal Fibonacci numbers grow in an obvious way, a Subprime Fib might be larger than, smaller than, or in between its predecessors. John did many calculations, and asked me to run computer experiments with the sequence. I plucked up the courage to ask him why he was so interested in something that was so clearly stupid (but rather addictive).

John answered right away. He explained that almost all mathematical questions cannot have mathematical solutions (they are unprovable) but the miracle is that mathematicians recognize the ones with solutions and work on them. He was captivated by phenomena like the Subprime Fibs that might lie on either side of the borderline. He explained his lifelong search for simple structures that exhibit universal computation. The Game of Life had been a fruit of this search. In John's opinion, a sequence would either have a mathematical explanation or could encode universal computation. The Subprime Fibs, for which it is so hard to understand even the most basic questions like whether they are bounded, ought to provide a model of the latter. Other mathematicians avoid things like the Subprime Fibs, precisely because they are looking for solutions. John's lifelong courage to face problems rather than collect solutions led him to many inventions.

More recently, over the last dozen years, I spent about a day of every week in Princeton with John. Mathematical problems would emerge whenever he came across an unusual pattern, structure, or idea. The utility or importance of a problem did not ever concern him; he expected to find interesting mathematics wherever he found interesting structures. As John grew older, his greatest disappointment was that he had not found a way to calculate with the Monster. He was certain that the ability to calculate in the Monster would be the first step towards really understanding it. In their tributes, Rob Curtis and Richard Borcherds discuss two big breakthroughs with the Conway group: Rob explains John's pleasure when they discovered a good way to calculate with it. That advance allowed John to grasp the group, eventually leading to the simple definition praised by Richard. John wanted to achieve something similar for the Monster. He said that the method described in his paper with Arvan Pritchard needed just one lustral idea to fit the bill. Unfortunately, more than thirty years passed without that idea.

I always found it strange that John viewed the Surreal Numbers as his most important discovery. The theory arose from John's more general theory of partisan games—John noticed that certain games have numerical values and turned this around to view all numbers as games. The result was a starkly beautiful definition, whose origin as a recreational topic was invisible in the final theory. The Surreals are among the least Conway-like of his discoveries (in my opinion). It is great mathematics, clearly done by a great mathematician, but it has none of the startling combinatorics that drives so many of John's other inventions. I believe that John rated the theory highly in deference to the mainstream valuation, taking the side of serious research mathematics (but perhaps also secretly happy that it had come out of a completely recreational investigation). John liked to explain that he was two handshakes away from one of his heroes, Georg Cantor. (Both had met Cecilia Tanner, the daughter of William and Grace Chisolm Young.) John would add that if only the distance had been one handshake, he would have told Cantor about the Surreal Numbers, and perhaps he might have overcome Cantor's rejection of theories of infinitesimals.

I think of the groupoid M_{13} as an invention that John did not need to sign—it could not have come from anyone else. It is a piece of serious mathematics (although, by John standards, not especially important mathematics) that becomes interesting and memorable because of its link to a recreational puzzle. John was intrigued by overlapping subgroups in the sporadic group M_{12} and the linear group $L_3(3)$. These had been noted by others and it was well known that a larger group would not explain them. However, John was persistent, he was not one to let a trail of evidence go cold. Eventually, he realized that he could form a groupoid (a structure in which some pairs of elements do not have a product) that exhibited M_{12} as a subgroup and $L_3(3)$ as automorphisms. As he experimented with the groupoid, he made a fantastic observation—its elements can be viewed as the positions of a version of Sam Loyd's 15-puzzle, played on the projective plane of order 3. In John's puzzle, counters occupy 12 of the 13 points of this plane. A legal move takes place on any of the 4 lines through the unoccupied point. It moves one of the counters on the line into the hole and switches the other pair of counters on the line. This wonderful and simple puzzle encodes all of the complexity of the sporadic group M_{12}.

John's inventions often had the simplicity that comes from insight into essence. All mathematicians search for this simplicity, but it can only be unlocked by the right key ideas. John was someone who would discover these ideas. John pursued and prized subtle results and—although he was well aware of standard proofs—he would deliberately consider unconventional approaches. He was very pleased with his proof of quadratic reciprocity—an argument that did not refer to either squares or primes! Although John's successes depended on his great imagination, they also required determination and effort. When I worked with John, I came to see that he was not satisfied when a problem was solved. He insisted on looking further to get a better feeling for the entire situation. In particular, whenever we had a choice of two successful options, he would pause to see if either had any advantage. Surprisingly, often one would prove to be a little better. Once, as he pushed an argument in a completely unexpected direction, I asked "Where did that come from?" He replied quoting Glendower: "I can call spirits from the vasty deep." He was disappointed when I did not respond at once: "Why, so can I, or so can any man. But will they come when you do call for them?" He explained that Hotspur's reply well summarized his feelings (and Shakespeare's too). He conveniently dodged the question of how he had just summoned a spirit.

John's mathematical thought was a force, and he wielded it with composure. Physical circumstances around him were only constraints. On a February day in 2016, he left Princeton to give a colloquium talk. At that time he was already in his late seventies and moved only with difficulty after a series of strokes some years earlier. The colloquium talk went well, but because the weather was deteriorating he was dropped off at the airport immediately afterwards. However, as snow fell heavily, his flight was repeatedly delayed until eventually the flight was canceled as the airport closed for the night. John carried no real money, had no credit card and could not book a hotel room. Instead, he took a free shuttle to the station downtown, where he planned to stay for the night. Unfortunately, at some point, the police cleared the station and John found himself outside in a heavy snowstorm. A passerby suggested that he could spend the night at a restaurant for the price of a coffee. The next morning, John returned to the airport and flew back to Newark. It was only then, when help was no longer needed, that he called me and other friends to tell the tale. The tale had become a solved puzzle that he wanted to explain. His own hunger, cold, fatigue, and danger never registered as important, and were not told. The force of John's thought made things happen John's way, and took us with him.

John was a generous mathematician. Generosity was not his virtue but his business. In fact, John's business was mathematics, and he did it not only by discovering results, but by making us do his mathematics. John's generosity was in his making it his business to make us do his mathematics. He drove other people, feeding them ideas, prompting them to think, pushing them in useful directions. John knew no higher priority than to attend to anyone who wanted to learn mathematics. He would design a route through his ideas and guide his companion, to bring them to his own point of view. John lived to do mathematics, and his mathematics was done not when it looked good on paper or in his own mind, but when it found its proper place in the minds of others. He held himself to this obligation to all of his audiences: doctoral students and professional mathematicians, readers, conference audiences, classes, summer camps, or breakfast time companions at Small World Coffee in Princeton. Jane Gilman recalls the electricity of co-teaching a course on *Geometry and the Imagination* with John, Bill Thurston, and Peter Doyle. Louis Kauffman recounts how a simple conversation with John changed his outlook on knot theory, and Aaron Siegel explains that John reshaped the perspective of the coauthors of *Winning Ways*. Time spent doing mathematics with John felt as if it was spent doing mathematics as John.

Articles in this tribute remember John as a friend. John's graduate students, whether his own (like Rob Curtis and Richard Borcherds) or adopted (like Manjul Bhargava and Dierk Schleicher) are well aware of the significance of John's presence in their lives. These eminent mathematicians, who have a measure of John's greatness as mathematician, have emphasized not the mathematics that John made but John's influence that helped make them as mathematicians. John's generosity eclipsed his greatness.

John's friend and (adopted) student Simon Norton (1952–2019) had extraordinary mathematical ability, but needed John's guidance to flourish. Together they produced exceptional work, which required their combined talents. This included the Moonshine conjectures and other fundamental results about the Monster. However, when Simon remained on his own after John left Cambridge, he seemed to lose his momentum. Simon retained his astounding brilliance to the end of his days, yet his work never approached its former significance, once he lost John's immediate counsel. While John let his choice of problems be driven by the thrill of exploring interesting structures, he had an eye so good that his chosen problems did lead to beautiful and important solutions. Everything that John chose to look at turned into something interesting. Without John's unfailing sense before him, Simon was like a blind giant whose strength was sadly undirected.

Another unusual Cambridge mathematician who prospered under John's guidance is Richard Parker. Almost every day Richard would appear with a new "loony" idea, and John would listen with half an ear. On rare occasions John heard something interesting and those ideas turned into important mathematics. For instance, Richard proposed that the 23 Niemeier lattices must correspond to the deep holes in the Leech lattice (of which only a few were conjectured at the time). This led to their important classification of deep holes and calculation of the covering radius of the Leech lattice. Later, Richard's proposal for a double cover of the Golay code turned in John's hands into a simple construction of the Monster group. Richard's imagination was overwhelming—John's imagination stood firm in judgment to regulate Richard's.

John's books and papers uncovered many different subjects and were written in different styles with different intended audiences. John was always a meticulously conscientious author. He wrote with a feeling of responsibility, almost affectionate concern, for all of his (unknown) readers. He liked to say that "any extra time spent writing is justified if it can save any time at all for the potentially infinite number of future readers." He insisted on developing notation to encapsulate rules, and to push others to work in a particular way. Of course, such notation empowers its users and feels obvious once it exists, but its creation is a task that requires John's sort of rare touch. John was also famous for the names he introduced for mathematical objects and concepts. He called this his "colorful word trick." It was one of the ways that he made certain that his audience noticed essential ideas. We could hardly imagine the Free Will Theorem under any other name, yet at the time when John named it, even his coauthor Simon Kochen needed some time to get used to it.

John's lectures were deliberately unrehearsed and appeared improvised. Nevertheless, John explained that they rested on years of contemplation. While audiences were always happy, John could be his own sternest critic. He opened a Moonshine meeting in Edinburgh in 2004 with a lecture on his simple construction of the Monster. I left the talk pleased to have followed the ideas of one of the most complex constructions in group theory. I believe that others in the audience felt the same way. However, John was unhappy. He had wanted not only to impart the idea but to carry his listeners through the intricacies of the construction. He insisted that the organizers schedule a repeat. Between the talks he was closeted with eager interested participants, testing different explanations. Everyone attended his second talk. I left it with the feeling that I had seen the Monster through John's eyes.

I well remember my return from that Edinburgh conference, when I happened to be on the London train with John. He explained his ideas for a new approach to the Monster, and then introduced me to the Pascal Mysticum (about which we wrote later), and he told me about his knot polynomial. When John left the train at Peterborough, the journey was almost complete—the remaining part was only about 45 minutes. Yet, the last, relatively short part felt so much longer, even though I kept busy writing notes about what John had told me. Of the first part, which went by in a flash, I remember the mathematics and nothing else. After so many years, I recall exactly what was done and in which order. We must have been in the restaurant car because all the work was done on napkins, but I have no recollection of anything or anybody around us, nor of the passing of time. John could do that to us—take over our minds and hold them, moving us to a different state, somewhat like his own. We could readily believe that we had done ourselves what John let us see.

John held audiences in his lectures in a similar way. In a report about the Zurich ICM of 1994, Vladimir Arnold wrote a long summary of John's invited address and concluded: "Eccentric as it was, Conway's was one of the most understandable talks in the Congress."

John always thought about mathematics, but he often said that he wanted to know everything. He was partial to astronomy and etymology. John was always alert to puzzles and solutions, and kept finding them in everyday things. I remember pushing him in a wheelchair, something that he only allowed in absolute emergencies, to make certain that he reached a guest lecture in New York on time. As we crossed the lines between paving stones, he instructed me to push at an angle so that the front wheels passed the slight barrier at different times, minimizing the jolt. He then proposed a new design of wheelchair wheels, which would accomplish the same goal automatically. This design was never implemented. After his last major stroke, in November 2017, John never left the wheelchair and never returned home. He died on April 11, 2020, in New Jersey.

He made so much difference to so many people, to each in their own way but always somehow grounded in mathematics. Now that he has left us, we will each lose some link we had to mathematics that we took from him.

R. T. Curtis

In 1964, Conway became a Junior Fellow at Sidney Sussex College, having recently been appointed to a University Lectureship, the same year that I started there as an undergraduate. The courtyard lawns were strictly out of bounds to us mere students, but fellows had the right to use them and I have a vivid memory of John striding purposefully across Chapel Court in the snow his long hair and gown flowing out behind him, followed by his wife and a retinue of little girls. He was already at that time a colourful character. He became my tutor, and I and Morwen Thistlethwaite, now an established knot-theorist, had weekly supervisions with him. These were never dull. On one occasion, struggling to cope with a rickety table, he insisted on proving to us that, so long as the four feet were on a plane and the surface of the floor was differentiable, one could always find a stable position. More easily proved theoretically than empirically! On another occasion our worksheet contained a misprint and question number 22 was missing; we spent more time discussing what a suitable answer to this question should be than we spent on either 21 or 23. At that time, he and his family lived in a College flat in Sussex Street, across the road from the main College; he would have been delighted to find that there is now a Bridge of Sighs-type structure connecting his old home to the main complex.

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As an undergraduate at Gonville and Caius College, where he matriculated in 1956, he had already been engaged in mathematical research, albeit of a somewhat extracurricular nature. This was often carried out with his friend Mike Guy. For instance, together they had worked out all possible solutions to Piet Hein's 3x3x3 SOMA cube; they drew a graph whose 240 vertices were the solutions, joined by an edge if you could move from one to the other by withdrawing, twisting, and reinserting two pieces of the puzzle. Naturally enough, they christened this graph the SOMAP; it had a connected region of 239 vertices and an isolated point. He and Guy later enumerated all 64 convex uniform 4-polytopes in their 1965 paper, see [1]. He was also devising methods for classifying knots with up to 10 crossings at this stage; it is a delight to me that Thistlethwaite has carried on this tradition. As always with Conway's work, notation is paramount: it should convey all the information required of it and no more, and he would hone it over several iterations until he was satisfied that he had the best possible version.

Having completed his first degree, in 1959 Conway embarked on a PhD under the supervision of the eminent number theorist Harold Davenport. His dissertation was to be on Waring's problem for fifth powers, that is to say proving that every integer can be expressed as the sum of 37 fifth powers. He proved the theorem, which involved a combination of powerful analytic techniques for large numbers combined with ad hoc methods for the smaller integers, but his interest had by this time moved on to logic and in particular transfinite numbers and the work of Cantor. His final PhD thesis, entitled "Homogeneous Ordered Sets," was completed in 1964, shortly before he took up his Fellowship at Sidney. Indeed his first few PhD students worked in the broad area of logic and set theory. His first student was Adrian Mathias, whom he supervised jointly with Ronald Jensen, and who himself went on to become a Lecturer in Logic at Cambridge for five years, before embarking on a fascinating academic career in the South Seas. His second student was Don Pilling who had come to Cambridge on a US Navy scholarship. Don's notes of a lecture course that Conway gave on regular algebras became the core of Conway's book *Regular Algebra and Finite Machines* (1971). Don returned to the States after completing his PhD and progressed in the Navy to become a four-star Admiral; he sadly died in 2008.

At the same time that Conway was entering his Fellowship at Sidney Sussex College, mathematical developments were taking place elsewhere which would affect his whole career. John Leech had taken a degree in Mathematics in 1950 while he was a student at Kings College, Cambridge, but he had moved into computing, which was at that time in its infancy. He worked for some years in the Maths Lab, which later came to house the massive Titan computer, but in 1968 he became Head of Computer Science at the newly established University of Sterling in Scotland. A few years earlier in 1965, he had published a paper in which he described a new 24-dimensional lattice, based on the Mathieu group M_{24} and the binary Golay code, which afforded a remarkably dense packing of spheres, now known to be optimal. John McKay, who had obtained his degree in Mathematics from Manchester and was a research student at Edinburgh University in 1968, was aware of this discovery and it occurred to him that the new *Leech lattice* might have an interesting group of symmetries. He attempted to interest a number of mathematicians, both in Oxford and in Cambridge, before lighting on Conway who grabbed it with both hands. In a beautifully elegant piece of work he produced a new element which demonstrated that the lattice had far more symmetries than those built into it by its construction, and he obtained the order of the new group. Conway's old friend Mike Guy, who was by this time working in the Maths Lab himself, helped with computing, but the final construction is all done by hand. Further investigation showed that not only were many of the recently discovered new finite simple groups involved in this large Conway group, which he called ·O or "dotto," but the subgroups fixing vectors of the two shortest types, those of type 2 and type 3, were also new simple groups.

It was at precisely this point, in 1968, that Conway took me on as his research student and inevitably guided me in the direction of simple groups. At that time, people felt that there must be a large number of sporadic finite simple groups waiting to be discovered and I remember sitting with Conway and John Thompson when they were discussing whether there were an infinite number of sporadic finite simple groups or just a very large finite number. The truth is nobody knew. Of course the fact that so many of the other new groups occurred in the Conway group was evidence that there may not be that many more, but that was certainly not a clinching argument. Indeed, the first project Conway gave me was to classify the finite subloops of the Cayley algebra, nowadays known as the *octonions*, in the hope that I would find an erstwhile unknown finite loop whose multiplication group was a new finite simple group. Sadly, I was able to prove that there was no such new loop!

But Conway was not only thinking about groups. In fact the year 1968–69, which he later referred to as his *annus mirabilis*, was the most mathematically productive period of his career. Playing around with counters on a Go board he had come up with rules governing when cells would die through isolation or through overcrowding, or a new cell would be born, which led to the Game of Life. In the early days, all investigations were carried out by hand on a Go board, but once the process had been set up on a computer things really took off. Mike Guy's father Richard, who had himself been a student at Caius College but who had emigrated to Canada some years before and worked in the University of Calgary, was a regular visitor to the department. He worked with Conway on the theory of mathematical games and, together with Elwyn Berlekamp they later produced the wonderful *Winning Ways for your mathematical plays*. Richard was involved in the early experiments with *Life* and it was he who noticed a small configuration which regained its original shape after four generations, having moved diagonally across the board. This became the famous "glider." Conway had already contributed on several occasions to Martin Gardner's celebrated column on recreational mathematics in the *Scientific American*, and now he sent across the rules of *Life* and issued a challenge: he would give \$50 to anyone who produced a configuration which could be shown to increase in population size indefinitely. A cult was formed and it is estimated that millions of hours of computer time around the world were spent running *Life* programs. Eventually Bill Gosper of MIT produced a configuration which sat there shimmering away and shot out a glider every 23 generations. Gosper won the bet and happily for Conway *Scientific American* paid the debt, as \$50 was a substantial sum of money in those days. More importantly these discoveries gave *Life* scientific respectability as it could now be shown that it could simulate a Turing machine, and itself be used to perform calculations. Nowadays it is celebrated as an early example of a cellular automaton.

Conway, though, had very mixed feelings about the Game of Life as he feared that that would be the main thing he would be remembered for. Amazingly, in the very same year that he had found his groups and created *Life* he also produced the mathematical construction which he would most like to be his legacy: the *surreal numbers*. The name was in fact coined by Donald Knuth who used it as the title of a novel; Conway simply referred to them as *numbers*, as opposed to *games*, in his 1976 book *On Numbers and Games*. In effect, he combines the methods of Cantor to define the ordinal numbers, and Dedekind cuts to produce the reals, to create a rich system containing finite, infinite, and infinitesimal numbers with remarkable properties. Conway himself believed that these surreal numbers would acquire great significance in the years to come and I for one hope that he was right.

Meanwhile I myself was working on the Leech lattice and the Conway group, and I realised that to understand these objects properly I needed to be very familiar with the Mathieu group M_{24} and, in particular, the Steiner system $S(5,8,24)$. Conway knew that I was playing with patterns on graph paper which would enable me immediately to recognise when a set of 8 points of the 24 was an *octad* of the system. One night in a pub called the Cricketers' Arms, whilst having a beer with a bemused non-mathematical friend and playing around with these patterns, I realised how to do it. The next morning, I went into the department and announced to Conway that these 35 patterns tell the whole story. He was thrilled to bits and immediately took over. We decided to draw the patterns neatly by hand and get them printed off on cards. We would use graph paper but didn't want the lines to show up, so we needed a glass-topped table with a light underneath and the graph paper turned upside down. Well, we didn't have such a table in the department but the librarian's office had a sliding glass window which suited our purposes. We removed the window and placed it between two chairs with an anglepoise lamp in a litter bin underneath it. We worked in this way until late into the evening when Richard Guy came by and suggested we pick up some wine and finish the job off in the Guest Room at Caius College where he and his wife Louise were staying. We were there till the early hours of the morning and a hilarious time was had by one and all. In the morning, I cycled around to a printer and when he had shrunk it down to a quarter of its original size all the imperfections due to our shaky hands and general inebriation miraculously disappeared. The resulting Miracle Octad Generator or MOG, see [2], gave Conway and me great pleasure as we competed with one another to complete a given five numbers, called out by a neutral umpire, to an octad. It was also immensely useful for more serious work!

The origins of the Leech lattice led to Conway's great interest in and contribution to linear codes and sphere-packing in n-dimensions. He worked on these areas over many years with Neil Sloane, who regularly visited Cambridge, and their fruitful collaboration resulted in the monumental *Sphere-packing, Lattices and Groups*.

By this time Conway had acquired international fame. He had resigned from Sidney in 1970 when he felt that the appointment of a new Master had been conducted improperly; he moved back to Caius which after all had been his undergraduate college. Marshall Hall Jr, who was also a regular visitor to the department, invited him to spend a sabbatical at Caltech, and he was allowed to bring me as a GRA. I recall a hellish journey which took some 30 hours and entailed a stopover of 6 hours in an unfurnished room at Philadelphia airport; but Conway, his wife Eileen, his four daughters Suzie, Rosie, Ellie and Ann-Louise, and I eventually arrived at Los Angeles airport blinking in the bright sunlight. It was early morning on the 3rd of January 1972 and the hills behind LA were so crisp and clear that they appeared to be just streets away.

I lived with the Conway family during our time in California in a house in Pasadena which they had rented from a history professor. Each morning John and I would walk down to Caltech and regularly be attacked by a fiercely territorial mocking bird which would fly right into us. During that time, I wrote up my thesis and he worked with David Wales on constructing the Rudvalis group. Arunas Rudvalis had predicted the existence of this group; he knew its order, its class list and character table, but a group is not known to exist until it has been constructed. A number of people were attempting to build it, but Conway, who was nothing if not competitive, was determined to be first. Accordingly, when they had succeeded in building the group, John and David announced the precise hour of the day on which the construction was shown to work.

Conway gave many lectures around the States at that time, mainly on the groups and the Leech lattice, but I remember accompanying him to UCLA when he gave a very well-received talk on *Life*. One of these trips was to Baton Rouge and the night before, back at home, he said to me "Why do you think it's called Baton Rouge?" I replied that I supposed that when the early French settlers arrived they found an Indian village with a large red pole at its centre. He said, "Oh, you think so!" took down the encyclopedia and read "When the early French settlers arrived. . . ." That was a distinct success!

When it came time to leave, Caltech made Conway a very generous offer of a permanent position with a salary many times his income in Cambridge. However, John was determined to return to his familiar stomping ground and back he went.

During his investigations of the Conway groups, John had become acutely aware that there was no common source where one could look up a particular group and discover its main properties. This led him to the concept of an *Atlas of Finite Groups*, the idea being that the families of simple groups would be the continents of a geographical atlas, and you would hone in on the individual groups which corresponded to particular countries. I was awarded a Science Research Council research fellowship and we set to work energetically on this daunting project. It was apparent that character tables would be at the heart of the operation and it is fair to say that most of the tables we inherited from other sources contained errors. The fact that we included power maps and Frobenius-Schur indicators often showed us that something was wrong and helped us put it right. We worked in my office, which naturally enough became known as *Atlantis*, and after a few months we were joined by an uninvited guest in the form of Simon Norton. At first John was very unhappy about this intrusion and said to me "If that man keeps coming into our room, I'm going to give up the whole project." But within a few weeks we had come to recognise the huge contribution Simon was making and we invited him to become a co-author. We would at times approach Mike Guy when we wanted a specific computing job done, usually at 3 o'clock in the morning when he could be relied on to be in the Maths Lab, but the project desperately needed a permanent computer-savvy input. This was supplied by Richard Parker who joined the project after I left; now orthogonality relations and much more could be checked efficiently and the output became remarkably accurate. The last author to join the project was Robert Wilson, who made an immense contribution in working out the classes of maximal subgroups for many of the Atlas groups, and in pulling the whole thing together for publication. Further computing support was supplied by John Thackray. The ATLAS was finally published in 1985 and is much used by all those whose work involves symmetries of finite configurations, not least by its five authors.

Richard Borcherds

Like many people, I first heard of John Conway through Martin Gardner's "mathematical games" column in Scientific American. Here are some of the stories I remember or was told about John Conway. Some of them may be true.

John was a student of Harold Davenport at about the same time as Alan Baker. Harold's wife Anne told me that what would typically happen is that each week Harold Davenport would discuss some hard problem with them. A week later Alan Baker would come back with a solution to the problem, while John Conway would come with a solution to a totally unrelated problem. John Conway said that when he solved one of the problems that Davenport had suggested (something related to Waring's problem), Davenport said to him something like "Well Mr Conway, what we have here is good enough for a poor PhD thesis. Now go and do something better."

John Conway would sometimes give evening lectures to some of the undergraduate mathematical societies. One that I remember was on sporadic simple groups. (As it often was with Conway, this was not the official title: he held a poll at the start of the lecture on what the topic should be, and groups won.) He mentioned the Monster group with its 196,883 dimensional representation, and John McKay's strange observation that the elliptic modular function $q^{-1} + 744 + 196884q + \ldots$ had a similar coefficient. I remember being completely flabbergasted by this; it was the first time I had heard of "Monstrous Moonshine," the name coined by Conway for these strange relations

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between finite groups and modular functions. Soon after, he and Simon Norton ran a seminar on Robert Griess's recent formidable construction of the Monster, which I attended but was rather lost in. A few years later John Conway managed to simplify Griess's construction (more than 100 pages of hard computations) down to the point where it would fit into two pages (admittedly very large ATLAS pages).

I took a graduate course on group theory with him, covering Mathieu groups and the simple groups he had discovered. He told us the story of how he found this group. John Leech had been going around trying to interest people in calculating the automorphism group of the lattice he had recently discovered. John Conway was at a bit of a loose end at the time and thought he might as well try, expecting to spend a year or so on the project. Instead it took him only a few hours to find the order. He rang up John Thompson to tell him the result of his calculation, and John Thompson called him back a few minutes later to tell him it was a double cover of a new simple group. (The standing joke at the time was that a good way to search for new sporadic groups was to ring up John Thompson, give him a random number as a group order, and see what he would say.) By probing further John Conway found a couple more groups, and also showed that most of the known sporadic groups could be found inside his largest simple group. He called his groups .1, .2, and .3, and explained to us that whenever you discover something you should give it a really bad name, so that people would have to choose a different name and with luck would name it after you. The other memorable event in his lectures was the time he tried to make a random decision by tossing a coin, which rolled around until it stopped by a wall, ending up on its edge (a possibility sometimes overlooked in probability courses).

I ended up working with John Conway almost by accident. After graduating, I should have found a PhD advisor but had not had much luck and was drifting. John Conway walked up to me in the library one day and asked me if I wanted to work with him. (I think he had heard that I was in trouble; he had a habit of helping graduate students in difficulties, and there were at least two other people at that time who worked with Conway after not getting very far with a different supervisor.)

The common room at the Cambridge math department was a large room surrounded by offices, three of which were occupied by John Conway. Officially he only had one office and the others were things like an office for his ATLAS project. In practice what happened was that whenever his office got too full of piles of old papers to use he would move on to an empty one. He came close to taking over the common room as well: it contained the original *Atlas of Finite Groups* (a huge scrapbook bought by Conway), and when Conway's coworkers were not arguing over some esoteric group extension in it they could be often found nearby playing one of the many games Conway had invented.

John Conway invented several rather esoteric computers. His ones based on fractions (FRACTAN) and his Game of Life are fairly well known, but his computer inspired by toilets seems to have received less attention. Conway observed that the flushing mechanism of toilets is a sort of logical gate that gives no output until it gets a fixed number of inputs, at which point it gives an output and resets. He claimed that he once built a primitive computer using this principle, which was not very successful as it had an unfortunate habit of having rather too literal overflow errors whenever it was exhibited.

His style of mathematics was very computational: if he wanted to understand something, whether it was knots, representations of E_8, or quadratic forms, he would start by calculating explicitly a large number of examples. The first research problem he suggested to me was a similar problem about enumerating certain configurations in the Leech lattice, which at first sight seemed to me to be rather pointless, but as Conway doubtless knew, doing this calculation gave much better insight into the Leech lattice than reading any number of abstract theorems about it. This calculation gave me the idea of forming a Kac-Moody algebra out of the vectors of the Leech lattice. When I excitedly told this to Conway he politely explained that had already written a paper about it. I found out later that he quietly added my name as a coauthor, so my first published paper was one that I contributed nothing to and had not even seen.

Unfortunately he had some minor strokes a few years ago. He explained that having a stroke had one unexpected benefit: he was now ambidextrous, having been forced to learn to use his left hand when his right hand was temporarily unusable.

The result of Conway's I will discuss in detail is his calculation of the automorphism group of the 26-dimensional Lorentzian lattice $II_{1,25}$. It is related to, but not nearly as well known as his earlier sensational discovery of the Conway groups that "explained" most of the sporadic groups known at the time. At first glance it seems to be a rather obscure and unimportant topic, but I consider it to be one of his deepest and most fundamental results: it underlies several areas such as Monstrous Moonshine and infinite dimensional generalized Kac-Moody algebras. Most of my published papers depend in some way on this result of Conway's and could not have been written without it. It even turns up in algebraic geometry: several authors have used Conway's result to calculate or study the automorphism groups of K3 surfaces.

Conway's result states that the automorphism group of $II_{1,25}$ is the product of -1 and an extension $R.\Lambda.Aut(\Lambda)$, where R is the reflection subgroup, Λ is the Leech lattice (the unique even unimodular lattice in 24 dimensions with no roots), and $Aut(\Lambda)$ is the double cover of Conway's largest sporadic simple group Co_1. This is already a rather astonishing result: a sporadic simple group has seemingly appeared out of nowhere. In fact this is the only example I know of where a sporadic simple group appears in the automorphism group of something ($II_{1,25}$) that is "trivial" to construct. In the course of his proof, Conway showed that the Leech lattice appears as the Dynkin diagram of the reflection group of $II_{1,25}$. This statement sounds like nonsense: a Dynkin diagram is a graph, so claiming that it is a lattice in Euclidean space seems to make no sense. What this means is that the Dynkin diagram is essentially the set of simple roots of the reflection group, and the set of simple roots is isometric to the Leech lattice.

Conway's proof of his result is long and difficult. It depends on Conway's discovery (with Parker and Sloane) that the Leech lattice has covering radius $\sqrt{2}$, which in turn depends on the rather bizarre fact that there are exactly 23 orbits of "deep holes" of radius $\sqrt{2}$, which correspond to the 23 Niemeier lattices in the sense that the vertices of the deep holes form the affine Dynkin diagrams of the Niemeier lattices. This calculation is hard and takes about 50–100 pages even with many routine details missing. (Somewhat shorter proofs have since been found, but they are all difficult and roundabout.)

Conway's result implies a sort of phase change in the behaviour of lattices at 26 dimensions. Roughly speaking, below this dimension lattices behave in a controlled way and can be classified without too much trouble, while beyond this dimension lattices go "wild." The 26-dimensional case is a sort of borderline case. For example, the numbers of even positive definite unimodular lattices in dimension at most 24 are 1 in dimension 8, 2 in dimension 16, 24 in dimension 24, but more than a billion in the next dimension 32. This can be explained by Conway's result: roughly speaking, the good behavior of its automorphism group can be used to control the lattices in dimension at most 24. (The analogous cases $Aut(II_{1,9})$ and $Aut(II_{1,17})$ are much easier as shown earlier by Vinberg: the reflection group has index 2 or 4 in the full automorphism group, while the next group $Aut(II_{1,33})$ is ridiculously complicated.)

Before Conway's result there seems to have been no hint that the behavior of lattices or quadratic forms changes dramatically at 26 dimensions. For example, O'Meara's standard book on quadratic forms has no mention of this. In string theory it had been observed shortly before that 26 is a critical dimension of Lorentzian space-time. This is closely related to Conway's result that 26 is a sort of critical dimension for Lorentzian lattices (and in fact Conway's result has occasionally been used in string theory).

Manjul Bhargava

Professor John Conway was truly one of the giants of mathematics. I feel honored and privileged that John was my first-year advisor in graduate school—I worked (and played) a great deal with him during my first year at Princeton.

I learned very quickly (not that I hadn't expected it—his reputation preceded him!) that it was futile to try and make an appointment with him. He did not make appointments—and when he did, he did not remember or show up for them. However, he was nevertheless very easy to find—not in his two big offices, which were both too full of beautiful and colorful mathematical toys to have any actual space for meetings—but in the mathematics department common room area, where he would spend morning till evening playing games like Go, Backgammon, and other rather more offbeat games that he devised. I also learned very quickly that playing games and working on mathematics were closely intertwined activities for him, if not actually the same activity. His attitude resonated with and affirmed my own thoughts about math as play, though he took this attitude far beyond what I ever expected from a Princeton math professor, and I loved it.

I thus spent hours in the common room watching him and joining him playing games. He would occasionally break into mathematics conversation in the midst of these games, and these moments were gems not to be missed.

That year, through such moments, I learned about his playful way of looking at knot theory, group theory, surreal numbers, and number theory, which certainly influenced my own thinking in the future. I also had my very first collaboration with him in the common room while we were all looking at seqfan, a mailing list for number theorists and indeed all fans of and contributors to the Online Encyclopedia of Integer Sequences. On this mailing list, David Wilson had asked the question of whether *twin peaks* exist, i.e., two distinct whole numbers that have the same least prime factor, but where every number in between has a strictly smaller least prime factor. While I do not remember who first brought up this question in the common room, it became a widespread topic of discussion all afternoon. Computationally, several students checked in the computer room next door that there were no twin peaks into the billions and beyond. As a

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consequence, while playing games, John, Derek Smith, and I were discussing various arguments to try and show that twin peaks did not exist. But none of those arguments were working, and no one else in the common room had succeeded either in proving the nonexistence of twin peaks. It dawned on us by evening that maybe what we were trying to show was not true, and that our argument could perhaps be turned on its head to show that twin peaks do exist!

That's when the real work started. Since that was the usual time when John and most professors would go home, we assumed that we would continue working the next day. Instead, John moved us to the blackboard in the hallway to try and construct a twin peak. His enthusiasm also attracted Johan de Jong, a brilliant new exuberant young professor who had joined the department recently and who was wondering what all the fuss was about. He quickly and excitedly caught up and joined us in this adventure. Through dinnertime and until night, without even thinking of food or the late time, the four of us worked at the blackboard in what was a relatively brief but very intense and productive collaboration. By night, led by John's vision and enthusiasm, we had produced the first known twin peak. The numbers in the twin peak were so large—having 45 digits each!—that it was understandable why no one had yet found such beasts simply by performing a computer search. This pleasant experience of my very first collaboration with John is still firmly in my memory, and it further affirmed to me the joy and playfulness of mathematics; it was a major source of encouragement to me to continue on the journey of mathematical research.

The day after that experience, John came to me in the common room and said, "I can tell from our work yesterday that you are a very clever guy." I told him that was a huge compliment given who it was coming from—I'm not sure I've ever met someone cleverer than John!

From then on, I talked math with him regularly. He told me about a theorem that he had proved with William Schneeberger, called the "15-Theorem," which states that an integer-matrix positive-definite quadratic form represents all positive integers if and only if it represents the positive integers up to 15. He also told me about his much harder 290-Conjecture, which states that if 15 is replaced by 290, then the statement would apply to any integer-coefficient quadratic form. I was so intrigued with that theorem and the associated conjecture that I stopped studying for general exams and started thinking about that and why it was true. I learned about the arithmetic of quadratic forms from his beautiful book *The Sensual Quadratic Form*, and the approach of his book is still how I think about quadratic forms; e.g., while most mathematicians think in terms of the traditional "Hilbert Symbol" (a certain p-adic invariant of the quadratic form), I still think in terms of the elegant "Conway Symbol" as introduced by John in his book and used in our discussions. My first-year work on the 15-Theorem was very influential in my life. In particular, it inspired work and a long-time collaboration with my good friend and fellow graduate student at the time, Jonathan (Jon) Hanke, who was working under the advisorship of Shimura. This collaboration eventually led to a proof of Conway's 290-Conjecture, one of Jon and my very proudest achievements. Years later, Jon and I presented the theorem to John, wrapped in a bow, and John said it was the best present he ever received as he did not think it could ever be proved in his lifetime.

Other than mathematics, I learned a great deal from John about mathematical outreach and mathematical games. His public lectures, so clear and accessible, were inspiring—something that I myself would aspire to in the future. I also learned a number of tricks that he would perform on little kids—as well as full grown adults—to communicate mathematical principles.

We also discussed the history of math, especially the *Disquisitiones Arithmeticae* of Gauss and other famous mathematical works written in Latin. I told him how impressed I was at his knowledge of the history of math—but that I felt it was somewhat Eurocentric. I told him about some of the early works of India that I was familiar with due to my family background. He was very intrigued, particularly by Brahmagupta's *Brahmasphutasiddhanta* and the notions of composition that were already in there more than 1000 years before Gauss. He said it was too late for him to learn Sanskrit, but that he hoped I would bring these works into the mainstream in the future for a more complete view of the history of math. I still think of his sincere request and hope to follow up one day.

For my PhD thesis, under the advisorship of Andrew Wiles, I worked on the theme of composition in the style of Brahmagupta and Gauss as best I could, and its modern connections with algebraic number theory. John came to my Ph.D. defense on "Higher composition laws" quite excitedly, saying he was looking forward to hearing about it. I began the defense by making a claim, without proof, that the discriminants of three quadratic forms that I had constructed from a 2 x 2 x 2 matrix of numbers were the same. Next thing I knew, John had stopped paying attention, and instead was frantically scribbling something on a piece of paper for the next half hour. I was saddened at the thought that I had bored him. But afterwards he told me that he was sorry but he simply *had* to check that remarkable assertion I had made at the beginning of the defense (which he had actually verified by hand was true during the defense), and so he was unable to listen to most of the rest of the defense. I filled him in later on what he missed as best I could.

A couple of years later, when I joined the faculty at Princeton as his colleague, I would regularly teach general-audience courses on mathematics and mathematics outreach. The first year I taught it with one of my best friends from college, Lenny Ng, who also happened to be at Princeton that year. Any time I taught a course like that, I always arranged to have guest speakers, and of course John was at the top of any such list. John was always the favorite guest visitor among the students, and always sprung some surprise or other that would shock the audience. The first time I taught such a course, it had about 100 students. John performed a trick where he has four students tangle two ropes together, however they like, within certain parameters, and for however long they like—with the tangle itself tied up by a large plastic bag so that we cannot see the actual tangle. After watching the complex tangling process, John would then ask the four students to perform certain moves, and proclaim that he has untangled the tangle! As the grand shocking reveal, he would ravenously shred the large plastic bag with his teeth!! The audience gasped. When the shredding was complete, the tangle was indeed seen to be untangled.

Other tricks he did for his classes involved: being able to say immediately the day of the week of any date in history (and then would teach his famous Doomsday algorithm); rotate a hanger quickly around his index finger with a penny balanced delicately in its inside edge; contort his tongue into various shapes; and more. He taught the notion of noncommutativity by taking off his shoes and socks in front of the whole class; then on one foot he would put on his sock and then his shoe, and on the other he would put on his shoe and then his sock—and then he'd proclaim "You see, the outcomes are different!"

I used to visit USA/Canada Mathcamps sometimes to speak to students. Once I visited with John, which was super-fun; we attended each other's lectures and played games together with the attendees. I remember, sometime during his talk there, he said to the students: "Let me be modest. I know more than 100 times as much math as any of you." One of the students said: "That's modest??!!" Conway said, "Yes, that's being very modest! If I was being accurate and not modest, I would have said that I know more than 1000 times as much math as any of you!"

The only difficulty and tension that I had with him over the years is that he didn't take good care of himself. That was tough on me. He wouldn't eat well, he wouldn't exercise. He wouldn't go to therapy after his stroke; he gave excuses about not having enough money (we'd all offered to pay for it), that he forgot (we said we'll put a reminder on his phone, which he refused), or that he just didn't feel like it. That was tough to deal with for the people who loved and cared about him during his last several years, especially during this recovery period when taking care of himself was of the utmost importance. Richard Guy, another very close friend and collaborator of John—but 20 years older than him—used to say worryingly that, with the way John ignored his health, he was afraid that John would pass away sooner than him despite being 20 years younger. In the end, Richard's fear did not come to be, though only by a margin of one month. Richard left us in March 2020, reaching the age of 103; John left us about one month later.

I have enjoyed every moment with John over the years. He was an advisor, teacher, collaborator, colleague, inspiration, and friend. I was not ready to let him go so soon and so suddenly. There were still many games left to play. I will miss him.

Dierk Schleicher

One of John's unique characteristics was that for him, there was little difference between research and play, or between talking to senior colleagues and high school students, and he enjoyed both. He actively participated in numerous student camps, including the "Modern Mathematics" summer schools that I co-organized in Germany and France. For six years from 2011 to 2017, he came to all of them from the first day to the very last day, which was almost two weeks. He mingled with the participants from breakfast until late at night, if not early morning. He always radiated his untiring spirit of lively interaction, always assembling a happy crowd of followers.

On the first morning of our first camp, in 2011, after his overnight travel, we knocked on his door to show him the way to breakfast (at 2:00 AM Princeton time!). We worried when he didn't answer. It turned out he had gotten up all by himself and had already found some kids to talk to, untouched by any jet lag. When we went to bed after midnight, he was still with other kids, sharing his stories until the last listener went to bed. The kids might have had a hard time getting up in the morning, while John was sitting there already with his "morning shift" of kids.

He gave talks on some of his unique contributions to math, of course without preparation (sometimes the students could vote at the beginning of a session what he should talk about), and he kept everyone's attention for every minute. His brilliant talks were legendary, from the great show in the plenary talk at the ICM 1994 in Zurich to ordinary undergraduate classes where he could get all excited about having proved a truly phenomenal theorem, that "1+1=2" (pointing out a fundamental step in building

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up arithmetic). At our schools, he was just as interested in all the other talks, came to all of them, and usually asked witty questions or suggested additions.

He had a unique talent for challenging students with whatever problems he was thinking about. In several cases this led to joint publications (such as the Mathematical Intelligencer papers on the "Prague Clock problem" and the "Immobilizer problem," written with Bremen participants).

I came to realize much later that my very first encounter with him was quite typical. I was a visiting student in Princeton when he had just moved there from Cambridge. He told me he was a bit uncertain as to whether he would fit in with the department where everyone else was such a serious mathematician. While holding court on one of his legendary sofas, he mentioned that he had come across a problem that he had once solved but he was unable to reconstruct. He offered me a modest award of $10 for a proof. Of course I accepted the challenge and returned a few days later with a detailed proof. He took my papers, tossed them almost immediately (keeping track of papers was not a concept for him), and handed me 5 dollars, watching my reaction. What was I to say—me, the young student in front of the legendary Conway? Of course he was fully aware. After letting me struggle for a while he resolved things by saying that meanwhile he too had found a solution, so we would share the prize. He then waved his hands into the air, drawing intersecting triangles of double summands, making the key idea obvious, ignoring the details. Others may have earned a greater bounty from him, but I was so proud to have shared a prize with Conway!

I had the good fortune to be involved in some of his late publications. He told me about a paper he was writing, in the spirit that "although we all know there are true but unprovable results, we tend to think they must be obscure and far away; however, in reality 'the beast is just around the corner.' " He got worried when a reader of an early version had commented "old Conway has finally gone off his rocker." I convinced him to publish "On unsettleable arithmetical problems" (his "Amusical Permutations") anyway. From the moment it was published, he remained enormously grateful to me forever. This paper was highly praised and eventually selected for the 2014 edition of "The Best Writing on Mathematics," published by Princeton University Press.

His last publication that appeared just before his death was the "Moscow paper" that had been an unpublished manuscript for decades. I received a copy when I visited him in Princeton: he retrieved it from under the cushions of a couch where he kept some of his important papers, and I was allowed to make a copy—and much later edit it so that it could be published. When it finally appeared, I called him up in early April. He clearly enjoyed the news very much that it was eventually published, and he told me (once again) the story of how it came about. We then discussed one more paper that he wanted to get published, and he gave me clear instructions about details of how it should be edited. This was a long and good-humored discussion. I was later told that this made me the last mathematician to have spoken to him.

Both of these papers were related to the idea that "the beast is just around the corner" in every area of mathematics that he had touched upon throughout his life. In Roberts's biography, he calls this issue "his Jugendtraum." He invented the Game of Life because he felt that virtually every non-trivial cellular automaton must be Turing universal and hence undecidable. He invented the "Fractran" language because he felt that there must be very simple sequences that are undecidable; and "Amusical Permutations" were an attempt to find even simpler such sequences (including the notorious Collatz problem). In general, many simple (at least recursively defined) theorems should be undecidable (and the proof of this claim should be undecidable itself, and so on). Similarly, he told me repeatedly that if a monkey had access to a large warehouse with many logical gates of NOR type and started connecting them at random, it would eventually build a computer as powerful as any humans could build. While it is a well known observation that a "random monkey" working away on a typewriter would eventually reproduce Shakespeare's works, Conway's monkey can actually be expected to succeed in very finite time (it might take very much longer until a device comes about that we can understand and prove to be universal, while undecidability or universality themselves might come very much faster). I tried many times to encourage him to bring all these ideas together into a coherent text, but even though he greatly enjoyed these issues throughout his life, he refused to do this and called this topic "my obsession." Eventually, we settled on "my obsession with his obsession" (for a topic he had earlier called his Jugendtraum).

John was full of humor, often about himself. He liked to say, "my only shortcoming is my modesty; if I weren't so modest I'd be perfect."

Of course we all miss him sadly, as an ingenious mathematician, as a good-humored friend, (yes, he could be a friend independently of being a mathematician), as an exciting person to play with, as an inventor of sequences and games, and so much more. Many times I felt the urge to ask John about one of the many things he knew so well, or discuss an idea with him that I had. But I would rather close with a witty remark I found on the internet: a remark that alludes to his famous Game of Life and that keeps raising my spirits: "if three of us stand next to each other around an empty square, will he come back to us?" I like the spirit, and in a sense, this is very true: he will remain with us for the rest of our lives, as a source of inspiration and surprise, as a mathematician, and as a friend.

Jane Gilman

It is natural that the contributors to this memorial write about what a great mathematician John Conway was. In addition to being seemingly supremely confident, he was also a sensitive human being who appreciated approval.

In 1991, I team taught an experimental mathematics course with him (along with Bill Thurston and Peter Doyle) twice, one semester at Princeton and once during the summer at the now defunct Geometry Center at the University of Minnesota. The course was an experiment in trying to engage students in mathematics by using unusual methods—chopping up vegetables to demonstrate curvature and using a flashlight flat-end running over an object with its light projecting on to the ceiling and walls to demonstrate the Gauss map. In the summer course the object was Conway's robust stomach used as he lay flat out on his back on a table. In later years he lost weight and was no longer a good subject for such demonstrations.

I don't know the extent to which the students in the summer class consisting of graduate students, undergraduates, post-docs, and high-school students found this exhilarating, but the four instructors were very taken with what we were doing and talked far into the night planning the next day's class and talking about mathematics.

John talked about his early years. He described his early life in a small town as being pegged as an introverted math nerd. He explained that when he left that small town for Cambridge University, he decided he could present any persona he chose, since nobody knew him. He chose to be outgoing, even flamboyant and to be the center of attention from an admiring crowd that often surrounded him. I remember seeing him in the midst of such a crowd during the summer of 1969. That is, he became the charismatic center of things. Those who met him later would think of him as self-confident and assured.

When the four of us team-taught these classes, we often argued in our pre-class preparations and did not agree on what we wanted to do or were going to do when we all walked into class together. In such paniced times, John could start off a class with a 5–10-minute lecture on some topic that popped into his mind, not related to the possibilities we had all been discussing, and would entertain the students and the three faculty members with engaging and beautiful talks on whatever. Perhaps these were pieces of talks he had given before, but in any case, they were all very fluid and focused on an idea that could be presented quickly.

When I was at Princeton again in 2017, John talked about Bill Thurston dying and about how those team-teaching days were one of the happier times. He would sit in a small cubby hole by a window on the third floor of Fine Hall and collaborate by cell phone with mathematicians. John followed my description of my current research even when his vision was not so great for looking on the small cubby hole whiteboard. He was also eager to share his research. And, of course, typically Conway, he shared the details of the bi-monthly injections given directly into his eyes to help maintain his vision.

When I was preparing to teach the Geometry and Imagination course at Rutgers, I went to see him for a lesson on how to make the five Platonic solids out of construction paper and instead of instructing me, he made the models, which ever since have hung on the walls of my office when not in classroom use. Last Spring the models began to crumble and had to be taken down, perhaps an omen of things to come.

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Aaron Siegel

Conway's career is marked again and again by astounding leaps of mathematical imagination that opened previously hidden avenues of investigation. Perhaps nowhere was his inventiveness on such playful and creative display as in the study of combinatorial games. His interest in the subject began through a fateful chance meeting: in 1960 Michael Guy, whose father Richard was to become one of Conway's most enduring collaborators, entered Cambridge as an undergraduate. He and John soon became friends, and Michael introduced Conway to some of his father's work on games.

At that time "combinatorial games" meant, with few exceptions, *impartial* games such as Nim, in which both players have exactly the same moves available at all times. In the classical impartial theory, each component G in such a game is assigned a *nim value* $m \in \mathbb{N}$; the celebrated Sprague–Grundy Theorem then reduces *any* impartial game to Nim, played on the nim values of its components. Richard Guy's pioneering work demonstrated that nim values, and hence winning strategies, for a wide class of impartial games can be efficiently computed.

Conway was immediately intrigued by this fascinating intersection of games and mathematics, two of his longstanding passions. Michael Guy soon introduced

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Conway to his father, and in 1969, at an Oxford number theory conference, Richard in turn introduced him to Elwyn Berlekamp, who had previously proposed to Richard that they collaborate on a book on games. Elwyn and John promptly skipped the conference for a day, retreated to a local pub, and spent the next ten hours or so playing games. It was clear that writing a book together was a good idea.

Ever since Michael Guy first showed him the impartial theory, Conway had wondered about a partizan analog. Various authors had attempted to consider partizan games, but typically by using numbers to quantify partizan game values—necessarily yielding approximate, rather than exact, solutions. If there were to be a complete partizan theory, what then would take the place of nim values? It was not until 1970, with work on *Winning Ways* already under way, that Conway would formulate a satisfactory answer. In his words:

> We had the British Go champion [Jon Diamond] in our math department at Cambridge, and I used to watch him play games in the vain hope that someday I would understand them. I never did. But I did see that in the end, a Go game decomposed into a sum of little games, and I thought it was a good idea to study this kind of sum.

So inspired, Conway in short order cracked the nut wide open. Rather than seek to quantify partizan games numerically, he introduced new algebraic values, obtained by equating games that behave identically in sums, and spun them into a novel axiomatic theory. The Simplest Form Theorem, demonstrating that every (finite) partizan game has a unique simplest form, made the new theory rigid. The world it opened up was initially mysterious, and Berlekamp, Conway, and Guy—assisted by a host of excellent graduate students—would spend the next ten years piecing together the theory and its applications, giving rise to an entirely new mathematical subject.

In hindsight, it is perhaps a happy accident that the classical impartial theory so neatly embeds in $\mathbb{N}$, and this elegant coincidence may have been a distraction in the search for a complete partizan theory: in the end, partizan values are not numbers at all. But in a remarkable twist, Conway soon realized that the universe of partizan game values contains *within it* a new type of number, an extraordinary synthesis of Dedekind's reals and Cantor's ordinals. Later dubbed *surreal numbers* by Donald Knuth, their discovery—and the path that led to them—came as a big surprise to Conway, and they would consume a great deal of his attention in the early 1970s.

Conway pushed the boundaries of combinatorial game theory in other directions as well. Impartial games in *misère play*—where the last player to move *loses*, inverting the usual convention—were long known to be substantially more difficult than in normal play. Conway demonstrated just *how much* more difficult, by showing that misère impartial games admit simplest forms, and that no simplifications are possible beyond the most trivial. This observation, if initially dispiriting, laid the foundation for work in misère games that continues to this day.

His imagination was on display once again in the study of *loopy games*, in which repetition is permitted (so that their associated game graphs may contain cycles). Together with his student Simon Norton (better known for his work in group theory), Conway showed that much of the partizan theory can be extended to a restricted (but large) class of loopy games. Thus in the ten years between 1970 and 1980, three major classes of games—partizan, misère, and loopy, all heretofore poorly understood—were given a clear and firm foundation.

One can sense a common thread across all these discoveries: Conway was never content to accept the confines of an existing theoretical framework. Instead he asked what would happen if the prevailing assumptions were discarded, forged ahead to seek a new theory, and usually succeeded. Then, having done so, he sought to distill the new theory to its essence, casting off distracting or unnecessary formalisms. And ever-present in Conway's work and writings is a characteristic wit, a sense that he *had fun* in the process. He later reflected:

> I used to feel guilty in Cambridge that I spent all day playing games, while I was supposed to be doing mathematics. Then, when I discovered surreal numbers, I realized that playing games *is* mathematics.

Conway's work on games serves as a reminder that great mathematics can rise from unexpected quarters, and it stands as one of his most unique and fascinating achievements.

Louis H. Kauffman

John would sit you down and show you a magic trick and then proceed to teach you how to do it (if you could follow such things). John would sit you down and give you a short lucid lecture on skein theory (as he did to many of us in the 1970s) and you would think—this is clever. And it changed your life.

That is, in short, my experience of John Conway.

I met him for the first time in the 1970s, when he was visiting Vera Pless at the University of Illinois at Chicago

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(where I taught since 1971 and am now emeritus). He gave a talk about the Alexander polynomial and how it could be computed by 'skein theory'—recursions involving only the knot and link diagrams. No other machinery was needed. The way topologists learned about the Alexander polynomial involved structures like the Fox Free Differential Calculus, homology groups, modules, the infinite cyclic covering space of the complement of the knot, the fundamental group, group presentations, a special arsenal of algebraic topology. This magician explained to us how to obtain the Alexander polynomial from pure combinatorics that a high school student could understand. (In fact, it was said that he had discovered this when he was in high school.) He further explained that he had told us this back in 1969, but very few had listened. Somehow we began to listen this time.

I met him again in Ann Arbor, Michigan, where I was visiting, and he told us a little more about the skein. I asked him questions that were very hard and looked opaque to the skein approach and he would say "It is all in the skein." And I would believe him. Eventually, I found a model for what he told us, using the work of Seifert from the 1930s and then I found another model using the work of Alexander from the 1920s. Alexander's work rearranged itself in my mind and became a special combinatorics that was also analogous to certain structures in statistical physics (partition functions), and I wrote a book about those new ways to handle the Alexander polynomial [3]. But I did not suspect the depths that were right there, and neither did Conway. In 1983, Vaughan Jones (led forward by ideas from von Neumann algebras and statistical physics) discovered a new knot invariant that had a skein identity that was a slight variation of Conway's identity for the Alexander polynomial. Conway could have discovered it if he had asked what would happen if one changed a coefficient or a sign in his formula for the Alexander polynomial! At that point many people, including myself, jumped upon this idea and a new era of knot theory began. There was the insight into physics and the Conway insight into diagrams and recursion. They have led us in a devil's dance for all the years since that time. The world changed.

John was a deep mathematician and a performing magician who wanted to involve you in being surprised by his magic, and he wanted to involve you in the production and design of that magic. He was an innovator and a master teacher. He would work very hard to bring a bit of mathematics to the point where it was highly non-trivial and yet could be communicated in a few strokes to almost anyone. This is how it is with the Game of Life, the skein theory, the rope tricks, the audio-activity, the surreal numbers and games and many other things that he loved. This labor of love, turning mathematics into creative magic was the most characteristic aspect of John Horton Conway. It will live forever in the hearts of everyone that he touched.

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Credits

Figure 1 is courtesy of Dierk Schleicher.

Remembering M. S. Narasimhan (1932–2021)

Arnaud Beauville, Oscar García-Prada, Nigel Hitchin, Chandrashekhar Khare, Shrawan Kumar (Editor), Herbert Lange, Nitin Nitsure, M. S. Raghunathan, T. R. Ramadas, and S. Ramanan

Narasimhan was born on June 7, 1932 in Tandarai, a small village in Tamil Nadu with hardly any infrastructure. After his early education in a rural part of the country, he joined Loyola College in Madras for his undergraduate education. Narasimhan joined the Tata Institute of Fundamental Research (TIFR), Bombay, for his graduate studies in 1953. He obtained his PhD in 1960 (granted by Bombay University) under the supervision of K. Chandrasekharan (a well-known number theorist).

His initial area of focus at TIFR was partial differential operators. During this time, he visited France under the invitation of Laurent Schwartz and was exposed to the works of other French mathematicians including Jean-Pierre Serre, Claude Chevalley, Élie Cartan, and Jean Leray. Following some fundamental works of D. Mumford, he started to study the moduli of vector bundles on curves (with Seshadri and later with G. Harder and S. Ramanan) where he made pioneering contributions. His most important ground-breaking work jointly with C. S. Seshadri is known as the *Narasimhan-Seshadri theorem* giving a topological characterization of stable vector bundles on smooth projective curves, thus building a bridge between topology and algebraic geometry. With Harder, he introduced what is known as the *Harder-Narasimhan filtration of vector bundles*, which is of fundamental importance in a variety of topics. Some of his later works are explained in the following pages.

On the administrative side, he became the first chair of the National Board for Higher Mathematics and introduced many initiatives. In the early 1990s, after retiring from TIFR, he decided to join the International Centre for Theoretical Physics in Trieste and worked for over ten years as the head of its mathematical division.

He was an inspiring advisor and guided a number of graduate students including: K. Gowrisankaran, M. S. Raghunathan, S. Ramanan, M. K. V. Murthy, V. K. Patodi, G. A. Swarup, R. R. Simha, R. Parthasarathy, S. Kumaresan, T. R. Ramadas, N. Nitsure, S. Subramanian, and F. Coiai; many of whom went on to become outstanding mathematicians in their own right.

He won several awards in India as well as internationally. Among others, he was recipient of the Shanti Swarup Bhatnagar Award (1975); Third World Academy of Sciences Prize for Mathematics (1987); Srinivasa Ramanujan Medal (1988); Chevalier de Ordre National du Mérite of France (1989); and King Faisal International Prize for Science (2006) (jointly with S. Donaldson). He was a Fellow of all three of the Indian Academies of Science. He was elected Fellow of the Royal Society in 1989. He was awarded Padma Bhushan in 1990 (India's third highest civilian honor).

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Figure 1. Family photo: with son Mohan, daughter Shobhana, and wife Sakuntala, January 2020.

Apart from mathematics, he liked Modern Art (particularly Impressionism), Carnatic music, and modern Tamil literature.

He continued to be mathematically active till literally the last day of his life. He wrote a paper with Gallego and García-Prada which appeared on the ArXiv on May 13, 2021. Battling with cancer, he passed away on May 15, 2021.

S. Ramanan

Recently M. S. Narasimhan, one of the finest mathematicians in India, passed away. Coming on the heels of the sad demise of another stalwart C. S. Seshadri, it has been a big loss. I had the good fortune to have had close friendship with both, academically as well as socially. Narasimhan was my PhD advisor, colleague, and collaborator for over three decades.

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Narasimhan was born in a small village with hardly any infrastructure. There was no high school there and not even any bus service to the nearest town where he went to study. Narasimhan often recalled how he had to travel to school by bullock cart. Although his family was reasonably well to do, his father passed away early and they had to manage with some difficulty.

His performance at the final examination at school was excellent, and he got into Loyola College, one of the highly rated colleges in Madras at the time. He was very happy with the standard of teaching he received there. The head of the mathematics department was a Jesuit, Father Racine. Unlike most professors of those days, Fr. Racine was au courant with contemporary mathematics. Although he was not rated a great lecturer, he took keen interest in students of high calibre and encouraged them to go into research. There was another lecturer, Professor Krishnamurthy who taught Real Analysis, and Narasimhan was appreciative of his efforts to inculcate a deep interest in the subject.

Fr. Racine was in touch with the development of the incipient Tata Institute of Fundamental Research (TIFR) in Bombay, and he recommended bright students to do their research at the fledgling institution. Accordingly, some students like Seshadri and Narasimhan, became graduate students at TIFR. The school of mathematics there had as its head, Professor K. Chandrasekharan (KC), who realized the best way to introduce modern mathematics to the Indian scene. KC invited top mathematicians from all over the world, and asked each to give introductory lectures on a current subject. One of the graduate students was assigned the task of writing up the notes. The Fields medalist Laurent Schwartz, known for his theory of distributions, was one of the invitees. The note taker of his lectures was Narasimhan. This proved to be fortuitous, for Schwartz was impressed with the keenness and ability of Narasimhan, as well as a few other graduate students. When Schwartz returned to France, he persuaded some of his students, like Jacques Lions, Bernard Malgrange, etc. to visit TIFR and give courses. On his visit, Lions posed a question connected with the limits of partial differential operators on manifolds, and Narasimhan solved the question in the affirmative.

KC soon realized the high quality of the graduate students and felt that there was no one to mentor them in India. Consequently, with the help of Schwartz, he delegated Narasimhan, among a couple of others, to visit France for a few years and work on such topics. In Paris, Narasimhan came in contact with a Japanese graduate student, Keisuke Kotake, and proved a nice result on elliptic operators, in collaboration with him.

Unfortunately, he contracted pleurisy while in Paris and had to be hospitalized. He looked upon it not as a disaster, but as an opportunity to be with "real Parisians." He also told me that his spoken French improved manyfold as a consequence.

On his return to India, he obtained his doctorate and soon was appointed Associate Professor. It is remarkable that he could start advising students for their doctorate so early in his career.

I was then a graduate student at TIFR and he warmed up to me when he came to know that I was conversant with the work of Kodaira and Spencer. I became his first student and we soon wrote up a paper, "Universal Connections," where we proved that the classifying space for principal bundles with compact structure group also had a connection which was universal for bundles with connections. This was very well received, and later became useful for many developments in differential geometry as well as in theoretical physics.

Seshadri and he jointly wrote a few papers in which the theory of vector bundles over a smooth projective algebraic curve was the main thrust and soon it culminated in what is now known as "the Narasimhan-Seshadri theorem." This provided a deep understanding of the classification of vector bundles over curves and led to many later developments. Narasimhan occupied himself with the study of the moduli space so constructed, and it was my good fortune to be able to work jointly with him in this enterprise.

A distinguishing feature of Narasimhan's research was his ability to come to grips with questions, even in an area in which he had no previous expertise, and bring new ideas to the questions, and then solve them. He would fill in the details later, often running a seminar on the topic. This explains the versatility of his research, and his propensity to collaborate with different types of mathematicians, from young graduate students to famous achievers. Thus, apart from his work with Indian colleagues like Ramadas, Seshadri, Simha, and myself, he worked jointly with Kotake, Harder, Beauville, Hirschowitz, Lange, Okamoto, and a host of others. Much of this research spawned new directions, which are still of value decades later. But even more remarkable is the fact that these collaborations span a wide range of fields including Number theory, Differential Equations, Differential Geometry, Lie groups, Algebraic Geometry, and even Theoretical Physics.

Mathematics was not his only interest. He liked Modern Art, particularly Impressionism, thanks to his French connection. He was also fond of books. We used to frequent the Strand Book Stall and buy books. He liked to read contemporary Tamil books. Even after he retired and settled in Bengaluru, he tried to time his visits to Chennai in order to buy books at the Annual Tamil Book Fair.

On the administrative side, when I was the dean of the school of mathematics at TIFR, there were efforts elsewhere to institute a mathematical establishment along the lines of the Council of Scientific and Industrial Research. Since 'Higher Mathematics' came under the Department of Atomic Energy, with the active help of the then Chair of the Atomic Energy, the National Board for Mathematics (NBHM) was established. Narasimhan became the first Chair of NBHM and it was my pleasure to be its first secretary and collaborate with him in this endeavor as well. He introduced many initiatives and NBHM has now become the principal funding agency for mathematical research in India. In the 90s, after decades of research and mentoring, he decided to join the International Centre for Theoretical Physics in Trieste and worked for over ten years as the head of its mathematical division. There he mentored students which, he mentioned to me, gave him great satisfaction as he always wished to make some contribution to science in the developing world.

He returned to India in the late 1990s. Although he had retired, he was full of ideas pertaining to moduli space and its generalizations. Apart from mathematics, he was interested in Carnatic (South Indian classical) music, as well as modern Tamil literature. However, his deep interest in mathematics never waned. A few months after he contracted the cursed disease, he wrote to me that our work had new ramifications and that they are now talking about 'Narasimhan-Ramanan branes'! Very close to his final days, he even helped a student of our Spanish friend Oscar García-Prada with some ideas.

Fortunately, his stature did not go unrecognized, in India as well as internationally. He was awarded Padma Vibhushan, the Bhatnagar prize and was a Fellow of all three Institutes of Science. He was elected Fellow of the Royal Society, was honored with Chevalier d'Ordre du Merite of France, the Abdul Kalam prize, and so on.

I am fortunate to have been his close friend till the very end.

Nigel Hitchin

My first encounter with Narasimhan was at a conference in July 1974 in Durham, England. The bus taking us on an excursion had broken down outside a pub (not serving beer because of restricted licensing hours in those days) so we sat at tables outside. I had given a talk about vanishing theorems in differential geometry which are often called Weitzenböck formulae and he asked me if I had read Weitzenböck's 1923 book *Invariantentheorie*. I had

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skimmed through it in the library but he told me to look at the Foreword and take the first letter from each sentence. It spells

N-I-E-D-E-R-M-I-T-D-E-N-F-R-A-N-Z-O-S-E-N.

"Down with the French!" Not that he shared that view (Weitzenböck was living in the French-occupied Rhineland at the time) because he followed the comment with interesting accounts of his experiences in the Paris of Serre, Cartan, and Schwartz which was clearly a formative experience for him. His talk at the symposium was about moduli of vector bundles on curves. At that time I took little notice, but somewhat later our interests converged much more.

In the intervening years it was more a question of action at a distance. When I got interested with Michael Atiyah in Yang-Mills theory and instantons in the late 1970s, he would send his younger collaborators to Oxford to find out what was going on, and then I got to know his entourage better. But it was the 1983 paper of Atiyah and Bott [AB] and Simon Donaldson's first paper [D] in the same year which attracted my attention. It brought gauge theory down from four to two dimensions and applied it to the famous theorem of Narasimhan and Seshadri about stable vector bundles on curves. Framed as it was in terms of moment maps and symplectic geometry it led to my investigation into what are now called Higgs bundles. I sent Narasimhan a preprint of my paper [H] and I subsequently found that we shared a common view on what were the interesting offshoots from this, though our motivations might well have been different.

To put it in context, the 1965 theorem of Narasimhan and Seshadri asserts that if a holomorphic bundle E on a compact Riemann surface C is stable—an open condition relating to its subbundles—then it admits a projectively flat unitary connection. A Higgs bundle is an extension of this idea, considering a pair of holomorphic objects (E, Φ) where $\Phi : E \to E \otimes K$ is holomorphic, K being the canonical line bundle, and the stability condition must hold only for Φ-invariant subbundles. In this case a theorem asserts that there is a unitary connection A with curvature F_A such that $F_A + [\Phi, \Phi^*] = 0$. When $\Phi = 0$ one recovers the original theorem but for other choices one can prove, for example, the uniformization theorem for Riemann surfaces, and also describe flat non-unitary connections with this data. My approach was differential-geometric but Narasimhan sent his young colleague Nitin Nitsure to Oxford who helped develop an algebro-geometric version, and the interaction was most useful for us both.

One feature, which had little to do with the equations, but arose naturally when I was writing the paper, was the use of the spectral curve – the covering of C defined by the characteristic equation $\det(x - \Phi) = 0$. It provided a way of constructing a Higgs bundle from a line bundle on the spectral curve. My treatment in the paper was rather ad hoc (though I had read about direct images as a student) but, together with Beauville and Ramanan, Narasimhan gave a much better account [BNR] which led to the consideration of non-abelian theta functions through spectral curves. Subsequent work in India especially, with Narasimhan looking on, clarified many aspects of the theory.

Over the past 30 years we saw each other on many occasions. A memorable one was his joint 60th birthday conference with Seshadri at the Tata Institute. But we met in Trieste for committees, which often ended in mathematical discussions in a restaurant overlooking the sea, and also amongst the community of researchers into vector bundles on curves who held conferences around the world. Many of these encounters were in Spain where Oscar García-Prada organized meetings to capitalize on the expertise from the Indian school of algebraic geometry. The familiar dark-suited figure was always a welcome sight, often accompanied by his long-time collaborator Ramanan. I valued both his knowledge about specific points and his view of the changing trends in mathematics and how one could react to them. His presence will be greatly missed.

Chandrashekhar Khare

The Narasimhan-Seshadri theorem as inspiration. I met Professor M. S. Narasimhan just a few times spread out over a couple of decades. The last few times were in the city of Bangalore where he then lived, at the prize ceremonies of the Infosys Science Foundation, and also at a Commonwealth Science Congress. He had a natural charisma, presence, intensity, an aura around him for me because of his renown.

Much to my regret, I did not get to know him well, either personally or mathematically. In this tribute to him, I will focus on his role as an inspiring figure—because of the importance of the work he had done, where he had done it, and when he had done it—to a young person (like myself) trying to do research in pure mathematics in India in the 1990s.

TIFR had established itself in the world of pure mathematics through important theorems proved by mathematicians working there, through the decades from the 1950s onwards, and of these there was none more celebrated than the Narasimhan-Seshadri theorem. It was in an area of algebraic geometry and differential geometry that was

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Figure 2. Passport photo of Narasimhan, around 1974.

far away from my area of work which was in number theory.

I had come back to India immediately after my thesis in 1995, and joined TIFR as a Visiting Fellow (the entry level postdoctoral position available to someone after finishing their PhD).

I did not know the mathematical content of the theorem. Many of my senior colleagues at TIFR worked in different aspects of the mathematical specialty—vector bundles on curves—that had been decisively impacted by the Narasimhan-Seshadri theorem. It continued to be the focus of much of the research done at TIFR decades after the theorem had been proven.

For me, the influence of the Narasimhan-Seshadri theorem was more indirect but still psychologically quite important. The discovery of such an influential theorem by two young brilliant Indian mathematicians, in their early thirties, working in Bombay at TIFR in the 1960s, led to putting TIFR on the world map of mathematics. It also made one feel that as someone working at the same Institute, one had to try and live up to the high repute of the place, live up to that theorem in a way. Further it gave a sense that it was possible to do first-class mathematics working at a place somewhat distant from the traditional, mainly Western, centers of mathematical research, especially as their work had been done in Bombay in times when the world was far less connected than now.

I read later in an interview with Narasimhan that he felt it could be advantageous to work somewhere at a distance from the main mathematical centers, and follow the latest developments from this distance. One could then work on one's own ideas, partly inspired by the work happening at these centers, without being overwhelmed by the influence of the leaders in the subject. Such direct influence, while it could be greatly beneficial for some, might dissuade others from trying out ideas that could seem unpromising to the experts, but that one could not give up on internally.

Although I did not know Professor Narasimhan's quote at that time, I found for myself that this awareness from a distance of important developments in one's area, while working independently on one's own, worked quite well for me in the roughly 10 years I spent working at TIFR.

After arriving in India in 1995, I worked on trying to generalize the work I had done in my thesis, which used ingredients that overlapped with the astonishing work of Andrew Wiles on Fermat's Last Theorem. I arrived in a faltering way at a satisfactory generalization of my thesis work over a period of a few years after I returned to India. Temperamentally I am drawn to tilting at windmills, and I started doing, more or less simultaneously, more open-ended work inspired by Serre's modularity conjecture. Particular cases of this conjecture had been used by Wiles in his work on Fermat's Last Theorem, but the general case of the conjecture was wide open.

I kept musing about questions suggested by Serre's conjecture, carrying them in my mind, experiencing mainly frustration, but also small eureka moments, making small observations that I wrote up as short papers. It was a little like kicking the ball around on a field, with the goalposts obscured by a thick fog. As there was no way of reasonably aiming to kick at the goal which was smothered in the fog, one just kicked the ball around and chased after it in bursts of somewhat random, sporadic, but still intense, activity.

Later, I read another piece of Professor Narasimhan's advice to young people, which was to work "off the top": work on something without necessarily knowing precisely all the background required, getting by on a sense of the subject, impressionistic to begin with, which could be deepened as one continued thinking (continually!) about the subject. This way one would not get bogged down and overwhelmed by the myriad technical details right at the beginning, which could have a paralyzing effect on a novice, and instead learn them as one needed to.

On reading Professor Narasimhan's interview, I realized that I had been unconsciously following his advice of working "off the top" all along. In my work, I reached my "natural boundary" (as defined by Eilenberg, and referenced by Narasimhan in the interview) very quickly, and could go past it only by obsessing about a piece of mathematics and living with it in my mind.

The fact that the Narasimhan-Seshadri theorem had been proven when India was still young as an independent country (not yet twenty years old) was also fascinating. TIFR was founded by Homi Bhabha who had filled it with paintings by contemporary Indian artists (through the 1950s and 1960s), many of them working in Bombay, not so far away from Navy Nagar where TIFR is located. The works of the members of the Bombay Progressive Artists' Group were amply present on the walls of the Institute. The modernity the paintings represented, made in a newly independent country, which had its own ancient culture and tradition of art, architecture, music, and dance, married these civilizational influences with what was happening in the contemporary art world then. Many of the artists whose paintings Bhabha collected (with discernment and a remarkable sense or intuition for what was vital in the art made in India then) had spent time in Paris and returned to produce work which was influenced by what they had absorbed in their time there. Narasimhan and Seshadri also had been deputed to Paris, from 1957 to 1960 as I learnt from interviews of Professor Narasimhan, absorbed new ideas and influences there, and after returning to India proved their landmark theorem.

It was a different India that I lived in during my years working at TIFR (post the economic liberalization of 1991), but the example of Professor Narasimhan, Professor Seshadri, and their colleagues, who had worked at TIFR and proved path breaking theorems decades earlier, lived on as an inspiration, present in the air, setting a certain tone, holding me and my colleagues accountable, pushing us to try and live up to their formidable legacy.

Nitin Nitsure

I knew M. S. Narasimhan for nearly four decades, first as my thesis advisor, and then as a friend. The first volume of the collected papers of Narasimhan [N] begins with two excellent review articles on his research corpus—an unsigned article, and an article by C. S. Seshadri. I have addressed the story of the Narasimhan-Seshadri theorem in my recent memorial article on Seshadri [Ni] where there is also some material on Narasimhan. The videos of the memorial meetings for Narasimhan in Mumbai and Bangalore [NMM] in June 2021 have a lot of interesting material, including reminiscences by many mathematicians from different parts of the world, together with a rapid account of Narasimhan's entire mathematical career. But beyond his celebrated research contributions and his substantial role in nurturing and running institutions, Narasimhan had a deep impact on a large number of younger people with whom he interacted, and I am going to focus on the aspects of his persona and his behaviour that helped bring this about.

When Narasimhan became my thesis advisor, he was about twice my age, and was a distant and formidable figure. This was in early 1983, when I was a graduate student in the School of Mathematics, Tata Institute of Fundamental Research (TIFR), Mumbai. The first problem he gave me was to construct a relative Picard space in the holomorphic category—a problem that had been posed by Grothendieck—and then left me alone till I came back with a solution two months later. Unfortunately, my effort was wasted as it turned out that the result had already been proven by Bingener. This was indeed a disappointment for me, but it must have given Narasimhan some confidence in my ability. After that, our interactions became more frequent, and he gave me another interesting problem. My thesis was completed in 1986, but we continued to discuss mathematics. Before I went to Oxford for a postdoc in 1987, he asked me to look at Hitchin's latest papers. Somewhat later he said that I must study Grothendieck's FGA. These suggestions turned out to be very important for me. The student-teacher relationship was central and lifelong in the ancient Indian scholarly tradition (as in Indian musical, spiritual, or artisanal traditions), and traces of that attitude have clearly endured. Over the decades as the ratio between our ages improved, we became good friends, which continued till the end.

In the 1980s, Narasimhan was much feared not just by most students and postdocs, but also by some of the younger faculty members. He did not mix easily with others, and even a casual observer would have noticed his unrelenting single-minded seriousness, and refusal to indulge in any small talk. So it may appear surprising at first sight that a continuous stream of talented students did their PhD with Narasimhan, and became his lifelong well-wishers and friends. I will say more about this later, but let me note that for all his apparent lack of sociability, he was perfectly well mannered when he actually engaged with anybody, if the other party kept to the business at hand. He was a patient listener who rarely interrupted others, but allowed others to interrupt him quite easily, which is certainly unusual among people of his eminence. When I asked him about it many years later, he said that it is because he already knows his own thoughts, but is curious to know what the other party thinks! I never saw him either raise his voice or lose his composure in the nearly four decades that I knew him. Once when a disgruntled faculty member known for a bad temper raised his voice while arguing with Narasimhan, who was then the Dean, Narasimhan is known to have calmly said "You do not appear to be in the right frame of mind for a discussion now, please come back later." The other person left without further argument.

The rapid emergence of modern mathematics in TIFR in the 1950s owes a lot to the active support of French mathematicians such as Laurent Schwartz and like-minded colleagues. These were idealistic leftists, some of whom had strong revolutionary beliefs as followers of Lenin and Trotsky. They believed not just in universal functorial properties within mathematics but in universal human values, and took the trouble to spend months at a time visiting India, which must surely have appeared as a backward, poor, dirty, and inconvenient place to them, to help spread modern mathematics. (Interestingly, the undergraduate teacher who introduced Narasimhan and Seshadri to modern mathematics was a French Jesuit missionary called

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Father Racine, another practitioner of universal values.) Narasimhan said that not just his mathematics, but his understanding of political matters, developed rapidly during his three year postdoctoral visit to Paris during 1958–1961. He became a lifelong leftist.

What practical shape did Narasimhan's socialist ideology take? This has a simple answer: he made it his mission to nurture mathematical excellence at the highest level in all parts of humanity. He believed in the universality of mathematical talent, and was certain that among the teeming millions of the underprivileged all over the world, of all races, nationalities, religions, and genders, there were a huge number of potential top mathematicians, who needed to be detected and educated. But he did not think that a talented mathematician should, for example, go to a slum to teach school dropouts, since that job can be done as well by many others. What only a front line mathematician can do is to mentor talented young mathematicians who have somehow survived and risen from backward places or countries. They need help at the doctoral and postdoctoral stage to get into the really interesting directions of research and make a lasting contribution to mathematics, otherwise their talent is in danger of being wasted. Consistent with this, soon after his retirement from TIFR in 1992 at the age of 60, Narasimhan took up the position of the Head of Mathematics at the International Center for Theoretical Physics (ICTP), Trieste, at the invitation of its founder-director Abdus Salam. Numerous workshop participants and postdoctoral visitors come to ICTP every year from all over the developing world. Narasimhan engaged with a large number of them, and he had the uncanny ability to suggest interesting problems and directions which would suit their ability and inclination. With age, his social style also changed—he became a relaxed, reassuring figure that the young would gather around (which surprised some TIFR people who only knew his younger avatar). Finding doable interesting problems for the young from very diverse areas is not easy. Narasimhan made the necessary effort to read current literature in different areas and consulted experts. You will find many first-rate mathematicians from all over the world who say they owe him a lot for giving them a research problem, a useful idea, or simply a nudge in the right direction at a crucial stage in their career.

When it came to institutions, Narasimhan was not a revolutionary, but a true conservative. He always emphasized how difficult it is to build institutions and how easy it is to destroy them. So even here, stability (or at least semi-stability) was of paramount importance to him. Narasimhan put up with the many imperfections that he saw around him, and kept his focus on his chosen mission: to do top-level mathematical research himself and to help others do it. His integrity shone through all of his actions. As one of the large number of young mathematicians that he helped and influenced, I will say that he has earned our undying admiration and gratitude.

M. S. Raghunathan

I saw Narasimhan for the first time some time in my first year as a graduate student at TIFR in 1960–1961. He was dressed formally in a suit, which to a student of my background, suggested a stiff, unapproachable persona. I was soon disabused of that perception, when one day he stopped me in the corridor to compliment me on my proof of a result that was making the rounds in the School of Mathematics.

In my third year, I attended a seminar in Differential Geometry run by Narasimhan and Ramanan (then a senior student). Towards the end of the seminar Narasimhan, Ramanan, and I had frequent informal discussions. In these interactions, many of them over coffee, I learnt a great deal, and not just mathematics. My understanding of political and social issues came to acquire some sophistication. Narasimhan had an abiding interest in Tamil literature and I learnt many things about contemporary Tamil writing from him. In the course of a few walks along the seashore in TIFR, he explained to me the entire Kodaira Spencer Deformation Theory of Complex Structures. He was an outstanding teacher, excelling, especially in one-on-one communication of mathematical ideas.

He suggested a problem (connected with the Kodaira Spencer theory) for me to work on. When I eventually solved it he told me that the work was adequate for a PhD thesis and asked me to register for the degree with him as the thesis advisor, which I did. He interceded with the University of Bombay to get the waiting time for the submission of the thesis reduced to two years from the mandatory three. He got me invited to speak at the prestigious International Mathematical Colloquium (on Differential Analysis) held in Bombay in January 1964, and made me rehearse my talk with him. That resulted in a well-received lecture, surprising colleagues, who knew of my poor track record as a speaker.

Around this time I toyed with the idea of quitting mathematics to join my father in the family business. Narasimhan got wind of this and when he happened to meet a friend of my family, told her that my quitting would be a loss for mathematics. This reached my family soon,

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Figure 3. With M. S. Raghunathan in Bangalore, January 2019.

and they promptly put an end to my idea of changing my vocation. Narasimhan's role in my career thus went well beyond that of a mentor in mathematics. And this would also be true for many of his other students.

After I wrote my thesis, my mathematical interests moved away from his and our mathematical interactions were of a general nature and not intense. Nevertheless each of us had a pretty good idea of what the other was doing in mathematics. Narasimhan's approach to mathematics was very French, Bourbakian, in fact. Riemann and Poincaré headed the pantheon of the greats whom he admired. Among his contemporaries Kodaira held pride of place in his admiration and somewhat later Grothendieck joined him, or maybe, even displaced him. He once told me that the idea of the proof of the Narasimhan-Seshadri theorem was inspired by Poincaré's method of "balayage." He told me that Grothendieck was amazing, saying something to the effect that any time you come up with a problem in the subject, you find that Grothendieck has already said something profound about it.

In 1966, I was inducted into the Mathematics Faculty of the Tata Institute of Fundamental Research. From day one, Narasimhan treated me as an equal, but it was necessarily an asymmetric relationship; there was no way I could forget that Narasimhan was my teacher. Our views on most issues were almost identical, but when we differed, I would invariably defer to his views. Our mathematical interactions continued, but were less frequent and less intense as the focus of my mathematical interests had drifted away from his.

Narasimhan took considerable interest in promoting mathematics in the country at large. Already in the sixties he served on many committees of universities and other institutions of higher learning. He was of course one of the principal architects of the rise of the School of Mathematics of TIFR from a minor player to international eminence in mathematics. In the early seventies, Jawaharlal Nehru University wanted to start a Department of Mathematics with him as the Head. He was not averse to the idea, perhaps because he wanted to build another centre rivaling the TIFR school in the country. Fortunately for TIFR, the project did not materialize and Narasimhan continued to guide the School for another 20 years.

When in 1983, the DAE set up the National Board of Higher Mathematics, an agency for the promotion of mathematics, he was the natural choice to head it. Under his leadership, the Board took many initiatives which went a long way towards fulfilling its mandate. As a fellow faculty member and a member of NBHM, I had the good fortune of observing him at close quarters and I learnt a great deal about science administration. He was a stickler for dignified and correct conduct on all occasions; he would visibly wince when someone failed on that front. He planned and conducted meetings meticulously. His decisions were always taken after considerable thought. His letters were written with great care; he not only wanted to say the right things but wanted them said the right way.

During his ICTP days, my contacts with him were few and far between. I paid a few short visits to ICTP at his invitation and they were very enjoyable. After he retired to Bangalore, the contacts, which were mostly on the telephone, became more regular.

Narasimhan—unlike many men of comparable stature—was accessible; and interaction with him, whether professional or otherwise, was easy and pleasant; his style though was not the deliberate informality of the corporate world. He had wide interests, beyond mathematics, and had interesting things to say on a variety of subjects. His sartorial preferences were conservative: I have never seen him in jeans and he was seen in a T-shirt only rarely. He was not averse to dressing formally if the occasion demanded it. But he was no conservative on social and political issues. He was of a strongly leftist persuasion and was unequivocal in his condemnation of the caste system and the Hindu right.

In recent years he was much distressed at the decline, in our country, of the values that he held dear; mathematics, his magnificent obsession, kept him from sinking into greater despondency. The manner in which he kept up with recent developments in mathematics right till the end, was truly amazing. He was of course a professional mathematician, yet his excitement with mathematics was reminiscent of the little boy on the seashore in Newton's famous self-assessment.

His passing away is a great personal loss to me: he was a close friend and mentor.

T. R. Ramadas

Narasimhan had an abiding curiosity about most aspects of mathematics. He firmly believed that mathematics has an internal dynamic, and that intelligent and insightful interrogation yields the deepest insights of the subject. But he was very open to insights that came from allied fields, particularly Physics.

I joined TIFR in 1977 as a graduate student in theoretical physics, with a good training from the Indian Institute of Technology, Kanpur. I knew of the work of Aharanov-Bohm, Wu-Yang, and others on the geometric aspects of gauge theories. The quantization of these theories involves an integration over the space of "gauge potentials," i.e., space of connections on a principal bundle on space-time. This space is infinite-dimensional, and this leads to the usual, and still largely unresolved, mathematical problems of quantum field theory. In the case of gauge-theories, there is an additional complication, due to the fact that the integrand is invariant under the infinite-dimensional group of gauge-transformations, i.e., automorphisms of the principal bundle. This is dealt with by "gauge-fixing." The Russian physicist V. N. Gribov had pointed out that there were ambiguities in this procedure, and speculated on physical consequences of this.

Although Narasimhan is best known as an algebraic geometer, his early training was as a complex geometer and analyst, and he had a deep knowledge of both the analytical and formal aspects of differential geometry, in particular the theory of connections on bundles. The theorem on universal connections due to Narasimhan and S. Ramanan is a fundamental insight. (The second of their joint papers on the subject contains an elegant proof of the basic result in Chern-Weil theory. This is now the standard proof, and rarely credited to its discoverers.)

I explained to Narasimhan my rather naive understanding of these matters. Narasimhan very quickly brought all the geometry into focus. He insisted on the "correct" analytical setting for infinite-dimensional geometry, and our work contained the earliest construction of the space of connections as an infinite-dimensional principal bundle modeled on a suitable Sobolev space. "Gauge-fixing," taken literally, would mean choosing a section for this bundle, and the point is that it is rarely trivial, except possibly in the case of connections on a principal bundle with abelian structure group. This had been proved independently and earlier by I. M. Singer, whose paper appeared after we finished our manuscript.

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In the early sixties, Narasimhan and Seshadri had made the fundamental discovery that stable vector bundles (an algebro-geometric notion introduced by D. Mumford) on a compact Riemann surface correspond to irreducible unitary representations of its fundamental group (possibly with a "puncture," depending on the degree of the bundle). The study of these moduli spaces was a major preoccupation of Narasimhan in the next two decades. His work with S. Ramanan and later with G. Harder provided a template for much of the later work on moduli.

In the early eighties, Atiyah and Bott realized that the Theorem of Narasimhan and Seshadri could be interpreted as an infinite-dimensional version of Kempf-Ness theory, with the curvature of a connection interpreted as the moment map. Their starting point was an investigation of a toy version of gauge theory, with a Riemann surface replacing space-time, and the norm (squared) of the moment map playing the role of "action functional."

All this was "classical mechanics." In the late eighties and nineties, physicists realized that certain quantum field theories in two and three space-time dimensions had as their quantum "state spaces," spaces of (holomorphic) sections of line bundles on moduli spaces of vector bundles on Riemann surfaces. It is fair to say that this revealed aspects of linear series on these spaces that were entirely new. In analogy with the classical case of Jacobians, these are called generalized theta functions.

Narasimhan was particularly intrigued by the beautiful formulae derived by E. Verlinde for the dimensions of these linear series on these moduli spaces. These matters remained a major preoccupation from then on.

Narasimhan and J.-M. Drézet developed the basic theory of the theta bundle on moduli spaces of vector bundles of arbitrary rank and degree. Narasimhan and I then worked out a proof of the Verlinde formula in purely algebro-geometric terms. We had to give a careful construction of parabolic moduli spaces on singular curves, (which we did à la Simpson), definition of theta bundle thereon, a vanishing theorem (in a context where Kodaira vanishing could not be immediately applied), and finally a geometric proof of "factorization."

It is worth taking stock of what the physics ingredient was in these matters. In general terms, the realization that there was a rich structure to the theory of generalized theta functions. Specifically, first the insight that degeneration techniques and the incorporation of "parabolic structures" made possible an inductive expression for the dimensions. Second, Verlinde's ingenious introduction of his algebra

Figure 4. With wife Sakuntala and daughter Shobhana, Bombay, 1964.

and its use in deriving an explicit formula. Third, the role of Kac-Moody groups (clarified largely through the work of Tsuchiya-Ueno-Yamada) in the context.

Narasimhan enlisted S. Kumar and A. Ramanathan in the first of a series of works that carefully elucidated the relationship between the definitions of conformal blocks in algebra-geometric terms and in terms of loop groups. These papers, technically difficult and carefully written, remain standard references.

In more recent times, Narasimhan remained engaged in developments in moduli theory, and followed the work on Bridgeland stability particularly closely. His last works were devoted to derived categories of coherent sheaves on his beloved moduli spaces, using the Hecke transform, a pioneering tool invented with Ramanan decades earlier.

Arnaud Beauville

My interest in vector bundles on curves was triggered by the work of Narasimhan and Ramanan. In the early 80s, I was interested in the Schottky problem. Recall that one associates to a curve C of genus g a complex torus J of dimension g, the *Jacobian variety*, which one can view as parameterizing line bundles L of degree $g-1$ on C (by choosing one of these as the origin). Then the locus Θ of those line bundles which admit a nonzero section is a hypersurface in J, the *Theta divisor*. The pair (J,Θ) is what we call a *principally polarized abelian variety* – p.p.a.v. for short.

As soon as $g \geq 4$, the p.p.a.v.'s depend on more parameters than the Jacobians; the *Schottky problem* asks for a characterization of Jacobians among all p.p.a.v.'s (A,Θ). There was a flurry of activity around this in the early 80s; most approaches involve the linear system $|2\Theta|$ (that is, the projective space of divisors linearly equivalent to 2Θ). For $a \in A$, the divisor $\kappa(a) := (\Theta+a)+(\Theta-a)$ belongs to this linear system; the map $\kappa : A \to |2\Theta|$ embeds the *Kummer variety* $\mathrm{Km}(A) := A/i$ into $|2\Theta|$, where i is the involution $a \mapsto -a$ of A. One of the approaches to the Schottky problem characterizes Jacobians by the existence of trisecants to their Kummer variety in $|2\Theta|$.

I was studying these questions when I discovered that Narasimhan and Ramanan had found a remarkable connection between $|2\Theta|$ and rank 2 vector bundles. Let $\mathcal{M}$ be the moduli space of semi-stable, rank 2 vector bundles on C with trivial determinant. Given $E \in \mathcal{M}$, the locus of line bundles $L \in J$ such that $E \otimes L$ admits a nonzero section is an element $\theta(E)$ of $|2\Theta|$; we thus get a map $\theta : \mathcal{M} \to |2\Theta|$, which maps the singular locus of $\mathcal{M}$ exactly onto the Kummer variety. In the beautiful paper [NR], Narasimhan and Ramanan work out completely the genus 3 case: θ is an embedding, and its image turns out to be the unique quartic hypersurface in $|2\Theta| \cong \mathbf{P}^7$ singular along $\mathrm{Km}(J)$. This hypersurface had been discovered by Coble long ago through algebraic manipulations; its geometric interpretation via vector bundles was entirely new.

This result inspired me to study the map $\theta : \mathcal{M} \to |2\Theta|$ in higher genus [B]. I noticed that there is a kind of duality between J and $\mathcal{M}$: given $L \in J$, one defines a divisor Θ_L on $\mathcal{M}$ as the locus of vector bundles E such that $E \otimes L$ has a nonzero section; the line bundle $\mathcal{L} = \mathcal{O}_{\mathcal{M}}(\Theta_L)$ does not depend on L, and there is a natural isomorphism $|\mathcal{L}|^* \xrightarrow{\sim} |2\Theta|$ which identifies θ with the map $\varphi_{\mathcal{L}}$ defined by the global sections of $\mathcal{L}$.

For vector bundles of rank $r \geq 3$, the analogous statements make sense (with a rational map $\mathcal{M} \dashrightarrow |r\Theta|$), but the proof in rank 2 is not directly adaptable. After a few months I found a way to do it. Shortly after I met Narasimhan and Ramanan at the AMS Summer conference on theta functions (in 1987), we realized we had had the same ideas—using the Hitchin fibration and the notion of very stable vector bundle introduced by Drinfeld. So we decided to write the joint paper [BNR]. Narasimhan invited me to the Tata Institute; I spent one month there in 1988. By that time, we had essentially finished the paper, but I had many lively discussions with Narasimhan, both mathematical and non-mathematical—he was a very cultured man, with interesting points of view on a wide range of subjects.

The key ingredient in [BNR] was the computation of the dimension of the space of global sections $\Gamma(\mathcal{L})$—often called "generalized theta functions." At about that time, word spread among mathematicians that physicists had a formula for the dimension of $\Gamma(\mathcal{L}^{\otimes k})$ for all k, in a much more general setting including for instance moduli spaces of G-bundles for all semi-simple groups G. This

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Verlinde formula soon became a challenge for algebraic geometers, and half a dozen proofs appeared in the following years, including one by Narasimhan with Kumar and Ramanathan [KNR] and one by Laszlo and myself [BL]. Our (independent) proofs were actually quite close: both papers used infinite-dimensional algebraic geometry, with the language of infinite-dimensional manifolds in [KNR] and of algebraic stacks in [BL].

I had no other opportunity to collaborate with Narasimhan. We met briefly in a few conferences; on one occasion he came to Nice, we had a nice lunch with a very pleasant conversation. Narasimhan liked good food, art, and music. He was a great mathematician and a colleague of high human quality.

Shrawan Kumar

I am saddened by Professor Narasimhan's passing away. It is a personal loss to me and a great loss for the mathematical community and all associated with him. I hold him in the utmost respect not only for all the beautiful mathematics he created (of which I have learnt only a tiny bit), but his integrity, administrative capability, and care for all those associated with him.

On a personal note, I vividly remember that one day while I was a graduate student, he called me to his office and explained a problem to me. But more importantly, he told me how to go about working on a problem. To this day, I try to follow his advice when I think about a problem. He was the one who introduced me to the problems surrounding the Verlinde formula, which became among one of the important projects I pursued for a while (mostly jointly with him and also some with A. Ramanathan and A. Boysal) and recently I wrote a book on the subject. I visited Professor Narasimhan at I.C.T.P. during 1994. He was very warm and caring.

I had the privilege of writing two papers with Professor Narasimhan, which I briefly describe here. Let Σ be a smooth projective irreducible s-pointed ($s \geq 1$) curve of any genus $g \geq 0$ with marked points $\vec{p} = (p_1, \dots, p_s)$ and let G be a simply connected simple algebraic group with Lie algebra $\mathfrak{g}$. We fix a positive integer ℓ called the *level* and let P_ℓ be the set of dominant integral weights of $\mathfrak{g}$ of level at most ℓ. We attach weights $\vec{\lambda} = (\lambda_1, \dots, \lambda_s)$ (each $\lambda_i \in P_\ell$) to the marked points $\vec{p}$ respectively. Associated to the triple $(\Sigma, \vec{p}, \vec{\lambda})$, there is the space $\mathcal{V}_\Sigma^\dagger(\vec{p}, \vec{\lambda})$ of *conformal blocks* (also called *space of vacua*), which is a certain finite dimensional space of $\mathfrak{g} \otimes \mathbb{C}[\Sigma \setminus \vec{p}]$-invariants of a tensor product of s-copies of integrable highest weight modules with highest weights $\vec{\lambda}$ and level ℓ of the affine Kac-Moody Lie algebra $\hat{\mathfrak{g}}$ associated to $\mathfrak{g}$. This space is a basic object in Rational Conformal Field Theory arising from the Wess-Zumino-Witten model associated to G. Now, E. Verlinde gave a remarkable conjectural formula for the dimension of $\mathcal{V}_\Sigma^\dagger(\vec{p}, \vec{\lambda})$ in 1988. This conjecture was "essentially" proved by a pioneering work of Tsuchiya-Ueno-Yamada [TUY], wherein they proved the *Factorization Theorem* and the *invariance of dimension* of the space of conformal blocks under deformations of the curve Σ, which allow one to calculate the dimension of the space of conformal blocks for a genus g curve from that of a genus $g-1$ curve. Thus, the problem gets reduced to a calculation on a genus 0 curve, i.e., on $\Sigma = \mathbb{P}^1$. The corresponding algebra for $\Sigma = \mathbb{P}^1$ is encoded in the fusion algebra associated to $\mathfrak{g}$ at level ℓ, which gives rise to a proof of an explicit Verlinde dimension formula for the space $\mathcal{V}_\Sigma^\dagger(\vec{p}, \vec{\lambda})$.

Classical theta functions can be interpreted in geometric terms as global holomorphic sections of a certain determinant line bundle on the moduli space $\mathrm{Pic}^{g-1}(\Sigma)$ of line bundles of degree $g-1$ on Σ (a smooth curve of genus g). This has a natural non-abelian generalization, where one replaces the line bundles on Σ by principal G-bundles on Σ to obtain the parabolic moduli space (or stack) $\mathrm{Parbun}_G(\Sigma)$ and certain determinant line bundles over $\mathrm{Parbun}_G(\Sigma)$. Holomorphic sections of these determinant line bundles over $\mathrm{Parbun}_G(\Sigma)$ are called the generalized theta functions (generalizing the classical theta functions). The Verlinde dimension formula attracted considerable further attention from mathematicians and physicists when it was realized that the space of conformal blocks admits an interpretation as the space of generalized theta functions. This interpretation was rigorously established in the "non-parabolic" case in my joint work with Narasimhan-Ramanathan (for general G) [KNR]. It was independently established around the same time by Faltings (for general G) and Beauville-Laszlo (for the special case $G = \mathrm{SL}_n$); and in the case of parabolic space by Pauly (for the special case $G = \mathrm{SL}_n$) and for parabolic stacks by Laszlo-Sorger (for general G).

In a second joint paper with Professor Narasimhan, we proved that the moduli *space* $M_G(\Sigma)$ of semistable G-bundles over a smooth irreducible projective curve Σ (of any genus) is Gorenstein and has its Picard group isomorphic with the group of integers [KN], thus generalizing the corresponding result for $G = \mathrm{SL}_n$ by Drezet-Narasimhan. We also proved the vanishing of higher cohomology of $M_G(\Sigma)$ with coefficients in positive line bundles.

It is simply amazing that inspite of all the pain and suffering he was going through towards the end of his life,

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he wrote a paper with Gallego and García-Prada which appeared on the ArXiv on May 13, 2021, just a couple of days before he passed away. It is a testimony to his utmost devotion to mathematics!

Oscar García-Prada

As a graduate student in Oxford in the late 1980s, working under the supervision of Nigel Hitchin and Simon Donaldson, I was very much influenced by the theorem of Narasimhan and Seshadri, and the important generalizations that were inspired by this theorem around that time. Published in 1965, this theorem captures the interconnection between various branches of geometry, topology, and theoretical physics, and was the basis for later fundamental works by numerous mathematicians, including Atiyah, Bott, Donaldson, Uhlenbeck, Yau, Hitchin, Simpson, and many others. After briefly recalling the theorem of Narasimhan and Seshadri, I will comment here on some generalizations related to representations of the fundamental group of a compact Riemann surface, with particular reference to works that are close to my own interests and research.

Upon his return from Paris to the Tata Institute for Fundamental Research (TIFR) in Bombay in 1960, Narasimhan embarked on an intense collaboration with Seshadri that resulted in their famous theorem. Inspired by some remarks in the 1938 paper of A. Weil on "Généralisation des fonctions abéliennes," in 1961–1962, Narasimhan and Seshadri started looking at unitary vector bundles. A unitary representation ρ of dimension n of the fundamental group of a compact Riemann surface X defines a holomorphic vector bundle E_ρ of rank n and degree 0, which is referred to as a *unitary vector bundle*. This is called an *irreducible unitary vector bundle* if ρ is irreducible. They showed that the infinitesimal deformations of a unitary vector bundle E_ρ as a holomorphic bundle can be identified with the infinitesimal deformations of the representation ρ. From this, they deduced that the set of equivalence classes of unitary vector bundles had a natural structure of a complex manifold, and were able to compute the expected dimension.

A breakthrough came with the work of Mumford on Geometric Invariant Theory. In the 1962 International Congress in Stockholm, he introduced the notion of stability of a vector bundle on a compact Riemann surface, and proved that the set of equivalence classes of stable bundles of fixed rank and degree has a natural structure

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Figure 5. Narasimhan with Oscar García-Prada at ICMAT, Madrid, 2017.

of a non-singular quasi-projective algebraic variety, projective if the rank and degree are coprime. After Narasimhan and Seshadri became aware of Mumford's work, the relation with unitary bundles was clear to them. They proved that an irreducible unitary bundle is stable. For arbitrary degree they showed that the stable vector bundles on X are precisely the vector bundles on X which arise from certain irreducible unitary representations of suitably defined Fuchsian groups acting on the unit disc and having X as quotient.

The result that they proved in [NS] can be easily reformulated as saying that a holomorphic vector bundle over X is stable if and only if it arises from an irreducible projective unitary representation of the fundamental group of X. From this, one deduces that a reducible projective unitary representation of the fundamental group corresponds to a direct sum of stable holomorphic vector bundles of the same slope, where the slope of a vector bundle is the quotient of its degree by its rank, (what is nowadays referred as a *polystable* vector bundle). One can observe that the projective unitary representations lift to unitary representations of a certain *central extension* of the fundamental group of X. The gauge-theoretic point of view of Atiyah and Bott [AB], using the differential geometry of connections on holomorphic bundles, and the new proof of the Narasimhan–Seshadri theorem given by Donaldson [D] following this approach, brought new insight and new analytic tools into the problem. In this approach, a projective unitary representation of the fundamental group is the holonomy representation of a unitary projectively flat connection.

A very natural question to ask is whether there is a holomorphic interpretation of representations of the fundamental group of X in $\mathrm{GL}(n,\mathbb{C})$ that are not unitary. The answer to this required the introduction of new holomorphic objects on the Riemann surface X called *Higgs bundles*. These objects, introduced by Hitchin in [H], are pairs (E, Φ) consisting of a holomorphic vector bundle E over X and a homomorphism $\Phi : E \to E \otimes K$, where K is the canonical bundle of X. There is a notion of stability similar to that of vector bundles, and corresponding moduli spaces. The correspondence between stable Higgs bundles and irreducible representations of the fundamental group of X (or its universal central extension if the degree is different from zero) in $\mathrm{GL}(n, \mathbb{C})$ was proved in the above mentioned paper by Hitchin [H] for $n = 2$ and by Simpson (1988) for arbitrary n (and in fact, for higher dimensional Kähler manifolds). The correspondence in the case of $\mathrm{GL}(n, \mathbb{C})$ needed an extra ingredient—not present in the compact case—having to do with the existence of twisted harmonic maps from X into the symmetric space $\mathrm{GL}(n, \mathbb{C})/\,\mathrm{U}(n)$. This theorem was provided by Donaldson (1987) for $n = 2$ and by Corlette (1988) for arbitrary n (who also proved it for higher dimensional compact Riemannian manifolds).

It turns out that the theory of Higgs bundles is also central in the study of representations of the fundamental group of X in non-compact real forms $G_{\mathbb{R}} \subset \mathrm{GL}(n, \mathbb{C})$. Indeed, the case of the *split* real form $G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{R})$ (more precisely $\mathrm{SL}(n, \mathbb{R})$) was studied by Hitchin (1992). Using Morse-theoretic techniques he counted the number of connected components of the moduli space. He also identified special components, now known as *Hitchin components*, whose representations, as shown by Labourie (2006) using concepts from dynamical systems, have similar properties to those in the Teichmüller space of the surface, regarded (Goldman, 1980) as a topological component of the moduli space of representations in $\mathrm{SL}(2, \mathbb{R})$.

The case of the pseudo-unitary groups $G_{\mathbb{R}} = \mathrm{U}(p, q)$ with $p \neq 0 \neq q$ and $p + q = n$ is in a sense closer to the case of the unitary group $\mathrm{U}(n)$ studied by Narasimhan and Seshadri, since these real forms are inner equivalent to $\mathrm{U}(n)$. This situation was investigated by Bradlow and Gothen in collaboration with the author [BGG]. Here, a Higgs bundle (E, Φ) corresponding to a representation in $\mathrm{U}(p, q)$ is of the form $E = V \oplus W$, where V and W are holomorphic vector bundles of rank p and q respectively, and Φ, in terms of this decomposition, has zeros in the diagonal. There is a topological invariant, called the *Toledo invariant*, defined as $\tau = 2\frac{aq - bp}{p+q}$, where a and b are the degrees of V and W respectively, for which the semistability of (E, Φ) implies the so-called *Milnor–Wood inequality* $0 \leq |\tau| \leq 2\min\{p, q\}(g - 1)$. In [BGG] it is proved that for any value of the degrees a and b so that τ satisfies the Milnor–Wood inequality the moduli space of stable $\mathrm{U}(p, q)$-Higgs bundles is non-empty and connected. This is very much in contrast with the case of $\mathrm{U}(n)$ and $\mathrm{GL}(n, \mathbb{C})$, for which there are no constraints on the topological invariant, and for which for any value of the degree there exists a non-empty connected component. Moreover, when n is even and $p = q$, the representations in the component with maximal Toledo invariant have properties similar to those in the Hitchin component, as shown by Burger–Iozzi–Labourie–Wienhard (2005). The study of these components and the Hitchin components is part of the content of the recent field of *higher Teichmüller theory*.

To complete the list of real forms of $\mathrm{GL}(n, \mathbb{C})$, when n is even one can consider the group $\mathrm{U}^*(n)$, the non-compact dual of $\mathrm{U}(n)$. This has been treated by Oliveira and the author (2011). In this case a Higgs bundle (E, Φ) is such that E is equipped with a holomorphic symplectic structure, with respect to which Φ is symmetric. Using the Morse-theoretic techniques introduced by Hitchin, the main result proved here is that the moduli space of $\mathrm{U}^*(n)$-Higgs bundles is non-empty and connected.

I first met Narasimhan quite soon after having completed my doctoral thesis in 1991. From the very beginning, he was very kind to me, and extremely generous in the exchange of ideas. Our mathematical and personal friendship grew over the years and we had the opportunity to meet many times in Europe and India. He visited our institute in Madrid on several memorable occasions, including Nigel Hitchin's 60th birthday conference and S. Ramanan's 70th birthday conference, as well as a conference in his honor on the occasion of his 80th birthday.

In addition to discussing mathematics and Indo-European collaboration schemes, Narasimhan and I very much liked to enjoy a glass (or two!) of good red wine, very often in company of our common friend and collaborator Ramanan, and other good friends. I last saw Narasimhan in person in Bangalore in February 2020, during a meeting at the International Centre for Theoretical Sciences (ICTS). After the ICTS meeting, I went for few days to Chennai for a visit at the Chennai Mathematical Institute (CMI), where as a matter of fact I saw C. S. Seshadri for the last time. During the last year of Narasimhan's life we were very actively in contact working on a joint project with him and my student Guillermo Gallego on a generalization of the Hitchin system. A paper on this work [GGN] saw the light just a few days before his passing.

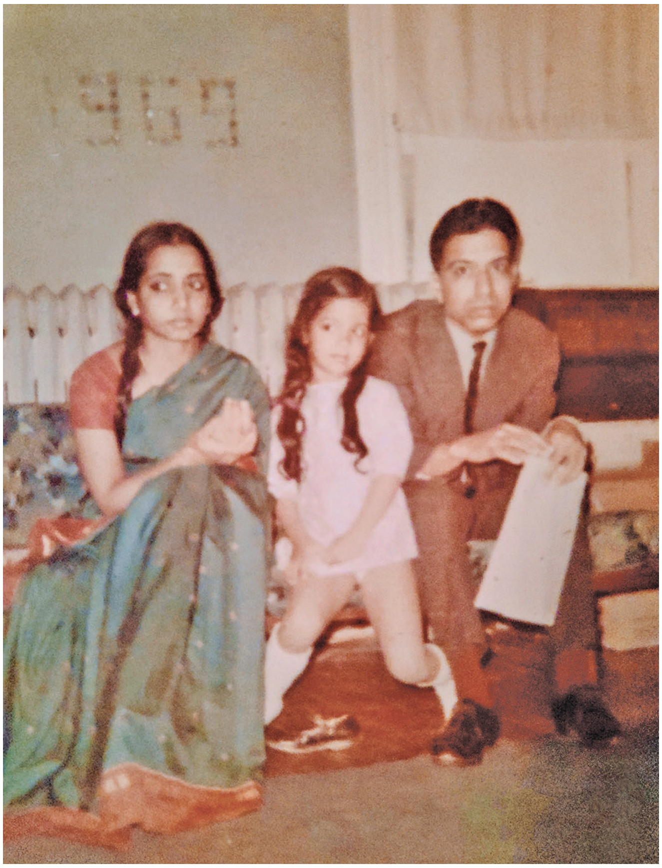

Figure 6. With wife Sakuntala and daughter Shobhana, 1969.

Herbert Lange

First of all I would like to say that it always was a great pleasure to work with Narasimhan. We met many times, at TIFR, in Chennai and Bangalore, at ICTP, in Erlangen, and at many conferences. Whenever we met, of course we mainly discussed Mathematics, but also other subjects, even personal ones.

I got to know Narasimhan as a very generous person, not only in mathematics. He was not just a colleague, but also a friend. Although he certainly was the better mathematician, he never let me feel it. Apart from publishing two papers together, the main results of which I will describe below, he also had an influence on some of my other papers. Moreover, some of the results of our discussions we did not publish.

First we met in Nice, where I spent some months and he a whole year and where we shared an office. Soon we found a problem, on vector bundles on curves, which led to our joint paper [LN1], the main results of which are as follows:

Let X be a smooth irreducible curve of genus g over an algebraically closed field of characteristic 0. For any vector bundle E of rank 2 over X define the invariant $s(E) := \deg E - 2\max\deg(L)$ where the maximum is taken over all line subbundles of E. E is stable if and only if $s(E) \geq 1$ and it is well known that $s(E) \leq g$. Denote by $M(E)$ the subscheme of $\mathrm{Pic}(X)$ formed by the maximal subbundles of E. Maruyama proved that $\dim M(E) = 1$ if $s(E) = g$ and conjectured that $M(E)$ is finite whenever E is not of the form $L \oplus L$ and $s(E) \leq g-1$. Our main result is the following

Theorem.

$s(E) = 2$: *If* $\deg E \equiv 0 \mod 2$, *then* $\dim M(E) = 0$ *for every* E *of rank* 2 *if and only if* X *is not double elliptic. In the double elliptic case every double elliptic cover yields a* g*-dimensional subspace of the moduli space of stable* E *of rank* 2 *with* $\dim M(E) = 1$.

$s(E) = 3$: *For every* $g \geq 4$ *there is a curve* X *of genus* g *which admits a vector bundle* E *with* $s(E) = 3$ *and* $\dim M(E) = 1$.

In each case, explicit examples are given. The method of proof is to translate the problem into a problem of projective geometry: namely to determine curves of genus g and degree 2g in the projective space $\mathbb{P}^g$ and points in $\mathbb{P}^g$, not on the curve, through which infinitely many secant lines of the curve pass. The maximal subbundles correspond to the secant lines passing through the point.

Our second paper was written during a visit of Narasimhan in Erlangen. This time the subject was not vector bundles, but abelian varieties.

According to a classical theorem of Lefschetz the n-th power of an ample line bundle of an abelian variety is very ample for any $n \geq 3$. The paper [LN2] deals with the analogous question for the second power of an ample line bundle. The results are not new, however, most of the proofs are. So for an ample line bundle L on X consider the map

$$\Phi = \Phi_{L^2} : X \to \mathbb{P}^N = P(H^0(L^2)).$$

To state the main theorem, let

$$(X, L) = (X_1, L_1) \times \cdots \times (X_s, L_s)$$

denote the decomposition of the polarized abelian variety (X, L) into a product of irreducible polarized abelian varieties. Suppose that (X_ν, L_ν) for $\nu = 1, \dots, r$ are principally polarized and for $\nu = r+1, \dots, s$ are not principally polarized. For $\nu \leq r$ let $K_\nu = X_\nu / \pm id$ denote the Kummer variety of X_ν with canonical projection $p_\nu : X_\nu \to K_\nu$.

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Define $K = K_1 \times ... K_r \times X_{r+1} \times \cdots \times X_s$ and $p = p_1 \times \cdots \times p_r \times id_{X_{r+1}} \times \cdots \times id_{X_s}$. So Φ factorizes as

$$\begin{tikzcd} X \arrow[rr, "\Phi"] \arrow[dr, "p"'] & & \mathbb{P}^N \\ & K \arrow[ur, "\psi"'] & \end{tikzcd}$$

with a holomorphic map ψ. The main result is,

Theorem. *ψ is an embedding.*

So Φ is of degree 2^r onto its image. In particular, if none of the (X_i, L_i) is principally polarized, Φ is an embedding. On the other hand, in the case of an irreducible principally polarized abelian variety, Φ embeds the Kummer variety. The main contributors of this topic are Wirtinger, Andreotti-Mayer, Sasaki, Ramanan and Ohbuchi.

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Credits

Figures 1–4 and 6 are courtesy of Shobhana Narasimhan. Figure 5 is courtesy of Oscar García-Prada.

The STaR Program for New Doctorates in Mathematics Education is in its Second Decade

Barbara Reys and Robert Reys

In 2010 the first cohort of STaR Fellows gathered in Park City, UT. The program, now in its second decade, is designed as an induction experience for doctoral graduates in mathematics education who are in their first- or second-year appointment in an institution of higher education. It was initially modeled on Project NExT, a successful program for mathematicians entering careers in institutions of higher education. The STaR Program was funded for its first four years by the National Science Foundation (Grant 0922410), but thanks to continuing contributions from individuals and philanthropic groups, it continues under the oversight of the Association of Mathematics Teacher Educators (AMTE). This article provides an update on the program that derives its name (STaR) from the important work of higher education faculty in mathematics education: Service, Teaching and Research.

The STaR Program is intended to serve as many new mathematics educators in higher education as possible. However, there are generally more applicants that can be accommodated – about 25–35 people each year. Participants are about equally distributed between those with academic appointments in mathematics departments and colleges/schools of education. Originally the STaR Program coincided with the Park City Mathematics Institute so that some speakers and experiences could be shared. The STaR Program continues to meet for one week annually in Park City, except in 2020 and 2021 when it was done remotely.

What happens during the STaR institute? Prior to the five-day institute, selected participants are asked to identify their teaching and research interests. While research interests vary, most fall into areas such as teacher knowledge/beliefs, teacher preparation, student learning, instructional materials/curriculum, and equity/diversity. Teaching interests/responsibilities include courses (content and/or methods) for elementary, middle, or secondary teacher education candidates (undergraduate and graduate level).

The summer institute consists of plenary sessions on teaching, research, and service that are led by a staff of experienced mathematics educators. In addition to the plenary sessions, special interest groups are established that focus on research and teaching. For example, participants teaching courses (content or methods) that are targeted toward middle school teachers meet to discuss what they are doing, share syllabi, discuss challenges they face, and exchange ideas about teaching these courses. These special interest groups continue to dialogue during the year as they teach comparable courses. In a similar manner, research groups allow STaR Fellows to meet with other mathematics educators sharing similar research interests and goals. The work of these groups often results in STaR Fellows designing and conducting joint research efforts across multiple institutions, and sometimes this work evolves into proposals for funding to support their collaborative research.

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Each participant also shares at least one of their academic manuscripts with other participants and staff. These manuscripts are then reviewed and discussed within a small cohort group, providing valuable feedback. Continued work on these manuscripts has resulted in many scholarly publications by the STaR Fellows, some of which are coauthored with other STaR Fellows.

Fireside Chats are optional informal conversations hosted by experienced mentors/staff about topics of current interest and held during specified times outside of the Research, Teaching, and Service-related sessions they all attend. Fellows can choose among different available chats to attend or can do their own thing during these times. There is also free time during the institute that allows for continued interaction among Fellows on topics of mutual interest.

The personal networking established during the institute and continued throughout the years following the institute is a particular strength of the program, as many STaR Fellows work in isolation at their home institution (e.g., they may be the only mathematics educator in their department). Fellows also meet at the next annual meeting of AMTE following the institute, providing additional opportunities for networking and collaboration.

The STaR Program is designed and delivered by senior faculty members in mathematics education, usually led by co-organizers. In addition to the co-organizers, 3–5 other senior level mathematics educators serve as staff to mentor Fellows.

The success of the initial funding of the STaR Program by NSF served as a catalyst to build, launch, and establish the program. The commitment of AMTE to continue oversight of the program has allowed it to become established as part of the culture of the profession. STaR Program leadership has regularly changed to ensure that the STaR Program is vibrant and does not become dependent on one or two people. Table 1 reports the organizers that provided major leadership in shaping the experiences for the STaR Fellows.

Where did the STaR Fellows graduate? Since 2010, a total of 399 STaR Fellows have completed the program. These STaR Fellows earned their doctorate in mathematics education from 102 different institutions. The ten institutions producing the most STaR Fellows are shown in Table 2. STaR Fellows graduated from each of the Big 10 institutions and a total of 37 institutions that are members of the American Association of Universities (AAU). Most of the STaR Fellows earned their doctorate from AAU institutions.

Where are the STaR Fellows employed? STaR Fellows are employed in 247 different institutions of higher education across 45 states. The ten institutions hiring the most STaR Fellows are shown in Table 2. About 50 percent of the STaR Fellows are employed in mathematics departments. Nine STaR Fellows are now employed in private enterprise, K–12 school districts, or in institutions of higher education in other countries.

Year(s)	Co-Organizers
2010–13	Barbara & Robert Reys, University of Missouri
2014–15	Denise Spangler, University of Georgia; Jeff Wanko, Miami University
2016	Denise Spangler, University of Georgia; Karen Hollebrands, North Carolina State University
2017	Karen Hollebrands, North Carolina State University; Jeffrey Shih, University of Nevada-Las Vegas
2018	Jeffrey Shih, University of Nevada-Las Vegas; Keith Leatham, Brigham Young University
2019	Keith Leatham, Brigham Young University; Beth Herbel-Eisenmann, Michigan State University; Marta Civil, University of Arizona
2020	Beth Herbel-Eisenmann, Michigan State University; Marta Civil, University of Arizona; Maria Fernandez, Florida International University
2021	Maria L. Fernandez, Florida International University; R. Judith Quander, Houston University
2022	R. Judith Quander, Houston University; Mathew Felton-Koestler, Ohio University
2023	Mathew Felton-Koestler, Ohio University; Dorothy White, University of Georgia

Table 1. Leadership organizers for the STaR Program.

Table 3 shows the Carnegie Basic Classification of institutions of higher education in the United States where the STaR Fellows are currently employed. More specifically, Table 3 shows that slightly over one-half of the STaR Fellows are employed in doctoral granting institutions, and about 40 percent are employed in regional institutions offering a range of degrees.

How to become a STaR Fellow? The current application process requires a vita, a letter of recommendation from the applicant's doctoral advisor, a letter of support from current department chair or dean, and a personal statement describing current work and how participation in the STaR Progam will advance their career. The complete application process is on the AMTE website at `https://amte.net/star/apply`.

Institutions graduating the most STaR Fellows from 2010–2022	#	Institutions employing the most STaR Fellows	#
Michigan State University	24	University of Alabama	8
University of Georgia	24	North Carolina State University	7
North Carolina State University	19	Kennesaw State University	7
University of Missouri	16	University of Northern Iowa	7
University of California, Berkeley	13	Georgia Southern University	6
Indiana University	12	Appalachian State University	5
Stanford University	11	East Carolina University	5
University of Delaware	11	University of South Carolina	5
Illinois State University, University of Michigan, University of Northern Colorado	9	Georgia Southern University, Iowa State University, James Madison University, Montclair State University	4
San Diego State University, University of California, San Diego	9	San Diego State University, Texas State University, University of Nebraska, West Chester University	4

Table 2. The ten institutions graduating the most STaR Fellows and the ten institutions employing the most STaR Fellows.

How is the STaR program currently supported? NSF funded the first four years of the program producing 150 STaR Fellows. Since that time, a four-part strategy has been used to continue support of the program:

- Administrative/organizational oversight facilitated by AMTE.
- Donations from a variety of professional organizations (state, regional, and national), foundations and companies.
- Donations from individual members of the mathematics education community, including STaR Fellows and mid- and senior-level faculty.
- Travel support to attend the summer institute is provided by the home institution of each Fellow.

Carnegie Basic Classification of Institutions	#	%
Doctoral Universities: Very High Research Activity	127	32.6
Doctoral Universities: High Research Activity	86	22.1
Doctoral/Professional Universities	40	10.3
Master's Colleges & Universities: Larger Programs	87	22.3
Master's Colleges & Universities: Medium Programs	12	3.1
Master's Colleges & Universities: Small Programs	12	3.1
Baccalaureate Colleges: Diverse Fields	11	2.8
Baccalaureate Colleges: Arts & Sciences Focus	10	2.6
Baccalaureate/Associate's Colleges: Mixed Baccalaureate/Associate's	1	0.0
Associate's Colleges: High Transfer-Mixed Traditional/Nontraditional	4	1.0

Table 3. Percent of the 390 STaR Fellows currently employed in institutions of higher education reflecting their Carnegie Basic Classification.

The organizations/foundations/companies that have supported the STaR Program include: American Statistical Association (ASA); AERA Special Interest Group/Research in Mathematics Education (SIG/RME); Brookhill Foundation/Mathematics Institute of Wisconsin; Connected Mathematics Project/Michigan State University; College Preparatory Mathematics; Educational Advancement Project; National Council of Teachers of Mathematics (NCTM); Psychology of Mathematics Education-North America (PME-NA); and state organizations including the Association of Mathematics Teachers Educators-Texas (AMTE-TX); Association of Maryland Mathematics Teacher Educators AMMTE; Georgia Mathematics Teachers Educators (GAMTA); Kentucky Association of Mathematics Teacher Educators (KAMTE); Michigan Association of Mathematics Teacher Educators (MI-AMTE); Mississippi Association of Mathematics Teacher Educators (MAMTE); Pennsylvania Association of Mathematics Teacher Educators (PAMTE); Hoosier Association of Mathematics Teacher Educators

(HAMTE); and the Utah Association of Mathematics Teacher Educators (UAMTE). For those interested in providing support for the STaR Program, see: `https://amte.net/civicrm/contribute/transact?reset=1&id=13`.

What is the future of the STaR Program? Each generation of scholars in a discipline has a responsibility to help educate and prepare their successors in the discipline – to serve as stewards of the discipline (3). The STaR Program is an effort to help initiate the next generation of mathematics educators in institutions of higher education, providing support for them to develop networks that can help them launch and establish successful and productive careers. We appreciate the vision of the National Science Foundation for supporting the establishment of the STaR Program, the many groups that have provided financial support, the mathematics educators that have served as organizers and institute staff, and the commitment of AMTE for continuing the effort. Donations, along with continued service through leadership of the program, will ensure the continuation of a vibrant, successful, and remarkably far-reaching induction program for higher education faculty in mathematics education.

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Robert and Barbara Reys

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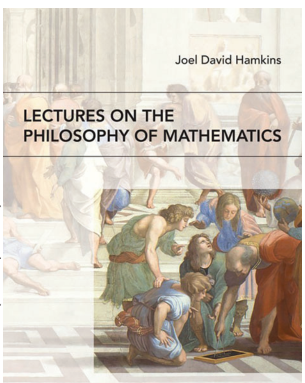
Photo of Robert and Barbara Reys is courtesy of Robert and Barbara Reys.

BOOK REVIEW

Lectures on the Philosophy of Mathematics

Reviewed by Daniel J. Velleman



Lectures on the Philosophy of Mathematics
Joel David Hamkins

Most mathematicians probably view mathematics as a subject in which we establish eternal truths with absolute certainty. What is there for philosophers to do, when contemplating mathematics, other than stand back and admire its beautiful perfection? And yet, when one looks more closely, one discovers that mathematics is full of philosophical mysteries.

For example, what are numbers? No one has ever seen one. (Of course, you can see the mark "5" on a piece of paper, but that is no more the number five than I am the sequence of letters "Dan.") G. H. Hardy, in his *A Mathematician's Apology*, famously claimed that mathematical objects do not belong to physical reality, but rather to "another reality, which I will call 'mathematical reality'" [Har92, p. 123]. What is the nature of this reality?

If mathematical objects belong to a reality that is separate from physical reality, how is it that their properties are so useful for understanding physical reality? As Eugene Wigner put it, how do we explain "the unreasonable effectiveness of mathematics in the natural sciences" [Wig85]?

How do we discover facts about mathematical reality? It appears that we do it by pure thought. But how is it that, just by thinking, we can discover properties of mathematical reality? Perhaps mathematical objects exist only in our minds. But if so, does each of us have our own, private

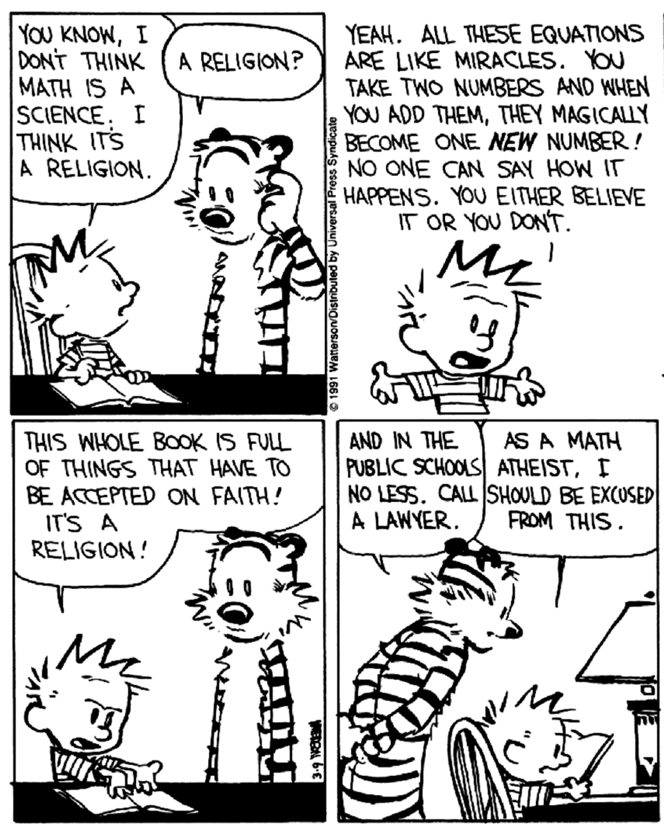


Figure 1. Calvin and Hobbes, by Bill Watterson.

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version of mathematical reality, and are they different? And if not—if, as Hardy believed, "mathematical reality lies outside us" [Har92, p. 123]—how does thought give us access to that reality?

And what are we to make of mathematical statements, such as the continuum hypothesis, that are known to be neither provable nor disprovable? Does the continuum hypothesis have an objective truth value? If not, does that mean that mathematical reality has an indeterminate quality? But if it has an objective truth value, then it seems to be a truth value that neither our senses nor our thoughts give us access to. Belief in such truth values begins to seem more like religious faith than science; perhaps Calvin is right about mathematics!

In the preface to his book *Lectures on the Philosophy of Mathematics*, Joel Hamkins says his aim is "to present a mathematics-oriented philosophy of mathematics." This orientation is reflected in the organization of the book into eight chapters, each focusing on a different mathematical topic: numbers, rigor, infinity, geometry, proof, computability, incompleteness, and set theory. In each chapter, Hamkins presents mathematical results of foundational significance, but these mathematical results lead naturally to discussions of a variety of philosophical issues and positions.

For example, in the chapter on numbers, Hamkins sketches constructions of the different number systems of mathematics. He introduces the logicist program of Gottlob Frege and explains how Frege tried to define the natural numbers. For the case of the real numbers, he describes constructions using both Dedekind cuts and equivalence classes of Cauchy sequences. Of course, these two constructions yield isomorphic structures. One way to see this is to observe that both constructions lead to a complete ordered field, and all complete ordered fields are isomorphic. These mathematical results motivate the philosophical position of *structuralism*, according to which, for example, there is nothing more to be said about what the real numbers are other than that they are a complete ordered field. This characterizes the real numbers up to isomorphism, which is all that mathematics requires. For the structuralist, the purpose of the constructions of the real numbers is not to identify what the real numbers "really" are—there is no such thing—but rather to give the existence half of the proof that there exists a unique (up to isomorphism) complete ordered field. Hamkins discusses several versions of structuralism, advocated by various philosophers. Most mathematicians will find some version of structuralism appealing, but in my view it requires a background theory, such as set theory, that is strong enough to allow us to prove the necessary existence and uniqueness theorems. Thus, it does not completely resolve the philosophical problems associated with defining numbers; rather, it pushes those problems back to the background theory.

In Chapter 2, Hamkins tells the story of how calculus was made rigorous by the introduction of the ϵ-δ definition of limits, and how this led to the understanding of important subtle distinctions in analysis, such as the difference between convergence and uniform convergence. The fact that calculus is used in almost all branches of science leads naturally to the philosophical argument, put forward by Hilary Putnam and Willard Van Orman Quine, that the indispensability of mathematical objects for science is evidence for the existence of those objects. Hamkins then describes the opposing argument by Hartry Field that mathematical objects are not actually indispensable for science, but rather can be considered to be convenient fictions.

In the next chapter Hamkins presents Cantor's theory of different sizes of infinity, the infinite ordinal and cardinal numbers, and the continuum hypothesis. Cantor's work brings up two philosophical questions. The first is whether infinity should be regarded as actual or potential; that is, whether infinite collections can be treated as completed totalities, or whether it is better to think of the infiniteness of a collection as merely the unending potential to produce more and more elements of the collection. This distinction was introduced by Aristotle, who believed all infinity was potential. Cantor's work was initially controversial, in part because he treated infinity as actual. The second question is whether a proof of the existence of a mathematical object should actually produce the object asserted to exist; that is, whether existence proofs should be constructive. It is sometimes claimed that Cantor's proof of the existence of transcendental numbers is nonconstructive, but, as Hamkins explains, a careful analysis of the proof shows that it is actually constructive.

Chapter 4 describes Euclid's axiomatic approach to geometry. What is the purpose of geometry? Is it to study what can be proven from Euclid's axioms, or perhaps what can be constructed by straightedge and compass? Or is it to discover the truth about space? Hamkins discusses the views of Kant and Hume about geometry. The story of the discovery of non-Euclidean geometry leads to the modern view, as expressed by Poincaré, that all geometries are equally mathematically correct. No geometry can be picked out as the one true geometry, although physical experiments might determine that one geometry is most useful as a representation of physical space.

The subject of mathematical proof is taken up in the next chapter. Hamkins explains several theorems of formal logic, such as the soundness and completeness of classical first-order logic. And he addresses a number of philosophical questions about proofs. What is the purpose of a mathematical proof? Is it merely to certify that a theorem is true, or is it to explain why it is true? Which is more convincing, a formal proof of a theorem, every step of which can (and must!) be checked, or an informal proof that conveys an understanding of why the theorem is true? As

Hamkins points out, the answer to this last question may change as proof assistant software becomes more powerful and its use becomes more widespread. (Proof assistants are programs, such as Isabelle, Coq, and Lean, that can help a mathematician write a formal proof and verify that the proof is correct. Further information about proof assistants can be found in [Avi18, Hal08].) Hamkins also discusses alternatives to classical logic, such as intuitionistic logic, although he does not explain in detail how one must revise many mathematical theorems if one pursues mathematics using intuitionistic rather than classical logic. For more on this, see [GV02] or [Hey71].

Chapter 6 is devoted to computability theory. Hamkins presents the Turing machine model of computation, explains the undecidability of the halting problem, and introduces the study of Turing degrees and complexity theory. He also discusses the Church-Turing thesis, which says that the Turing model accurately captures the intuitive notion of computability. Might a computing machine constructed to take advantage of some strange quantum mechanical or relativistic effect be able to go beyond the limits of Turing computability?

Hamkins discusses Hilbert's program in Chapter 7. Hilbert made a distinction between finitary and infinitary mathematics, and he proposed that infinitary mathematics should be viewed as a formal symbol-manipulation game that is not meaningful itself but can be used to establish meaningful finitary truths. Hilbert hoped to establish, by finitary means, the reliability of this formalist version of infinitary mathematics by proving the consistency of infinitary mathematics.

It might have been helpful if Hamkins had given a little more detail about how such a consistency proof would have guaranteed the reliability of infinitary methods for establishing finitary truths. To take an example, consider Wiles's proof of Fermat's last theorem. If, following Hilbert, we view this proof as just a meaningless arrangement of symbols that obeys the formal rules of infinitary mathematics, why should we trust the conclusion that Fermat's last theorem is true? One answer is that if there were a counterexample to the theorem, then that counterexample, together with Wiles's proof, would constitute a contradiction in infinitary mathematics. Thus, if we knew that infinitary mathematics was consistent, then Wiles's proof, even when viewed as a meaningless arrangement of symbols, would assure us that no such counterexample could exist. (Hilbert used Fermat's last theorem as an example in his 1927 lecture Die Grundlagen der Mathematik, an English translation of which can be found in [vH67, pp. 464–479]. For more details, see [Vel97].)

Unfortunately, Hilbert's hopes were dashed by Gödel's incompleteness theorems. Hamkins sketches the proofs of the incompleteness theorems and explains related results concerning decidability and definability.

Finally, in Chapter 8, Hamkins explains how set theory can serve as a foundation for all of mathematics. He presents the cumulative hierarchy of sets and the ZFC axioms (the Zermelo-Fraenkel axioms, including the axiom of choice) that describe that hierarchy. Then he explains that much of modern research in set theory is concerned with showing that various statements cannot be proven from the ZFC axioms (assuming those axioms are consistent), or that they are independent of the axioms—that is, they are neither provable nor disprovable. This includes, for example, the continuum hypothesis, as well as a long list of statements asserting the existence of various kinds of large infinite cardinal numbers. This raises a host of philosophical questions. Should any of these statements be accepted as new axioms? How do we choose axioms? Must axioms be statements that are intuitively evident, or can the fruitful consequences of a statement count as evidence that it should be accepted as an axiom? Which is a better foundation for mathematics, a weak theory that assumes only those axioms in which we have the most confidence, or a strong theory that describes mathematical reality more fully through the inclusion of additional axioms and therefore allows us to prove more? Is there a single "real" universe of sets, in which independent statements have their "correct" truth values? Or are there multiple universes of set theory—what Hamkins calls a "multiverse"—some in which the continuum hypothesis is true and some in which it is false, some containing various kinds of large cardinals and some with none?

Each chapter ends with a list of "questions for further thought." Some of these questions ask for solutions to mathematical problems, and some ask for philosophical reflection.

In the preface, Hamkins says that he has "tried to provide something useful for everyone, always beginning gently but still reaching deep waters." My impression is that, for the most part, Hamkins has student readers in mind; he often devotes considerable space to explaining ideas, such as the construction of the number systems, rigor in calculus, or the distinction between countable and uncountable sets, that will be familiar to all professional mathematicians. But there is still plenty of material that professionals will find interesting. To give just one example, he presents the standard proof of the irrationality of $\sqrt{2}$, which will be familiar to all mathematicians, but he follows that with a lovely pictorial proof, due to Stanley Tennenbaum, that I had not seen before. And, of course, even readers who are already familiar with the mathematical topics will find Hamkins's philosophical discussions stimulating and thought-provoking.

As the summary above makes clear, the book discusses a very wide range of topics. For most topics, Hamkins summarizes the main ideas and then provides suggestions for further reading, and this approach usually works well. For example, when discussing the construction of the number systems, Hamkins explains how rational numbers can be defined as equivalence classes of pairs of integers (the second of which is nonzero) and how the basic arithmetic operations can be defined on these equivalence classes, but he doesn't go through the tedious verification that these operations satisfy the field axioms. Similarly, the explanation of the incompleteness theorems gives a very good idea of how the proofs go without getting bogged down in the details of Gödel numbering.

However, occasionally Hamkins skips over enough details that readers may struggle a bit. For example, in the chapter on infinity, Hamkins introduces the infinite cardinal numbers $\beth_\alpha$ by saying that "at each stage we apply the power set operation $\beth_{\alpha+1} = 2^{\beth_\alpha}$," but as far as I can tell, the exponentiation operation on cardinal numbers was never defined. Will readers figure out that $\beth_{\alpha+1}$ is the cardinality of the power set of a set of cardinality $\beth_\alpha$?

Another example occurs when Hamkins is describing the cumulative hierarchy of sets. He says that the hierarchy is built up in stages, with each stage containing sets whose elements were constructed at earlier stages, but he doesn't say that the stages are numbered by the ordinal numbers, he doesn't introduce the notation V_α for the αth stage of the construction, and he doesn't introduce the terminology that the *rank* of a set is the stage at which it is constructed. As a result, readers may have a hard time understanding what he means a few pages later when he refers to "the set $V_{\omega+\omega}$, the rank-initial segment of the cumulative hierarchy up to rank $\omega+\omega$." Hamkins also sometimes makes use of the distinction between sets and proper classes, but I was unable to find a place where this distinction was explained.

While most of the book is written at a level that an undergraduate student will understand, Hamkins occasionally refers to more advanced topics. For example, when explaining why some geometric constructions cannot be done with straightedge and compass, he uses some ideas from the theory of field extensions. And every so often he can't resist quoting an interesting advanced idea that will go over the heads of many student readers. Thus, students will need to be willing to skip over the occasional passage that is not explained at their level. The book would work well as a set of readings for a seminar course, where the professor could help students deal with these occasional advanced topics.

The book is written in an engaging, conversational style, with numerous well-drawn figures. There are very few typographical errors. (The only one I found that could cause any confusion was the use of the word "transfinite" on p. 258 where Hamkins clearly meant "transitive.")

Of course, Hamkins does not resolve all of the philosophical puzzles he discusses. So it is appropriate that he ends his book, not with a conclusion, but with an invitation: "Please join us in what I find to be a fascinating conversation." The book can serve as an interesting and enjoyable introduction to this conversation for a wide range of readers.

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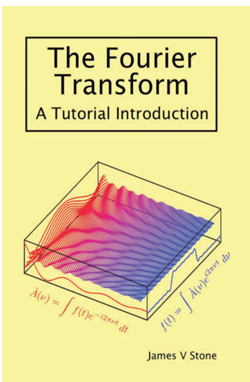

Courtesy of James V Stone.

The Fourier Transform
A Tutorial Introduction
By James V Stone

Visible and invisible waves are all around us. Fourier transforms allow us to break waves, or any periodic function, into unique sinusoidal expressions, allowing for more accessible analysis and study. Fourier transforms are used in applications such as CAT scans, data compression, and quantum mechanics, just to name a few.

This book is a self-contained guide to Fourier transforms. It includes a review of required material such as trigonometric identities and arithmetic involving complex numbers, making it accessible to someone early in their math journey. Topics in this book are explained in a clear, informal fashion with many visual aids that support the ideas discussed. The book does not include any proofs, and instead focuses on providing the reader with an understanding of how Fourier transforms work and what they are used for. The book begins with a discussion of how to find the Fourier transform for a real-valued function and then repeats the process for complex-valued functions. Equipped with this information, the author introduces content such as Parseval's Theorem, convolutions, and the Fourier transform of a Gaussian. Once the theoretical groundwork is in place, the author explains how Fourier transforms are applied in fields such as data compression and quantum physics. Interested readers may access a GitHub repository containing Python and MatLab code for various examples.

The Fourier Transform is well-suited to someone who wants to know what Fourier transforms are, their mathematical structure, and some examples of how they are used. At a minimum, someone reading this book would need to have completed two semesters of calculus. More advanced undergraduates or mathematicians interested in understanding some basics about Fourier transforms would still find this a valuable read and would be able to skip the sections with review content in them.

What's the Use
How Mathematics Shapes Everyday Life
By Ian Stewart

Most of us have had the experience of being met with negativity when we tell someone that we are a mathematician. In addition, we have all had the experience of students reacting to a required course or even a topic within a course by asking "When will I ever need this?" Thanks to Stewart's engaging book, we have a plethora of relevant examples to add to our artillery of responses.

What's the Use focuses on instances where math has been applied to solve a problem that it was not developed to solve. For example, there is a chapter dedicated to the ways in which space filling curves, which were discovered in the 1890s, are currently being used to help Meals on Wheels deliver meals. Stewart acknowledges that as technology has advanced, often using math in the background, it has become easier for people to dismiss math as obsolete. Throughout the book he works to show this is not the case, giving examples of how math is used to help make CGI motion smooth and to develop medical imaging techniques, among many other applications. In doing so, he touches on graph theory, number theory, and quantum mechanics, to name a few.

This book is a great read for anyone, regardless of mathematical background, interested in how mathematics is applied in critical ways on a daily basis. It is not mathematically deep and would be an excellent companion read to any liberal arts math course as it shows students how the topics they are learning (such as the bridges of Konigsberg) are connected to modern uses of math (like navigating the kidney transplant list). It can also be used as inspiration for outreach programs with an aim of exposing people to math topics that are relevant and easy to motivate yet outside the standard K–12 curriculum.

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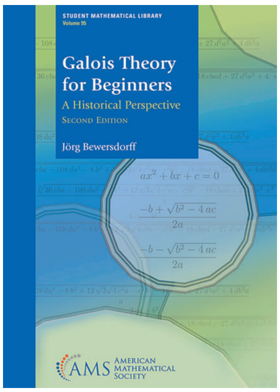

Galois Theory for Beginners

A Historical Perspective, Second Edition

By Jörg Bewersdorff

To a student, the sources of mathematical knowledge can seem mysterious. How could anyone ever conceive of the idea of attaching a group to a polynomial in a productive way? It appears that ideas like this only come to geniuses with access to sources of divine inspiration. Of course, we know that inspiration only arrives after prolonged, difficult struggle through murky convoluted details. But, this can be difficult to convey to students who are reading textbooks that contain beautifully constructed royal roads to deep truths. We rarely explain to our students that these clean approaches are works of massive hindsight. A teacher might, if he or she were brave, or foolhardy, attempt to illustrate how inspiration arrives by leading one's students along a path that retraces the historical evolution of a mathematical discovery. But, there are two obstacles to this. Most of us only vaguely know the historical path to most of the mathematics we teach. And, more importantly, those paths are full of misadventures and mistakes and misleading detours. Surveying the landscape would take so much time and generate so much confusion, you'd never get to the big picture.

Fortunately for us, Jörg Bewersdorff is both brave and foolhardy; he is also clever enough and knowledgeable enough to clear away the confusion while revealing the essence. His goal in this volume is to explain what Galois theory is and how Galois came to think of it. He carefully lays out, mostly by way of detailed examples, developments in solving polynomials starting with the work of Cardano and his contemporaries. The examples are, in most cases, drawn directly from the original sources. The computations are carried out in enough detail that we are led to see the importance of intermediate steps in sparking future insights. The notion of symmetry under permutations of roots arises naturally as Bewersdorff describes the work of Lagrange and Ruffini. After a brief digression into Gauss's construction of the heptadecagon, Bewersdorff describes exactly what Galois did. Only later are abstract groups and fields and the standard point of view introduced, by which point it all seems thoroughly natural.

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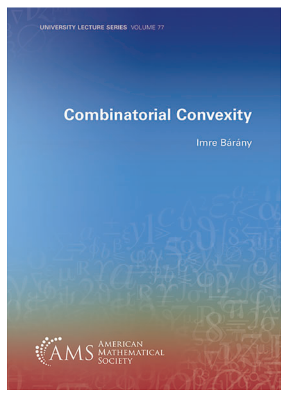

Combinatorial Convexity

By Imre Bárány

If you have a collection of three or more intervals on the real line with the property that any two have non-empty intersection, then the entire collection has non-empty intersection. The d-dimensional analogue, called Helly's Theorem, replaces "intervals" with "convex sets" and "any two" with "any $d + 1$." That is, if a collection of $n \geq d + 1$ of convex sets in R^d has the property that any $d + 1$ of them have non-empty intersection, then the entire collection has non-empty intersection. This result, first proved in 1913 by Helly, is paradigmatic of the results assembled in this slim volume.

The book makes the case that there are regions of discrete geometry that can be completely illuminated with just a little bit of exploitation of convexity and a touch of combinatorics or graph theory. A serious, and successful, effort to avoid deep technicalities is made with the result that the book is accessible to undergraduate students. The emphasis is on understanding the geometric content and implications. There are enough exercises so that it could be used as a textbook for a special topics course. The presentation is welcoming and intuitive and it gets to some very modern results, thus it should appeal to graduate students and researchers as well as undergraduates. In fact, the book grew out a series of lectures delivered by the author to audiences including mathematicians from undergraduates to senior researchers at a number of different venues around the world.

Towards a Fully Inclusive Mathematics Profession—One Year Later

Tasha Inniss, Jim Lewis, Irina Mitrea, Kasso Okoudjou, Adriana Salerno, Francis Su, and Dylan Thurston

Introduction

During the tumultuous summer of 2020, George Floyd, an unarmed African American man, was killed by policemen in Minneapolis. The tragic death of Mr. Floyd generated national outrage and brought into plain sight the discrimination and the obstacles faced by African Americans in many spheres of life in our country. The mathematics profession has not been an exception. Following this event, AMS President Jill Pipher appointed us to serve on the AMS Task Force on Understanding and Documenting the Historical Role of the AMS in Racial Discrimination, co-chaired by Kasso Okoudjou and Francis Su. The creation of the Task Force (as we shall henceforth call it) was endorsed by the AMS Council in June 2020 with this charge:

> *(1) To help the mathematical community understand the historical role of the AMS in racial discrimination; (2) To consider and recommend actions addressing the impact of such discrimination to the AMS Council and Board of Trustees.*

Given the nature of the work, the two co-chairs, after consultations with the AMS President, chose as members of the Task Force a group of racially diverse mathematicians at a variety of institutions, who had leadership experience (including several current or past members of the AMS Council) and who would be able to conduct interviews with the sensitivity required.

The Task Force reviewed AMS archives, talked with AMS staff, interviewed mathematician colleagues, and sought direct input from the wider mathematics community. Our work was assisted by math historian Michael Barany and by AMS staff, especially Abbe Herzig and Andrea Williams. We produced the report *Towards a Fully Inclusive Mathematics Profession*, released in March 2021, that documented AMS policies, practices, and actions that have had discriminatory impact and described ways that the mathematics profession remains unwelcoming, especially to mathematicians of color. We also cataloged more recent efforts by many in the profession and the AMS to become more welcoming, and we made several recommendations for ways the AMS can improve how it fulfills the part of its mission that advocates for "the full participation of all individuals."

Since the report's release, we've heard a number of reactions from mathematician colleagues through direct conversation as well as on social media. These reflect the diversity of viewpoints within the ranks of the Society. Some feel the report does not go far enough to castigate

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the AMS for its role in the lack of racial diversity in the profession. Others felt the exact opposite and suggested that the AMS is moving away from its core mission of research and scholarship. But it is clear to us that there is no contradiction for the AMS to "further the interests of mathematical research and scholarship, serve the national and international community through its publications, meetings, advocacy and other programs" as well as to make the profession more inclusive. Most would agree that there is a lack of diversity in the profession. Where disagreements begin is when questions of "why" and "what to do about it" are asked. Our report made several recommendations to try to address these issues.

You can see a partial list of actions taken on the Task Force recommendations and the Executive Summary of findings and recommendations from the report at the end of this article.

The full report is available for download from the Task Force website at `https://www.ams.org/understanding-ams-history`. If you do not have time to read it all, we encourage you to read Chapter 1, which gives context for each of the findings and recommendations in the Executive Summary.

Below, we reflect on what we took away from the experience, what we found surprising, what might be missing from our report, and how the work can be used by math departments. We've ordered the reflections in a way that may make the most sense given what each of us chose to discuss, since some introduce events to which others later make reference.

Irina Mitrea

In a very challenging year from so many perspectives, including historic, academic, and personal ones, the work on the AMS Task Force was an incredible roller coaster of experiences that is difficult to describe. Being asked to serve on this group is without a doubt the most significant AMS assignment I have ever been part of, counting here a previous three-year term on the AMS Council. While engaged in the work on the report, I always had on my mind the weight of the responsibility of the Task Force's charge, the burden of the immense loss of talent due to lack of diversity and inclusion in mathematics, the vastness of the relevant directions that had to be considered, the very painful personal and professional experiences revealed in the process, the insight and incredible generosity of the mathematicians who have agreed to be interviewed, the unabated commitment of my Task Force colleagues, the genuine team camaraderie that has ensued, the heightened hope for change, and the dispiriting thought that our work might simply end up shelved and ignored. Time will tell, but meanwhile I am very grateful for everything I learned along the way.

Dylan Thurston

For me personally, one aspect of the Task Force work that hit home hard was learning about the involvement of my father, William Thurston, in these issues around diversity and inclusion many years ago. This was most obvious in reading through the report from a 1996 task force, which he was a member of. It also came through in interviews. Multiple people mentioned him as a positive force, especially in his role as director of MSRI. For instance, in 1995 MSRI hosted the first Conference for African-American Researchers in the Mathematical Sciences (CAARMS), while my father was director there. It was bittersweet to learn about his work, while being unable to ask him about it.

The other thing that stands out to me is the importance of listening to voices that have so often been shut out. We must learn from them and work to fix the issues, and reflect on our own actions that have had negative influences. Are we really listening to our graduate students? What is their conference experience really like? Do you know? There is always more to learn!

It should be clear that the report is not a solution; it will only lead to positive change as far as people take these lessons to heart. After the 1996 report, many people worked hard at implementing these suggestions, but ultimately many of those efforts fell short, and I very much do not want that to happen again. Read the report; share it among your department, your mathematical circles, and your friends; discuss it; and then, most importantly, look for ways to implement solutions all around you. I guarantee you will find some way to improve things!

Tasha Inniss

Participating on the AMS Task Force to understand and document the historical role of AMS in racial discrimination was a very rewarding experience. We were all sincere in our desire to make our professional community one that is welcoming and supportive of all mathematicians. It would be good if departments agreed to read the report and then collaboratively develop strategies they feel would work best for their context, needs, and goals.

Our time was very limited, but if we had more time, I think developing an action plan with proposed strategies for implementation would have been incredibly useful for those departments who are serious about being inclusive. It may also have been good to do a survey of math department chairs to see if any are working on diversity initiatives and implementing equitable practices to support all math undergraduate and graduate students. Hearing from them about their goals and processes related to diversity, equity, and inclusion may have been illuminating.

I appreciate working with a great group of people who helped to create a safe space for us to complete our task and report. It was a pleasure working with each of them!

Adriana Salerno

First of all, the experience itself was much more emotional than I thought it was going to be (and I thought it was going to be plenty emotional from the offset). I have grown to really love and respect the other members of the Task Force. We brought different strengths, experiences, and perspectives to the table.

It was also emotional because we were interviewing dozens of mathematicians, many of whom recounted some hard and painful experiences caused by their profession and professional society. It was hard to even ask, because the process itself felt extractive—"give us your stories, and we hope someone out there develops some more empathy because of it." I can only hope that the community and the AMS understand that there is a moral imperative to change. I am well aware that many in our profession don't believe that racism is a problem that mathematicians have to contend with, and some even go as far as to say that there is not racism in mathematics. I was surprised to see the reactions of some mathematicians (mainly in our surveys) saying that dealing with this problem was bad for mathematics. Some said that if the AMS cares more about humans, then we will value research less. Some said that "we will basically become the MAA." This false dichotomy has always been there, but it felt so much more callous because there are humans literally saying "you are hurting me, and that makes my math worse." For some, the takeaway after reading the report was "OK, so Claytor was treated poorly, but the real reason that there are not that many Black mathematicians is that white people are better at math," and in the same breath, call themselves not racist.

Another thing that really struck me was our discovery of a 1996 report, a report written by a task force much like ours, composed to deal with the issues of racism in mathematics. We read it, and many of the recommendations they made in that report were similar to the recommendations we were writing up. And the kicker: very little had changed since 1996. For me, that was a gut punch. I suddenly saw myself in the "20 years later" flash forward at the end of the movie, in which another task force is formed, in space I presume, and that they are charged to detail the racist history of the AMS, and that our report surfaces and people ask "whatever happened to that? Why has nothing changed?" We were all shocked, and a little depressed, when we found this older report.

But this actually presented an opportunity — the opportunity to say to the AMS, the membership, the leadership, and mathematicians more generally, that a task force and a document on their own do nothing. That if there's no accountability, and no commitment to change things, the world will stay how it is, for the most part. I really hope that, in 20 years, people look back at this document and say that it was the start of something good.

Jim Lewis

A surprise was that the Task Force worked so well together. Producing a report on such an important and sensitive subject in six months was a major ask. (I served on one AMS Task Force that took seven years to write its report.) Starting July 1, 2020, we met almost every week for six months. In addition, there were lots of interviews to gather information, reports on interviews to write, etc., and eventually there was our report to write. So the time demand was significant. One thing that stood out to me was that we really listened to each other during our meetings. Another is that we really focused on identifying steps the AMS could take that have the potential to make a meaningful difference over the next 5–10 years. However, one aspect we did not look at was an analysis of who gets a PhD in mathematics that focuses on US citizens and permanent residents—I suspect it would reveal just how little progress we have made with respect to increasing the number of underrepresented people of color who earn a PhD in mathematics. Because mathematicians respond to data, that might increase our collective sense of urgency that we must do more.

Discovering the 1996 Task Force Report was another surprise. It had a big impact on our Task Force. I am convinced that the AMS (especially the professional staff) really tried to implement the 1996 recommendations. But ultimately, the impact of the report was limited for reasons discussed in the report. To make things different this time will take sustained effort from the AMS leadership.

But I am optimistic that real change is possible this time. It is significant that about 80% of the AMS leadership (Council + Board of Trustees) who responded to our survey said they viewed racism as a concern in mathematics and 90% said that the AMS has a role in addressing racism in the profession.

While our report focused on what the AMS should do as an organization, real change must come in our academic institutions. My department has created a Diversity Committee (in Fall 2020) and my Department Chair appointed me to chair the committee. I sense that the time is right to make changes with respect to an inclusive approach to teaching undergraduates, recruiting, and mentoring graduate students, and recruiting faculty. For certain we will try.

Francis Su

Some have asked about our work: "why look back and dig up the problems of the past? Can't we just let it go and try to move forward in an inclusive way?"

Being on the Task Force—reading through the historical record and interviewing Black mathematicians about their current experiences—has helped me draw the connection between the two. When I read that the AMS Council passed a non-discrimination motion in 1951, and yet that AMS meetings technically avoided running afoul of that by not having official social events (yet still holding informal ones to which Black mathematicians were not welcome), and I see a *Notices* ad for a 1958 meeting list a "colored" option for hotels, I could not help but feel that good intentions are not enough. When I read how the AMS missed several opportunities throughout the 1960's to the 1990's to improve the climate for mathematicians of color, it helps me see that the misguided desire of some AMS members to "think of only the math" is actually harming our Black colleagues' ability to think of only the math. When I hear about the current climate for mathematicians of color from our interviews and the slights they continually face in doing their work, I think to myself: the AMS (which exists to promote mathematical research) is actually seeing less research done because some of our colleagues cannot do their research without dealing with this other stuff.

In our report is this remarkable statistic: Historically Black Colleges and Universities (HBCUs) produce nearly *half* of all mathematics bachelor's degrees earned by African-Americans in the United States, even though they represent just 3% of all colleges and universities in this country. Think about that. This statistic not only shows that HBCUs can be credited with producing many African-American mathematicians, but it also reveals the extent to which mathematics departments at non-HBCUs are failing African-Americans and not fulfilling their missions to educate all students.

I felt that I learned a lot in our work on the Task Force, which will personally help me think about what being inclusive means. Research mathematicians are, first and foremost, human beings. The practice of doing research involves choices about whom to collaborate with, whom to invite to give talks, whom we talk with at social events where informal research discussions happen. We are failing our colleagues if we continue to practice business as usual.

I hope many will read our report and take action. Being welcoming and inclusive should not be an add-on. It should be integral to everything we do in our roles as researchers and as mathematical educators.

Kasso Okoudjou

Working on the Task Force would have been a daunting undertaking in normal circumstances, but was even more so in the middle of a pandemic. It was both emotionally draining and time consuming. Today I view it as both a rewarding experience as well as a big leap of faith. Faith in our capacity to seize the moment and act so as to make the profession truly inclusive of all. However, part of me is worried that the sense of urgency we all felt during the summer of 2020 will gradually dissipate and we will revert to reactiveness or will continue to make modest changes without ever touching the main issues. The events since the release of the Task Force report seem to support both that sense of optimism but also that worry. It is apparent that despite some stumbles, the AMS is trying to make Equity, Diversity, and Inclusion a central component of all its other activities. I hope the fear of making mistakes or being criticized will neither inhibit nor slow down the Society's ongoing efforts to make the profession more welcoming.

I hope the Society can lead the profession to set bold goals in increasing the participation of historically underrepresented groups in the mathematical sciences. Those goals should come with policy actions and benchmarks to access progress. We could look for a model developed by our colleagues in Physics and Astronomy who wrote a data-driven report setting such goals for increasing African-American participation in their profession.

I also hope that the newfound attention and appreciation by the AMS and the profession to the outsized role the HBCUs are continuously playing in educating, mentoring, and nurturing generations of African Americans in the mathematical sciences will not be ephemeral. Finally, as I reflect on the Task Force work and its report, I regret that some voices were not not heard through our reports. These include people we tried to connect with but could not for some reasons, or people we simply did not think of reaching out to.

Postscript

Since the release of our report, several actions have already been taken on our recommendations. Many of our recommendations address changing AMS structures and practices, which may not sound exciting but are actually very important in improving the climate for mathematicians of color. For instance, the Board of Trustees approved the creation of an AMS staff position on Equity, Diversity, and Inclusion, as we recommended, and the position has been filled. Other Task Force recommendations are currently being considered or implemented by appropriate AMS entities. For accountability purposes, the newly formed AMS Committee on Equity, Diversity, and Inclusion (CoEDI) will ensure that our recommendations do not fall by the wayside and are properly considered.

We are heartened by this response, but we must not be naive and think that continuing the work of making our profession more inclusive will be easy. The AMS can provide leadership, but in fact, it is beyond the AMS's power to make our profession fully inclusive. Change must begin with us, the people of the profession in mathematics departments across the country, as we rethink the practices of our departments: how we teach mathematics, how we communicate mathematics, what mathematics we value, and whom we consider to be a mathematician. We hope our report will assist you in these efforts.

A partial list of actions taken on Task Force recommendations as of the writing of this article include:

- As a temporary measure, AMS President Ruth Charney will assign a current Vice President to serve on the Committee on Equity, Diversity, and Inclusion.

- In January 2022, the AMS hired Dr. Leona Harris as Director of Equity, Diversity, and Inclusion.
- AMS Ballot prompts have been modified to allow for the inclusion of a broader range of professional activities, effective with the 2022 election.
- Through wider outreach by the Committee on Committees and Nominating Committee, the AMS has renewed efforts to seek a diverse pool of candidates for committee appointments and roles in governance. The AMS Council has approved adding a statement to the charges of these committees and the Editorial Boards Committee to remind them to keep diversity of all kinds in mind when selecting candidates.
- In 2021, the AMS established the Claytor-Gilmer Fellowship, an annual award supporting the research and scholarship of mid-career Black mathematicians. Mohamed Omar (2021) and Ryan Hynd (2022) were the first recipients.
- The AMS Programs that Make a Difference Award, which recognizes work that brings more people from underrepresented backgrounds into the profession, was elevated from being a policy committee award to being an award of the Society, starting in 2022.
- Several recent AMS publications highlight the work of mathematicians of color. For instance, in 2021, the AMS published the book *Testimonios: Stories of Latinx and Hispanic Mathematicians*, and in 2022, the *Notices of the AMS* published the piece "Dr. Raymond L. Johnson: A Mathematical Journey and Some Reflections on African Americans in Graduate Mathematical Sciences Programs in the US." In addition, in 2023, the *Notices* will publish memorial articles on Gloria Ford Gilmer, Bob Moses, and Shirley M. McBay.
- The AMS Committee on Equity, Diversity, and Inclusion will provide annual updates on Task Force recommendations to Council beginning in April 2022.

Towards a Fully Inclusive Mathematics Profession

Report of
The Task Force on Understanding and Documenting the Historical Role of the AMS in Racial Discrimination

March 22, 2021

Executive Summary

Findings

- Racism is a concern of many mathematicians and leaders of the Society, and the AMS has a role in addressing racism in the profession.
- The effects of blatant discrimination in the mathematics community (and in the AMS) since its inception continue to have repercussions today in the development of Black mathematicians, the visibility and perceptions of their work, and the lack of recognition that further hinders their professional advancement.
- The AMS has missed several opportunities to improve the professional climate for mathematicians of color.
- Black mathematicians suffer from a lack of professional respect and endure microaggressions, even today.
- There is a profound lack of trust from Black mathematicians that the AMS represents them, speaks to them, hears them, and includes them in its decision making.
- Historically Black Colleges and Universities have an outsized influence on the production and the support of Black mathematicians, and providing outstanding models of successful mentoring.
- The history of the AMS has shown that sustained attention to problems has resulted in positive outcomes. Implementing sustainable change is challenging and requires intentionality and continual vigilance.

Recommendations

Governance-Related Recommendations

1. Establish a Vice President for Equity, Diversity, and Inclusion.
2. Create a high-level staff position on Equity, Diversity, and Inclusion, with an Office/Division of Minority Affairs under its purview.
3. Reform election procedures.
4. Reform appointment procedures.

Program-Related Recommendations

5. Develop and implement an engagement plan to welcome the participation of Black mathematicians in the AMS.
6. Create and support programs to further the career development of mathematicians of color.
7. Include equity, diversity, and inclusion in the AMS's professional development offerings.
8. Publicize the expertise of mathematicians of color.

Accountability-Related Recommendations

9. Request that the AMS provide annual updates on the status of these report recommendations.
10. Accept responsibility for not fulfilling the AMS's own commitment to increasing the participation of mathematicians of color in the profession, including Black mathematicians.

Full report available at
`www.ams.org/understanding-ams-history`


Tasha Inniss


Jim Lewis


Irina Mitrea


Kasso Okoudjou


Adriana Salerno


Francis Su


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Credits

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Ars Mathemalchemica: From Math to Art and Back Again

Susan Goldstine, Elizabeth Paley, and Henry Segerman

Mathemalchemy was on display at the National Academy of Sciences in Washington, DC, from mid-January to mid-June, 2022. For information about other venues, as well as supplementary information about the history, development, and mathematical and artistic content of the installation, see `mathemalchemy.org`.

Mathemalchemy is a collaborative multimedia art installation that celebrates the creativity and beauty of mathematics (Figure 1). The project is headed by mathematician Ingrid Daubechies, a professor at Duke University, and artistic director Dominique Ehrmann, a fiber artist based in Québec (Figure 2A). Twenty-four core mathematicians and artists created the installation, with additional contributions from "adjuvant," "coset," "adjoint," and "apprentice" mathemalchemists.

Reveling in both mathematics and art, *Mathemalchemy* depicts multiple narrative scenes. Among them, a cat and mouse prepare tessellating cookies in a bakery; a bird gathers shiny mathematical treasures in a curio shop; herons tag a new species of aquatic knots in the bay; a tortoise slowly but surely ambles toward the end of Zeno's path with her Sierpiński kite in tow; chipmunks explore number games in the playground; squirrels chatter about prime number algorithms in the garden; and an octopus slips between two- and three-dimensional space to paint graffiti under a lighthouse whose beams project into the sky. The installation is fabricated with myriad media: beadwork, ceramics, crochet, embroidery, knitting, leatherwork, needlefelting, origami, painting, polymer clay, 3D printing, quilting, sewing, stained glass, steel welding, light, temari, weaving, wire bending, and woodworking.

Figure 1. The installation at its first public venue.

The artwork developed out of extended conversations between our large group of collaborators, who brought their different backgrounds and perspectives from mathematics, art, and the sciences. A recurrent theme of these discussions was the relationship between mathematics and art and how they would intersect in the fantasia we were constructing. Indeed, the authors of this paper are two professional mathematicians and one professional

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(A) Ingrid Daubechies and Dominique Ehrmann. (B) A spherical photograph of the installation surrounded by a subset of the mathemalchemists.

Figure 2. The installation at the completion of its construction at Duke University.

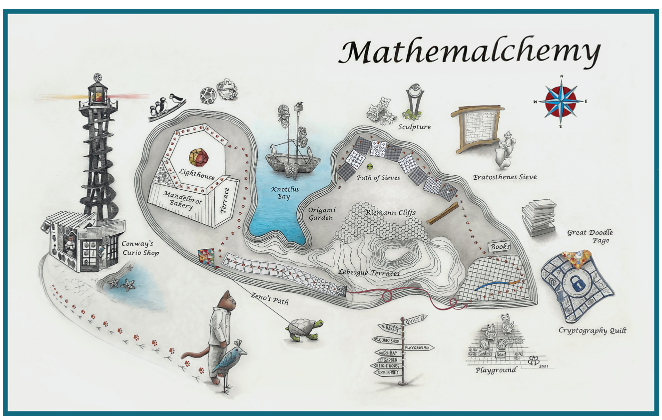


Figure 3. A map of the installation, by Bronna Butler. Compare with Figure 2B. Pawprints indicate the path that Harriet and Arnold take.

artist, though all three of us create art and explore mathematical ideas.

As we talked over how to present *Mathemalchemy* in this article, we kept returning to how our experiences within our respective fields had shaped our perceptions of this fabricated realm. In particular, the artist noted several colleagues in the visual and musical arts who self-deprecatingly deny having any aptitude for mathematics—despite using mathematics in their disciplines. Our discussions coalesced into an imagined dialogue between two of the characters that inhabit the installation, one responding to their world through a mathematical lens, the other through an artistic one. To highlight attitudes that we have internalized, we toy with some common tropes about mathematics and art—for example that art is accessible to all, while math is accessible only to some. Our characters interact with other creatures to reflect the challenges and pleasures of collaboration and communication across diverse fields of expertise. Their discussion also gives us an opportunity to tour parts of the exhibit. The map in Figure 3 shows the route that our protagonists take.

* * *

Harriet,[1] *a bowerbird, has channeled her propensity for collecting mathematically intriguing objects into a thriving business. Her fine-math gallery, Conway's Curios,*[2] *is located in the heart of Mathemalchemy's vibrant downtown (Figure 4).*

[1]*Harriet is named after the Harriss Spiral, discovered by Edmund Harriss* [4]. *The spiral adorns her feathers and features prominently on the Curio Shop sign and wall.*

[2]*Named after John H. Conway, with the permission of his family. Like Conway's office, the shop is full of mathematical curiosities.*

(A) Harriet in her fine-math gallery, Conway's Curios.

(B) Arnold in his bakery, with his assistant Mo[u]se.[3]

(C) Harriet's brother Horton[4] runs the terrace café, accessed via the lighthouse ramp.

Figure 4. Harriet and Arnold meet on the terrace.

One evening, Harriet meets her friend Arnold,[5] a cat, on the terrace above the shop. Arnold is the chef at the Mandelbrot Bakery next door, and he has brought some award-winning cookies to share.

ARNOLD: *Bon soir*, bowerbird. How's the new installation progressing?

HARRIET: Felicitations, my favorite feline; thanks for asking. I've just hung a new Alexander Horned Sphere next to the Möbius Strip in the window, opposite the Hopf Fibration of the Three-Sphere. You must have caught a glimpse on your way to the café. What do you think?

ARNOLD: Captivating! I don't know math, but I know what I like. You sure have an eye for it though, Harriet.

HARRIET: Why, you do too, Arnold. Consider your cookies—don't they tessellate?

ARNOLD: Oh, that's art, not math. I created the intertwining forms as a metaphor for the intertwining ingredients, plus it prevents overworking the dough. Artistic intuition and a practiced paw shaped these delectable *objets d'art*.

HARRIET: Form follows flavor! You know, I wish I were better at art. I used to be pretty good at it, but after I fledged, my art teacher told me my "little bird brain" was better suited to math, and my art ability simply evaporated.

ARNOLD: Alas, I've heard many a mathematician say the same thing. Yet you obviously understand art: your gallery is overflowing with it!

HARRIET: Art? You've seen my inventory—Klein bottles, interlocking sliced tori, rhombicosidodecahedra, double helices, fractal trees, borromean rings…These are all *objets de mathématique*. I fill my shop with reflections of fundamental truths, Arnold—that is what math is all about.

* * *

Mathemalchemy began when Ingrid saw Dominique's quilted installation, *Time to Break Free* [6], in which a fantastical machine converts the flat figures on a traditional quilt into fully three-dimensional characters (Figure 5). This inspired Ingrid to wonder whether something like this could similarly bring the beauty and creativity of mathematical ideas to life.

Figure 5. *Time to Break Free*, an installation by Dominique Ehrmann.

[3]*The unusually punctuated name of Arnold's assistant is a tip of the hat to Vladimir Arnold's research collaborator Jürgen Moser, recognized in the name of the KAM (Kolmogorov-Arnold-Moser) Theorem* [5].

[4]*Conway's middle name.*

[5]*The folding and stretching of pastry dough brought to mind the stretching and compressing of dynamical systems; thus Arnold was named after mathematician Vladimir Arnold* [1].

Figure 6. Mathemalchemists, from left to right, top to bottom: Emily Baker, Bronna Butler, Edmund Harriss, Elizabeth Paley, Kimberly Roth, Edward Vogel; Dorothy Buck, Rochy Flint, Li-Mei Lim, Kathy Peterson, Henry Segerman, Jake Wildstrom; Ingrid Daubechies, Faye Goldman, Sabetta Matsumoto, Samantha Pezzimenti, Jessica K. Sklar, Mary William; Dominique Ehrmann, Susan Goldstine, Vernelle A. A. Noel, Tasha Pruitt, Daina Taimina, Carolyn Yackel.

Ingrid and Dominique presented at the Joint Mathematics Meetings in January 2020, and a group coalesced around the idea (Figure 6). When COVID preempted the first planned in-person workshop in March 2020, the participants began meeting regularly via Zoom. From an initial cornucopia of concepts, Dominique created a vividly detailed quarter-scale maquette that served as the basis for further discussion, revisions, prototyping, adjustments, and collaborative construction from afar (facilitated by the United States and Canadian postal services).

After 18 months of planning and long-distance making, participants came together in July 2021 at Duke University to assemble, trouble-shoot, and fine-tune the installation.

* * *

Arnold and Harriet continue their discussion with a stroll, exiting the terrace via the lighthouse ramp (Figure 4C). Between them and the ground are a pair of puffins who tend the lighthouse, absorbed in conversation.

HARRIET: I do enjoy a helical hop around the lighthouse.

ARNOLD: I keep thinking I should make a lighthouse-shaped pastry to celebrate this edifice.

HARRIET: Flavor follows form this time? Tell me more.

ARNOLD: Naturally, the flavor and texture must inspire thoughts of light. I've tried my paw at an occasional phyllo tower, but it's a challenge to get free-standing pastry to support itself.

DEL: Pardon, friends—could not help hearing
That pastry treats need engineering.

NABLA: Could steel beams here perhaps inspire
A means to make dough towers higher?

DEL: The long straight girders 'round axis Z
are shaped like Ts from sky to sea.[6]

NABLA: Attach the ramp with L-shaped brace
and phyllo dough should stay in place.

ARNOLD: Your thoughts for food are food for thought… I'll play around with that perpendicularity idea to see if it helps with the strength.

HARRIET: Aha! Geometry *does* inform your art.

DEL: He's won awards for pastries fine
when art, taste, math all intertwine!

ARNOLD: *Touché*, Harriet. Can you puffins also help this bowerbird recognize the art in her math?

NABLA: We'd stay and chat, but must away;
We're changing shifts, *adieu*, good day!

[6]*With apologies to non-American readers.*

(A) The surface S, with lines of principal curvature γ drawn in red.
(B) Strips meet along the curves γ, forming T- or L- shaped cross-sections.
(C) The lighthouse stands nine feet tall.
(D) The steel frame was cut and assembled at the University of Arkansas.
(E) A Fresnel lens focuses a beam of light through the stained glass dodecahedron.

Figure 7. Design and construction of the lighthouse. The renders in Figures 7A and 7B are by Sabetta Matsumoto.

* * *

We designed the art in the installation to incorporate mathematical details that would appeal as much to inquisitive non-mathematicians as to those devoted to the field. The mathematics includes expositions of classical results, demonstrations of recent mathematical innovations,[7] and mathematics developed specifically in response to the challenges of constructing our vision.

The construction of the lighthouse (Figure 7) is an example of this last connection. The key idea comes from Edmund Harriss and Emily Baker [3]. We want to represent a surface S (the shape of the lighthouse) by a grid of curves on S. Each such curve γ is built by bending then welding together two strips of metal, forming either a T- or L-shaped cross-section. Thin sheets of metal can be bent out of plane without too much effort, but twisting them is very difficult. After welding, each strip reinforces the bends we make in its partner, resulting in a rigid structure. Deforming the welded strips would require twisting the metal (or bending within the plane of the metal sheet).

One of the line segments of the T- or L- shaped cross-section is tangent to S while the other is perpendicular. These two directions together with the tangent to the curve γ form the *Darboux frame* for γ. Keeping the metal from twisting corresponds to this frame having no torsion. It turns out that this restricts γ to being a *line of principal curvature* of S.

Sabetta Matsumoto designed a surface S with interesting lines of principal curvature that would produce an attractive lighthouse. It is a perturbed hyperboloid of one sheet, with one of the lines of principal curvature giving a gentle spiral ramp. The lighthouse was physically built by Emily Baker and her students.

[7]*For example, the stained glass dodecahedron (Figure 7E) atop the lighthouse illustrates that unlike the other four platonic solids, the dodecahedron admits geodesic paths from a vertex back to itself that don't cross any other vertices, a result that was published in May 2020* [2].

(A) Visiting knot theorists Adélie, Chinstrap, and Gentoo in the penguin's nest.
(B) The herons lead a knot tag-and-release program.

Figure 8. Teamwork is essential at sea (and in many other contexts).

* * *

Having reached the base of the lighthouse, Harriet and Arnold are delighted to hear music coming from the bay (Figure 8).

HERONS [SINGING]: Come put these knots in order
'cause they've tangled in the water,
Heave away, me jollies, heave away;
Come put these knots before us, hyperbolic, sat'lite, torus,
Heave away, me jolly birds, we'll all heave away!

PENGUINS [SHOUTING]: Ahoy, cousin bird, ahoy friend cat!

HARRIET: Look Arnold! Up in the crow's nest—penguins!

ADÉLIE[8]: Where we come from, we call it a penguin's nest, thank you very much.

ARNOLD: Have you traveled far, friend birds?

HERONS: I wrote me knot a letter, 'twas a Theta sweet and smooth,

[8]*Many of our characters are named after species. Chinstrap, Adélie, and Gentoo are species in the genus Pygoscelis.*

Heave away, me jollies, heave away;
Then along came Reidemeister—twist, poke, slide—he made his move.
Heave away, me jolly birds, we'll all heave away!

CHINSTRAP: Quite far! We're visiting scholars, knot theorists by trade. We're here to assist the herons with their wavebreaking baywork—

ADÉLIE: —That's groundbreaking fieldwork, as you land mammals say—

CHINSTRAP: They've started a critical tag-and-release program to study the behavior of—

HERONS: Sometimes a spotted Cinquefoil, sometimes striped Stevedore,
Heave away, me jollies, heave away;
But now we've found empirically what theorists thought were lore!
Heave away, me jolly birds, we'll all heave away!

GENTOO: What the herons sing is true. We did not expect that *amphibichiral* theta-curves[9] could realistically survive in brackish water, so when the herons notified us of their newly discovered invertiblate[10]—

ADÉLIE: —well naturally, we came to assist. They have a completely different perspective on the deck down below than we have up here from this tower. Thanks to their preliminary data—

CHINSTRAP: —we're now able to computationally model the spawning locations—

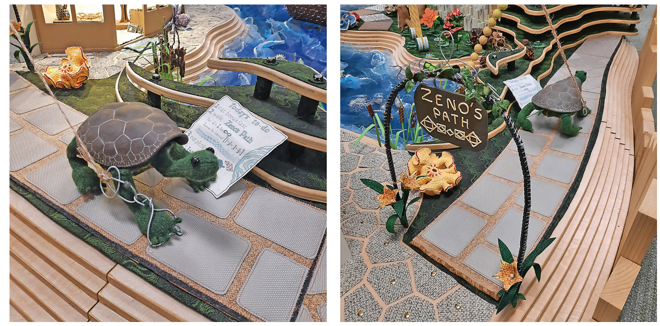
GENTOO: —and the herons are classifying new knots by the netload!—

HERONS: Heave away, me jolly birds, we'll all heave away!

* * *

From concept to realization, we benefited immensely from the diverse skills and expertise of our twenty-four core collaborators. The seeds planted through *Mathemalchemy*'s regular large and small group discussions germinated into playful layerings of mathematical concepts within scenes, as well as intersections and recurring themes across scenes. Mathematics inspired narrative elements that inspired artistic design. In tandem, artistic vision inspired narrative elements that shaped mathematical design.

At the same time, we faced real challenges in how to fit the contributions from so many different perspectives together. The mathematical content had to be correct and mathematically interesting, and the assembled work had to be structurally sound. But a cohesive whole also required making aesthetic and materials-related judgment calls. We learned to appreciate—or at least have patience with—one anothers' different disciplinary priorities and modes of communication. Compromise, flexibility, and trust were essential from conception to fabrication to installation. We embraced variety and abundance as part of our aesthetic, and relied on Dominique's keen artistic eye to keep us on track.



(A) Tess the tortoise and her to-do list. (B) Zeno's path.

Figure 9. Tess keeps herself busy by walking Zeno's Path every day.

* * *

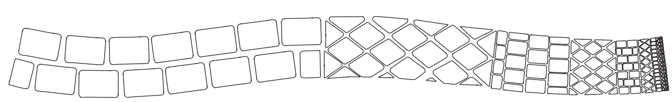
The two friends arrive at Zeno's Path and enter through the ornamental gate (Figure 9).

HARRIET: Oh, look, Tess is up ahead! I haven't talked to her in ages.

ARNOLD: Let's catch up!

HARRIET: Arnold, you try racing on Zeno's Path every time we come here, and it never works.

ARNOLD: I'm sure it will work this time—I've been training, and Tess walks so much slower than we do. Ready, set...



ARNOLD: Well, that's odd. We've been almost to the end for a while now, and we still haven't caught up.

HARRIET: This trail always takes forever. Why don't we just aim for the playground instead?

Some time later, Harriet and Arnold arrive at the edge of the garden, where chipmunks are enthusiastically arranging acorns on the playground (Figure 10).

HARRIET: How delightful—the chipmunks are playing their prime numbers game.

ARNOLD: Evening, kids! Who's winning?

UINTA: We both are. We're practicing for a doubles tournament, working on technique and artistry.

DURANGO: You might think Pickle-Prime is all about the numbers—

[9] *Knotted theta-curves* [8] *are a current topic of study. "Amphibichiral" is a made up word.*
[10] *This is also made up.*

Figure 10. Uinta and Durango play a game with Babylonian cuneiform numeral tiles and acorns.

UINTA: —determining whether a game tile is prime or composite—

DURANGO: —but acorn placement is becoming increasingly important to the judges.

UINTA: We've decided freestyle layout is just too …distracting. Makes us want to eat all the acorns. So we're going with this parallel placement instead.

HARRIET: And the braided rings?

DURANGO: They remind us not to eat the remainders.

* * *

The pandemic forced us to substantially extend the initial timeline for completing *Mathemalchemy*, as six months stretched into almost eighteen. That extra time allowed us to refine our underlying story-telling. The initial proposal for *Mathemalchemy* depicted studious mice at work, not play, and woe to a tearful mouse who struggled to understand prime and composite numbers. Our Garden subteam set about changing the narrative: now, two inquisitive chipmunks actively engage in collaborative, hands-on play with tangible three-dimensional objects, reveling in their discoveries as much as in the game's outcomes. We similarly hope *Mathemalchemy* itself offers viewers a vehicle for joyfully engaging with mathematics.

* * *

Harriet and Arnold leave the chipmunks to their practice and continue through the garden, stopping at the gaussian-integer stepping stones to watch the squirrels (Figure 11).

ARNOLD: Looks like the squirrels are getting ready for their annual Sieving of the Primes!

HARRIET: Gets bigger every year.

ARNOLD: How many sieves do they have? 2…3…5, and now 7 and 11 are rolling in…Where do they store them?

HARRIET: Behind the Riemann Cliffs. I used to lend them storage space in Conway's Curios, but they kept adding sieves. Even without the sieves, I hardly have room to turn around.

Figure 11. 3D printed polyhedra appear as stamens and pistils for origami flowers in the garden and as sea creatures in the reef. In the background, two squirrels discuss prime number algorithms in front of their Sieve of Eratosthenes.

* * *

Just like Harriet, we had to make some decisions about what to include in the installation. Our criteria included mathematical and artistic considerations. Despite our extra months, we were also restricted by time.

Over one hundred small, 3D printed models of polyhedra adorn *Mathemalchemy* (Figure 11). We have the platonic, archimedean, and Johnson solids, and the first few prisms and antiprisms, totalling over a hundred pieces. These could have been made in ceramics, paper, or wood, but it would have taken far longer. By 3D printing them, we were able to save time on the manufacturing end, which allowed more time for Tasha Pruitt to envision how they could be incorporated into the exhibit. Supplemented with paint, embroidery, and beadwork, they manifest as stamens and pistils, boulders, cushions, planters, jellyfish, and other imaginative adornments.

Decisions about how to optimize quality and quantity also played a role in the two arches of thread-wrapped balls that rise above the garden (Figure 12). One arch illustrates a geometric series, while the other illustrates a diverging series.[11] The visible portions of these series involve over 120 woven balls, a selection of which are embroidered following a form of Japanese folk art called *temari*. Early on, it became clear that embellishing all of the balls was neither feasible nor visually desirable, so the team faced the

[11] *The ratio of the* $(n+1)$*st ball diameter to the nth is* $(n/(n+1))^{2/3}$.

(A) The diverging ball arch plunges into the bay.

(B) The converging ball arch reaches upwards.

Figure 12. The ball arches alone required hundreds of hours of work.

mathematical, practical, and aesthetic question of which to embellish and which to leave plain. The solution was to embroider the 22 balls at the indices of the twin primes. This choice extends the theme of primes associated with the garden below, yields a manageable number of temari, and gives a pleasingly irregular arrangement.

* * *

ARNOLD: You do have a lot of mathematical pieces in the gallery. Is it difficult to curate such a large collection?

HARRIET: I try to make objective decisions, so I rely on basic principles of the field. Simplicity, clarity, parsimony. . . universal facts—I always try to have some of the classics on hand. Of course, it's also nice to have contemporary work; unexpected connections between distant fields. And elegance; I'm so fond of elegant proofs. . .

ARNOLD: What makes something elegant?

HARRIET: Something is elegant when. . . well, when. . . it's. . . elegant. Hmm. On reflection, elegance is not so easy to define. I know it when I see it.

ARNOLD: That doesn't sound terribly objective.

HARRIET: Now you mention it, topics do go in and out of vogue in mathematics. . . Rigor is particularly popular these days, possibly at the expense of intuitive understanding. . .

ARNOLD: Wait wait wait, aren't those all matters of aesthetics?

HARRIET: Aesthetics? Aesthetics. I suppose so...

ARNOLD: Why Harriet, I believe you have a mind for art after all.

* * *

In our imagined narrative, Harriet is about to rediscover her confidence in matters of art, overcoming the internalized lessons of her youth. In the world the authors inhabit, it is mathematics, not art, whose students often receive the message that this domain is not meant for them. *Mathemalchemy* harnesses art and narrative to invite viewers to participate in the joy of mathematics. This invitation is embodied by three human silhouettes experiencing mathematics at different stages of life: a child improvises music with her trumpet; a teenager surfs across a swirling vortex of ideas, trailing a pennant that whips in the wind; and an adult releases an exuberant cavalcade of mathematical notes and jottings into the world (Figures 13A and 13B). Whether they are observing *Mathemalchemy* or imagining it into existence is up to the viewer to decide.

The three silhouettes are deliberately coded as female. This choice stems from a desire on the part of the core mathemalchemists, twenty of whom are women, to support the cultural shift that is making mathematics a more gender-inclusive field. In the installation, we celebrate the mathematical work of many women. A "Great Doodle Page" quilt floats with the cavalcade above the playground (Figure 13C); it features drawings by women mathematicians, including Maryam Mirzakhani, to date the only woman awarded a Fields Medal. On the artistic side, *Mathemalchemy* incorporates a variety of art forms such as crochet, knitting, and cross stitch, that the fine arts community has traditionally dismissed as "crafts" or "women's work" but that are weaving their way into fine art. The quilted bakery floor is appropriately adorned with a

(A) Perched on a stack of books, the child blows into a trumpet.
(B) A cavalcade of mathematical pages flutters into the world.
(C) The Great Doodle Page features doodles by seven women mathematicians.

Figure 13. We inspire and are inspired by mathematics.

convex pentagonal tiling discovered by amateur mathematician Marjorie Rice in her kitchen. Of the now-proven complete list of fifteen pentagonal tiling types, Rice discovered four, working in her spare time [7]. Her story serves as a reminder that mathematicians can be anyone, anywhere. Our hope is that *Mathemalchemy* fulfills a similar purpose, inviting its audience to see mathematics as an enterprise open to all intrepid explorers.

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Credits

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Interview with Richard P. Stanley

The Wonderful World of Combinatorics

Interviewed by Shaoshi Chen

Richard Stanley has been Professor Emeritus of Mathematics at MIT since January 2018. He received his BS in mathematics from Caltech in 1966, and his PhD in mathematics from Harvard University in 1971, under the direction of Gian-Carlo Rota. He joined the MIT faculty in applied mathematics in 1973, and became a professor in 1979. Professor Stanley's research concerns problems in algebraic and enumerative combinatorics. Professor Stanley's distinctions include the SIAM George Pólya Prize in applied combinatorics in 1975, a Guggenheim fellowship in 1983, the Leroy P. Steele Prize for Mathematical Exposition in 2001, the Rolf Schock Prize in Mathematics in 2003, and the Leroy P. Steele Prize for Lifetime Achievement in 2022. Professor Stanley was the inaugural Levinson Professorship Chair of Mathematics at MIT, 2000–2010. He was appointed Senior Scholar at the Clay Mathematics Institute in 2004, and received an Honorary Doctorate from the University of Waterloo. In 2007, he received an Honorary Professorship from Nankai University. He is a Fellow of the American Academy of Arts & Sciences (1988) and a Member of the National Academy of Sciences (1995). He was an invited speaker at ICM1983 and a plenary speaker at ICM2006.

Figure 1. Richard Stanley in the memorial room of Professor Wen-tsun Wu.

Shaoshi Chen is an associate professor at the Key Laboratory of Mathematics Mechanization at the Academy of Mathematics and Systems Science, Chinese Academy of Sciences. His email address is schen@amss.ac.cn.

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This interview took place on October 14, 2019 at the National Center for Mathematics and Interdisciplinary Sciences, Chinese Academy of Sciences.

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Figure 2. Richard Stanley with the author in the memorial room of Professor Wen-tsun Wu.

SC (Shaoshi Chen): Thank you very much for accepting this interview. We would like to ask you some questions about you and your mathematics. Let's start with: When did you realize that you loved mathematics and wanted to be a mathematician?

RS (Richard Stanley): I think it was roughly at the age of thirteen when I was in the ninth grade. I moved to Savannah, Georgia. There was another student in my class (named Irvin Asher) who seemed to be doing something that I thought was very advanced mathematics. I was very interested in what he was doing and became inspired to learn more about mathematics. That was the start. I then began to go to the library and check out all of the popular math books in order to learn as much math as I could.

SC: Did anyone help you understand the books? Or did you just study by yourself?

RS: At that time in Savannah there were really no mathematicians, no good universities or anything similar, and I was on my own.

SC: When did you make the decision that you wanted to be a mathematician?

RS: Maybe in high school when I was thirteen or fourteen years old. I thought I would go into astronomy or physics, and after the experience with this classmate, then math became a possibility. By the time I graduated from high school, I was pretty sure that I preferred mathematics. In college, I took a lot of physics courses to see which I liked better, but I never really changed my mind about math.

SC: During your undergraduate study, was there a teacher who influenced you very much? Who made you understand or learn more about mathematics?

RS: I remember there were many good teachers at Caltech. Maybe the one with the most influence on me was Marshall Hall. He is a very well-known group theorist, but he was also interested in combinatorics and wrote one of the few books on combinatorics at that time. I was mainly interested in group theory from his influence, not combinatorics. In fact, I was not interested in taking a combinatorics course from Marshall Hall because I did not consider combinatorics to be a serious subject! I wanted to go to Harvard University to be a graduate student and work with the group theorist Richard Brauer. Finite group theory is like a combination of combinatorics and algebra, so you can see I had some nascent combinatorial interests. But when I graduated from Caltech I was thinking only of algebra, or perhaps number theory. Incidentally, I took a graduate group theory course from Marshall Hall, and in the same class was Michael Aschbacher who became very famous later for his role in the classification of finite simple groups. I should also mention that it was Marshall Hall who got me a summer job at JPL (the Jet Propulsion Laboratory, operated by Caltech for NASA, and responsible for the missions of unmanned extraterrestrial spacecraft). I spent around seven consecutive summers working in the coding theory group at JPL.

SC: Maybe I will ask a stupid question, namely, I know there is another group theorist named Phillip Hall. So is there any connection between those two Hall's?

RS: People always ask this question. The answer is that there is no connection.

SC: I know that the famous combinatorist Gian-Carlo Rota was your PhD supervisor. Can you say something about Rota and how he supervised you or how he influenced you during your graduate study?

RS: Yes, certainly his biggest influence was to get me to work in combinatorics. I never thought that it was really a serious subject, but I got interested in some combinatorial problems mainly from my job at JPL. I asked people at Harvard (where I was a graduate student) about my problems. They suggested that I should see Gian-Carlo Rota at MIT. Rota was very enthusiastic, suggesting all kinds of combinatorics that I should learn and convinced me that I should work with him in combinatorics. But I actually ended up doing most of the work on my own, just talking to him about other topics, not directly related to my research.

SC: What is combinatorics in your opinion?

RS: People are always asking me about that. The answer depends on their level of mathematics. Combinatorics deals with discrete structures. There are many different questions you can ask: what are the best ways or most efficient ways of arranging things and organizing them? How many ways are there to do it? How easy is it to find these solutions or prove that solutions exist or connect them with other things? It's very basic for much of mathematics to arrange discrete objects according to certain rules in the best and most elegant ways, or to see how many ways (or approximately how many ways, if an exact answer cannot be found) there are to do it, and to understand the structure formed by all these arrangements. Thus I think the problem of how to arrange discrete objects in certain interesting and elegant ways, and to understand how many of them there are and how they are related to each other, are basic questions in combinatorics.

SC: *On your MIT homepage, it says that your main research area is algebraic combinatorics. Can you give a brief introduction to algebraic combinatorics?*

RS: Well, it is basically the way in which combinatorics and algebra are connected with each other. I like both subjects: combinatorics and algebra—they are very natural to me. In algebraic combinatorics, you can go in either direction. You can apply algebra to combinatorics by embedding combinatorial objects into algebraic structures and using algebra to obtain information about the combinatorics objects. Or you can do it in the other direction, to understand the algebraic objects by looking at their combinatorial properties, and applying combinatorial reasoning. People were doing this since Euler, Jacobi, Cayley, and others. Of special importance to algebraic combinatorics today is the work of Frobenius, Young, Schur, and others on the representation theory of the symmetric group. These people never really thought of algebraic combinatorics as a separate subject, but they were very well-qualified to find some connections between algebra and combinatorics. The idea, mainly due to Rota, to try to develop algebraic combinatorics (also geometric combinatorics) in a systematic way really appealed to me.

SC: *Recently, I was reading the AMS book* The Mathematical Legacy of Richard P. Stanley, *which is in celebration of your 70th birthday. In this book, you wrote a very nice note on all your publications, briefly discussing how your papers started and how you finished them. You mentioned the "wishful thinking proof technique." I was curious about how this technique works.*

RS: Yes, this is the idea that when you're trying to prove something, you think about what is the best possible situation that could be true so that you could prove it or at least make further progress. It's a kind of wishful thinking that if only this were true, then I would have a chance of proving it. There's no reason to believe it's true at first, and most of the time it doesn't work. But several times in my career it actually worked amazingly well. For instance, a complicated power series in several variables arose in the theory of reduced decompositions of permutations. I had no idea on how to get any information about it unless it happened to be a symmetric function. It was just wishful thinking that it might be a symmetric function; I didn't have any reason to believe that it should be. I did some computations and saw that it seemed to be true. Once it seems to be true, then you can try to prove it and use it. I think it's a good idea when doing research to try to think about the best thing that could happen so you can make further progress, and then see if it works.

SC: *I think it's connected to your mathematical intuition. Before you really do the rigorous proof, you have intuition to believe it's true.*

RS: I call it wishful thinking because it is just a hope that might work. The key difference between wishful thinking and other kinds of conjecturing and guessing in mathematics is that in wishful thinking, you don't have any reason for believing your wish might come true. Intuition is not yet involved, but a good knowledge of possible tools and techniques is essential.

SC: *Many of your papers arise from some questions asked by non-mathematicians or mathematicians who are not doing combinatorics. Your answers to these questions increase the impact of combinatorics on other topics because it impresses people that combinatorics is a powerful tool to solve some questions that they want to answer. Can you say something about how combinatorics is connected to other areas, similar to how you already talked about the connection between algebra and combinatorics, according to your understanding?*

RS: Many subjects are connected with combinatorics. In the past, people worked on their problems and usually found the solutions by doing some combinatorial computations. They just leave it at that and think that's the answer. But combinatorics can actually push you much further. Solving combinatorial problems that arise in other areas gives you all kinds of insights into these areas. For instance in topology, which has concepts like tori, spheres, and ways of putting on handles and twisting space, you can easily imagine that some combinatorial structures are involved. Biology is another example that is full of combinatorial considerations, from phylogenetic trees to the genetic code to protein folding. The question is how to make this intuition rigorous and precise and related to serious combinatorics. I think that almost any branch of mathematics, as well as many other areas, has some connection to combinatorics. You could try to do some serious mathematics by explaining these connections in a precise way.

SC: You have written two classical books on enumerative combinatorics, Enumerative Combinatorics, *volumes 1 and 2 (known as* EC1 *and* EC2*) which now are perhaps the most influential books in combinatorics. When did you start this writing project?*

RS: It was in 1979, I was visiting UCSD (University of California at San Diego) when Pierre Leroux was visiting there from Montreal. We decided that it would be a good idea to write a book on enumerative combinatorics. I would write the text and he would make the problems. That was the original plan. But after I started to work on it, I realized that I wanted to do the whole thing by myself. I had a strong idea about how to do the problems and in many cases their solutions. So that was the start in 1979, and it took seven years before the first volume came out. That volume almost doubled in size in a second edition published in 2012. The second volume was published in 1999.

SC: In these books there are many interesting exercises which come from different papers. How did you collect so many interesting exercises? How did you organize them?

RS: Even before I started writing the book, maybe just after I got my PhD, I thought it would be a good idea to write down any kind of facts that I thought interesting or possibly useful. As you know there were no personal computers back then, so I got a lot of 3x5 index cards. I just wrote down some interesting fact on each card and put them in a file box. I still have them at my MIT office. When I started writing up exercises for my book, I used these cards as a basis. Any time I came across something interesting from reading, going to lectures, etc., I would just write it down on an index card. Thus this was my way to collect exercises.

SC: It's a very good habit. Could you talk about some of your work in graph theory and your opinion on the importance of the theory of graphs and networks?

RS: Well, concerning my own work in graph theory, I think it is purely algebraic and enumerative, and not too closely related to all these applications in network theory, operations research, etc. As an example of my work in graph theory, I earlier gave a talk in your institute about the chromatic polynomials of graphs. They are a special case of characteristic polynomials of hyperplane arrangements. They're connected with Mobius functions and partially ordered sets. I found all these connections very interesting. What could you say about chromatic polynomials or other polynomials arising from combinatorics? One special topic is about the reciprocity that arises when a polynomial is defined to have some nice combinatorial meaning for positive integers, and then turns out to have a nice meaning for negative integers. In a rather indirect way I was led to

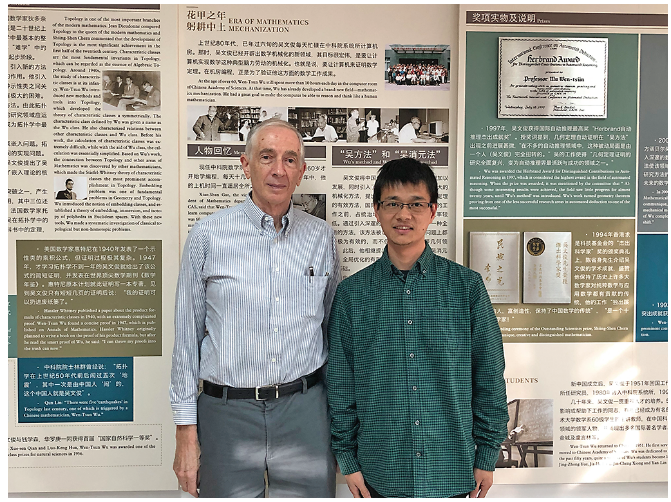

Figure 3. Richard Stanley with the author in the memorial room of Professor Wen-tsun Wu, in front of the bulletin board about how Wu promoted Computer Mathematics in China.

ask whether something like this happens with chromatic polynomials. That was one motivation for me to work in the area of graph theory.

SC: Nowadays we use computers every day and everywhere. Can you say something about the influence of computers on combinatorics, or on mathematics in general?

RS: Well, for just one thing, you can do experiments with a computer far beyond what you can do by hand. Sometimes you make discoveries that you just never could see by hand. In order to see a pattern in some data, you might have to work up to n=16, but by hand you can only go up to n=4. There are many examples like that. For myself, I use computers as a way to generate data for making conjectures. But as you know some people use computers to prove new theorems, like proving the four-color conjecture or Kepler's conjecture. I don't really go in that direction, but certainly these are very interesting developments which are different from just generating a lot of data and trying to make sense of it.

SC: I think you have visited China many times. When was the first time?

RS: In 1986, I attended a graph theory conference at Shandong University in Jinan. After the meeting I spent a few days in Beijing and visited Beijing University.

SC: Can you tell us some stories about your collaborations with Chinese mathematicians?

RS: Well, it started out with Bill Chen, my mathematical brother, that is, we have the same thesis adviser (Gian-Carlo

Figure 4. In the coffee room of Academy of Mathematics and Systems Science, CAS: Richard Stanley (sitting next to Atsuko Kida) was discussing the e-positivity conjecture with Dun Qiu, Arthur L.B.Yang, and Philip B. Zhang.

Rota). When Bill was a graduate student at MIT, I was a professor and taught him some classes. We started our collaboration then. Subsequently I had some Chinese PhD students. I didn't write joint papers with them when they were students (I like to have my students work as independently as possible), but I certainly collaborated with them on their thesis research. I guess you know Fu Liu, my Chinese student with whom I have most collaborated. I had very good collaborations with her resulting in two published papers. My Chinese graduate students (including students who grew up in Taiwan) and their degree dates are Bo-Yin Yang (1991), Wungkum Fong (2000), Fu Liu (2006), Jingbin Yin (2009), and Nan Li (2013). Other Chinese mathematicians I have worked with include Ruoxia (Rosena) Du, Yinghui Wang, Yuping (Eva) Deng, Huafei (Catherine) Yan, Xiaoying (Ellen) Qu, Xingmei (Sabrina) Pang, Beifang Chen, Xiaomei Chen, Xueshan (Teresa) Li, Lili Mu, Wuxing (Tommy) Cai, and Guoliang (David) Wang. Often I would first meet them in China at some meeting, and they would ask if they could visit MIT. I would make the arrangements, and then we would start the collaboration.

SC: *Yes, it's very helpful that people can go to MIT so that they can communicate with leading scholars, and thereby enlarge their research area. Up to now, you have supervised 60 PhD students in total. Many of them have become famous like Ira Gessel and Thomas Lam. Do you have any advice for young researchers or PhD students, especially in combinatorics?*

RS: Of course, giving some general advice to any students is not so easy, but I think it's very good that the students learn how to generate their own research problems or questions. Always keep your eyes open to something that is interesting, and if you think there's some way that you might be able to make a contribution, you shouldn't be afraid and say "that's not my area" or "people have already worked on this." You should, as long as you have some kind of "reasonable" idea, go ahead and pursue it. You need to have a bit of judgment about what is reasonable. For instance, I would not recommend working on the Riemann hypothesis unless you really think you have a new idea. When someone gives a talk and mentions some problem, if you find it interesting and even if it's not your exact area, I would say keep it in mind, go for it. Even if you don't get anywhere, you should try to remember everything, because you never know whether someday you might find something that you can use to eventually make progress or even solve the problem.

SC: *My last question. Which open problem in combinatorics is your favorite one? I think you have many in your mind. And you would like to see the solutions in the future?*

RS: Well, I could name two. The first one is definitely the g-conjecture for spheres (or certain more general simplicial complexes called Gorenstein). What can be the number of faces of each dimension of a triangulation of a sphere? Lou Billera, Carl Lee, and I solved this problem for simplicial convex polytopes in 1979. Since 1979, the g-conjecture is probably the biggest open problem in the area of the combinatorics of simplicial complexes. There has been some recent progress on this problem, in particular, by Karim Adiprasito, who posted a proof of the g-conjecture for spheres on the arXiv (see `https://arxiv.org/abs/1812.10454`), but so far no one has read the whole proof. It will be very exciting if it's correct.[1]

Another open problem I especially like is the e-positivity conjecture for certain chromatic symmetric functions, because it's connected with so many other areas like Kazhdan-Lusztig theory and Hessenberg varieties. I think there's all kinds of deep mathematics going on behind this conjecture, and I would really like to see what this looks like.

Credits

All figures are courtesy of the author.

[1] *In an email to me on March 17, 2022, Richard asked me to add a footnote here: "After this interview, Adiprasito's proof was carefully checked and is now accepted as correct."*

Artist Residency Sparks Fruitful Collaboration

Sophia D. Merow

Edward Crane and Liam-Taylor West began meeting in September 2020, sometimes via Zoom, sometimes at one of the glass whiteboards in the courtyard of the University of Bristol's Fry Building.

During these one-on-ones, Crane sketched—without recourse to equations, mind—the big ideas behind his research[1] on self-organized criticality. Consider such complex real-world phenomena as avalanches, financial crashes, and electrical cascades in the brain, he prompted. All involve a slow build-up of energy or potential followed by a large release, the precise timing of which is unpredictable. At scales large and small, Crane explained, the frequency of these releases is related to their size by a power law. Self-organized criticality aims to understand why.

As the two talked, Taylor-West tried to distill what Crane told him into a handful of pithy phrases, jotting them in his notebook to run by Crane for accuracy later. Until he had a solid grasp of the concepts discussed, Taylor-West did his best to refrain from considering how they might lend themselves to audiovisual artistry.

Then Crane shared some colorful videos showing simulations of forest fire models.

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[1]*See, for instance,* `https://arxiv.org/abs/1811.07981`.

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DOI: https://dx.doi.org/10.1090/noti2511

Taylor-West's conversations with Crane were part of his education as an artist-in-residence at the University of Bristol's School of Mathematics. A composer working toward a doctorate at the Royal College of Music in London, Taylor-West had been drawn to the residency—enabled by a partnership between the university and CREATE-REACT, an organization that pairs artists and scientists to foster interdisciplinary collaborations—because he had already begun injecting some mathematics into his music. He was incorporating randomization processes as a way of adding variation and unpredictability to how chords were built, for instance. "I felt that the ideas being studied by mathematicians would be an excellent source of inspiration for my work and would push me beyond my own simple experiments," he remembers.

More than 20 Bristol mathematicians expressed interest in exploring with Taylor-West how their research might inform or inspire sonic art. Crane, for one, hoped that working with Taylor-West might bolster his perhaps rusty ability to communicate with the mathematically uninitiated. "I think we owe it to the taxpayers who fund much of our research to explain something about what we're doing and why, in a non-technical way," he says. "The idea of being helped to express something about my research in a totally new and non-technical way appealed to me."

Figure 1. Liam Taylor-West's residency-culminating exhibition, *IN/FINITE: Order in the Unknown*, ran at Liberty House in Bristol January 19–30, 2022.

From the spark of Crane's simulation show-and-tell grew "Forest Fire," one of five[2] interactive light and sound installations included in Taylor-West's January 2022 exhibition *IN/FINITE: Order in the Unknown*. While a mathematician might recognize in the piece a very small instance of the Drossel-Schwabl forest fire model,[3] there's no mathematical prerequisite for viewer engagement with the 10-by-10 grid of LED lights. A green light represents a tree, a red light fire. Trees grow and fires start at random, and viewers can alter the probabilities of these events via a no-touch controller. As forests sprout, as fires ignite and spread to neighboring trees, cascades of color illuminate the grid.[4]

And there's a musical dimension too! Each of the grid's 100 squares can be dark, red, or green, and there's a sound associated with each of these possible states. A dark square is silent. When a square goes green, it triggers a slow melody that outlines a chord, which becomes more dissonant as it rises. When a square turns red, it produces a sharp, punch bass note; these notes form a rhythmic sequence as the forest goes up in flames. Taylor-West added variety to the simulation's score by recording several versions of the melody, some on clarinet (Lloyd Coleman) and some on synthesizer (Georgie Ward). The bass note associated with the fire also varies among three possibilities, switching whenever more than half of the grid is green.

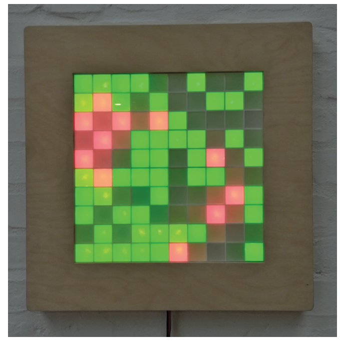


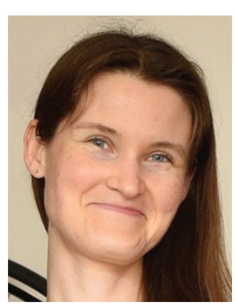
Figure 2. A freeze-frame of "Forest Fire" in action. See video—and hear sound!—at `https://bit.ly/33nz3Bc`.

Crane and Taylor-West both deem "Forest Fire" a success. Taylor-West thinks he accomplished what he set out to do with the piece: to render the workings of the forest fire model obvious to the general viewer and to convey the sense that random events can create structured, predictable systems. Crane, who vetted both the code that underlies the installation and the few sentences of descriptive text that accompany it, admires how Taylor-West identified the heart of the self-organized criticality matter and devised an intuitive and engaging way to present it to a general audience. "I love the way that by interacting physically with the piece you get a great sense of how the feedback works, which you wouldn't get from simply reading about it or looking at graphs," he says.

And both Taylor-West and Crane expect the residency to have a positive impact moving forward.

"Almost all of the work I have lined up is related to it in some way," Taylor-West reported in February. He was writing a piece for the BBC Concert Orchestra based on the structure and growth of aperiodic sequences (to which Bristol mathematicians Felix Flicker, Henna Koivusalo, and Demi Allen had introduced him); he and some of his School of Mathematics collaborators were hoping to secure grant funding for continued work together; "Forest Fire" and the other pieces from the *IN/FINITE* show were slated for exhibition at other venues.

For his part, Crane came away from the collaboration more motivated than ever to devote time and effort to explaining his work to non-mathematicians. "I think it is important that this is seen as part of our professional responsibility and not just a nice add-on," he says. "Working with Liam has reminded me that 'explaining my work' can mean a huge range of different things, depending on the audience."



Sophia D. Merow

Credits

Figure 1 is by Lloyd Coleman.
Figure 2 is by Liam Taylor-West.
Author photo is by Igor Tolkov.

[2] *Read more about the other installations at* `https://bit.ly/3h6KHn5`.

[3] *"It is quite closely related to site percolation on the square lattice, a model that a lot of mathematicians would be familiar with," says Crane. "The Drossel-Schwabl model is simple but it turns out to be very difficult to analyse mathematically, unlike the mean-field forest fire model (which I work on) and some one-dimensional forest fire models that are pretty well understood. The Drossel-Schwabl model hasn't been proven to display the key features of self-organized criticality, and big simulations suggest that maybe it doesn't. But it does make for a very nice visual model of the feedback mechanism."*

[4] *Watch footage of "Forest Fire" in action, and you'll spot a third color: yellow. Taylor-West added a process to the piece that was not present in the mathematical model that inspired it: tree aging. Green squares in the artwork become more yellow over time. "This means that if the probability of fire breaking out is really low," Taylor-West notes, "you eventually end up with a full grid of green lights slowly turning yellow, which adds to your sensation as an audience member that a fire MUST be about to start soon (which of course is not how random events work, but it is interesting that we expect a fire to be more likely after a period without any)."*

From the AMS Secretary

TO ALL AMS MEMBERS: PLEASE VOTE

Voting Information for the 2022 AMS Election

YOUR VOTE COUNTS!

Voting Online

AMS members who have chosen to vote online will receive an email on August 15, 2022.

The email will come from "AMS Election Coordinator" via **noreply@directvote.net** and the subject will be "AMS 2022 Election—login information below" (you may want to use this information to configure your spam filter to ensure delivery of this email). The body of the message will provide your unique voting login information and the web address of the voting website.

Please vote by midnight (US Eastern Time) on **November 1, 2022**. After midnight, the website will stop accepting votes.

Voting by Paper Ballot

AMS members who have chosen to vote by paper will receive their ballot by the middle of September. Unique voting login information will be printed on the ballot should you wish to vote online.

Paper ballots received after **November 1, 2022** will not be counted.

For Further Information

Additional information regarding the 2022 AMS Election is available in Frequently Asked Questions at: **www.ams.org/election-info.**

If you have questions, please email **election@ams.org** or call 800.321.4267, extension 4129 (US & Canada), 401.455.4129 (worldwide).

Thank you, and please remember to vote.

Boris Hasselblatt

AMS American Mathematical Society

Advancing research. Creating connections.

To learn more visit:

www.ams.org/election-info

Report of the Executive Director

Credit: Mike Nyman Photography

Catherine A. Roberts, Executive Director

As AMS Executive Director, my job is to manage and coordinate all Society operations and serve as the primary liaison between staff and volunteer bodies that govern the Society. In this report, I'll describe AMS activities and accomplishments from 2021. The mission of the AMS is to advance research and we do this mainly through conferences and publications. We also have a range of professional programs and services, and we advocate in DC on behalf of our mathematics community.

The AMS budget typically increases each year, although in 2021 we budgeted lower than the prior year in anticipation of negative impacts related to the COVID-19 pandemic. Nonetheless, the AMS was able to continue its practice of ending the year on a positive financial note, details of which will appear in a future issue of the *Notices*. We returned to our usual practice in 2022 with an increased budget of approximately \$33 million. Our primary revenue sources are derived from library subscriptions to AMS products such as MathSciNet® and our research journals. Our endowment, currently valued at about \$230 million, funds our many prizes and awards and provides operating income that supports a wide variety of professional programs and activities. We also receive membership dues, conference registration fees, and income from selling our books. Did you know that purchasing books directly from the AMS Bookstore (`https://bookstore.ams.org`) will both save you money and increase the amount that comes to the AMS over purchasing elsewhere?

All of our income, along with donation and grant support, helps us advance research and support the mathematics profession. We achieve this through AMS meetings and conferences, a multitude of AMS programs, our member journals, *Bulletin* and *Notices*, as well as our DC-based advocacy work. We have a robust development program to help raise funds for new prizes and awards and to further support initiatives of the Society. The budget for the AMS each year follows several key principles, including that we apply our revenue to keep our fees low. Did you know that the AMS subsidizes membership benefits, AMS conferences, and more in order to help keep prices low so as to benefit as many people as possible? Indeed, we publish books when we believe they will be of value to the mathematics community, even as we recognize that some may not be commercially viable. Our DC-based advocacy work ensures the voices and values of our community are represented on many important issues and is completely paid for by the AMS. Did you know that 70% of federal funding for mathematics comes from the National Science Foundation? We work with other scientific societies and Congress to increase support for the NSF, which helps advance research across our discipline. Indeed, in 2021, our AMS Director of Government Relations made some 75 visits with Congressional offices, many times accompanied by other mathematicians. This is vastly more than in preceding years, which was made possible because these were all virtual meetings due to the pandemic.

There have been several significant donations to the AMS in 2021 and I'll mention just a few here. One donor has agreed to underwrite a new translation effort to make widely available historically important transcripts of mathematics lectures from the University of Göttingen. The AMS received a pledge of $750,000 from another donor to support our BEGIN (Business, Entrepreneurial, Government, Industry, and Non-profit sectors) initiative, which will enhance employment connections for mathematicians outside of academia. We learned recently that we will receive an unrestricted bequest of roughly $3 million from the estate of a donor. Generous gifts such as these are testaments to the confidence our many donors have in the long-term mission and values of the AMS. Have you ever thought about contributing to a fund at the AMS in support of mathematics? If you have thoughts about helping the AMS enhance its ability to increase our prizes, awards, programs, and activities and would like to have a conversation with me, please reach out at `croberts@ams.org`.

It seems the pandemic affected just about everything at the AMS. And even today, we recognize that our work and mission have been impacted in ways we don't yet fully understand. AMS staff had to figure out how to accomplish our goals working from home. We replaced office desktop computers with portable laptops. We implemented cloud storage, sharing of files, and new collaboration tools. As an unsurprising sign of the times, the AMS undertook multiple security-related projects. This included software upgrades, addressing issues of external network penetration and web application scans, and upgrading our internal network.

Perhaps most notably, the entire Mathematical Reviews pipeline was converted from paper to digital during the first months of the pandemic. This herculean task continued throughout 2021 as this conversion has enabled us to update the Mathematical Reviews database itself, opening the door to a greater variety of future enhancements to MathSciNet.

In July 2021, employees started to return to our AMS office buildings part time (although I'll note that our printing and distribution center maintained in person operations throughout the pandemic). Our DC office completed its move from Dupont Circle to its new location within blocks of the US Capitol building. The space is spectacular for hosting AMS committee meetings, as well as convenings of other small groups. Did you know that if you contact us ahead of time, we can help arrange for you to meet members of your Congressional delegation and we can coach you on how to have an effective visit? We hope you will visit when you are in the area!

In the fall, as the pandemic threat lessened (and prior to the omicron variant that shut everything down again), three of Council's policy committees, as well as a meeting of the Executive Committee of the Council and the Board of Trustees (ECBT), were able to be held in person. The AMS organized an in-person Congressional briefing in December with the Mathematical Sciences Research Institute. It was one of the first such briefings in Congress in two years, and it was exciting to have mathematicians initiate the resumption of this important advocacy program in Congress. Dr. Cédric Villani spoke about *Mitigating climate change: science and policy*. Sadly, the resurgence of the pandemic at the end of 2021 forced us to postpone the in-person JMM 2022 and to offer it in a virtual format three months later. Throughout the two years of the pandemic, however, we successfully maintained one of our signature programs, the Mathematics Research Communities (MRCs) using virtual tools. We also ran several virtual sectional meetings, workshops, and webinars. We are now thinking deeply about how to engage our community of mathematicians effectively as we emerge from the pandemic. What are your ideas?

One of the most complex projects in 2021 involved finalizing plans for the reimagined Joint Mathematics Meetings[1] (JMM). In response to the AMS Council affirming that the JMM would be welcoming to all mathematicians, the Secretariat created a new expanded abstract proposal classification scheme[2] so as to permit a much wider range of topics. We introduced several new lectures, receptions, and other events. We created memoranda of understanding with our dozen (and growing!) partners. Our team rebranded the JMM with a new logo and developed promotional materials including emails, flyers, social media campaigns, signage, as well as the app and website. The new JMM Program Committee reviewed panel proposals and initiated

[1] `https://www.jointmathematicsmeetings.org`

[2] `https://www.ams.org/session-classifications`

a series of professional development opportunities called PEPs (Professional Enhancement Programs) in response to community feedback requesting more such programming. A communications campaign to introduce our partnership model and program additions has generated tremendous excitement. Will you join us in Boston in January 2023?

Significant advancements occurred in the publishing arena at the AMS in 2021. In October, *Communications of the AMS*, our brand new diamond open access primary journal, debuted its first article. The Mathematical Reviews database now has over 4 million items, having added a record number of new items (137,576) in 2021. Library subscriptions to MathSciNet remain stable, and use of the database continues to grow. In fact, MathSciNet had its busiest year ever in 2021, which continues the trend of an average increase in usage of over 9% per year over the previous five years. So, even with the pandemic, MathSciNet continues to be relied upon heavily worldwide. We published 75 new books, many of them research monographs. We have also made progress on accessibility to our content across our books and journals. For example, all 39 of the 2021 feature articles in *Notices of the AMS* were converted to HTML, which is an important step in our efforts to improve reader accessibility.

Next, some people news. In 2021 we on-boarded and supported several new members of governance and staff. In 2021, this included Dr. Doug Ulmer, Treasurer, Dr. Boris Hasselblatt, Secretary, Dr. Torina Lewis, Associate Executive Director of Meetings and Professional Services, and Lucy Maddock, Chief Financial Officer and Associate Executive Director of Administration. Our 2021–2022 Congressional Fellow is Dr. AJ Stewart, who is working for Senator Warnock from Georgia. We also welcomed Dr. Tyler Kloefkorn, Associate Director of the Office of Government Relations and Dr. Justin Eilersten, who joined us as an Associate Editor at Mathematical Reviews, with a specialty in numerical analysis and statistics. With 200 employees, the AMS has not been immune to the increased employee turnover and the severe labor shortage you have been hearing about. If you would like to join our talented team in support of the global mathematics community, our open positions are listed here: `https://www.ams.org/ams-jobs` and those requiring a doctorate can also be found on Mathjobs.org.

You may recall that part of implementation of our 2016–2020 Strategic Plan addressed communication. Effective communication remains an important priority for the AMS. We spent two years redesigning our AMS logo and propagating it throughout the organization. We made significant improvements to the printed and digital formats of the *Notices of the AMS*. We continue to progress through a multi-year project to upgrade ams.org. We upgraded our bi-weekly member newsletter, *Headlines & Deadlines*, as well as printed materials originating from Membership and Development. Did you know that you can log into your member profile at `https://www.ams.org` to set your preferences (e.g., paper or print *Notices* and *Bulletin*) and to share your demographic data to help us understand our membership better?

We established our first Communications Department, which has been fully staffed for one year. In its first year, this department helped centralize our pandemic communications and conducted campaigns around AMS priorities such as promoting JMM Reimagined. This department wrote and distributed 18 long-form feature articles highlighting the positive impact of AMS programs on mathematicians; several of these articles were among our most highly viewed and shared digital content of the year. Although we continue to develop and implement more robust and sophisticated approaches to our communication strategy and messaging, we have struggled as an organization to navigate some delicate situations. Given that we are living in a world where reasonable discourse seems to be diminishing daily, identifying resources of patience and grace as we transverse this rocky terrain is essential to moving forward. I am pleased that Dr. Leona Harris, our new Director of Equity, Diversity, and Inclusion, has expressed as one of her priorities leveraging AMS communications to better inform our community and help bring the community together. Working together, I believe we can serve all mathematicians.

In closing, it is my absolute pleasure to serve this mathematics community in my role as Executive Director. It is deeply rewarding to work with exceptional employees and hundreds of dedicated volunteers. The pandemic has been rough on everyone, including me. The support we give each other is so important. Even when we disagree on the best approaches, it is clear that we share a common desire to advance mathematics in the most impactful ways possible. Thank you.

Catherine A Roberts

Dr. Catherine A. Roberts
Executive Director
May 2022

Department Chairs Discuss Challenges and Solutions at Annual Workshop

Scott Hershberger

Mathematicians excel in learning from books and papers. But serving as an academic department chair presents a different set of challenges—nuanced interpersonal situations requiring soft skills no mathematics textbook can teach.

May Mei became chair as her department was restructuring its curriculum.

"We don't have any management training, [...] and all of a sudden we're middle management," said May Mei, who is starting her second year as the chair of the mathematics department at Denison University. "On the one hand, I now report to someone that's much higher up than me, and on the other hand, I'm in charge of people."

According to Mei, the only way to learn how to be an effective department chair is to talk with the leaders of other mathematics departments. To that end, the annual AMS Workshop for Department Chairs and Leaders provides a space for chairs from around the country to share their experiences and build a supportive community. The workshop is typically held in January immediately preceding the Joint Mathematics Meetings.

Around 50 participants gathered virtually on April 5 for this year's workshop, which focused on equity, diversity, and inclusion (EDI) in math departments. Attendees represented departments ranging in size from seven faculty and staff to more than 50 and spanning community colleges, liberal arts schools, and large research universities.

Scott Hershberger is an incoming master's student in science communication at the University of Wisconsin–Madison. When he wrote this piece, he was the communications and outreach content specialist at the AMS. His email address is `scotthersh42@gmail.com`.

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Making a Difference as Chair

Dave Kung began the workshop with an interactive session about diversifying the mathematics community. Kung, the director of policy at the Charles A. Dana Center and the director of MAA Project NExT, urged participants to consider how structures that seem neutral might actually hinder EDI. For instance, at many institutions, students with the most earned credits enroll before the rest of their peers each semester. Given that white students are more likely than Black students to enter college with credits, this system tends to push Black students out of the classes that they want or need in their first year, exacerbating an inequity that persists through graduation, he said.

Emille Davie Lawrence first attended the workshop in 2021 and co-organized it this year.

Next was a session on inclusive hiring practices co-led by Emille Davie Lawrence, the chair of the Department of Mathematics and Statistics at the University of San Francisco. "I feel like the conversation needs to start to shift and expand away from these very strict notions of what math faculty should look like," she said later. Rather than just considering the length and prestige of a candidate's publication list, she said, hiring committees should ask themselves, "[Does the candidate] reflect the students in your university? Would they potentially meet the needs of those students?"

Before becoming chair, Lawrence heard others in the position complain that it was too much work to manage the politics and personalities involved. But two years into the role, Lawrence has a different view: "I like trying to make a difference for the better. Being chair is a perfect opportunity to get a bird's-eye view of your department and try to fix the problems that you see."

As chair, Lawrence has worked to improve advisor-advisee relationships and successfully encouraged more incoming students to declare a math major or minor. She co-organized this year's chairs workshop and plans to do so for two more years, in large part to convey the joys of leading a department.

Ongoing Challenges

Along with Emille Davie Lawrence, Luca Capogna co-led a session on inclusive hiring practices.

Luca Capogna, a three-time co-organizer, first attended the workshop in 2014 and finds it valuable every year. Capogna served as department head at Worcester Polytechnic Institute from 2013 to 2020 and is now department chair at Smith College. Like Lawrence, he enjoys helping colleagues and students, though he acknowledges the many challenges that come with the role. Math departments are involved in research, teaching of majors, teaching of other STEM students, service to the college, and service to society at large. "Some of these needs are actually at odds with each other," he said. "It's a constant effort in trying to find an equilibrium."

Multiple workshop attendees noted that they carry substantial responsibility yet little authority. Limited resources, already a perennial concern, have taken center stage as a result of the COVID-19 pandemic. Capogna and others are struggling to fill gaps in the teaching staff. At schools like Denison, with increasing enrollment, but no budget for hiring new math faculty, "Our choice is basically to cut an upper-division course or to increase the size of [calculus] classes," Mei said. "Neither of those are comfortable options." Meanwhile, more faculty and students alike are struggling with their mental health.

All these stresses can lead to difficult conversations with students, faculty, staff, or university leadership. In the final workshop session, Kevin Knudson of the University of Florida and Anne Fernando of Norfolk State University advised attendees to approach these potentially fraught interactions from a place of empathy. First and foremost, Knudson said, a chair must listen and help people feel understood.

"A Deeply Human Endeavor"

Juan Gutiérrez is working to increase graduation rates at UTSA.

During session breaks, participants continued a lively discussion of the issues they faced. Juan Gutiérrez talked about his efforts to increase student success in introductory mathematics courses at the University of Texas at San Antonio (UTSA). Around 13,000 students—nearly one-third of the student body at this Hispanic-Serving Institution—take courses in the department each year.

When Gutiérrez became chair in 2019, he requested granular data on students' grades, choices of majors, and more. He found that success or failure in mathematics was the largest factor in whether students stayed in STEM majors or even graduated at all. "Mathematics truly is a measurable gateway to social mobility," he said.

For courses with a large standard deviation in the number of D's, F's, and withdraws (DFWs) across sections, the UTSA math instructors implemented tighter coordination of curriculum. Now a wiki holds lesson-by-lesson details for most courses. As a result of this and other data-driven efforts, the rate of DFWs in the department dropped from 38% to 25% in just two years.

For Gutiérrez and Lawrence, the workshop provided valuable insights into the broader trends in mathematics departments. "Once you are embedded in one single environment it is very difficult to know what is specific to you and what is general," Gutiérrez said. "The importance of knowing general problems is that they might permit general solutions, or at least the input of many people to try to find a solution." The event also fostered camaraderie and connections: The following week, Gutiérrez had one-on-one meetings with two fellow attendees to share more of the strategies that are working for him at UTSA.

As more than four dozen department leaders shared their perspectives, one common theme emerged. Whether a chair is negotiating with the dean, addressing student concerns, or sitting on a hiring committee, "it is a deeply human endeavor to try to lead other humans," said Mei.

Scott Hershberger

Credits

Photo of May Mei is courtesy of Aaron Conway, Denison University.
Photo of Emille Davie Lawrence is courtesy of Emille Davie Lawrence.
Photo of Luca Capogna is courtesy of Luca Capogna.
Photo of Juan Gutiérrez is courtesy of University of Texas at San Antonio.
Author photo is courtesy of Scott Hershberger.

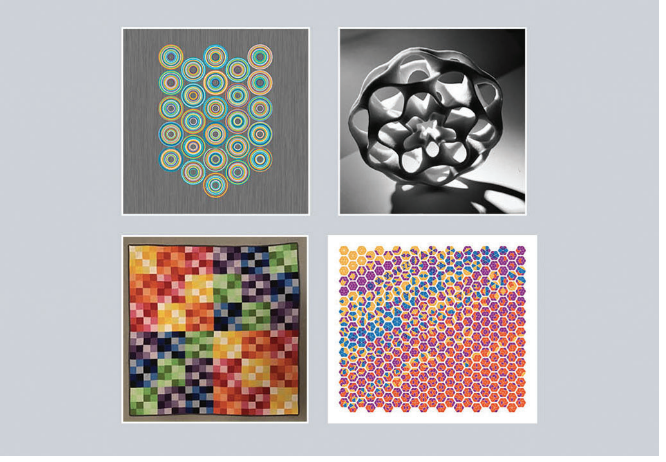
AMS Updates

2022 Mathematical Art Awards

The 2022 Mathematical Art Exhibit Awards were made in April at the Virtual Joint Mathematics Meetings (JMM) "for aesthetically pleasing works that combine mathematics and art." These awards were established in 2008 through an endowment provided to the AMS by an anonymous donor who wishes to acknowledge those whose works demonstrate the beauty and elegance of mathematics expressed in a visual art form.

The winners of the 2022 awards are:

- Best Photograph, Painting, or Print: David Reimann
- Best Textile, Sculpture, or Other: Laura Nica
- Co-Honorable Mention: Public Math and the University of Kentucky Math Lab (Nathan Fieldsteel, et al.)



Clockwise from top left: "Septenary Circles" by David Reimann, "Scherk Minimal Surfaces" by Laura Nica, "Is It Complete? Children's Solutions toThe Hexagon Challenge" by Public Math, "Symmetric group on 4 Letters" by University of Kentucky Math Lab.

View these and more than ninety other works from mathematicians and artists from around the world at `https://bit.ly/3P1RRc2`.

—AMS Programs Department

Anuraag Bukkuri Named 2022 AMS Mass Media Fellow

Anuraag Bukkuri

Anuraag Bukkuri, a PhD student in integrated mathematical oncology at the University of South Florida and Moffitt Cancer Center, is the 2022 AMS Mass Media Fellow. He is working this summer as a journalist for *The Miami Herald*, reporting on science and related topics.

In his research, Bukkuri applies perspectives from mathematics, ecology, evolution, and Earth history to cancer therapies. He studied mathematics as an undergraduate at Dartmouth College and the University of Minnesota. Growing up, he wrote short stories and poems, but his interests in science and writing did not overlap at first.

"In more recent years is when I saw a big gap in the mathematical field for bringing our ideas, our passion for math, [and] the applications of math in the real world to the public," Bukkuri said. In an article appearing in *The Conversation* last year, he described how evolutionary game theory could improve cancer treatment.

COVID-19, climate change, rising sea levels, the Everglades—all these scientific topics matter to the people of South Florida, Bukkuri said. He is glad to have the opportunity to talk with local experts and use his training as a mathematician to write for *Herald* readers in a clear and methodical style.

Bukkuri is one of 28 fellows embedded in newsrooms across the country this summer. "To join this group of truly distinguished people—not only in my cohort, but for decades—is truly an honor, and I hope to be able to learn from everyone over the years to come." After the fellowship, Bukkuri plans to pursue an academic research career while continuing to write.

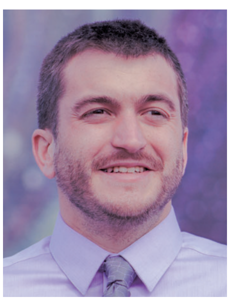
About the Fellowship

Organized by the American Association for the Advancement of Science (AAAS), the Mass Media Science and

Engineering Fellowship program improves public understanding of science and technology by placing advanced undergraduate, graduate, and postgraduate science, mathematics, and engineering students in media outlets nationwide. Fellows work for 10 weeks over the summer as reporters, researchers, and production assistants alongside media professionals to sharpen their communication skills and increase their understanding of the editorial process by which events and ideas become news. Now in its 47th year, the fellowship program counts over 750 scientists and science communicators as alumni. The AMS has sponsored a Mass Media Fellow most years since 1997. Learn more about the fellowship at `https://bit.ly/2tiZceq`.

—AMS Communications Department

Duncan Wright Named 2022–2023 AMS Congressional Fellow



Duncan Wright

Duncan Wright, a postdoc at Worcester Polytechnic Institute (WPI), will be the 2022–2023 AMS Congressional Fellow. He will spend a year working on the staff of a member of Congress or that of a congressional committee, assisting in legislative and policy areas that require scientific and technical input.

Wright earned his PhD in mathematics in 2019 at the University of South Carolina (USC), studying quantum information theory. At WPI, he collaborates with chemical engineers to model how neuronal cells retract after a change in the concentration of surrounding salts. He has also participated in STEM outreach to Massachusetts middle schoolers through WPI's STEM Education Center, STEAM Together, and AmeriCorps VISTA.

During his fellowship, Wright hopes to work on education policy, diplomacy, or foreign policy. "It's an honor to be able to try to effect change on a much larger scale than just my class of 50 [or] 100 students at a time," he said. "It's an incredible opportunity that I'm very excited for."

Wright became interested in policy while teaching at USC as a PhD student. Seeing some students fail college algebra multiple times, he knew that inequities throughout the educational system were at play, some of which could be addressed through better policies. He was also drawn to Washington by a desire to counter the increasingly toxic political environment.

"I grew up mediating amongst my family members. […] I was right in the middle, always trying to keep them calm," Wright said. "I want to bring a calm, steady voice to Washington to try to cut through some of the animosity and hatred." According to Wright, mathematicians' knack for logical reasoning can help ensure that policy decisions are made for legitimate reasons.

Wright is looking forward to learning about the inner workings of the legislative branch. He anticipates that the network he builds during the fellowship will facilitate his transition into the policy or nonprofit world.

About the Fellowship

Each year, the AMS sponsors a Congressional Fellowship in conjunction with the American Association for the Advancement of Science. The fellowship provides PhD mathematicians a unique public policy learning experience, demonstrates the value of science-government interaction, and brings a technical background and external perspective to the congressional decision-making process. In addition to working on the staff of a member of Congress or that of a congressional committee, fellows receive an orientation on congressional and executive branch operations and participate in a year-long seminar series on issues involving science, technology, and public policy. Learn more about the fellowship at `https://bit.ly/2X5Yi3D`.

—AMS Communications Department

Deaths of AMS Members

H. Bechtell, of Eatonton, Georgia, died on May 22, 2022. Born on April 26, 1929, he was a member of the Society for 65 years.

Herbert H. Diekhans, of Evanston, Illinois, died on December 23, 2019. Born on March 4, 1925, he was a member of the Society for 55 years.

Herbert I. Freedman, of the University of Alberta, died on November 21, 2017. Born on November 16, 1940, he was a member of the Society for 52 years.

Earl J. Taft, of New York, New York, died on August 9, 2021. Born on August 27, 1931, he was a member of the Society for 67 years.

Credits

2022 Mathematical Art Exhibit images are courtesy of the artists.
Photo of Anuraag Bukkuri is courtesy of Anuraag Bukkuri.
Photo of Duncan Wright is courtesy of Duncan Wright.

Mathematics People

Larremore Receives NSF Waterman Award


Daniel B. Larremore

Daniel B. Larremore of the University of Colorado Boulder has been named a corecipient of the 2022 National Science Foundation (NSF) Alan T. Waterman Award for his work in using mathematics to "track and understand the mechanisms by which human diseases spread." According to the prize citation, he has "used networks, modeling and mathematics in a diversity of topics, from infectious diseases and epidemiology to social inequalities and algorithms for network analysis." The citation reads in part: "Larremore's work during the COVID-19 pandemic used the tools of computational epidemiology to answer urgent questions about pandemic countermeasures. How should the first doses of a scarce vaccine be targeted to minimize deaths or infections? What role could widely available rapid testing play in mitigating viral transmission prior to the arrival of vaccines? In his research, Larremore combined mathematics and computation with real-world data to create new models that provide answers to globally important questions like these." Larremore received his PhD in applied mathematics from the University of Colorado Boulder in 2012. He was a postdoctoral fellow at the Harvard T. H. Chan School of Public Health (2012–2015) and an Omidyar Fellow at the Santa Fe Institute (2015–2017) before joining the faculty at Boulder, where he is an assistant professor in the Department of Computer Science and the BioFrontiers Institute. He tells the *Notices*: "I live with my wife, Liana, in Boulder, where we like to do stereotypically Colorado things like rock climbing, mountain biking, and skiing. My favorite days at work are the long afternoons spent with students and colleagues, taking turns at the whiteboard."

The Waterman Award annually recognizes an outstanding young researcher in any field of science or engineering supported by NSF. Researchers forty years of age or younger, or up to ten years post-PhD, are eligible. Awardees receive US$1 million distributed over five years.

—From an NSF announcement

Aggarwal and Tikhomirov Receive Rollo Davidson Prizes


Amol Aggarwal


Konstantin Tikhomirov

Amol Aggarwal of Columbia University and **Konstantin Tikhomirov** of the Georgia Institute of Technology have been awarded the 2022 Rollo Davidson Prizes. Aggarwal was honored "for fundamental contributions to random matrix theory and integrable probability." Tikhomirov was recognized "for deep new results on the singularity of random Bernoulli matrices." Aggarwal received his PhD from Harvard University in 2020 under the direction of Alexei Borodin. He was awarded a Clay Research Five-Year Fellowship in 2020 and the 2021 International Association of Mathematical Physics Early Career Award from the International Congress of Mathematical Physics. Tikhomirov received his PhD from the University of Alberta in 2016, advised by Nicole Tomczak-Jaegermann and Vlad Yaskin. From 2016–2018 he was an instructor at Princeton University, then he held a postdoctoral fellowship at the Mathematical Sciences Research Institute in Berkeley before joining the School of Mathematics at Georgia Tech. The Davidson Prize is awarded annually to early-career probabilists.

—James Norris
Chair, Rollo Davidson Trustees

DeMarco Named Noether Lecturer

Laura DeMarco

Laura DeMarco of Harvard University and the Radcliffe Institute for Advanced Study has been awarded the 2023 AWM–AMS Emmy Noether Lectureship of the Association for Women in Mathematics and the AMS. She will deliver the lecture at the Joint Mathematics Meetings in Boston in January 2023. The prize citation reads: "DeMarco has made fundamental and influential contributions to complex dynamics, arithmetic dynamics, and arithmetic geometry. In complex dynamics, she introduced the bifurcation current to study the stable locus in moduli spaces of rational maps and constructed a dynamically natural compactification of these spaces. Both groundbreaking ideas opened new directions of research in complex dynamics. She is a leading architect of the field of arithmetic dynamics. In her joint work with Matthew Baker, a far-reaching dynamical analog of the André–Oort conjecture in arithmetic geometry was formulated. Cases of the conjecture were proved using ingenious combinations of ideas from complex dynamics, logic, and number theory. In arithmetic geometry, her recent joint work with Krieger and Ye addressed a conjecture of Bogomolov, Fu, and Tschinkel on uniform bounds on the number of common torsion points on two elliptic curves, and they obtained the first uniform result for a complex family of curves in the Manin–Mumford Conjecture. This paper, published in *Annals of Mathematics* in 2000, won the 2020 Alexanderson Award of the American Institute of Mathematics."

DeMarco received her PhD from Harvard University in 2002 under the direction of Curtis McMullen. She has held positions at the University of Chicago (2002–2007), at the University of Illinois at Chicago (2007–2014), and at Northwestern University (2014–2020). She is an inaugural Fellow of the AMS. Other honors include an NSF CAREER Award (2008–2013), a Simons Foundation Fellowship (2015), and the AMS Ruth Lyttle Satter Prize (2017). She is active in serving the mathematics community, particularly undergraduate women in mathematics. She gave an invited address at the 2018 International Congress of Mathematicians. She is an elected member of the National Academy of Sciences.

—From an AWM announcement

Wang Awarded 2022 Brin Prize

Zhiren Wang

Zhiren Wang of Pennsylvania State University has been awarded the eleventh Michael Brin Prize in Dynamical Systems for his "fundamental contributions to the study of topological and measure rigidity of higher rank actions, and his proof of Möbius disjointness for several classes of dynamical systems." He received his PhD in 2011 from Princeton University under the supervision of Elon Lindenstrauss. He was Gibbs Assistant Professor at Yale University (2011–2014) before joining the faculty at Penn State in 2014 as an assistant professor. He has been associate professor since 2019. He was awarded an NSF CAREER grant for 2018–2023 and a von Neumann Fellowship at the Institute for Advanced Study in 2022.

The Brin Prize recognizes mathematicians who have made substantial impact in dynamical systems and related fields at an early stage of their careers. It carries a cash award of US$18,000.

—Giovanni Forni, Chair
Brin Prize Selection Committee

NCTM Awards Announced

The National Council of Teachers of Mathematics has announced its Lifetime Achievement Awards for the year 2021. The awardees are **Elizabeth Fennema** (University of Wisconsin-Madison), **Marta Civil** (University of Arizona), and **Steven Leinwand** (American Institutes for Research).

Elizabeth Fennema

Fennema received her PhD in education with emphasis on math education from the University of Wisconsin–Madison and taught at Madison from 1962 to 1997. She was a pioneer in mathematics education for women, among other accomplishments. According to the prize citation, she was "known for two field-changing bodies of work, either of which alone would be worthy of lasting recognition. First is her work about gender in mathematics. After publishing a review of gender differences literature in the *Journal for Research in Mathematics Education* in 1974, she teamed with Julia Sherman to produce what are now known as the Fennema–Sherman studies. With methodological rigor and new measurement tools (the Fennema–Sherman Scales), the pair redefined knowledge and perspectives on the intersection between gender and achievement in mathematics, showing that underperformance by females was sociocultural in nature and a function of opportunity, and not due to differences in biology. In the 1980s, Dr. Fennema joined Thomas Carpenter and others for another grand body of work that came to be known as cognitively guided instruction (CGI). The research program was a model for applying new theories of constructivism to children's mathematics learning. Dr. Fennema's work in CGI took equally seriously the development of professional development to empower teachers to use their findings to improve elementary mathematics education. This combination of understanding student cognition and developing teacher learning in the same research program was ambitious and relatively novel for the time. As a result, few, if any, mathematics research programs to date have been as comprehensive, rigorous, and beneficial to the field of mathematics education as CGI."

Fennema received numerous honors during her career for her work and service, including the Presidential Citation from the American Educational Research Association (AERA; 1997). She gave the inaugural awardee presentation for Outstanding Contribution to Research on Women and Education, and she was named a member of the National Academy of Education (NAE) in 1997. Elizabeth Fennema passed away in December 2021.


Marta Civil

Civil received her PhD in mathematics education from the University of Illinois at Urbana–Champaign. She joined the faculty of the Department of Mathematics at the University of Arizona in 1990 and currently holds the Roy F. Grasser Endowed Chair there. The prize citation reads in part: "One of her most notable achievements is that she served as the lead principal investigator for the National Science Foundation–funded Center for the Mathematics Education of Latinos. She is noted for her ability to continually organize a large and complex project that involved four research universities over the course of a decade. Dr. Civil's scholarly work has been and continues to be innovative and unique in mathematics education. She is nationally known for developing the project Math and Parent Partnerships in the Southwest, which was an NSF-funded project involving immigrant parents, primarily women whose home country is Mexico, in learning about mathematics education in the United States." She has contributed greatly to diversity, equity, and social justice in mathematics education, particularly in regard to the immigrant Mexican community, and is devoted to mentoring and supporting students of color. She is a prolific author of journal articles and book chapters, as well as coeditor of eleven books. She has served as an officer and on various committees for the National Academy of Sciences, the Association of Mathematics Teacher Educators, the NCTM Research Committee, TODOS–Mathematics for All, and Psychology in Mathematics Education. She enjoys traveling throughout the world and trying all kinds of food.


Steven Leinwand

Leinwand received an MS in educational supervision and administration from Central Connecticut State University in 1976. He taught high school for seven years in Connecticut, then served as a mathematics coordinator, as a project director for two National Science Foundation–sponsored summer institutes, and as a college instructor of secondary mathematics courses at Wesleyan University. As mathematics supervisor for the Connecticut Department of Education, he was responsible for "the development and oversight of a broad statewide program of activities in K–12 mathematics education, including the provision of technical assistance and professional development, the evaluation of programs, the assessment of student achievement and teacher competency, the dissemination of information, and the coordination of programs and activities that resulted in consistently high NAEP mathematics scores." He has been a member of the board of directors of the NCTM and a member and president of the National Council of Supervisors of Mathematics Board of Directors and has been involved in leadership of mathematics education in many capacities. He is currently the principal research analyst for the American Institutes for Research (AIR; 2002–present), working on projects that focus on high-quality mathematics instruction.

The NCTM Lifetime Achievement Awards are presented for achievement in leadership, teaching, and service for a minimum of twenty-five years of distinguished service to mathematics education.

—From NCTM announcements

Churchill Scholars Announced

The Winston Churchill Foundation of the United States has announced the recipients of the Churchill Scholarships for the academic year 2022–2023. Two mathematical scientists are among the awardees. **Andrew Burke** of the University of Notre Dame and **Steven Jin** of the University of Maryland, College Park, were both awarded scholarships in pure mathematics. The scholarships provide for one year of master's study at Churchill College, University of Cambridge.

—From a Churchill Foundation announcement

National Academy of Sciences Election

The National Academy of Sciences (NAS) has elected 120 new members and 30 international members for 2022. Following are the names and institutions of the mathematical scientists who are among the new members.

- **Fedor A. Bogomolov**, Courant Institute of Mathematical Sciences
- **Amir Dembo**, Stanford University
- **Michael Harris**, Columbia University
- **Svetlana Jitomirskaya**, University of California, Irvine
- **Mikhail Lyubich**, Stony Brook University
- **Michael Shelley**, Courant Institute of Mathematical Sciences
- **Karen Vogtmann**, University of Warwick

Elected as international members were:

- **Alice Guionnet**, Ecole Normale Supérieure de Lyon (France)
- **Yurii E. Nesterov**, Université Catholique de Louvain (Belgium and Russia)
- **Laure Saint-Raymond**, Institut des Hautes Etudes Scientifiques (France)
- **Sara A. van de Geer**, ETH Zürich (the Netherlands).

Harris, Lyubich, and Vogtmann are members and Fellows of the AMS. Jitomirskaya is a member of the AMS. Vogtmann previously served as AMS vice president and as a member of the AMS Board of Trustees.

—From an NAS announcement

AAAS Election

The American Academy of Arts and Sciences has elected 261 new members to the class of 2022. The following new members were elected to the Section on Mathematics, Applied Mathematics, and Statistics.

- **Roman Bezrukavnikov**, Massachusetts Institute of Technology
- **Panagiota Daskalopoulos**, Columbia University
- **Wilfrid Gangbo**, University of California, Los Angeles
- **Alice Guionnet** (International Honorary Member), Ecole Normale Supérieure de Lyon
- **Haruzo Hida**, University of California, Los Angeles
- **Mark Kisin**, Harvard University
- **Rafe Mazzeo**, Stanford University
- **Endre Szemerédi**, Rutgers University
- **Claire Voisin** (International Honorary Member), Centre National de la Recherche Scientifique
- **Sijue Wu**,University of Michigan

The following new members whose work also involves the mathematical sciences were elected to other sections.

- **Shamit Kachru**, Stanford University
- **Robert Calderbank**, Duke University
- **Adi Shamir** (International Honorary Member), Weizmann Institute of Science
- **Leslie Valiant**, Harvard University
- **Alberto Abadie**, Massachusetts Institute of Technology
- **Elie Tamer**, Harvard University
- **Bernard Harris**, National Math + Science Initiative

Gangbo, Hida, Kisin, Mazzeo, and Calderbank are members and Fellows of the AMS.

—From an AAAS announcement

Fellows of the Royal Society

The Royal Society of London has announced the names of its newly elected Fellows for 2022. Following are the new Fellows and Foreign Fellows whose work involves the mathematical sciences.

- **Luis Fernando Alday**, University of Oxford
- **Alain Goriely**, University of Oxford
- **Jane Hillston**, University of Edinburgh
- **Mark Newman**, University of Michigan
- **Yvonne Rogers**, University College London
- **Paul Seymour**, Princeton University

Elected as Foreign Fellows were:

- **Peter Scholze**, University of Bonn
- **Howard Stone**, Princeton University

—From a Royal Society announcement

MathWorks Math Modeling (M3) Challenge

The 2022 MathWorks Math Modeling competition asked students to use math modeling to predict the future of remote work, analyzing the percentage of jobs that are remote-ready and whether workers in those jobs will be willing or able to work remotely, then determining the percentage of workers who will go remote in a given city or metro area.

The Challenge Champion team prize of US$20,000 was awarded to a team from Homestead High School in

Mequon, Wisconsin. The team members were **Adam Garsha**, **Jacob Schmidman**, **Eric Wan**, and **Ethan Wang**. Their coach was Weizhong Wang.

The Challenge Runner Up team prize of US$15,000 went to a team from New Trier Township High School, Winnetka, Illinois. The team members were **Aruni Chenxi**, **Nika Chuzhoy**, **Max Hartman**, **Connor Lane**, and **Nathan Liu**. They were coached by Bradley Kuklis.

The Third Place team prize of US$10,000 was earned by a team from Pine View School in Osprey, Florida. The team members were **Nolan Boucher**, **Uday Goyat**, **John Halcomb**, **Max Rudin**, and **Lisa Zhang**. Their coach was Mark Mattia. The team was also awarded the Technical Computing Finals team prize of US$3,000.

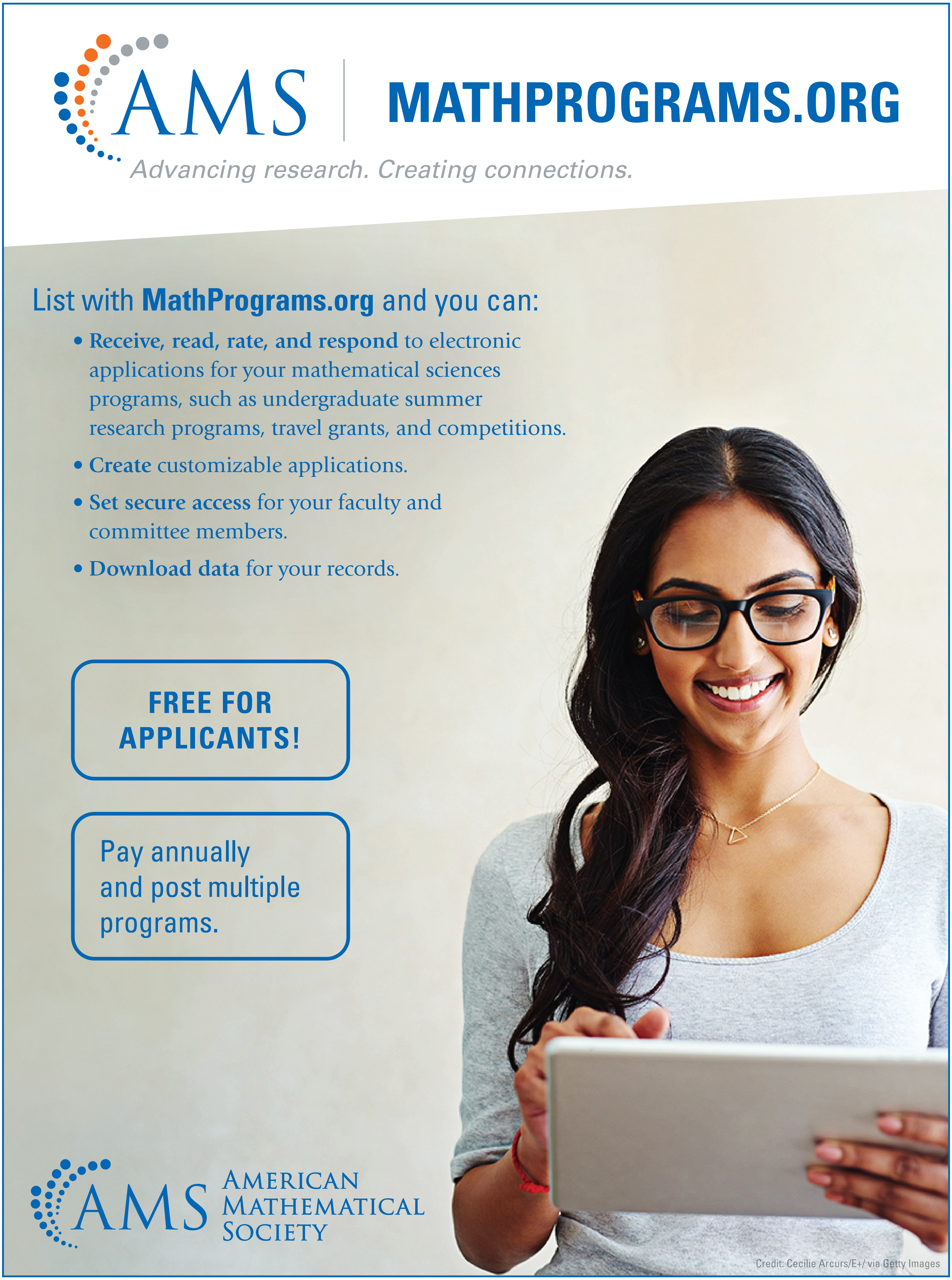
Finalist team prizes of US$5,000 were awarded to three teams. Two teams from Adlai E. Stevenson High School in Lincolnshire, Illinois, were honored. The first team consisted of **Jack Chen**, **Collin Fan**, **Aadit Juneja**, **Aayush Kashyap**, and **Nathan Ma**. The members of the second team from Stevenson were **Spandan Goel**, **Andrew Liu**, **Joy Qu**, **Greycen Ren**, and **Gabriel Visotsky**. Both teams were coached by Paul Kim. The third Finalist prize went to a team from High Technology High School in Lincroft, New Jersey. The members were **David Chang**, **Andrew Eng**, **Kevin Guan**, **Alexander Postovskiy**, and **Ivan Wong**. This team was also awarded a US$2,000 Technical Computing Runner Up prize. A team from New Century Tech High School in Huntsville, Alabama, received a US$1,000 Technical Computing Third Place award. The team consisted of **Ella Duus**, **Donal Higgins**, **Alexander Ivan**, and **Shreyas Puducheri**; their coach was Clifford Pate.

The M3 Challenge invites teams of high school juniors and seniors to solve an open-ended, realistic, challenging modeling problem focused on real-world issues. The competition is sponsored by MathWorks, a developer of computing software for engineers and scientists, and is organized by the Society for Industrial and Applied Mathematics (SIAM).

—From a MathWorks/SIAM announcement

Credits

Photo of Daniel B. Larremore is by Glenn Asakawa/University of Colorado.
Photo of Laura DeMarco is courtesy of Stephanie Mitchell.
Photo of Zhiren Wang is courtesy of Svetlana Katok.
Photo of Elizabeth Fennema is courtesy of University of Wisconsin–Madison.
Photo of Marta Civil is courtesy of Olga Yiparaki.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American Mathematical Society
Credit: Cecilie Arcurs/E+/ via Getty Images

Mathematics Opportunities

Listings for upcoming mathematics opportunities to appear in Notices *may be submitted to* notices@ams.org.

Call for Nominations for Popov Prize

The Vasil A. Popov Prize is awarded every three years to a young mathematician for outstanding research contributions in approximation theory and related areas of mathematics. The tenth prize will be awarded at the Foundation of Computational Mathematics International Conference in 2023. The deadline for nominations is **January 31, 2023**. See https://www.ljll.math.upmc.fr/popov-prize/pdf/flier_popov2023-2.pdf.

—Albert Cohen, Sorbonne University

Call for Nominations for AWM Prizes

The Association for Women in Mathematics (AWM) awards a number of prizes throughout the year. The following prizes are currently open for nominations.

The Ruth I. Michler Memorial Prize honors outstanding women who have recently been promoted to associate professor or an equivalent position in the mathematical sciences. It provides a research fellowship for a semester in the mathematics department at Cornell University without teaching obligations. The deadline for nominations is **October 1, 2022**.

The Sonia Kovalevsky Lectureship, sponsored by AWM and the Society for Industrial and Applied Mathematics (SIAM), recognizes significant contributions of women to applied or computational mathematics. The deadline for nominations is **October 1, 2022**.

The Etta Zuber Falconer Lectureship, sponsored by AWM and the Mathematical Association of America (MAA), honors women who have made distinguished contributions to the mathematical sciences or mathematics education. The deadline for nominations is **October 1, 2022**.

The Emmy Noether Lectureship, sponsored by AWM and the AMS, honors women who have made fundamental and sustained contributions to the mathematical sciences. The deadline for nominations is **October 1, 2022**.

The Alice T. Schafer Mathematics Prize is awarded to an undergraduate woman for excellence in mathematics. The nominee must be an undergraduate when nominated. The deadline is **October 1, 2022**.

The AWM Dissertation Prizes honor outstanding dissertations by women mathematical scientists defended within the previous two years. The deadline for nominations is **October 1, 2022**.

For further information on all these prizes, see https://awm-math.org/awards/.

—From an AWM announcement

Early Career Opportunity

Call for Nominations for Gerald Sacks Prize

The Association for Symbolic Logic (ASL) invites nominations for the Gerald Sacks Prize for the most outstanding doctoral dissertation in mathematical logic. The deadline for nominations is **September 30, 2022**. See the website http://aslonline.org/other-information/prizes-and-awards/sacks-prize-recipients/sacks-prize-nominations/.

—From an ASL announcement

Early Career Opportunity

Research Experiences for Undergraduates

The Research Experiences for Undergraduates (REU) program supports student research in areas funded by the National Science Foundation (NSF) through REU sites and REU supplements. See www.nsf.gov/funding/pgm_summ.jsp?pims_id=5517. The deadline date for proposals from institutions wishing to host REU sites is **August 24,**

2022. Dates for REU supplements vary with the research program (contact the program director for more information). Students apply directly to REU sites; see www.nsf.gov/crssprgm/reu/list_result.jsp?unitid=5044 for active REU sites.

—From an NSF announcement

NSF Focused Research Groups

The National Science Foundation (NSF) Focused Research Group program supports collaborative groups employing innovative methods to solve specific major research challenges in the mathematical sciences. The deadline for full proposals is **September 14, 2022**. See https://www.nsf.gov/funding/pgm_summ.jsp?pims_id=5671&org=NSF&sel_org=MPS&from=fund.

—From an NSF announcement

Joint DMS/NIGMS Initiative to Support Research at the Interface of the Biological and Mathematical Sciences

The National Science Foundation (NSF) and the National Institute of General Medical Sciences (NIGMS) at the National Institutes of Health (NIH) support research in mathematics and statistics on questions in the biological and biomedical sciences. The application period is **September 1–19, 2022**. For more information, see https://www.nsf.gov/funding/pgm_summ.jsp?pims_id=5300.

—From an NSF announcement

News from MSRI

The Mathematical Sciences Research Institute (MSRI) announces several upcoming programs.

Fall Scientific Workshops

MSRI plans to hold the following workshops in the fall of 2022. For further information, see the website www.msri.org/workshops.

August 25–26, 2022: Connections Workshop: Analytic and Geometric Aspects of Gauge Theory. Organizers: Lara Anderson (Virginia Polytechnic Institute and State University) and Laura Schaposnik (University of Illinois at Chicago).

August 29–September 2, 2022: Introductory Workshop: Analytic and Geometric Aspects of Gauge Theory. Organizers: Aleksander Doan (Trinity College; University College London), Laura Fredrickson (University of Oregon), and Michael Singer (University College London).

September 8–9, 2022: Connections Workshop: Floer Homotopy Theory. Organizers: Teena Gerhardt (Michigan State University), Kristen Hendricks (Rutgers University), and Ailsa Keating (University of Cambridge).

September 12–16, 2022: Introductory Workshop: Floer Homotopy Theory. Organizers: Sheel Ganatra (University of Southern California), Tyler Lawson (University of Minnesota Twin Cities), Robert Lipshitz (University of Oregon), and Nathalie Wahl (University of Copenhagen).

October 24–28, 2022: New Four-Dimensional Gauge Theories. Organizers: Andriy Haydys (Université Libre de Bruxelles), Lotte Hollands (Heriot-Watt University, Riccarton Campus), Eleny-Nicoleta Ionel (Stanford University), Richard Thomas (Imperial College, London), and Thomas Walpuski (Humboldt-Universität).

November 14–18, 2022: Floer Homotopical Methods in Low Dimensional and Symplectic Topology. Organizers: Mohammed Abouzaid (Columbia University), Andrew Blumberg (Columbia University), Jennifer Hom (Georgia Institute of Technology), Emmy Murphy (Northwestern University), and Sucharit Sarkar (University of California, Los Angeles).

2023 Summer Research in Mathematics

The goal of this program is to enhance the mathematical sciences by providing opportunities for research and career growth to individuals who may have been disproportionately affected by such obstacles as family obligations, professional isolation, or access to funding. The deadline for applications is **November 1, 2022**. For details about eligibility, application processes, and funding, see the website www.msri.org/summer.

Forthcoming Compendium from 2021 MSRI Workshop on Mathematics and Racial Justice

In June of 2021, MSRI held a virtual Workshop on Mathematics and Racial Justice, which convened nearly 300 mathematicians, statisticians, computer scientists, and STEM educators to critically examine the role that mathematics plays in today's movement for racial justice. A free and publicly available compendium on the workshop is expected to be released on the occasion of Juneteenth 2022.

Mathematics is often viewed as one of the main tools responsible for scientific progress, and developments in mathematics are behind some of society's most significant technological advancements. While mathematics has been used to push society forward, there are also well-documented instances of mathematics being used as a tool

The most up-to-date listing of NSF funding opportunities from the Division of Mathematical Sciences can be found online at www.nsf.gov/dms *and for the Directorate of Education and Human Resources at* www.nsf.gov/dir/index.jsp?org=ehr. *To receive periodic updates, subscribe to the DMSNEWS listserv by following the directions at* www.nsf.gov/mps/dms/about.jsp.

of racial oppression. The inequities faced by the Black community have become more and more difficult to ignore, and mathematicians have increasingly been answering the call to engage with issues of social justice within their research, their teaching, and in the broader scientific community. This workshop and the resulting compendium are a part of this movement and make the distinct contribution of centering issues of mathematics and racial justice, with focus on the Black community.

The 2021 keynotes by Robert Berry (University of Virginia) and Rediet Abebe (University of California, Berkeley) set the stage for this volume: Berry, by presenting a historically informed view of the way mathematics education as it is often implemented dehumanizes people of color, and Abebe, by demonstrating the power of data and computer science to study social problems and guide their solutions. Following their contributions, the workshop was divided into four primary thematic areas, which have also guided the organization of this compendium: Bias in Algorithms and Technology; Public Health Disparities; Racial Inequities in Mathematics Education; and Fair Division, Allocation, and Representation.

The 2021 workshop was organized by Omayra Ortega (Sonoma State University), Robin Wilson (California State Polytechnic University, Pomona), Caleb Ashley (Boston College), Ron Buckmire (Occidental College), Duane Cooper (Morehouse College), and Monica Jackson (American University) and supported by the American Mathematical Society (AMS), the Center for Minorities in the Mathematical Sciences (CMMS), the Mathematical Sciences Research Institute (MSRI), the National Association of Mathematicians (NAM), the National Science Foundation (NSF), and the Society for Industrial and Applied Mathematics (SIAM).

—MSRI announcements

Classified Advertising
Employment Opportunities

CHINA

Tianjin University, China
Tenured/Tenure-Track/Postdoctoral Positions at the Center for Applied Mathematics

Dozens of positions at all levels are available at the recently founded Center for Applied Mathematics, Tianjin University, China. We welcome applicants with backgrounds in pure mathematics, applied mathematics, statistics, computer science, bioinformatics, and other related fields. We also welcome applicants who are interested in practical projects with industries. Despite its name attached with an accent of applied mathematics, we also aim to create a strong presence of pure mathematics.

Light or no teaching load, adequate facilities, spacious office environment and strong research support. We are prepared to make quick and competitive offers to self-motivated hard workers, and to potential stars, rising stars, as well as shining stars.

The Center for Applied Mathematics, also known as the Tianjin Center for Applied Mathematics (TCAM), located by a lake in the central campus in a building protected as historical architecture, is jointly sponsored by the Tianjin municipal government and the university. The initiative to establish this center was taken by Professor S. S. Chern. Professor Molin Ge is the Honorary Director, Professor Zhiming Ma is the Director of the Advisory Board. Professor William Y. C. Chen serves as the Director.

TCAM plans to fill in fifty or more permanent faculty positions in the next few years. In addition, there are a number of temporary and visiting positions. We look forward to receiving your application or inquiry at any time. There are no deadlines.

Please send your resume to `mathjobs@tju.edu.cn`.

For more information, please visit `cam.tju.edu.cn` or contact Mr. Albert Liu at `mathjobs@tju.edu.cn`, telephone: 86-22-2740-6039.

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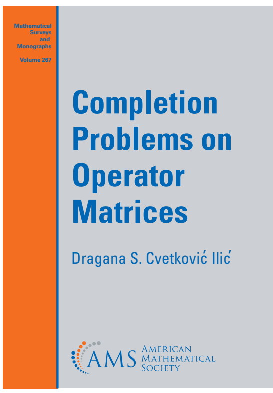
Upcoming deadlines for classified advertising are as follows: October 2022—July 22, 2022; November 2022—August 26, 2022; December 2022—September 23, 2022.

US laws prohibit discrimination in employment on the basis of color, age, sex, race, religion, or national origin. Advertisements from institutions outside the US cannot be published unless they are accompanied by a statement that the institution does not discriminate on these grounds whether or not it is subject to US laws.

Submission: Send email to `classads@ams.org`.

New Books Offered by the AMS

Analysis



Completion Problems on Operator Matrices

Dragana S. Cvetković Ilić, *University of Nis, Visegradska, Nis, Serbia*

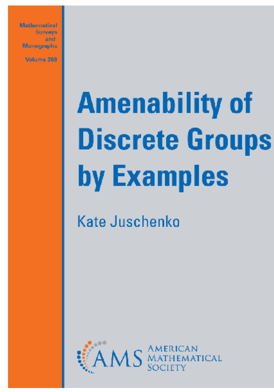
Completion problems for operator matrices are concerned with the question of whether a partially specified operator matrix can be completed to form an operator of a desired type. The research devoted to this topic provides an excellent means to investigate the structure of operators. This book provides an overview of completion problems dealing with completions to different types of operators and can be considered as a natural extension of classical results concerned with matrix completions.

The book assumes some basic familiarity with functional analysis and operator theory. It will be useful for graduate students and researchers interested in operator theory and the problem of matrix completions.

This item will also be of interest to those working in applications.

Mathematical Surveys and Monographs, Volume 267
September 2022, 170 pages, Softcover, ISBN: 978-1-4704-6987-0, 2010 *Mathematics Subject Classification*: 47A08, 47B01, 47B02, 47A53, 15A83, List US$125, **AMS members US$100**, **MAA members US$112.50**, Order code SURV/267

bookstore.ams.org/surv-267



Amenability of Discrete Groups by Examples

Kate Juschenko, *University of Texas, Austin, TX*

The main topic of the book is amenable groups, i.e., groups on which there exist invariant finitely additive measures. It was discovered that the existence or non-existence of amenability is responsible for many interesting phenomena such as, e.g., the Banach-Tarski Paradox about breaking a sphere into two spheres of the same radius. Since then, amenability has been actively studied and a number of different approaches resulted in many examples of amenable and non-amenable groups.

In the book, the author puts together main approaches to study amenability. A novel feature of the book is that the exposition of the material starts with examples which introduce a method rather than illustrating it. This allows the reader to quickly move on to meaningful material without learning and remembering a lot of additional definitions and preparatory results; those are presented after analyzing the main examples. The techniques that are used for proving amenability in this book are mainly a combination of analytic and probabilistic tools with geometric group theory.

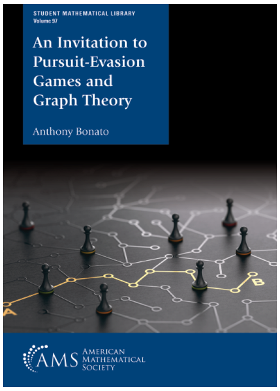
This item will also be of interest to those working in geometry and topology.

Mathematical Surveys and Monographs, Volume 266
August 2022, approximately 167 pages, Softcover, ISBN: 978-1-4704-7032-6, 2010 *Mathematics Subject Classification*: 20–03, 20L99, 22–02, List US$125, **AMS members US$100**, **MAA members US$112.50**, Order code SURV/266

bookstore.ams.org/surv-266

NEW BOOKS

Discrete Mathematics and Combinatorics



An Invitation to Pursuit-Evasion Games and Graph Theory

Anthony Bonato, *Toronto Metropolitan University, ON, Canada*

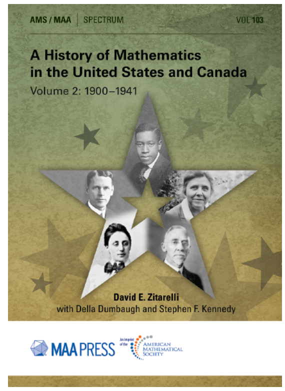
Graphs measure interactions between objects such as friendship links on Twitter, transactions between Bitcoin users, and the flow of energy in a food chain. While graphs statically represent interacting systems, they may also be used to model dynamic interactions. For example, imagine an invisible evader loose on a graph, leaving only behind breadcrumb clues to their whereabouts. You set out with pursuers of your own, seeking out the evader's location. Would you be able to detect their location? If so, then how many resources are needed for detection, and how fast can that happen? These basic-seeming questions point towards the broad conceptual framework of pursuit-evasion games played on graphs. Central to pursuit-evasion games on graphs is the idea of optimizing certain parameters, whether they are the cop number, burning number, or localization number, for example.

This book would be excellent for a second course in graph theory at the undergraduate or graduate level. It surveys different areas in graph searching and highlights many fascinating topics intersecting classical graph theory, geometry, and combinatorial designs. Each chapter ends with approximately twenty exercises and five larger scale projects.

Student Mathematical Library, Volume 97
September 2022, 254 pages, Softcover, ISBN: 978-1-4704-6763-0, 2010 *Mathematics Subject Classification*: 05C57, 05C05, 05C90, 68R10, List US$59, **AMS members US$47.20**, AMS institutional member US$47.20, **MAA members US$47.20**, Order code STML/97

`bookstore.ams.org/stml-97`

General Interest



A History of Mathematics in the United States and Canada

Volume 2: 1900–1941

David E. Zitarelli, **Della Dumbaugh**, *University of Richmond, VA*, and **Stephen F. Kennedy**, *Carleton College, Northfield, MN, and MAA Press, Providence, RI*

This is the first truly comprehensive and thorough history of the development of a mathematical community in the United States and Canada. This second volume starts at the turn of the twentieth century with a mathematical community that is firmly established and traces its growth over the next forty years, at the end of which the American mathematical community is pre-eminent in the world.

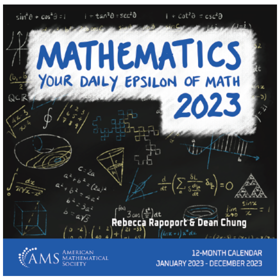
In the preface to the first volume of this work Zitarelli reveals his animating philosophy, "I find that the human factor lends life and vitality to any subject." History of mathematics, in the Zitarelli conception, is not just a collection of abstract ideas and their development. It is a community of people and practices joining together to understand, perpetuate, and advance those ideas and each other. Telling the story of mathematics means telling the stories of these people: their accomplishments and triumphs; the institutions and structures they built; their interpersonal and scientific interactions; and their failures and shortcomings.

One of the most hopeful developments of the period 1900–1941 in American mathematics was the opening of the community to previously excluded populations. Increasing numbers of women were welcomed into mathematics, many of whom—including Anna Pell Wheeler, Olive Hazlett, and Mayme Logsdon—are profiled in these pages. Black mathematicians were often systemically excluded during this period, but, in spite of the obstacles, Elbert Frank Cox, Dudley Woodard, David Blackwell, and others built careers of significant accomplishment that are described here. The effect on the substantial community of European immigrants is detailed through the stories of dozens of individuals.

In clear and compelling prose Zitarelli, Dumbaugh, and Kennedy spin a tale accessible to experts, general readers, and anyone interested in the history of science in North America.

Spectrum, Volume 103
September 2022, approximately 547 pages, Softcover, ISBN: 978-1-4704-6730-2, 2010 *Mathematics Subject Classification*: 01A05, 01A60, List US$120, **AMS members US$90**, AMS institutional member US$96, **MAA members US$90**, Order code SPEC/103

bookstore.ams.org/spec-103



Mathematics 2023: Your Daily Epsilon of Math

12-Month Calendar—January 2023 through December 2023

Rebecca Rapoport, *Harvard University, Cambridge, MA, and Michigan State University, East Lansing, MI* and **Dean Chung**, *Harvard University, Cambridge, MA, and University of Michigan, Ann Arbor, MI*

Keep your mind sharp all year long with *Mathematics 2023: Your Daily Epsilon of Math*, a 12″ x 12″ wall calendar featuring a new math problem every day and 12 beautiful math images!

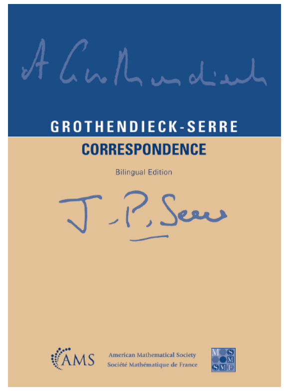
Let mathematicians Rebecca Rapoport and Dean Chung tickle the left side of your brain by providing you with a math challenge for every day of the year. The solution is always the date, but the fun lies in figuring out how to arrive at the answer, and possibly discovering more than one method of arriving there.

Problems run the gamut from arithmetic through graduate level math. Some of the most tricky problems require only middle school math applied cleverly. With word problems, math puns, and interesting math definitions added into the mix, this calendar will intrigue you for the whole year.

End the year with more brains than you had when it began with *Mathematics 2023: Your Daily Epsilon of Math*.

June 2022, approximately 16 pages, Softcover, ISBN: 978-1-4704-7107-1, 2010 *Mathematics Subject Classification*: 00A07, 00A09, 00A06, 00A08, 00A05, List US$18, **AMS members US$14.40**, **MAA members US$16.20**, Order code MBK/144

bookstore.ams.org/mbk-144



Grothendieck-Serre Correspondence

Bilingual Edition

Pierre Colmez, *Ecole Normale Superieure, Paris, France* and **Jean-Pierre Serre**, *College de France, Paris, France*, Editors

This extraordinary volume contains mathematical correspondence between A. Grothendieck and J-P. Serre, two central figures who were key to the development of algebraic geometry. This correspondence forms a vivid introduction to the topic during a period of years in which algebraic geometry went through a remarkable transformation. This book will interest all mathematicians who want to experience the unfolding of great mathematics; strongly recommended for booksellers.

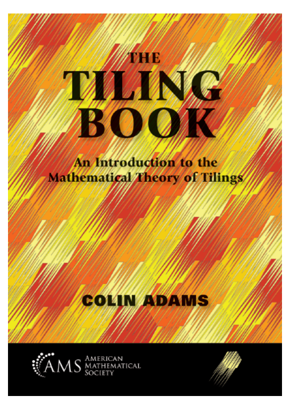
This item will also be of interest to those working in algebra and algebraic geometry.

This book is jointly published by the AMS and the Société Mathématique de France. SMF members are entitled to AMS member discounts.

May 2022, 600 pages, Softcover, ISBN: 978-1-4704-6939-9, LC 2003062815, 2010 Mathematics Subject Classification: 14–03; 01A25, List US$85, AMS members US$68, MAA members US$76.50, Order code CGS.S

bookstore.ams.org/cgs-s

Geometry and Topology



The Tiling Book

An Introduction to the Mathematical Theory of Tilings

Colin Adams, *Williams College, Williamstown, MA*

Tiling theory provides a wonderful opportunity to illustrate both the beauty and utility of mathematics. It has all the relevant ingredients: there are stunning pictures; open problems can be stated without having to spend months providing the necessary background; and there is both deep mathematics and applications.

Furthermore, tiling theory happens to be an area where many of the sub-fields of mathematics overlap. Tools can be applied from linear algebra, algebra, analysis, geometry, topology, and combinatorics. As such, it makes for an ideal capstone course for undergraduates or an introductory course for graduate students. This material can also be used for a lower-level course by skipping the more technical sections. In addition, readers from a variety of disciplines can read the book on their own to find out more about this intriguing subject.

This book covers the necessary background on tilings and then delves into a variety of fascinating topics in the field, including symmetry groups, random tilings, aperiodic tilings, and quasicrystals. Although primarily focused on tilings of the Euclidean plane, the book also covers tilings of the sphere, hyperbolic plane, and Euclidean 3-space, including knotted tilings. Throughout, the book includes open problems and possible projects for students. Readers will come away with the background necessary to pursue further work in the subject.

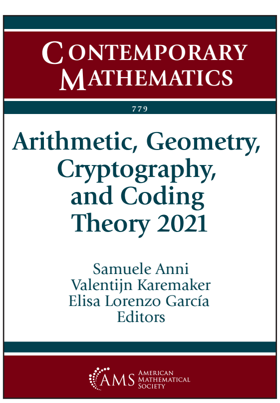
This item will also be of interest to those working in discrete mathematics and combinatorics.

September 2022, approximately 295 pages, Hardcover, ISBN: 978-1-4704-6897-2, 2010 *Mathematics Subject Classification*: 05–01, 05B45, 52–01, 52C20, 52C22, 52C23, List US$59, **AMS members US$47.20**, **MAA members US$53.10**, Order code MBK/142

bookstore.ams.org/mbk-142

New in Contemporary Mathematics

Algebra and Algebraic Geometry



Arithmetic, Geometry, Cryptography, and Coding Theory 2021

Samuele Anni, *Aix-Marseille Université, Cedex, France*, **Valentijn Karemaker**, *Universiteit Utrecht, The Netherlands*, and **Elisa Lorenzo García**, *Université de Neuchâtel, Switzerland*, Editors

This volume contains the proceedings of the 18th International Conference on Arithmetic, Geometry, Cryptography, and Coding Theory, held (online) from May 31 to June 4, 2021.

For over thirty years, the biennial international conference AGC2T (Arithmetic, Geometry, Cryptography, and Coding Theory) has brought researchers together to forge connections between arithmetic geometry and its applications to coding theory and to cryptography.

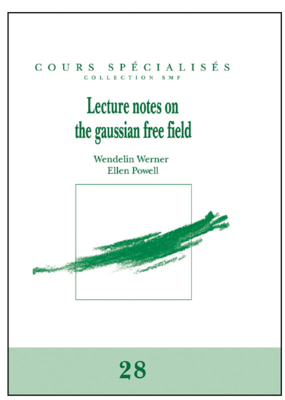
The papers illustrate the fruitful interaction between abstract theory and explicit computations, covering a large range of topics, including Belyi maps, Galois representations attached to elliptic curves, reconstruction of curves from their Jacobians, isogeny graphs of abelian varieties, hypergeometric equations, and Drinfeld modules.

This item will also be of interest to those working in applications.

Contemporary Mathematics, Volume 779
September 2022, approximately 192 pages, Softcover, ISBN: 978-1-4704-6794-4, 2010 *Mathematics Subject Classification*: 11G20, 11G30, 11G32, 11G40, 11T71, 14G10, 14H40, 14Q05, 20C20, 20G41, List US$125, **AMS members US$100**, **MAA members US$112.50**, Order code CONM/779

bookstore.ams.org/conm-779

New AMS-Distributed Publications



Lecture Notes on the Gaussian Free Field

Wendelin Werner, *ETH Zürich, Switzerland* and **Ellen Powell**, *Durham University, United Kingdom*

The Gaussian Free Field (GFF) in the continuum appears to be the natural generalisation of Brownian motion, when one replaces time by a multidimensional continuous parameter. While Brownian motion can be viewed as the most natural random real-valued function defined on $\mathbb{R}_+$ with $B(0)=0$, the GFF in a domain D of $\mathbb{R}^d$ for $d \geq 2$ is a natural random real-valued generalised function defined on D with zero boundary conditions on ∂D. In particular, it is not a random continuous function.

The goal of these lecture notes is to describe some aspects of the continuum GFF and of its discrete counterpart defined on lattices, with the aim of providing a gentle self-contained introduction to some recent developments on this topic, such as the relation between the continuum GFF, Brownian loop-soups and the Conformal Loop Ensembles CLE_4.

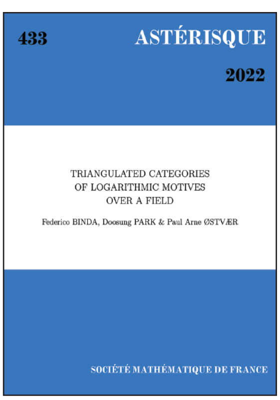
This is an updated and expanded version of the notes written by the first author (Wendelin Werner) for graduate courses at ETH Zürich (Swiss Federal Institute of Technology in Zürich) in 2014 and 2018. It has benefited from the comments and corrections of students, as well as of a referee. The exercises that are interspersed in the first half of these notes mostly originate from the exercise sheets prepared by the second author (Ellen Powell) for this course in 2018.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Cours Spécialisés—Collection SMF, Number 28
May 2022, 184 pages, Hardcover, ISBN: 978-2-85629-952-4, 2010 *Mathematics Subject Classification*: 60G60, 60G15, 60J67, 82B20, 82B21, List US$65, **AMS members US$52**, Order code COSP/28

bookstore.ams.org/cosp-28

Algebra and Algebraic Geometry



Triangulated Categories of Logarithmic Motives Over a Field

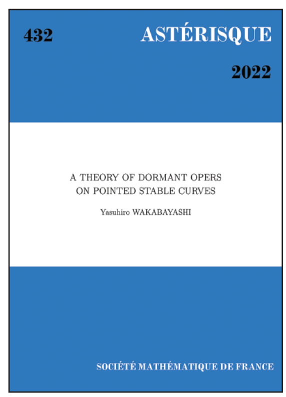
Federico Binda, *Universitá degli Studi di Milano, Italy*, **Doosung Park**, *Universität Zürich Winterthurerstrasse, Switzerland*, and **Paul Arne Østvær**, *University of Oslo, Norway*

In this work the authors develop a theory of motives for logarithmic schemes over fields in the sense of Fontaine, Illusie, and Kato. The authors' construction is based on the notion of finite log correspondences, the dividing Nisnevich topology on log schemes, and the basic idea of parameterizing homotopies by $\overline{\square}$, i.e., the projective line with respect to its compactifying logarithmic structure at infinity. The authors show that Hodge cohomology of log schemes is a $\overline{\square}$-invariant theory that is representable in the category of logarithmic motives. Their category is closely related to Voevodsky's category of motives and $\mathbb{A}^1$-invariant theories: assuming resolution of singularities, the authors identify the latter with the full subcategory comprised of $\mathbb{A}^1$-local objects in the category of logarithmic motives. Fundamental properties such as $\overline{\square}$-homotopy invariance, Mayer-Vietoris for coverings, the analogs of the Gysin sequence and the Thom space isomorphism as well as a blow-up formula and a projective bundle formula witness the robustness of the setup.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Astérisque, Number 433
May 2022, 280 pages, Softcover, ISBN: 978-2-85629-957-9, 2010 *Mathematics Subject Classification*: 14A21, 14A30, 14F42, 18N40, 18N55, 18F10, 18G35, 19E15, List US$82, **AMS members US$65.60**, Order code AST/433

bookstore.ams.org/ast-433



A Theory of Dormant Opers on Pointed Stable Curves

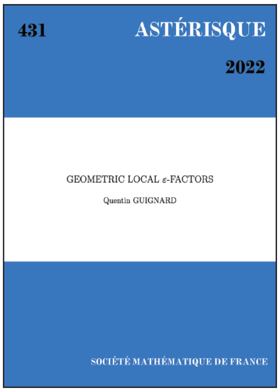
Yasuhiro Wakabayashi, *Tokyo Institute of Technology, Japan*

This manuscript presents a detailed and original account of the theory of opers defined on pointed stable curves in arbitrary characteristic and their moduli. In particular, it includes the development of the study of dormant opers, which are opers of a certain sort in positive characteristic. The author's goal is to give an explicit formula, conjectured by Joshi, for the generic number of dormant opers.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Astérisque, Number 432
May 2022, 296 pages, Softcover, ISBN: 978-2-85629-956-2, 2010 *Mathematics Subject Classification*: 14H10, 14H60, List US$90, **AMS members US$72**, Order code AST/432

bookstore.ams.org/ast-432



Geometric Local ε-Factors

Quentin Guignard, *Institut de Mathématiques de Jussieu-Paris Rive Gauche, France*

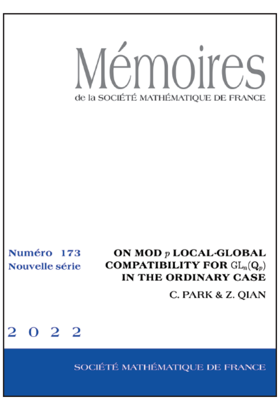
Inspired by the work of Laumon on local ε-factors and by Deligne's 1974 letter to Serre, the author gives an explicit cohomological definition of ε-factors for ℓ-adic Galois representations over henselian discrete valuation fields of positive equicharacteristic $p=\ell$, with (not necessarily finite) perfect residue fields. These geometric local ε-factors are completely characterized by an explicit list of purely local properties, such as an induction formula and the compatibility with geometric class field theory in rank 1, and satisfy a product formula for ℓ-adic sheaves on a curve over a perfect field of characteristic p.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Astérisque, Number 431
May 2022, 137 pages, Softcover, ISBN: 978-2-85629-953-1, 2010 *Mathematics Subject Classification*: 14F20, List US$65, **AMS members US$52**, Order code AST/431

bookstore.ams.org/ast-431

Number Theory



On Mod p Local-Global Compatibility for $GL_n(\mathbb{Q}_p)$ in the Ordinary Case

C. Park, *Ulsan National Institute of Science and Technology, Republic of Korea* and **Z. Qian**, *University of Toronto, Ontario, Canada*

In this paper, the authors show that the isomorphism class of $\bar{r}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_w)}$ is determined by $\mathrm{GL}_n(F_w)$-action on a space of mod p algebraic automorphic forms cut out by the maximal ideal of a Hecke algebra associated to $\bar{r}$.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Mémoires de la Société Mathématique de France, Number 173
March 2022, 150 pages, Softcover, ISBN: 978-2-85629-945-6, 2010 *Mathematics Subject Classification*: 11F80, 11F33, List US$65, **AMS members US$52**, Order code SMFMEM/173

bookstore.ams.org/smfmem-173

Probability and Statistics

Séminaire Bourbaki
Volume 2019/2021
Exposés 1166–1180

This 72nd volume of the Bourbaki Seminar gathers the texts of the fifteen survey lectures delivered from November 2019 to June 2021: Herman's positive entropy conjecture, pseudospectral estimates and stability of planar vortices, a closing C^∞-lemma, pseudospectrum and resolvents of non-selfadjoint operators, Hodge theory and o-minimality, forcing theory for surface homeomorphisms, nodal sets of eigenfunctions of the Laplace operator, Ratner-type phenomena for hyperbolic manifolds, the inverse theorem for Gowers norms, reconstruction of an algebraic variety from its topology, special values of Riemann's zeta function and polylogarithms, a fractal uncertainty principle, totally geodesic subvarieties of the moduli space of curves, asymptotic counting of minimal surfaces in hyperbolic manifolds, and continuous logic and property (T) of Roelcke-precompact groups.

A publication of the Société Mathématique de France, Marseilles (SMF), distributed by the AMS in the US, Canada, and Mexico. Orders from other countries should be sent to the SMF. Members of the SMF receive a 30% discount from list.

Astérisque, Number 430
May 2022, 565 pages, Softcover, ISBN: 978-2-85629-931-9, 2010 *Mathematics Subject Classification*: 60K35, 60F17, 60J67, 60G57, List US$120, **AMS members US$96**, Order code AST/430

bookstore.ams.org/ast-430

Meetings & Conferences of the AMS

August Table of Contents

The Meetings and Conferences section of the *Notices* gives information on all AMS meetings and conferences approved by press time for this issue. Please refer to the page numbers cited on this page for more detailed information on each event. *Paid meeting registration is required to submit an abstract to a sectional meeting.*

Invited Speakers and Special Sessions are listed as soon as they are approved by the cognizant program committee; the codes listed are needed for electronic abstract submission. For some meetings the list may be incomplete. Information in this issue may be dated.

The most up-to-date meeting and conference information can be found online at `www.ams.org/meetings`.

Important Information About AMS Meetings: Potential organizers, speakers, and hosts should refer to `https://www.ams.org/meetings/meetings-general` for general information regarding participation in AMS meetings and conferences.

Abstracts: Speakers should submit abstracts on the easy-to-use interactive Web form. No knowledge of LaTeX is necessary to submit an electronic form, although those who use LaTeX may submit abstracts with such coding, and all math displays and similarly coded material (such as accent marks in text) must be typeset in LaTeX. Visit `www.ams.org/cgi-bin/abstracts/abstract.pl`. Questions about abstracts may be sent to `abs-info@ams.org`. Close attention should be paid to specified deadlines in this issue. Unfortunately, late abstracts cannot be accommodated.

Associate Secretaries of the AMS

Central Section: Betsy Stovall, University of Wisconsin–Madison, 480 Lincoln Drive, Madison, WI 53706; email: `stovall@math.wisc.edu`; telephone: (608) 262-2933.

Eastern Section: Steven H. Weintraub, Department of Mathematics, Lehigh University, Bethlehem, PA 18015-3174; email: `steve.weintraub@lehigh.edu`; telephone: (610) 758-3717.

Southeastern Section: Brian D. Boe, Department of Mathematics, University of Georgia, 220 D W Brooks Drive, Athens, GA 30602-7403; email: `brian@math.uga.edu`; telephone: (706) 542-2547.

Western Section: Michelle Manes, University of Hawaii, Department of Mathematics, 2565 McCarthy Mall, Keller 401A, Honolulu, HI 96822; email: `mamanes@hawaii.edu`; telephone: (808) 956-4679.

Meetings in this Issue

The AMS strives to ensure that participants in its activities enjoy a welcoming environment. Please see our full Policy on a Welcoming Environment at `https://www.ams.org/welcoming-environment-policy`.

Meetings & Conferences of the AMS

IMPORTANT information regarding meetings programs: AMS Sectional Meeting programs do not appear in the print version of the *Notices*. However, comprehensive and continually updated meeting and program information with links to the abstract for each talk can be found on the AMS website. See `https://www.ams.org/meetings`.

Final programs for Sectional Meetings will be archived on the AMS website accessible from the stated URL.

New: Sectional Meetings Require Registration to Submit Abstracts. In an effort to spread the cost of the sectional meetings more equitably among all who attend and hence help keep registration fees low, starting with the 2020 fall sectional meetings, you must be registered for a sectional meeting in order to submit an abstract for that meeting. You will be prompted to register on the Abstracts Submission Page. In the event that your abstract is not accepted or you have to cancel your participation in the program due to unforeseen circumstances, your registration fee will be reimbursed.

Grenoble, France

AMS-SMF-EMS Joint International Meeting

Université de Grenoble-Alpes

July 18–22, 2022

Monday – Friday

Meeting #1168

Associate Secretary for the AMS: Brian D. Boe
Program first available on AMS website: Not applicable
Issue of *Abstracts*: Not applicable

Deadlines

For organizers: Expired
For abstracts: To be announced

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/internmtgs.html`.

Invited Addresses

Andrea Bertozzi, University of California, Los Angeles, USA, *Title to be announced*.
Peter Bühlmann, ETH Zürich, Switzerland, *Title to be announced*.
Maria Chudnovsky, Princeton University, USA, *Title to be announced*.
Hugo Duminil-Copin, Institut des Hautes Études Scientifiques (IHÉS), Bures-sur-Yvette, France, *Title to be announced*.
Alessio Figalli, ETH Zürich, Switzerland, *Title to be announced*.
Vincent Lafforgue, Université de Grenoble Alpes & CNRS, France, *Title to be announced*.
Peter Sarnak, Institute for Advanced Study (IAS), Princeton, USA, *Title to be announced*.
Claire Voisin, Collège de France, Paris, France, *Title to be announced*.
Simone Warzel, Technische Universität München (TUM), Munich, Germany, *Title to be announced*.

Special Sessions

Advances in Functional Analysis and Operator Theory, **Marat V. Markin**, California State University, Fresno, USA, **Igor Nikolaev**, St. John's University, USA, **Jean Renault**, Universite d'Orleans, France, and **Carsten Trunk**, Technische Universitat Ilmenau, Germany.

Algebraic Geometry (Associated with Plenary Speaker Claire Voisin), **Radu Laza**, Stony Brook University, USA, **Catriona Maclean**, Grenoble, France, and **Claire Voisin**, Paris, France.

Automorphic Forms, Moduli Spaces, and Representation Theory (Associated with Plenary Speaker Vincent Lafforgue), **Jean-François Dat**, Sorbonne Université, France, and **Bao-Chau Ngo**, University of Chicago, USA.

Classical and Quantum Fields on Lorentzian Manifolds, **Dietrich Häfner**, Université Grenoble Alpes, France, and **Andras Vasy**, Stanford University, USA.

Combinatorial and Computational Aspects in Topology, **Eric Samperton**, University of Illinois, USA, **Saul Schleimer**, University of Warwick, United Kingdom, and **Greg McShane**, Université Grenoble-Alpes, France.

Deformation of Artinian algebras and Jordan type, **Anthony Iarrobino**, Northeastern University, USA, **Pedro Macias Marques**, Universidade de Evora, Portugal, **Maria Evelina Rossi**, Universita degli Studi di Genova, Italy, and **Jean Valles**, Universite de Pau et des Pays de l'Adour, France.

Deformation Spaces of Geometric Structures, **Sara Maloni**, University of Virginia, USA, **Andrea Seppi**, Université Grenoble Alpes, France, and **Nicolas Tholozan**, Ecole Normale Superieure de Paris, France.

Derived Categories and Rationality, **Matthew Ballard**, University of South Carolina, USA, **Emanuele Macrì**, Université Paris-Saclay, France, and **Patrick McFaddin**, Fordham University, USA.

Differential Geometry in the Tradition of Élie Cartan (1869–1959), **Vincent Borelli**, Université Claude Bernard, **Bogdan Suceavă,**California State University, Fullerton, USA, **Mihaela B. Vajiac**, Chapman University, USA, **Joeri Van der Veken**, KU Leuven, Belgium, **Marina Ville**, Université Paris-Est Créteil, and **Luc Vrancken**, Université Polytechnique Hauts-de-France, Valenciennes, France.

Drinfeld Modules, Modular Varieties and Arithmetic Applications, **Tuan Ngo Dac**, CNRS Université Claude Bernard Lyon 1, France, **Matthew Papanikolas**, Texas A&M University, USA, **Mihran Papikian**, Pennsylvania State University, USA, and **Federico Pellarin**, Université Jean Monnet, France.

Financial Mathematics, **Beatrice Acciaio**, ETH Zürich, Switzerland, **Carole Bernard**, Grenoble Ecole de Management, Grenoble, France, and **Stephan Sturm**, Worcester Polytechnic Institute, USA.

Fractal Geometry in Pure and Applied Mathematics, **Hafedh Herichi**, Santa Monica College, USA, **Maria Rosaria Lancia**, Sapienza Universita di Roma, Italy, **Therese-Marie Landry**, University of California, Riverside, USA, **Anna Rozanova-Pierrat**, CentralSuplec, Universite Paris- Saclay, France, and **Steffen Winter**, Karlsruhe Institute of Technology, Germany.

Functional Equations and Their Interactions, **Guy Casale**, IRMAR, Université de Rennes 1, France, **Thomas Dreyfus**, IRMA, Université de Strasbourg, France, **Charlotte Hardouin**, IRMAR, Université de Toulouse 3, France, **Joel Nagloo**, CUNY, New York, USA, **Julien Roques**, Institut Camille Jordan, Université de Lyon 1, France, and **Michael Singer**, North Carolina State University, Raleigh, USA.

Graph and Matroid Polynomials: Towards a Comparative Theory, **Emeric Gioan**, LIRMM, France, **Johann A. Makowsky**, Israel Institute of Technology- IIT, Israel, and **James Oxley**, Louisiana State University, USA.

Groups and Topological Dynamics, **Nicolas Matte Bon**, University of Lyon, France, **Constantine Medynets**, United States Naval Academy, USA, **Volodymyr Nekrashevych**, Texas A&M University, USA, and **Dmytro Savchuk**, University of South Florida, USA.

Group Theory, Algorithms and Applications, **Indira Chatterji**, Université de Nice, France, **Francois Dahmani** and **Martin Deraux**, Institut Fourier, Université Grenoble, Alpes, France, and **Delaram Kahrobaei**, CUNY and NYU, USA.

History of Mathematics Beyond Case-Studies, **Catherine Goldstein**, CNRS, IMJ-PRG, France, and **Jemma Lorenat**, Pitzer College, USA.

Integrability, Geometry, and Mathematical Physics, **Luen-Chau Li**, Pennsylvania State University, USA, and **Serge Parmentier**, Universite Claude Bernard Lyon 1, France.

Inverse Problems, **Hanna Makaruk**, Los Alamos National Laboratory (LANL), USA, **Robert Owczarek**, University of New Mexico, Albuquerque and Los Alamos, USA, **Tomasz Lipniacki**, Polish Academy of Sciences, Poland, and **Piotr Stachura**, Warsaw University of Life Sciences-SGGW, Poland.

Low-Dimensional Topology, **Paul Kirk**, University Bloomington, USA, **Christine Lescop**, CNRS, Institut Fourier, Université Grenoble Alpes, France, and **Jean-Baptiste Meilhan**, Institut Fourier, Université Grenoble, Alpes, France.

Mathematical Challenges in Complex Quantum Systems (Associated with Plenary Speaker Simone Warzel), **Alain Joye**, Institut Fourier, Université Grenoble Alpes, France, **Jeffrey Schenker**, Michigan State University, USA, **Nicolas Rougerie**, Ecole Normale Supérieure de Lyon and CNRS, France, and **Simone Warzel**, Zentrum Mathematik, TU München, Germany.

Mathematical Knowledge Management in the Digital Age of Science, **Patrick Ion**, University of Michigan, Ann Arbor, USA, **Thierry Bouche**, Université Grenoble-Alpes, France, and **Stephen Watt**, University of Waterloo, Canada.

Mathematical Physics of Gravity, Geometry, QFTs, Feynman and Stochastic Integrals, Quantum/Classical Number Theory, Algebra, and Topology, **Michael Maroun**, AMS-MRC Boston, USA, **Pierre Vanhove**, EMS/SMF CEA Paris Saclay, France, and **Federico Zerbini**, EMS/SMF CEA Paris Saclay.

Modular Representation Theory, **Pramod N. Achar**, Louisiana State University, USA, **Simon Riche**, Universite Clermont Auvergne, France, and **Britta Spath**, Bergische Universitat Wuppertal, Germany.

Percolation and Loop Models (Associated with Plenary Speaker Hugo Duminil-Copin), **Ioan Manolescu**, University of Fribourg, Switzerland.

Quantitative Geometry of Transportation Metrics, **Florent Baudier**, Texas A&M University, USA, **Dario Cordero-Erausquin**, Sorbonne Universite, France, **Alexandros Eskenazis**, University of Cambridge, United Kingdom, and **Eva Pernecka**, Czech Technical University in Prague, Czech Republic.

Recent Advances in Diffeology and their Applications, **Jean-Pierre Magnot**, Université d'Angers, France, and **Jordan Watts**, Central Michigan University, USA.

Rough Path and Malliavin Calculus, **Fabrice Baudoin**, University of Connecticut, USA, **Antoine Lejay**, University of Lorraine, France, and **Cheng Ouyang**, University of Illinois at Chicago, USA.

Spectral Optimization, **Richard S. Laugesen**, University of Illinois at Urbana Champaign, USA, **Enea Parini**, Aix Marseille University, France, and **Emmanuel Russ**, Grenoble Alpes University, France.

Statistical Learning (Associated with Plenary speaker Peter Bühlmann), **Christophe Giraud**, Paris Saclay University, France, **Cun-Hui Zhang**, Rutgers University, USA, and **Peter Bühlmann**, ETH Zürich, Switzerland.

Sub-Riemannian Geometry and Interactions, **Luca Rizzi**, CNRS, Institut Fourier, Grenoble, France, and **Fabrice Baudoin**, University of Connecticut, USA.

El Paso, Texas

University of Texas at El Paso

September 17–18, 2022

Saturday – Sunday

Meeting #1179

Central Section
Associate Secretary for the AMS: Betsy Stovall

Program first available on AMS website: August 5, 2022
Issue of *Abstracts*: Volume 43, Issue 3

Deadlines

For organizers: Expired
For abstracts: July 26, 2022

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Caroline Klivans, Brown University, *Title to be announced*.

Brisa N. Sánchez, Drexel University, *Measuring spatial access to community amenities and its association with health.*

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Algebraic, Geometric, and Topological Combinatorics (Code: SS 17A), **Art M. Duval**, The University of Texas at El Paso, **Caroline J. Klivans**, Brown University, and **Jeremy L. Martin**, University of Kansas.

Algebraic Structures in Topology, Logic, and Arithmetic (Code: SS 4A), **John Harding**, New Mexico State University, and **Emil D. Schwab**, The University of Texas at El Paso.

Banach Fixed Point Theorem: 100th year Celebration (Code: SS 5A), **Parin Chaipunya**, King Mongkut University of Technology, Thailand, and **Mohamed A. Khamsi**, **Osvaldo Mendez**, and **Julio C. Urenda**, The University of Texas at El Paso.

Eliiptic and Parabolic PDEs in Complex Fluid and Free Boundary Problems (Code: SS 6A), **Alaa Haj Ali**, Arizona State University, and **Hengrong Du**, Vanderbilt University.

From points to neighborhoods and beyond: frames and locales and their applications (Code: SS 14A), **Julio Urenda** and **Francisco Avila**, The University of Texas at El Paso, and **Angel Zaldivar** and **Miriam Borcardo**, Universidad de Guadalajara.

Geometry of Submanifolds (Code: SS 19A), **Hung Tran**, Texas Tech University, **Stephen McKeown**, UT Dallas, **Alvaro Pampano**, Texas Tech University, and **Magdalena Toda**, Texas Tech.

High-Frequency Data Analysis, Complex Datasets, and Applications (Code: SS 3A), **Maria Christina Mariani** and **Michael Pokojovy**, The University of Texas at El Paso, **Ambar Sengupta**, University of Connecticut, **Osei K. Tweneboah**, Ramapo College of New Jersey, and **Maria Pia Beccar Varela**, The University of Texas at El Paso.

Interactions between Combinatorics and Commutative Algebra (Code: SS 16A), **Louiza Fouli**, New Mexico State University, **Christopher Eur**, Harvard University, and **Jonathan Montano**, New Mexico State University.

Low dimensional topology and knot theory (Code: SS 15A), **Luis Valdez-Sanchez**, The University of Texas at El Paso, and **Ross E. Staffeldt**, New Mexico State University.

Mathematical and Computational Methods in Omics Research (Code: SS 13A), **Ming-Ying Leung** and **Jonathon E. Mohl**, The University of Texas at El Paso.

Methods and Applications in Data Science (Code: SS 7A), **Xiaogang Su**, **Ming-Ying Leung**, and **Amy Wagler**, The University of Texas at El Paso.

Numerical Partial Differential Equations and Applications (Code: SS 11A), **Son Young Yi** and **Xianyi Zeng**, The University of Texas at El Paso.

Ordered Structures (Code: SS 2A), **Piotr Wojciechowski**, University of Texas at El Paso.

Recent advances in scientific computing and applications (Code: SS 12A), **Natasha Sharma**, The University of Texas at El Paso, and **Annalisa Quaini**, University of Houston.

Statistical Methodology and Applications (Code: SS 10A), **Ori Rosen**, **Suneel Chatla**, **Asim Dey**, and **Abhijit Mandal**, The University of Texas at El Paso.

Stochastic Analysis and Applications (Code: SS 9A), **Adina Oprisan** and **Dante DeBlassie**, New Mexico State University, and **Robert Smits**, New Mexico State.

Stochastic dynamics: Theory and Applications in Biology (Code: SS 18A), **Tuan Anh Phan**, Institute for Modeling Collaboration and Innovation, University of Idaho, and **Jianjun Paul Tian**, New Mexico State University.

The Intersection of Number Theory and Combinatorics (Code: SS 8A), **Katie Anders**, University of Texas at Tyler, and **Timothy Huber** and **Brandt Kronholm**, University of Texas Rio Grande Valley.

Topics in Applied Analysis (Code: SS 1A), **Behzad Djafari-Rouhani**, University of Texas at El Paso, and **Gisele Goldstein** and **Jerome Goldstein**, University of Memphis.

Amherst, Massachusetts

University of Massachusetts-Amherst

October 1–2, 2022

Saturday – Sunday

Meeting #1180

Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub

Program first available on AMS website: August 18, 2022
Issue of *Abstracts*: Volume 43, Issue 4

Deadlines

For organizers: Expired
For abstracts: August 16, 2022

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Melody Chan, Brown University, *Title to be announced*.
Steven J. Miller, Williams College, *Title to be announced*.
Tadashi Tokieda, Stanford University, *Title to be announced*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Algebraic and Analytic theory of Elliptic Curves (Code: SS 1A), **Alina Cojocaru**, University of Illinois, Chicago, **Seoyung Kim**, Grand Valley State University, **Steven J. Miller**, Williams College, and **Jesse A Thorner**, University of Florida.

Combinatorial Algebraic Geometry (Code: SS 18A), **Melody Chan** and **Madeline Brandt**, Brown University, and **Juliette Bruce**, University of California Berkeley.

Connections Between Theoretical and Applied Dynamical Systems: a session in honor of the 60th birthdays of Renato Feres and Boris Hasselblatt (Code: SS 19A), **Timothy Chumley**, Mount Holyoke College, and **Yao Li** and **Hongkun Zhang**, University of Massachusetts.

Game-Theoretic and Agent-Based Approaches to Modeling Biological and Social Systems (Code: SS 7A), **Olivia Chu**, Dartmouth College, and **Daniel Cooney**, University of Pennsylvania.

Geometric Aspects of algebraic Combinatorics (Code: SS 10A), **Theo Douvropoulos**, University of Massachusetts, **Edward Richmond**, Oklahoma State University, and **Vasu Tewari**, University of Hawaii.

Higher Structures and Homotopical Algebra (Code: SS 11A), **John Berman**, University of Massachusetts, **Michael Ching** and **Ivan Contreras**, Amherst College, and **Owen Gwilliam** and **Martina Rovelli**, University of Massachusetts.

Iwasawa Theory (Code: SS 6A), **Robert Pollack**, Boston University, **Anwesh Ray**, University of British Columbia, and **Tom Weston**, University of Massachusetts.

Lagrangian and Legendrian Submanifolds (Code: SS 2A), **Dani Alvarez-Gavela**, Massachusetts Institute of Technology, and **Mike Sullivan**, University of Massachusetts.

Latinx and Hispanics in Combinatorics, Number Theory, Geometry and Topology (Code: SS 9A), **Iván Contreras**, Amherst College, **Pamela E. Harris**, Williams College, **Alejandro Morales**, University of Massachusetts, and **Geremías Polanco Encarnación**, Smith College.

Machine Learning Methods for PDEs (Code: SS 15A), **Yulong Lu**, University of Massachusetts, and **Wuzhe Xu**, University of Minnesota.

Math and Democracy: perspectives in research and teaching (Code: SS 17A), **Ben Blum-Smith**, New York University and The New School, and **Stanley Chang** and **Andrew Schultz**, Wellesley College.

Non-Abelian Hodge Theory and Minimal Surfaces (Code: SS 4A), **Robert Kusner**, **Charles Ouyang**, and **Franz Pedit**, University of Massachusetts.

Nonlinear waves and Applications: a Celebration of Dimitri Frantzeskakis 60th Birthday (Code: SS 5A), **Ricardo Carretero**, San Diego State University, and **Panos Kevrekidis**, University of Massachusetts.

Nonsmooth Analysis and Geometry (Code: SS 12A), **Ryan J. Alvarado**, Amherst College, and **Armin Schikorra**, University of Pittsburgh.

Ramsey Theory (Code: SS 3A), **Louis DeBiasio**, Miami University, and **Gábor Sárközy**, Worcester Polytechnic Institute and Alfréd Rényi Institute of Mathematics.

Recent Advances in Causal Inference (Code: SS 20A), **Haben Michael**, University of Massachusetts.

Some Tantalizing Conjectures in Discrete Mathematics (Code: SS 13A), **Laura Colmenarejo**, North Carolina State University, **Nadia Lafreniére**, Dartmouth College, and **Annie Raymond**, University of Massachusetts.

Structure-preserving machine learning (Code: SS 16A), **Wei Zhu**, University of Massachusetts.

The Combinatorics and Geometry of Jordan type and Commuting Varieties (Code: SS 21A), **Peter Crooks**, **Iva Halacheva**, and **Anthony Iarrobino**, Northeastern University, and **Leila Khatami**, Union College.

Topics in PDEs and Harmonic Analysis (Code: SS 8A), **Zongyuan Li**, Rutgers University, **Weinan Wang**, University of Arizona, **Xueying Yu**, University of Washington, and **Zhiyuan Zhang**, New York University.

Young Voices in Combinatorics (Code: SS 14A), **Laura Colmenarejo** and **Jianping Pan**, North Carolina State University.

Chattanooga, Tennessee

University of Tennessee at Chattanooga

October 15–16, 2022

Saturday – Sunday

Meeting #1181

Southeastern Section
Associate Secretary for the AMS: Brian D. Boe

Program first available on AMS website: September 1, 2022
Issue of *Abstracts*: Volume 43, Issue 4

Deadlines

For organizers: Expired
For abstracts: August 23, 2022

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Giulia Saccà, Columbia University, *Title to be announced*.

Chad Topaz, Williams College, *Mathematical and Computational Approaches to Social Justice*.

Xingxing Yu, Georgia Institute of Technology, *Graph structure and graph coloring*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Active Learning Methods and Pedagogical Approaches in Teaching College Level Mathematics (Code: SS 6A), **Hashim Saber**, University of North Georgia.

Applied Knot Theory (Code: SS 1A), **Jason Cantarella**, University of Georgia, **Eleni Panagiotou**, University of Tennessee at Chattanooga, and **Eric Rawdon**, University of St Thomas.

Boundary Value Problems for Differential, Difference, and Fractional Equations (Code: SS 9A), **John R Graef** and **Lingju Kong**, University of Tennessee at Chattanooga, and **Min Wang**, Kennesaw State University.

CANCELED: Ends and Boundaries of Groups: On the Occasion of Mike Mihalik's 70th Birthday (Code: SS 11A), **Craig Guilbault**, University of Wisconsin-Milwaukee, and **Kim Ruane**, Tufts University.

Combinatorial Commutative Algebra (Code: SS 8A), **Michael Cowen**, **Hugh Geller**, **Todd Morra**, and **Keri Sather-Wagstaff**, Clemson University.

Deterministic and Stochastic PDEs: Theoretical and Numerical Analyses (Code: SS 16A), **Hakima Bessaih**, Florida International University, and **Pelin Guven Geredeli**, Iowa State University.

Enumerative Combinatorics (Code: SS 10A), **Miklós Bóna** and **Vince Vatter**, University of Florida.

Geometric and Topological Generalization of Groups (Code: SS 4A), **Bikash C Das**, University of North Georgia.

Geometry and Arithmetic of Hyperkähler Manifolds (Code: SS 12A), **Giulia Saccà**, Columbia University, and **Laure Flapan**, Massachusetts Institute of Technology.

Interactions Between 3-Manifolds and 4-Manifolds (Code: SS 13A), **Jonathan Simone**, Georgia Institute of Technology, **Bulent Tosun**, University of Alabama, and **Hannah Turner**, Georgia Institute of Technology.

Modern Applied Mathematics and Spectral Analysis (Code: SS 18A), **Boris Belinskiy** and **Roger Nichols**, University of Tennessee at Chattanooga.

Multiplicative Ideal Theory and Arithmetical Properties of Monoids and Domains (Code: SS 15A), **Scott Chapman**, Sam Houston State University, **Jim Coykendall**, Clemson University, and **Richard Hasaenauer**, Northeastern State University.

New Developments in Operations Research and Management Sciences (Code: SS 20A), **Aniekan Ebiefung** and **Lakmali Weerasena**, University of Tennessee at Chattanooga.

Nonstandard Elliptic and Parabolic Regularity Theory with Applications (Code: SS 2A), **Hongjie Dong**, Brown University, and **Tuoc Phan**, University of Tennessee, Knoxville.

Probability and Statistical Models with Applications (Code: SS 5A), **Sher Chhetri**, University of South Carolina, Sumter, and **Cory Ball**, Florida Atlantic University.

Qualitative Analysis and Control Theory of Evolutionary Partial Differential Equations (Code: SS 19A), **George Avalos**, University of Nebraska-Lincoln, and **Weiwei Hu**, University of Georgia.

Quantitative Approaches to Social Justice (Code: SS 7A), **Chad Topaz**, Williams College.

Recent Advances in Mathematical Biology (Code: SS 14A), **Xiunan Wang** and **Jin Wang**, University of Tennessee at Chattanooga.

Reliable and Efficient Machine Learning for Scientific Forward and Inverse Problems (Code: SS 21A), **Anuj Abhishek**, University of North Carolina at Charlotte, **Feng Bao**, Florida State University, **Lan Gao**, University of Tennessee at Chattanooga, **Taufiquar Khan**, University of North Carolina at Charlotte, and **Jin Wang**, University of Tennessee at Chattanooga.

Structural and Extremal Graph Theory (Code: SS 3A), **Hao Huang**, Emory University, and **Xingxing Yu**, Georgia Institute of Technology.

Topological, Measurable, and Symbolic Dynamics, and Interactions with Geometry (Code: SS 17A), **Chris Johnson**, Western Carolina University, and **Martin Schmoll**, Clemson University.

Salt Lake City, Utah

University of Utah

October 22–23, 2022

Saturday – Sunday

Meeting #1182

Western Section
Associate Secretary for the AMS: Michelle Ann Manes

Program first available on AMS website: September 8, 2022
Issue of *Abstracts*: Volume 43, Issue 4

Deadlines

For organizers: Expired
For abstracts: August 30, 2022

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Bhargav Bhatt, University of Michigan, *Title to be announced*.
Jonathan Brundan, University of Oregon, *Title to be announced*.
Mariel Vazquez, UC Davis, *Title to be announced*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Algebraic Combinatorics and Applications in Harmonic Analysis (Code: SS 3A), **Joseph Iverson** and **Sung Y. Song**, Iowa State University, and **Bangteng Xu**, Eastern Kentucky University.

Approximation Theory and Numerical Analysis (Code: SS 2A), **Vira Babenko**, Drake University, and **Akil Narayan**, University of Utah.

Arithmetic Dynamics (Code: SS 28A), **Jamie Juul**, Colorado State University, **Bella Tobin**, Oklahoma State University, and **Bianca Thompson**, Westminster College.

Building Bridges Between Commutative Algebra and Nearby Areas (Code: SS 5A), **Benjamin Briggs** and **Josh Pollitz**, University of Utah.

Commutative Algebra (Code: SS 4A), **Adam Boocher**, University of San Diego, **Eloísa Grifo**, University of California, Riverside, and **Jennifer Kenkel**, University of Michigan.

Data, Parameters, and Inverse Problems for Dissipative Systems (Code: SS 7A), **Jared Whitehead**, Brigham Young University, **Adam Larios**, University of Nebraska - Lincoln, and **Vincent Martinez**, CUNY.

Extremal Graph Theory (Code: SS 1A), **József Balogh**, University of Illinois, and **Bernard Lidický**, Iowa State University.

Fractal Geometry, Dimension Theory, and Recent Advances in Diophantine Approximation (Code: SS 9A), **Alexander M. Henderson**, University of California, **Machiel van Frankenhuijsen**, Utah Valley University, and **Edward K. Voskanian**, The College of New Jersey.

Free Boundary Problems Arising in Applications (Code: SS 14A), **Mark Allen**, Brigham Young University, **Mariana Smit Vega Garcia**, Western Washington University, and **Braxton Osting**, University of Utah.

Geometry and Representation Theory of Quantum Algebras and Related Topics (Code: SS 6A), **Mee Seong Im**, United States Naval Academy, Annapolis, **Bach Nguyen**, Xavier University of Louisiana, and **Arik Wilbert**, University of Georgia.

Graphs and Matrices (Code: SS 11A), **Emily Evans**, **Mark Kempton**, and **Ben Webb**, Brigham Young University.

Heegaard Floer Homology in Topology, Algebra, and Physics (Code: SS 21A), **Aaron Lauda**, University of Southern California, and **Andrew Manion**, NC State University.

Higher Topological and Algebraic K-Theories (Code: SS 18A), **Agnès Beaudry**, University of Colorado Boulder, **Jonathan Campbell**, Duke University, and **John Lind**, California State University, Chico.

Hypergeometric functions and q-series (Code: SS 25A), **Howard S. Cohl**, National Institute of Standards and Technology, **Robert Maier**, University of Arizona, and **Roberto S. Costas-Santos**, Universidad de Alcalá.

Inverse Problems (Code: SS 12A), **Hanna Makaruk**, Los Alamos National Laboratory, and **Robert Owczarek**, University of New Mexico.

Mathematical Modeling of Biological and Social Systems (Code: SS 26A), **Daniel Cooney**, **Daniel Gomez**, and **Hyunjoong Kim**, University of Pennsylvania.

Mathematics of Collective Behavior (Code: SS 10A), **Daniel Lear** and **Roman Shvydkoy**, University of Illinois at Chicago.

Partitioning and Redistricting (Code: SS 22A), **Wesley Hamilton**, University of Utah, and **Tyler Jarvis**, BYU.

Quantum groups, Hopf Algebras and Applications: In honor of Professor Earl J. Taft (Code: SS 24A), **Susan Montgomery** and **Siu-Hung Ng**, University of Southern California.

Recent Advances in Algebraic Geometry and Commutative Algebra in or Near Characteristic p (associated with the Invited Address by Bhargav Bhatt) (Code: SS 8A), **Bhargav Bhatt**, University of Michigan, and **Karl Schwede**, University of Utah.

Recent Advances in the Theory of Fluid Dynamics (Code: SS 17A), **Elaine Cozzi**, Oregon State University, and **Magdalena Czubak**, University of Colorado Boulder.

Recent Advances of Numerical Methods for Partial Differential Equations with Applications (Code: SS 16A), **Joe Koebbe** and **Jia Zhao**, Utah State University, and **Yunrong Zhu**, Idaho State University.

Recent Developments in Inverse Problems for PDEs and Applications (Code: SS 20A), **Loc Nguyen**, University of North Carolina at Charlotte, **Dinh-Liem Nguyen**, Kansas State University, and **Fernando Guevara Vasquez**, University of Utah.

Several Complex Variables: Emerging Applications, Connections, And Synergies (Code: SS 13A), **Jennifer Brooks**, Brigham Young University, and **Dusty Grundmeier**, Harvard University.

Special Session on 4-dimensional topology (Code: SS 15A), **Mark Hughes**, Brigham Young University, and **Maggie Miller** and **Patrick Naylor**, Princeton University.

Topics in Graphs, Hypergraphs and Set Systems (Code: SS 19A), **John Engbers**, Marquette University, **David Galvin**, University of Notre Dame, and **Cliff Smyth**, The University of North Carolina at Greensboro.

Topology and geometry of multi-stranded nucleic acids (associated with the Invited Address by Mariel Vazquez) (Code: SS 23A), **Mariel Vazquez**, University of California Davis, **Christine E. Soteros**, University of Saskatchewan, **Margherita Maria Ferrari**, University of South Florida, and **Javier Arsuaga**, University of California Davis.

Boston, Massachusetts (JMM 2023)

John B. Hynes Veterans Memorial Convention Center, Boston Marriott Hotel, and Boston Sheraton Hotel

January 4–7, 2023

Wednesday – Saturday

Meeting #1183

Associate Secretary for the AMS: Steven H. Weintraub
Program first available on AMS website: To be announced
Issue of *Abstracts*: Volume 44, Issue 1

Deadlines

For organizers: Expired

For abstracts: September 13, 2022

Atlanta, Georgia

Georgia Institute of Technology

March 18–19, 2023

Saturday – Sunday

Meeting #1184

Southeastern Section
Associate Secretary for the AMS: Brian D. Boe
Program first available on AMS website: January 26, 2023
Issue of *Abstracts*: To be announced

Deadlines

For organizers: August 18, 2022
For abstracts: January 17, 2023

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Advanced Topics in Graph Theory and Combinatorics (Code: SS 1A), **Songling Shan**, Illinois State University, and **Guangming Jing**, Augusta University.

Commutative Algebra and its Interactions with Algebraic Geometry (Code: SS 10A), **Michael K. Brown** and **Henry K. Schenck**, Auburn University.

Fractal Geometry and Dynamical Systems (Code: SS 11A), **Scott Kaschner**, Butler University, and **Mrinal Kanti Roychowdhury**, University of Texas Rio Grande Valley.

Groups, Geometry, and Topology (Code: SS 4A), **Dan Margalit** and **Yvon Verberne**, Georgia Institute of Technology.

Knots, Skein Modules and Categorification (Code: SS 8A), **Rhea Palak Bakshi** and **Józef H. Przytycki**, George Washington University, **Radmilla Sazdanovic**, North Carolina State University, and **Marithania Silvero**, Universidad de Sevilla.

Recent Developments in Commutative Algebra (Code: SS 5A), **Florian Enescu**, Georgia State University, and **Thomas Polstra**, MSRI and University of Virginia.

Recent Developments on Analysis and Computation for Inverse Problems for PDEs (Code: SS 2A), **Dinh-Liem Nguyen**, Kansas State University, and **Loc Nguyen** and **Khoa Vo**, University of North Carolina at Charlotte.

Representation Theory of Algebraic Groups and Quantum Groups: A Tribute to the Work of Cline, Parshall and Scott (CPS) (Code: SS 6A), **Chun-Ju Lai**, Academia Sinica, Taiwan, **Daniel K. Nakano**, University of Georgia, and **Weiqiang Wang**, University of Virginia.

Singer-Hopf Conjecture in Geometry and Topology (Code: SS 9A), **Luca Di Cerbo**, University of Florida, and **Laurentiu Maxim**, University of Wisconsin-Madison.

Stochastic Analysis and its Applications (Code: SS 7A), **Parisa Fatheddin**, Ohio State University, Marion, and **Kazuo Yamazaki**, Texas Tech University.

Topology and Geometry of 3- and 4-Manifolds (Code: SS 3A), **Siddhi Krishna**, Georgia Institute of Technology and Columbia University, **Miriam Kuzbary**, Georgia Institute of Technology, and **Beibei Liu**, Max Planck Institute for Mathematics and Georgia Institute of Technology.

Spring Eastern Virtual Sectional Meeting

Meeting virtually, EDT (hosted by the American Mathematical Society)

April 1–2, 2023

Saturday – Sunday

Meeting #1185

Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub

Program first available on AMS website: February 9, 2023
Issue of *Abstracts*: To be announced

Deadlines

For organizers: September 1, 2022
For abstracts: January 31, 2023

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Sigal Gottlieb, University of Massachusetts, Dartmouth, *Title to be announced*.

Samuel Payne, University of Texas, *Title to be announced*.

Cincinnati, Ohio

University of Cincinnati

April 15–16, 2023

Saturday – Sunday

Meeting #1186

Central Section
Associate Secretary for the AMS: Betsy Stovall, University of Wisconsin-Madison

Program first available on AMS website: February 23, 2023
Issue of *Abstracts*: To be announced

Deadlines

For organizers: September 15, 2022
For abstracts: February 14, 2023

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Nathaniel Whitaker, University of Massachusetts Amherst, *2023 AMS Einstein Public Lecture in Mathematics*.

Fresno, California

California State University, Fresno

May 6–7, 2023

Saturday – Sunday

Meeting #1187

Western Section
Associate Secretary for the AMS: Michelle Ann Manes

Program first available on AMS website: March 16, 2023
Issue of *Abstracts*: To be announced

Deadlines

For organizers: October 1, 2022
For abstracts: March 7, 2023

The scientific information listed below may be dated. For the latest information, see `https://www.ams.org/amsmtgs/sectional.html`.

Invited Addresses

Sami Assaf, University of Southern California, *Title to be announced*.
Natalia Komarova, University of California, Irvine, *Title to be announced*.
Joseph Teran, University of California, Los Angeles, *Title to be announced*.

Special Sessions

If you are volunteering to speak in a Special Session, you should send your abstract as early as possible via the abstract submission form found at `https://www.ams.org/cgi-bin/abstracts/abstract.pl`.

Advances by Scholars in the Pacific Math Alliance (Code: SS 22A), **Andrea Arauza Rivera**, California State University, East Bay, **Mario Banuelos**, California State University, Fresno, and **Jessica De Silva**, California State University, Stanislaus.

Advances in Functional Analysis and Operator Theory (Code: SS 6A), **Michel L. Lapidus**, University of California, Riverside, **Marat V. Markin**, California State University, Fresno, and **Igor Nikolaev**, St. John's University.

Algebraic Structures in Knot Theory (Code: SS 4A), **Carmen Caprau**, California State University, Fresno, and **Sam Nelson**, Claremont McKenna College.

Algorithms in the Study of Hyperbolic 3-manifolds (Code: SS 26A), **Robert Haraway, III** and **Maria Trnkova**, University of California, Davis.

Analysis of Fractional Differential and Difference Equations with its Application (Code: SS 20A), **Bhuvaneswari Sambandham**, Dixie State University, and **Aghalaya S. Vatsala**, University of Louisiana at Lafayette.

Artin-Schelter Regular Algebras and Related Topics (Code: SS 27A), **Ellen Kirkman**, Wake Forest University, and **James Zhang**, University of Washington.

Combinatorics Arising from Representations (associated with the Invited Address by Sami Assaf) (Code: SS 16A), **Sami Assaf**, University of Southern California, **Nicolle Gonzalez**, University of California, Los Angeles, and **Brendan Pawloski**, University of Southern California.

Complexity in Low-Dimensional Topology (Code: SS 14A), **Jennifer Schultens**, University of California, Davis, and **Eric Sedgwick**, DePaul University.

Data Analysis and Predictive Modeling (Code: SS 8A), **Earvin Balderama**, California State University, Fresno, and **Adriano Zambom**, California State University, Northridge.

Inverse Problems (Code: SS 5A), **Hanna Makaruk**, Los Alamos National Laboratory, and **Robert Owczarek**, University of New Mexico, Albuquerque and University of New Mexico, Los Alamos.

Math Circle Games and Puzzles that Teach Deep Mathematics (Code: SS 13A), **Maria Nogin** and **Agnes Tuska**, California State University, Fresno.

Mathematical Biology: Confronting Models with Data (Code: SS 21A), **Erica Rutter**, University of California, Merced.

Mathematical Methods in Evolution and Medicine (associated with the Invited Address by Natalia Komarova) (Code: SS 1A), **Natalia Komarova** and **Jesse Kreger**, University of California, Irvine.

Methods in Non-Semisimple Representation Categories (Code: SS 11A), **Eric Friedlander**, University of Southern California, Los Angeles, **Julia Pevtsova**, University of Washington, Seattle, and **Paul Sobaje**, Georgia Southern University, Statesboro.

Recent Advances in Mathematical Biology, Ecology, Epidemiology, and Evolution (Code: SS 10A), **Lale Asik**, Texas Tech University, **Khanh Phuong Nguyen**, University of Houston, and **Angela Peace**, Texas Tech University.

Research in Mathematics by Early Career Graduate Students (Code: SS 7A), **Doreen De Leon**, **Marat Markin**, and **Khang Tran**, California State University, Fresno.

Scientific Computing (Code: SS 19A), **Changho Kim**, University of California, Merced, and **Roummel Marcia**.

The Use of Computational Tools and New Augmented Methods in Networked Collective Problem Solving (Code: SS 18A), **Mario Banuelos**, California State University, Fresno, **Andrew G. Benedek**, Research Centre for the Humanities, Hungary, and **Agnes Tuska**, California State University, Fresno.

Women in Mathematics (Code: SS 12A), **Doreen De Leon**, **Katherine Kelm**, and **Oscar Vega**, California State University, Fresno.

Zero Distribution of Entire Functions (Code: SS 9A), **Tamás Forgács** and **Khang Tran**, California State University, Fresno.

Buffalo, New York

University at Buffalo (SUNY)

September 9–10, 2023

Saturday – Sunday

Meeting #1188

Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub

Program first available on AMS website: July 27, 2023
Issue of *Abstracts*: To be announced

Deadlines

For organizers: February 9, 2023
For abstracts: July 18, 2023

Omaha, Nebraska

Creighton University

October 7–8, 2023

Saturday – Sunday

Meeting #1190

Central Section
Associate Secretary for the AMS: Betsy Stovall, University of Wisconsin-Madison

Program first available on AMS website: August 17, 2023
Issue of *Abstracts*: To be announced

Deadlines

For organizers: March 7, 2023
For abstracts: August 8, 2023

Mobile, Alabama

University of South Alabama

October 13–15, 2023

Friday – Sunday

Meeting #1189

Southeastern Section
Associate Secretary for the AMS: Brian D. Boe

Program first available on AMS website: August 17, 2023
Issue of *Abstracts*: To be announced

Deadlines

For organizers: March 6, 2023
For abstracts: August 8, 2023

Albuquerque, New Mexico

University of New Mexico

October 21–22, 2023

Saturday – Sunday

Meeting #1191

Western Section
Associate Secretary for the AMS: Michelle Ann Manes
Program first available on AMS website: August 31, 2023
Issue of *Abstracts*: To be announced

Deadlines

For organizers: March 21, 2023
For abstracts: August 22, 2023

Auckland, New Zealand

December 4–8, 2023

Monday – Friday
Associate Secretary for the AMS: Steven H. Weintraub
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

San Francisco, California (JMM 2024)

Moscone West Convention Center

January 3–6, 2024

Wednesday – Saturday
Associate Secretary for the AMS: Michel L. Lapidus
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

Washington, District of Columbia

Howard University

April 6–7, 2024

Saturday – Sunday
Eastern Section
Associate Secretary for the AMS: Steven H. Weintraub
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

San Francisco, California

San Francisco State University

May 4–5, 2024

Saturday – Sunday
Western Section
Associate Secretary for the AMS: Michelle Ann Manes
Program first available on AMS website: To be announced
Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced
For abstracts: To be announced

Palermo, Italy

July 23–26, 2024

Tuesday – Friday

Associate Secretary for the AMS: Brian D. Boe

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced

For abstracts: To be announced

Riverside, California

University of California, Riverside

October 26–27, 2024

Saturday – Sunday

Western Section

Associate Secretary for the AMS: Michel L. Lapidus

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced

For abstracts: To be announced

Washington, District of Columbia (JMM 2026)

Walter E. Washington Convention Center and Marriott Marquis Washington DC

January 4–7, 2026

Sunday – Wednesday

Associate Secretary for the AMS: Betsy Stovall

Program first available on AMS website: To be announced

Issue of *Abstracts*: To be announced

Deadlines

For organizers: To be announced

For abstracts: To be announced

MATHEMATICAL MOMENTS

Deblurring Images

This image was deblurred using code available at https://www.mathworks.com/help/images/ref/deconvwnr.html

Imagine snapping a quick picture of a flying bird—the image is likely to come out blurry. But thanks to mathematics, you might be able to use software to improve the photo. Scientists often deal with blurry pictures, too. Linear algebra and clever numerical methods allow researchers to fix imperfect photos in medical imaging, astronomy, and more. In a computer, the pixels that make up an image can be represented as a column of numbers called a vector. Blurring happens when the light meant for each pixel spills into the adjacent pixels, changing the numbers in a way that can be mathematically represented as an enormous matrix. But knowing that matrix is not enough if you want to reconstruct the original (non-blurry) image.

That's because pixels in the blurred image will have extra errors (or "noise") resulting from the physical process of taking the photo. If you don't account for them, trying to recreate the original image amplifies these errors. Mathematicians have developed various approaches to get rid of the noise while still retaining as much correct information as possible. The best way to do so depends on the cause of the blur, whether the original image had sharp edges or was smooth, and the physics underlying how the image was captured. In ongoing research, experts are working to speed up the necessary computer calculations and store vast amounts of image data efficiently. Whether you're getting an MRI scan of your body or admiring a photo of a distant galaxy, mathematics helped make the image crystal-clear.

For More Information: "The Image Deblurring Problem: Matrices, Wavelets, and Multilevel Methods," D. Austin, M. Espanol, M. Pasha, *Notices of the American Mathematical Society* **69**, 2022 (forthcoming).

Watch an interview with an expert!

MM/161

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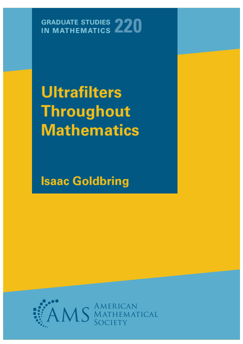
NEW RELEASES from the AMS

◆ = Textbook

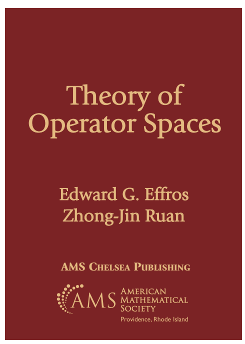


Ultrafilters Throughout Mathematics ◆

Isaac Goldbring, *University of California, Irvine, CA*

Ultrafilters and ultraproducts provide a useful generalization of the ordinary limit processes which have applications to many areas of mathematics. Typically, this topic is presented to students in specialized courses such as logic, functional analysis, or geometric group theory. In this book, the basic facts about ultrafilters and ultraproducts are presented to readers with no prior knowledge of the subject and then these techniques are applied to a wide variety of topics.

Graduate Studies in Mathematics, Volume 220; 2022; 408 pages; Hardcover; ISBN: 978-1-4704-6900-9; List US$125; AMS members US$100; MAA members US$112.50; Order code GSM/220

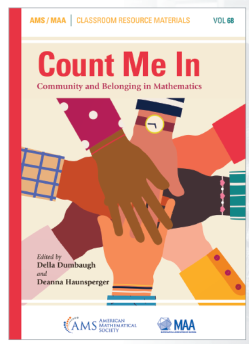


Theory of Operator Spaces ◆

Edward G. Effros, and **Zhong-Jin Ruan**, *University of Illinois at Urbana-Champaign, IL*

This book provides the main results and ideas in the theories of completely bounded maps, operator spaces, and operator algebras, along with some of their main applications. It requires only a basic background in functional analysis to read through the book. The descriptions and discussions of the topics are self-explained. It is appropriate for graduate students new to the subject and the field.

AMS Chelsea Publishing, Volume 386; 2022; 358 pages; Softcover; ISBN: 978-1-4704-6505-6; List US$60; AMS members US$48; MAA members US$54; Order code CHEL/386



Count Me In
Community and Belonging in Mathematics

Della Dumbaugh, *University of Richmond, VA*, and **Deanna Haunsperger**, *Carleton College, Northfield, MN*, Editors

This groundbreaking work explores the powerful role of communities in mathematics. It introduces readers to twenty-six different mathematical communities and addresses important questions about how they form, how they thrive, and how they advance individuals and the group as a whole. The chapters celebrate how diversity and sameness bind colleagues together, showing how geography, gender, or graph theory can create spaces for colleagues to establish connections in the discipline.

Classroom Resource Materials, Volume 68; 2022; 241 pages; Softcover; ISBN: 978-1-4704-6566-7; List US$65; AMS members US$48.75; MAA members US$48.75; Order code CLRM/68

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