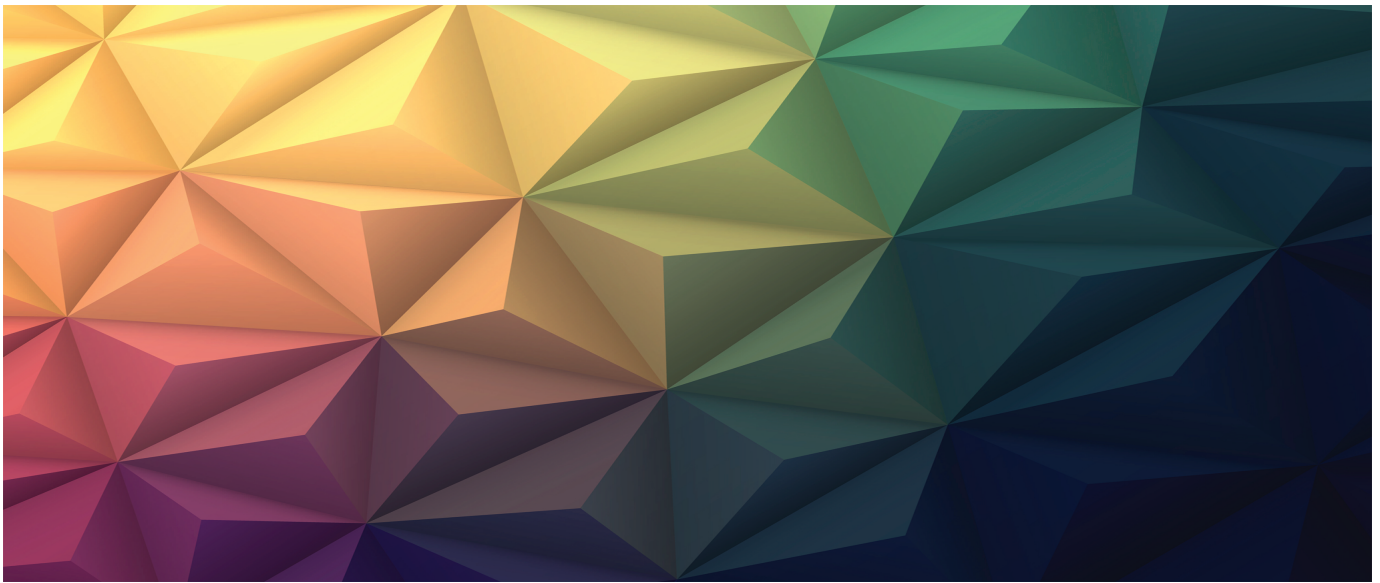# A.T.V.'s for (Geometric) Off-roading: A Gentle Introduction to Arithmetic Toric Varieties

Patrick K. McFaddin

1. Introduction

A fundamental problem in mathematics is how to determine whether a given finite collection of polynomials has a common solution, and how to quantify and qualify the shape of this solution set in a meaningful way. Indeed, linear algebra is the study of polynomials of degree one, and the theory of quadratic forms investigates polynomials of degree two. In general, this is the staring point of *algebraic geometry* and the theory of *algebraic varieties*. Over the past few centuries, algebraic geometers have developed a wide array of tools called *invariants* for studying such geometric objects. Invariants are pieces of data (numbers, sets, groups, vector spaces, categories) associated to mathematical objects which produce the same output on objects which are effectively identical, i.e., those which are *isomorphic* in the appropriate sense. In other words, invariants are those associations which are invariant under isomorphism.

Invariants help us distinguish algebraic varieties which have distinct geometric properties, greatly improving our ability to classify them. To study varieties, it is necessary to study their invariants, how to effectively compute them, and what information they reflect about the algebra, geometry, and arithmetic of a given variety. Often one gains a great deal of insight by computing invariants on large classes of particularly nice varieties. Toric varieties defined over the complex numbers have proved to be an extremely useful class of geometric objects to test the flexibility and robustness of a slew of algebro-geometric invariants. Our understanding of divisor class groups, Picard groups, algebraic *K*-theory, derived categories, moduli problems,
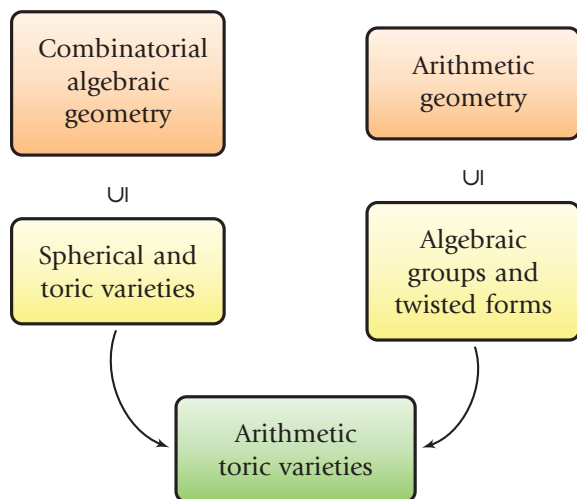
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geometric invariant theory, and the minimal model program has been greatly improved by investigating these invariants for toric varieties [CLS11].

To investigate the utility and limitations of these invariants in the arithmetic case, i.e., for varieties over non-algebraically closed fields, one aims to find a similarly useful class of varieties. Simply taking the naive, analogous definition of toric varieties over arbitrary fields yields a theory which is too similar to the complex case and fails to faithfully reflect arithmetic. Instead, one looks to *twisted forms* of toric varieties, which we call *arithmetic toric varieties*. For such varieties, the combinatorial nature of complex toric varieties persists, but additional Galois-theoretic analysis plays a crucial role. The study of arithmetic toric varieties is then reduced to understanding combinatorial objects (fans, cones, etc.) and how Galois groups associated to certain field extensions act on these objects. In this way, arithmetic toric varieties serve as a convenient bridge between arithmetic and combinatorial algebraic geometry, which we hope to exploit to study derived invariants. By veering off of the geometrically beaten path with our A.T.V.'s, we hope to gain deeper insight into various invariants and their limitations in reflecting arithmetic data. Here we highlight two properties of interest in the arithmetic setting: the existence of *rational points* and *rationality*. The first is the problem of determining whether a collection of polynomials admits a common solution. The second focuses on how to parametrize the solution set of a collection of polynomials using only rational functions.

The purpose of this manuscript is to give a small sampling of the theory of arithmetic toric varieties, focusing on simple examples in low dimension using explicit polynomial equations. Even with this restricted "cheat-sheet" type viewpoint, there is hope that the reader may glean a general sense of the analysis of such objects. While some topics are only glossed over, and some terminology is defined in a colloquial manner, we hope that this manuscript

serves as an invitation to the reader to look more deeply into these topics and ideas.

A second goal is to highlight the class of arithmetic toric varieties as the most natural testing ground to study the limitations of algebro-geometric invariants in the arithmetic setting. There is already a rich history of such work, including the analysis of del Pezzo surfaces and Severi-Brauer varieties. Using more recent work which systematically investigates arithmetic toric varieties using Galois (and non-ablelian) cohomology [Dun16, ELFST14, MP97], we hope to gain new insights into this class of objects. We point out one particular success story in this line of investigation concerning the invariant given by the *coherent derived category*. It has been found that unless the derived category has a fine decomposition into very simple pieces, it does not generally reflect rationality properties well. This yields a broader understanding of the derived category as an arithmetic invariant. On the other hand, one might share the author's disappointment that the derived category is not better suited to this task. C'est la vie! Our hope is that the class of arithmetic toric varieties will be used to probe other invariants in an analogous fashion.

Organization. We begin by recalling some basic terminology from the theory of algebraic varieties and their arithmetic, including a discussion of algebraic groups and twisted forms. In Section 3, we discuss toric varieties defined over the complex numbers, introducing the combinatorial description of these geometric objects in low dimension. In Section 4, arithmetic toric varieties are introduced and examples are provided. In Section 5, we report on recent results which provide insight into the use of arithmetic toric varieties in studying the limitations of the derived category as an arithmetic invariant of smooth projective varieties.

Notation and conventions. All rings are associative with identity and all vector spaces are finite-dimensional. A *monoid* is a set together with an associative binary operation which admits an identity element. Given a field k, a *k-algebra* is a ring which also admits the structure of a k-vector space, and these operations are compatible. A *division ring* (resp., *division k-algebra*) is a ring (resp., k-algebra) in which every non-zero element has a multiplicative inverse. Given two objects A and B in the same category, we use $\mathrm{Hom}(A, B)$ to denote the collection of all morphisms (i.e., structure preserving functions) from A to B. Throughout, we let $\mathbb{C}$ denote the field of complex numbers, and $k \subseteq \mathbb{C}$ a Galois extension of fields.

2. Varieties, Groups, and Twisted Forms

Let us begin by giving an overview of varieties defined over the field of complex numbers. An *affine $\mathbb{C}$-variety* is a subset of $\mathbb{C}^n$ of the form

$$\mathbf{V}(I) = \{\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{C}^n \mid f(\mathbf{a}) = 0 \text{ for all } f \in I\}$$

for some ideal $I \trianglelefteq \mathbb{C}[x_1, \ldots, x_n]$. Given a finite set of polynomials $f_1, \ldots, f_m \in \mathbb{C}[x_1, \ldots, x_n]$, we similarly define the variety given by the set of simultaneous zeros of all the f_i, i.e., $\mathbf{V}(f_1, \ldots, f_n) = \{\mathbf{a} \in \mathbb{C}^n \mid f_i(\mathbf{a}) = 0 \text{ for all } f_i\}$. This coincides with the variety associated to the ideal generated by the f_i, and any variety is given by the vanishing of finitely many polynomials, a consequence of the Hilbert Basis Theorem [Eis95, Thm. 1.2].

Conversely, given any affine $\mathbb{C}$-variety $V \subseteq \mathbb{C}^n$, we obtain *the defining ideal* of V

$$\mathbf{I}(V) = \{f \in \mathbb{C}[x_1, \ldots, x_n] \mid f(\mathbf{a}) = 0 \text{ for all } \mathbf{a} \in V\}.$$

The quotient ring $\mathbb{C}[V] := \mathbb{C}[x_1, \ldots, x_n]/\mathbf{I}(V)$ is the *coordinate ring* of V. Notice that if $W \subseteq V$ is an inclusion of affine $\mathbb{C}$-varieties, then $\mathbf{I}(V) \subseteq \mathbf{I}(W)$. The constructions $\mathbf{I}$ and $\mathbf{V}$ are are inverses of one another if we restrict to a nice class of ideals. Indeed, for any ideal $I \trianglelefteq \mathbb{C}[x_1, \ldots, x_n]$, we have

$$\mathbf{I}(\mathbf{V}(I)) = \sqrt{I} = \{f \in \mathbb{C}[x_1, \ldots, x_n] \mid f^m \in I \text{ for some } m \in \mathbb{Z}\}$$

by Hilbert's Nullstellenstatz [Eis95, Thm. 1.6]. This result, which crucially relies on the fact that $\mathbb{C}$ is algebraically closed, yields a bijective correspondence between affine varieties and radical ideals (i.e., ideals satsifying $I = \sqrt{I}$). In other words, the geometric object, given by solution sets of polynomials, completely determines the underlying algebraic data, and vice versa.

The natural maps between affine $\mathbb{C}$-varieties are given by polynomial functions. If $V \subseteq \mathbb{C}^n$ and $W \subseteq \mathbb{C}^m$ are affine $\mathbb{C}$-varieties, a *morphism* or *regular function* $\varphi : V \to W$ has both geometric and algebraic descriptions:

1. Geometric: A function $\varphi : V \to \mathbb{C}$ is *regular* if there is a polynomial $F \in \mathbb{C}[x_1, \ldots, x_n]$ such that $\varphi(x) = F(x)$ for all $x \in V$. A function $\varphi : V \to W$ is *regular* if there exist regular functions $\varphi_1, \ldots, \varphi_m : V \to \mathbb{C}$ such that $\varphi(x) = (\varphi_1(x), \ldots, \varphi_m(x))$ for all $x \in V$.
2. Algebraic: A regular function $V \to W$ is equivalent to the data of a $\mathbb{C}$-algebra homomorphism $\mathbb{C}[W] \to \mathbb{C}[V]$. If $\varphi : V \to W$ is a regular function, the corresponding $\mathbb{C}$-algebra homomorphism is defined as $\varphi^*(f) = f \circ \varphi$, for $f \in \mathbb{C}[W]$.

An *isomorphism* of affine $\mathbb{C}$-varieties is a regular map which has a regular inverse. Equivalently, an isomorphism of affine $\mathbb{C}$-varieties V and W is given by an isomorphism of $\mathbb{C}$-algebras $\mathbb{C}[W] \xrightarrow{\sim} \mathbb{C}[V]$. A general $\mathbb{C}$-variety is a space which locally looks like an affine $\mathbb{C}$-variety, i.e., each point has a neighborhood which is isomorphic to an affine $\mathbb{C}$-variety. If V and W are $\mathbb{C}$-varieties, a function $\varphi : V \to W$ is *regular* if for each $x \in V$ and any affine open subset $V \subseteq \mathbb{C}^m$ containing $\varphi(x)$, there is a neighborhood $x \in U \subseteq V$ such that $\varphi(U) \subseteq V$ and $\varphi : U \to V$ is regular. This is analogous to the definition of a manifold, which is locally Euclidean space.

In the case where the base field is not algebraically closed, the Nullstellensatz no longer holds, and the geometric data of the variety is not enough to fully determine its underlying algebraic structure. For this reason, our definition of k-variety centers on algebraic data. For our purposes, we adopt (a simplified version of) the approach given in [Bor91, Spr98] using k-structures on varieties.

A k-structure on a $\mathbb{C}$-algebra A is a k-subalgebra $A_k \subseteq A$ such that the homomorphism $\mathbb{C} \otimes_k A_k \to A$ is an isomorphism. If A is a $\mathbb{C}$-algebra with k-structure A_k and $J \trianglelefteq A$, then J is *defined over k* if $J \cap A_k$ generates J as an ideal. If $\varphi : A \to B$ is a homomorphism of $\mathbb{C}$-algebras with k-structures A_k and B_k, then f is *defined over k* if $\varphi(A_k) \subseteq B_k$. An affine $\mathbb{C}$-variety $V \subseteq \mathbb{C}^n$ is *defined over k* if there exist $f_1, \ldots, f_m \in k[x_1, \ldots, x_n]$ such that $V = \mathbf{V}(f_1, \ldots, f_m)$. From this perspective, k-varieties are given by $\mathbb{C}$-varieties whose defining polynomials are elements of $k[x_1, \ldots, x_n]$, and we use this perspective onward.

Definition 2.1 (Affine k-variety). An *affine k-variety* is a finite collection of polynomials

$$X = \{f_i \in k[x_1, \ldots, x_n]\}.$$

Such a collection gives rise to the following algebraic and geometric data:

1. Algebraic: The defining ideal $\mathbf{I}(X) = (f_1, \ldots, f_m)$ and coordinate ring $\mathbb{C}[X]$, which admits a k-structure given by $k[X] := k[x_1, \ldots, x_n]/\mathbf{I}(X) \subseteq \mathbb{C}[X]$. Relative to this k-structure, the ideal $\mathbf{I}(X)$ is defined over k.
2. Geometric: The set of k-rational points (or simply k-points) is given by

$$X(k) := \{\mathbf{a} \in k^n \mid f_i(\mathbf{a}) = 0 \text{ for all } f_i\} \subseteq k^n.$$

More generally, for any field extension $k \subseteq F$, the set of F-points is given by

$$X(F) := \{\mathbf{a} \in F^n \mid f_i(\mathbf{a}) = 0 \text{ for all } f_i\}.$$

Notice that the solution set of the defining equations of a variety is just one piece of information that we can extract. The ability to consider the set of solutions over extensions of k allow us to probe deeper questions about the arithmetic of varieties.

We may refer to a variety by describing its set of points, but will often remind the reader of its full variety structure when necessary, i.e., we write $X = \{f_i = 0\}$ to denote a variety X defined by $\{f_i\}$. In certain cases, e.g., the pointless conic of Example 2.4, varieties may have no points at all, yet still provide interesting examples of "geometric objects," via their algebraic and arithmetic data. In these cases (especially in Section 4) it is necessary to include the full variety structure.

The sets given by the collection of common zeros of finitely many polynomials are the closed sets in a topology on k^n called the *Zariski topology*. If we view a

polynomial $f \in k[x_1, .., x_n]$ as a function $k^n \to k$, then the Zariski topology is the coarsest topology in which all polynomials are continuous [Kem11, Rmk 3.3(g)]. Algebraic varieties are often described in terms of sets together with this topology. For instance, the set k^n together with its Zariski topology is the variety $\mathbb{A}_k^n$ called *affine n-space*. We can also describe it using a (quite trivial!) polynomial equation.

Example 2.2 (Affine space). Consider the k-variety $\mathbb{A}_k^n$ whose defining polynomial is the 0 polynomial in $k[x_1, \ldots, x_n]$. Since every element of k^n vacuously satisfies this equation $0 = 0$, the k-points of $\mathbb{A}_k^n$ are given by $\mathbb{A}_k^n(k) = k^n$. We will often denote this variety simply by k^n, e.g., $\mathbb{C}^n$ denotes the variety $\mathbb{A}_\mathbb{C}^n$ and $\mathbb{R}^n$ denotes $\mathbb{A}_\mathbb{R}^n$.

Note that any affine variety X can be viewed as a *subvariety* of affine space. This is easiest to see on points. Indeed, a point of X is a simultaneous solution of its defining equations. But such solutions are elements of $k^n = \mathbb{A}_k^n$ by definition.

Example 2.3 (Circle group). Consider the $\mathbb{R}$-variety $X = \{x^2 + y^2 - 1 = 0\}$. Then $X(\mathbb{R})$ is given by the unit circle in $\mathbb{R}^2$. For that reason, we denote this variety by S^1. It also has the structure of a group, which comes from the action of rotation matrices on the coordinates x, y of the ambient space $\mathbb{R}^2 = \mathbb{A}_\mathbb{R}^2$, and we call this variety the *circle group*. We will encounter a few other *algebraic groups* below (Example 2.5 and Subsection 2.1).

Example 2.4 (Pointless real conic). Consider the $\mathbb{R}$-variety $X = \{x^2 + y^2 + 1 = 0\}$. Then $X(\mathbb{R}) = \emptyset$, so there exist $\mathbb{R}$-varieties which have no $\mathbb{R}$-points, a fact we all remember from solving equations in grade school! Even though this variety has no points, it is still useful to view it as a one-dimensional geometric object, i.e., a curve. This is the benefit of remembering its defining equation (i.e., its full structure as a variety). This reflects the arithmetic complexity of $\mathbb{R}$ versus that of $\mathbb{C}$, particularly the fact that $\mathbb{C}$ is algebraically closed while $\mathbb{R}$ is not. We expect arithmetically complicated fields to admit many examples of such varieties.

Example 2.5 (Multiplicative group). Consider the $\mathbb{R}$-variety $X = \{xy - 1 = 0\}$. The set of $\mathbb{R}$-points of X is given by a hyperbola in $\mathbb{R}^2$. Projecting this hyperbola onto the x-axis gives us $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. This variety is called the *multiplicative group* and is an algebraic group, i.e., it simultaneously has the structure of an affine variety and a group. When we wish to remember its structure as a variety (i.e., its defining equation), we will denote it by $\mathbb{G}_m$ or $\mathbb{G}_{m,\mathbb{R}}$. We will also denote this variety by $\mathbb{R}^*$, since this set defines its $\mathbb{R}$-points.

The defining equation of $\mathbb{G}_{m,\mathbb{R}}$ is equivalent to $y = x^{-1}$, and so this variety is often described by the polynomial ring $\mathbb{R}[x, x^{-1}]$. This gives an intuitive understanding of how polynomial functions define our varieties. When we restrict our attention to non-zero elements of $\mathbb{R} = \mathbb{A}_\mathbb{R}^1$, both x and $\frac{1}{x}$ are well-defined functions which satisfy the relation $x \cdot \frac{1}{x} = 1$. Conversely, the one-dimensional geometric space on which both x and $\frac{1}{x}$ are well-defined is given by the non-zero elements of $\mathbb{R}$. We also realize $\mathbb{G}_{m,\mathbb{R}}$ as an open subset of the affine line. In the Zariski topology, the polynomial function $f(x) = x$ has zero set $\{0\}$, which is a closed set. Its complement is then the open set $\mathbb{R}^*$, a *quasi-affine variety*.

We may equally well view X as a variety over $\mathbb{Q}$ or $\mathbb{C}$, or any field k; they all contain 0 and 1, and this is all that is needed for its defining equation. In these cases, we write $\mathbb{G}_{m,\mathbb{Q}}$, $\mathbb{G}_{m,\mathbb{C}}$, or $\mathbb{G}_{m,k}$ but will also use $\mathbb{Q}^*$, $\mathbb{C}^*$, and k^*, respectively. We also refer to the multiplicative group as the *one-dimensional torus*. This moniker is discussed in Section 3.

Definition 2.6. Let X and Y be affine k-varieties. A *morphism $X \to Y$* is given by the data of a $\mathbb{C}$-algebra homomorphism $\varphi : \mathbb{C}[Y] \to \mathbb{C}[X]$ which is defined over k, i.e., φ satisfies $\varphi(k[Y]) \subseteq k[X]$. A morphism $X \to Y$ is an isomorphism if the corresponding map of $\mathbb{C}$-algebras is an isomorphism defined over k.

A morphism $X \to Y$ of k-varieties defined over k induces a function $X(F) \to Y(F)$ on F-points for any $k \subseteq F$, which makes senses even if the domain has no points (by including the empty set as a subset). A nice class of examples is provided by *twisted-linear subvarieties* of Severi-Brauer varieties [GS06, §5.1]. If the domain has a point, the image of this point is then a point in the codomain. This is a useful fact that can be used to argue the non-existence of morphisms between k-varieties, e.g., any $\mathbb{R}$-variety which has $\mathbb{R}$-points cannot admit a morphism to the pointless $\mathbb{R}$-conic in Example 2.4.

Examples 2.3 and 2.4 above show us the difficulty in determining when a k-variety admits rational points. Indeed, the circle group and pointless conic have nearly identical defining equations, but starkly contrasting algebro-geometric properties; one has lots of $\mathbb{R}$-points and a group structure while the other has neither.

A related problem is to determine when a given variety contains a dense open subset which is isomorphic to an open subset of affine space $\mathbb{A}_k^n$. Such varieties are called *rational*, and this property is equivalent to having a parametrization by rational functions. Such varieties have lots of rational points, corresponding to all the points of $\mathbb{A}_k^n$. Again, S^1 and the pointless conic are quite distinct in this regard. The variety S^1 is rational (stereographic projection onto a line is one way to see this), while the pointless conic is not rational since it does not have any rational points.

Determining whether a given variety is rational (or unirational, stably rational, retract rational) is a long-standing problem in arithmetic and algebraic geometry. Recent work has invoked the use of sophisticated (co)homological techniques, including Chow groups, unramified cohomology, and derived categories (see [AB17, Pir18] for surveys on these topics). In Section 5 we provide a discussion on a particular success story in the theory of derived categories.

2.1. Algebraic groups.

Since our main objects of interest are certain varieties which admit a group structure (and partial compactifications thereof), we give a formal definition of algebraic groups, their morphisms, and additional examples.

Definition 2.7. An *algebraic group* or *group variety* over k is a k-variety G which also admits the structure of a group. In other words, an algebraic group is a variety together with the following regular maps (morphisms in the category of varieties)

1. group multiplication $m : G \times G \to G$
2. inclusion of the identity element $e : * \to G$
3. inversion $\text{inv} : G \to G$.

Here, $*$ denotes the variety which consists of a single point, e.g., $\{x = 0\} \subseteq \mathbb{A}^1_k$. These maps are then required to satisfy additional compatibility properties, mimicking the usual group axioms and which we leave to the reader. We note that if G is an algebraic group, its set of points $G(k)$ has a group structure (in the usual sense of abstract groups), but this does not fully define the algebraic group structure on G. Given algebraic groups G and H, an *algebraic group morphism* $\varphi : G \to H$ is a regular map which preserves the algebraic group structure, i.e., a homomorphism of groups.

Example 2.8 (General linear group). Consider the polynomial ring $R = k[x_{11}, x_{12}, \ldots, x_{nn}, y]$ in $n^2 + 1$ variables, and let $X = (x_{ij})$ be the matrix of indeterminates. The determinant of X is a polynomial, so that $\det(X)y - 1 \in R$. Let $\text{GL}_{n,k}$ denote the affine k-variety defined by this polynomial. Notice that its k-points $\text{GL}_n(k)$ are given by pairs (M, d) where M is an invertible $n \times n$ matrix with entries in k and $d = \det(M)^{-1}$. Since invertibility of a matrix is equivalent to its determinant being non-zero, the defining equation recovers the usual definition encountered in group theory. Moreover, this description realizes $\text{GL}_{n,k}$ as a subvariety of $\mathbb{A}^{n^2+1}_k$.

Example 2.9 (Algebraic tori). It was mentioned above that $\mathbb{G}_{m,k} = k^*$ is also an algebraic group. In fact, we can realize it as a subvariety of $\text{GL}_{n,k}$. Indeed, if we take our matrix of variables X, to be the diagonal matrix $\text{diag}(x_{11}, 1, \ldots, 1)$, then the equation $\det(X)y = 1$ becomes $x_{11}y = 1$. But this is exactly the defining equation of $\mathbb{G}_{m,k} = k^*$ (apart from the names of our indeterminants). More generally,

$\mathbb{G}^n_{m,k} = \mathbb{G}_{m,k} \times \cdots \times \mathbb{G}_{m,k}$ is the subvariety of $\text{GL}_{n,k}$ where we take our matrix of variables to be $X = \text{diag}(x_{11}, x_{22}, \ldots, x_{nn})$.

Example 2.10 (Determinant, special linear group). The determinant map $\det : \text{GL}_{n,k} \to \mathbb{G}_{m,k}$ defines a morphism of algebraic groups and has kernel $\text{SL}_{n,k}$, the affine k-variety given by $\det(X) = 1$. If we consider the k-points of each of these algebraic groups, we recover our usual determinant map $\det : \text{GL}_n(k) \to k^*$, which has kernel $\text{SL}_n(k)$.

2.2. Twisted forms.

For $k \subseteq \mathbb{C}$, any k-variety may be considered as a $\mathbb{C}$-variety simply by viewing the defining equations as having coefficients in $\mathbb{C}$. We denote this $\mathbb{C}$-variety by $X_\mathbb{C}$. The process of enlarging the field of coefficients of a given variety is called *base extension* or *extension of scalars*.

Example 2.11. Let $\mathbb{R}^* = \{uv - 1 = 0\}$ be the real multiplicative group and let $S^1 = \{x^2 + y^2 - 1 = 0\}$ be the circle group. As $\mathbb{R}$-varieties, these are non-isomorphic. Indeed, one may consider the invariant given by the number of connected components of $\mathbb{R}$-points. On the other hand, we note that the defining equations yield isomorphic varieties when we view them as polynomial equations over $\mathbb{C}$, i.e., after *extending scalars*. Indeed, the equation $uv - 1 = 0$ over $\mathbb{C}$ defines the variety $\mathbb{G}_{m,\mathbb{C}} = \mathbb{C}^*$, and we have an isomorphism $\mathbb{C}^* \cong S^1_\mathbb{C}$ given by $u \mapsto (x + iy)$ and $v \mapsto (x - iy)$.

Since varieties defined over $\mathbb{C}$ are completely determined by their geometric structure, we view $X_\mathbb{C}$ as a geometric "shadow" of X, and use its nicer geometric properties to extract information about X. For this reason, properties of k-varieties and morphisms which hold over $\mathbb{C}$ are often labelled as *geometric*. In the above example, $\mathbb{R}^*$ and S^1 are *geometrically isomorphic* since they are isomorphic over $\mathbb{C}$.

How can we get a handle on systematically studying examples of geometrically isomorphic varieties? One solution is to parametrize the collection of all such varieties in a way that reduces the study of k-varieties X to understanding their geometric avatar $X_\mathbb{C}$, and the arithmetic relationship between k and $\mathbb{C}$. This can be achieved using the theory of *Galois cohomology*. Given a Galois field extension $k \subseteq \mathbb{C}$, the *Galois group* $\text{Gal}(\mathbb{C}/k) = \{\sigma \in \text{Aut}(\mathbb{C}) \mid \sigma|_k = \text{id}_k\}$ is the set of those field automorphisms of $\mathbb{C}$ which fix k pointwise. If X is a k-variety, the action of the Galois group on $\mathbb{C}$ induces an action on $X_\mathbb{C}$, where an automorphism $\sigma \in \text{Gal}(\mathbb{C}/k)$ is applied to the coefficients of the defining polynomial equations.

Example 2.12. Consider the $\mathbb{R}$-variety $X = \{x^2 + y^2 = 0\}$. The $\mathbb{R}$-points of X are given by $X(\mathbb{R}) = \{(0,0)\}$. If we consider X as a $\mathbb{C}$-variety, then we get a factorization of its defining equation. That is, $X_\mathbb{C} = \{(x + iy)(x - iy) = 0\}$. This variety consists of two pieces, given by $\{x = iy\}$ and $\{x = -iy\}$. The Galois group $\text{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}/2$ acts on $X_\mathbb{C}$ via

conjugation, and this action swaps its two components. It also acts on the $\mathbb{C}$-points, e.g., $(i, -1) \mapsto (-i, -1)$.

Definition 2.13 (Twisted form). Let X be a k-variety (resp., k-algebra). A *twisted form* of X is a k-variety (resp. k-algebra) Y such that $X_{\mathbb{C}} \cong Y_{\mathbb{C}}$. We let $\mathrm{Tw}(X)$ denote the set of isomorphism classes of twisted forms of X. This set is *pointed*, i.e., it comes with a canonical distinguished element given by X itself.

Example 2.14. As we saw in Example 2.11, the variety S^1 is a twisted form of $\mathbb{R}^*$. Indeed, we have $S^1_{\mathbb{C}} \cong \mathbb{C}^* \cong (\mathbb{R}^*)_{\mathbb{C}}$. This is a complete list, so that $\mathrm{Tw}(\mathbb{R}^*) = \{\mathbb{R}^*, S^1\}$, and the distinguished element is $\mathbb{R}^*$.

There is a natural (in the technical sense of the word) way to parametrize the collection of twisted forms of a given variety. This comes from the theory of profinite group cohomology and its application in the case of Galois groups associated to field extensions provides an invaluable tool in the study of twisted forms.

Theorem 2.15 ([Ser79, Ser02]). *There is a bijection of pointed sets* $\mathrm{Tw}(X) \cong H^1(k, \mathrm{Aut}(X))$.

Let us roughly define $H^1(k, \mathrm{Aut}(X))$, following [Jah00]. Given a finite group G, a G-*group* M is a group with an action of G such that $g \cdot (ab) = (g \cdot a)(g \cdot b)$ for all $a, b \in M$ and $g \in G$, so that the group structure of M is compatible with the group action of G. A *cocycle* from G to M is a function $\varphi : G \to M$ such that $\varphi(gh) = \varphi(g)g \cdot \varphi(h)$. This may be viewed as a twisted version of a group homomorphism, where the twisting occurs when moving an application of φ past g. Two cocycles φ and ψ are *cohomologous* if there is an element $b \in M$ so that $\psi(g) = b^{-1}\varphi(g)g \cdot b$ for all $g \in G$ (a "twisted conjugate"). This defines an equivalence relation on cocycles, and the set of equivalence classes is denoted $H^1(G, M)$. This set is pointed by the *trivial cocycle* given by $\varphi : G \to M$ via $\varphi(g) = e$ for all $g \in G$. We extend this definition to *profinite* groups, i.e., (projective) limits of finite groups, by taking the colimit of inflation maps over all open normal subgroups of G. Given a Galois extension $k \subseteq \mathbb{C}$, the group $\mathrm{Gal}(\mathbb{C}/k)$ is profinite. If X is a k-variety, the automorphism group $\mathrm{Aut}(X_{\mathbb{C}})$ of $X_{\mathbb{C}}$ is a $\mathrm{Gal}(\mathbb{C}/k)$-group, so we can consider its first group cohomology $H^1(\mathrm{Gal}(\mathbb{C}/k), \mathrm{Aut}(X_{\mathbb{C}}))$, which we denote by $H^1(k, \mathrm{Aut}(X))$ as above.

The advantage of the above result is that we have a purely algebraic and functorial method to parameterize the collection of all twisted forms of a given variety (as long as it is nice enough; Theorem 2.15 holds for quasiprojective varieties where $\mathrm{Aut}(X)$ is an algebraic group). This makes analysis of such twisted forms much more systematic.

Example 2.16. A classical example of the above framework is the bijective correspondence between

Severi-Brauer varieties and central simple algebras. A *Severi-Brauer variety* X is a twisted form of projective space, i.e., $X_{\mathbb{C}} \cong \mathbb{P}^{n-1}_{\mathbb{C}}$. A *central simple algebra* A is a twisted form of a matrix algebra, i.e., $A \otimes_k \mathbb{C} \cong M_n(\mathbb{C})$. The automorphism group of both the matrix algebra $M_n(\mathbb{C})$ and projective space $\mathbb{P}^{n-1}_{\mathbb{C}}$ is the projective linear group $\mathrm{PGL}_n(\mathbb{C})$. Thus, we get bijections

$$\mathrm{Tw}(M_n(\mathbb{C})) \cong H^1(k, \mathrm{PGL}_n) \cong \mathrm{Tw}(\mathbb{P}^{n-1}_{\mathbb{C}}),$$

and therefore a bijection between Severi-Brauer varieties and central simple algebras. This assoiation has an explicit description in low dimension (see Example 4.6).

3. Toric Varieties

We discuss the beautiful class of toric varieties via low-dimensional examples. Our goal is to put forth the systematic combinatorial description of these varieties using cones and fans. The standard references for this material are [Ful93, CLS11]. Throughout this section we only work over the field $\mathbb{C}$, and so will almost solely work with rational points. As such, we let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ denote the multiplicative group $\mathbb{G}_{m,\mathbb{C}}$, which we also call the *one-dimensional algebraic torus*. More generally, we let $(\mathbb{C}^*)^n = \mathbb{G}^n_{m,\mathbb{C}}$ denote the n-dimensional algebraic torus. The reason we use the term "torus" in this context (at the risk of ignoring the true etymology) is that the topological circle is a deformation retract of $\mathbb{C}^*$. Since products of the topological circle are called tori, it is natural to view products of $\mathbb{C}^*$ as algebraic tori.

Definition 3.1 (Toric variety). A *torus* is any algebraic group T isomorphic to $\mathbb{G}^n_{m,\mathbb{C}} = (\mathbb{C}^*)^n$. A *toric variety* is a variety X that contains a torus T as a dense open subset so that the natural group multiplication $T \times T \to T$ extends to an action $T \times X \to X$ of T on X.

The fact that X contains a torus as a dense open set allows us to view toric varieties as (partial) *compactifications* of tori, i.e., varieties which look nearly identical to $(\mathbb{C}^*)^n$, but which have some or all of the holes filled in. It turns out that our most basic examples of algebraic varieties are toric.

Example 3.2 (Affine space). Affine n-space has the structure of a toric variety. We have a dense open inclusion $(\mathbb{C}^*)^n \subseteq \mathbb{C}^n = \mathbb{A}^n_{\mathbb{C}}$. The action of $(\mathbb{C}^*)^n$ is via scaling, i.e., for $(\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n$ and $(a_1, \ldots, a_n) \in \mathbb{A}^n_k$, we have $(\lambda_1, \ldots, \lambda_n) \cdot (a_1, \ldots, a_n) = (\lambda_1 a_1, \ldots, \lambda_n a_n)$. This clearly extends the group multiplication of $(\mathbb{C}^*)^n$.

Example 3.3 (Projective space). Projective n-space is defined via an equivalence relation on punctured affine space $\mathbb{A}^{n+1}_{\mathbb{C}} \setminus \{\mathbf{0}\}$, identifying any two points which are scalar multiples of one another. Points of $\mathbb{P}^n_{\mathbb{C}}$ are denoted $[x_0 : x_1 : x_2 : \cdots : x_n]$, so that $[x_0 : \cdots : x_n] = [\lambda x_0 : \cdots : \lambda x_n]$ for any $\lambda \in \mathbb{C}^*$. Two points of $\mathbb{A}^{n+1}_{\mathbb{C}}$ which lie on the same line

yield the same point of $\mathbb{P}_\mathbb{C}^n$, so we view projective space as the collection of lines in $\mathbb{A}_\mathbb{C}^{n+1}$ passing through the origin.

Projective space is a k-variety since it is locally an affine variety. Indeed, any point $[x_0 : \cdots : x_n]$ with $x_0 \neq 0$ is equal to $[1 : X_1 : X_2 : \cdots : X_n]$, where $X_i = \frac{x_i}{x_0}$. The set of points of this form defines a copy of $\mathbb{A}_\mathbb{C}^n$ with coordinates given by $X_1, \ldots, X_n$. Proceeding with $x_1 \neq 0$, $x_2 \neq 0$, etc., we arrive at a union $\mathbb{P}_\mathbb{C}^n = \mathbb{A}_\mathbb{C}^n \cup \cdots \cup \mathbb{A}_\mathbb{C}^n$ of $n+1$ copies of $\mathbb{A}_\mathbb{C}^n$ which cover $\mathbb{P}_\mathbb{C}^n$. This description holds for any base field k, and this defines the (points of the) variety $\mathbb{P}_k^n$.

Projective space compactifies affine space. Indeed, in the case of the projective line we can write $\mathbb{P}_\mathbb{C}^1 = \{[a : 1] \mid a \in \mathbb{C}\} \cup \{[1 : b] \mid b \in \mathbb{C}\} = \mathbb{A}_\mathbb{C}^1 \cup \mathbb{A}_\mathbb{C}^1$. The first set in this union is only missing the point $[1 : 0] \in \mathbb{P}_k^1$ since any point of the form $[1 : b]$ with $b \neq 0$ can already be written as $[a : 1]$, via $[1 : b] = [\frac{1}{b}1 : \frac{1}{b}b] = [\frac{1}{b} : 1]$. The projective line is thus a union $\mathbb{P}_\mathbb{C}^1 = \{[a : 1] \mid a \in \mathbb{C}\} \cup \{[1 : 0]\}$. The first subset is $\mathbb{A}_\mathbb{C}^1$ and the second is the "point at ∞." The real projective line $\mathbb{P}_\mathbb{R}^1$ has points which are given by the circle. The complex projective line is given by the Riemann sphere (the one-point compactification of $\mathbb{C}$). The action of the torus $\mathbb{C}^*$ on $\mathbb{P}_\mathbb{C}^1$ is given by $\lambda \cdot [x : y] = [\lambda x : y]$, extending the action of $\mathbb{C}^*$ on itself.

Example 3.4 (Products). Products of toric varieties are toric varieties, stemming from the fact that products of tori are tori. If X and Y are toric varieties, then we have dense inclusions $(\mathbb{C}^*)^n \subseteq X$ and $(\mathbb{C}^*)^m \subseteq Y$. Thus, we also have a dense inclusion $(\mathbb{C}^*)^{n+m} \cong (\mathbb{C}^*)^n \times (\mathbb{C}^*)^m \subseteq X \times Y$. One checks that the induced action extends the group multiplication of $(\mathbb{C}^*)^{n+m}$. In particular, the variety $\mathbb{P}_\mathbb{C}^1 \times \mathbb{P}_\mathbb{C}^1$ is a toric variety. See also Figure 2, as well as Examples 4.7 and 4.9.

3.1. Extracting combinatorial data.
The amazing fact about toric varieties is that their geometry can be encoded combinatorially via *fans*, i.e., collections of cones in affine/Euclidean space. This comes from considering their *characters* and *cocharacters*, which are simply algebraic group morphisms (see Definition 2.7) to and from the one-dimensional torus $\mathbb{G}_{m,\mathbb{C}} = \mathbb{C}^*$.

Definition 3.5. Let G be an algebraic group. A *character* of G is an algebraic group morphism $\chi : G \to \mathbb{C}^*$. We let $\widehat{G} = \mathrm{Hom}(G, \mathbb{C}^*)$ denote the set of all characters of G. A *cocharacter* (sometimes called a *one-parameter subgroup*) of G is an algebraic group morphism $\lambda : \mathbb{C}^* \to G$. We let $\Lambda(G) = \mathrm{Hom}(\mathbb{C}^*, G)$ denote the collection of all cocharacters of G.

Proposition 3.6. *Let $T = (\mathbb{C}^*)^n$ be a torus*

1. *For each $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$, we have a character $\chi^m : (\mathbb{C}^*)^n \to \mathbb{C}^*$ defined by $\chi^m(t_1, \ldots, t_n) = t_1^{m_1} \cdot t_2^{m_2} \cdots t_n^{m_n}$. All characters of T are of the form χ^m for some $m \in \mathbb{Z}^n$, yielding a group isomorphism $\widehat{T} \cong \mathbb{Z}^n$.*

2. *For each $u = (b_1, \ldots, b_n) \in \mathbb{Z}^n$, we have a cocharacter $\lambda^u : \mathbb{C}^* \to (\mathbb{C}^*)^n$ defined by $\lambda^u(t) = (t^{b_1}, \ldots, t^{b_n})$. All cocharacters of T are of the form λ^u for some $u \in \mathbb{Z}^n$, yielding a group isomorphism $\Lambda(T) \cong \mathbb{Z}^n$.*

Remark 3.7. Given a torus $T = (\mathbb{C}^*)^n$, it is customary to use M and N to denote $\widehat{T}$ and $\Lambda(T)$, respectively. By the very definition of these *lattices*, duality yields a pairing $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$. If compatible bases are chosen for both M and N, this is just given by the dot product.

Remark 3.8. Characters $\chi : T \to \mathbb{C}^*$ define regular functions on T, and *rational functions* (regular functions defined on open dense subsets) on any toric variety $X \supseteq T$. Similarly, cocharacters $\lambda : \mathbb{C}^* \to T$ define curves in T and any toric T-variety X. The geometry of X is thus easier to extract from subsets of $N = \Lambda(T)$, while the algebraic properties of X are more easily extracted from subsets of $M = \widehat{T}$.

3.2. One-dimensional toric varieties (toric curves).
Let us exhibit the process for producing toric varieties from subsets of $N = \Lambda(T)$, beginning with the case of curves. Our focus will be on *smooth* or *nonsingular* varieties. If $T = \mathbb{C}^*$ is the one-dimensional torus, then $M = \widehat{T} \cong \mathbb{Z}$ and $N = \Lambda(T) \cong \mathbb{Z}$.

Step 1. Choose a *cone* in $N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R} \cong \mathbb{R}$.

Definition 3.9. Given an $\mathbb{R}$-vector space V, a *cone* in V is is a subset of the form

$$\sigma = \{\alpha_1 v_1 + \cdots + \alpha_s v_s \mid \alpha_i \in \mathbb{R}^{\geq 0}\}.$$

We can view cones as "vector half-spaces." For certain technical reasons, we only consider those cones which are *strongly convex*, *rational*, and *polyhedral* [Ful93, §1]. In dimension one, there are only three such cones σ_1, σ_0, and σ_{-1} in $N_\mathbb{R} = \mathbb{R}$, given by the non-negative real numbers, $\{0\}$, and the non-positive real numbers, respectively. We view these as those cones generated by 1, 0, and -1, respectively. While other cones do exist (e.g., the cone generated by 2), these do not yield smooth varieties.
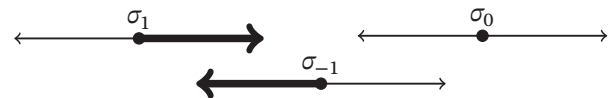
Step 2. Compute the *dual cones* in $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R} \cong \mathbb{R}$.

Definition 3.10. Given a cone $\sigma \subseteq N_\mathbb{R}$, its *dual cone* is $\sigma^\vee = \{v \in M_\mathbb{R} \mid \langle v, u \rangle \geq 0 \text{ for all } u \in \sigma\}$.

The dual cones of σ_1, σ_0 and σ_{-1} are given below. We view these cones as being generated by powers of x, obtained by exponentiating the generators of σ_i. That is, $\sigma_1^\vee$ is generated by x^1 since $1 \in \mathbb{R} = M \otimes \mathbb{R}$ generates σ_1. Similarly, $\sigma_0^\vee = \langle x, x^{-1} \rangle$ and $\sigma_{-1}^\vee = \langle x^{-1} \rangle$.
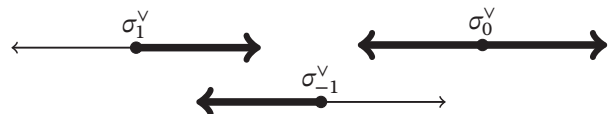
Step 3. Having utilized the vector spaces $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$ for cone geometry, we now produce functions on our resulting toric variety, i.e., elements of M. For each cone $\sigma \subseteq N$, we can form the *monoid* $S_\sigma = \sigma^\vee \cap M \subseteq M$. In our one-dimensional case, these are given by

$$S_{\sigma_1} := \{1, x, x^2, \dots\} \cong \mathbb{N},$$

$$S_{\sigma_0} = \{\dots, x^{-2}, x^{-1}, 1, x, x^2, \dots\} \cong \mathbb{Z},$$

$$S_{\sigma_{-1}} = \{1, x^{-1}, x^{-2}, \dots\} \cong \mathbb{N}.$$

Step 4. The *monoid algebra* $\mathbb{C}[S_{\sigma_i}]$ associated to S_{σ_i} generalizes the common notion of the group algebra or group ring. These algebras have basis given by the elements of S_{σ_i}, so that a general element of $\mathbb{C}[S_{\sigma_i}]$ is given by a finite sum $\sum a_g g$ with $g \in S_{\sigma_i}$ and $a_g \in \mathbb{C}$. Addition is given component-wise and multiplication is the usual convolution product. In our case

$$\mathbb{C}[S_{\sigma_1}] = \mathbb{C}[\mathbb{N}] = \mathbb{C}[x],$$

$$\mathbb{C}[S_{\sigma_0}] = \mathbb{C}[\mathbb{Z}] = \mathbb{C}[x, x^{-1}],$$

$$\mathbb{C}[S_{\sigma_{-1}}] = \mathbb{C}[\mathbb{N}] = \mathbb{C}[x^{-1}].$$

Step 5. We determine generators and relations for our $\mathbb{C}$-algebras which yield polynomial equations, and in turn define affine varieties. We have

$$V_{\sigma_1} := \mathrm{Spec}\,\mathbb{C}[S_{\sigma_1}] = \mathrm{Spec}\,\mathbb{C}[x] = \mathbb{A}^1_{\mathbb{C}},$$
$$V_{\sigma_0} := \mathrm{Spec}\,\mathbb{C}[S_{\sigma_0}] = \mathrm{Spec}\,\mathbb{C}[x, x^{-1}]$$
$$= \mathrm{Spec}\,\mathbb{C}[x, y]/(xy - 1) = \mathbb{C}^*,$$
$$V_{\sigma_{-1}} := \mathrm{Spec}\,\mathbb{C}[S_{\sigma_{-1}}] = \mathrm{Spec}\,\mathbb{C}[x^{-1}] = \mathbb{A}^1_{\mathbb{C}}.$$

Step 6. We build new toric varieties from these affine ones by gluing. This is prescribed by the data of a collection of cones and the way in which they fit together.

Definition 3.11. A *fan* Σ in $N_{\mathbb{R}}$ is a collection of cones such that

1. each face of a cone in Σ is also a cone in Σ
2. the intersection of two cones in Σ is a face of each.

In our case, we take $\Sigma = \{\sigma_{-1}, \sigma_0, \sigma_1\}$, and we realize σ_0 as the intersection $\sigma_1 \cap \sigma_{-1}$. The fan Σ defines a toric variety $X(\Sigma)$ which is built from the affine pieces V_{σ_i} and the gluing data is encoded in the combinatorics of the fan structure. This defines the variety $\mathbb{P}^1_{\mathbb{C}}$ as the union of affine spaces from Example 3.3:

$$\begin{array}{ccccc}
V_{\sigma_{-1}} & \supseteq & V_{\sigma_0} & \subseteq & V_{\sigma_1} \\
\| & & \| & & \| \\
\mathbb{A}^1_{\mathbb{C}} & \supseteq & \mathbb{C}^* & \subseteq & \mathbb{A}^1_{\mathbb{C}}
\end{array}$$

(A) The fan for $\mathbb{A}^2_{\mathbb{C}}$ is given by a single cone, generated by $(1, 0)$ and $(0, 1)$. These correspond to the variables x and y in the lattice M.

(B) The fan for $\mathbb{P}^2_{\mathbb{C}}$. Note that it contains the cone for $\mathbb{A}^2_{\mathbb{C}}$. In fact, you can see the three copies of $\mathbb{A}^2_{\mathbb{C}}$ which give an affine cover described in Example 3.3.

Figure 1.

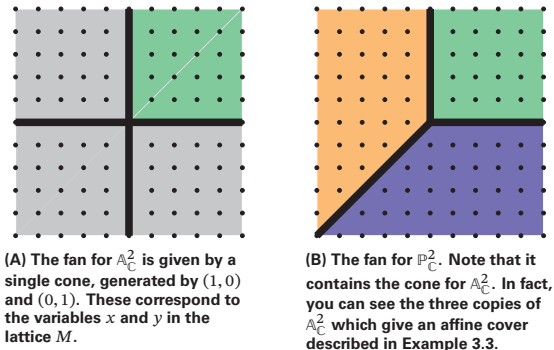### 3.3. Two-dimensional toric varieties (toric surfaces).
In dimension 2, our torus is $T = (\mathbb{C}^*)^2$, so that $M \cong N \cong \mathbb{Z}^2$.

Example 3.12 (Affine plane, Figure 1A). Consider the cone σ in $N_{\mathbb{R}} \cong \mathbb{R}^2$ given by the first quadrant, i.e., generated by $(1, 0)$ and $(0, 1)$. The dual cone $\sigma^\vee$ is then given by the first quadrant in $M_{\mathbb{R}} \cong \mathbb{R}^2$ since this is the collection of vectors which have non-negative dot product with elements of σ. Intersecting this cone with M gives the generators x, y, so that the associated affine toric variety is given by $V_\sigma = \mathrm{Spec}\,\mathbb{C}[x, y] = \mathbb{A}^2_{\mathbb{C}}$.

Example 3.13 (Projective plane, Figure 1B). The fan given by three cones $\sigma_1 = \langle (1, 0), (0, 1) \rangle$, $\sigma_2 = \langle (0, 1), (-1, -1) \rangle$, and $\sigma_3 = \langle (-1, -1), (1, 0) \rangle$, as shown in Figure 1B, defines the variety $\mathbb{P}^2_{\mathbb{C}}$. Indeed, each cone gives a copy of $\mathbb{A}^2_{\mathbb{C}}$, and these are precisely the three affine patches described in Example 3.3. These are glued to one another according to the combinatorics of the cone intersections.

Example 3.14. Consider the fan generated by the four cones depicted in Figure 2. This yields the variety $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, which comes from the fact that products of fans correspond to products of the associated toric varieties. That is, given fans $\Sigma, \Delta \subseteq N_{\mathbb{R}}$, the fan $\Sigma \times \Delta := \{\sigma \times \tau \mid \sigma \in \Sigma, \tau \in \Delta\} \subseteq N \times N$ defines the toric variety $X(\Sigma \times \Delta) = X(\Sigma) \times X(\Delta)$. Since $\mathbb{P}^1_{\mathbb{C}}$ has an associated fan consisting of two cones (the positive and negative real lines) given in Step 6 above, the description of the fan of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ follows.
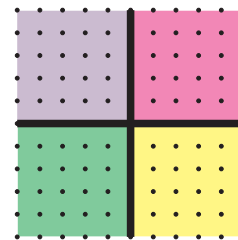

Figure 2. The fan for $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ consists of 4 cones, corresponding to the product of the fans which define $\mathbb{P}^1_{\mathbb{C}}$ as described in Example 3.14.

Example 3.15 (Blowups, Figures 3A and 3B). Recall that the *blowup* $\mathrm{Bl}_{\mathbf{0}}(\mathbb{A}_{\mathbb{C}}^2)$ of $\mathbb{A}_{\mathbb{C}}^2$ at the origin is a subvariety of $\mathbb{A}_{\mathbb{C}}^2 \times \mathbb{P}_{\mathbb{C}}^1$ given by the equation $xw_1 = yw_0$, where (x, y) are coordinates on $\mathbb{A}_{\mathbb{C}}^2$ and $[w_0 : w_1]$ are coordinates on $\mathbb{P}_{\mathbb{C}}^1$. Covering this variety by its affine pieces $\{w_0 \neq 0\} = \mathbb{A}_{\mathbb{C}}^2 \times \mathbb{A}_{\mathbb{C}}^1 \cong \mathbb{A}_{\mathbb{C}}^3$ and $\{w_1 \neq 0\} \cong \mathbb{A}_{\mathbb{C}}^3$, our defining equation becomes $y = x\frac{w_1}{w_0}$ and $x = y\frac{w_0}{w_1}$ on these respective sets. We can thus generate our corresponding $\mathbb{C}$-algebras by x and $\frac{w_1}{w_0} = x^{-1}y$ on the first affine subset (since their product determines y) and by y and $\frac{w_0}{w_1} = xy^{-1}$ on the second (since their product determines x).

Consider the fan given in Figure 3A. A simple system of equations gives a description of the corresponding dual cones as $\langle x, x^{-1}y \rangle$ and $\langle y, xy^{-1} \rangle$, which are precisely the same generators that arise in the blowup construction! We see from this example that blowups can be described by subdivision of a cone.

The *del Pezzo surface of degree 6* over $\mathbb{C}$ is the blowup of $\mathbb{P}_{\mathbb{C}}^2$ at three non-colinear points. As such, we can recover its fan by staring with the fan of $\mathbb{P}_{\mathbb{C}}^2$ (see Figure 1B), and dividing each of its three cones. This results in the fan given in Figure 3B.
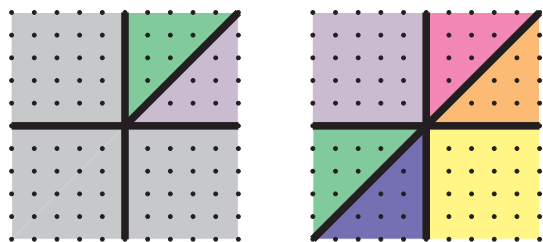
(A) The fan for the blowup of $\mathbb{A}_{\mathbb{C}}^2$ at the origin. The blowup construction has the effect of subdividing cones.

(B) The fan for the del Pezzo surface of degree 6, which is the blowup of $\mathbb{P}_{\mathbb{C}}^2$ at 3 non-colinear points. This fan is obtained from the fan of $\mathbb{P}_{\mathbb{C}}^2$ by subdividing each of its three cones.

Figure 3.

4. Arithmetic Toric Varieties

The advantage of working with toric varieties over $\mathbb{C}$ is the combinatorially encoded geometry, which allows one to effectively compute various algebro-geometric invariants using the data of fans. On the other hand, these varieties are geometrically and arithmetically simple since they are rational varieties and $\mathbb{C}$ is algebraically closed. We treat the appropriately analogous class of varieties over $k \subseteq \mathbb{C}$. The computationally effective understanding of the geometry of complex toric varieties allows one to corner arithmetic aspects of this new class of varieties defined over arbitrary fields. We revisit Example 2.11 to motivate a more general notion of "torus."

Example 4.1 (Non-isomorphic tori of equal dimension). The real one-dimensional torus $\mathbb{R}^* = \{uv = 1\}$ and the circle group $S^1 = \{x^2 + y^2 - 1 = 0\}$ are non-isomorphic $\mathbb{R}$-varieties but become isomorphic after extending scalars to $\mathbb{C}$. The isomorphism $\mathbb{C}^* \xrightarrow{\sim} S_{\mathbb{C}}^1$ is given by $u \mapsto x + iy$ and $v \mapsto x - iy$.

This example encourages us to enlarge our definition of "algebraic tori" to include those varieties which become isomorphic to $(\mathbb{C}^*)^n$ after base extension, i.e., *twisted forms* of the algebraic tori $(\mathbb{C}^*)^n = \mathbb{G}_{m,\mathbb{C}}^n$.

Definition 4.2 (Arithmetic toric variety). A *k-torus* (also called a $\mathbb{G}_{m,k}^n$-*torsor*) is a group variety T such that $T_{\mathbb{C}} \cong (\mathbb{C}^*)^n = \mathbb{G}_{m,\mathbb{C}}^n$. If T is a k-torus such that $T \cong (k^*)^n$, then we call T a *split torus*. An *arithmetic toric variety* is a variety with a faithful action of a torus T with dense open orbit. Given an arithmetic toric variety X with torus T, we say X is *split* if T is a split torus.

Remark 4.3. Example 4.1 shows that both $\mathbb{R}^*$ and S^1 are $\mathbb{R}$-tori, where the former is a split torus.

As in the case of toric varieties over $\mathbb{C}$, arithmetic toric varieties should be viewed as (partial) compactifications of algebraic tori (in this more general sense). The study of arithmetic toric varieties is relatively new, having been taken up over the past few decades. They were first used as tools to study general tori via their compactifications [Kun82, Kun87, Vos98]. A systematic classification and analysis via Galois cohomology was only treated within the last decade or two [Dun16, ELFST14]. The K-theory of tori and arithmetic toric varieties was also studied in [MP97], and the analysis therein was thematically aligned with the theory of non-commutative motives.

Example 4.4 (Split toric varieties). Over any field k, the split tori $\mathbb{G}_{m,k}^n = (k^*)^n$ provide a geometrically rich class of tori. The analysis of $(k^*)^n$-toric varieties via the associated combinatorial data of fans and cones is nearly identical to the complex case. One example of this phenomenon arises via the analysis of the divisor class group and Picard group. If X is a split toric variety with torus $T = \mathbb{G}_{m,k}^n$, there are canonical isomorphisms $\mathrm{Cl}(X) \cong \mathrm{Cl}(X_{\mathbb{C}})$ and $\mathrm{Pic}(X) \cong \mathrm{Pic}(X_{\mathbb{C}})$ [Dun16, §4].

Example 4.5 (Real projective space). The variety $\mathbb{P}_{\mathbb{R}}^1$ admits the structure of an arithmetic toric variety for two non-isomorphic tori, $\mathbb{R}^*$ and S^1. Viewed as an $\mathbb{R}^*$-toric variety, $\mathbb{P}_{\mathbb{R}}^1$ is a split arithmetic toric variety. When we extend scalars to $\mathbb{C}$, both tori become isomorphic to $\mathbb{C}^*$, and $(\mathbb{P}_{\mathbb{R}}^1)_{\mathbb{C}} \cong \mathbb{P}_{\mathbb{C}}^1$.

Example 4.6 (Pointless real conic). The variety $X = \{x^2 + y^2 + z^2 = 0\}$ is an arithmetic toric variety with torus $S^1 = \{x^2 + y^2 = 1\}$, where the S^1-action is given by rotation matrices. Note that the pointless conic of Example 2.4 is

an affine open subset in X. The variety X is not a toric variety for the split torus $\mathbb{R}^*$. Again, the torus S^1 is a twisted form of $\mathbb{C}^*$, and $X_\mathbb{C} \cong \mathbb{P}^1_\mathbb{C}$. This gives our first example of a (truly) non-split arithmetic toric variety. That is, no choice of torus gives X the structure of a split toric variety.

The above example arises naturally in the study of *central simple algebras*. Indeed, the variety X given above is the *Severi-Brauer variety* $\mathrm{SB}(\mathbb{H}) = \{x^2 + y^2 + z^2 = 0\}$ associated to Hamilton's quaternion $\mathbb{R}$-algebra $\mathbb{H}$. This is the four-dimensional $\mathbb{R}$-algebra given by

$$\mathbb{H} = \langle 1, i, j, k \mid i^2 = j^2 = -1, ij = -ji = k \rangle.$$

In fact, to any *quaternion k-algebra* $(a, b)_k := \langle 1, i, j, k \mid i^2 = a, j^2 = b, ij = -ji = k \rangle$ we have the *associated conic* $C(a, b) = \{ax^2 + by^2 = z^2\}$, which is a one-dimensional k-subvariety of $\mathbb{P}^2_k$ and a twisted form of $\mathbb{P}^1_\mathbb{R}$ [GS06]. Note that $\mathbb{H} = (-1, -1)_\mathbb{R}$ and $M_2(\mathbb{R}) = (1, 1)_\mathbb{R}$.

The associated conic is a special case of the general construction of the Severi-Brauer variety associated to a central simple algebra A, encountered in Example 2.16 via Galois cohomology. Such varieties are all examples of arithmetic toric varieties, although one needs to additionally specify a torus to completely determine their toric structure (just as in the case of $\mathbb{P}^1_\mathbb{R}$).

Example 4.7 (Weil restriction). Given a $\mathbb{C}$-variety X, we may view it as an $\mathbb{R}$-variety of twice the dimension by rewriting the defining equations of X using an $\mathbb{R}$-basis of $\mathbb{C}$. This process restricts our field of scalars to a subfield, and is called the *Weil restriction of scalars* [Vos98, §3.12]. For instance, taking $\{1, i\}$ as an $\mathbb{R}$-basis of $\mathbb{C}$, where $i = \sqrt{-1}$, any function $f \in \mathbb{C}[z]$ can be written in terms of the variables x_1, x_2 given by $z = x_1 + x_2 i$, and thus may be viewed as an element of $\mathbb{R}[x_1, x_2]$. We use the notation $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(X)$ to denote the Weil restriction. Since the 0-polynomial in $\mathbb{C}[z]$ (the defining equation of $\mathbb{A}^1_\mathbb{C}$) corresponds to the 0-polynomial in $\mathbb{R}[x_1, x_2]$ in this basis, we have $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{A}^1_\mathbb{C}) = \mathbb{A}^2_\mathbb{R}$.

The Weil restriction is best described using points. If X is a $\mathbb{C}$-variety and $k \subseteq \mathbb{C}$, then the k-variety $\mathrm{R}_{\mathbb{C}/k}(X)$ has k-points given by

$$\mathrm{R}_{\mathbb{C}/k}(X)(k) = X(k \otimes_k \mathbb{C}) = X(\mathbb{C}),$$

i.e., the $\mathbb{C}$-points of X. Each $\mathbb{C}$-point must be specified using a k-basis of $\mathbb{C}$, so that $X(\mathbb{C})$ has dimension $[\mathbb{C} : k] \cdot \dim_\mathbb{C}(X(\mathbb{C}))$ as a k-variety. This is consistent with our example of the Weil restriction of the affine line above. More generally, we have $\mathrm{R}_{\mathbb{C}/k}(\mathbb{A}^n_\mathbb{C}) = \mathbb{A}^{n[\mathbb{C}:k]}_\mathbb{R}$.

Weil restrictions of tori are tori. For instance, $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$ has $\mathbb{R}$-points given by $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})(\mathbb{R}) = \mathbb{G}_{m,\mathbb{C}}(\mathbb{R} \otimes_\mathbb{R} \mathbb{C}) = \mathbb{C}^*$. Since $[\mathbb{C} : \mathbb{R}] = 2$, we have $\mathbb{C} = \mathbb{R}^2$ as sets, and taking units gives $\mathbb{C}^* = (\mathbb{R}^*)^2$. Extending scalars to $\mathbb{C}$, we obtain $(\mathbb{C}^*)^2$, and thus $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$ is a twisted form of $\mathbb{G}^2_{m,\mathbb{C}} = (\mathbb{C}^*)^2$. Similarly, one can show that $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{P}^1_\mathbb{C})$ is a twisted form of $\mathbb{P}^1 \times \mathbb{P}^1$.

This gives us many new examples of arithmetic toric varieties. The varieties $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{A}^n_\mathbb{C})$ and $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{P}^n_\mathbb{C})$ are arithmetic toric varieties with torus $T = \mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}^n_{m,\mathbb{C}})$. More generally, if X is a complex toric variety with torus T, then $\mathrm{R}_{\mathbb{C}/k}(X)$ is an arithmetic toric variety with torus $\mathrm{R}_{\mathbb{C}/k}(T)$.

Example 4.8 (Norm tori, [Vos98, p. 53]). A related construction is given by the *norm-one torus*. For $k \subseteq \mathbb{C}$ a Galois extension, the Weil restriction $\mathrm{R}_{\mathbb{C}/k}(\mathbb{G}_{m,\mathbb{C}})$ has k-points given by $\mathbb{G}_{m,\mathbb{C}}(k \otimes \mathbb{C}) = \mathbb{G}_{m,\mathbb{C}}(\mathbb{C}) = \mathbb{C}^*$. There is a natural map $N : \mathbb{C}^* \to k^*$ given by the field norm

$$z \mapsto \prod_{\sigma \in \mathrm{Gal}(\mathbb{C}/k)} \sigma(z).$$

This induces an algebraic group morphism $\mathrm{R}_{\mathbb{C}/k}(\mathbb{G}_{m,\mathbb{C}}) \to \mathbb{G}_{m,k}$. Its kernel is denoted $\mathrm{R}^{(1)}_{\mathbb{C}/k}(\mathbb{G}_{m,\mathbb{C}})$, called the norm-one torus.

The circle group S^1 arises as such a torus. If we apply the above construction in the case $k = \mathbb{R}$, real points of $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^*)$ are $\mathbb{C}^*$, and the field norm $\mathbb{C}^* \to \mathbb{R}^*$ is given by the product of a non-zero complex number and its conjugate $z \mapsto z\bar{z} = |z|^2$. This induces a group variety morphism $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}}) \to \mathbb{G}_{m,\mathbb{R}}$. Thus $\mathrm{R}^{(1)}_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^*)$ consists of all those complex numbers of modulus 1. This yields the unit circle S^1.

Example 4.9 (Products [Dun16, Ex. 5.8 and 6.9]). Just as in the split case (Example 3.4), products of arithmetic toric varieties are arithmetic toric varieties. This stems from the fact that products of twisted forms of tori are twisted forms of tori. We have already encountered one example of a twisted form of $\mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C}$ with its standard torus $(\mathbb{C}^*)^2$ in Example 4.7, given by $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{P}^1_\mathbb{C})$ with torus $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$. Let us dive further into this analysis for examples of products.

Of course, we have the split example $\mathbb{P}^1_\mathbb{R} \times \mathbb{P}^1_\mathbb{R}$ with torus $(\mathbb{R}^*)^2$. We can use this same variety together with the non-split tori $S^1 \times \mathbb{R}^*$ and $S^1 \times S^1$. Let $X = \{x^2 + y^2 + z^2 = 0\}$ be the (projectivized) pointless conic encountered in Example 4.6. Then we have the arithmetic toric variety $X \times \mathbb{P}^1_\mathbb{R}$ with torus $S^1 \times \mathbb{R}^*$ or $S^1 \times S^1$. We can also replace both factors of $\mathbb{P}^1_\mathbb{R}$, to obtain $X \times X$ with torus $S^1 \times S^1$. This gives all possible twisted forms of $\mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C}$.

Example 4.10 (See [Dun16, ELFST14]). Arithmetic toric varieties are twisted forms of split toric varieties, so they may be described via Galois cohomology, as in Theorem 2.15 and Example 2.16. If X is an arithmetic toric variety over k and T is the split torus of the split toric variety $X_\mathbb{C}$, the Galois group $\mathrm{Gal}(\mathbb{C}/k)$ acts on $X_\mathbb{C}$, so its elements can be viewed as automorphisms of $X_\mathbb{C}$. Since a toric variety is completely determined by its associated fan $\Sigma \subseteq N_\mathbb{R} = \Lambda(T)_\mathbb{R}$, these automorphisms of $X_\mathbb{C}$ determine automorphisms of Σ. This gives $\mathrm{Aut}(\Sigma)$ the structure of a $\mathrm{Gal}(\mathbb{C}/k)$-group, and the particular form X can be associated to a cocycle $\beta \in H^1(k, \mathrm{Aut}(\Sigma))$. This still leaves the

choice of torus for the arithmetic toric variety structure on X, which is determined by a cocycle $\alpha \in H^1(k, {}^\beta T)$, where ${}^\beta T$ denotes the twisted form of T corresponding to the cocycle β. In general, any twisted form of X corresponds to a cocycle $\gamma \in H^1(k, T \rtimes \mathrm{Aut}(\Sigma))$.

The above cohomological description shows that the study of arithmetic toric k-varieties may be reduced to understanding split toric varieties $X_{\mathbb{C}}$ together with the action of the Galois group $\mathrm{Gal}(\mathbb{C}/k)$ on the fan associated to $X_{\mathbb{C}}$. In many cases, the automorphism group of the fan is easy to identify. For instance, the automorphism group of the fan of $\mathbb{P}^2_{\mathbb{C}}$ (Figure 1B) is S_3 given by permuting each of the three (maximal) cones. The automorphism group of the fan associated to $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ (Figure 2) is D_4, the dihedral group of order 8. And the automorphism group of the fan associated to dP6 (Figure 3B) is D_6, the dihedral group of order 12.

5. A.T.V.'s Are What Derive Us

Arithmetic toric varieties provide a wonderful class of objects on which to test the capabilities of certain invariants to reflect arithmetic and geometric data. One interesting invariant is given by the *coherent derived category* $\mathsf{D}^b(X) = \mathsf{D}^b(\mathrm{coh}(X))$, a triangulated category which probes the geometry of a variety through sheaves of modules. This should be viewed as a globalization of the study of rings via their modules. An affine variety X may be recovered from its defining ideal $\mathbf{I}(X)$ or associated coordinate ring $R = k[x_1, \dots, x_n]/\mathbf{I}(X)$ (Defn. 2.1). In this case, $\mathsf{D}^b(X)$ has objects given by chain complexes of finitely-generated R-modules, where two complexes are identified if they are quasi-isomorphic, i.e., if their associated homology groups are isomorphic in all degrees. For a general variety, one considers the category of coherent $\mathcal{O}_X$-modules.

As an invariant, the derived category sits between algebra and topology. Viewing the derived category as an algebraic invariant, our interest lies in its decompositions into indecomposable pieces in the vein of vector spaces, modules, and representations. On the other hand, this category is non-abelian, and its triangulated structure allows one to use more topological approaches in the study of these decompositions. One such decomposition is given by an *exceptional collection*.

Definition 5.1. Let X be a k-variety and D a division k-algebra. An object $E \in \mathsf{D}^b(X)$ is D-*exceptional* if (1) $\mathrm{End}(E) = D$, concentrated in degree 0, and (2) $\mathrm{Hom}(E, E[n]) = 0$ if $n \neq 0$ (here $E[n]$ denotes the shift of the complex E by n). An exceptional object is *étale* if D is a field extension of k. A totally ordered set $\{E_1, \dots, E_n\}$ of exceptional objects in $\mathsf{D}^b(X)$ is an *exceptional collection* if $\mathrm{Hom}(E_j, E_i[n]) = 0$ for all n whenever $j > i$. It is *full* if

the only triangulated subcategory of $\mathsf{D}^b(X)$ which contains every E_i is all of $\mathsf{D}^b(X)$.

Remark 5.2. Throughout the literature, the term "exceptional" often means "k-exceptional," as we have defined above. This more restrictive definition is appropriate in the case where k is algebraically closed. The definition given here is a strict generalization. If k is algebraically closed, these definitions coincide since there are no non-trivial division algebras over such fields.

From the algebraic perspective, such a collection is analogous to decomposing a vector space via an (semi-)orthonormal basis. Indeed, the objects E_i should be viewed as basis elements of $\mathsf{D}^b(X)$, which are semi-orthonormal relative to the (categorified) bilinear form $\mathrm{Hom}(-, -)$. From the topological viewpoint, such a collection provides a decomposition of the derived category into a collection of subcategories with maps in only one direction. One may view such maps as the instructions for gluing these subcategories together, analogous to a simplicial/cell complex.

The use of derived categories to study questions of rationality in the arithmetic setting was motivated by the success in the geometric case for rationality of three- and four-dimensional varieties. The use of $\mathsf{D}^b(X)$ in determining whether a variety admits rational points was motivated by a question of H. Esnault. This was answered in the negative in [AAFH19]. Related work was carried out in [AKW17], showing that twisted forms cannot always be distinguished by the derived category, even in nice situations in low dimension. Examples in both cases were non-toric. Thus, it turns out the derived category is generally not an appropriate invariant to determine if a collection of polynomials has a common solution, although there is evidence that higher-categorical invariants might do the trick.

To better understand the derived category of an arithmetic toric variety, we would like to leverage the combinatorial tools available over $\mathbb{C}$. The following theorem shows that the Galois-theoretic philosophy described above holds for the derived category. Indeed, the existence of an exceptional collection on a k-variety X is guaranteed by the existence of an exceptional collection on $X_{\mathbb{C}}$ where the Galois group action permutes the elements of the exceptional collection (*stable*).

Theorem 5.3 (Descent for exceptional collections, [BDM19, Thm. 1.3]). *Let $k \subseteq \mathbb{C}$ be a Galois extension and let X be a smooth projective k-variety. Then X admits a (full, strong) exceptional collection if and only if $X_{\mathbb{C}}$ admits a $\mathrm{Gal}(\mathbb{C}/k)$-stable (full, strong) exceptional collection.*

Recently, arithmetic toric varieties have been successfully used to determine the limitations of the derived category in the arithmetic setting in regards to questions

of rationality. In particular, while the existence of a *k*-exceptional collection is sufficient to determine that an arithmetic toric variety is rational, the existence of a general exceptional collection is not sufficient to determine whether a variety is even *retract rational*, a much weaker property.

Theorem 5.4 ([BDLM, Thm. 1 and 2]). *(1) Let X be a smooth projective arithmetic toric variety over a field k with $X(k) \neq 0$. If $\mathsf{D}^b(X)$ admits a full k-exceptional collection, then X is k-rational. (2) There exists a smooth threefold X defined over $\mathbb{Q}$ which admits a full étale exceptional collection but is not k-rational.*

With this success story in hand, it is our hope that other invariants may be investigated using the class of arithmetic toric varieties. Indeed, such varieties are so well-suited to act as a proving ground for analyzing geometric invariants and their ability to faithfully reflect arithmetic information.

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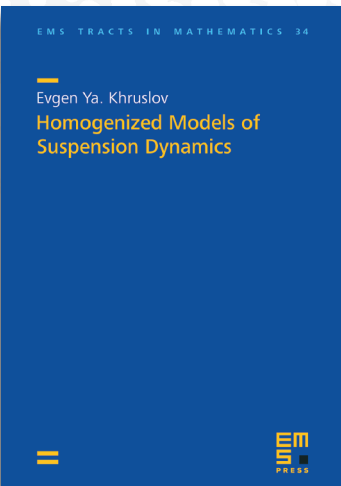
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