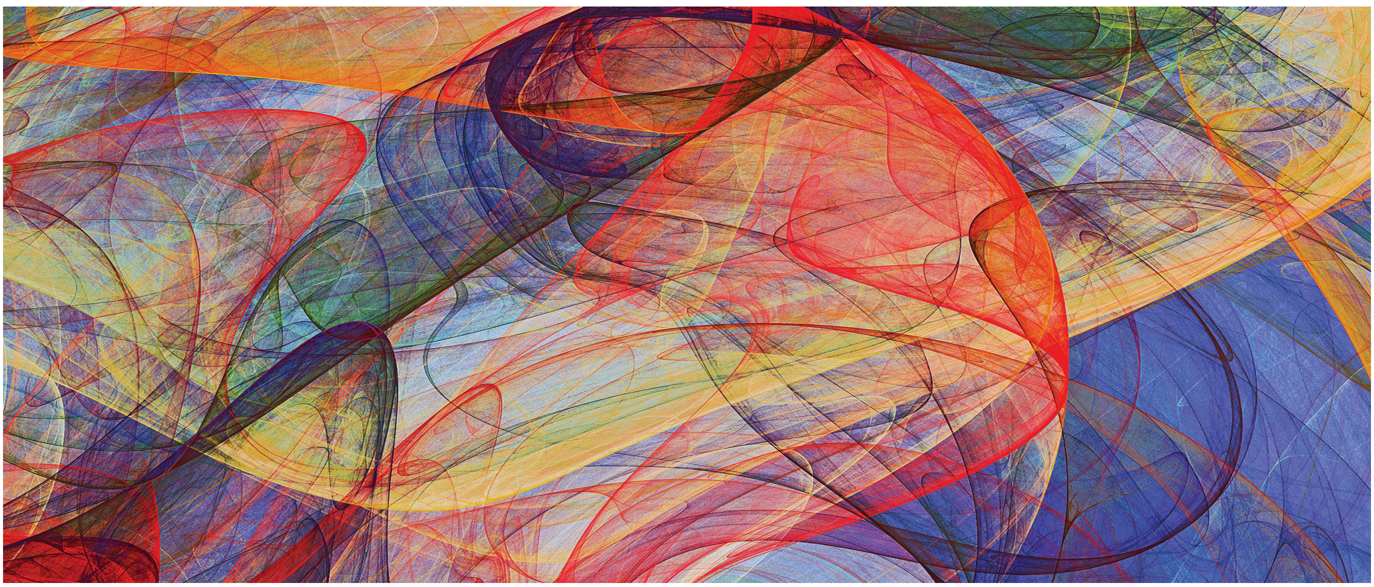
K-stability: The Recent Interaction Between Algebraic Geometry and Complex Geometry



Chenyang Xu

Solution spaces of polynomial equations in $\mathbb{C}^n$ or its natural compactification $\mathbb{CP}^n$ form the basic objects to study in algebraic geometry. Because such a space, called *an algebraic variety*, has an underlying topology inherited from the Euclidean topology of the complex spaces, it can also be studied through complex differential geometry. The interplay between these two kinds of geometries has been investigated by some of the greatest minds in mathematical history, started from Abel, Jacobi, Riemann, Weierstrass, Enriques, Lefschetz, Hodge, Weil, Chern, Kodaira, to more recently Serre, Mumford, Griffiths, Siu, Deligne, Yau, Mori, Kollár, Donaldson, Demailly, Tian, Voisin etc, and it leads to monumental progress in mathematics.

In this article, we want to convince the reader as with previous landmark theorems, in the last decade, there is another major step forward along this direction. The research originates from the searching of Einstein metrics on a manifold, with a root from physics. This topic has prevailed in geometry for more than a century, and on a complex variety, one should consider Kähler metrics. The most challenging case is when the variety has a positive Chern class, called *a Fano variety*. The research results in purely algebro-geometric theorems such as the construction of moduli spaces parametrizing Fano varieties, which is far beyond the original scope of the field. In fact, an astonishingly huge amount of mathematical topics, including geometric analysis, metric geometry, pluripotential theory, geometric invariant theory, non-archimedean geometry, higher dimensional geometry etc., are linked together by one theme.

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The central notion is called *K-stability*. The term was coined by Tian, following Mabuchi's definition of K-energy, where "K" stands for the first letter of the German word "Kanonisch." It has been known for decades that the solution of the Kähler-Einstein Problem has striking consequences in algebraic geometry, as exhibited famously in Yau's work. The relation between the existence of canonical metrics on algebraic objects and its stability from the Geometric Invariant Theory has also been intensively investigated. In the more recent episode, to understand the Kähler-Einstein Problem of Fano varieties, one crucial shift in viewpoint is the realization that one has to go beyond the Geometric Invariant Theory and invoke the tools provided by the Minimal Model Program. This bridges two previously well-studied areas. Discovering it not only sheds new light on longstanding central questions and eventually leads to their complete solutions, but also points toward unseen research directions.

In this article, we will discuss some of the main ideas involved in K-stability theory stemmed from different parts of mathematics, with a focus on the recently founded algebraic part of the story.

Canonical metrics and stability. A smooth complex variety X admits a class of metrics which are called *Kähler*. More precisely, for any Riemannian metric g, there is an associated 2-form $\omega(X, Y) = g(JX, Y)$ where J is the complex structure on the tangent bundle TX. The Kähler condition is equivalent to saying that ω is a closed 2-form. While a given Kähler metric ω is analytic, its class $[\omega] \in H^2(X, \mathbb{R})$ is a topological invariant. For a Kähler form ω, one can attach the Ricci form $\mathrm{Ric}(\omega)$. It is also a closed two form and a remarkable fact is that its class $[\mathrm{Ric}(\omega)]$ is the first Chern class $c_1(X)$.

Around the 50s, two fundamental questions became central in complex geometry. The first one is called the Calabi Conjecture, which claims

Given a compact Kähler manifold (X, ω) together with a 2-form R representing $c_1(X)$, one can always find a Kähler form $\tilde{\omega}$ such that $[\omega] = [\tilde{\omega}]$ and $\mathrm{Ric}(\tilde{\omega}) = R$.

This conjecture was proved in Yau's famous work in the late 70s. The second question, called the *Kähler-Einstein Problem*, first asked by Kähler in the 30s and later advertised by Calabi, considers whether there exists a Kähler metric satisfying the Einstein equation. This requires the existence of a Kähler form whose class is proportional to $c_1(X)$, and that the Kähler-Einstein Problem is equivalent, to ask whether the equation $c_1(X) = \lambda \cdot [\omega]$ ($\lambda = -1, 0$ or $+1$) can be lifted to the level of form, i.e.,

Does there always exist a Kähler form ω_{KE} on X, such that $\mathrm{Ric}(\omega_{\mathrm{KE}}) = \lambda \cdot \omega_{\mathrm{KE}}$?

A solution ω_{KE} yields a *Kähler-Einstein metric*.

> When $\dim_{\mathbb{C}} X = 1$, the Kähler-Einsten metric is given by the Poincaré Uniformization Theorem.

One immediately sees that when $\lambda = 0$, X always has a Kähler-Einstein metric by the solution of the Calabi Conjecture, since one can choose the form representing $c_1(X)$ to be 0. It was also proved, by Aubin and Yau independently, when $\lambda = -1$, X admits a Kähler-Einstein metric. These results have tremendous influence on people's understanding of algebraic varieties with $c_1(X)$ being zero or negative.

When $\lambda = 1$, X is called *Fano* (named after Italian mathematician Fano). In this case, the Kähler-Einstein Problem becomes more subtle as there is no definitive answer. The first obstruction was found in the late 50s, when Matsushima showed that a Fano manifold X has a Kähler-Einstein metric would require $\mathrm{Aut}(X)$ to be reductive. This can be applied to, e.g., the blow up X of a point on $\mathbb{CP}^2$ to conclude it does not admit a Kähler-Einstein metric. In the early 80s, Futaki found a deeper obstruction which is the vanishing of $\mathrm{Fut}(v)$ for any vector field v on X. Here $\mathrm{Fut}(v)$ is a numerical invariant that we will discuss more later.

In the late 80s, Tian proved that for a compact complex Fano surface X, $\mathrm{Aut}(X)$ being reductive is the necessary and sufficient condition for admitting a Kähler-Einstein metric. However, the proof was a case-by-case study and there was no reason to believe this would be the general situation.

In fact, Yau had a profound speculation that for a Fano manifold X, having a Kähler-Einstein metric should be equivalent to deeper algebraic properties of X, namely certain kind of stability. There are two reasons to make the speculation. Firstly, based on the solution of the Calabi Conjecture, it was believed that a variant of C^0-estimate would be the key analytic input, but such an estimate should be governed by algebraic data. Secondly, prior to the Kähler-Einstein Problem, this philosophy of canonical metric/stability correspondence had already appeared in a different setting: by the work of Donaldson and Uhlenbeck-Yau, a vector bundle on a compact Kähler manifold admits a Hermitian-Einstein metric if and only if it is slope polystable, which is now called *the Hitchin-Kobayashi Correspondence*. Here the Hermitian-Einstein on a vector bundle has a similar nature to the Kähler-Einstein metric. However, the non-linear feature of varieties makes the Kähler-Einstein Problem manifestly harder, and to deal with the difficulty, there is a major shift of viewpoint in regard to how to understand the stability condition in algebraic geometry.

In the late 90s, Tian [Tia97] proposed the precise statement and defined the notion of *K-stability*: For a given Fano manifold X, Tian considers $\mathbb{C}^*$-equivariant families

$\mathcal{X} \to \mathbb{C}$, such that the fiber $\mathcal{X}_t \cong X$ for any $t \neq 0$, and the special fiber X_0 is *mildly singular* (we will discuss this technical assumption later), so that one can define *the (generalized) Futaki invariant* $\mathrm{Fut}(\mathcal{X}) := \mathrm{Fut}(X_0, v)$ where v is the vector field on X_0 induced by the $\mathbb{C}^*$-action. Then Tian showed that X admits a Kähler-Einstein metric implies $\mathrm{Fut}(\mathcal{X}) \geq 0$ for all $\mathcal{X}$ and the equality holds only when $\mathcal{X} \cong X \times \mathbb{C}$. This latter condition posted on X, which is of a global nature, is called *K-polystablity*.

Later Donaldson [Don02] considered a similar setting of $\mathbb{C}^*$-equivariant degenerations but for all polarized projective varieties (X, L) with no assumptions on the degeneration, and defined the Futaki invariant *algebraically*. Under this setting, one can extend the study to the existence of a constant scalar curvature metric, but we will only focus on Fano varieties in this article.

One major step forward in the field is that the converse direction of Tian's theorem is also true.

Theorem (Yau-Tian-Donaldson Conjecture). *Let X be a Fano variety, then X admits a Kähler-Einstein metric if and only if X is K-polystable.*

The Yau-Tian-Donaldson Conjecture was first proved for smooth Fano manifolds by Chen-Donaldson-Sun [CDS15] and Tian [Tia15], following the strategy called the Cheeger-Colding-Tian theory in Riemannian geometry.

Later based on many earlier works of understanding the geometry of the space of Kähler metrics, notably by Mabuchi, Tian, Donaldson, Chen, Eyssidieux, Guedj, Zeriahi, Berndtsson, Darvas, and many others, Berman-Boucksom-Jonsson [BBJ21] proceeded along a completely different route, namely the variational approach, which focuses on (the completion of) the space of Kähler metrics. The argument was extended by Li-Tian-Wang and later to all singular cases by Li [Li22]. It relies on far less differential geometry input than the Cheeger-Colding-Tian theory. However, on the algebraic side it has to assume a stronger condition called *uniform K-stability*.

Therefore, it remains to verify the subtle equivalence between uniform K-stability and K-stability, which is an algebraic question. Built on the powerful machinery developed in the Minimal Model Program, the equivalence is eventually established by Liu-Xu-Zhuang [LXZ21], as part of the project to construct the moduli space of Fano varieties. (The easier direction that the existence of Kähler-Einstein metric implies K-polystability was achieved earlier in [Ber16].) In the rest of this article, we will survey these ideas, with a focus on the algebraic theory.

For now let us explain why this kind of condition is called *stability*. The term "stability" used in algebraic geometry is often related to the construction of moduli spaces. While the connection between K-stability and moduli space is a profound topic that we will extensively discuss later, we first compare it with a more classical notion in the moduli theory, which is the *geometric invariant theory (GIT) stability*.

As far as we know, the notion of stability in algebraic geometry first implicitly appeared in Hilbert's work on invariant theory, and later was explicitly coined by Mumford in the geometric invariant theory, which considers the setting of a reductive group G acting on a projective variety M. To form the quotient space, which is the 'moduli space' parametrizing orbits, one has to throw away some *unstable* points on M. To make the setting pragmatical, one often also considers a very ample line bundle L over M such that the action on G can be lifted to L as linear maps $L_x \to L_{g(x)}$ for any x. Then the Hilbert-Mumford criterion says that to check whether $x \in M$ is stable, one only needs to consider for any one parameter subgroup $\mathbb{C}^* \subset G$, the sign of the weight of the $\mathbb{C}^*$-action on $L|_{x_0} \cong \mathbb{C}$ where $x_0 := \lim_{t \to 0} t \cdot x$.

Now let X be a Fano variety, and $[X]$ the corresponding point in the Hilbert scheme M, under an embedding $X \to \mathbb{P}^N$ induced by $|-rK_X|$ for some r. Let $\mathbb{C}^*$ be any subgroup of $\mathrm{PGL}_{\mathbb{C}}(N+1)$. The closure $\overline{\mathbb{C}^* \cdot [X]}$ yields a morphism $\mathbb{C} \to M$, and the pull-back of the universal family induces a $\mathbb{C}^*$-equivariant family $(\mathcal{X}, \mathcal{L})$ over $\mathbb{C}$. This is precisely the notion of a *test configuration* (with index r). Then the Futaki invariant for a test configuration $\mathcal{X}$ can be considered as the weight of a $\mathbb{C}^*$-action on the restriction of a line bundle, namely the *CM line bundle*, over the point $[X_0]$ where $[X_0] := \lim_{t \to 0} t \cdot [X]$. There is a deeper reason to adopt the GIT stability into the consideration, proposed by Donaldson, following the famous work of Atiyah-Bott. We first recall Kempf-Ness's interpretation of stability: let G be the complexification of a compact group K, and assume a projective manifold (M, L) admits a G-action with a K-invariant norm $\|\cdot\|$ on L. Define the function

$$f : G/K \to \mathbb{R}, \ \ x \to \log \|g \cdot \hat{x}\|,$$

where $\hat{x}$ is a non-zero lift of x. This function is convex along any geodesic $\mathbb{C}^* \to G/K$, therefore f has a minimum on G/K if and only if the limit $\lim_{t \to \infty} f'(e^{it\xi} \cdot x)$ at infinity is positive for any $\xi \in \mathrm{Lie}(K)$. The latter is precisely the weight of the $\mathbb{C}^*$-action on $L|_{x_0}$, so combining with the Hilbert-Mumford criterion, we know

$$x \text{ is stable if and only if } f \text{ has a minimum.}$$

In the Kähler-Einstein Problem, one can view the space $\mathcal{H}$ of Kähler metrics with the same class as an infinite dimensional analogue of symmetric spaces, and the existence of the Kähler-Einstein metric is equivalent to the existence of a minimum for certain geometric functional, e.g., the Mabuchi functional, on $\mathcal{H}$. For a test configuration $(\mathcal{X}, \mathcal{L})$, the pullbacks ω_t of the (normalized) Fubini-Study metric along $X_t \subseteq \mathbb{P}^N$ gives a ray in $\mathcal{H}$. The derivative of the Mabuchi functional on ω_t at infinity is the Futaki invariant. This clear analogy makes the comparison of Kähler-Einstein Problem with the finite dimensional GIT fruitful.

However, there are two fundamental differences between the K-stability and the standard GIT: since we consider all $\mathcal{X}$, a priori we need to take into account all sufficiently large r; moreover, the CM line bundle is not ample. Therefore, the usual geometric invariant theory fails to apply here. With the benefit of hindsight, it is the more intrinsic Minimal Model Program theory which remedies the study of K-stability as an algebro-geometric subject and enables people to go beyond the GIT.

Fano varieties. As one of the three building blocks of an arbitrary variety, the class of Fano varieties has been the central subject in higher dimensional geometry for more than a century. The only Fano manifold in dimension one is $\mathbb{CP}^1$. In dimension two, it is a classical result that there are 10 families of them, namely blowing up m general points on $\mathbb{CP}^2$ for $0 \leq m \leq 8$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$. For dimension three, in the 80s Iskovskikh and Mori-Mukai classified that there are 105 smooth families. Fano manifolds often have rich geometry, as they could have many interesting birational models. However, this also makes the study of them intricate. Moreover, from the view of Minimal Model Program, one needs to also consider Fano varieties with mild singularities. In particular, these singular Fano varieties naturally appear as degenerations of Fano manifolds.

Example. To see a basic example, we look at the family given by the equation

$$(tx_0^d + \sum_{i=1}^{n} x_i^d = 0) \subset \mathbb{CP}^n \times \mathbb{C} \ \ (2 \leq d < n). \qquad (1)$$

If $t \neq 0$, we get a Fano manifold X_t isomorphic to the Fano Fermat hypersurface $\sum_{i=0}^{n} x_i^d = 0$; and if $t = 0$, the Fano variety X_0 is a cone over a Fano hypersurface of one dimensional lower, with a singularity at $(1, 0, \ldots, 0)$. Therefore, this family of Fano varieties is trivial over $\mathbb{C}^*$, but the algebraic structure changes over $t = 0$. We note that as a comparison, this phenomenon can not occur when $c_1(X) \leq 0$. Moreover, let $f\colon Y \to X_0$ be the blow up at $(1, \ldots, 0)$. The exceptional divisor is E, and a simple calculation yields

$$f^*(\omega_{X_0})((n-d-1)E) = \omega_Y.$$

Definition. Let E be a divisor over X, i.e., E is a divisor on a normal birational model $\mu\colon Y \to X$. We define *the log discrepancy along E* to be

$$A_X(E) = (\text{the coefficient of } K_{Y/X} \text{ along } E) + 1.$$

We say X has *Kawamata log terminal (klt)* singularities if $A_X(E) > 0$ for any divisor E over X. It is known it suffices to check $A_X(E) > 0$ for divisors E appearing on a fixed log resolution.

In particular, X_0 in (1) is a klt Fano variety.

Admittedly, it is not easy to see at the first glance why klt singularities are important. Nevertheless, built on four decades' experience accumulated by people working in the Minimal Model Program theory, it becomes clear that, from various viewpoints in higher dimensional geometry, klt varieties form the right class of singular varieties to be studied.

For instance, one non-trivial property, established in [LX14], is that klt Fano varieties are closed under degeneration: Let $X^\circ \to \Delta^\circ$ be a family of klt Fano varieties over a punctured disc $\Delta^\circ = \Delta \setminus \{0\}$, then after a possible base change of Δ°, we can fill in the limit with a klt Fano variety.

New ideas from algebraic geometry. The exploration of the connection between K-stability and the MMP was initiated in the work in Odaka [Oda13] and Li-Xu [LX14] around the early 2010s. After a small gap of a few years, it started to blossom around 2015. The meeting of these two subjects brings up many new insights to both sides. One main advantage is that this algebraic method provides a stronger tool to treat Fano varieties with singularities, which are currently beyond the reach of the original metric geometry approach.

There are *five* fundamental new ideas coming out to advance the study of K-stability via higher dimensional geometry. These new ideas are related to each other. Nevertheless, they are somewhat different. In my later discussions, I will elaborate three of them:

1. Various equivalent characterizations of K-stability.
2. The uniqueness of K-polystable degeneration.
3. The higher rank finite generation.

I will only briefly mention the last two:

4. The connection of the K-stability of fibers and the positivity of the CM line bundle on the base, via Harder-Narashimhan filtration.
5. The local K-stability theory via the normalized volume function.

Valuative criterion of K-stability. One fundamental development of the algebraic theory of K-stability, underlying the further progress in various directions, is the equivalent description of the notions of K-stability, using valuations over the function field $K(X)$.

Let X be a Fano variety with klt singularities, a prime divisor E yields a valuation ord_E given by the vanishing multiplicity along E for any meromorphic function, i.e., for any meromorphic function f,

$$\mathrm{ord}_E(f) \coloneqq \mathrm{mult}_E(\mathrm{div}(f)).$$

Definition. Let E be a prime divisor over X, i.e., E is a prime divisor on a normal birational model $\mu\colon Y \to X$. Let $\dim H^0(-mK_X) = N_m$, we can choose a basis

$\{s_1, \dots, s_{N_m}\}$ compatible with the filtration on $H^0(-mK_X)$ induced by ord_E, i.e., for any λ, the subspace

$$\begin{aligned} & \mathcal{F}_E^\lambda H^0(-mK_X) \\ = \; & \{s \in H^0(-mK_X) \mid \mathrm{ord}_E(s) \geq \lambda\} \end{aligned} \qquad (2)$$

is generated by s_i of $\{s_1, \dots, s_{N_m}\}$ with $s_i \in \mathcal{F}_E^\lambda H^0(-mK_X)$. We define

$$S_m(E) := \lim_{m\to\infty} \frac{1}{mN_m} \sum_{i=1}^{N_m} \mathrm{ord}_E(s_i),$$

and the *expected vanishing order*

$$S_X(E) = \frac{1}{(-K_X)^n} \int_0^\infty \mathrm{vol}(\mu^*(-K_X) - tE)dt.$$

Then $S_X(E) = \lim_{m\to\infty} S_m(E)$. We also denote *the stability threshold* by

$$\delta(X) = \inf_E \frac{A_X(E)}{S_X(E)},$$

where E runs through over all prime divisors over X.

Theorem (Valuative criterion)**.** *We have the following equivalent characterization of notions in K-stability.*

1. *X is K-semistable if and only if $\delta(X) \geq 1$;*
2. *X is uniformly K-stable if and only if $\delta(X) > 1$; and*
3. *X is K-stable if and only if $A_X(E) > S_X(E)$ for any E.*

The Statement 1 is proved by Fujita [Fuj19] and Li [Li17], which we will explain below. A similar argument is used to prove Statement 2 in [Fuj19], as well as Statement 3 in [Fuj19, Li17] with a technical assumption, which was later removed by Blum-Xu [BX19].

Before explaining the ideas of proving the valuative criterion, to exemplify its strength, we first give an application to the Bishop-Gromov type Comparison Theorem for Kähler manifolds.

Example (Fujita)**.** Let X be an n-dimensional K-semistable Fano variety, e.g., X admits a Kähler-Einstein metric. Then $(-K_X)^n \leq (n+1)^n$.

This can be seen in the following way. Let $\mu : Y \to X$ be the blow up of a smooth point $x \in X$ with the exceptional divisor E. Then

$$H^0(\mu^*(-mK_X) - tmE) = H^0(\mathcal{O}_X \otimes \mathfrak{m}_x^{[tm]}),$$

which implies that

$$\mathrm{vol}(-\mu^* K_X - tE) \geq (-K_X)^n - t^n.$$

The valuative criterion implies that

$$n(-K_X)^n \geq \int_0^{((-K_X)^n)^{\frac{1}{n}}} \left((-K_X)^n - t^n\right)\mathrm{d}t.$$

Therefore, $(-K_X)^n \leq (n+1)^n$.

What is striking about the valuative criterion is that K-stability notions are checked by looking at valuations which are apparently different from test configurations as in the original definition. A connection is observed by Boucksom-Hisamoto-Jonsson: for any irreducible component F of the special fiber X_0 of a test configuration $\mathcal{X}$, since $\mathcal{X}$ is birational to $X \times \mathbb{A}^1_t$, $K(\mathcal{X}) \cong K(X)(t)$, the restriction of ord_F on the subfield $K(X) \subseteq K(\mathcal{X})$ is of the form $a \cdot \mathrm{ord}_E$ for some divisor E over X, as long as F is not from $X \times \{0\}$.

A deeper result, proved in [LX14], says that started with any test configuration $(\mathcal{X}, \mathcal{L})$ we can use the Minimal Model Program techniques to run a process to obtain a new normal test configuration $\mathcal{Y}$ such that $-K_{\mathcal{Y}}$ is relatively ample and the central fiber X_0 is also a klt Fano variety. These test configurations are called *special*. Moreover, up to a base change $\mathbb{C} \xrightarrow{z^d} \mathbb{C}$,

$$\mathrm{Fut}(\mathcal{X}, \mathcal{L}) \geq \mathrm{Fut}(\mathcal{Y}, -K_{\mathcal{Y}}) := \mathrm{Fut}(\mathcal{Y}).$$

As a consequence, verifying K-stability on all test configurations as in Donaldson's definition, is equivalent to restricting over special test configurations. This subclass is smaller than the one allowed in Tian's definition, therefore, for Fano varieties, Donaldson's definition is indeed equivalent to Tian's.

Given a non-trivial special test configuration $\mathcal{X}$, let $a \cdot \mathrm{ord}_E$ be the valuation on $K(X)$ induced by the unique component X_0 as above. One can calculate

$$\mathrm{Fut}(\mathcal{X}) = a(A_X(E) - S_X(E)),$$

and we immediately conclude that if $\delta(X) \geq 1$, then X is K-semistable.

There are two different approaches to prove the converse direction. The first one was given in [Fuj19, Li17]. It needs the notion of Ding stability, which is algebraically defined by Berman [Ber16], by looking at the sign of the *Ding invariant* $\mathrm{Ding}(\mathcal{X}, \mathcal{L})$ for all test configurations $(\mathcal{X}, \mathcal{L})$. Berman-Boucksom-Jonsson [BBJ21] and Fujita [Fuj19] observed that the Minimal Model Program process in [LX14] also can be used to show testing Ding stability notions on general test configurations and special test configurations are equivalent. As a consequence, one immediately sees that K-stability notions are equivalent to the corresponding Ding-stability ones, as Futaki invariants are identical to Ding invariants on special test configurations.

Then Fujita makes the key observation that the definition of Ding invariant can be extended from test configurations to any linearly bounded (decreasing) multiplicative filtration $\mathcal{F}^\bullet := (\mathcal{F}^t R)_{t\in\mathbb{R}}$ on the anti-canonical ring $R = \bigoplus_m H^0(-mK_X)$, and Ding semistability implies $\mathrm{Ding}(\mathcal{F}^\bullet) \geq 0$ for any such filtration $\mathcal{F}^\bullet$. For a divisor E over X, one can define the filtration $\mathcal{F}_E^\bullet$ by (2). Moreover, we have

$$A_X(E) - S_X(E) \geq \mathrm{Ding}(\mathcal{F}_E^\bullet).$$

Therefore, K-semistability, which is equivalent to Ding semistability, implies that $\delta(X) \geq 1$.

Complements and log canonical places. The second approach reinterprets special test configurations using valuations with a concrete geometric description, following [BLX19]. For this we need to first introduce the concept of *complements* defined by Shokurov.

Definition. For a Fano variety X, a $\mathbb{Q}$-divisor D is an N-complement, if $D = \frac{1}{N}\Gamma$ for some divisor $\Gamma \in |-NK_X|$ and (X, D) is log canonical.

We say D is a $\mathbb{Q}$-complement, if it is an N-complement for some N.

At first sight it is merely a technical concept. However, the following deep result, conjectured by Shokurov and proved by Birkar, sheds light on Fano varieties: there exists a uniform N which only depends on $\dim(X)$, such that X always has an N-complement. This is surprising since n-dimensional Fano varieties are unbounded.

Below, we will explain how complements play an important role in the study of K-stability, through the construction of the *basis type divisor*. For a sufficiently divisible m, we say D is a m-basis type divisor if

$$D = \frac{1}{m \cdot N_m}\big(\mathrm{div}(s_1) + \cdots + \mathrm{div}(s_{N_m})\big),$$

where $N_m = \dim H^0(X, -mK_X)$ and $\{s_1, \ldots, s_{N_m}\}$ forms a basis of $H^0(X, -mK_X)$. This concept was introduced by Fujita-Odaka, and they considered

$$\delta_m(X) = \inf_D \mathrm{lct}(X, D) = \inf_D \inf_E \frac{A_X(E)}{\mathrm{ord}_E(D)}$$

where D runs through over all m-basis type divisors, and E runs through over all prime divisors over X. Together with Jonsson-Blum's work, they show

$$\lim_{m\to\infty} \delta_m(X) = \delta(X). \tag{3}$$

We note that this gives us a way to verify K-stability for explicit Fano varieties, by estimating $\delta_m(X)$.

For a Fano variety X and a $\mathbb{Q}$-complement D, we say a prime divisor E over X is *a log canonical place* of D if $A_X(D) = \mathrm{ord}_E(D)$. Then using (3), one can show that

$$\text{if } \delta(X) \leq 1, \delta(X) = \inf_E \frac{A_X(E)}{S_X(E)},$$

where E runs through over all log canonical places of a $\mathbb{Q}$-complement.

An observation made in [BLX19] is that any divisor E arises from a special test configuration only if it is a log canonical place of a $\mathbb{Q}$-complement. In particular, if $\delta(X) < 1$, then there exists a special test configuration $\mathcal{X}$ with

$$\mathrm{Fut}(\mathcal{X}) = A_X(E) - S_X(E) < 0,$$

i.e., X is K-unstable. Additionally, by using the minimum norm function $\|\mathcal{X}\|_{\mathrm{m}}$ introduced by Dervan, a more precise relation can be given when $\delta(X) \leq 1$:

$$\delta(X) := \inf_E \frac{A_X(E)}{S_X(E)} = 1 + \inf_{\mathcal{X}} \frac{\mathrm{Fut}(\mathcal{X})}{\|\mathcal{X}\|_{\mathrm{m}}}. \tag{4}$$

Then [BLX19] applied Birkar's theorem on the existence of bounded complements to investigate log canonical places of $\mathbb{Q}$-complements, and proved that there exists a *uniform* N, which only depends on $\dim(X)$, such that if E is a log canonical place of $\mathbb{Q}$-complement, it is indeed a log canonical place of an N-complement D. As all N-complements form a bounded family, their log canonical places are toroidal divisors over a bounded family of log resolutions. Both $A_X(E)$ and $S_X(E)$ are deformation invariant, so we can assume $\delta(X) = \lim_{E_i} \frac{A_X(E_i)}{S_X(E_i)}$ for a sequence of lc places E_i of N-complements, which is toroidal over *the same* log resolution. Thus after a rescaling of ord_{E_i} and passing to a subsequence, there exists a limiting quasi-monomial valuation v and it satisfies that $\delta(X) = \frac{A_X(v)}{S_X(v)}$. (There are straightforward generalization of various definitions and invariants from divisors to quasi-monomial valuations v over X.)

Higher rank finite generation. In the above discussion, the valuation v has its toroidal coordinates the limit of the ones of a rescaling of ord_{E_i}. In particular, it could be $\mathbb{Q}$-linearly independent. In other words, v could be a quasi-monomial valuation of higher rational rank. For any valuation v, its values on $R = \bigoplus_m H^0(-mK_X)$ form a monoid Φ contained in $\mathbb{R}_{\geq 0}$. For any $\lambda \in \Phi$, we define $I_\lambda \subseteq R$ to be the ideal consisting of sections $s \in R$ with $v(s) \geq \lambda$; and $I_{>\lambda} \subseteq R$ to be the ideal of sections s with $v(s) > \lambda$. Then the *associated graded ring* of R with respect to v is

$$\mathrm{gr}_v(R) := \bigoplus_{\lambda\in\Phi} I_{\geq\lambda}/I_{>\lambda}.$$

To further advance the theory, we need another major recipe which is the following finite generation theorem, proved by Liu-Xu-Zhuang [LXZ21].

Theorem (Higher rank finite generation). *Let X be a Fano variety with $\delta(X) \leq 1$. Let v be the quasi-monomial valuation constructed above satisfying that $\delta(X) = \frac{A_X(v)}{S_X(v)}$. Then the associated graded ring $\mathrm{gr}_v(R)$ is finitely generated.*

In the above theorem, if v arises from a divisor, the finite generation easily follows from the work of Birkar-Cascini-Hacon-McKernan [BCHM10], as v is a log canonical place of a $\mathbb{Q}$-complement. However, it does not always hold that

$\mathrm{gr}_v(R)$ is finitely generated if a higher rational rank valuation v is merely assumed to be a log canonical place of a $\mathbb{Q}$-complement. Therefore, finer properties have to been extracted from the quasi-monomial valuation v computing $\delta(X)$, and the proof needs various new recipes, especially deep results on the boundedness of Fano varieties.

Revisit stability. As we already emphasized, roughly speaking, *stability* can be thought of as a condition to pick out the *optimal* degeneration for a family. As seen in Kempf-Ness's classical theorem, the theory often can be extended to associating every unstable object an "optimal destabilization." An example is to degenerate an unstable sheaf to the direct summand of the graded sheaves of its Harder-Narasimhan filtration. The notion is coined into the term Θ-*stratification* by Halpern-Leistner.

Our finite generation theorem yields such a theory for K-stability. In fact, one can show from it that there exists a divisorial valuation $a \cdot \mathrm{ord}_E$ sufficiently close to v, with $\delta(X) = \frac{A_X(E)}{S_X(E)}$ and E induces a special test configuration. As a consequence, we have

Theorem (Optimal Destabilization Theorem)**.** *Let X be a Fano variety with $\delta(X) \leq 1$. Then there exists a divisor E which induces a special test configuration and satisfies $\delta(X) = \frac{A_X(E)}{S_X(E)}$. In particular, if X is K-stable then it is uniformly K-stable.*

In particular, X is K-stable if and only if $\delta(X) > 1$.

The "optimal" in our theorem is with respect to the minimum norm as (4) shows. Székelyhidi explored the optimal degeneration with respect to L^2-norm for toric varieties. Nevertheless, for a general Fano variety, the minimum norm fits better into our machinery. To help the reader to understand the features of different approaches, we would like to give a panoramic comparison in each approach of how the stability is applied to the degeneration process.

> Philosophically speaking, a key point to studying K-stability is understanding how to get the *limit* of a sequence of Fano varieties X_t with extra conditions, or similarly a sequence of test configurations of a fixed Fano variety, which approximates the infimum of the normalized Futaki invariants.

In the Cheeger-Colding-Tian theory, one starts with a Fano manifold X without a Kähler-Einstein metric, and it suffices to construct a test configuration destabilizing X. Donaldson introduced the idea of using an auxiliary boundary divisor and conic metrics along it. More precisely, fix any smooth divisor $\Gamma \in |-mK_X|$, one can find a sequence $\beta_i \in (0, \frac{1}{m})$ with $\beta = \lim_i \beta_i$ such that $(X, \beta_i \cdot \Gamma)$ admits a conical Kähler-Einstein metric ω_i, but $(X, \beta \cdot \Gamma)$ does not. After passing through a subsequence, the *Gromov-Hausdorff limit* X_∞ of the conical Kähler-Einstein Fano manifolds (X, ω_i) exists as a metric space. Then it took a large endeavor to show that X_∞ is indeed the underlying space of the Chow limit when one uses an orthonormal basis (with respect to ω_i) of $|-NK_X|$ (for some $N \gg 0$) to embed X into a projective space. Moreover if we take Γ_∞ to be the Chow limit of Γ, then $(X_\infty, \beta \cdot \Gamma_\infty)$ has a conical Kähler-Einstein metric. Then we obtain a test configuration degenerating X to X_∞ with a negative Futaki invariant, which implies X is K-unstable. However, for now this kind of analytic argument is restricted to the case when X is smooth.

The variational approach focuses on the space of smooth Kähler metrics and its various completions. The existence of a Kähler-Einstein metric is equivalent to the Mabuchi (or Ding) functional having a critical point. Following the infinite dimensional GIT picture, one should concentrate on the functional along *geodesic rays* by the analogue of the Hilbert-Mumford criterion in this setting. For simplicity, here we restrict ourselves to the case that the automorphism group is finite. It is shown that the existence of a critical point of the Ding functional follows from the *coercivity*, which means for any geodesic $U : \mathbb{R} \to \mathcal{E}^1$ in the space of Kähler metrics $\mathcal{E}^1$ with finite energy plurisubharmonic (psh) potentials, the Ding functional $\mathbf{D}$ grows at least linearly. After establishing the convexity of $\mathbf{D}$ along the geodesic ray, it is sufficient to understand the growth $\frac{1}{t}\mathbf{D}(U_t)$ when $t \to \infty$, where a translation dictionary between the archimedean and nonarchimedean setting is established. We note that the space of non-archimedean metrics contains all test configurations, so we have a chance to connect the algebraic condition of K-stability and the analytic condition of the existence of Kähler-Einstein metrics.

The positivity of Ding invariants on non-archimedean metrics (with finite energy) implies coercivity, as when $t \to \infty$,

$$\lim_{t\to\infty} \frac{\mathbf{D}(U_t)}{t} \geq \mathrm{Ding}(\phi) > 0$$

for a non-archimedean metric ϕ attached to the geodesic ray U. Moreover, while ϕ might not come from a test configuration, it can be approximated by a sequence of non-archimedean metrics ϕ_i coming from test configurations (constructed by Demailly's regularization) such that the Ding invariants satisfy $\lim_{i\to\infty} \mathrm{Ding}(\phi_i) \leq \mathrm{Ding}(\phi)$. However, to have positivity of the limiting Ding invariant $\mathrm{Ding}(\phi)$, positivity of $\mathrm{Ding}(\phi_i)$ is not enough, but we have to assume uniform Ding stability.

As we mentioned before, uniform K-stability is known to be the same as uniform Ding stability. It remains to show the equivalence between K-stability and uniform K-stability, which is addressed by the Optimal Destabilization Theorem, obtained with deep results from the

Minimal Model Program. In other words, the algebro-geometric approach provides the necessary compactness result.

In the work of stability, this kind of study on the limit of a family of Fano varieties is clearly related to moduli spaces. In fact, the progress of our understanding on the fundamental concepts of K-stability intertwines with the construction of moduli spaces of K-polystable Fano varieties. We will continue to discuss the latter topic.

Moduli of Fano varieties. It is one of the most fundamental questions in algebraic geometry to construct moduli spaces of varieties. For instance, for $g \geq 2$ the moduli space $\mathcal{M}_g$ of curves and its compactification $\overline{\mathcal{M}}_g$ are among the most studied objects in algebraic geometry. Built on the machinery of the Minimal Model Program and progress on understanding the moduli problem of higher dimensional varieties, KSB (Kollár-Shephard-Barron) stability gives a satisfying theory to generalize the construction of $\overline{\mathcal{M}}_g$ to higher dimensions. In particular, an appropriate definition of families of arbitrary dimensional varieties (or even log pairs) was given by Kollár. As the output, we get a projective scheme which parametrizes KSB stable varieties X, whose canonical class K_X is ample.

The opposite case when $-K_X$ is ample had been mysterious for a long time. It is trivial when $\dim(X) = 1$ as $X = \mathbb{CP}^1$. However, in higher dimensions, it is much trickier. As (1) shows, an isotrivial family X_t could jump to X_0 with a different complex structure, which does not occur for a KSB stable family. So it has been well understood for a long time that the Minimal Model Program itself is unlikely to be sufficient to provide a satisfactory moduli theory for Fano varieties. However, in the example (1), the central fiber is K-unstable, so there is a chance that if we restrict ourselves to Fano varieties with K-stability assumptions, this pathology does not occur.

Various results from the analytic side, especially "the partial C^0-estimate" in the works of Tian and Donaldson-Sun, suggest that the condition on the existence of Kähler-Einstein metrics could sort out a class of Fano varieties to be parametrized by well-behaved moduli spaces. As K-stability is the corresponding algebraic condition, it is natural to hope that it yields a robust moduli theory. Speculation is also inspired by the successful role that the Hitchin-Kobayashi correspondence plays in the study of compact moduli spaces parametrizing them. However, for families of K-stable Fano varieties it was far from clear how to use the definition of K-stability itself to actually establish the list of properties needed to construct the moduli space. This could be the reason that such a topic did not attract people's attention in the decade right after the inventions of the notion. Li-Xu [LX14] studied families of Fano varieties from the perspective of K-stability and provided a hint that K-stability may lead to an algebraic moduli theory. Nevertheless, only until people have obtained enough fundamental knowledge on K-stability, e.g., the valuative criterion, we could start to get definitive results.

The main theorem is the following.

Theorem (K-moduli). *Fix a positive integer n, and a rational number V. Then*

1. *There exists an Artin stack $\mathfrak{X}^{\mathrm{kss}}_{n,V}$ of finite type, which parametrizes families of n-dimensional K-semistable Fano varieties $X \to T$, with $(-K_{X_t})^n = V$ for any $t \in T$.*
2. *The stack $\mathfrak{X}^{\mathrm{kss}}_{n,V}$ admits a proper good moduli space $\mathfrak{X}^{\mathrm{kss}}_{n,V} \to X^{\mathrm{kps}}_{n,V}$, whose points correspond to K-polystable Fano varieties. Moreover, the CM line bundle Λ_{CM} is ample on $X^{\mathrm{kps}}_{n,V}$.*

The proof of the above theorem is a combination of many people's contributions. We will explain below the content of the K-moduli Theorem, and sketch the circle of ideas involved in the proof.

Example. Consider cubic hypersurfaces

$$X \subset \mathbb{P}^{n+1} \quad (n \geq 2). \tag{5}$$

It is not hard to verify K-stability for some special cubic hypersurface, e.g., the Fermat $X = (x_0^3 + \cdots + x_{n+1}^3) = 0$. As we will see this implies that there is a Zariski open locus of the parametrizing space, such that any corresponding cubic hypersurface is K-semistable.

It is shown by Odaka-Spotti-Sun for $n = 2$, Liu-Xu for $n = 3$, and Liu for $n = 4$, that in these cases, the GIT stability of cubic hypersurfaces is exactly the same as K-stability. In other words, the K-moduli is given by the GIT moduli space.

K-moduli stack. By definition, the *K-moduli stack* $\mathfrak{X}^{\mathrm{kss}}_{n,V}$ admits a universal family $\mathfrak{U}^{\mathrm{kss}}_{n,V} \to \mathfrak{X}^{\mathrm{kss}}_{n,V}$, such that for any scheme T, all pull-back families under morphisms in $\mathrm{Hom}(T, \mathfrak{X}^{\mathrm{kss}}_{n,V})$ precisely correspond to all isomorphic classes of families of n-dimensional K-semistable Fano varieties with volume V over T.

To prove the first part of the K-moduli theorem, we first need to show the boundedness which says that there exists a uniform $M = M(n, V)$ such that for any n-dimensional K-semistable Fano variety, $-MK_X$ is very ample, i.e., the linear system $|-MK_X|$ induces an embedding of X into a same projective space $\mathbb{CP}^N$. This was first proved by Jiang, who reduced the question to Birkar's proof of the Borisov-Alexeev-Borisov Conjecture.

We consider (X, f) for X with an embedding $f : X \to \mathbb{P}^N$ as above. These pairs are parametrized by a locus Z of the Hilbert scheme $\mathrm{Hilb}(\mathbb{P}^N)$. It suffices to prove Z is locally closed in $\mathrm{Hilb}(\mathbb{P}^N)$, since

$$\mathfrak{X}^{\mathrm{kss}}_{n,V} = [Z/\mathrm{PGL}_{\mathbb{C}}(N+1)], \tag{6}$$

as two embeddings of X differ by an element in $\mathrm{PGL}_{\mathbb{C}}(N+1)$.

The locus in $\mathrm{Hilb}(\mathbb{P}^N)$ parametrizing Fano subvarieties of $\mathbb{P}^N$ is locally closed, so it remains to prove for a family of Fano varieties $X \to T$, the locus $U \subseteq T$ parametrizing K-semistable fibers is open. One way to get this is using the fact that the function $t \to \min\{\delta(X_t), 1\}$ is constructible and lower semi-continuous. This was first proved in [BLX19] for $\delta(X_{\bar{t}})$ instead of $\delta(X_t)$, where $\bar{t}$ is the geometric point given by t. Then Zhuang verified $\delta(X_{\bar{t}}) = \delta(X_t)$.

K-moduli space. From the above discussion, we see that for any fixed positive number $\delta_0 \in (0,1]$, we still get an Artin stack of finite type if we replace the K-semistability condition for $\mathfrak{X}^{\mathrm{kss}}_{n,V}$ by $\delta(X_t) \geq \delta_0$. However, a much more delicate property which distinguishes $\mathfrak{X}^{\mathrm{kss}}_{n,V}$ is that it admits a space which parameterizes points up to orbit closure equivalence, i.e., the S-equivalence after Seshadri, called *the good moduli space*.

One difficulty in constructing moduli space of Fano varieties is the fact that a Fano variety X may have a positive dimensional automorphism group $\mathrm{Aut}(X)$. This contrasts to the case of the KSB stability, where the moduli is a separated Deligne-Mumford stack for which one can apply Keel-Mori's theorem to conclude the existence of a coarse moduli space. When a stack has points with positive dimensional stabilizer groups, it usually does not admit any coarse moduli space. Nevertheless, a replacement construction initiated by Alper, called the *good moduli space*, provides a good framework to treat this more complicated case, at least in characteristic 0.

Definition. Let $\mathfrak{X}$ be a stack. We say X is a good moduli space of $\mathfrak{X}$, if the quasi-compact morphism $\phi\colon \mathfrak{X} \to X$ satisfies

1. The push-forward functor ϕ_* is exact on quasi-coherent sheaves;
2. There is an isomorphism $\mathcal{O}_X \to \phi_*(\mathcal{O}_{\mathfrak{X}})$.

Example. Let $\mathbb{C}^*$ act on $\mathbb{C}$ by $t \cdot x \to tx$, then the stack $[\mathbb{C}/\mathbb{C}^*]$ admits a good moduli space, which is a point.

However, if we replace $\mathbb{C}$ by $\mathbb{CP}^1$, with the action $t \cdot [x_0, x_1] \to [x_0, tx_1]$, then the stack $[\mathbb{CP}^1/\mathbb{C}^*]$ has no good moduli space.

The definition of good moduli space is simple, however it has strong implications. For a quotient stack $[Z/G]$ if it admits a good moduli space, it implies that for any $z \in Z$, the closure $\overline{G \cdot z} \subset Z$ has a unique minimal orbit $G \cdot z^*$. Moreover, the stabilizer of z^* is reductive. For $\mathfrak{X}^{\mathrm{kss}}_{n,V}$ (see (6)), this means that for any K-semistable Fano variety X, there is a unique K-polystable Fano variety Y which is S-equivalent to X, and $\mathrm{Aut}(Y)$ is reductive. The latter was Matsushima's theorem when Y is a smooth Kähler-Einstein Fano manifold.

A prototype example of good moduli space is given by $\mathfrak{X} = [\mathrm{Spec}(A)/G]$ where G is a reductive group acting on an affine variety $\mathrm{Spec}(A)$ (over characteristic 0), where the good moduli space is $X = \mathrm{Spec}(A^G)$. More generally, for a polarized projective variety (Y, L) with a reductive group action, then one can take $\mathfrak{X} = [Y^{\mathrm{ss}}/G]$ where Y^{ss} is the GIT semistable locus and the good moduli space of $\mathfrak{X}$ is the GIT quotient $X = Y/sslashG$. Nevertheless, there is no known way to interpret the K-moduli problem as a GIT problem, so we have to rely on a more general abstract method. Fortunately, a valuative criterion of the existence of good moduli space, which is a version of Keel-Mori's theorem for Artin stacks, is established in [AHLH18]. Applying their work, built on earlier analysis in [LWX21] and [BX19], the existence of the good moduli space and its separatedness are proved in [ABHLX20].

While the existence of the good moduli space $X^{\mathrm{kps}}_{n,V}$ should justify that K-stability gives the right notion of moduli theory of Fano varieties, arguably the most remarkable property of $X^{\mathrm{kps}}_{n,V}$ is its properness.

> Let $X^{\circ} \to \Delta^{\circ}$ be a family of K-semistable Fano varieties over a punctured curve, then after a base change of Δ°, there exists a unique K-polystable Fano filling.

For a family whose general fibers are smooth with Kähler-Einstein metrics, the existence of the limit is given as the deepest consequence from the Cheeger-Colding-Tian theory. For the general case, the compactness result essentially relies on the Optimal Destabilization Theorem.

Another fundamental property of $X^{\mathrm{kps}}_{n,V}$ is its projectivity, i.e., $X^{\mathrm{kps}}_{n,V}$ is a subscheme of $\mathbb{P}^N$ for some N. The CM line bundle Λ_{CM}, defined in various works with growing generality and eventually by Paul-Tian in its current form, gives a candidate of an ample bundle, because over the locus parametrizing smooth Kähler-Einstein Fano manifolds the positivity of Λ_{CM} can be seen analytically via a metric on Λ_{CM}, whose curvature form is the Weil-Petersson metric. However, as usual there is essential difficulty to extend the analytic argument to the locus parametrizing singular Fano varieties. Later [CP21] initiated an algebraic way to attack the positive problem, and their method was completed by [XZ20].

Explicit K-stable Fano varieties. With the vast progress on K-stability theory, it is natural to study explicit examples.

The valuative criterion provides a way to verify K-stability of a Fano variety X by estimating $\delta(X)$. Using $\delta(X) = \lim_m \delta_m(X)$, Abban-Zhuang invent an approach of applying inversion of adjunction to reduce the estimate of the log canonical thresholds $\delta_m(X)$ to low dimensional varieties. As a result, they prove any degree d smooth hypersurface X in $\mathbb{P}^{n+1}$ is K-stable if $n+2-n^{1/3} \leq d \leq n+1$.

Following Tian's work in dimension 2 which completes the Kähler-Einstein Problem for smooth Fano surfaces, in a recent joint work of Araujo-Castravet-Cheltsov-Fujita-Kaloghiros-Martinez-Garcia-Shramov-Suss-Viswanathan, they go through the list of 105 families in Iskovskikh-Mori-Mukai classification of smooth Fano threefolds, and determine in each family whether a general member is K-(semi,poly)stable. The arguments involve many different techniques.

K-moduli spaces provides many explicit examples of moduli spaces. Mabuchi-Mukai and Spotti-Odaka-Sun gave explicit descriptions of the K-moduli spaces which compactify the moduli spaces parametrizing smooth K-stable Fano surfaces. In higher dimensions, there is only a small number of cases for which we have a complete explicit description of the K-moduli space. We have seen in (5) for cubic hypersurfaces of dimension at most 4, the K-moduli spaces are identical to the GIT moduli spaces. The same statement is expected to hold for cubic hypersurfaces in any dimension. However, for hypersurfaces of a degree larger than 3, the K-moduli usually is not the same as the GIT moduli. While we expect all smooth Fano hypersurfaces are K-(poly)stable, there is not even a conjectural picture on what should appear as their limits.

It is also interesting to consider the case of log pairs, for which all the previous main theorems still hold. A research project, led by Ascher-DeVleming-Liu, studies moduli spaces whose general members parametrize K-semistable log Fano pairs (X, tD) for a Fano variety X, $D \in |-rK_X|$ and a constant $0 \leq t < \frac{1}{r}$. When one varies t, the K-moduli spaces are connected by wall-crossings. This framework is applied to $(X, D) = (\mathbb{P}^3, \text{quartic K3})$. The K-stability moduli theory provides interpolating moduli spaces between the Satake compactification of the moduli of quartic K3 surfaces which corresponds to $t \to 1$ and the GIT moduli space which corresponds to $t \to 0$. As a consequence, they confirm a conjecture of Laza-O'Grady, who study these moduli spaces from a different perspective.

Analogous problems. Given the fundamental importance of Kähler-Einstein metrics, it is probably not surprising that people try to develop similar studies in other geometric settings. Many parts of our discussion above can be extended accordingly.

For any Fano variety X, we know it does not necessarily have a Kähler-Einstein metric. As a replacement, people study whether it admits a more general kind of canonical metrics, namely *Kähler-Ricci soliton*. Combining the works of Han-Li and Blum-Liu-Xu-Zhuang, *any* klt Fano variety X is known to admit a unique two-step degeneration process to a Kähler-Ricci soliton. More precisely, X first degenerates to (X_0, ξ_0) where ξ_0 is a vector field on X_0, such that (X_0, ξ_0) is K-semistable (in the sense of a Fano variety with a vector field). Moreover, (X_0, ξ_0) has a unique K-polystable degeneration (Y, ξ_Y), which admits a Kähler-Ricci soliton metric. When X is smooth, this degeneration of X to Y is also given by the Gromov-Hausdoff limit of the Kähler-Ricci flow, upon the Hamilton-Tian Conjecture solved by Tian-Zhang in dimension three, Chen-Wang and Bamler in a general dimension.

One can also consider the local setting. There has been a well-studied local analogue of Kähler geometry, namely the *Sasaki geometry*, which corresponds to the link of a cone singularity. The corresponding Yau-Tian-Donaldson Conjecture, which claims the existence of a *Sasaki-Einstein metric* on a Fano cone singularity $(x \in X, \xi)$ is equivalent to the K-polystability of $(x \in X, \xi)$, was proved by Collins-Székelyhidi when the singularity is isolated, and by Li and Huang in the general case.

A more challenging local question is to consider an *arbitrary* klt singularity $x \in X$. As the combination of a number of conjectural statements, made by Li and Li-Xu, the *Stable Degeneration Conjecture* predicts that any klt singularity $x \in X$ has a degeneration to a K-semistable Fano cone singularity $(x_0 \in X_0, \xi)$ and such a degeneration comes from the unique (up to rescaling) minimizer of the normalized volume function. We recall that the normalized function $\widehat{\mathrm{vol}}$, invented by Li, is defined on the space of valuations $\mathrm{Val}_{X,x}$ which consists of all valuations on $K(X)$ centered on x. There are many works, by Blum, Li, Liu, Xu, and Zhuang, which solved various parts of the Stable Degeneration Conjecture. As of the writing of this article, there is only one part remaining open. That is the claim that any minimizing valuation v of $\widehat{\mathrm{vol}}$, which is shown by Xu to be quasi-monomial, has a finitely generated associated graded ring. In other words, as the writing of this article, the local higher rank finite generation is still unknown!

What's next? I wish by now, with the complete solution of the Yau-Tian-Donaldson Conjecture for all Fano varieties, as well as the construction of the K-moduli spaces and many other results, it becomes clear that the interaction between the Kähler-Einstein Problem and the Minimal Model Program provides an exceptionally beautiful example to mathematician's dream of viewing mathematics as one unified discipline.

As an important chapter is about to be closed, there are good reasons to be optimistic about the future of the field. Many exciting directions remain unexplored. For instance, we lack a good understanding of the algebraic properties of polarized manifolds with a constant scalar curvature metric, whereas in differential geometry many deep results have been obtained for it as a natural extension of the Kähler-Einstein Problem. It is also tempting to find out whether the new K-stability perspective of higher dimensional geometry can shed light on the longstanding questions in the Minimal Model Program, e.g., termination of flips and the Abundance Conjecture. Nevertheless, as always, only time can tell how far the interaction will take us.

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References

[ABHLX20] Jarod Alper, Harold Blum, Daniel Halpern-Leistner, and Chenyang Xu, *Reductivity of the automorphism group of K-polystable Fano varieties*, Invent. Math. **222** (2020), no. 3, 995–1032, DOI 10.1007/s00222-020-00987-2. MR4169054

[AHLH18] Jarod Alper, Halpern-Leistner, and Jochen Heinloth, *Existence of moduli spaces for algebraic stacks*, arXiv:1812.01128 (2018).

[Ber16] Robert J. Berman, *K-polystability of ℚ-Fano varieties admitting Kähler-Einstein metrics*, Invent. Math. **203** (2016), no. 3, 973–1025, DOI 10.1007/s00222-015-0607-7. MR3461370

[BBJ21] Robert J. Berman, Sébastien Boucksom, and Mattias Jonsson, *A variational approach to the Yau-Tian-Donaldson conjecture*, J. Amer. Math. Soc. **34** (2021), no. 3, 605–652, DOI 10.1090/jams/964. MR4334189

[BCHM10] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. **23** (2010), no. 2, 405–468, DOI 10.1090/S0894-0347-09-00649-3. MR2601039

[BLX19] Harold Blum, Yuchen Liu, and Chenyang Xu, *Openness of K-semistability for Fano varieties*, arXiv:1907.02408, to appear in Duke Math. J. (2019).

[BX19] Harold Blum and Chenyang Xu, *Uniqueness of K-polystable degenerations of Fano varieties*, Ann. of Math. (2) **190** (2019), no. 2, 609–656. MR3997130

[CDS15] Xiuxiong Chen, Simon Donaldson, and Song Sun, *Kähler-Einstein metrics on Fano manifolds. III: Limits as cone angle approaches* 2π *and completion of the main proof*, J. Amer. Math. Soc. **28** (2015), no. 1, 235–278, DOI 10.1090/S0894-0347-2014-00801-8. MR3264768

[CP21] Giulio Codogni and Zsolt Patakfalvi, *Positivity of the CM line bundle for families of K-stable klt Fano varieties*, Invent. Math. **223** (2021), no. 3, 811–894, DOI 10.1007/s00222-020-00999-y. MR4213768

[Don02] S. K. Donaldson, *Scalar curvature and stability of toric varieties*, J. Differential Geom. **62** (2002), no. 2, 289–349. MR1988506

[Fuj19] Kento Fujita, *A valuative criterion for uniform K-stability of ℚ-Fano varieties*, J. Reine Angew. Math. **751** (2019), 309–338, DOI 10.1515/crelle-2016-0055. MR3956698

[Li17] Chi Li, *K-semistability is equivariant volume minimization*, Duke Math. J. **166** (2017), no. 16, 3147–3218, DOI 10.1215/00127094-2017-0026. MR3715806

[Li22] Chi Li, *G-uniform stability and Kähler-Einstein metrics on Fano varieties*, Invent. Math. **227** (2022), no. 2, 661–744. MR4372222

[LWX21] Chi Li, Xiaowei Wang, and Chenyang Xu, *Algebraicity of the metric tangent cones and equivariant K-stability*, J. Amer. Math. Soc. **34** (2021), no. 4, 1175–1214. MR4301561

[LX14] Chi Li and Chenyang Xu, *Special test configuration and K-stability of Fano varieties*, Ann. of Math. (2) **180** (2014), no. 1, 197–232. MR3194814

[LXZ21] Yuchen Liu, Chenyang Xu, and Ziquan Zhuang, *Finite generation for valuations computing stability thresholds and applications to K-stability*, arXiv:2102.09405, to appear in Ann. of Math. (2) (2021).

[Oda13] Yuji Odaka, *The GIT stability of polarized varieties via discrepancy*, Ann. of Math. (2) **177** (2013), no. 2, 645–661, DOI 10.4007/annals.2013.177.2.6. MR3010808

[Tia97] Gang Tian, *Kähler-Einstein metrics with positive scalar curvature*, Invent. Math. **130** (1997), no. 1, 1–37. MR1471884

[Tia15] Gang Tian, *K-stability and Kähler-Einstein metrics*, Comm. Pure Appl. Math. **68** (2015), no. 7, 1085–1156, DOI 10.1002/cpa.21578. MR3352459

[XZ20] Chenyang Xu and Ziquan Zhuang, *On positivity of the CM line bundle on K-moduli spaces*, Ann. of Math. (2) **192** (2020), no. 3, 1005–1068, DOI 10.4007/annals.2020.192.3.7. MR4172625

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